Graph Algorithms: Breadth First Search (BFS)

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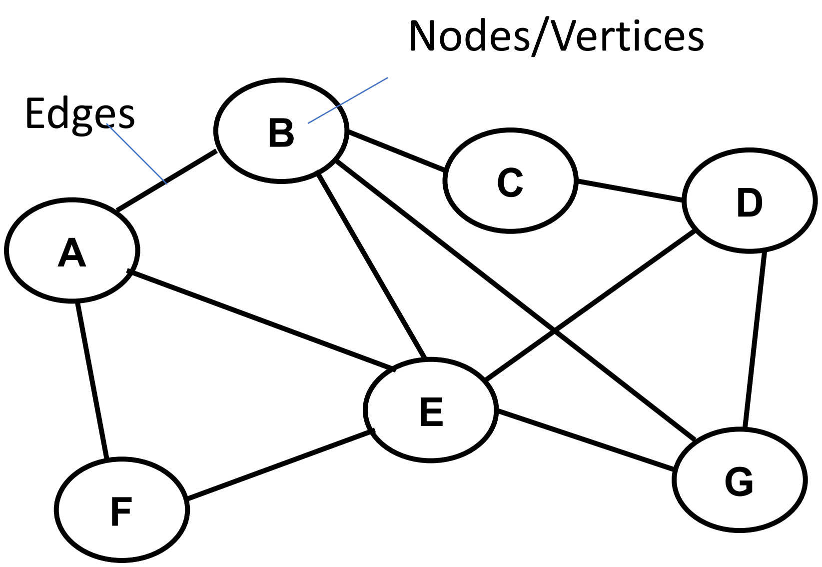
CSC 212

Based on slides from Prof. Carolina Ruiz at WPI

Graphs

- Data structures that connect a set of objects to form a kind of a network
- Objects are called **"Nodes" or "Vertices"**
- Connections are called **"Edges"**
- **Unlike trees graphs may have paths that form loops like "A->B->C-> A"**



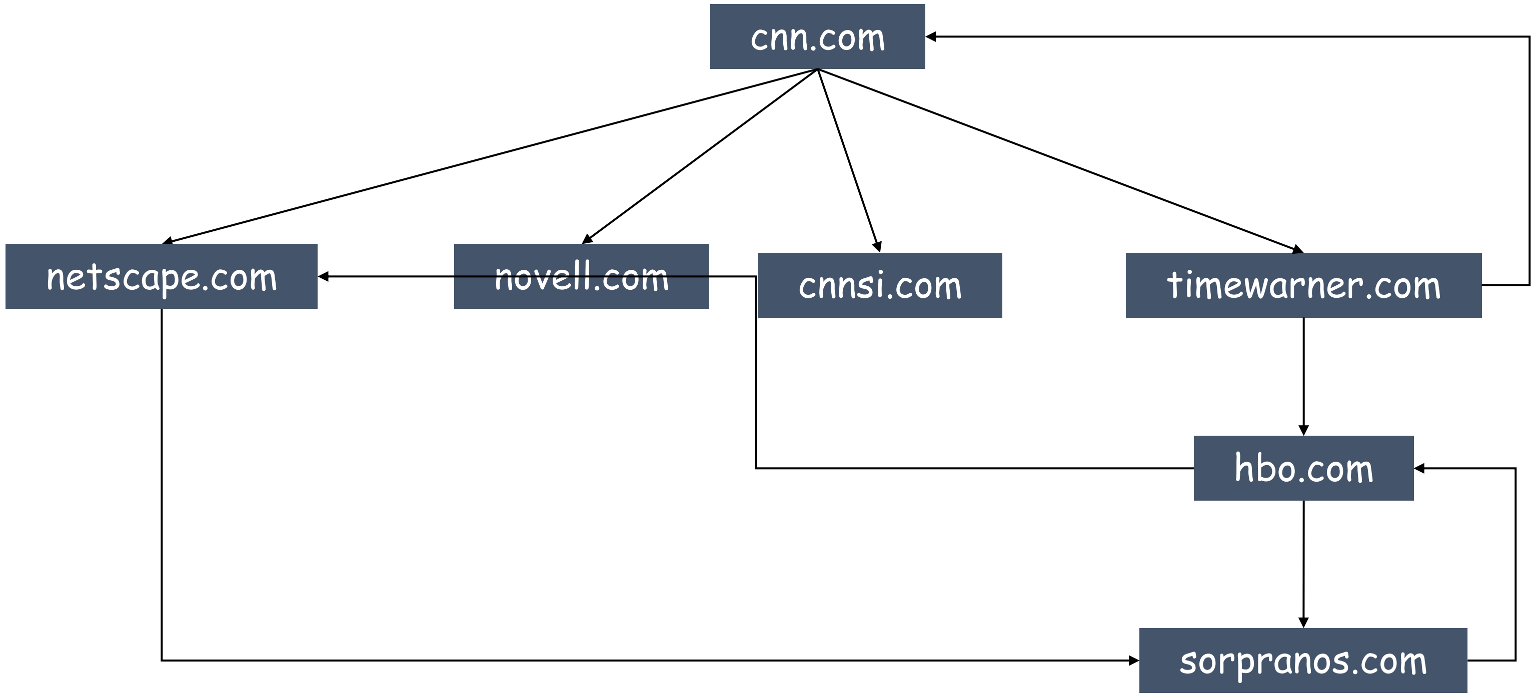
Some Graph Applications

Graph	*Nodes*	*Edges*
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires

World Wide Web

Web graph.

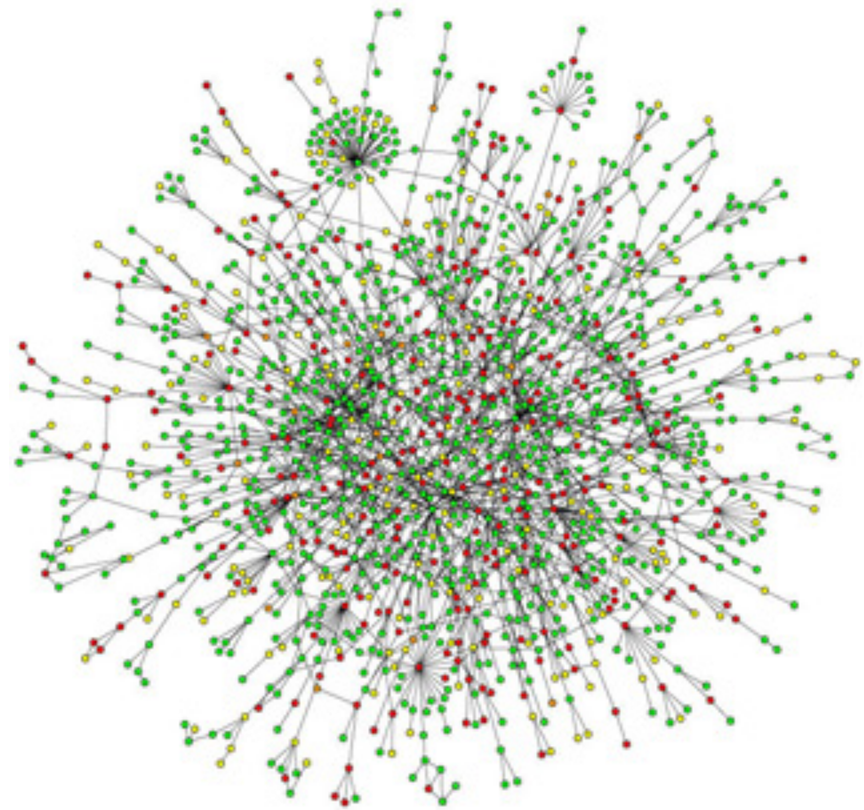
- Node: web page.
- Edge: hyperlink from one page to another.



Protein Networks

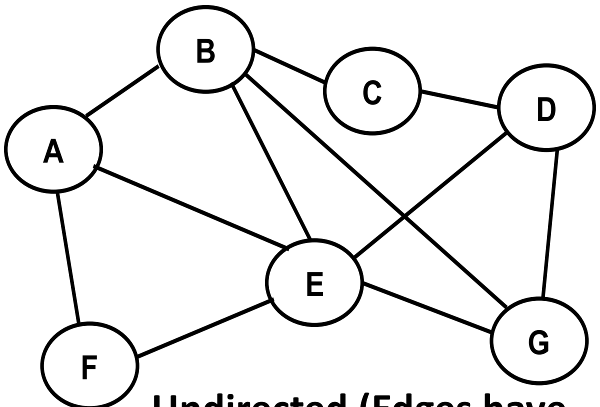
Nodes are proteins

Edges are connections (interaction between proteins)

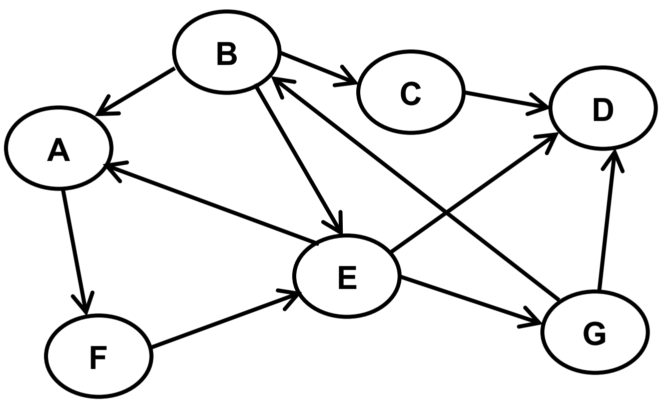


Types of Graphs

Directed vs. undirected

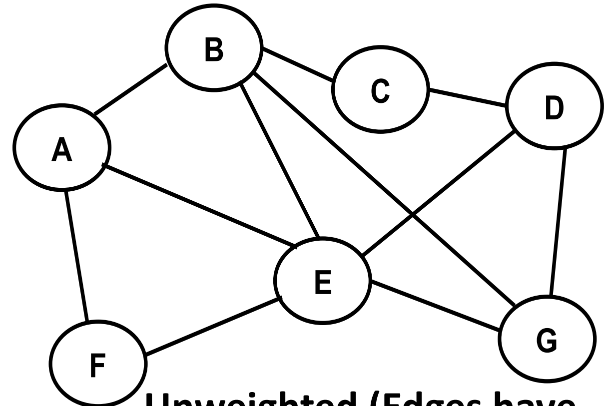


Undirected (Edges have no direction)

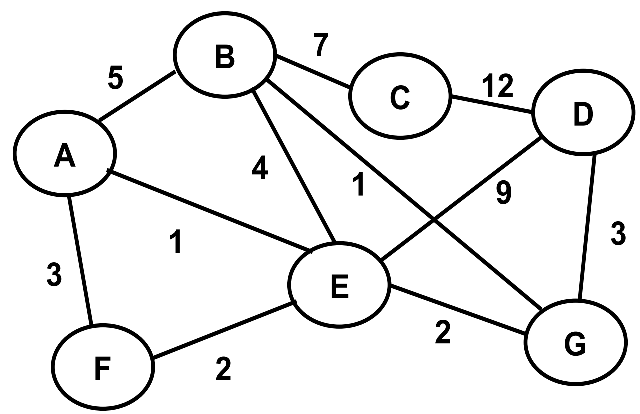


Directed (Edges have directions)

Weighted vs. unweighted



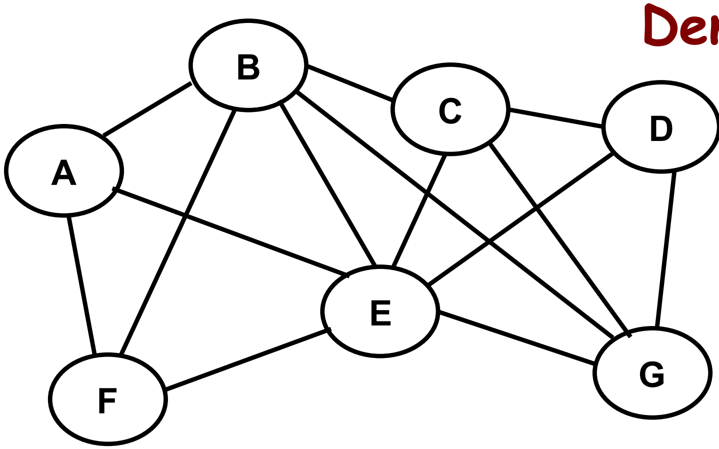
Unweighted (Edges have no cost/weight)



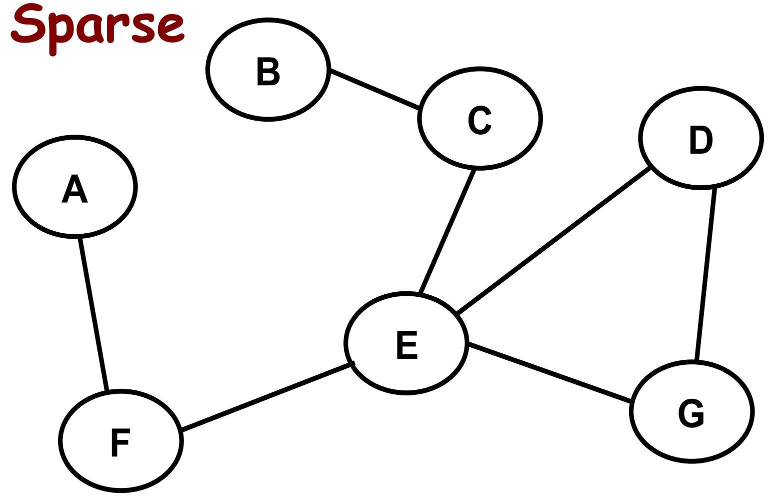
Weighted (Edges have associated cost/weight)

Types of Graphs (Cont'd)

Dense vs. Sparse



Dense graphs (many edges between nodes)



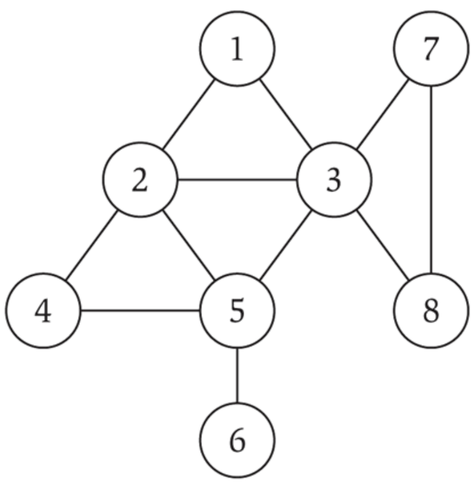
Sparse graphs (few edges between nodes)

- **If the graph has n vertices (nodes) ➔ Maximum # of edges is $(n^2\text{-}n)/2 = O(n^2)$**
- **In dense graphs number of edges is close to $O(n^2)$**
- **In sparse graphs number of edges is close to $O(n)$**

Undirected Graphs

Undirected graph. G = (V, E)

- V = set of nodes or vertices
- E = edges between pairs of nodes.
- Graph size parameters: n = |V|, m = |E|.



