Heap Sort

Instructor: Krishna Venkatasubramanian

CSC 212

Announcements

- Repeated submissions for assignment 1
  - You can make multiple submissions on Gradescope, the last submission will determine your final grade
  - Make sure the assignment file is **"assignment1.py"** and the method names match the specifications.
- Assignment 1 deadline moved to Sunday (Oct 27) 11:59PM
- Quiz 3 on Thursday (Oct 24)
  - Will cover material from classes on Oct 8, Oct 10, and Oct 22 (i.e., including today's stuff)
- Assignment 2 OUT on Thursday

Sorting: Problem Definition

- **Sorting takes an unordered collection and makes it an ordered one.**

Input:

1	2	3	4	5	6
77	42	35	12	101	5

Output:

1	2	3	4	5	6
5	12	35	42	77	101

How can we sort an array using divide and conquer approach?

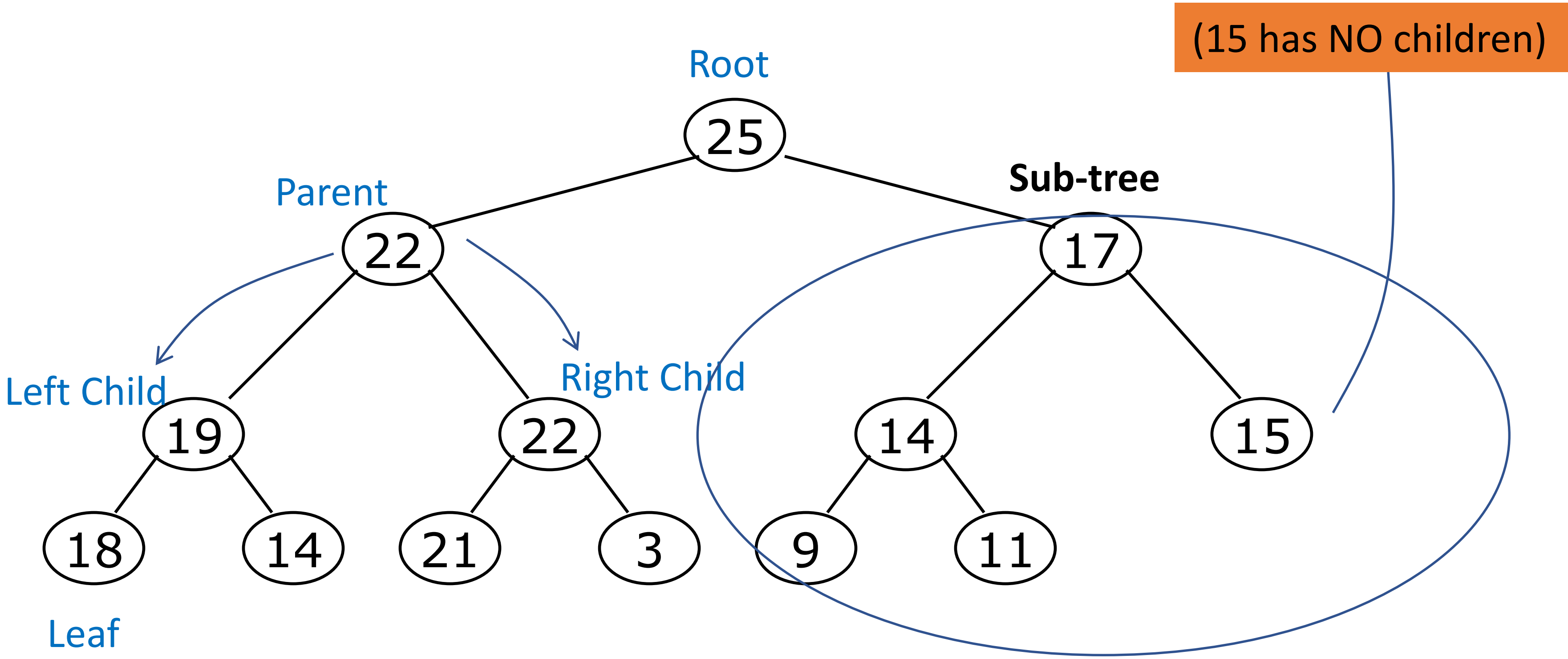
Sorting Algorithms

- *Insertion Sort --- covered already*
- *Bubble Sort --- covered already*
- *Selection Sort --- covered already*
- *Merge Sort --- covered already*
- *Quick Sort --- covered already*
- *Linear-Time Sort --- covered already*
- **Heap Sort**

Why study Heapsort?

- It is a well-known, traditional sorting algorithm you will be expected to know
- Heapsort is *always* **O(n log n)**
    - Quicksort is usually O(n log n) but in the worst case slows to $O(n^2)$
    - **Quicksort is generally faster, but Heapsort has the guaranteed O(n log n) time** and can be used in time-critical applications
- Heapsort is a *really cool* algorithm!

We start with: A Binary Tree

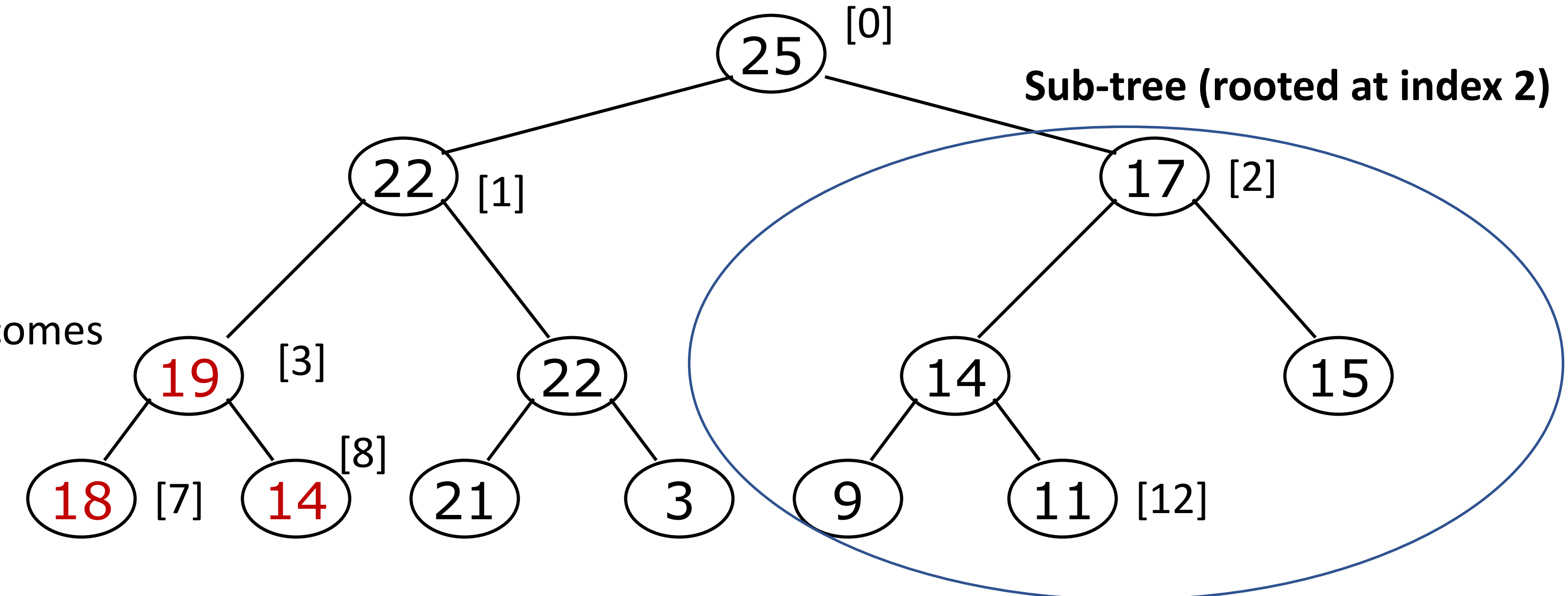


Tree is called binary – because every element has two children

Leaf nodes have children which are basically Null pointers (empty).

We just don't show them.

An array and a Binary Tree



A[0] becomes root

0	1	2	3	4	5	6	7	8	9	10	11	12
25	22	17	19	22	14	15	18	14	21	3	9	11

Array -- A

A[1] left-child of root

A[2] right-child of root

- The **left child** of index i is at index **2*i+1**
  - **[left child's parent** is index **floor(i/2)]**
- The **right child** of index i is at index **2*i+2**
  - **[right child's parent** is **index floor(i/2)-1]**
- Example: the children of node 3 (19) are 7 (18) and 8 (14)

HeapSort and Binary Trees

- HeapSort works on arrays (lists), but to do the actual sorting we use a **mental model of a binary tree**

- This means we view the array as a binary tree
  - Like the previous slide

- This DOES NOT mean that we are converting the array into a tree structure in memory
  - We just view it as such --- that is we track the parent/child indices

- **Once we view the array as a binary tree, we then understand the sorting through the tree structure**

Again: Things to Remember

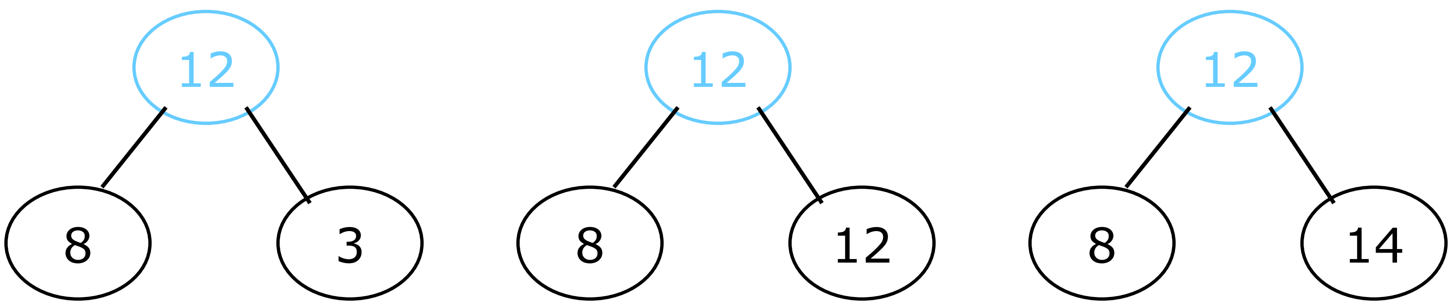
- The rest of the slides will **focus on the BinaryTree view of the array**

- **Even though we are manipulating the BinaryTree structure, it's not an actual tree structure in memory**

- In **reality we are changing the input array**
  - It's just easier to see what's happening when we view the array as a BinaryTree

The MAX-Heap property

- A node has the **MAX-Heap property** if the **value in the node is >= the values in its children**



Blue node <u>has</u> MAX-heap property

Blue node <u>has</u> MAX-heap property

Blue node <u>does not have</u> MAX-heap property

- All **leaf nodes automatically have the MAX-Heap property**
- A **binary tree is a MAX-heap if *all* nodes in it have the MAX-heap property**
- <u>You can similarly have MIN-heap(with the smallest element as parent)</u>

MAX-Heap and Min-Heaps are generally called HEAPS

HeapSort: Plan of attack

- First, we **turn any <u>portion</u> of a binary tree** (which we imagine our array to be) **into a MAX-*<u>heap</u>* (data structure)**

- Next, we will learn how to **turn a binary tree *back* into a MAX-heap after it has been changed** in a certain way

- Finally we will see how to use these ideas to sort an *array*

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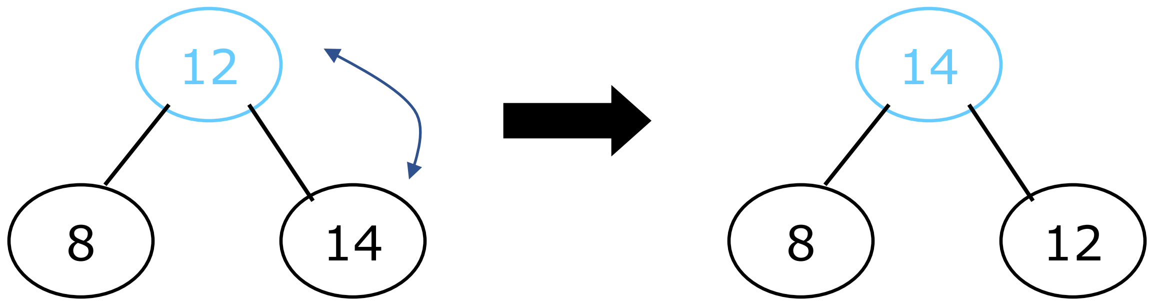
MAX-Heapifying Step

- Given a node that does not have the MAX-heap property, you can give it the MAX-heap property by exchanging its value with the value of the larger child



Blue node does not have MAX-heap property

Blue node has MAX-heap property

- Notice that the **child may have *lost* the MAX-heap property**
- **We may to do this more than once**
- **This <u>is a single step</u> in converting an binary tree into a MAX-Heap**

MAX-Heapify

- Given the MAX-Heapifying step (for a triad of parent and it's two children)

