Shortest Path Algorithms

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CSC 212

Based on slides from Prof. Carolina Ruiz at WPI

Announcements

- Final exam: Thursday (Dec 19) at 11:30am.
  - It will be held in this classroom (WHITE HALL 205)

- Topics for the final exam have been sent to everyone.

- Assignment 3 is due tonight by 11:59pm.

Shortest-Path Variants

Single-Source Single-Destination (1-1) - No good solution that beats 1-M variant - Thus, this problem is mapped to the 1-M variant	**Single-Source All-Destination (1-M)** - Need to be solved (several algorithms) - We will study this one
All-Sources Single-Destination (M-1) - Reverse all edges in the graph - Thus, it is mapped to the (1-M) variant	**All-Sources All-Destinations (M-M)** - Need to be solved (several algorithms) - Will not be covered in this class

Shortest Path

Shortest Path = Path of minimum weight

$$d(u,v)= \begin{cases} \min\{\omega(p) : u \overset{p}{\rightsquigarrow} v\}; & \text{if there is a path from u to v,} \\ \infty & \text{otherwise.} \end{cases}$$

Important Idea: Initialization

- Maintain d[*v*] for each *v* in *V*
- d[*v*] is called *shortest-path weight estimate*

$INIT(G, s)$

  for each $v \in V$ do

    $d[v] \leftarrow \infty$

    $\pi[v] \leftarrow NIL$

  $d[s] \leftarrow 0$

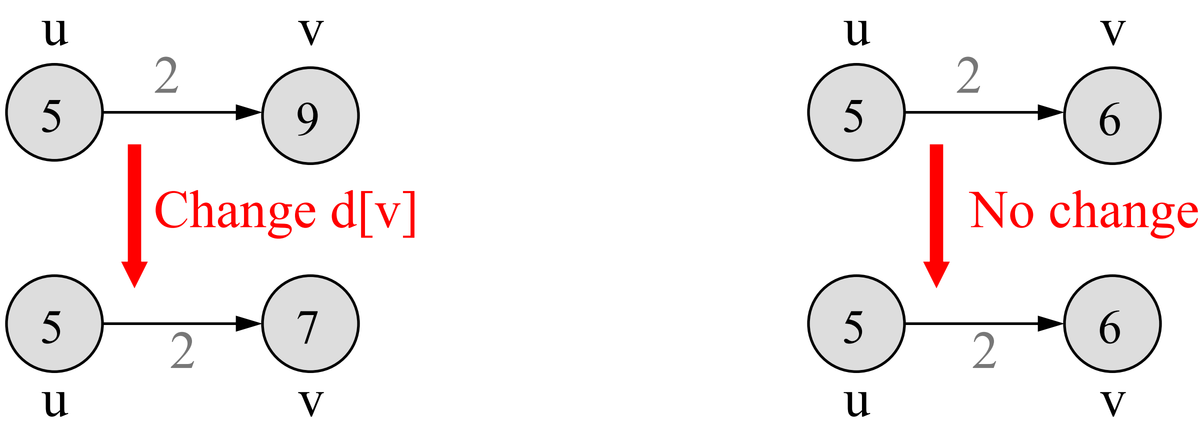
Upper bound

Parent node

Important Idea: Relaxation

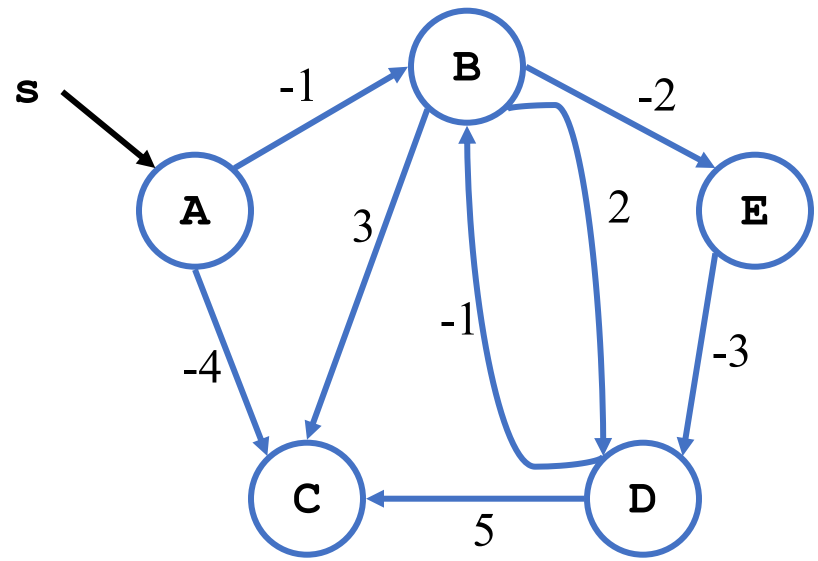
RELAX(u, v)
 if d[v] > d[u]+w(u,v) then
  d[v] ← d[u]+w(u,v)
  π[v] ← u