V = { 1, 2, 3, 4, 5, 6, 7, 8 }

E = { 1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6 }

n = |V| = 8

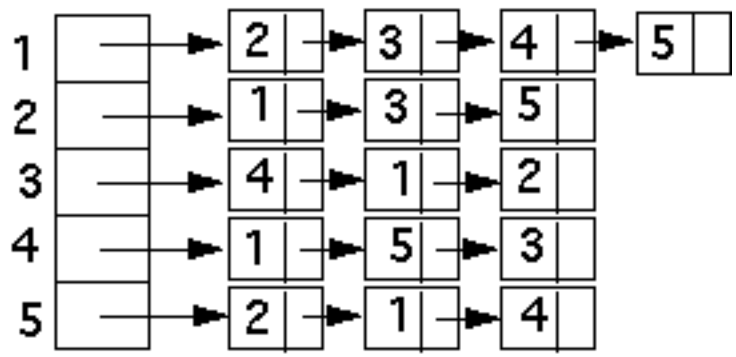
m = |E| = 11

Graph Representation

Two main methods

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	0	0	1	1
C	1	0	0	0	0	1
D	1	0	0	0	0	0
E	0	1	0	0	0	0
F	0	1	1	0	0	0

Adjacency Matrix

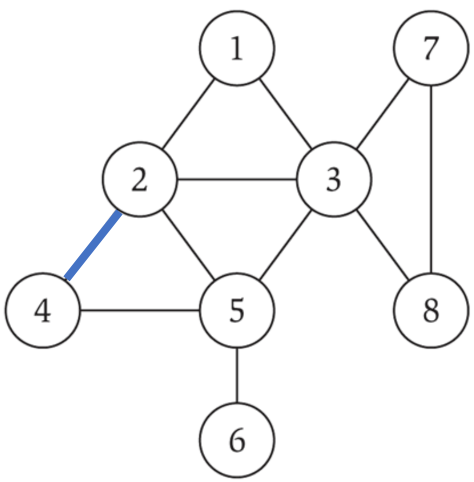


Adjacency List

Adjacency Matrix

Adjacency matrix. |V|-by-|V| matrix (A)

- **A[i, j] = 1 if exists edge between node i and node j**
- **Space proportional to $|V|^2$**
- **Checking if (u, v) is an edge takes O(1) time.**
- **Identifying all edges takes $O(|V|^2)$ time.**
- **For undirected graph → matrix is symmetric across the diagonal**



Vertices

Vertices

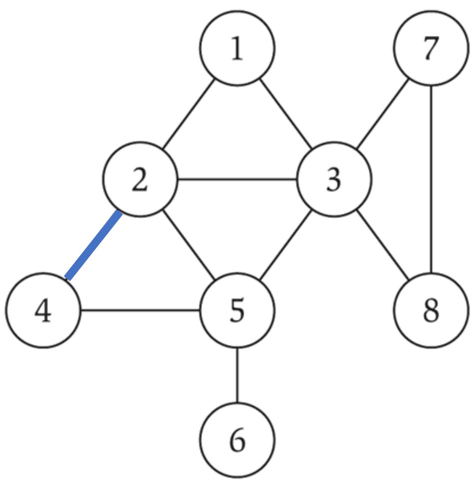
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	1	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

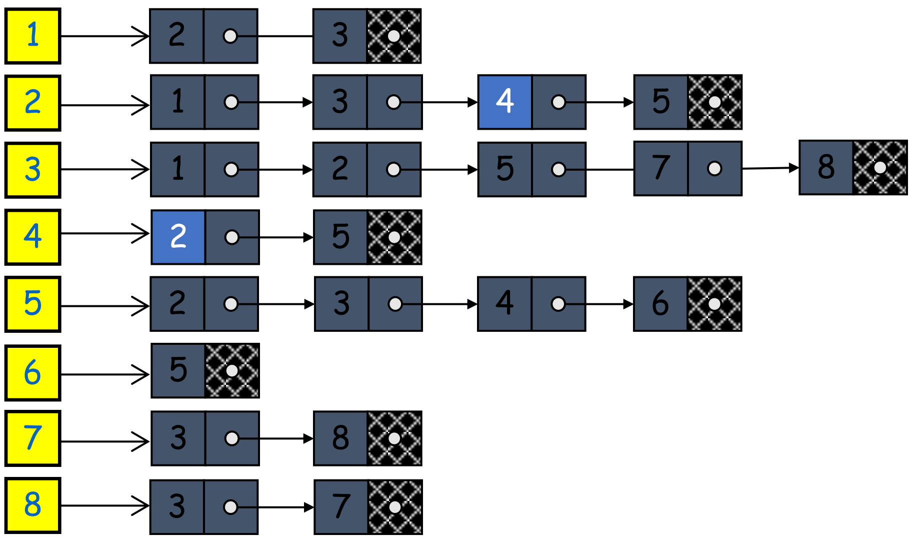
Adjacency List

Adjacency list. Node indexed array of lists.

- Two representations of each edge.
- Space proportional to **O(E + V).**
- Checking if (u, v) is an edge takes O(deg(u)) time.
- Identifying all edges takes **O(E + V) time.**

degree = number of neighbors of u



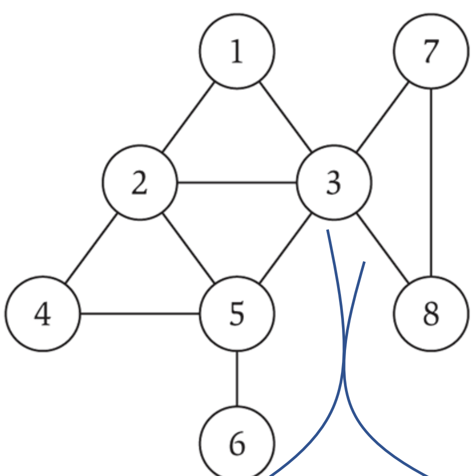


Degree of a Node

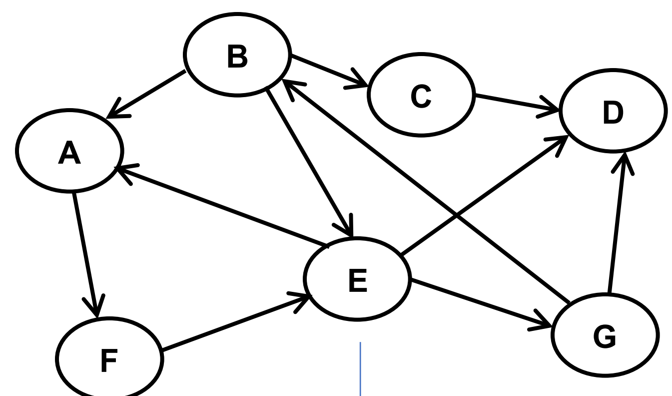
In-degree(v): Number of edges coming to (entering) node v

Out-degree(v): Number of edges getting out (leaving) node v

For Undirected graphs ➔ In-degree = Out-degree



In-degree(3) = Out-degree(3) = 5

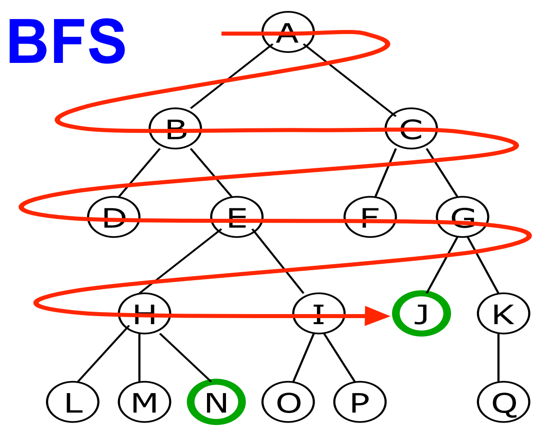


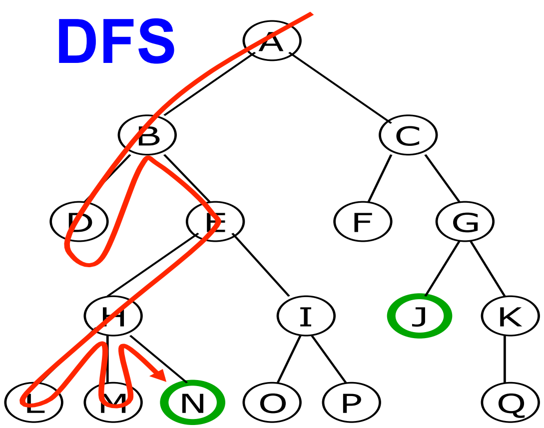
In-degree(E) = 2
Out-degree(E) = 3

Each vertex will have different In-Degree and Out-Degree

Graph Traversal

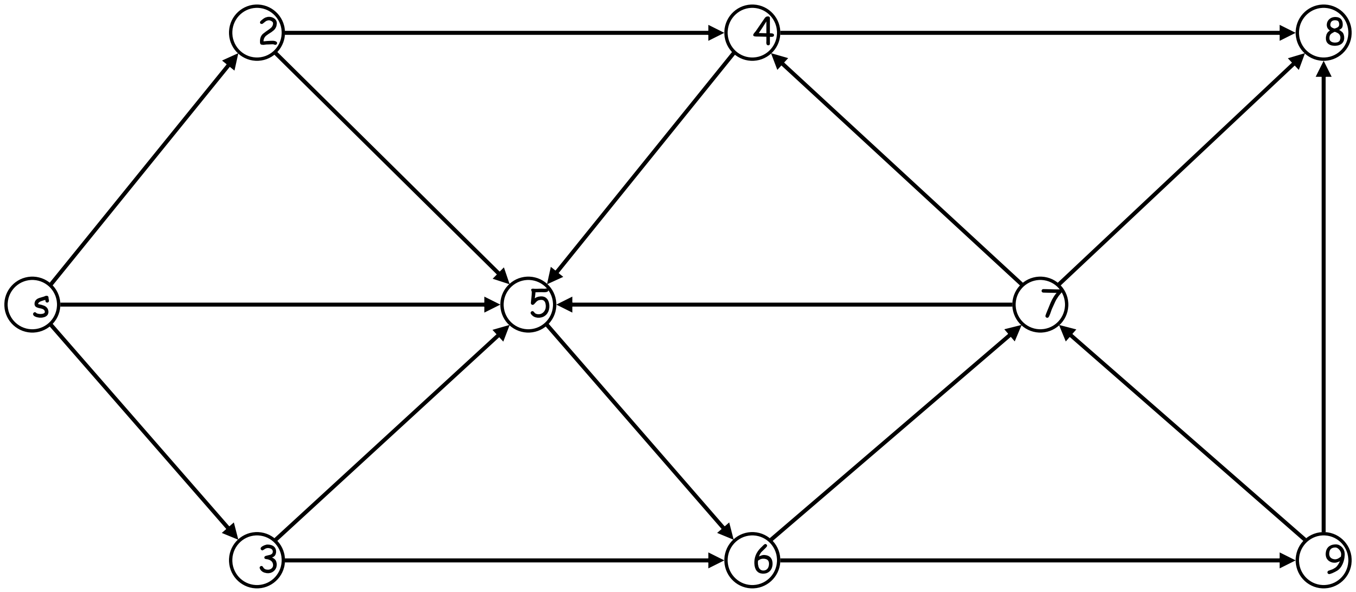
- **Graph Traversal means visiting each node in the graph**
- **There is a starting node (s)**
- **Two main types of traversal**
  - **Breadth-First-Search (BFS)**
  - **Depth-First-Search (DFS)**
- **Both are applicable for *directed* and *undirected graphs***



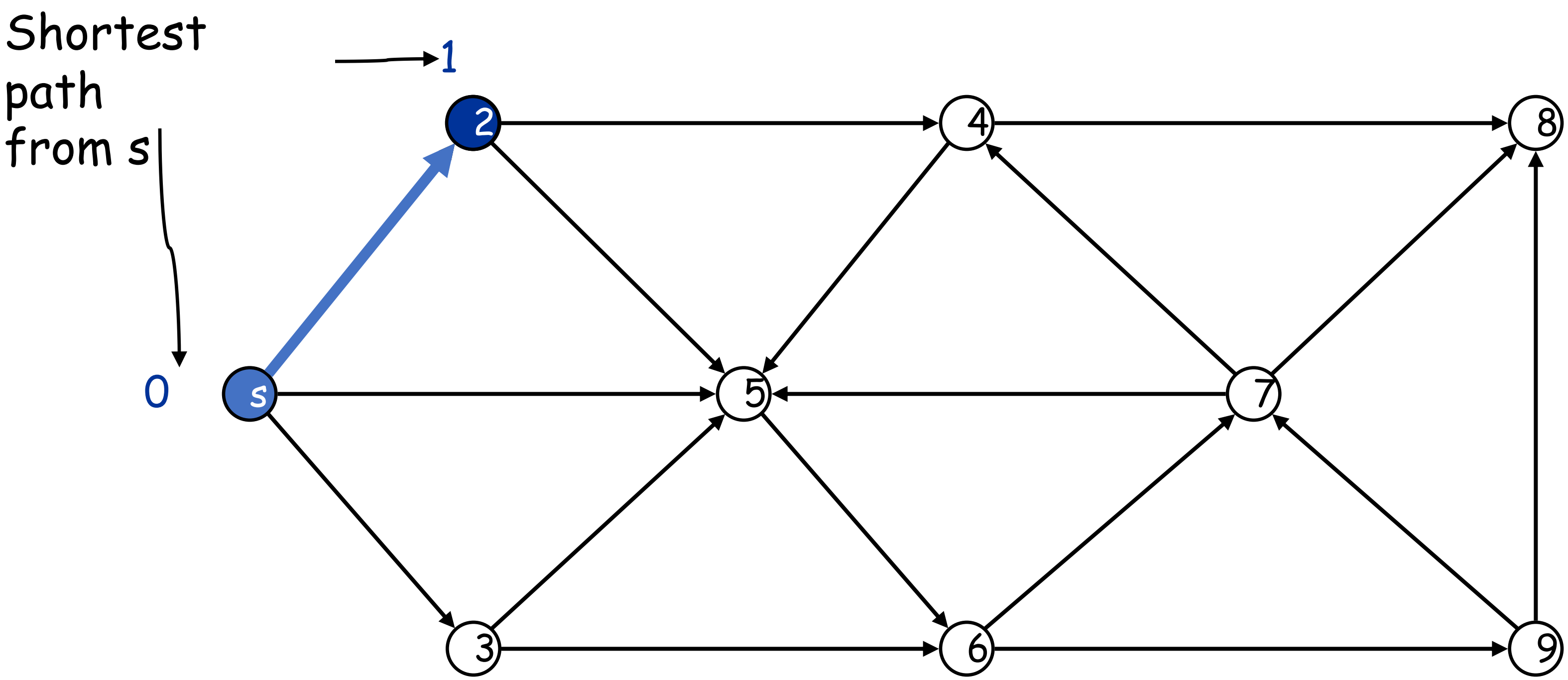


BFS i.e., Breadth-First Traversal

- **Even though the approach is called Breadth-First Search (BFS) it is essentially a traversal algorithm.**
  - **The idea is to traverse the graph is a specific way, which can be leverage for search**
    - **As we are traversing the graph we can check if the node we are on has what we want**
- **Visit the nodes one-level at a time**
- **Requires a queue (First-come-first-served)**