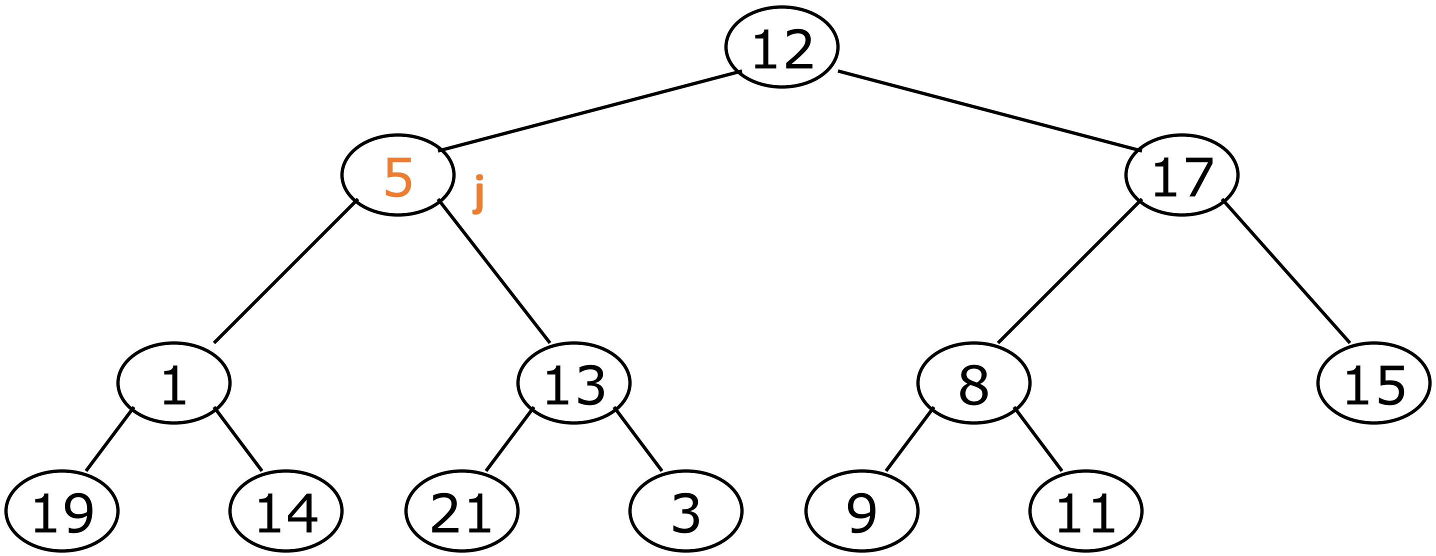
- We now, present MAX-Heapify algorithm, that takes an **Array (A)** and an **index (j)** and produces

  a MAX-Heap for the sub-tree rooted index j

MAX-Heapify

j

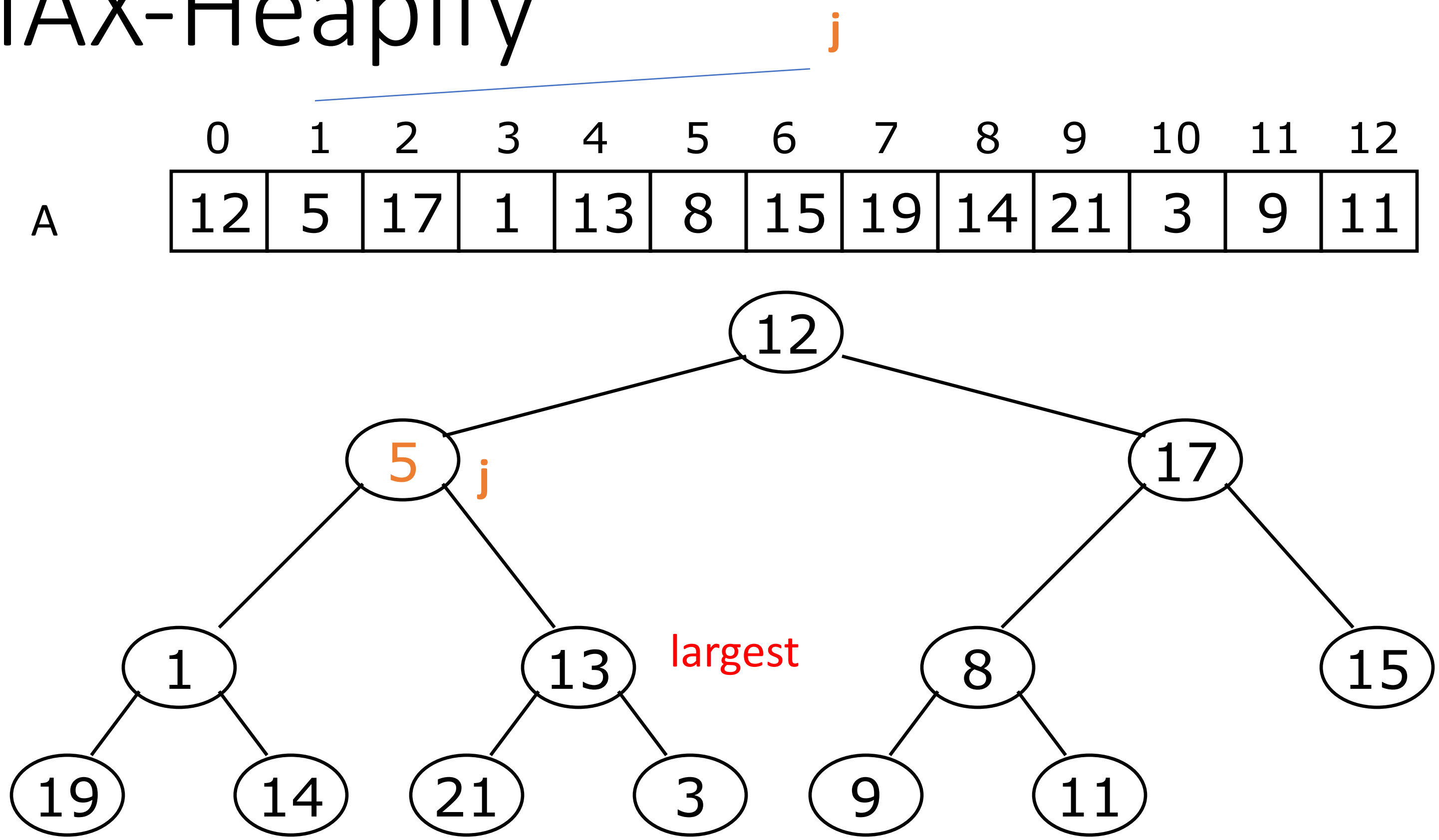
	0	1	2	3	4	5	6	7	8	9	10	11	12
A	12	5	17	1	13	8	15	19	14	21	3	9	11



Function call **MAX-*Heapify((A, j)***

- Compare A[j] (if not leaf) and it's children
  - find index **largest** of the largest of the three
- If **largest NOT j**, then **swap A[largest] and A[j]**
- **MAX-Heapify(A,largest)**

MAX-Heapify



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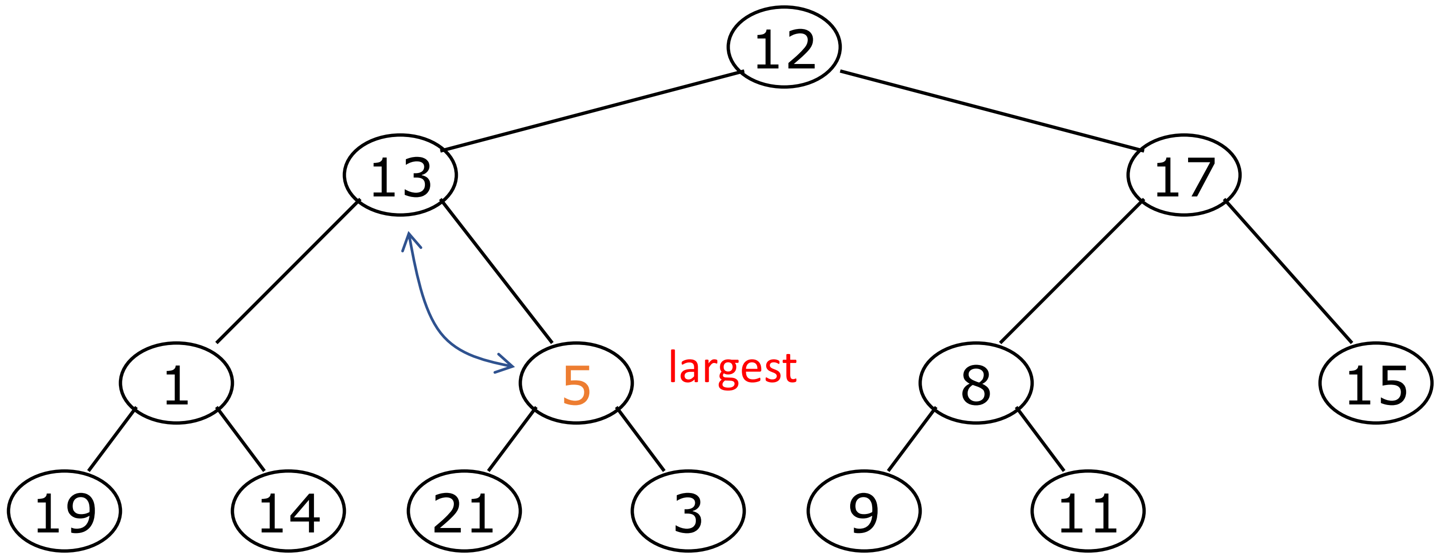

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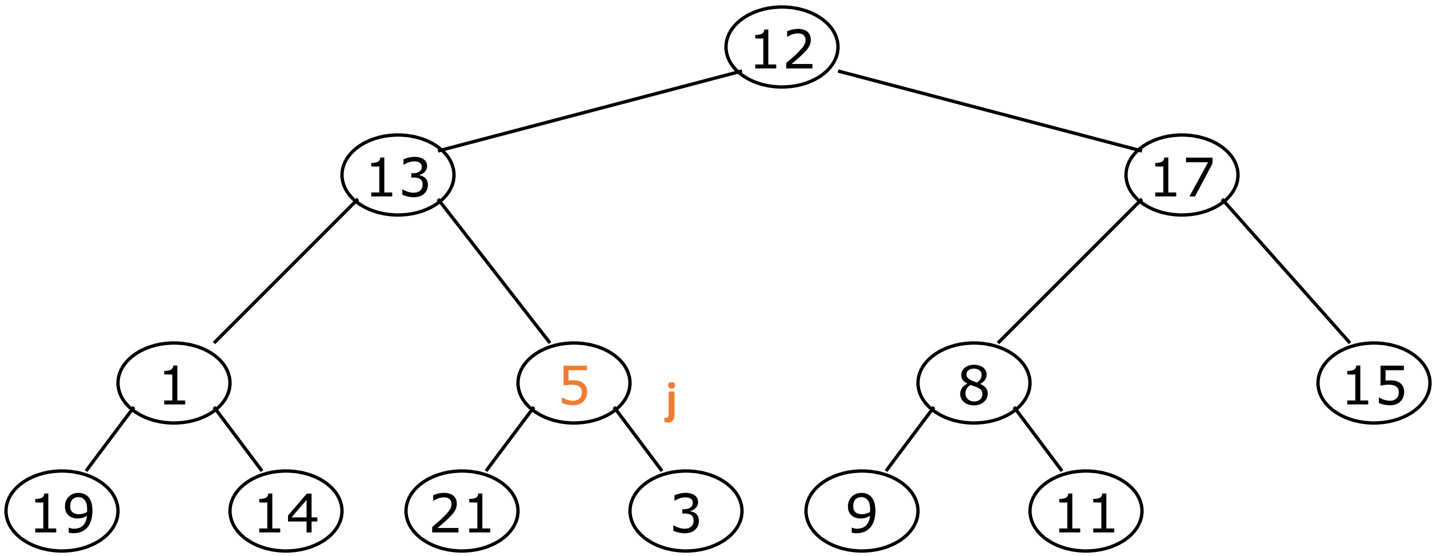

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MAX-Heapify

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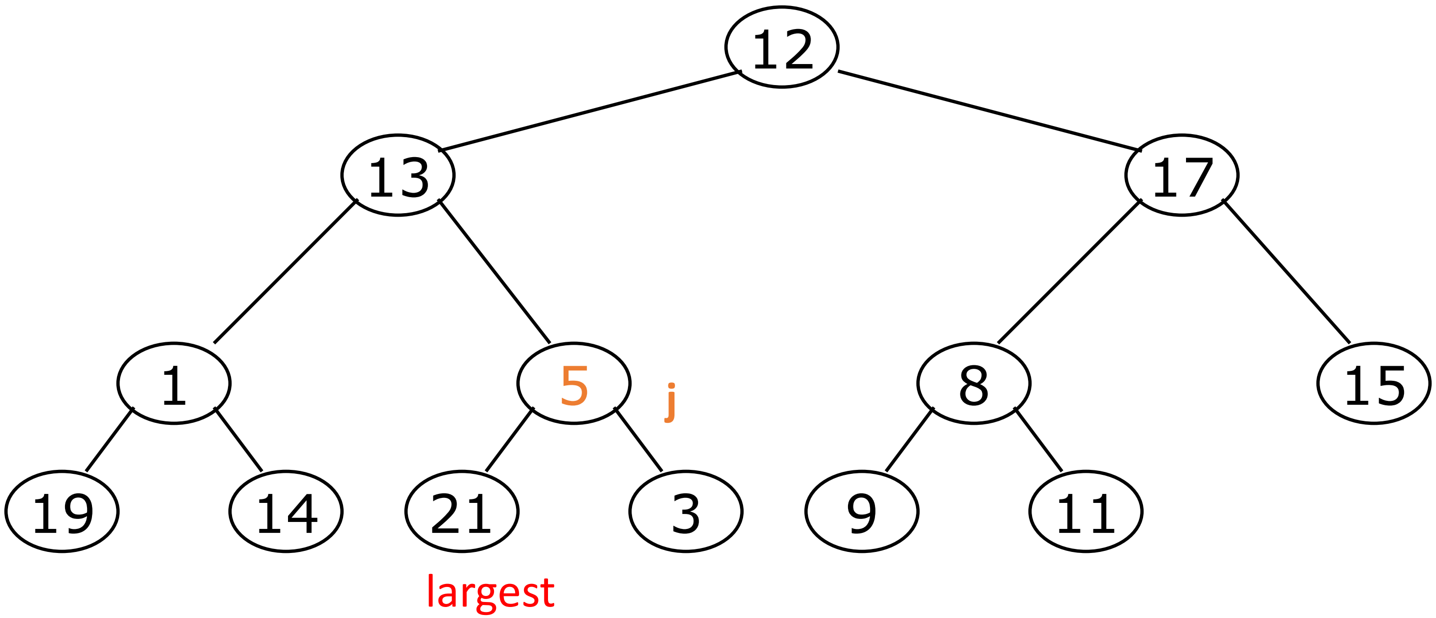

