Parametric Curves

Reading

Required

- Hearn & Baker, 10.6 -10.9
- Foley, 11.2

Optional

- Bartels, Beatty, and Barsky. An Introduction to Splines for use in Computer Graphics and Geometric Modeling. 1987.
- Farin. Curves and Surfaces CAGD: A Practical Guide. 4th ed. 1997.

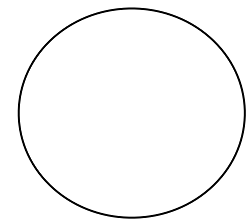
Curves Before Computers

- The loftsman’s or carpenter’s spline:
  - long, narrow strip of wood or metal
  - shaped by lead weights called “ducks”
  - gives curves with second-order continuity, usua
- Used for designing cars, ships, airplanes etc.
- But curves based on physical artifacts cannot be replicated well, since there is no exact definition of what the curve is.
- Around 1960, a lot of industrial designers were working on this problem.

Motivation for curves

- What do we use curves for?
  - building models
  - movement paths
  - animation

Mathematical Curve Representation



- Explicit $y = f(x)$
  - what if the curve is not a function?
- Implicit $f(x,y,z) = 0$

  $x^2 + y^2 - R = 0$
  - hard to work with
- Parametric $(x(u), y(u))$
  - easier to work with

$x(u) = \cos 2\pi u$

$y(u) = \sin 2\pi u$

Parametric Polynomial Curves

- We'll use parametric curves where the functions are all polynomials in the parameter.

$$x(u) = \sum_{k=0}^{n} a_k u^k$$

$$y(u) = \sum_{k=0}^{n} b_k u^k$$

- Advantages
  - easy (and efficient) to compute
  - infinitely differentiable

Cubic Curves

- Fix n=3
- For simplicity we define each cubic function with the range

$$0 \le t \le 1$$

$$\mathbf{Q}(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}$$

x(t) = $Q_x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$

y(t) = $Q_y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$

z(t) = $Q_z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$

Compact Representation

- Place all coefficients into a matrix:

$$\mathbf{C} = \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix} \quad \mathbf{T} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}$$

$$Q(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} = \mathbf{T} \cdot \mathbf{C}$$

$$\frac{d}{dt} Q(t) = Q'(t) = \frac{d}{dt}(\mathbf{T} \cdot \mathbf{C}) = \frac{d}{dt}\mathbf{T} \cdot \mathbf{C} + \mathbf{T} \cdot \frac{d}{dt}\mathbf{C} = \begin{bmatrix} 3t^2 & 2t & 1 & 0 \end{bmatrix} \cdot \mathbf{C}$$

Controlling the Cubic

- **Q**: How many constraints do we need to specify fully to determine the cubic **Q**(t)?

Constraining the Cubics

- Redefine **C** as a product of the **basis matrix M** and the 4-element column vector of constraints or **geometry vector G**.

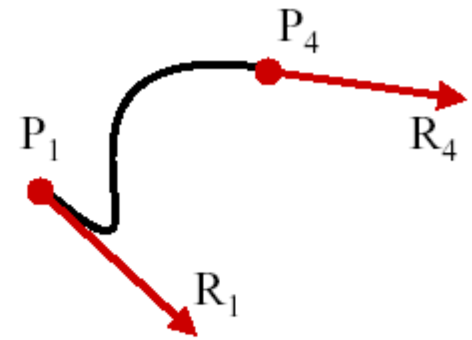
$$\mathbf{C} = \mathbf{M} \cdot \mathbf{G}$$

$$\begin{aligned}
\mathbf{Q}(t) &= \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix} \\
&= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} G_{1x} & G_{1y} & G_{1z} \\ G_{2x} & G_{2y} & G_{2z} \\ G_{3x} & G_{3y} & G_{3z} \\ G_{4x} & G_{4y} & G_{4z} \end{bmatrix} \\
&= \mathbf{T} \cdot \mathbf{M} \cdot \mathbf{G}
\end{aligned}$$

Hermite Curves

- Determined by
  - endpoints P_1 and P_4.
  - tangent vectors at the endpoints R_1 and R_4.
- So,

$$\mathbf{Q}(t) = \mathbf{T}\ \mathbf{M}_h\ \mathbf{G}_h$$



where

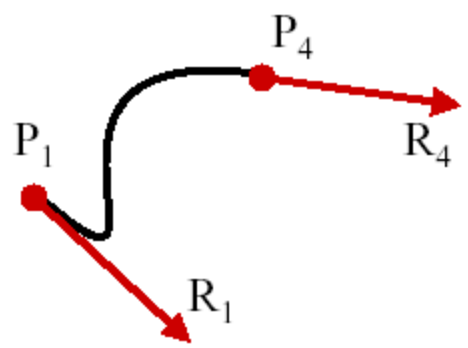
$$\mathbf{G}_h = \begin{array}{c@{\quad}l} \begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{4x} & P_{4y} & P_{4z} \\ R_{1x} & R_{1y} & R_{1z} \\ R_{4x} & R_{4y} & R_{4z} \end{bmatrix} & \begin{array}{l} \leftarrow P_1 \\ \leftarrow P_4 \\ \leftarrow R_1 \\ \leftarrow R_4 \end{array} \end{array}$$