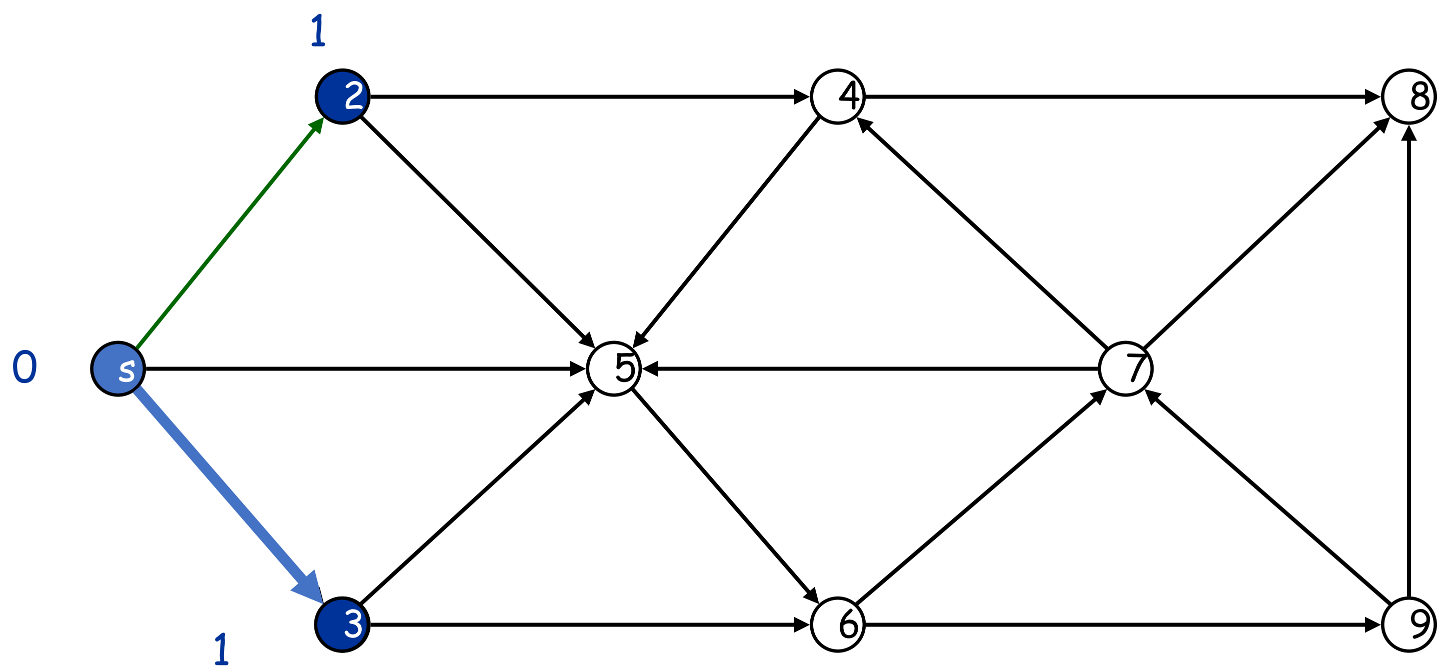
BFS Example



Undiscovered
Discovered
Top of queue
Finished

Queue: s

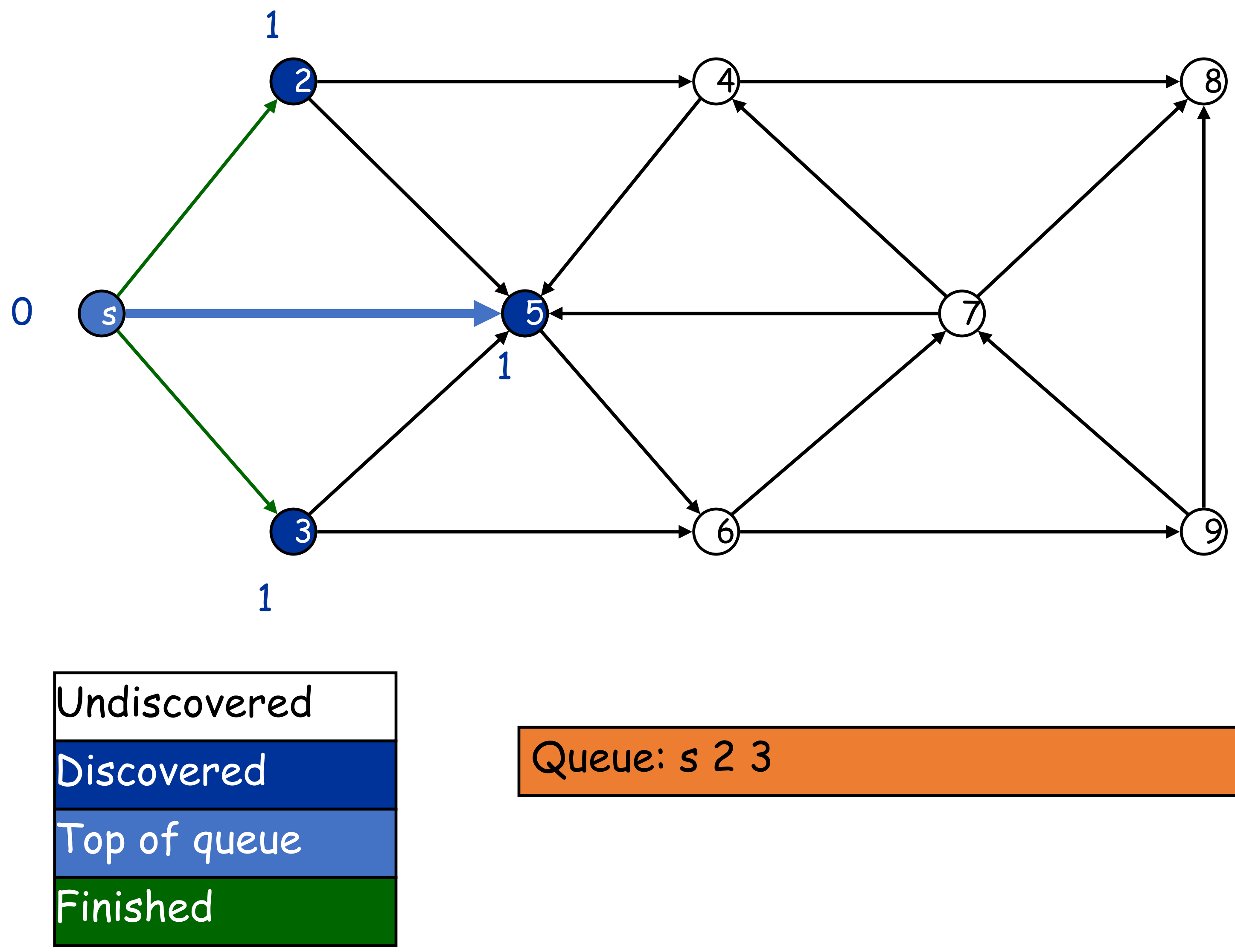
BSF Example



Undiscovered
Discovered
Top of queue
Finished

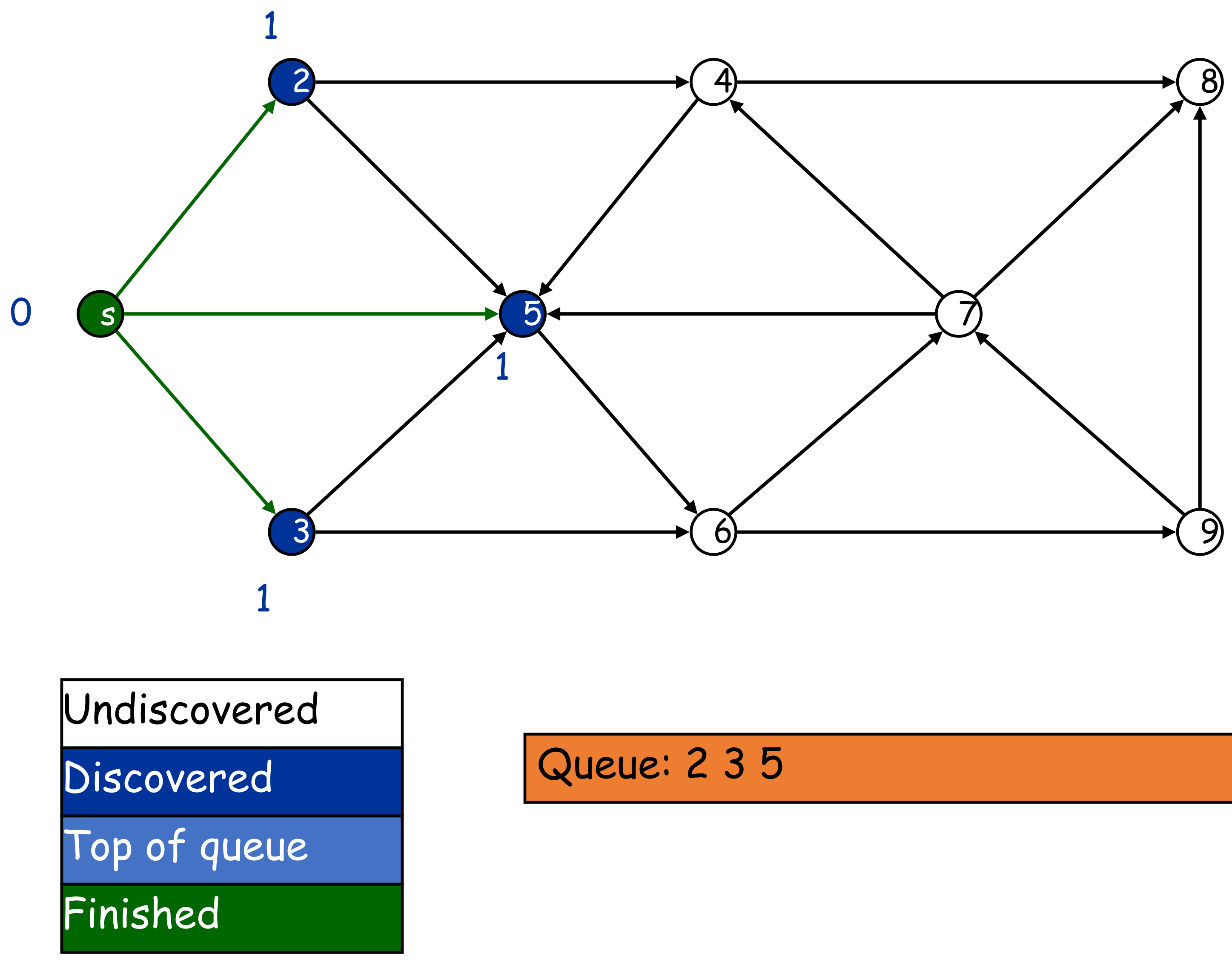
Queue: s 2

BFS Example



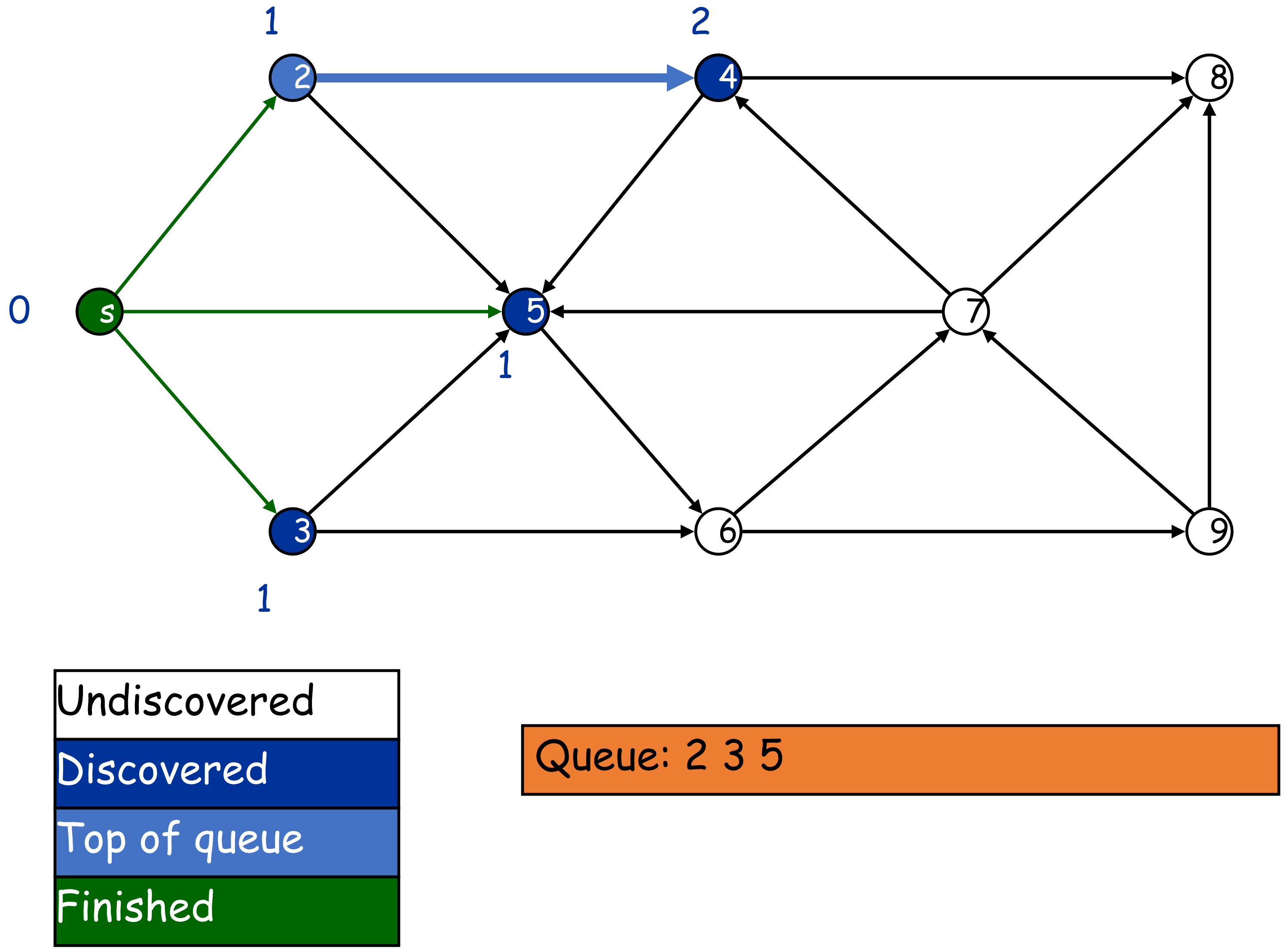
Queue: s 2 3

BFS Example



Queue: 2 3 5

BFS Example


1
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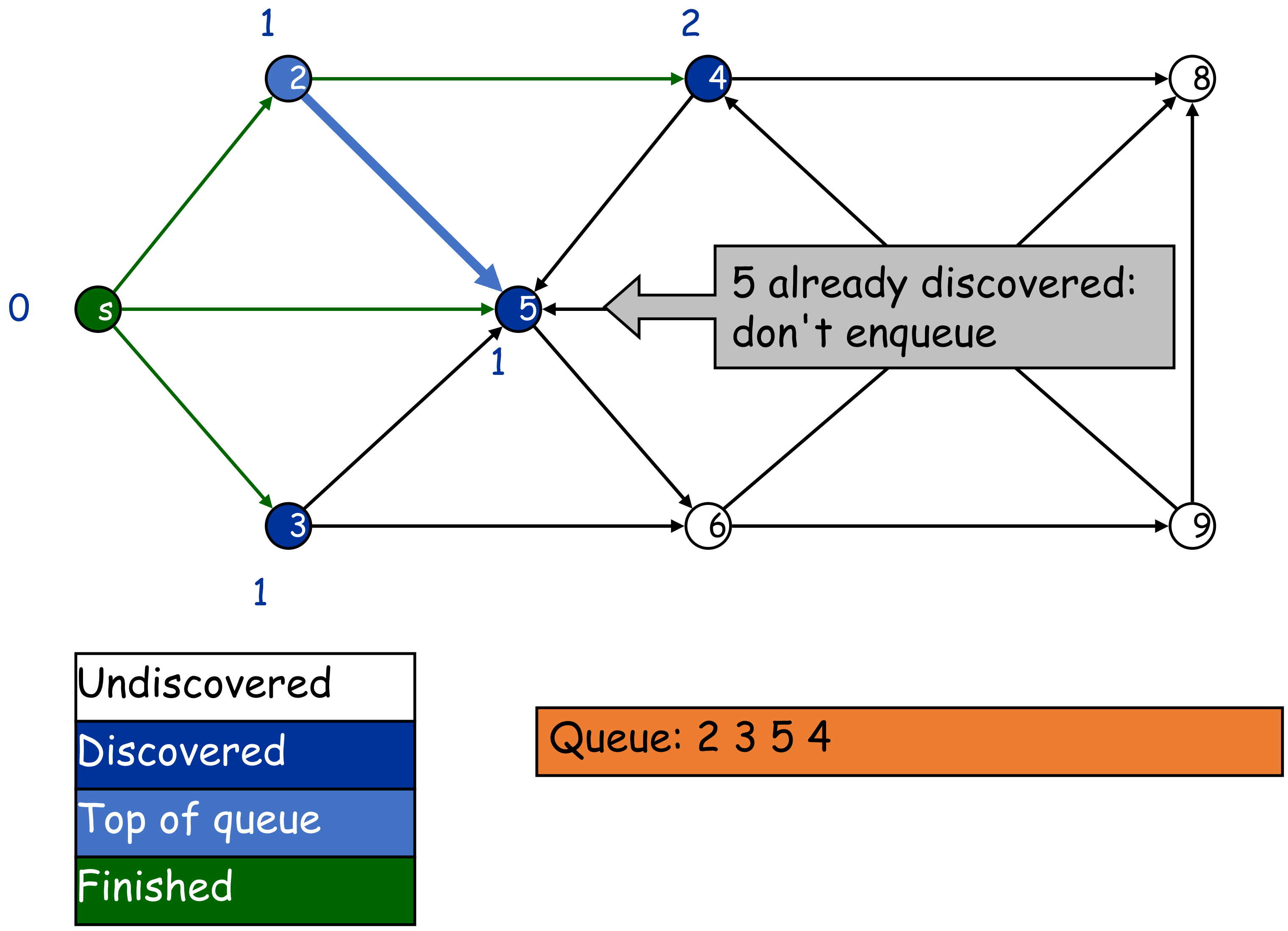
Undiscovered
Discovered
Top of queue
Finished
Queue: 2 3 5

