CS 341: Computer Graphics – Spring 2003
HW#3
Assigned : May 5, 2003

Due : 23:59 May 20, 2003

(Collection box outside 3561)

Please provide short written answers to the questions in the space provided. If you require extra space, you may staple additional pages to the back of your assignment. Feel free to talk over the problems with classmates but please answer the questions on your own.

Name: Key

1 Parametric Curves (10 pts, 30 min)

(a) (2 pts) In class, we establish the relation between the Hermite and Bezier geometry vectors. Given two endpoints P_0, P_3, and their respective endpoint tangents R_0, R_3, we can derive the Hermite basis matrix. In Bezier curves, we specify tangent vectors by two intermediate points. The Bezier control points are given as an ordered list of points $\{V_0, V_1, V_2, V_3\}$.

Let $\mathbf{G_h} = \begin{bmatrix} P_0 \\ P_3 \\ R_0 \\ R_3 \end{bmatrix}$ and $\mathbf{V} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$, we have $\mathbf{G_h} = \mathbf{M_{hb}}\mathbf{V}$.

Write $\mathbf{M_{hb}}$.

Ans:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

(b) (3 pts) Bezier curve is a special case of Hermite curve. Let $\mathbf{Q}(t)$ be a cubic Hermite curve, $\mathbf{T} = [t^3\ t^2\ t\ 1]$ where $t \in [0, 1]$. A Hermite curve $\mathbf{Q}(t)$ is define as

$$\mathbf{Q}(t) = \mathbf{T M_h G_h} \tag{1}$$

where $\mathbf{M_h}$ is the Hermite basis matrix. Given $\mathbf{M_h} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, use Equation (1) to derive the Bezier basis matrix $\mathbf{M_b}$. Show your working steps clearly.

Ans:

$$\begin{aligned} \mathbf{Q}(t) &= \mathbf{T M_h G_h} \\ &= \mathbf{T M_h M_{hb} V} \\ &= \mathbf{T M_b V} \end{aligned}$$

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

(c) One undesirable feature of Bezier curve is that the resulting curve does not interpolate the given control points. **Catmull-Rom** curve possesses this property. This problem asks you to derive the basis matrix for a Catmull-Rom curve, by establishing the relation between Bezier control points and Catmull-Rom control points.

Let $\mathbf{P} = \{P_0, P_1, P_2, P_3\}$ be an ordered set of 4 Catmull-Rom control points. We can express the relation between the Bezier and Catmull-Rom control points by:

$$
\begin{aligned}
V_0 &= P_1 \\
V_1 &= P_1 + \frac{\tau}{3}(P_2 - P_0) = -\frac{1}{6}P_0 + P_1 + \frac{1}{6}P_2 \\
V_2 &= P_2 - \frac{\tau}{3}(P_3 - P_1) = \frac{1}{6}P_1 + P_2 - \frac{1}{6}P_3 \\
V_3 &= P_2
\end{aligned}
$$

Let $\mathbf{V} = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix}$ and $\mathbf{P} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$.

Assume the tension control $\tau = \frac{1}{2}$.

(i) (3 pts) Write the 4×4 transformation matrix $\mathbf{X}$ such that

$$\mathbf{V} = \mathbf{X}\mathbf{P}$$

Ans:

$$
\begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ V_3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 & 6 & 0 & 0 \\ -1 & 6 & 1 & 0 \\ 0 & 1 & 6 & -1 \\ 0 & 0 & 6 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}
$$