Computing Hermite Basis Matrix

- The constraints on Q(0) and Q(1) are found by direct substitution:

$$P_1 = \mathbf{Q}(0) = [\ 0\ 0\ 0\ 1\ ]\ \mathbf{M}_h\ \mathbf{G}_h$$

$$P_4 = \mathbf{Q}(1) = [\ 1\ 1\ 1\ 1\ ]\ \mathbf{M}_h\ \mathbf{G}_h$$



- so constraints on tangents are

$$R_1 = \mathbf{Q'}(0) = [\ 0\ 0\ 1\ 0\ ]\ \mathbf{M}_h\ \mathbf{G}_h$$

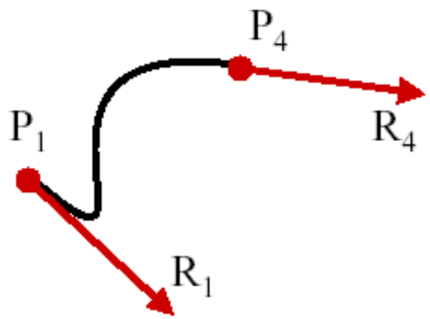
$$R_4 = \mathbf{Q'}(1) = [\ 3\ 2\ 1\ 0\ ]\ \mathbf{M}_h\ \mathbf{G}_h$$

- Note: tangents are defined by

$$\mathbf{Q'}(t) = [\ 3t^2\ \ 2t\ \ 1\ 0\ ]\ \mathbf{M}_h\ \mathbf{G}_h$$

Computing Hermite Basis Matrix

- Collecting all constraints we get



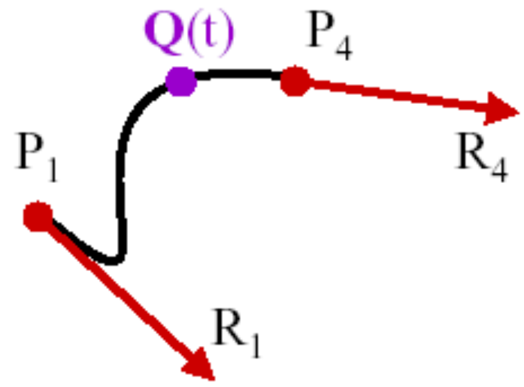
$$\begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{4x} & P_{4y} & P_{4z} \\ R_{1x} & R_{1y} & R_{1z} \\ R_{4x} & R_{4y} & R_{4z} \end{bmatrix} = \mathbf{G}_h = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \mathbf{M}_h \cdot \mathbf{G}_h$$

- So

$$\mathbf{M}_h = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Computing a Point

- Given two endpoints (P_1,P_4) and two endpoint tangents (R_1,R_4):



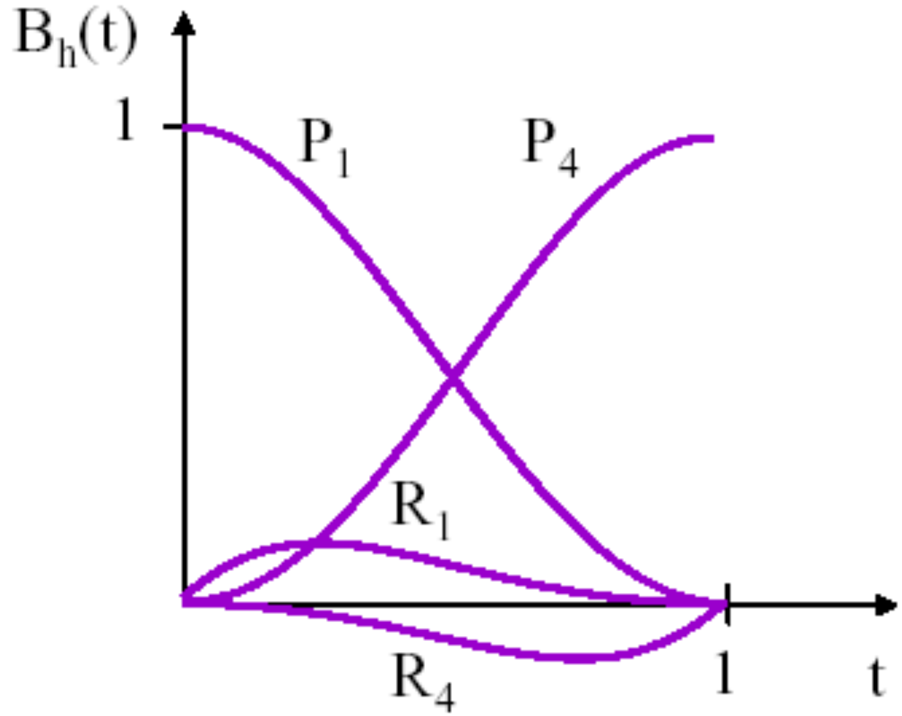
$$\mathbf{Q}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix}$$