BFS Example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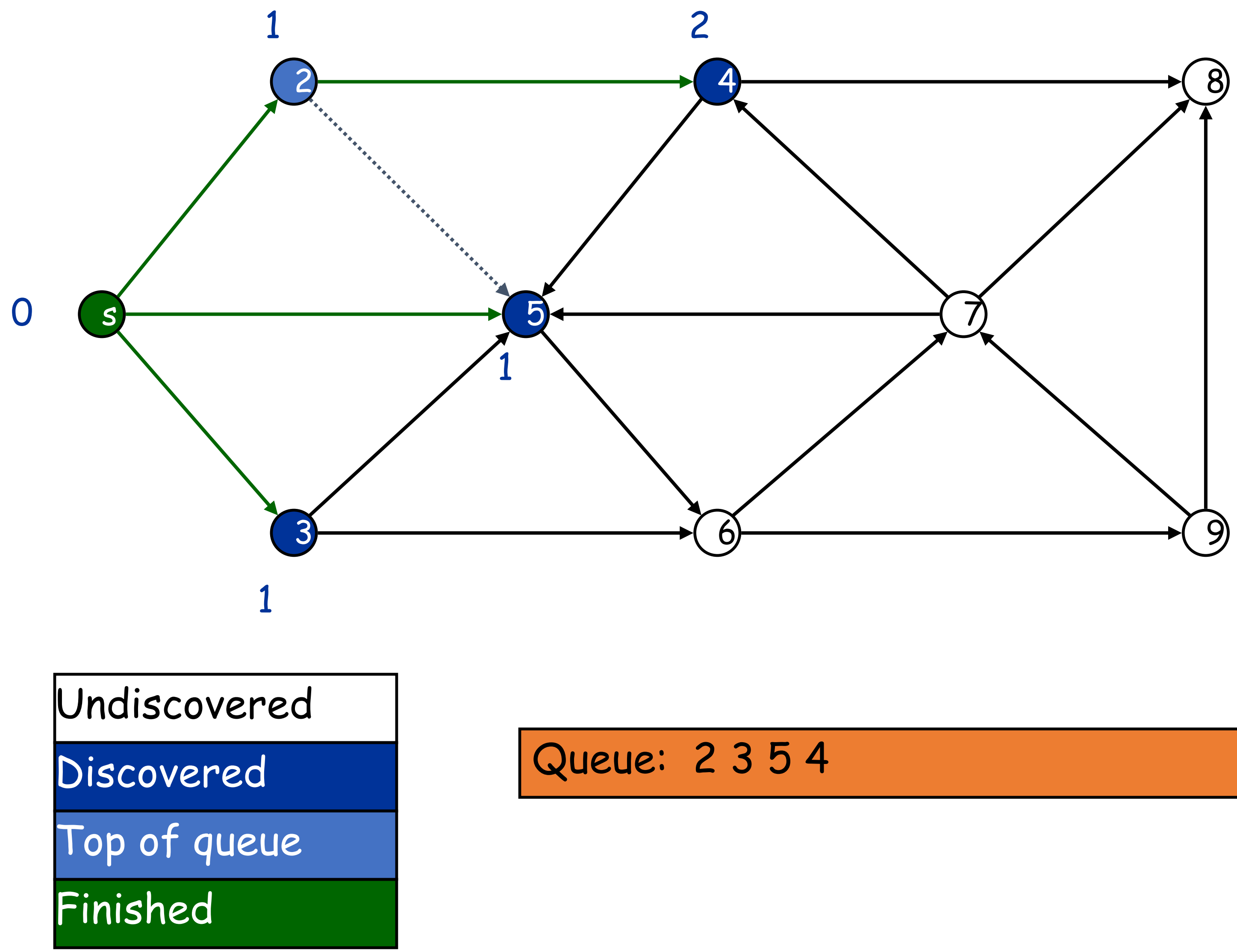
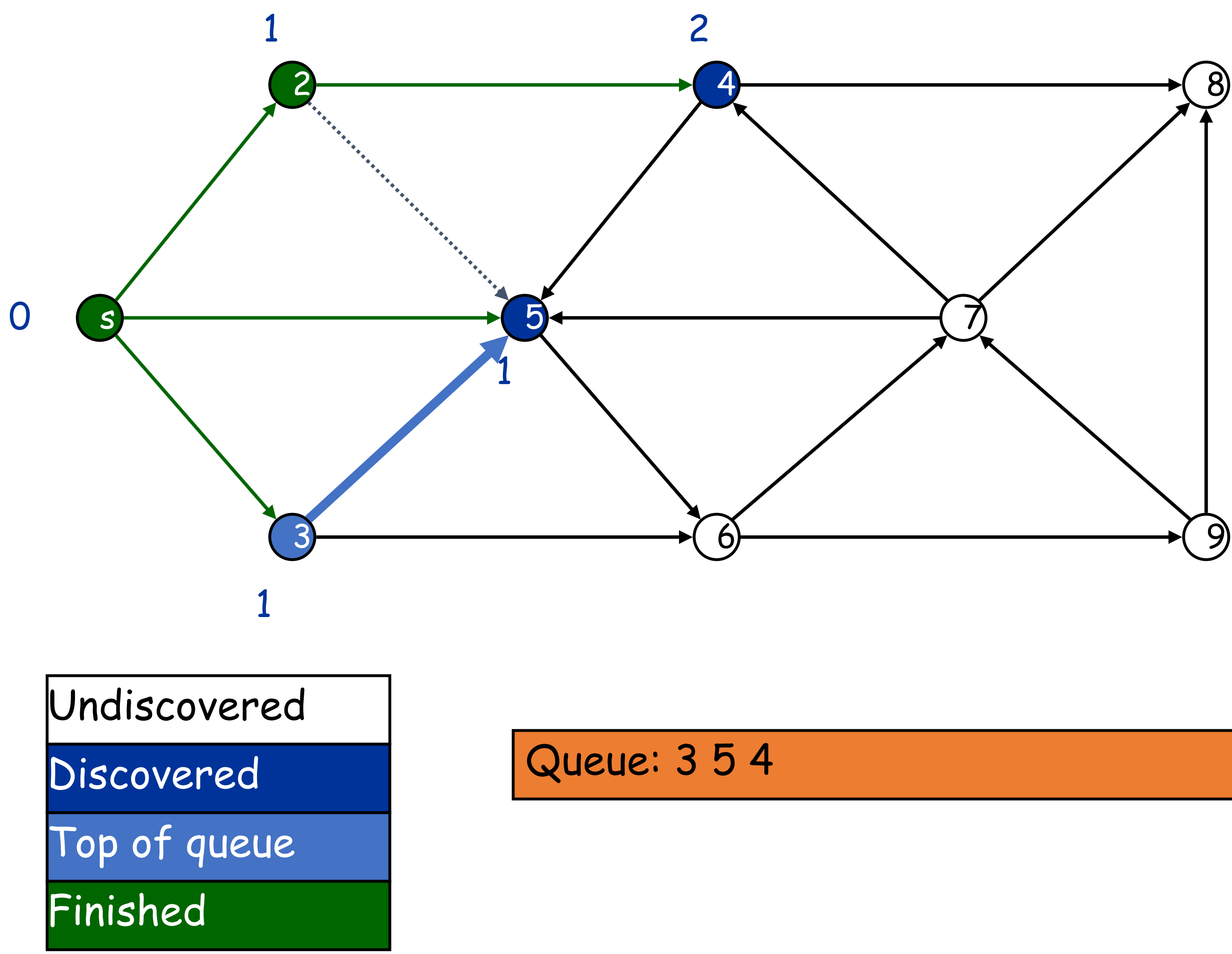
5 already discovered: don't enqueue
Undiscovered
Discovered
Top of queue
Finished
Queue: 2 3 5 4