When you find an edge (u,v) then check this condition and relax d[v] if possible



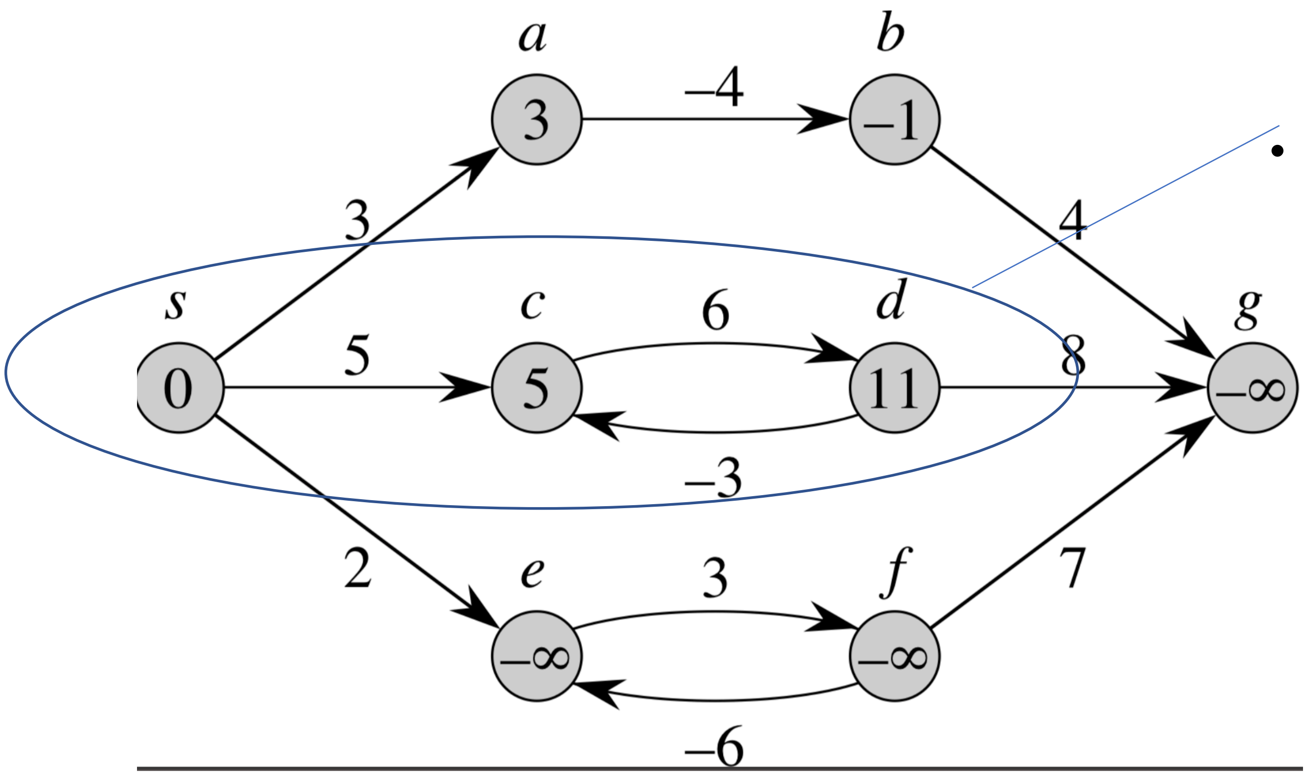
Graphs Edges and Negative Weights

- Graph edges can have negative weights
- The meaning of the negative weights depends upon the context
  - E.g., decrease in traffic in a network at midnight compared to noon!
- **One can still find <u>1-M shortest paths</u> to vertices with edges have negative weights**
  - **But not always (see next two slides)**



Example

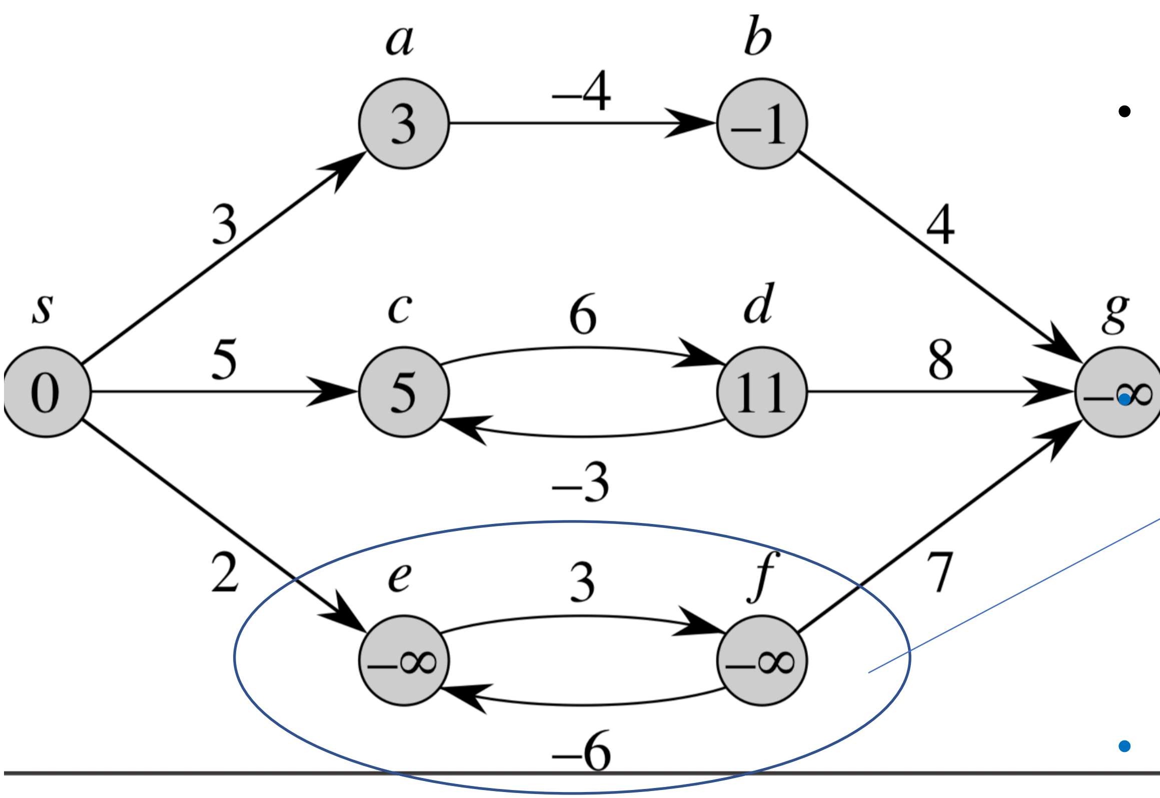
- In graphs with negative edge weights shortest paths can be found:



- There are infinitely many paths from to s-c. <s,c> <s,c,d,c>, <s,c,d,c,d,c>…
- However, since the cycle <c,d,c> has a weight of 6+(-3) > 0, d(c) = 5 and d(d) = 11 and will not be affected by the negative weights

Example

- In graphs with negative weight cycles, shortest path to some nodes will not exist:



- There are infinitely many paths from to s-e. <s,e> <s,e,f,e>, <s,e,f,e,f,e>…
- However, since the cycle <e,f,e> has a weight of 3+(-6) < 0, d(e) = -inf and d(d) = -inf

No matter what d[e] starts with, going to vertex f and coming back to vertex e will reduce d[e] further.

- **The same goes for d[f]**

Cycles in Shortest Path?

- **Negative weight cycles* cannot be in a shortest path**

  *That is, the entire cycle path adds up to a negative weight

- **Positive weight cycles will not be in a shortest path**
  - You can remove the cycle from the path and it will produce a better shortest path
  - See previous slide 8. We just ignore the cycle.

- **<u>IMPORTANT POINT: When we find 1-M shortest path, there are no cycles in any of the paths</u>**

Shortest Path Algorithms

- **Bellman-Ford**
  - Solve single source shortest path algorithm in a graph **with negative edge weights**

- **Dijkstra's**
  - Solve single source shortest path algorithm in a graph **with positive edge weights (only)**
  - **Faster**

Bellman-Ford Algorithm

```
BellmanFord()
   for each v ∈ V
      d[v] = ∞;
   d[s] = 0;
   for i=1 to |V|-1
      for each edge (u,v) ∈ E
         Relax(u,v, w(u,v));
   for each edge (u,v) ∈ E
      if (d[v] > d[u] + w(u,v))
           return "no solution";
```