Blending Functions

- Polynomials weighting each element of the geometry vector

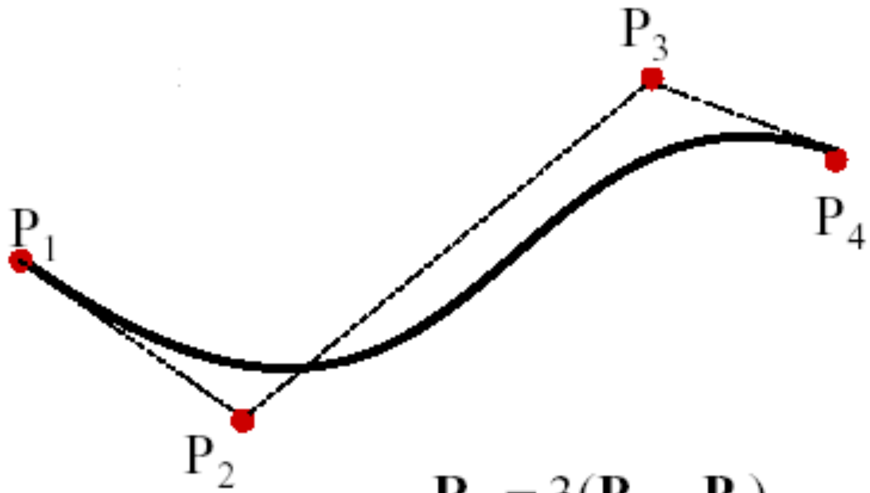
$$\mathbf{Q}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix}$$

$$= \mathbf{B}_h(t) \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix}$$



Bezier Curves

- Indirectly specify the tangent vectors by specifying two intermediate points.



$$\mathbf{R}_1 = 3(\mathbf{P}_2 - \mathbf{P}_1)$$

$$\mathbf{R}_4 = 3(\mathbf{P}_4 - \mathbf{P}_3)$$

$$\mathbf{G}_b = \begin{bmatrix} P_{1x} & P_{1y} & P_{1z} \\ P_{2x} & P_{2y} & P_{2z} \\ P_{3x} & P_{3y} & P_{3z} \\ P_{4x} & P_{4y} & P_{4z} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix}$$

Bezier Basis Matrix

- Establish the relation between the Hermite and the Bezier geometry vectors:

$$\mathbf{G}_h = \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_4 \\ \mathbf{R}_1 \\ \mathbf{R}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix} = \mathbf{M}_{hb} \mathbf{G}_b$$

Bezier Basis Matrix

$$\mathbf{Q}(t) = \mathbf{T} \cdot \mathbf{M}_h \cdot \mathbf{G}_h = \mathbf{T} \cdot \mathbf{M}_h \cdot \left(\mathbf{M}_{hb} \cdot \mathbf{G}_b\right)$$
$$= \mathbf{T} \cdot \left(\mathbf{M}_h \cdot \mathbf{M}_{hb}\right) \cdot \mathbf{G}_b = \mathbf{T} \cdot \mathbf{M}_b \cdot \mathbf{G}_b$$