BFS Example



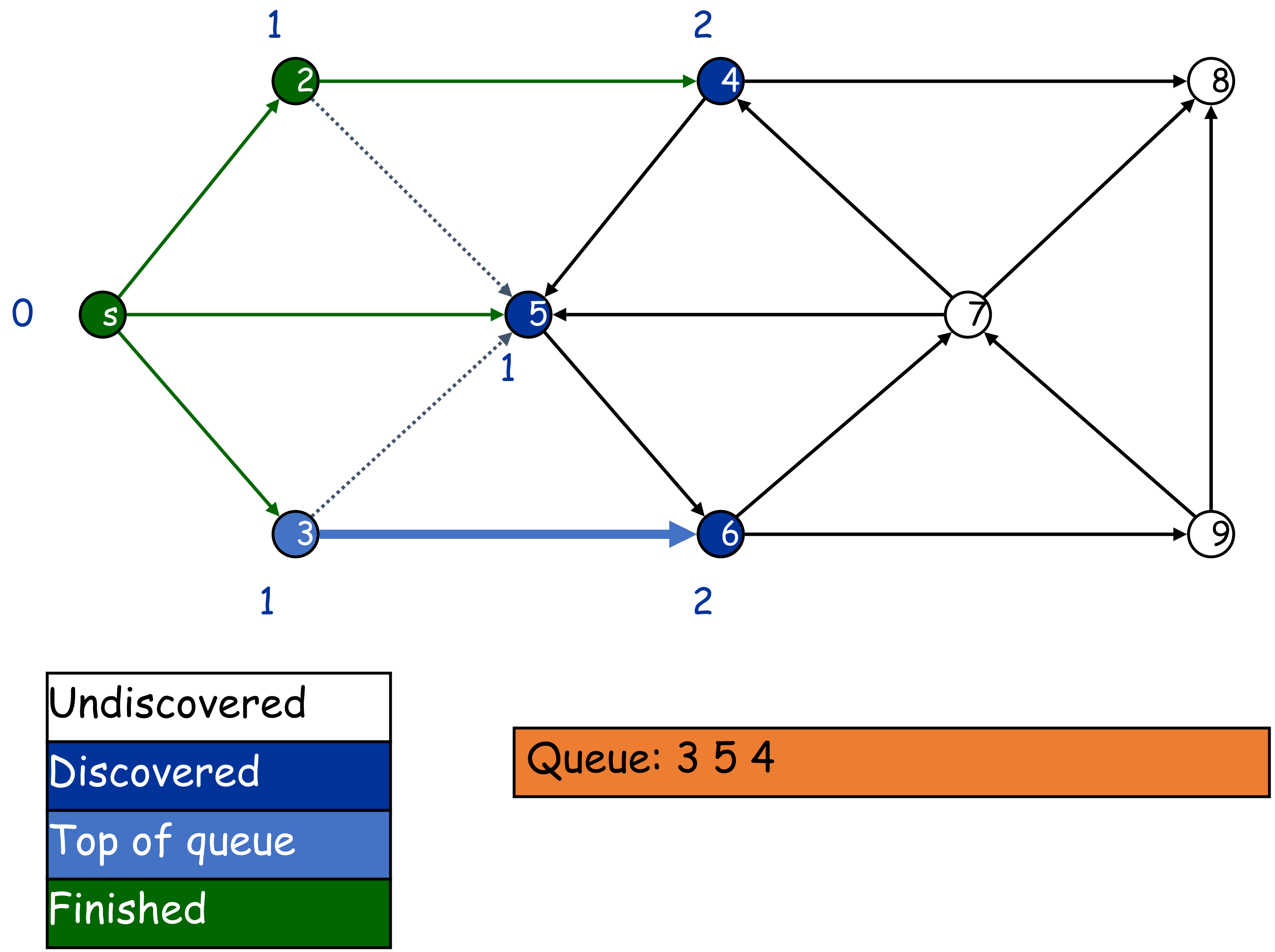
Queue: 2 3 5 4

BFS Example



Queue: 3 5 4

BFS Example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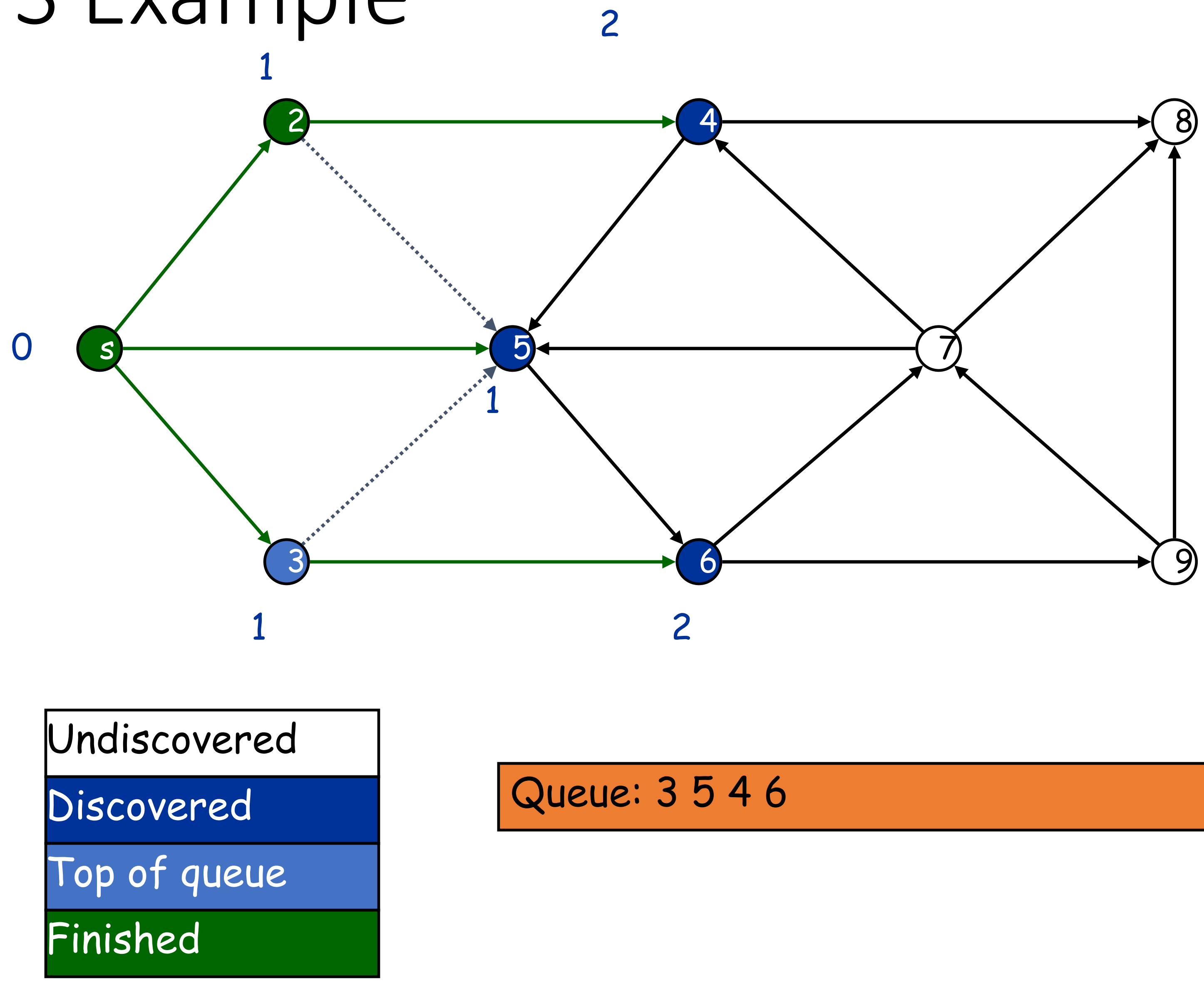
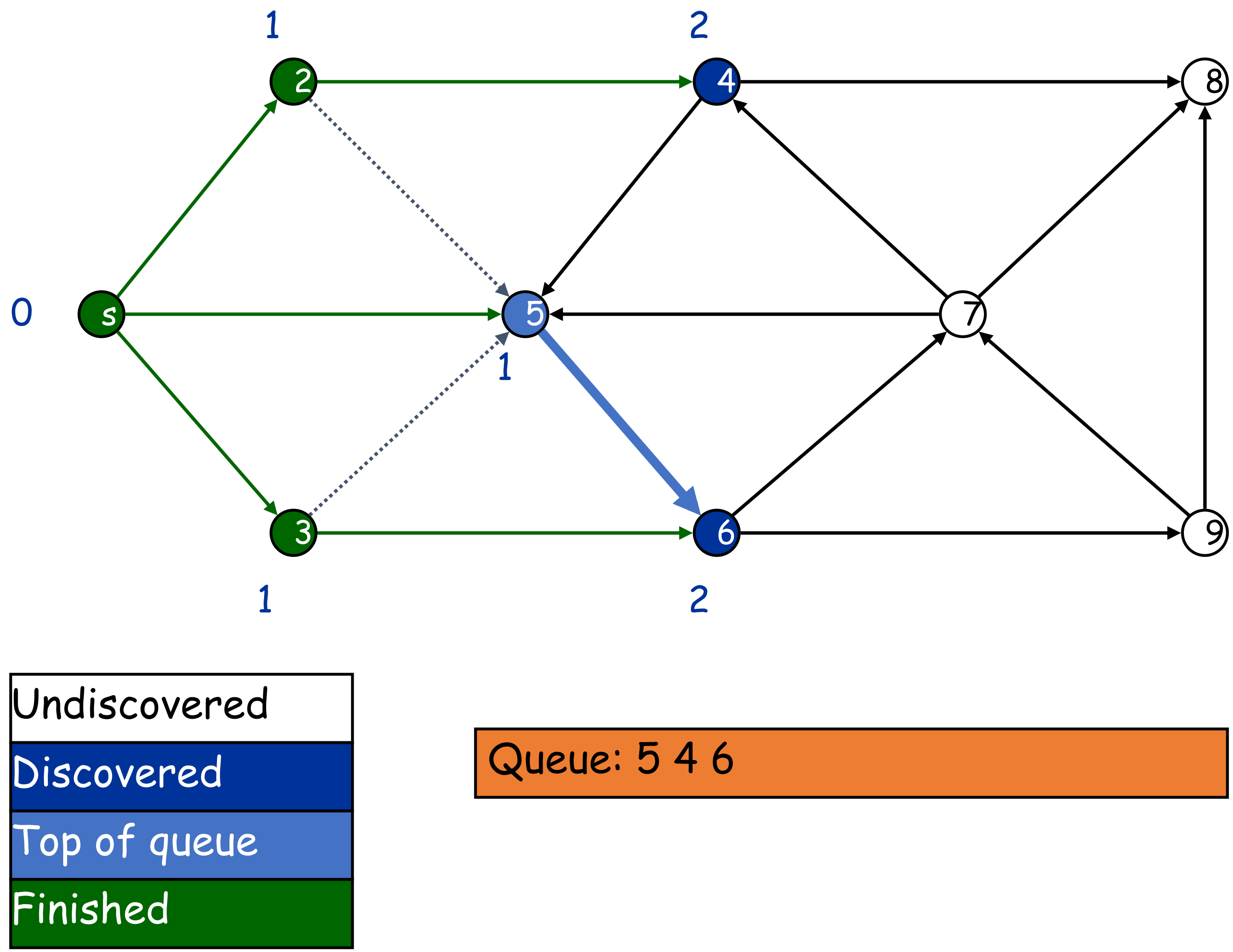
Undiscovered
Discovered
Top of queue
Finished
Queue: 3 5 4

BFS Example



Queue: 3 5 4 6

BFS Example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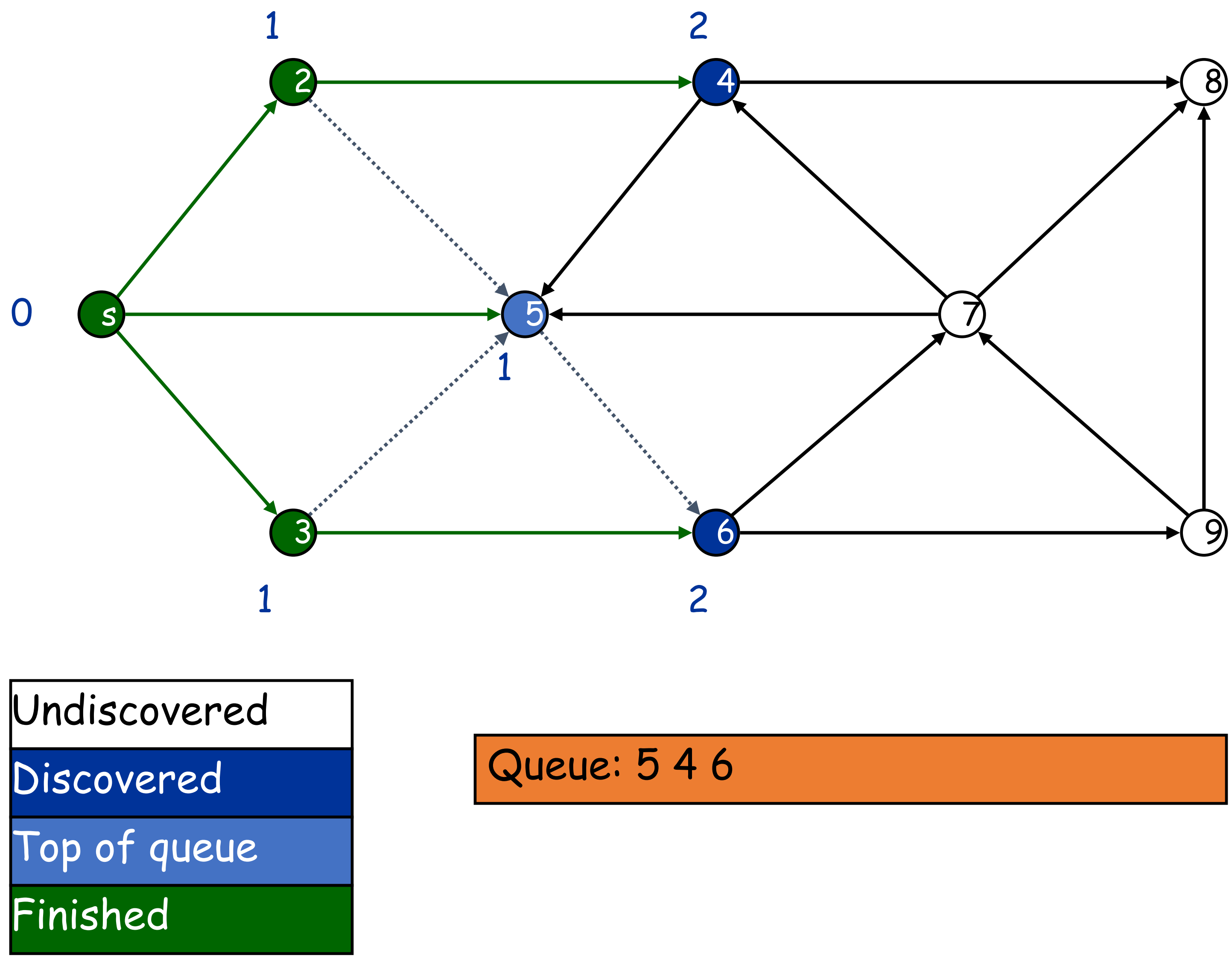
Undiscovered
Discovered
Top of queue
Finished
Queue: 5 4 6

BFS Example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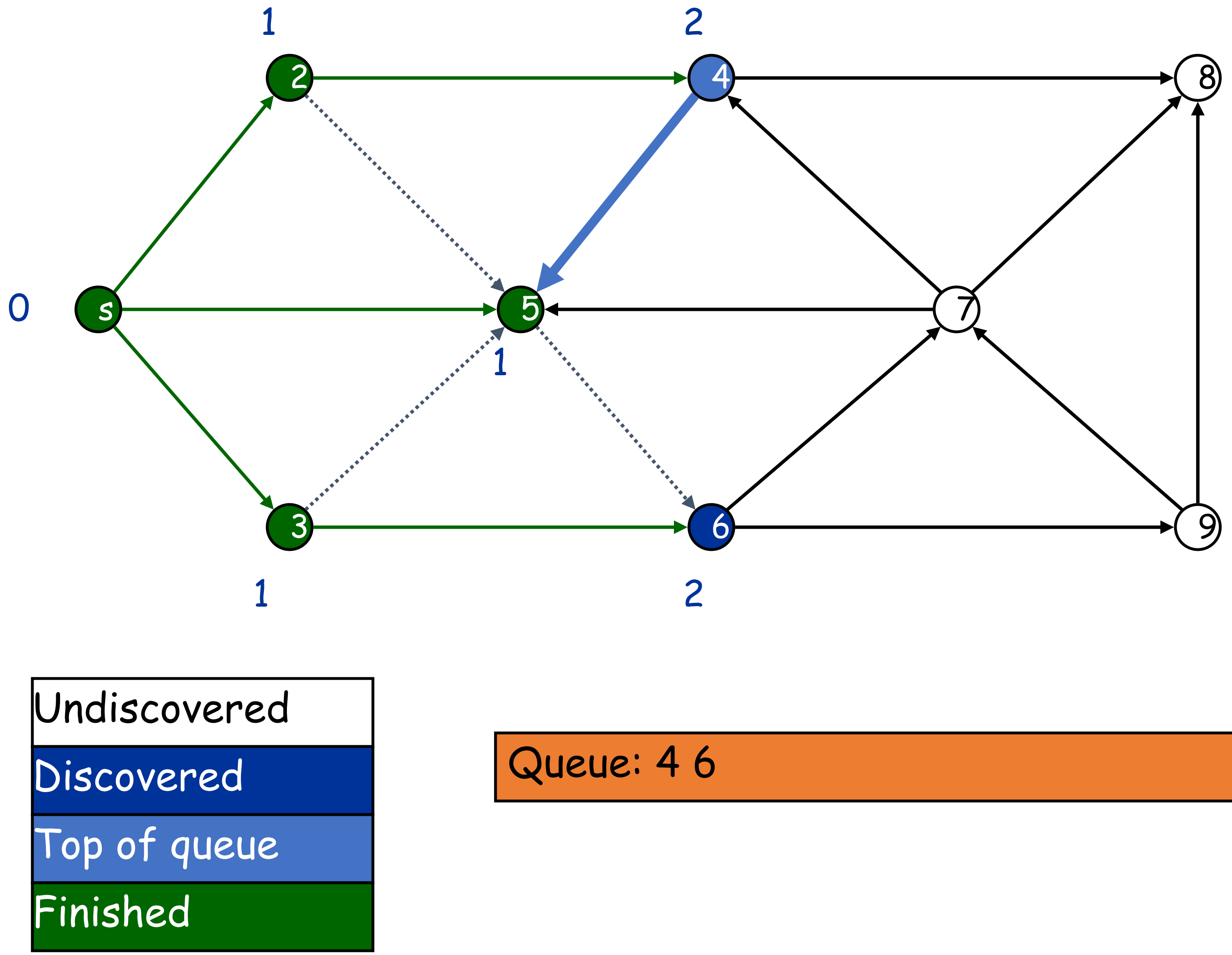
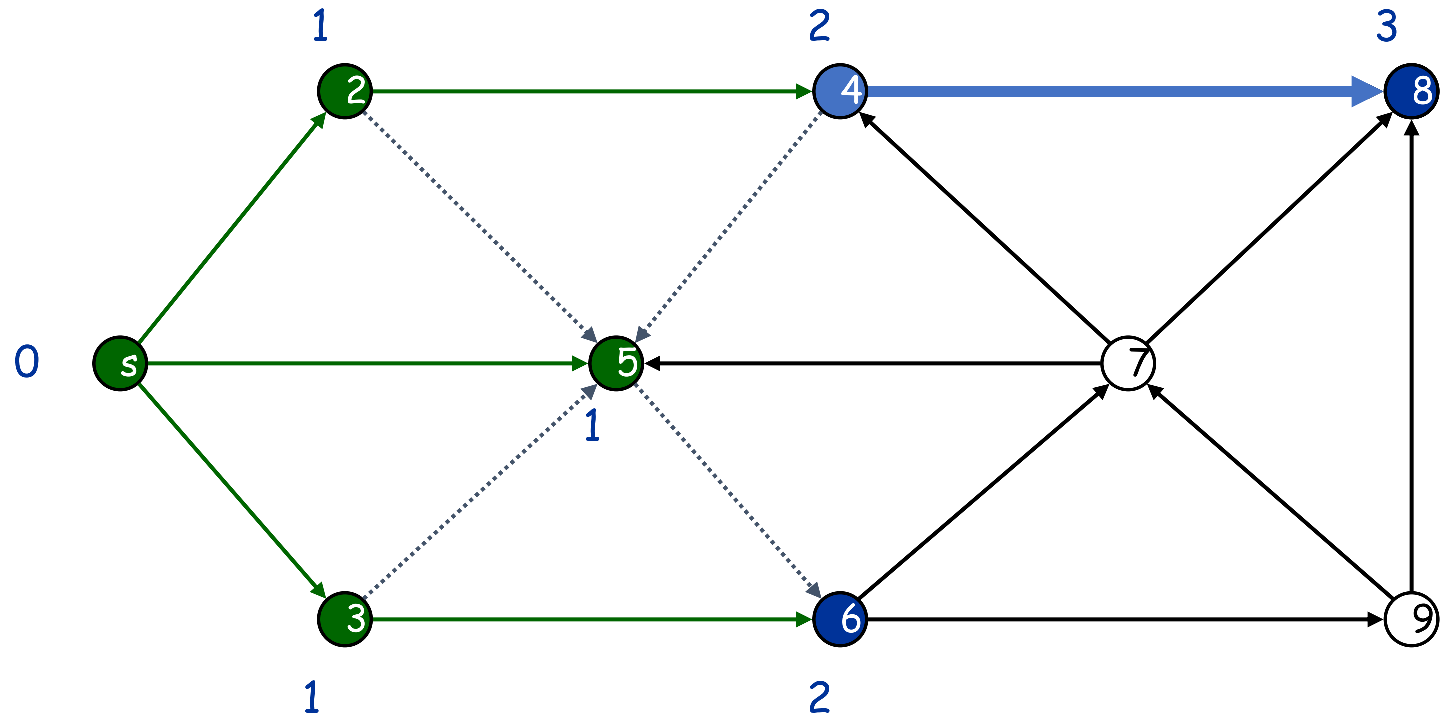
Undiscovered
Discovered
Top of queue
Finished
Queue: 5 4 6