(ii) (2 pts) Use Equation (1) and your answer in (i) to show that the basis matrix for a Catmull-Rom curve is

$$\frac{1}{2}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

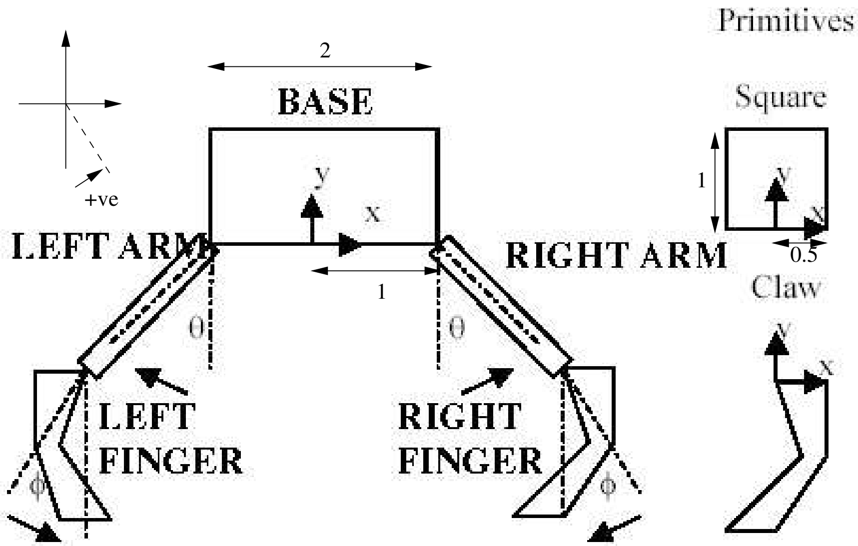
Ans:

$$\begin{aligned} \mathbf{Q}(t) &= \mathbf{T}\mathbf{M_h}\mathbf{G_h} \\ &= \mathbf{T}\mathbf{M_b}\mathbf{V} \\ &= \mathbf{T}\mathbf{M_b}\mathbf{X}\mathbf{P} \\ &= \mathbf{T}\mathbf{M_{CR}}\mathbf{P} \end{aligned}$$

$$\begin{aligned} \mathbf{M_{CR}} &= \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 6 & 0 & 0 \\ -1 & 6 & 1 & 0 \\ 0 & 1 & 6 & -1 \\ 0 & 0 & 6 & 0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \end{aligned}$$

2 Hierarchical Modeling (13 pts, 30 min)

You are modeling a hanging pincer as shown below. There are two primitives available to you: **Square** and **Claw**. Please note the local coordinate system for each primitive.



The transformations available are:

- $R(\theta)$ – rotate by θ degrees (counter-clockwise).
- $T(t_x, t_y)$ – translate by t_x, t_y.
- $S(s_x, s_y)$ – scale by s_x, s_y.
- R_y – reflect about the y-axis.

The arrows in the figure indicate the direction of rotation that θ and ϕ for each piece. The base of the pincer is twice as wide (along the x-axis) as the **Square**. Each arm is $\frac{1}{3}$ as wide as the square, and $\frac{3}{2}$ as long.

(a) (4 pts) Define a draw function each for BASE and ARM in OpenGL. For example, since the FINGER uses an unscaled claw, the draw function looks like:

```
finger() {
  glPushMatrix ();
  Claw ();
  glPopMatrix ();
}
```