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MAX-Heapify

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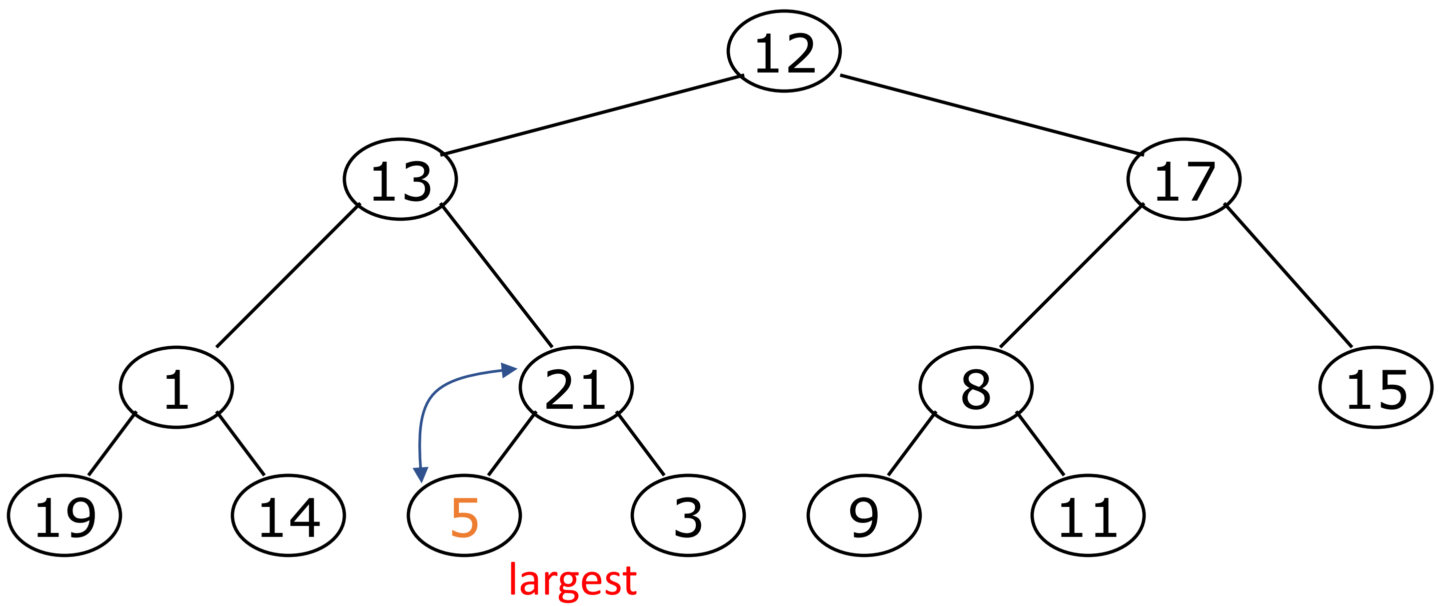

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HeapSort: Plan of attack

- First, we **turn any portion of a binary tree** (which we imagine our array to be) **into a MAX-*heap* (data structure)**

- Next, we will learn how to **turn a binary tree *back* into a MAX-heap after it has been changed** in a certain way

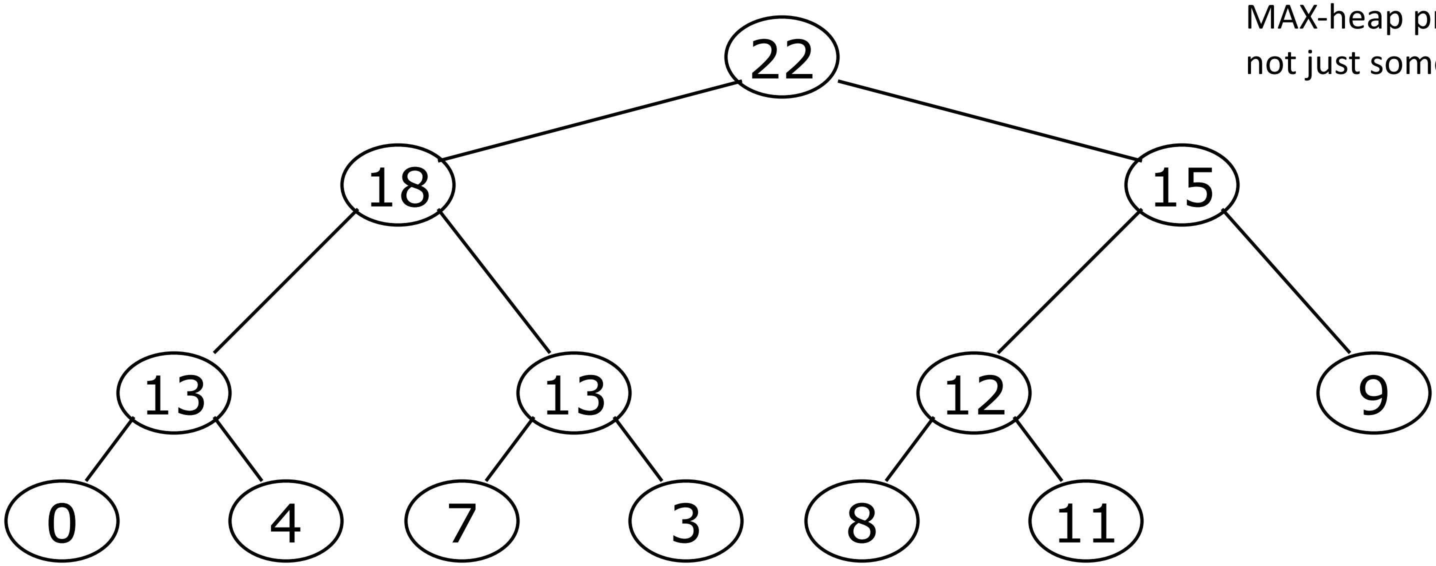
- Finally we will see how to use these ideas to sort an *array*

A sample heap

- Here's a **sample binary tree after it has been MAX-HEAPIFIED**

(different tree than in previous slides)

The whole tree satisfies the MAX-heap property here, not just some sub-tree

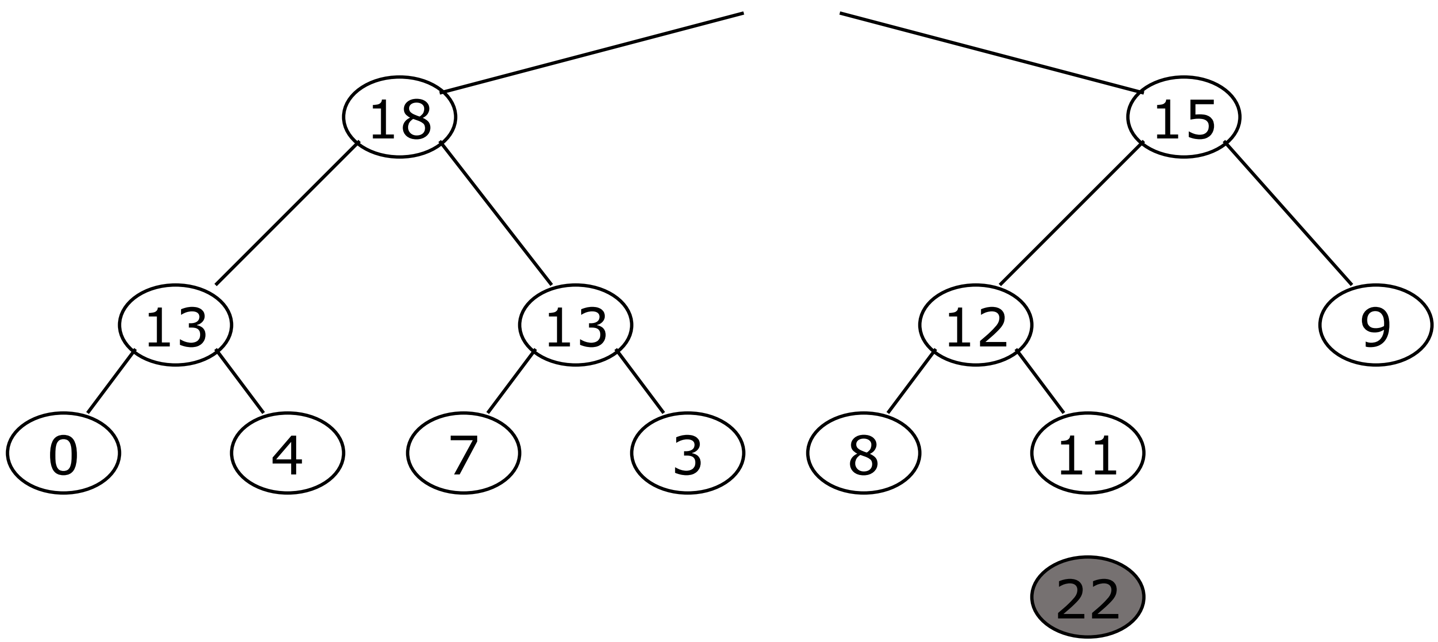


- Notice that **MAX-heapified does *not* mean sorted**
- MAX-Heapifying **does *not* change the shape of the binary tree**; this **binary tree is balanced** because it started out that way

The height difference between left sub-tree and right-subtree is at most 1

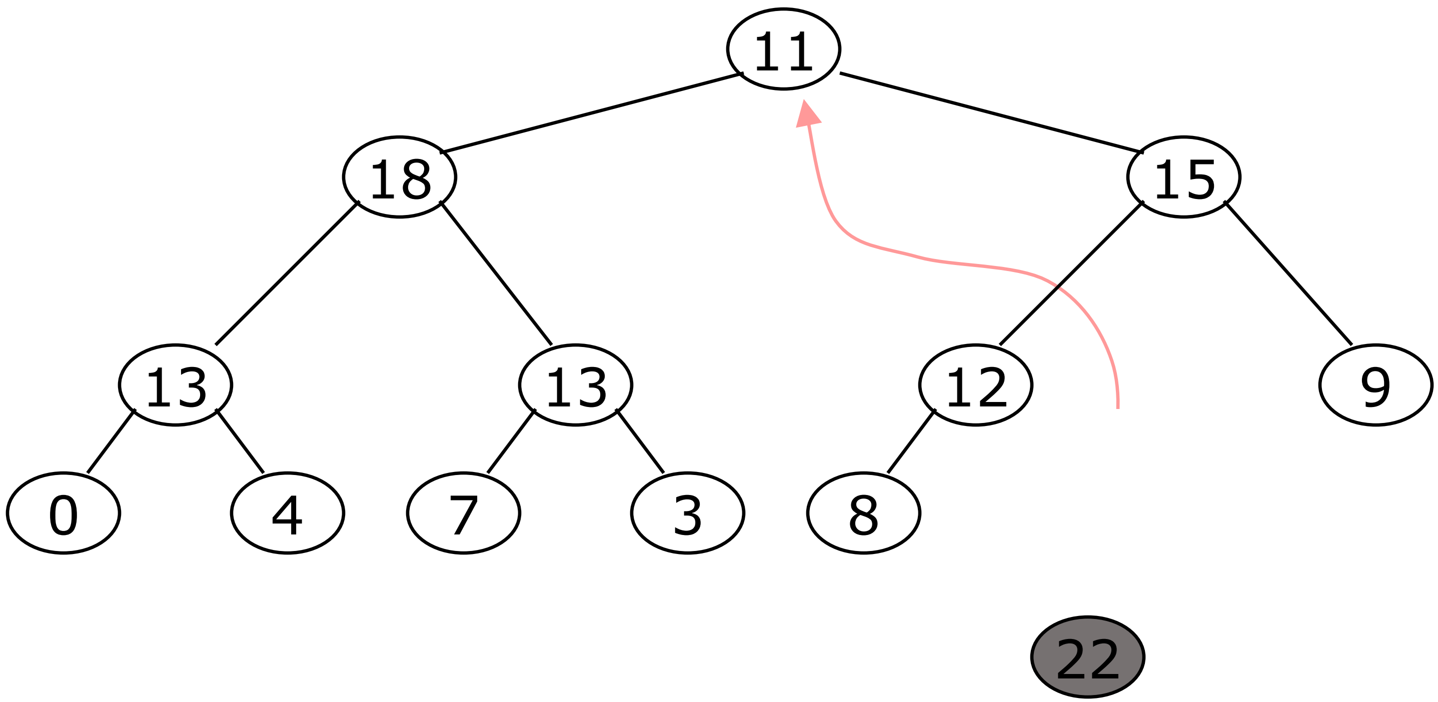
Removing the Root

- Notice that the largest number is now in the root
- Suppose we *discard* the root:



Replacing the Root

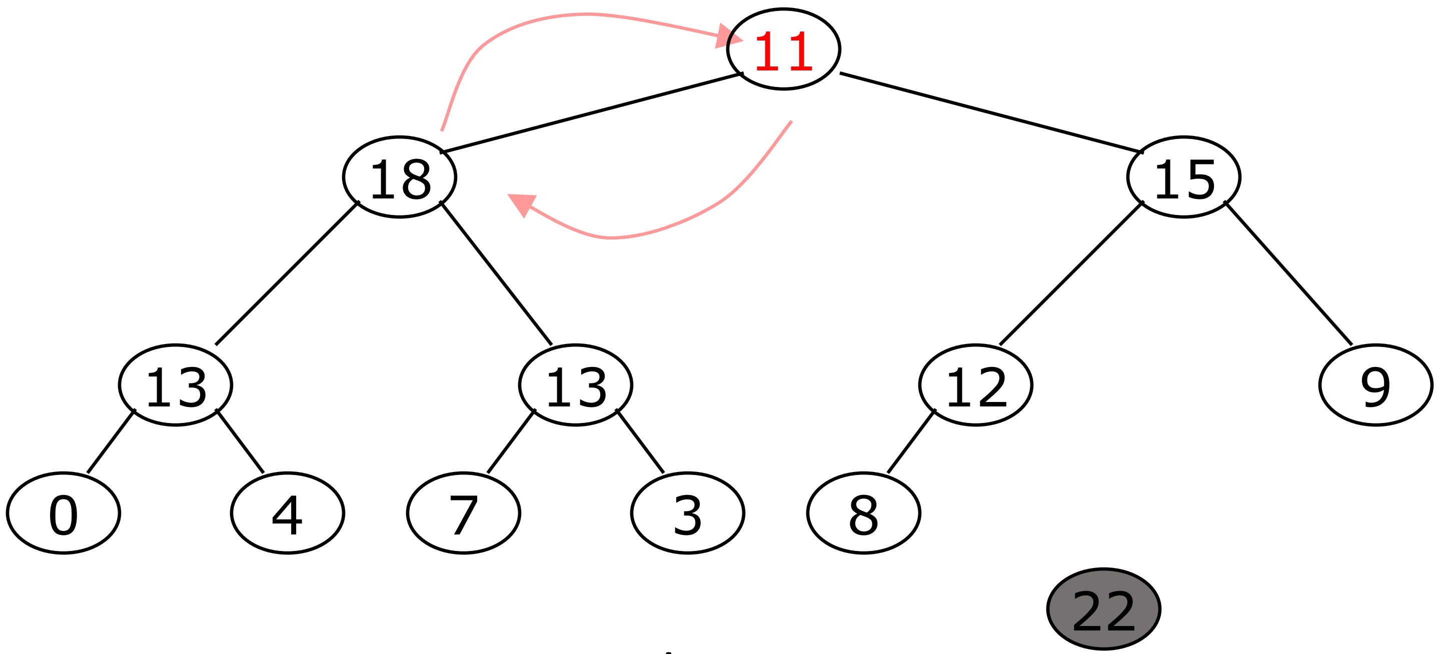
- Remove the rightmost leaf at the deepest level and use it for the new root



- The MAX-heap property of this “new” tree is now lost!

MAX-Heapify (I)

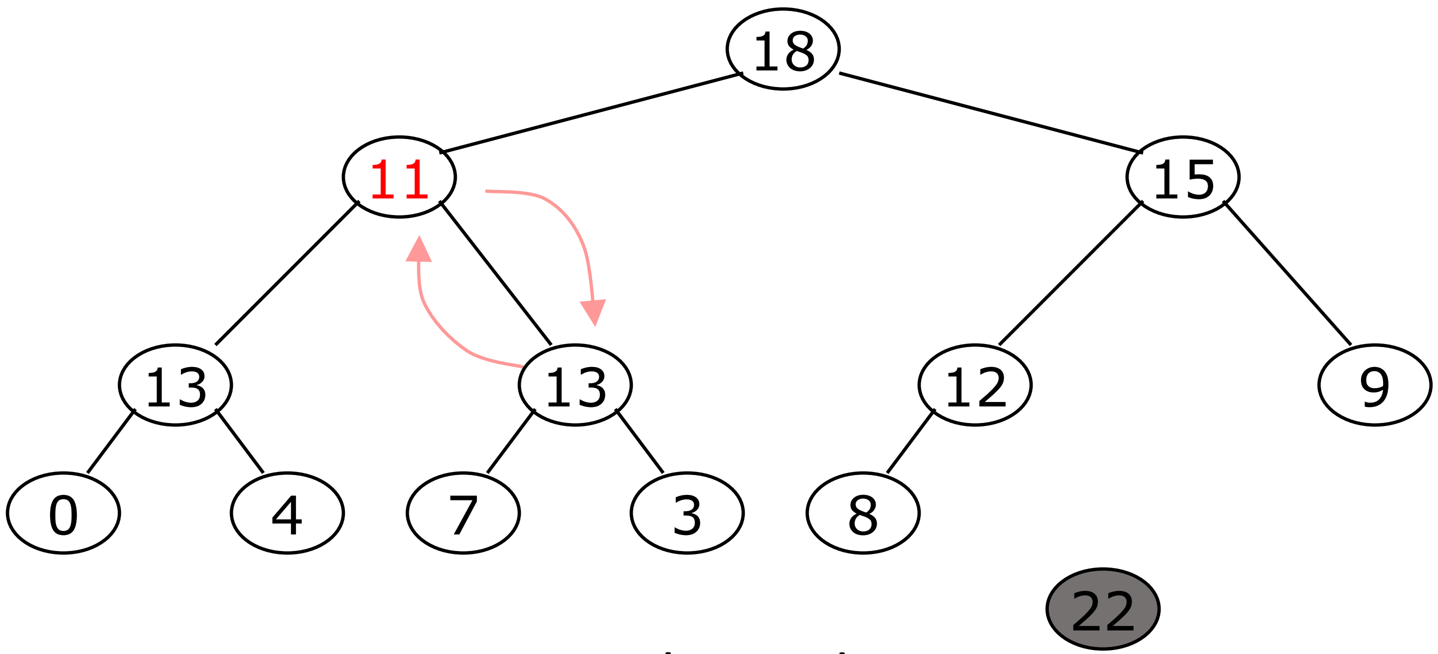
- However, *only the root* lacks the MAX-heap property



- We can MAX-Heapify() the root
- After doing this, one and only one of its children may have lost the MAX-heap property

MAX-Heapify (II)

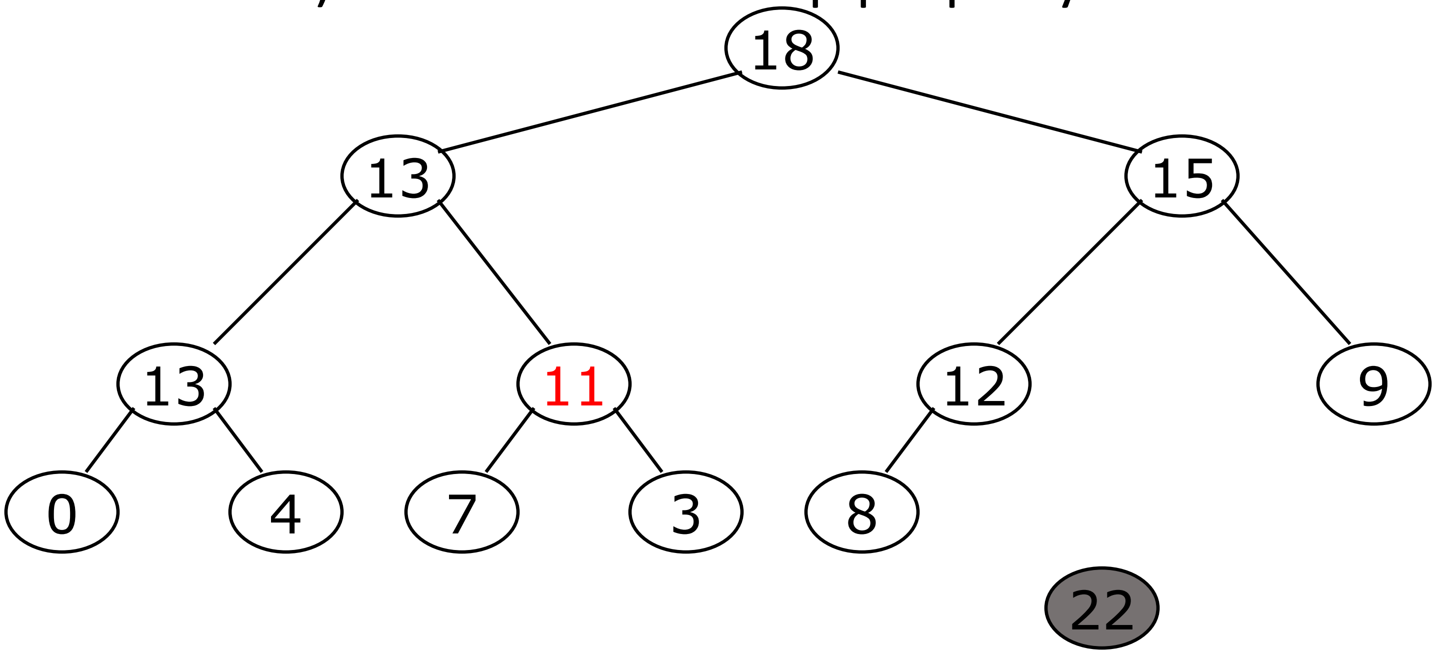
- Now the left child of the root (still the number 11) lacks the MAX-heap property



- We can MAX-Heapify() this node
- After doing this, one and only one of its children may have lost the MAX-heap property

MAX-Heapify (III)

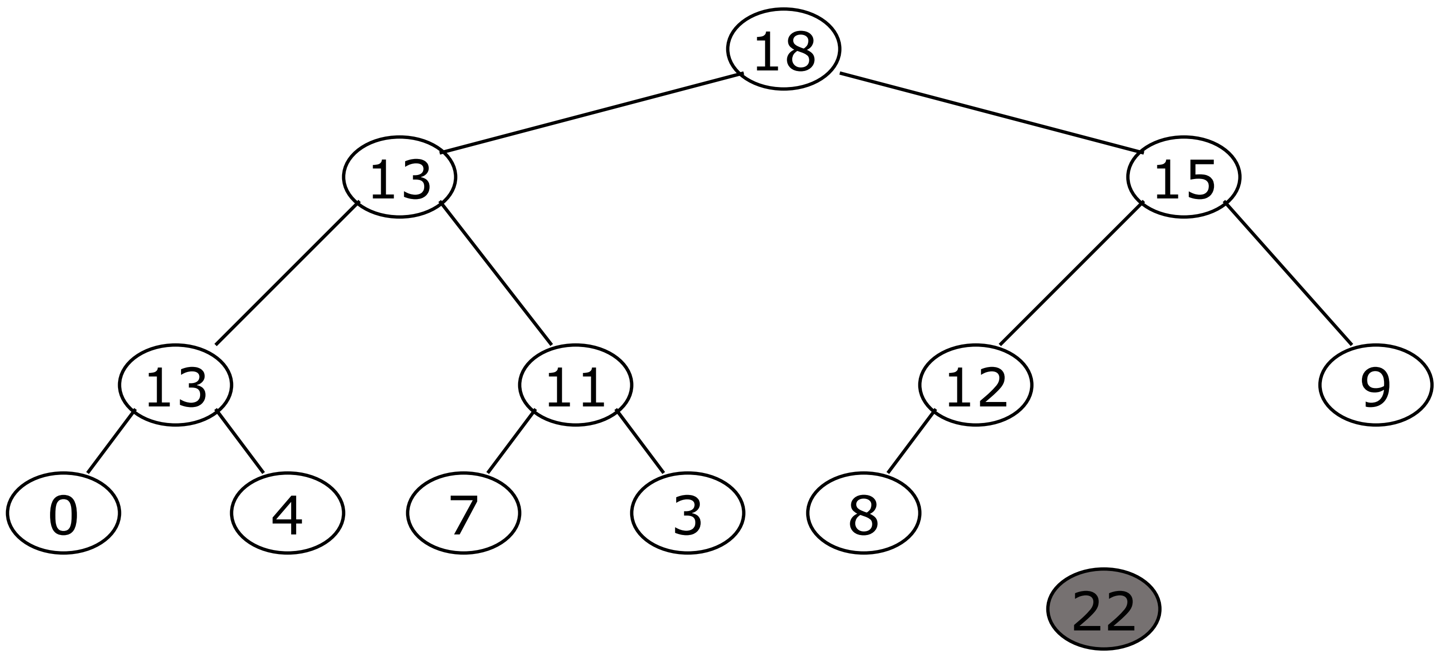
- Now the right child of the left child of the root (still the number 11) HAS the MAX-heap property:



- At thus stage the MAX-heap property is satisfied.