BFS Example



Queue: 4 6

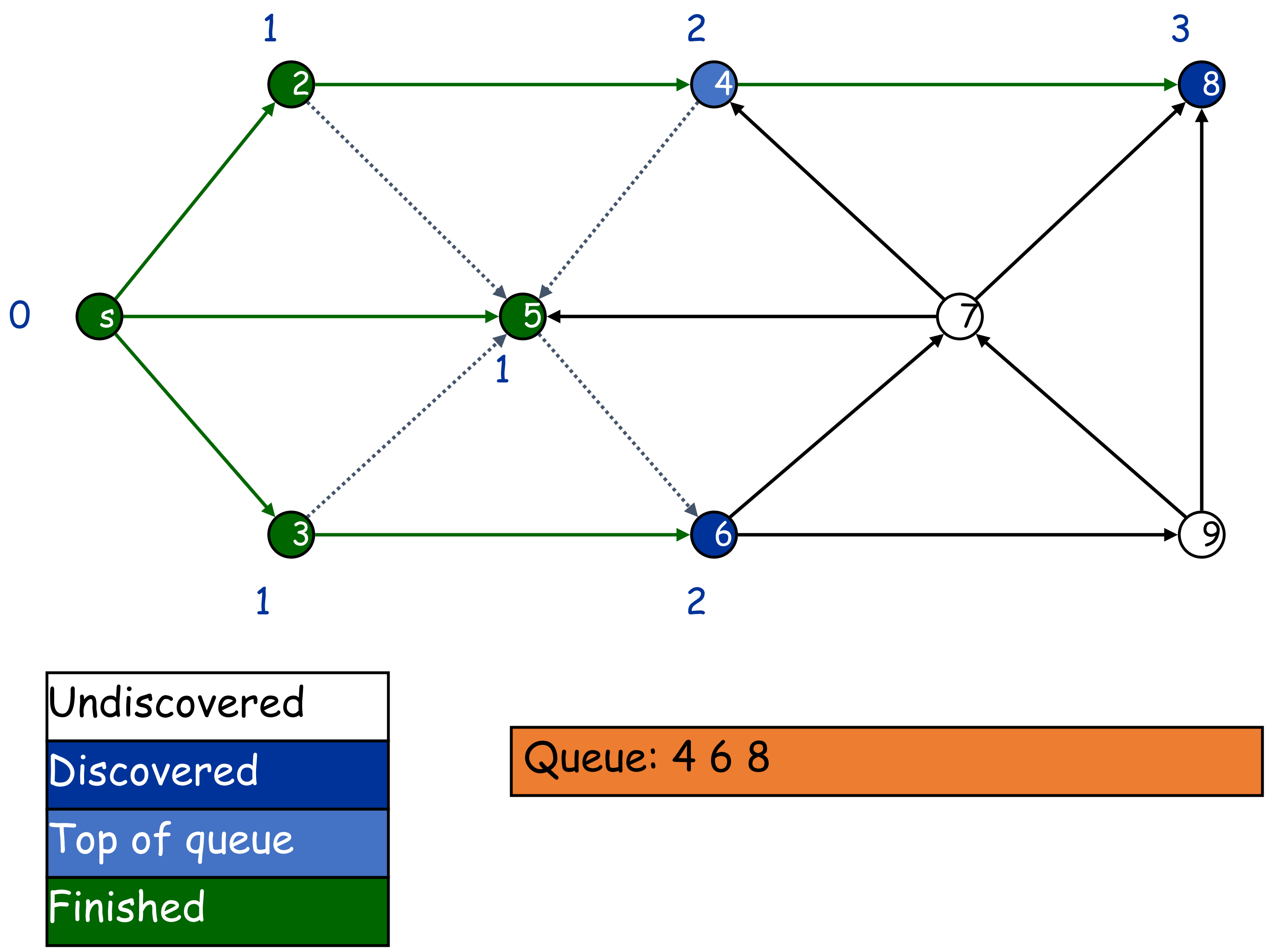
BFS Example



Undiscovered
Discovered
Top of queue
Finished

Queue: 4 6

BFS Example



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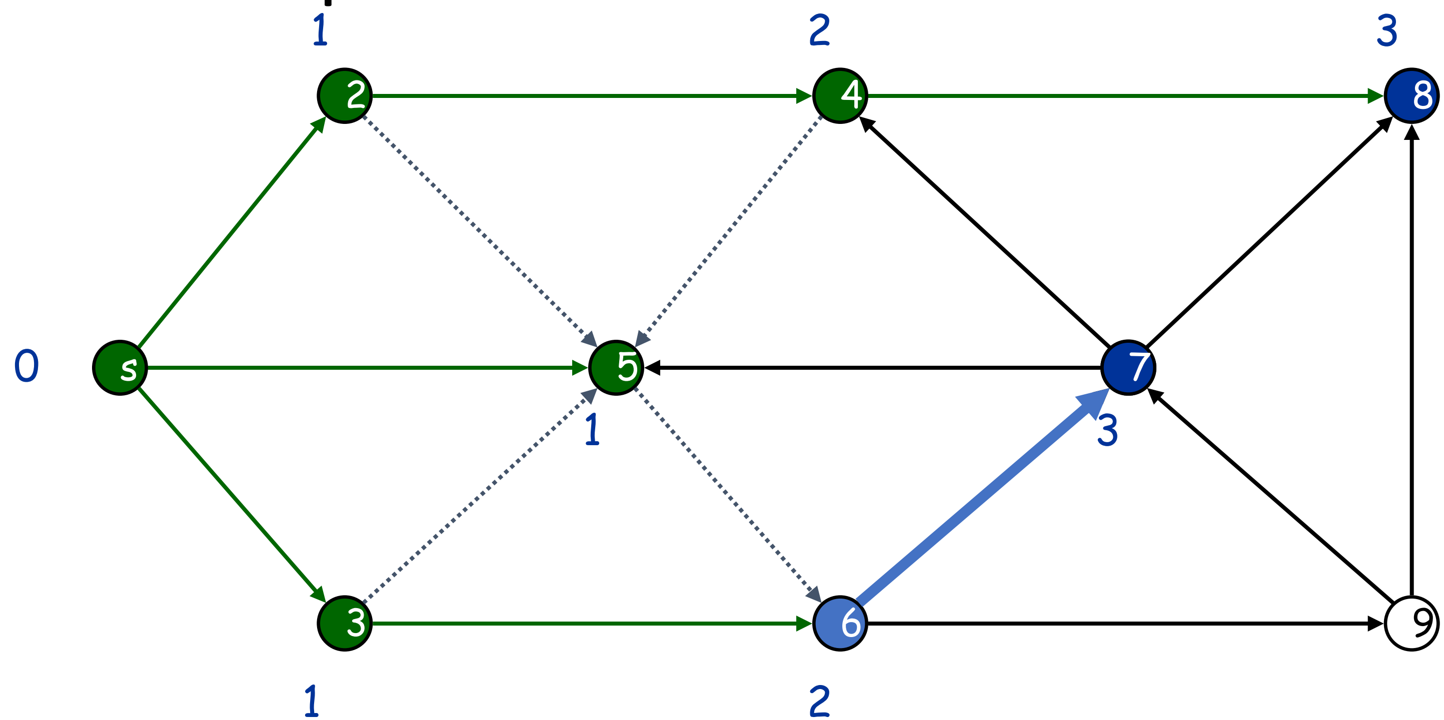
Undiscovered
Discovered
Top of queue
Finished
Queue: 4 6 8