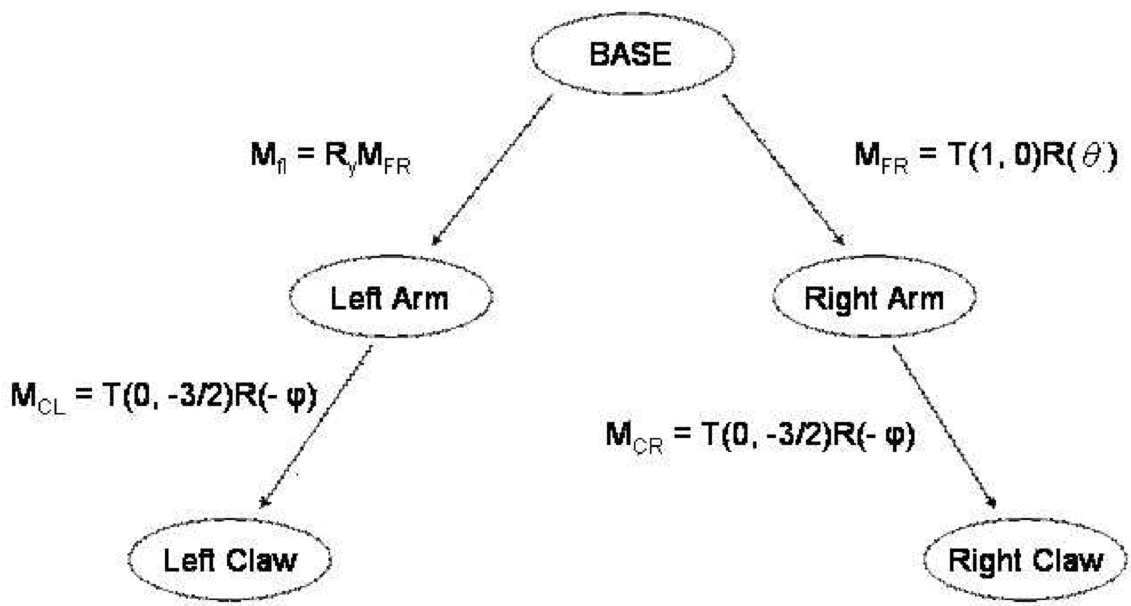
Ans:

```
Base() {
  glPushMatrix ();
  glScale (2, 1, 1);
  Square ();
  glPopMatrix ();
}

Arm() {
  glPushMatrix ();
  glScale (1/3, 3/2, 1);
  Square ();
  glPopMatrix ();
}
```

(b) (7 pts) Draw a directed acyclic graph (DAG) to specify the pincer. For each edge in your DAG, write the expression for the transformation using only the symbols given on the previous page. This transformation should be the transformation required to position the piece relative to the next node, as taught in class. Each node in the DAG should be one of the primitives given to you. Leave θ and ϕ as symbols, these are the parameters you intend to animate. Assume **Square** is one unit in size.

To guarantee full credits, you need to give the simplest expression. (Hint: Please use R_y in one of the transformation.)



(c) (2 pts) What is the entire sequence of transformation applied to the LEFT FINGER? You do not need to multiply the matrices. Please give the expression.

Ans:

$\mathbf{R_y}\mathbf{T}(1,0)\mathbf{R}(\theta)\mathbf{T}(0,-\frac{3}{2})\mathbf{R}(-\phi)$

3 Perspective Projection (20 pts, 45 min)

Suppose we are observing the perspective projection of a 3-D line segment P_1P_2 originating at some $P_1 = [a_1\ a_2\ a_3]^T$, of length δ and of direction $d = [d_x\ d_y\ d_z]^T$, with $||d|| = 1$.

(a) (5 pts) What is the parametric equation of this 3-D line segment? Use only the symbols defined above.

Ans:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = \mathbf{P_1} + \lambda d$$

$$0 \leq \lambda \leq \delta$$

(b) (5 pts) Let the observed projection of P_1 be $p_1 = [u_1\ v_1]^T$, and P_2 be $p_2 = [u_2\ v_2]^T$. Hence, p_1 and p_2 are the image points. What are the coordinates of p_1 and p_2 in the *3D camera coordinate system*, if the image plane is at a distance f in front of the center of projection, which is the origin of this coordinate system?

Write your answer in the space provided. (Hint: this is simple. Don't complicate things!)

Your answer:

$$p_1 = (u_1, v_1, f)^T$$

$$p_2 = (u_2, v_2, f)^T$$

(c) (10 pts) We want to express the 3D coordinates of P_1 and P_2 from their projections p_1 and p_2 and knowledge of d and δ (i.e., p_1, p_2, d, and δ are known). Here, the focal length is unknown. Please derive *the 4 equations* so that the 4 unknowns a_1, a_2, a_3, f can be solved. Show your work and write your 4 equations in the space provided below. Give the simplest form to guarantee full credits.