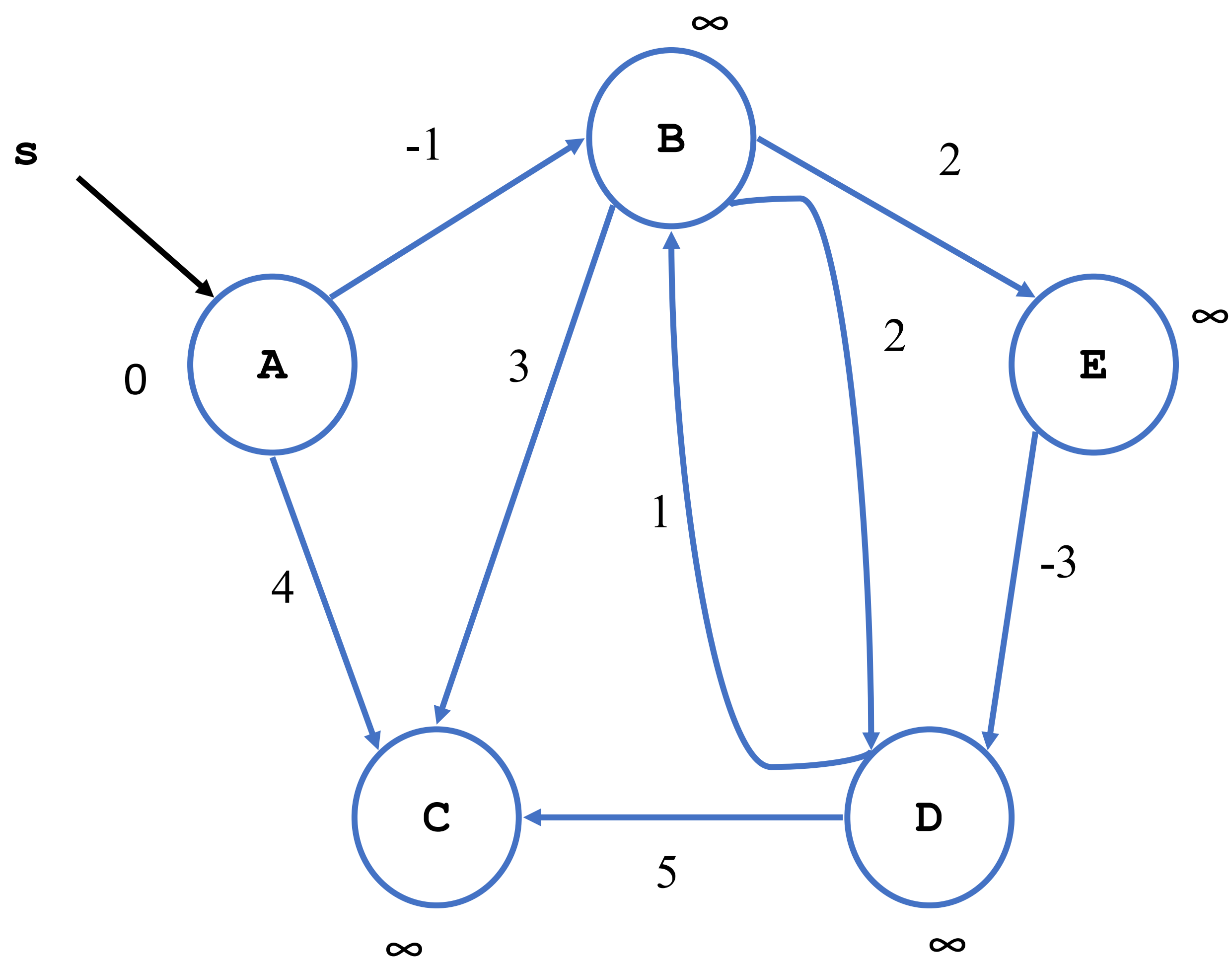
Initialize d[], which will converge to shortest-path value δ

Relaxation: Make |V|-1 passes, relaxing each edge

Test for solution
Under what condition do we get a solution? --- NO Cycles in the final shortest path!

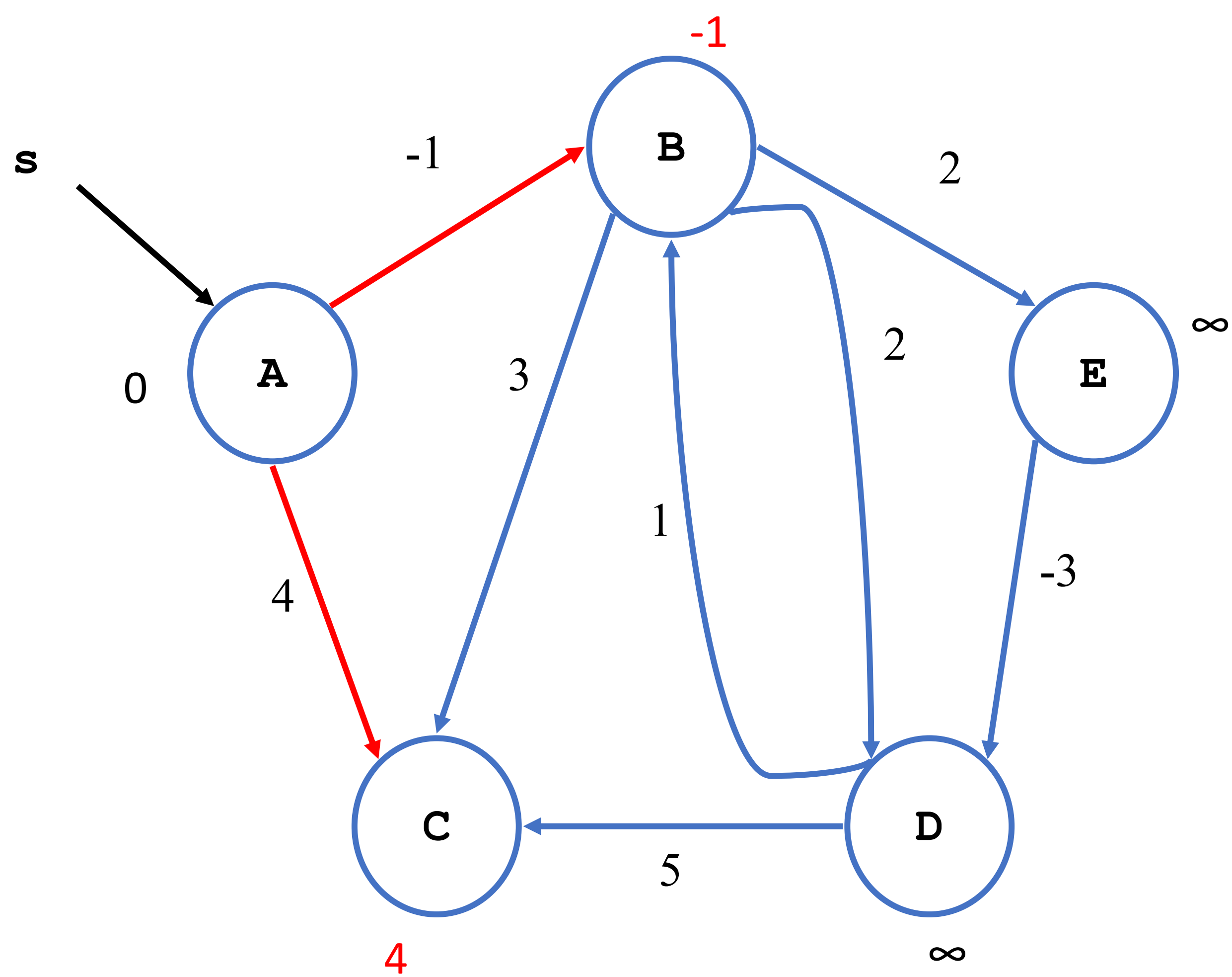
```
Relax(u,v,w): if (d[v] > d[u]+w) then d[v]=d[u]+w
```