BFS Example



Undiscovered

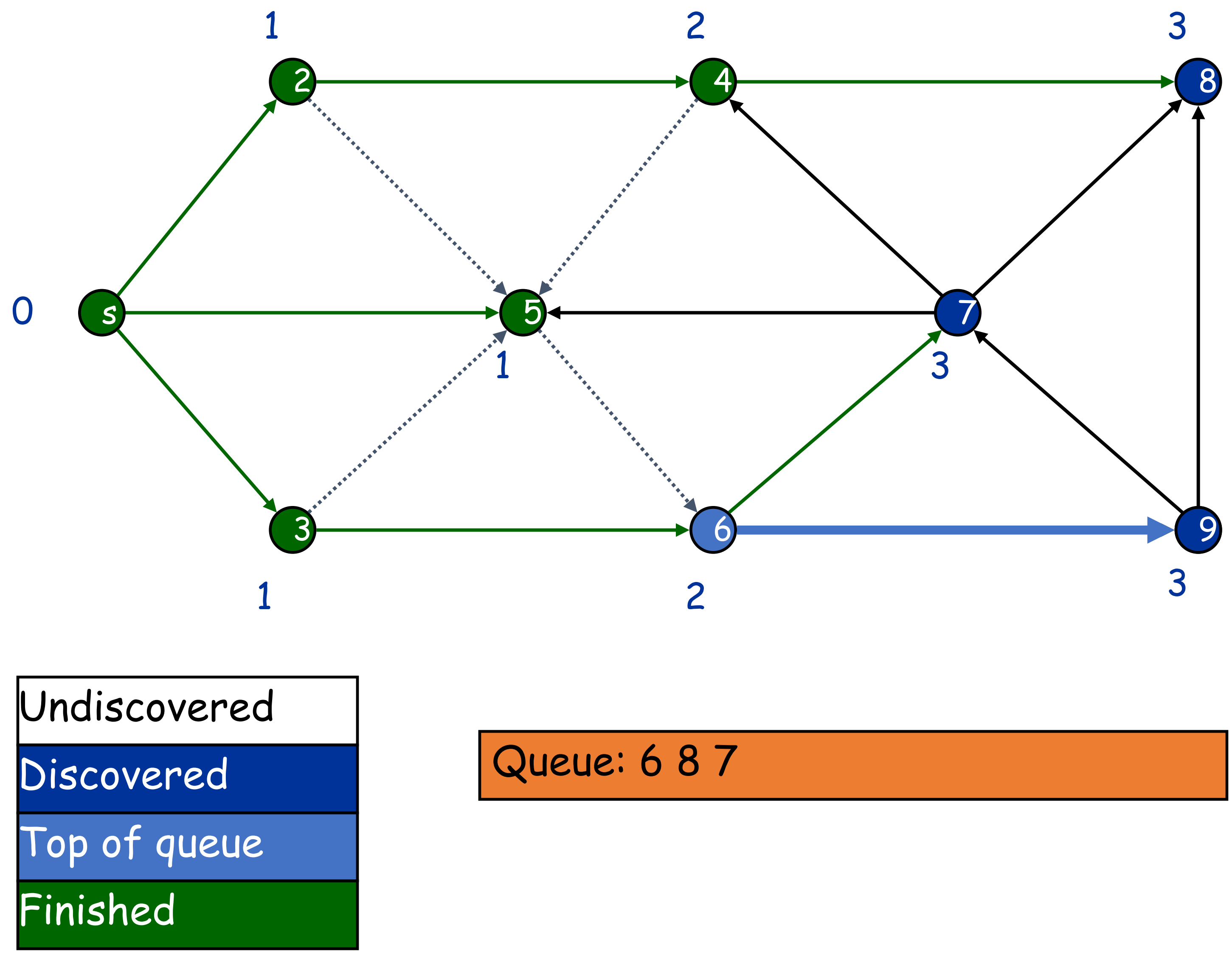
Discovered

Top of queue

Finished

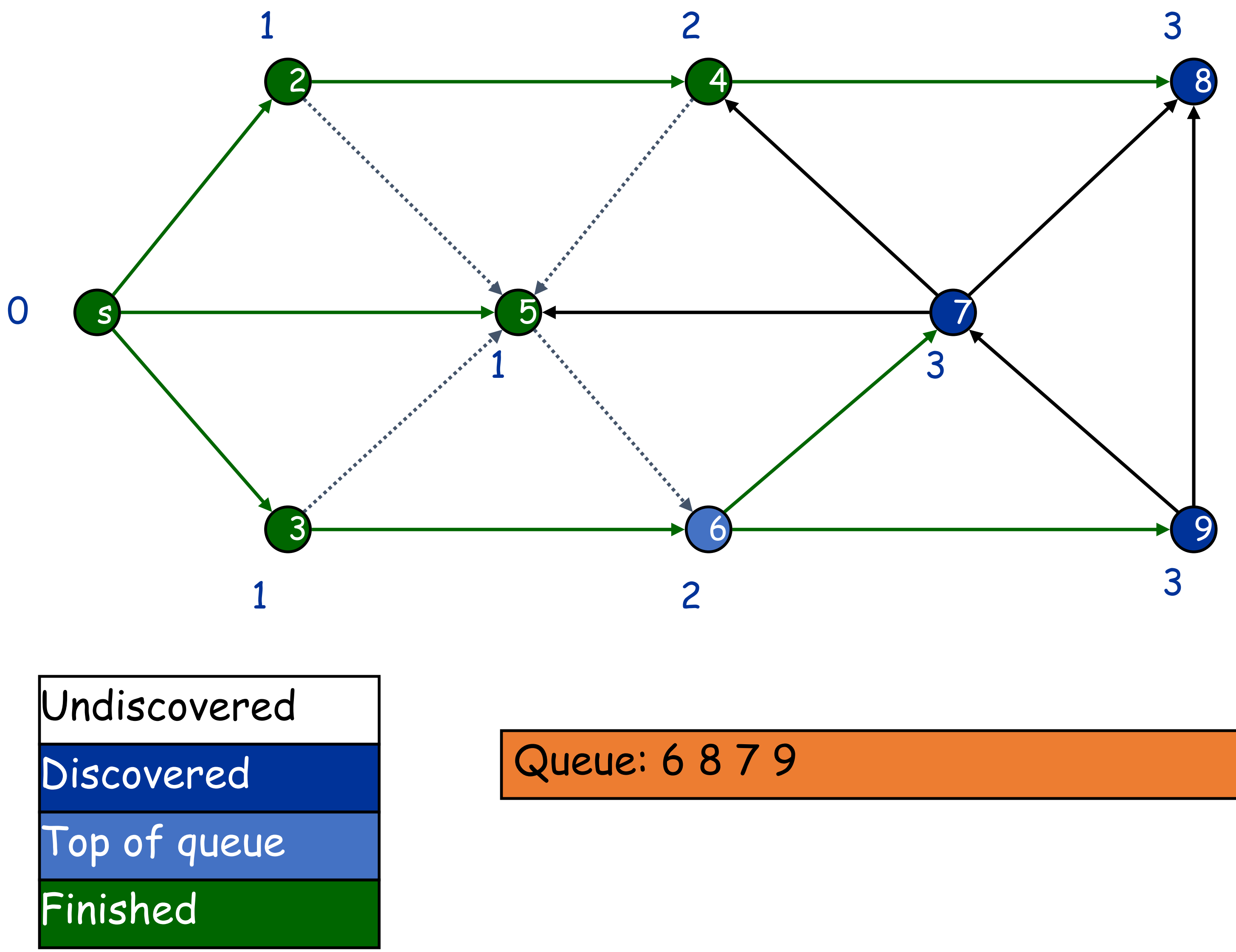
Queue: 6 8

BFS Example



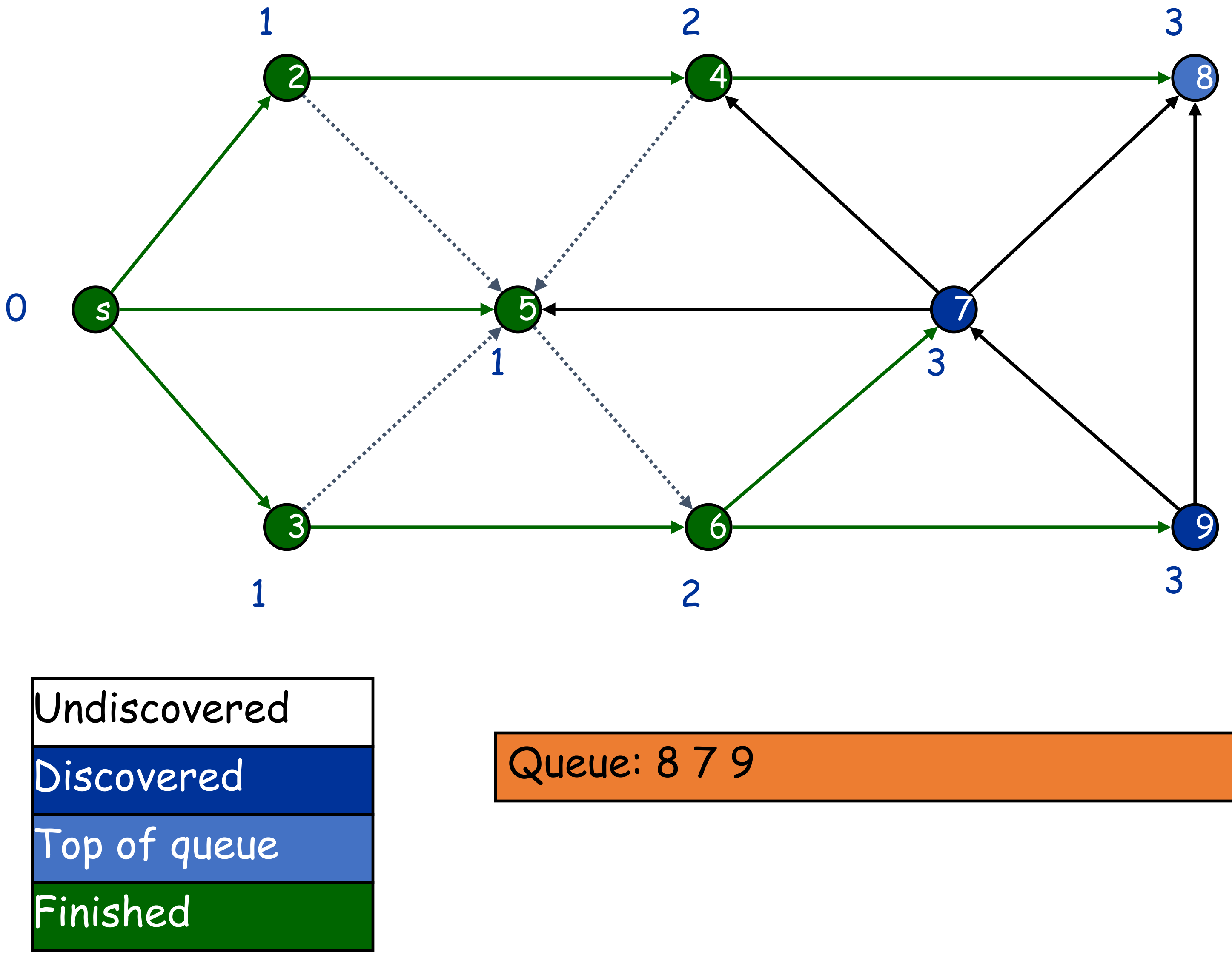
Queue: 6 8 7

BFS Example



Queue: 6 8 7 9

BFS Example



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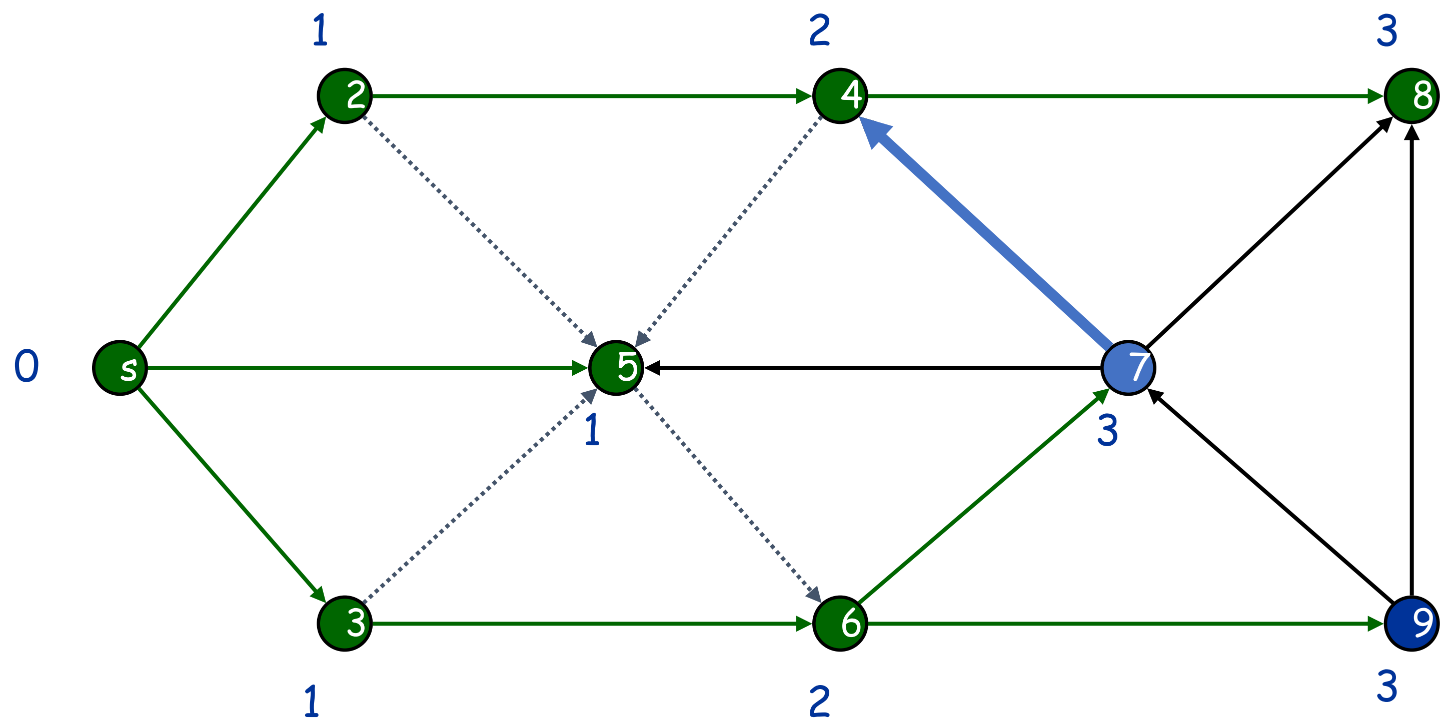
Undiscovered
Discovered
Top of queue
Finished
Queue: 8 7 9