Your answer:

$$\begin{aligned} f &= (u_2 v_1 - v_1 u_2) d_z \\ a_1 &= u_1 a_3 / f \\ a_2 &= v_1 a_3 / f \\ a_3 &= \frac{v_2 d_x - u_2 d_y}{(u_2 - u_1) d_y - (v_2 - v_1) d_x} \delta d_z \end{aligned}$$

From:

$$\begin{aligned} u_1 &= f \frac{a_1}{a_3} \\ v_1 &= f \frac{a_2}{a_3} \\ u_2 &= (\frac{a_1 + \delta d_x}{a_3 + \delta d_z}) f \\ v_2 &= (\frac{a_2 + \delta d_y}{a_3 + \delta d_z}) f \end{aligned}$$

4 Chaikin's Algorithm and Quadratic B-Splines (17 pts, 45 min)

In class, we also learned subdivision curves. It can be shown that **Chaikin's algorithm** produces the equivalent **quadratic B-spline**. The problem asks you to prove this result step-by-step.
A quadratic B-spline is given by

$$\mathbf{S}(t) = [1\ t\ t^2]\mathbf{M} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

To make your life easier, the quadratic B-spline basis matrix and its inverse are given as follows:

$$\mathbf{M} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}, \mathbf{M}^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix}$$

To prove the result, we re-parameterize $\mathbf{S}(t)$, and divide $\mathbf{S}(t)$ into two halves: $\mathbf{S}_{[0,\frac{1}{2}]}(t)$ and $\mathbf{S}_{[\frac{1}{2},1]}(t)$. In (a), we only consider $\mathbf{S}_{[0,\frac{1}{2}]}(t)$. The other case is similar.

$$\mathbf{S}_{[0,\frac{1}{2}]}(t) = \mathbf{S}(\frac{t}{2}) = [1\ \frac{t}{2}\ (\frac{t}{2})^2]\mathbf{M} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} \tag{2}$$

(a) (9 pts) Starting from Equation (2), show that $\mathbf{S}_{[0,\frac{1}{2}]}(t)$ can be expressed as

$$\mathbf{S}_{[0,\frac{1}{2}]}(t) = [1\ t\ t^2]\mathbf{M} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}$$

where $\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix} = \mathcal{M}_L \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$. What is $\mathcal{M}_L$ (which is known as the "left splitting matrix")?

You can show your derivation on the next empty page since there is not much space left on this page.

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Ans:

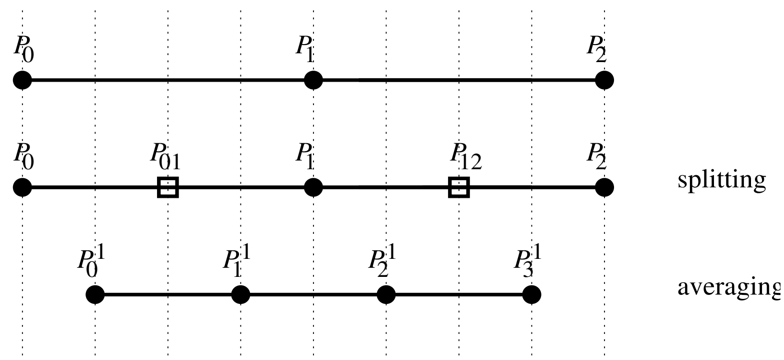
$$\begin{aligned}
\mathbf{S}_{[0,\frac{1}{2}]}(t) &= \mathbf{S}(\frac{t}{2}) \\
&= [1\ \frac{t}{2}\ (\frac{t}{2})^2]\mathbf{M}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} \\
&= [1\ t\ t^2]\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}\mathbf{M}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} \\
&= [1\ t\ t^2]\mathbf{M}\mathbf{M}^{\mathbf{-1}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}\mathbf{M}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} \\
&= [1\ t\ t^2]\mathbf{M}\mathbf{M}_{\mathbf{L}}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{M}_{\mathbf{L}} &= \mathbf{M}^{\mathbf{-1}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}\mathbf{M} \\
&= \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}
\end{aligned}$$

(b) (5 pts) Now, consider the Chaikin's algorithm on subdivision curves. Given a set of 3 vertices $\{P_0, P_1, P_2\}$, $P_0 \neq P_2$, after the splitting and averaging step we discussed in class, a new set of 4 vertices is obtained. Let this set be $\{P_0^1, P_1^1, P_2^1, P_3^1\}$ (Thus, the superscript i denotes the i-th subdivision.)