Bellman-Ford Algorithm: Example



Order in which edges evaluated: {(A,B),(A,C),(B,C),(B,E),(B,D),(E,D),(D,B),(D,C)}

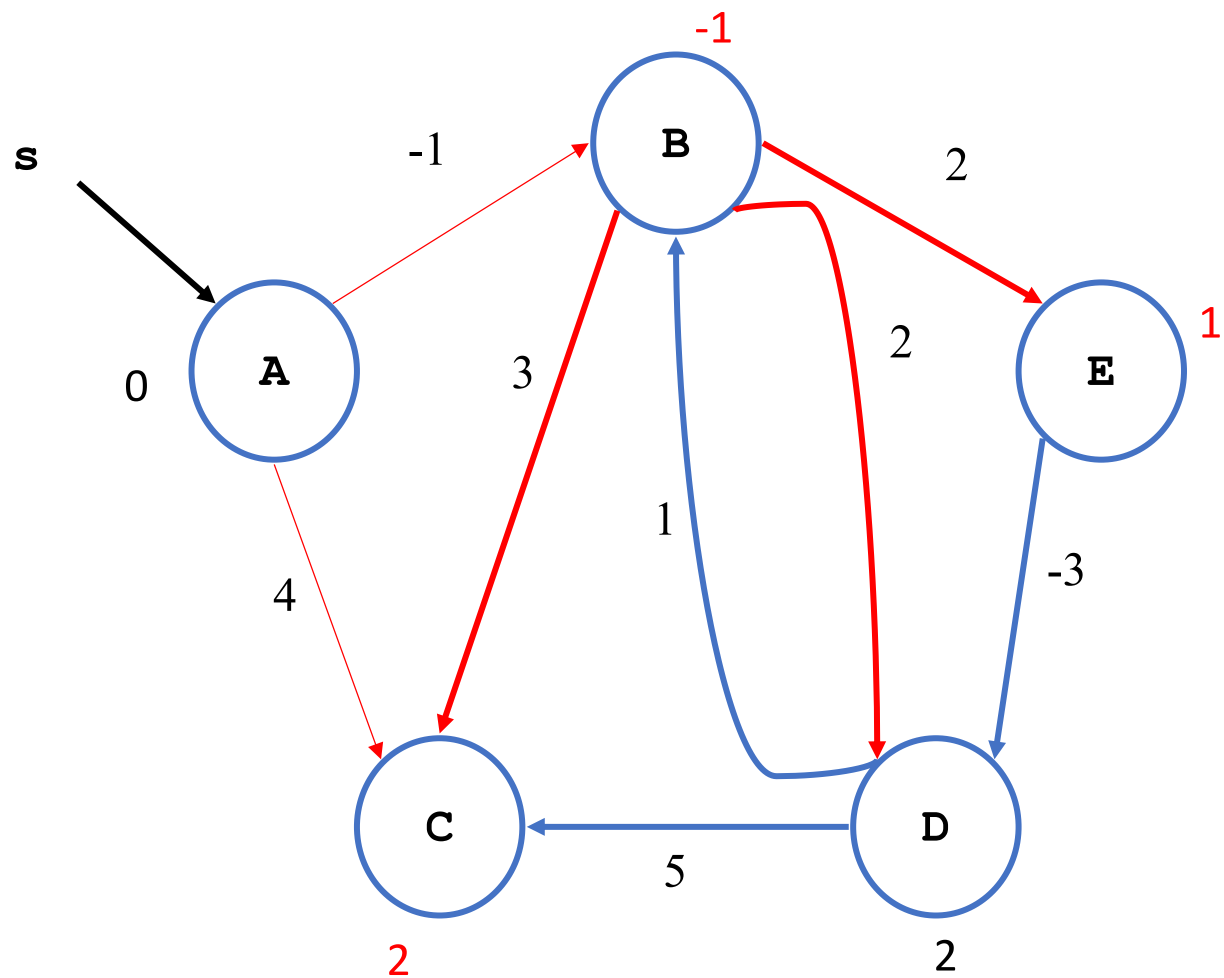
Bellman-Ford Algorithm: Example



Order in which edges evaluated: {(A,B)(A,C),(B,C),(B,E),(B,D),(E,D),(D,B),(D,C)}

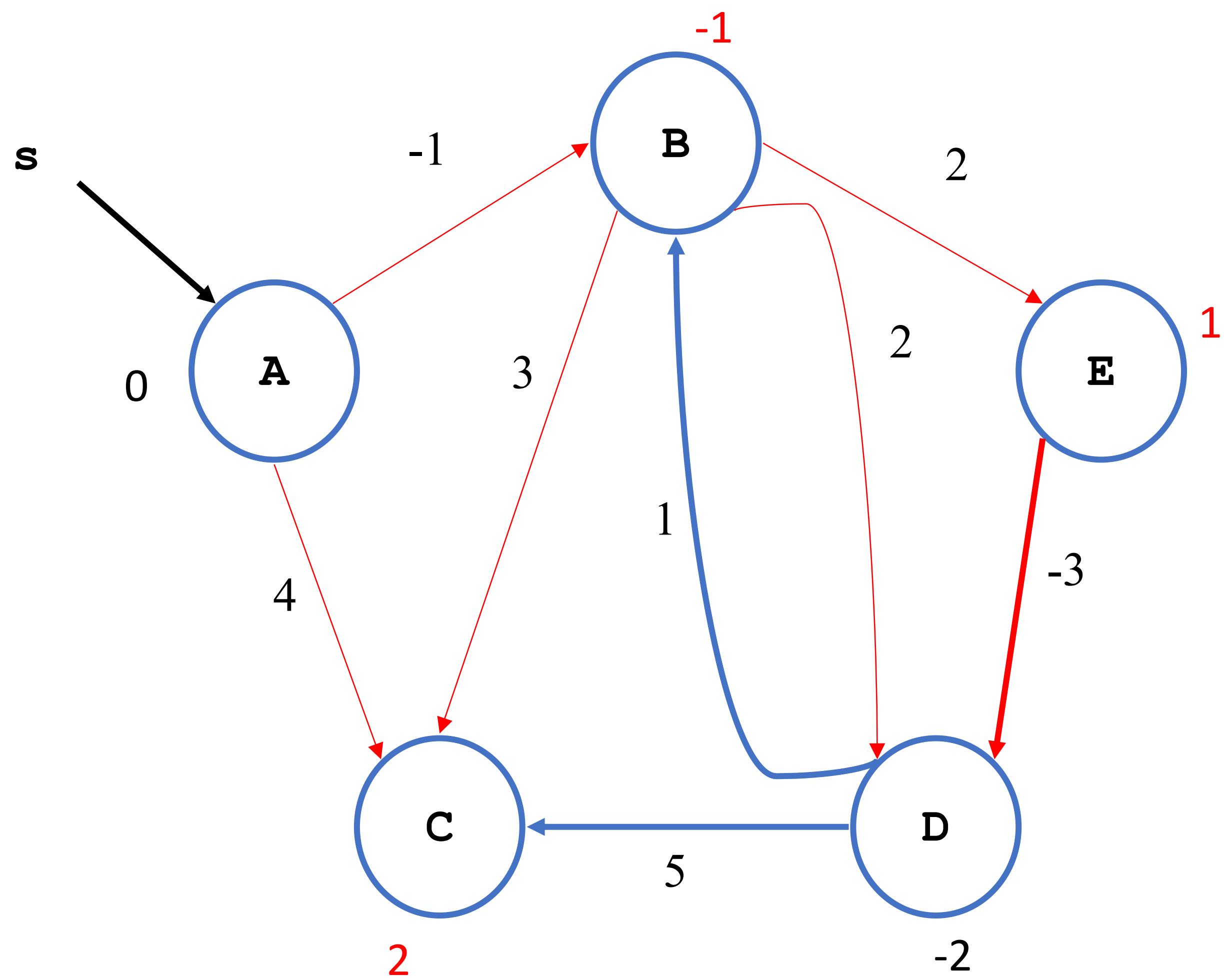
I have shown the relaxation of two edges at a time to speed up the slides. You should look at each edge in the list one by one, and relax it