New Heap

- Our tree is once again a heap, because every node in it has the MAX-heap property



- Once again, the largest (or *a* largest) value is in the root

- **We can repeat this process until the tree becomes empty**
  - This **produces a sequence of values in order largest to smallest**

HeapSort: Plan of attack

- First, we **turn any portion of a binary tree** (which we imagine our array to be) **into a *MAX-heap* (data structure)**
- Next, we will learn how to **turn a binary tree *back* into a MAX-heap after it has been changed** in a certain way
- Finally we will see how to use these ideas to sort an *array*

HeapSort

To sort:

Build MAX-HEAP
while the array isn’t empty
    remove the root
    replace the root with the last leaf node
    MAX-HEAPIFY(A,0)

Build MAX-HEAP from an array (A):

if A.length() %2 == 0 **then** MID = floor(A.length()/2)-1
else MID = floor(A.length()/2)
for i in MID downto 1:
        MAX-Heapify(A,i)

This is called initially when a new array is received for sorting via HeapSort

HeapSort

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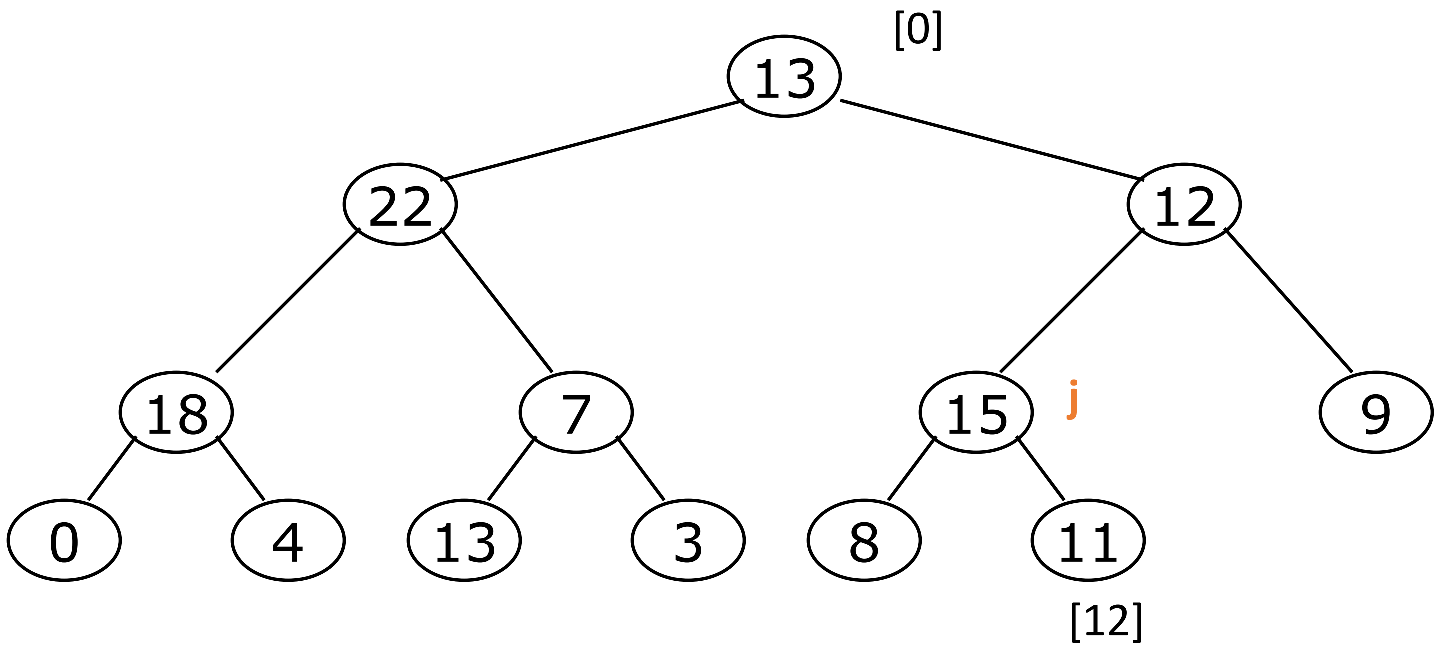
```

This is called initially when a new array is received for sorting via HeapSort

Build MAX-HEAP

Input Array

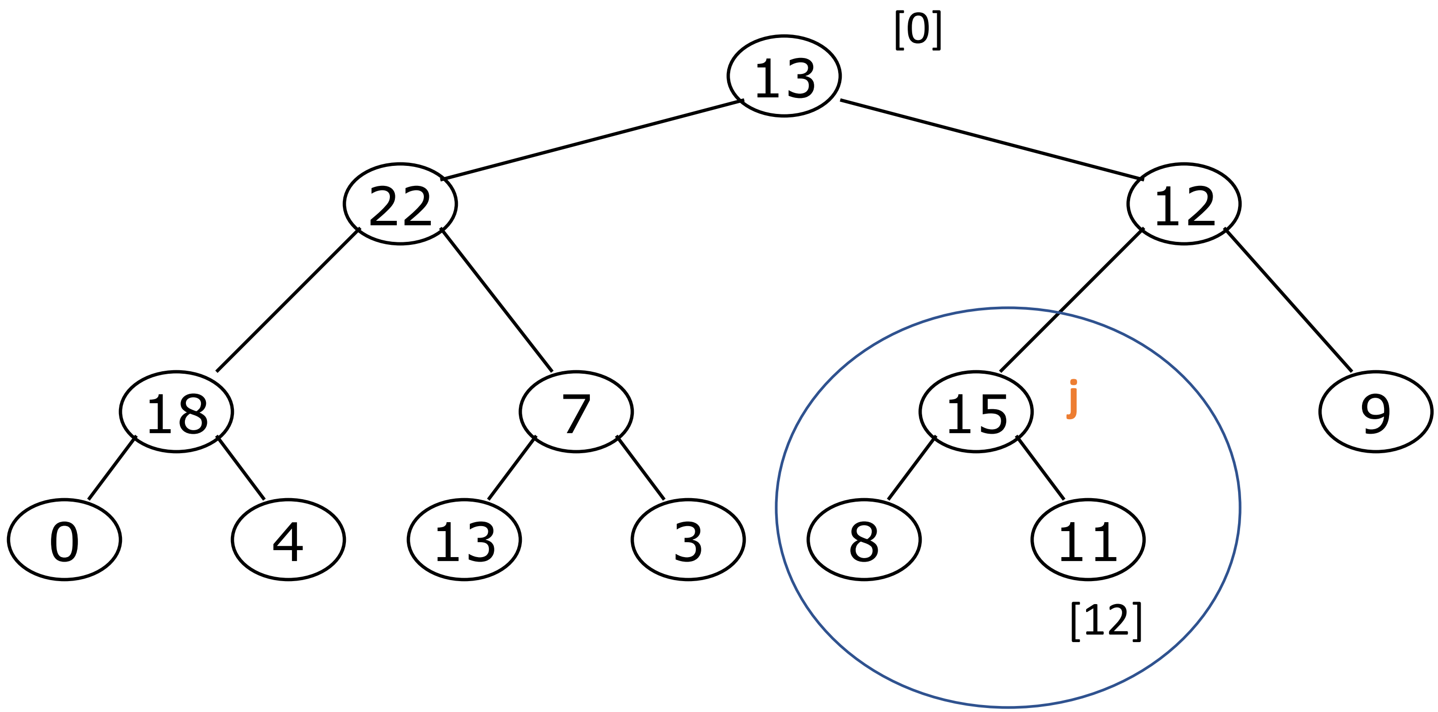
0	1	2	3	4	5	6	7	8	9	10	11	12
13	22	12	18	7	15	9	0	4	13	3	8	11



Binary Tree view of the input array

Build MAX-HEAP

	0	1	2	3	4	5	6	7	8	9	10	11	12