Let $\begin{bmatrix} P_0^1 \\ P_1^1 \\ P_2^1 \\ P_3^1 \end{bmatrix} = \mathcal{C} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$. Hence, $\mathcal{C}$ is a 4×3 transformation matrix. What is $\mathcal{C}$?

(Hint: Consider the following illustration that shows that relation between the old and new vertices)



Ans:

$$
\begin{aligned}
P_0^1 &= \frac{3}{4}P_0 + \frac{1}{4}P_1 \\
P_1^1 &= \frac{1}{4}P_0 + \frac{3}{4}P_1 \\
P_2^1 &= \frac{3}{4}P_1 + \frac{1}{4}P_2 \\
P_3^1 &= \frac{1}{4}P_1 + \frac{3}{4}P_2
\end{aligned}
$$

$$
\begin{bmatrix} P_0^1 \\ P_1^1 \\ P_2^1 \\ P_3^1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}
$$

(c) (3 pts) Compare the results in (a) and (b) of this problem. In (a), we only consider $\mathbf{S}_{[0,\frac{1}{2}]}(t)$ and produce a 3×3 basis matrix $\mathcal{M}_L$. By comparing the 3×3 matrix $\mathcal{M}_L$ and the 4×3 matrix $\mathcal{C}$, what is the 3×3 basis matrix $\mathcal{M}_R$ for $\mathbf{S}_{[\frac{1}{2},1]}(t)$, the right splitting matrix?

Ans:

$$
\begin{aligned}
\mathbf{M_R} &= \frac{1}{4}\begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \\
&= lower\ 3 \times 3\ submatrix\ of\ \mathbf{C}
\end{aligned}
$$

5 Bézier and Hermite curve

Suppose the equations relating the Hermite geometry to Bezier geometry were of the form $R_1 = \beta(P_2 - P_1)$, $R_4 = \beta(P_4 - P_3)$. Consider the four equally spaced Bezier control points $P_1 = (0,0), P_2 = (1,0), P_3 = (2,0), P_4 = (3,0)$. Show that, for the parametric curve $Q(t)$ to have constant velocity from P_1 to P_4, the coefficient β must be equal to 3.

In order to have constant velocity throughout $Q(t)$, $t \in [0,1]$, the acceleration must be zero for all t.
A Hermite curve is given by

$$
\begin{aligned}
Q(t) &= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -2 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} \\
Q''(t) &= \begin{bmatrix} 6t & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -2 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix} \\
&= \begin{bmatrix} 12t-6 & -12t+6 & 6t-4 & 6t-2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_4 \\ R_1 \\ R_4 \end{bmatrix}
\end{aligned}
$$

In order to make this equation zero for all t, β must be 3.

6 Catmull-Rom Splines

Find the blending functions for the Catmull-Rom splines we learned in class. Do they sum to 1, and are there everyone nonzero? Show your derivation. If not, the spline is not contained in the convex hull of the points.

$$
\begin{aligned}
Q(t) &= \begin{bmatrix} t^3 & t^2 & t^1 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \\
&= \mathbf{B_c}(t) \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \\
\mathbf{B_c}(t) &= \frac{1}{2} \begin{bmatrix} -t^3 + 2t^2 - t \\ 3t^3 - 5t^2 + 2 \\ -3t^3 + 4t^2 + t \\ t^3 - t^2 \end{bmatrix}
\end{aligned}
$$

Sum the four blending functions:

$$
\begin{aligned}
& \frac{1}{2}(-t^3 + 2t^2 - t + 3t^3 - 5t^2 + 2 - 3t^3 + 4t^2 + t + t^3 - t^2) \\
= \; & 1
\end{aligned}
$$