Bellman-Ford Algorithm: Example



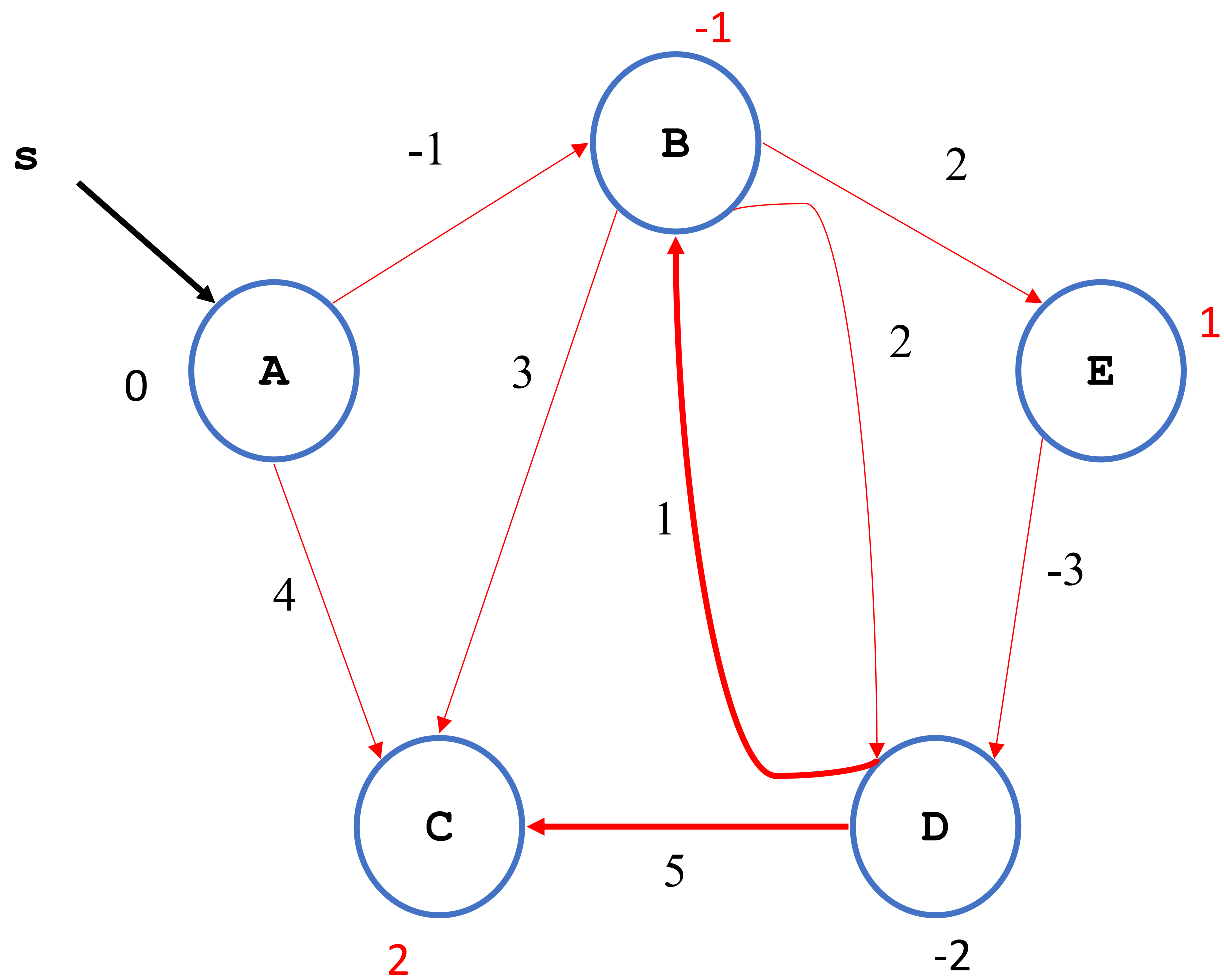
Order in which edges evaluated: {(A,B),(A,C),(B,C),(B,E),(B,D),(E,D),(D,B),(D,C)}