Input Array	13	22	12	18	7	15	9	0	4	13	3	8	11

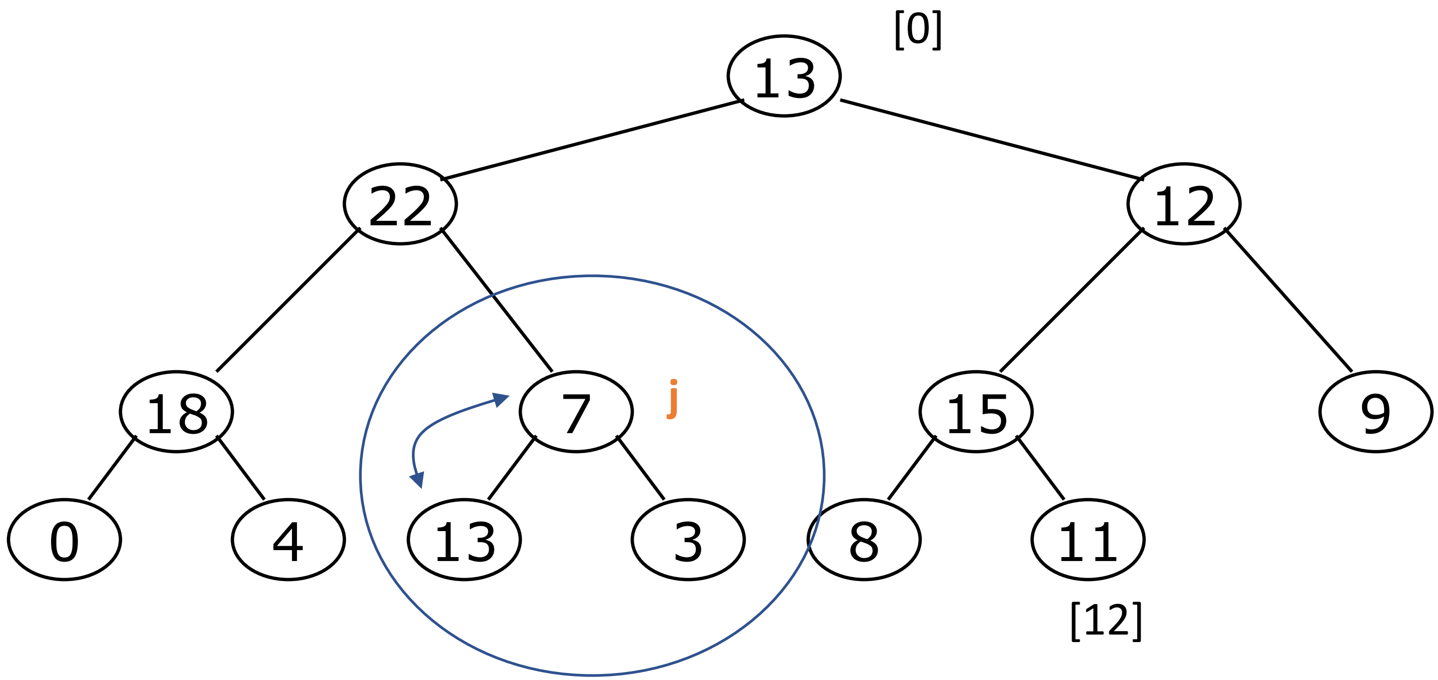


j is the index of "mid" point of the array, given length A is even

Build MAX-HEAP

Input Array

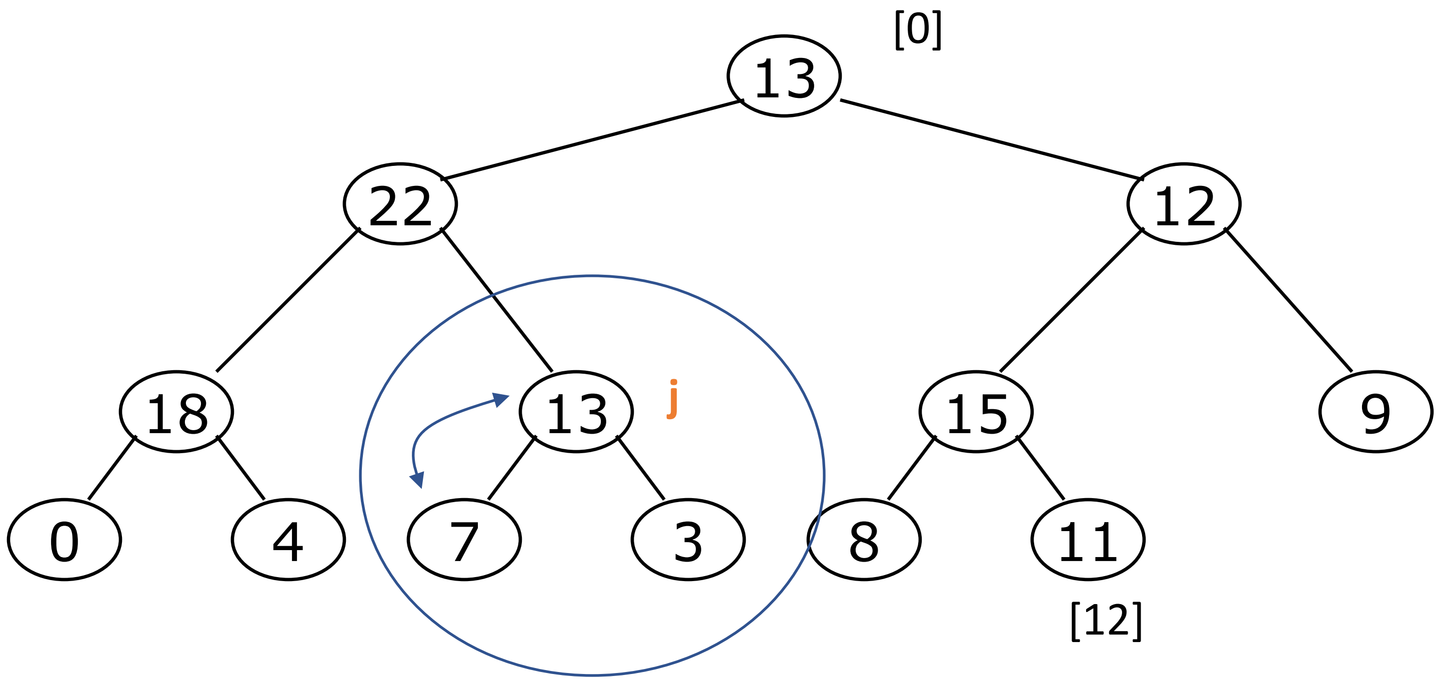
0	1	2	3	4	5	6	7	8	9	10	11	12
13	22	12	18	7	15	9	0	4	13	3	8	11



Run the for loop in Build-MAX Heap by considering the previous element to 15

Build MAX-HEAP

	0	1	2	3	4	5	6	7	8	9	10	11	12
Input Array	13	22	12	18	13	15	9	0	4	7	3	8	11



Build MAX-HEAP

Input Array

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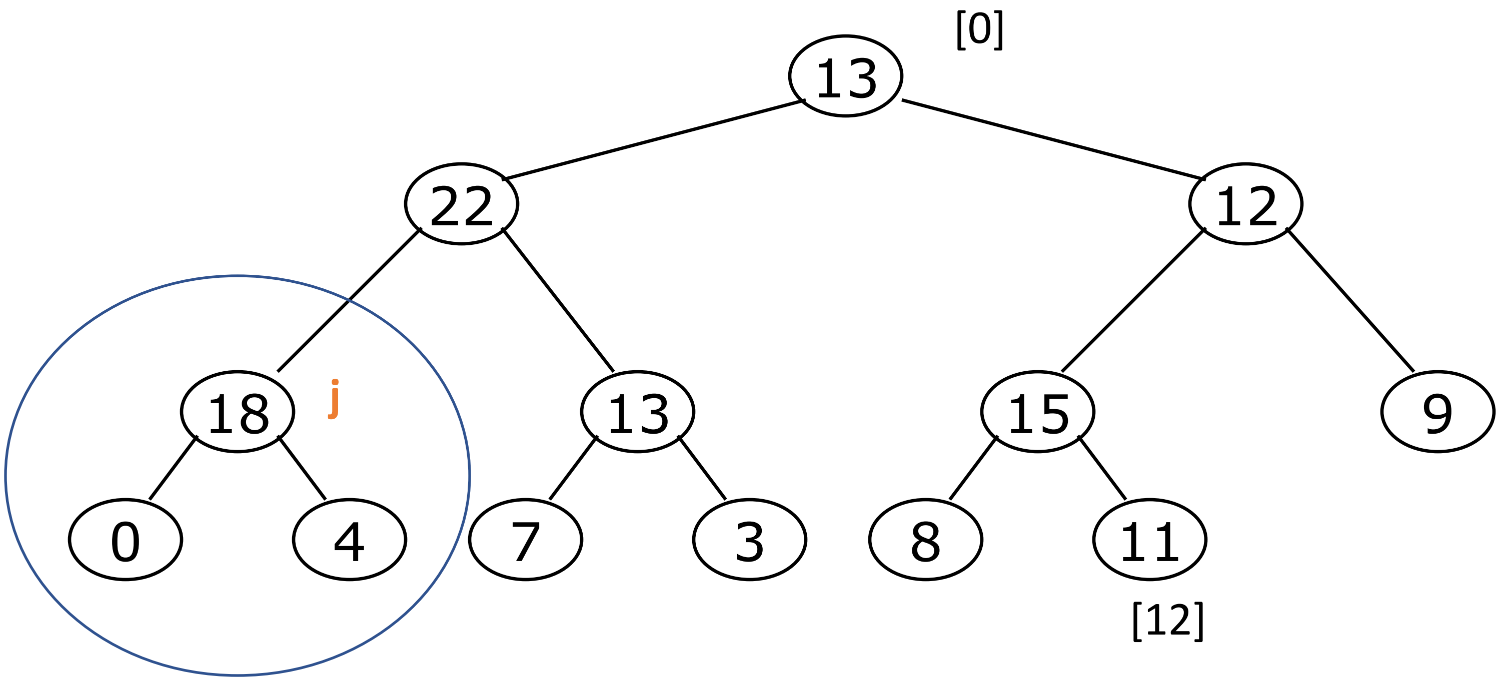

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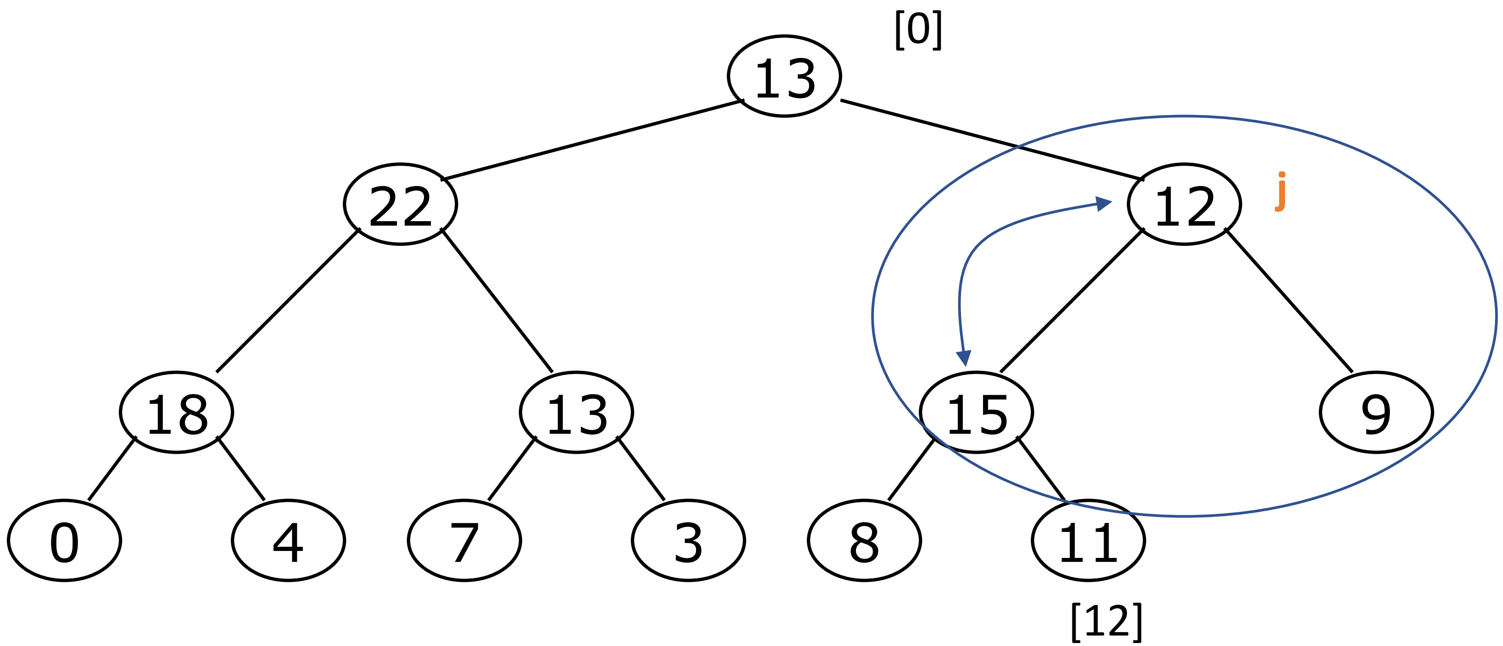

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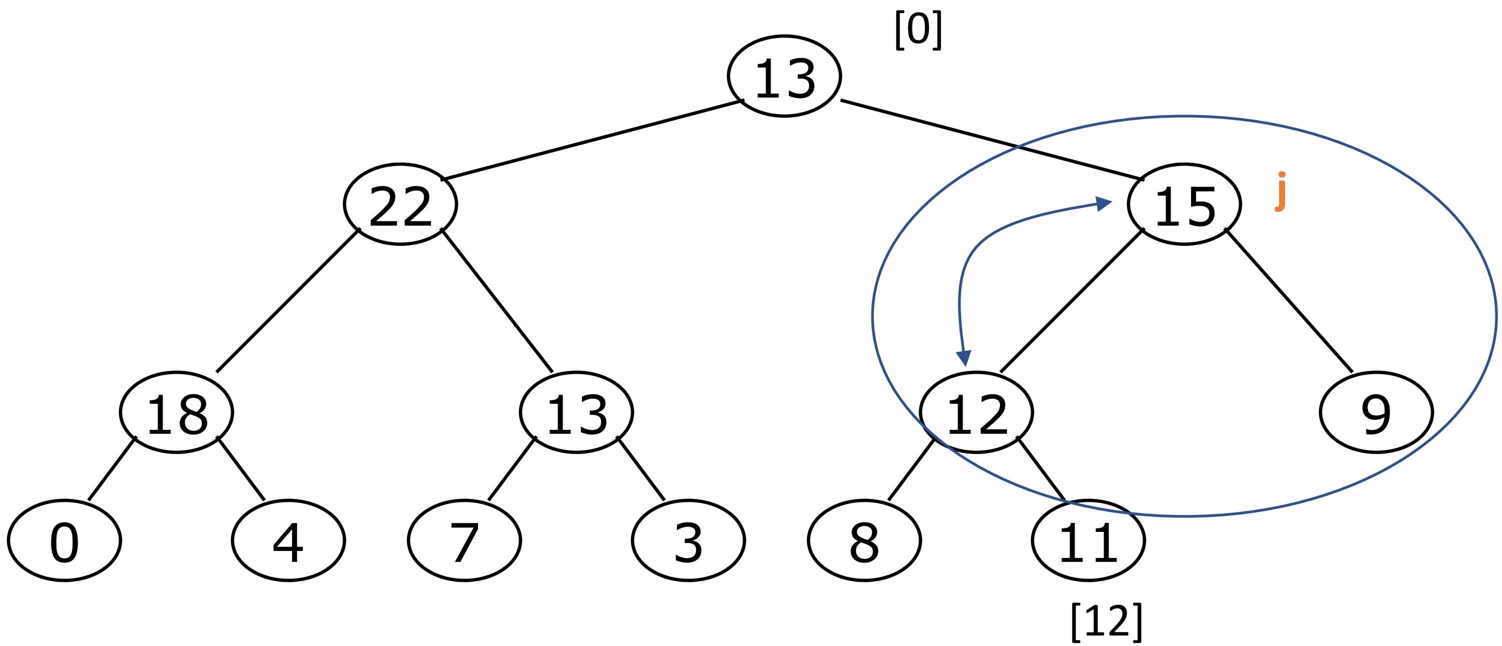

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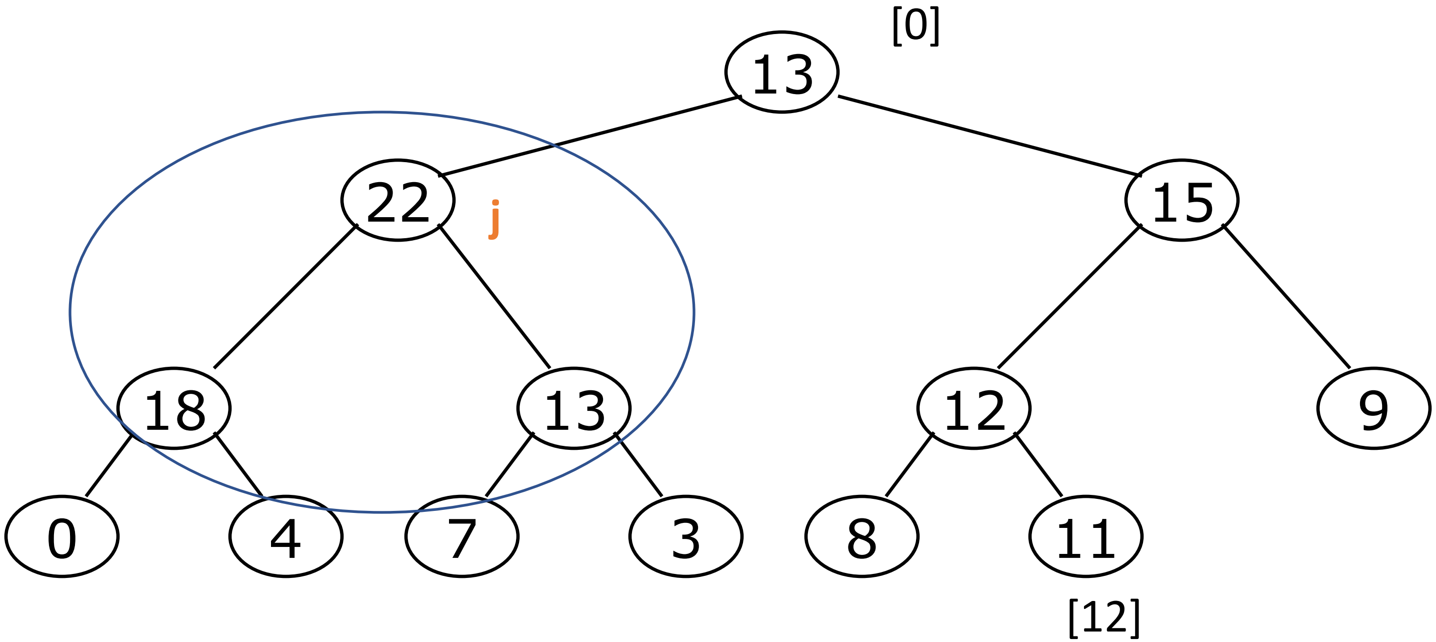

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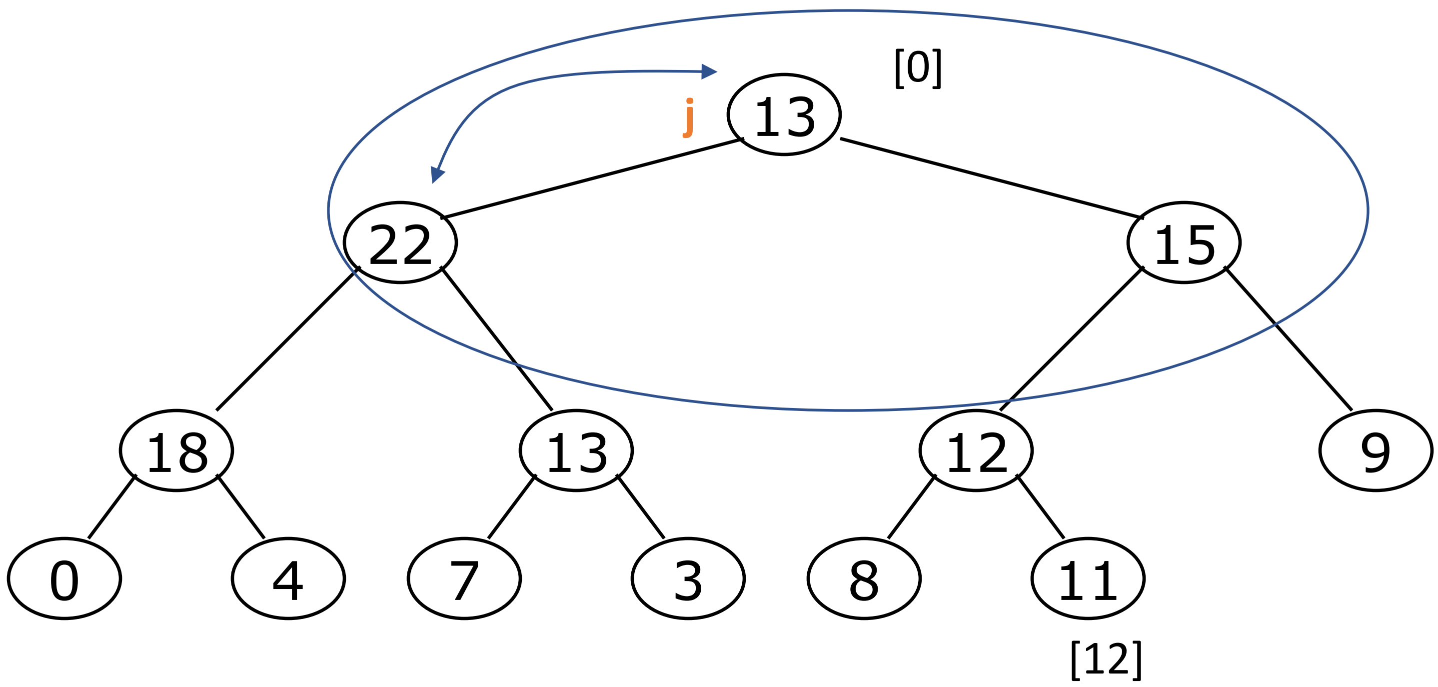

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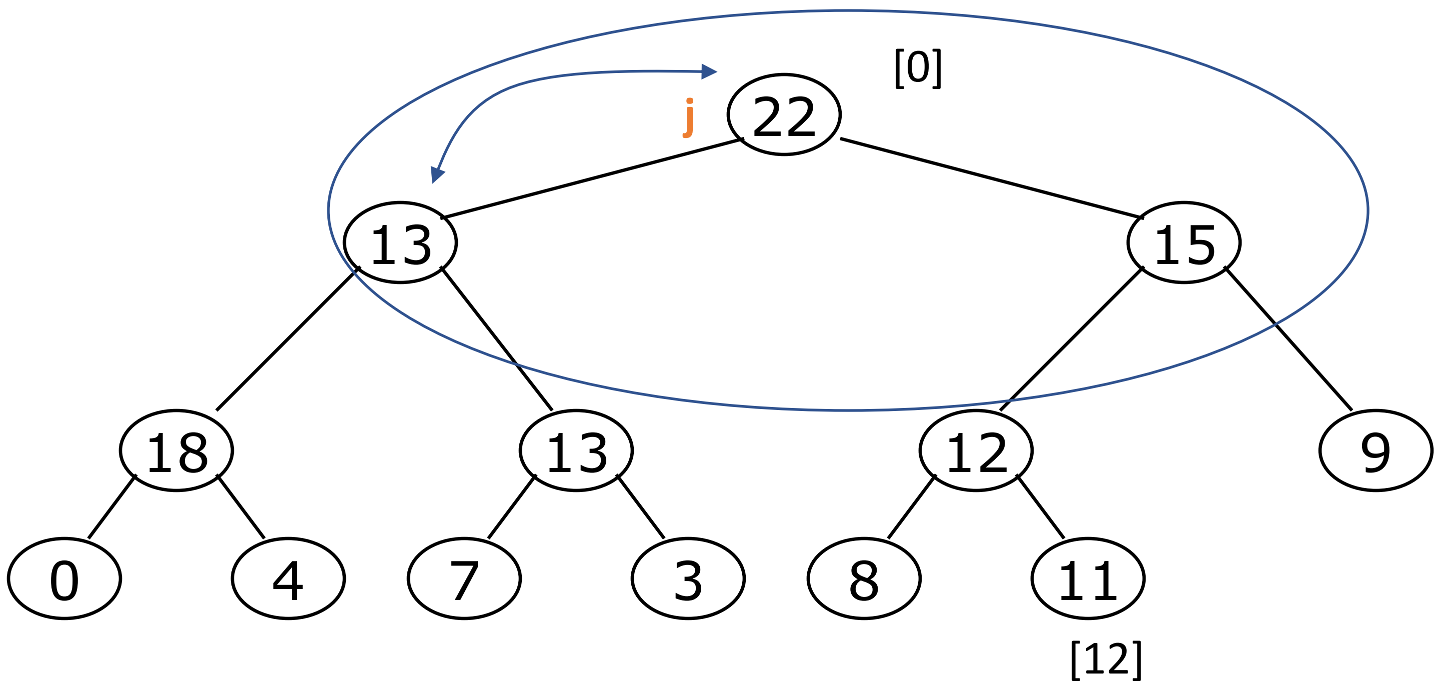

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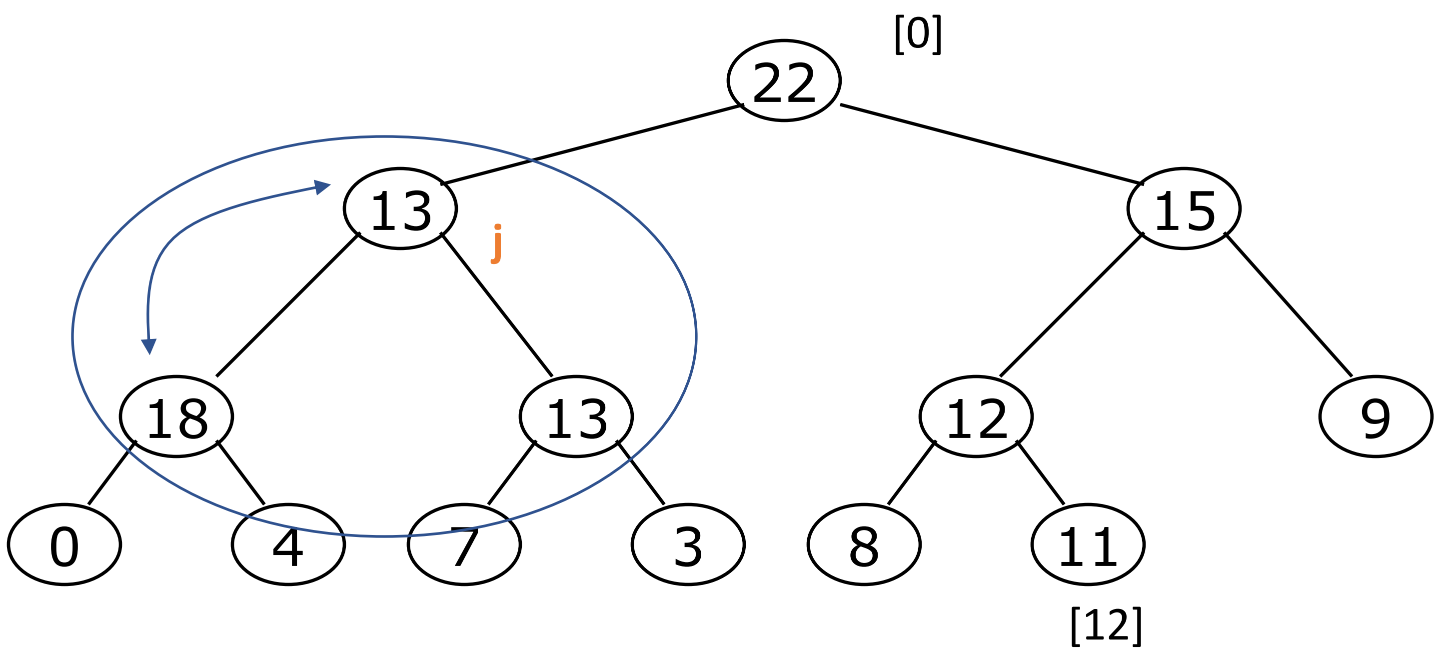

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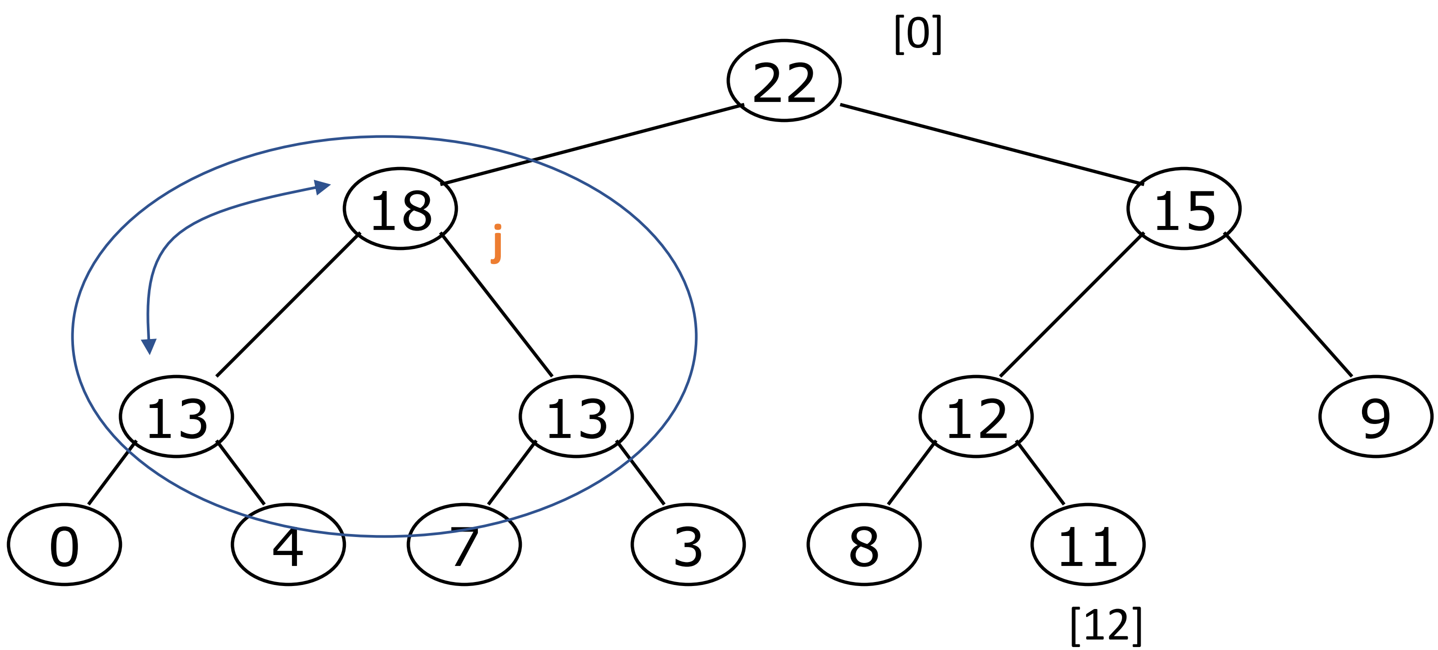

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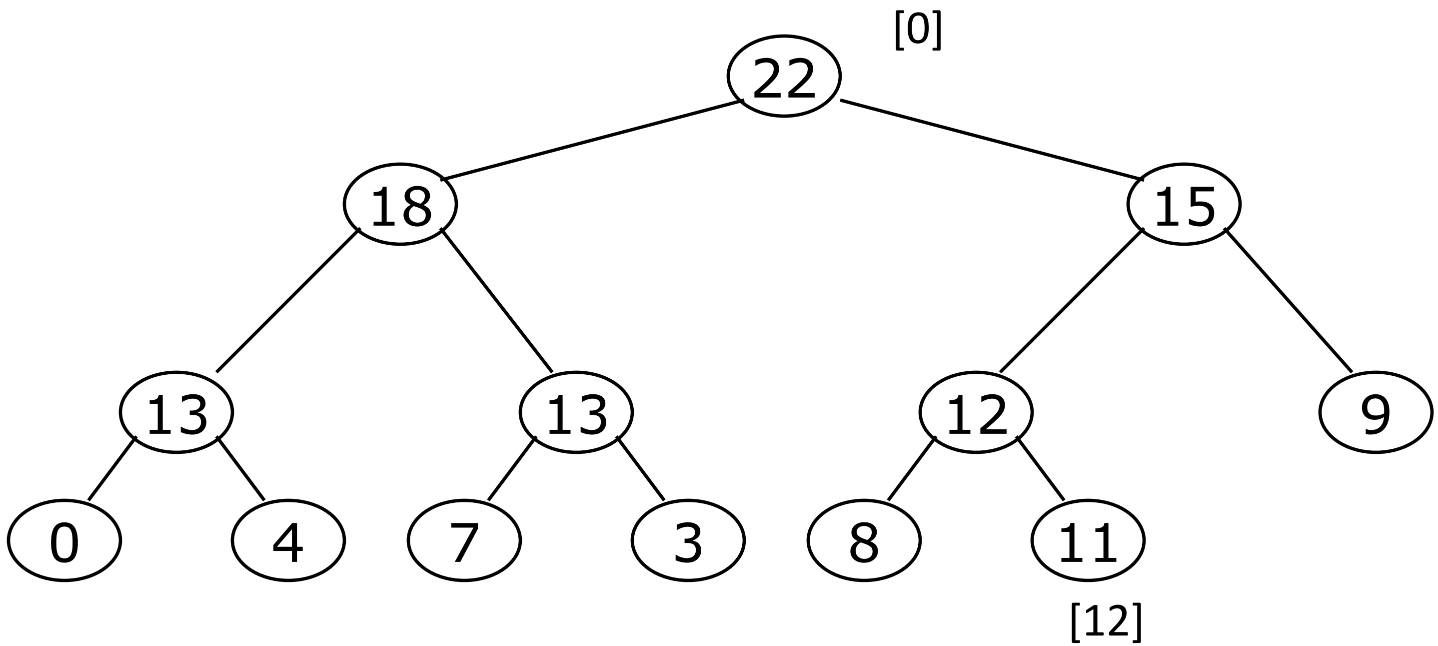


The array is a MAX-HEAP, but not sorted, which we do next

HeapSort

To sort:

Build MAX-HEAP