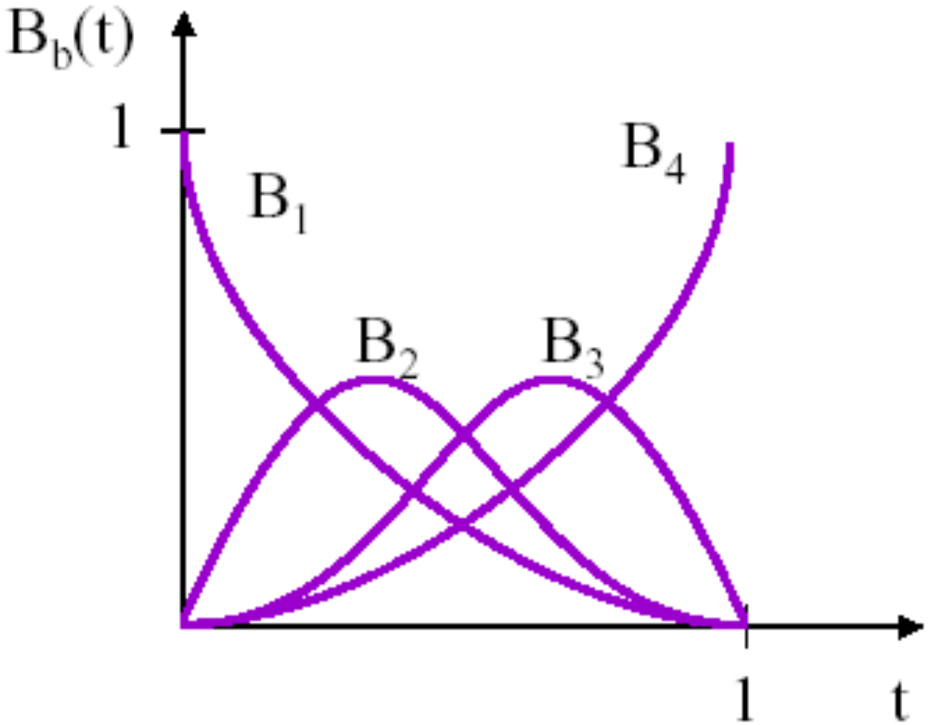
$$\mathbf{M}_b = \mathbf{M}_h \mathbf{M}_{hb} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{Q}(t) = \mathbf{T} \cdot \mathbf{M}_b \cdot \mathbf{G}_b$$

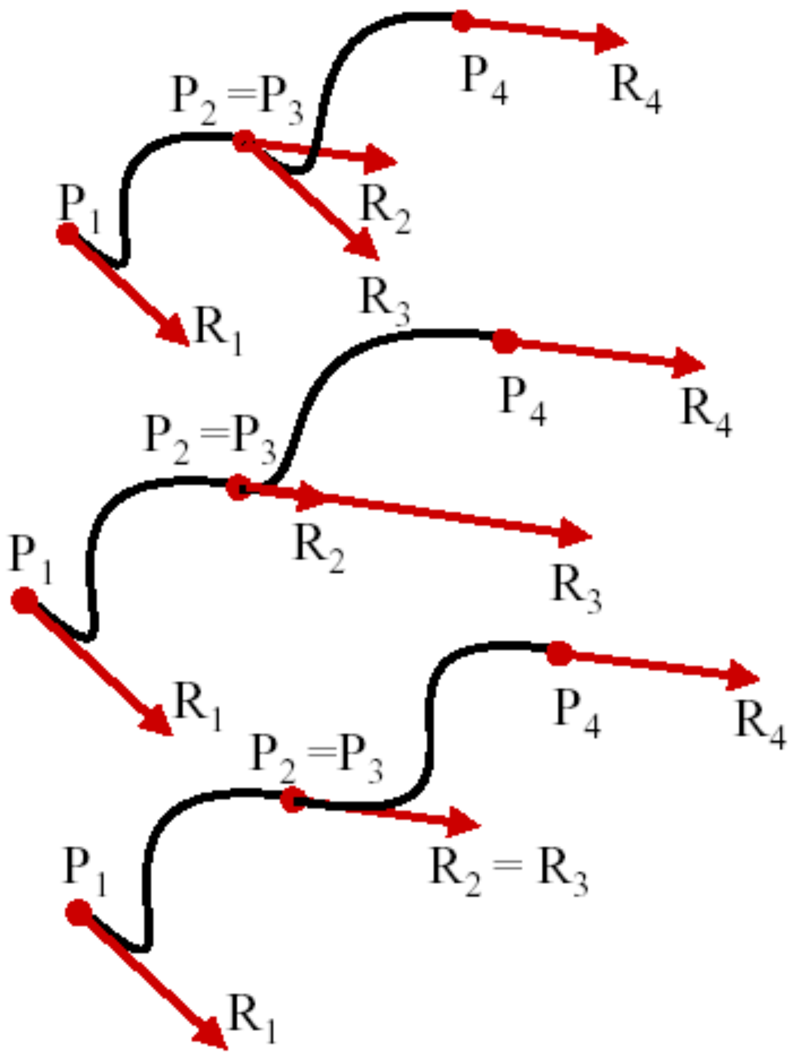
Bezier Blending Functions

- A.k.a. Bernstein polynomials

$$\mathbf{Q}(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix} = \mathbf{B}_b(t) \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{bmatrix}$$



Continuity of Splines



- Splines: 2 or more curves are concatentated together
- C^0: points coincide, velocity don't
- G^1: points coincide, velocities have the same direction.
- C^1: points and velocities coincide.
- **Q**: What's C^2?

Summary

- Use of parametric functions for curve modeling
- Enforcing constraints on cubic functions
- The meaning of basis matrix and geometry vectors
- General procedure for computing the basis matrix
- Properties of Hermite and Bezier splines
- The meaning of blending functions
- Enforcing continuity across multiple curve segments