Bellman-Ford Algorithm: Example



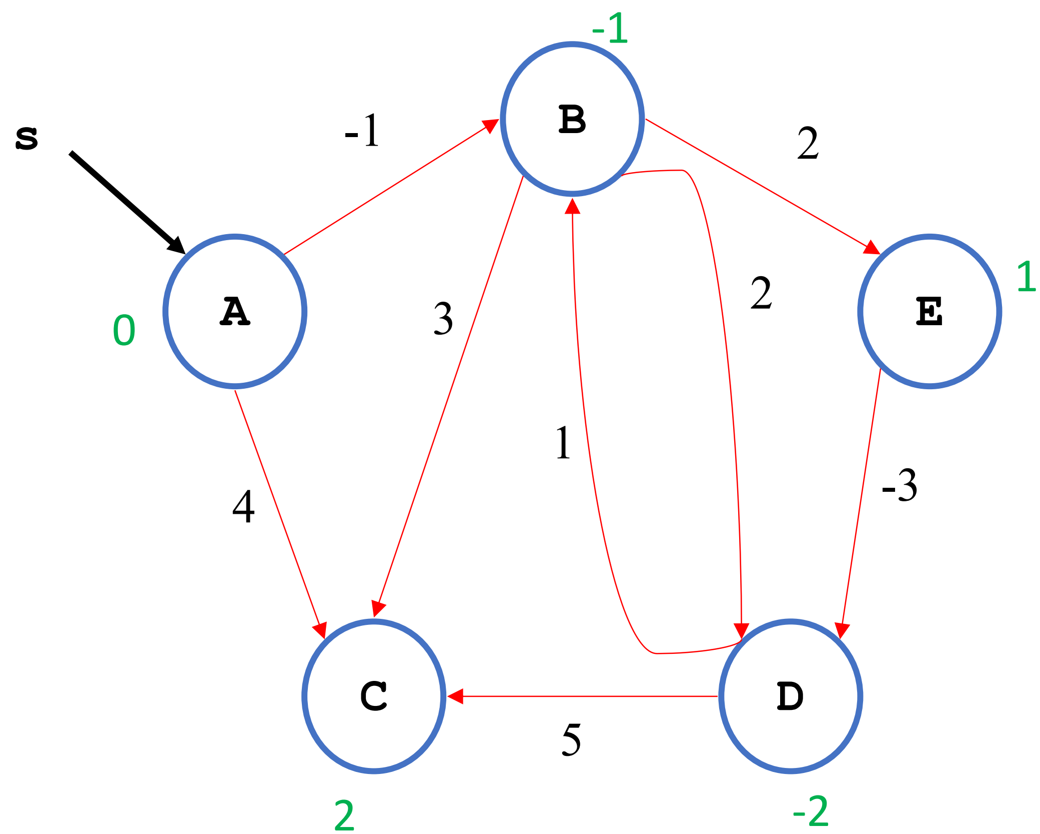
Order in which edges evaluated: {(A,B),(A,C),(B,C),(B,E),(B,D),(E,D),(D,B),(D,C)}

Bellman-Ford Algorithm: Example



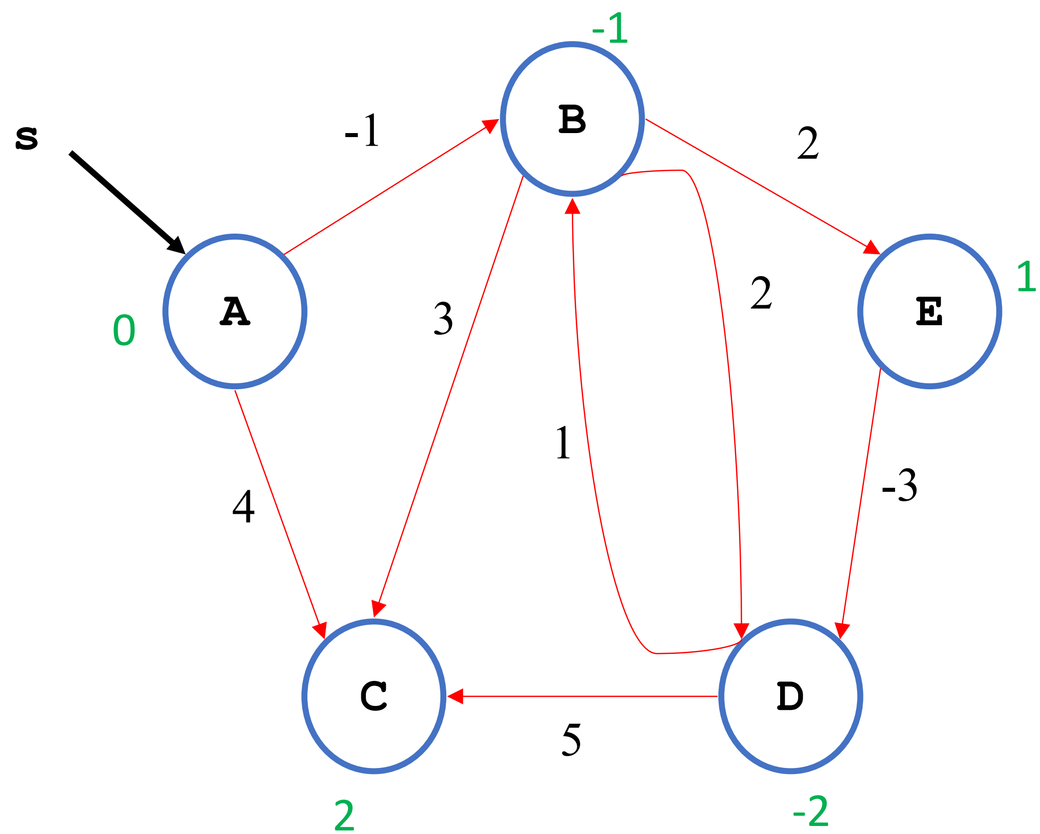
Order in which edges evaluated: {(A,B),(A,C),(B,C),(B,E),(B,D),(E,D),(D,B),(D,C)}

Bellman-Ford Algorithm: Example



- Do this relaxation V-1 (4) times. Again and again given “weights” in every round
- After V-1 rounds, we will be done.
- **The order of edge evaluation does not matter (try at home). The result (of distance from s) should be the same.**

Bellman-Ford Algorithm: Example



Are there cycles?

..Nope

- **Check to see if there are cycles in this graph?**
- **For each edge check**
  - **if d[v] > d[u] + w(u,v)**
  - **If so. we say there are cycles in the graph and shortest path cannot be found for all nodes in the graph from 's'.**

Bellman-Ford Algorithm

```
BellmanFord()
   for each v ∈ V
      d[v] = ∞;
   d[s] = 0;
   for i=1 to |V|-1
      for each edge (u,v) ∈ E
         Relax(u,v, w(u,v));
   for each edge (u,v) ∈ E
      if (d[v] > d[u] + w(u,v))
           return "no solution";
```

What will be the running time?

A: O(VE)

We look at all E edges V-1 times.

Can we do better?