while the array isn't empty

remove the root

replace the root with the **last** leaf node

MAX-HEAPIFY(A,0)

Build MAX-HEAP from an array (A):

if A.length() %2 == 0 **then** MID = floor(A.length()/2)-1

else MID = floor(A.length()/2)

for i in MID downto 1:

MAX-Heapify(A,i)

This is called initially when a new array is received for sorting via HEapSort

Removing and replacing the root

- The "root" is the first element in the array
- The "rightmost node at the deepest level" is the last element
- Swap them...

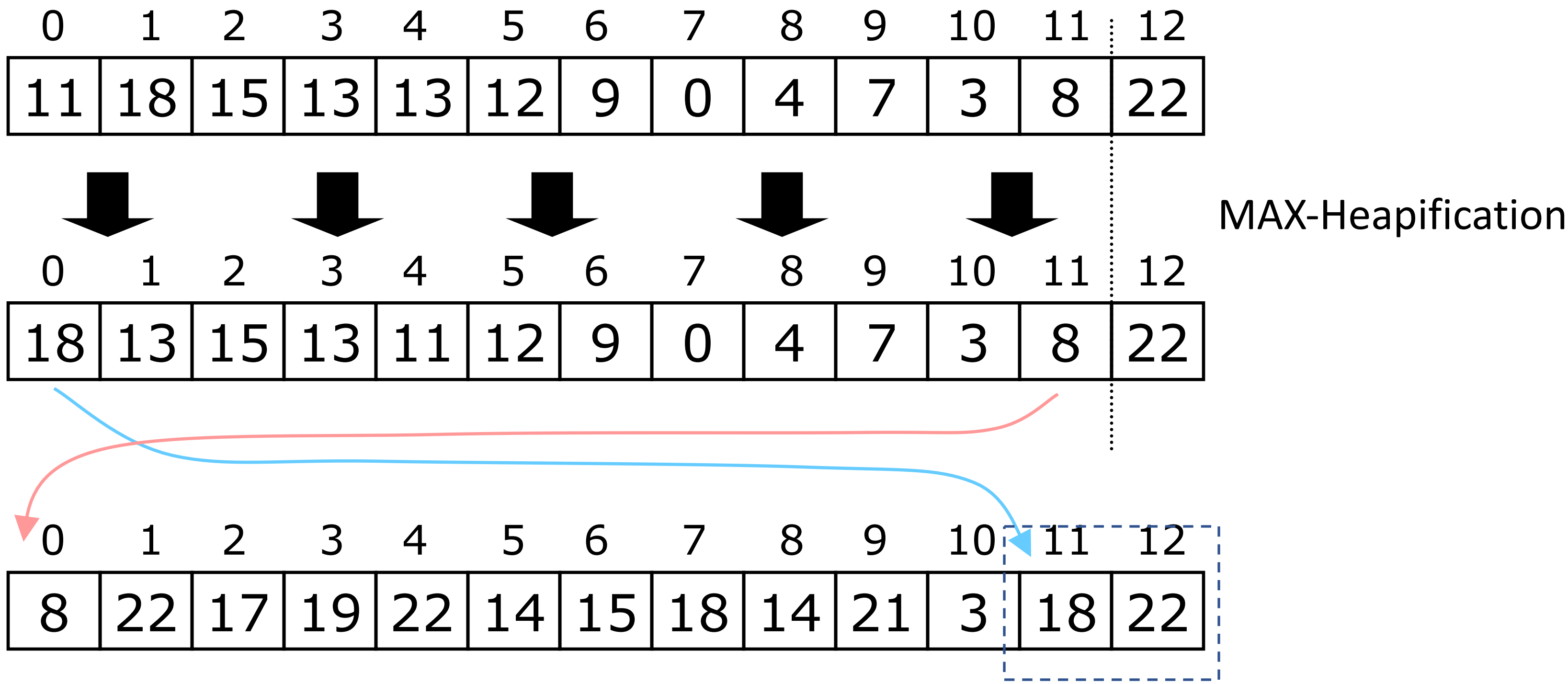
0	1	2	3	4	5	6	7	8	9	10	11	12
22	18	15	13	13	12	9	0	4	7	3	8	11

0	1	2	3	4	5	6	7	8	9	10	11	12
11	18	15	13	13	12	9	0	4	7	3	8	22

- ...And pretend that the last element in the array no longer exists—that is, the "last index" is 11 (9)
- **We reduce the length of the array by 1, as 22 is in it's correct position**

Repeat and MAX-Heapify

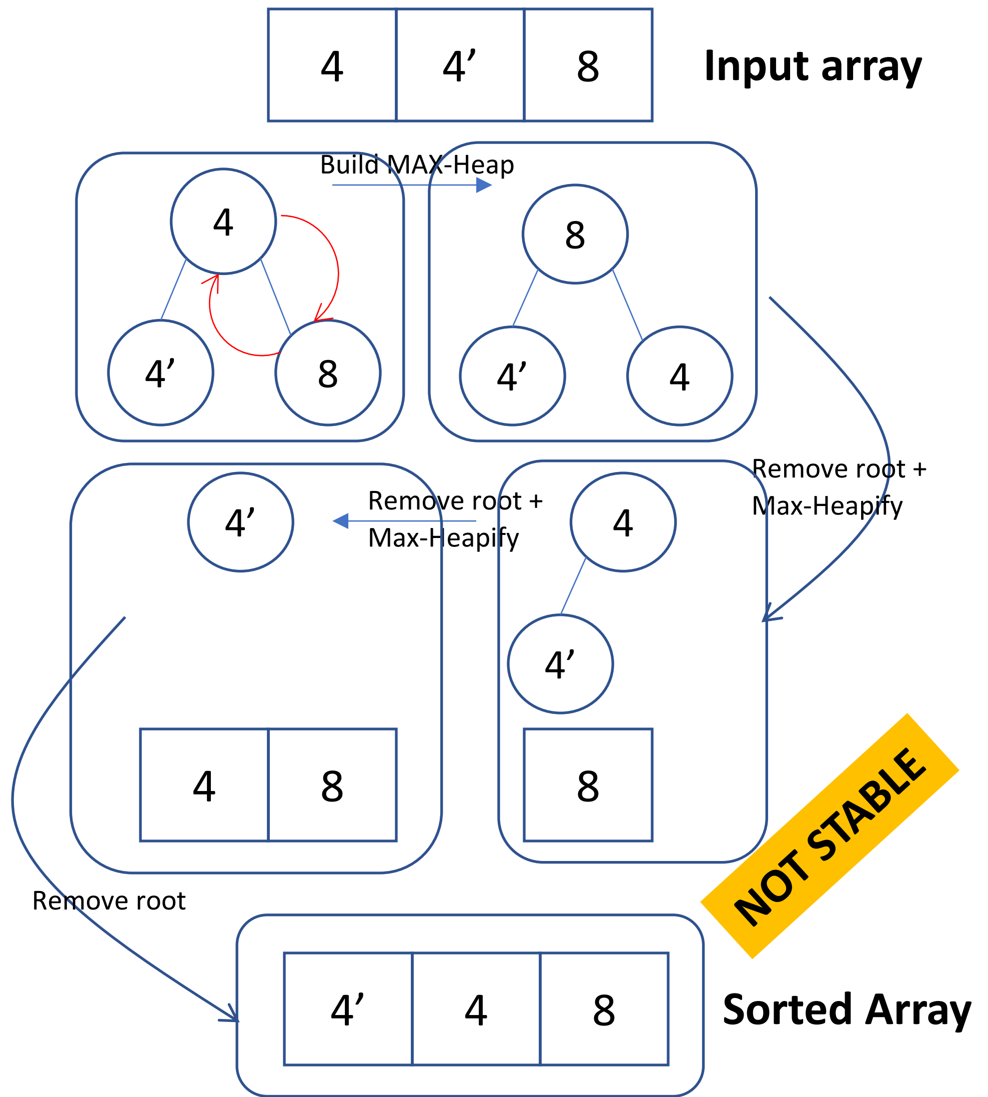
- MAX-Heapify the root node (index 0, containing 11)...



- ...And again, remove and replace the root node
- Remember, though, that the “last” array index is changed
- Repeat until the last becomes first, and the array is sorted!

Memory Requirements + Stability

- HeapSort can sort the array **in-place**
- Extra memory required during the sorting process is **O(1)**
  - **Only small amount needed to swap variables**



Analysis: Build MAX-HEAP

- Build MAX- Heap for the array
  - We start at the middle of the array and MAX-heapify at every node until first element of the array
  - The MAX-Heapifying itself takes
    - O(lg n) time
  - Since we do this (aprox.) n/2 times (i.e., O(n))
    - MAX-Heapify takes O(n)*O(lg n) time
- **Running time of Build MAX-HEAP is**

  O(n lg n)

Analysis: MAX-Heapify

- To MAX-Heapify the root node, we have to follow *one path* from the root to a leaf node (and we might stop before we reach a leaf)

- Therefore, this path is O(lg n) long
  - And we only do O(1) operations at each node

- Therefore, MAX-Heapify takes

  O(lg n)

Analysis: HeapSort

- Here's the rest of the algorithm:

```
while the array isn't empty
    remove the root
    replace the root with the last leaf node
    MAX-HEAPIFY(A,0)
```

- We do the while loop n times (actually, n-1 times), because we remove one of the n nodes each time
- Removing and replacing the root takes O(1) time

- **Therefore, the total time is n times however long it takes the MAX-Heapify method**

O(n lgn)

Analysis

- Here's the algorithm again:

  Build MAX-HEAP
  while the array isn't empty
      remove the root
      replace the root with the last leaf node
      MAX-Heapify(A,0)

  - We have seen that Build MAX-HEAP takes O(n lg n) time
  - The while loop (with MAX-Heapify) takes O(n lg n) time
  - The total time is therefore O(n lg n) + O(n lg n)

- **HeapSort therefore takes: O(n log n) time**


That's all Folks!
Any Question?