– Yes, with caveats

```
Relax(u,v,w): if (d[v] > d[u]+w) then d[v]=d[u]+w
```

Dijkstra's Algorithm For Shortest Paths

- **Non-negative edge weight**
- Like BFS: If all edge weights are equal, then use BFS, otherwise use this algorithm
- Use Q = min-priority queue keyed on d[v] values

Dijkstra's Algorithm For Shortest Paths

```
DIJKSTRA(G, s)
  INIT(G, s)
    S←Ø                  > set of discovered nodes
    Q←V[G]               > put all vertices in a queue
  while Q ≠Ø do
      u←EXTRACT-MIN(Q)          PRIORITY QUEUE OPERATION
      S←S U {u}                 (Yet Again)
      for each v in Adj[u] do
          RELAX(u, v)     > May cause
                          > DECREASE-KEY(Q, v, d[v])
```

Example: Initialization Step

```
DIJKSTRA(G, s)
    INIT(G, s)
    S←Ø                 > set of discovered nodes
    Q←V[G]
    while Q ≠Ø do
        u←EXTRACT-MIN(Q)
        S←S U {u}
        for each v in Adj[u] do
            RELAX(u, v)  > May cause
                         > DECREASE-KEY(Q, v, d[v])
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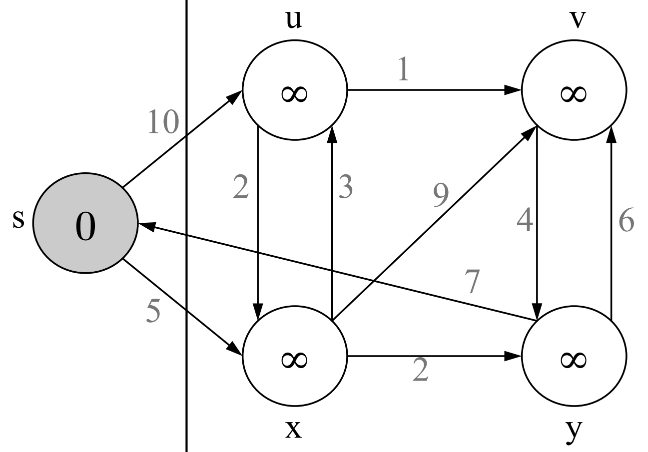

Example

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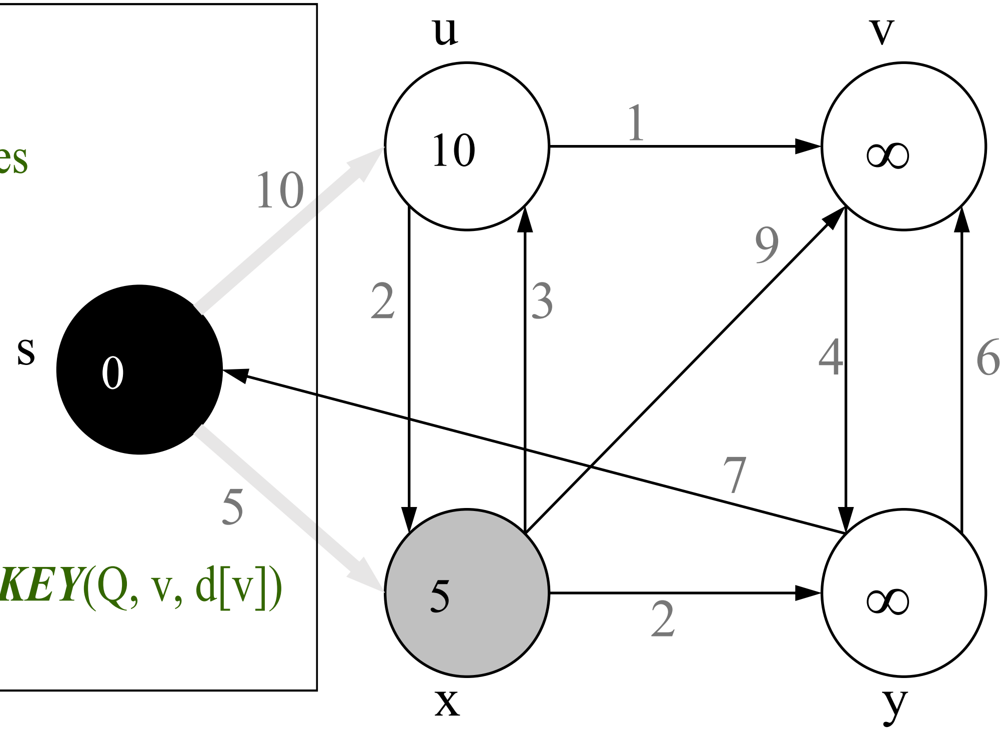

Example

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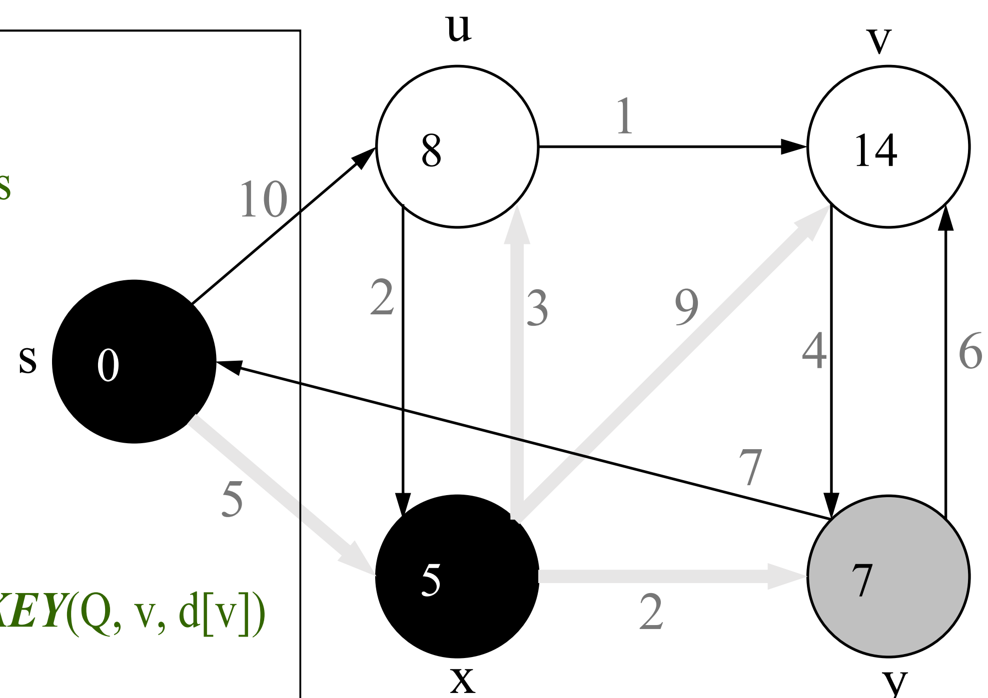

Example

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DIJKSTRA(G, s)
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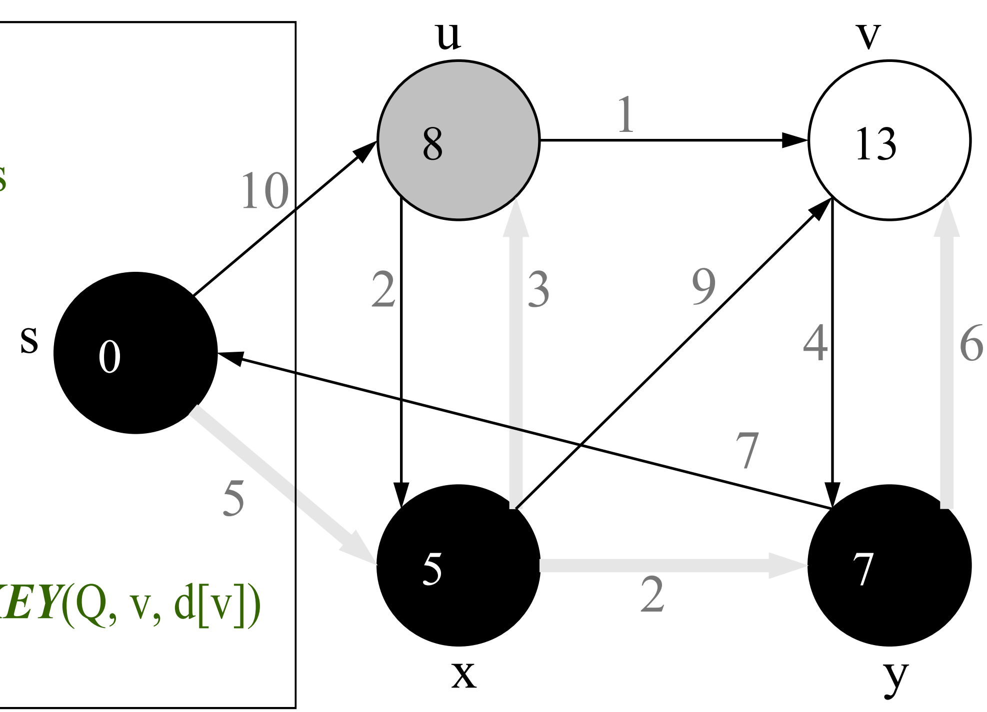

Example

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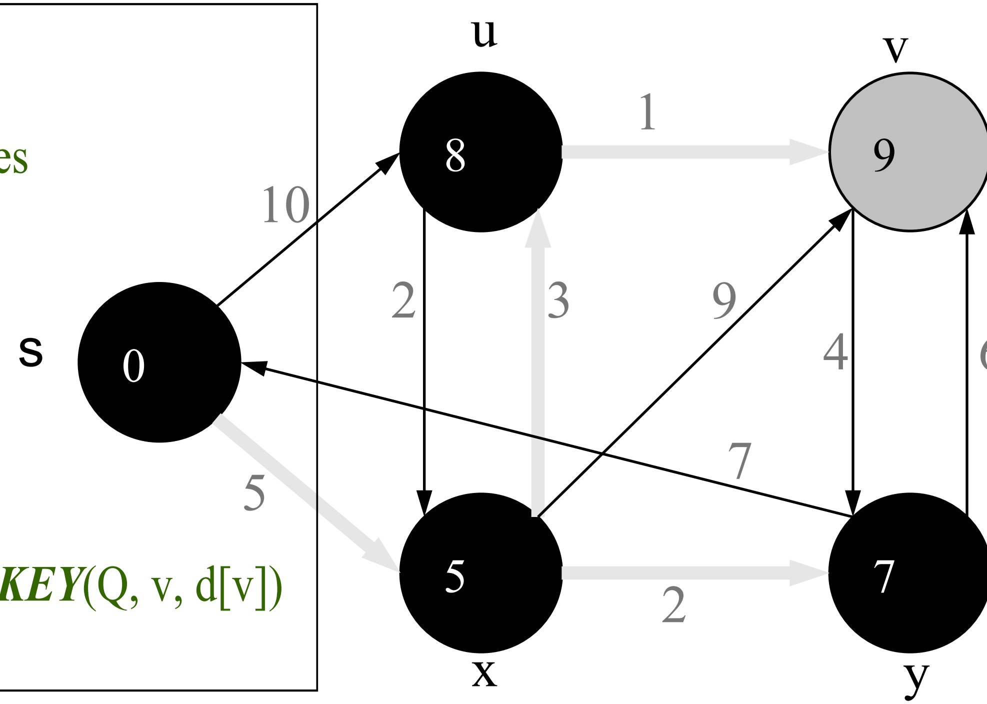
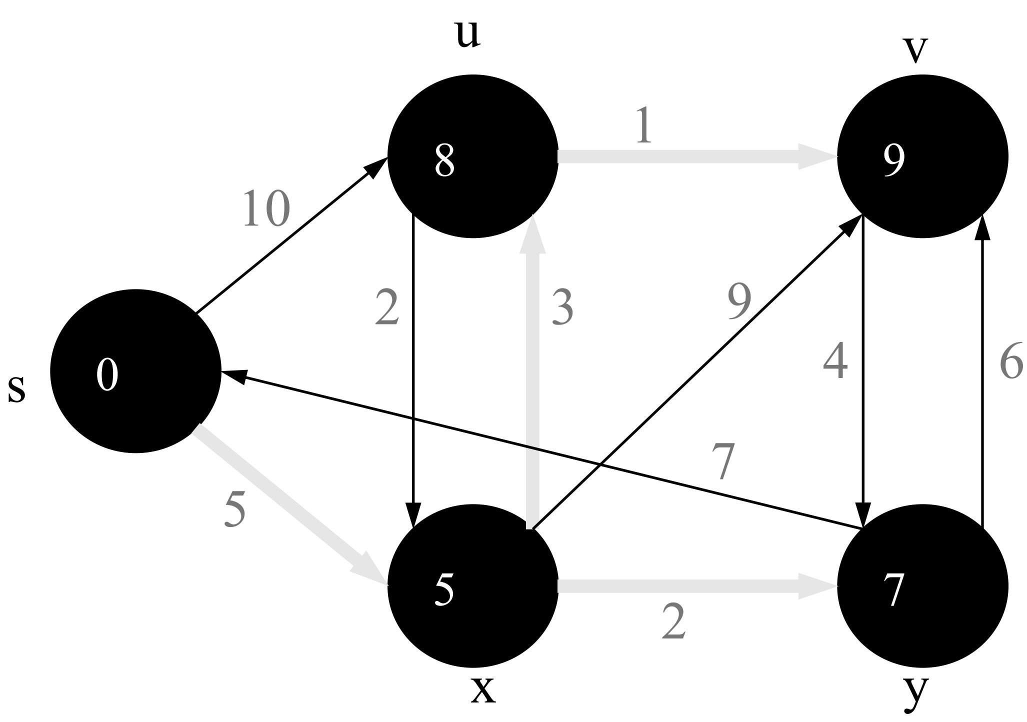


Example



Dijkstra's Algorithm Analysis

O(V)

```
DIJKSTRA(G, s)
    INIT(G, s)
    S←Ø              > set of discovered nodes
    Q←V[G]
    while Q ≠Ø do
        u←EXTRACT-MIN(Q)
        S←S U {u}
        for each v in Adj[u] do
            RELAX(u, v)   > May cause
                          > DECREASE-KEY(Q, v, d[v])
```

Total in the loop: O(V log V)

Total in the loop: O(E log V)

O(log V) for each call

Time Complexity: O (E log V)


That's all Folks!
Any Question?