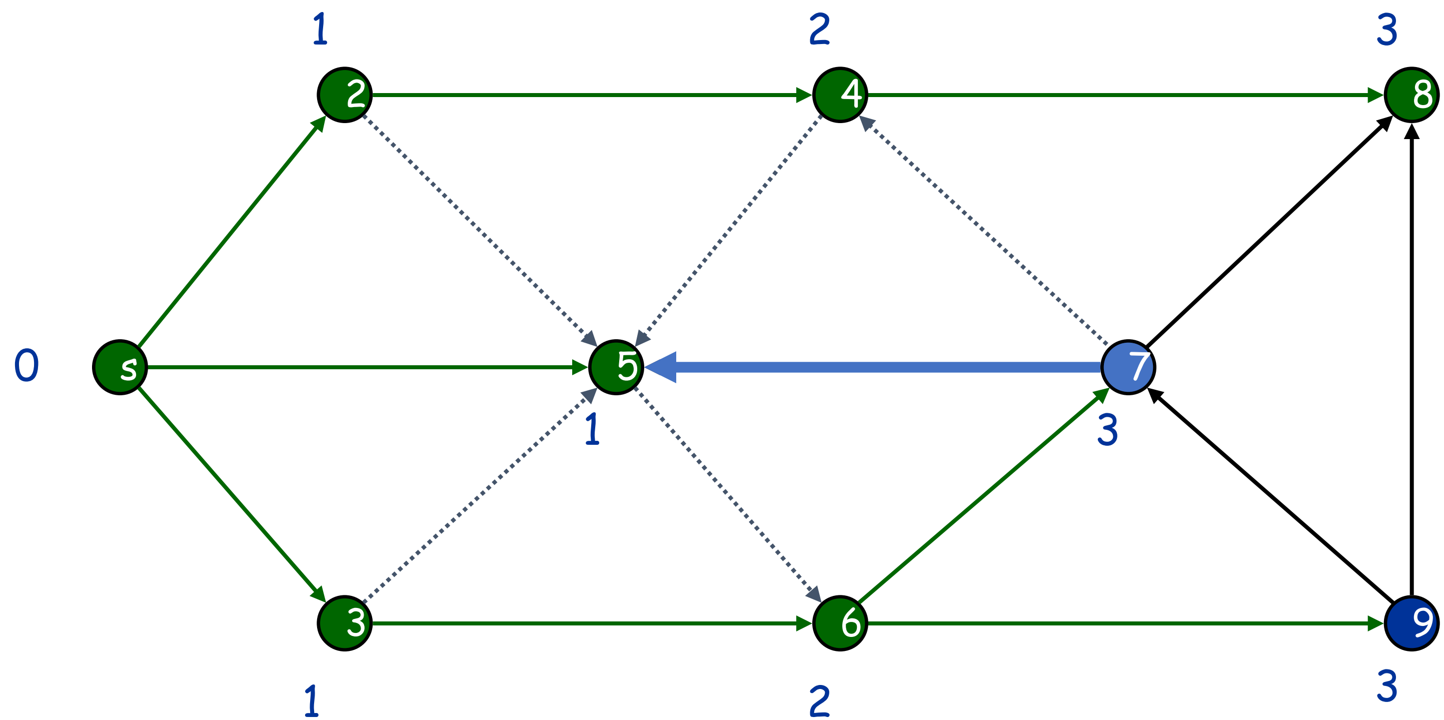
BFS Example



Undiscovered
Discovered
Top of queue
Finished

Queue: 7 9

BFS Example



Undiscovered

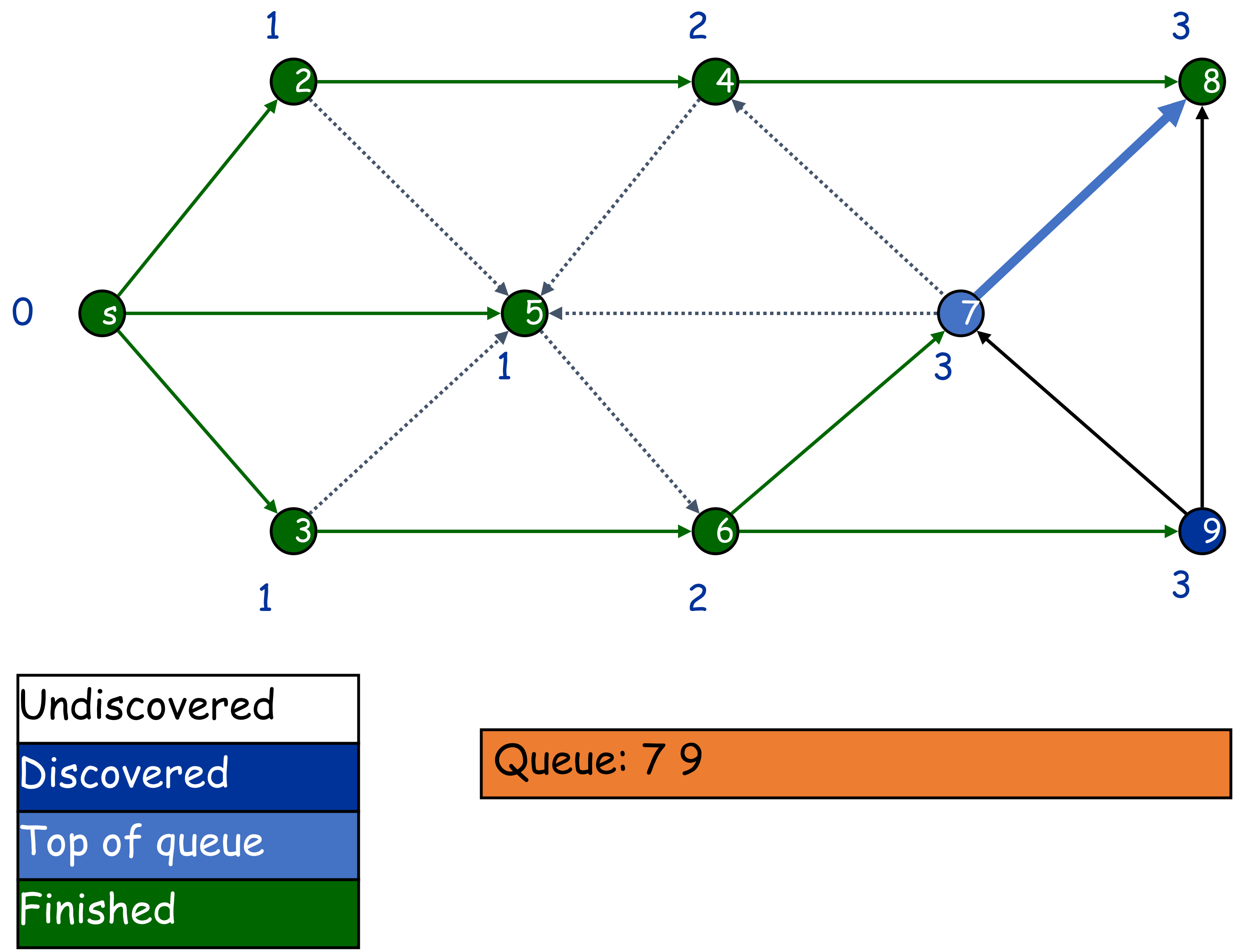
Discovered

Top of queue

Finished

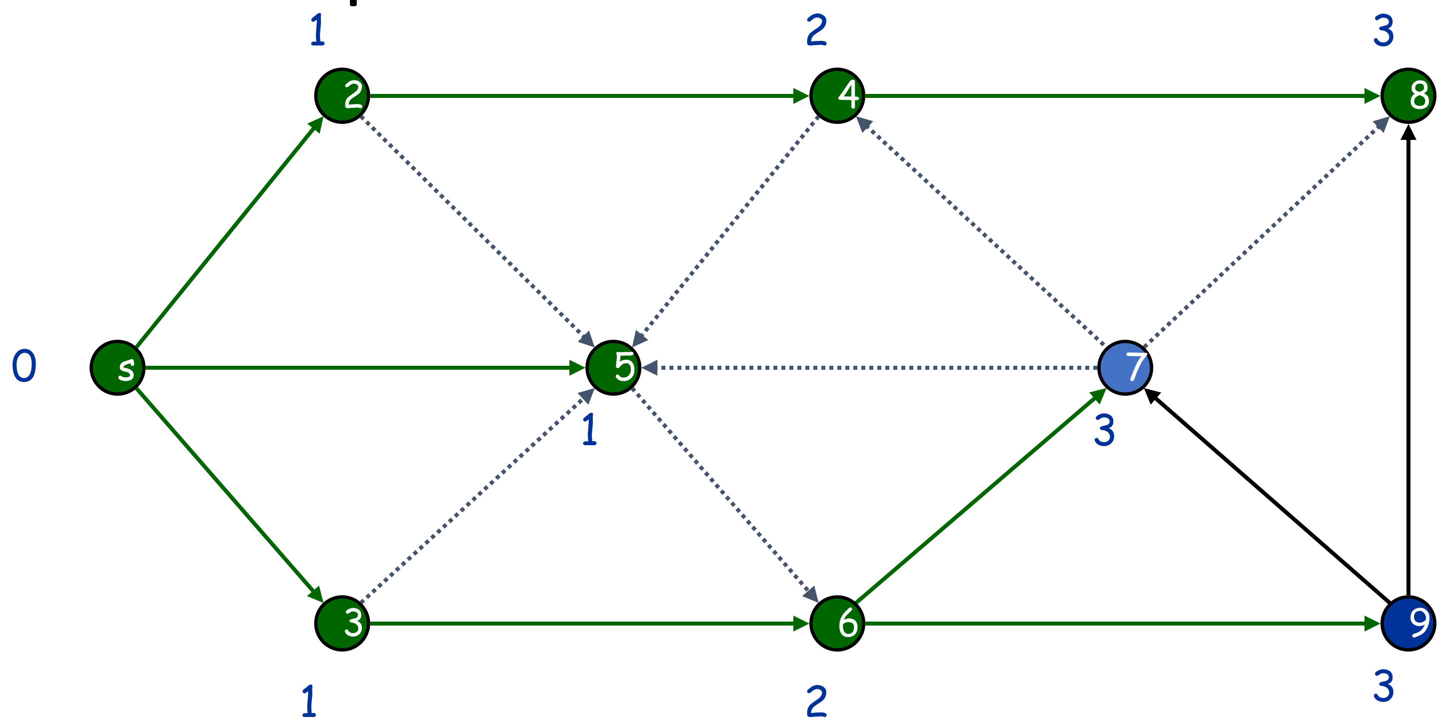
Queue: 7 9

BFS Example



Queue: 7 9

BFS Example



Undiscovered

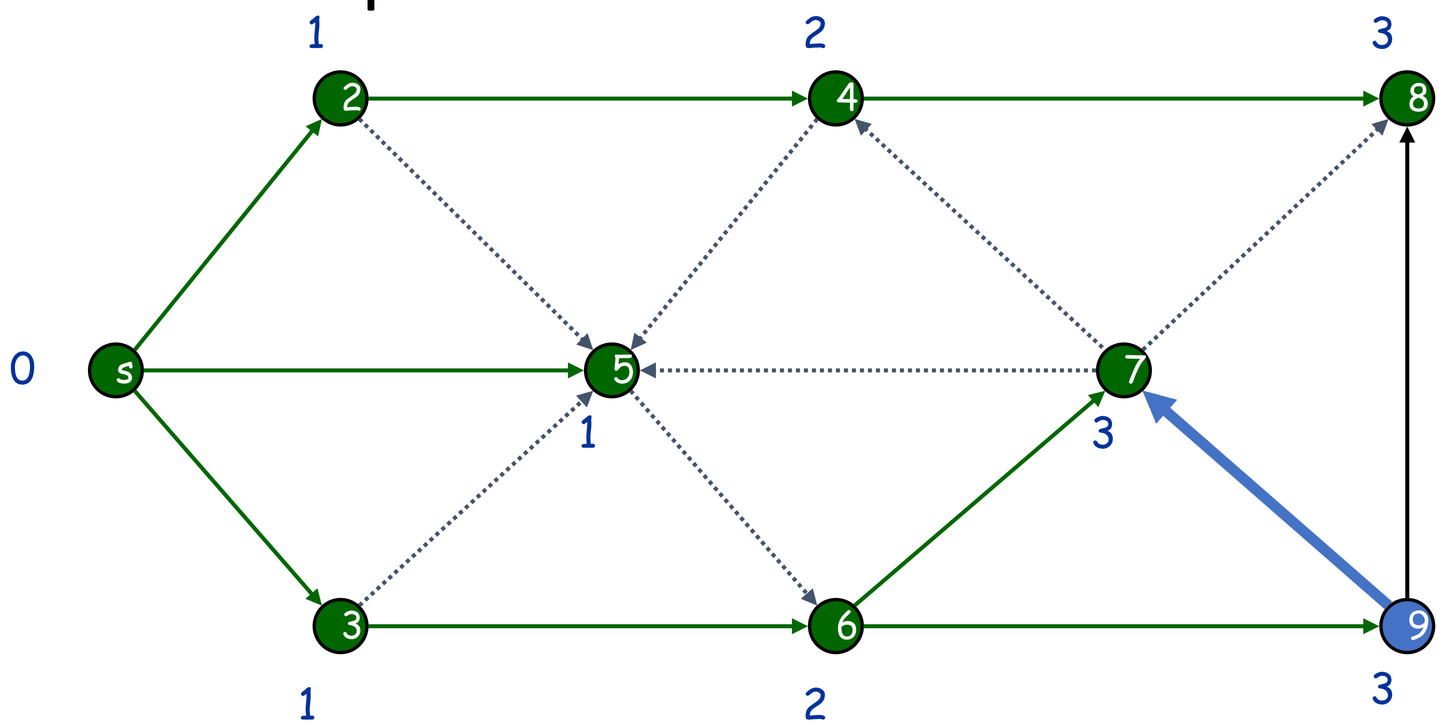
Discovered

Top of queue

Finished

Queue: 7 9

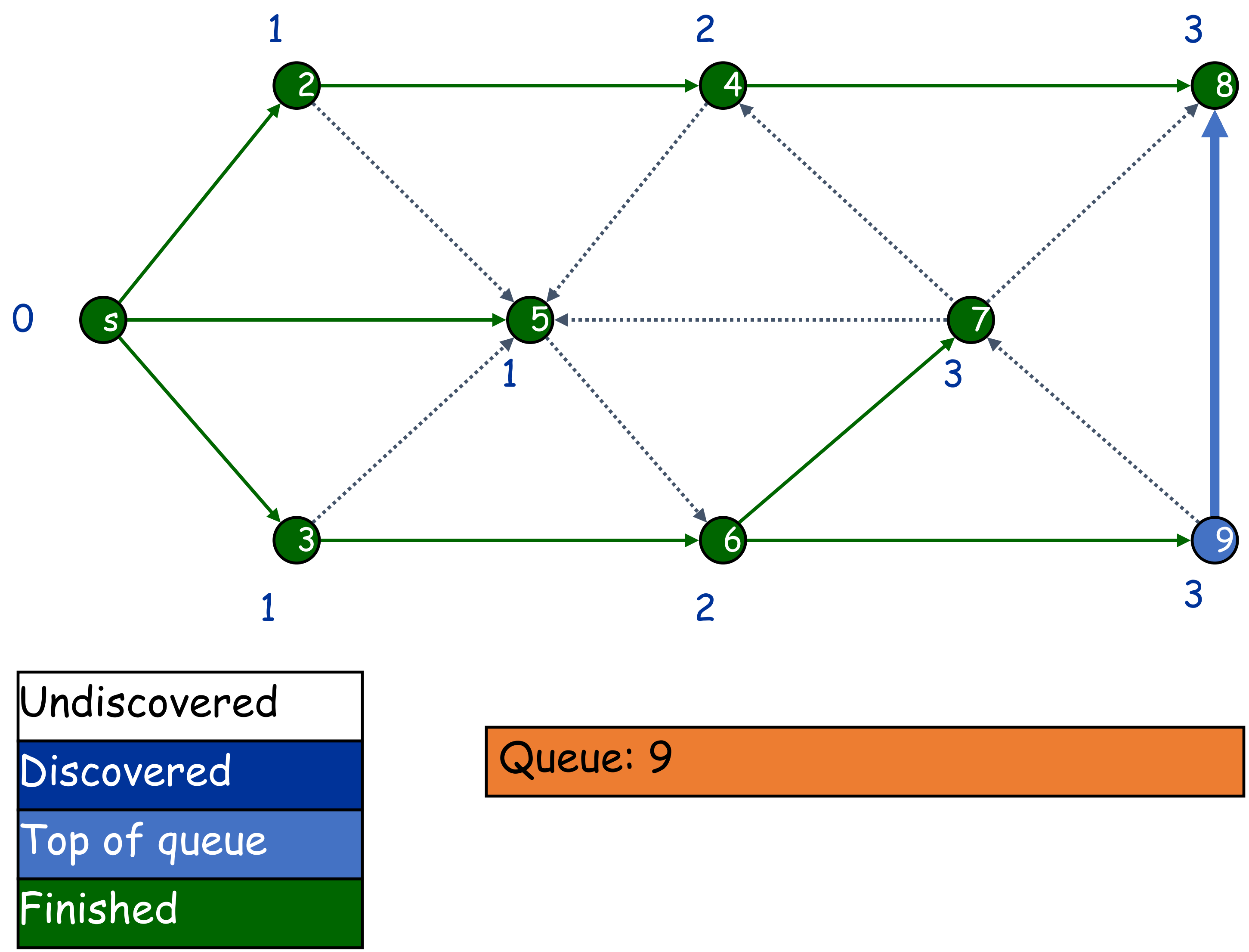
BFS Example



Undiscovered
Discovered
Top of queue
Finished

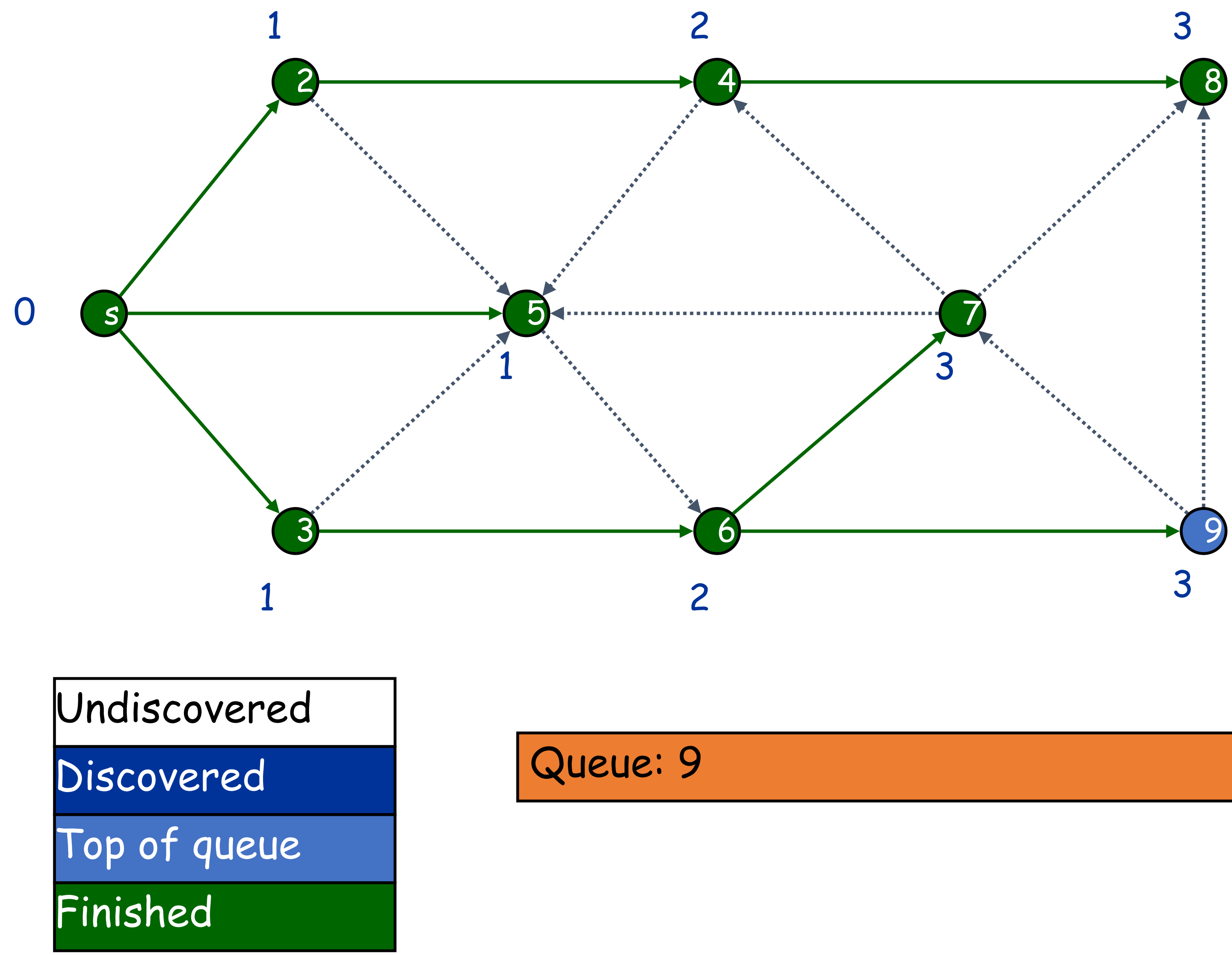
Queue: 9

BFS Example



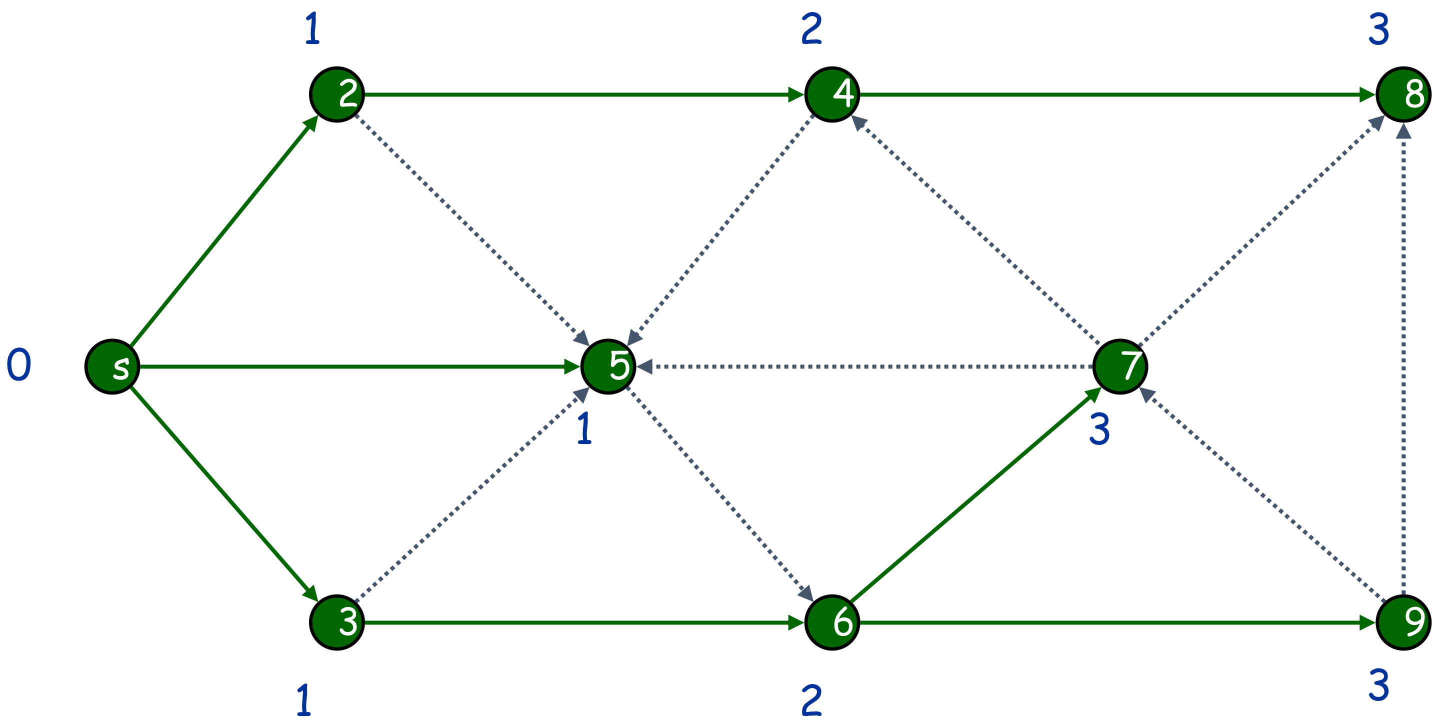
Queue: 9

BFS Example



Queue: 9

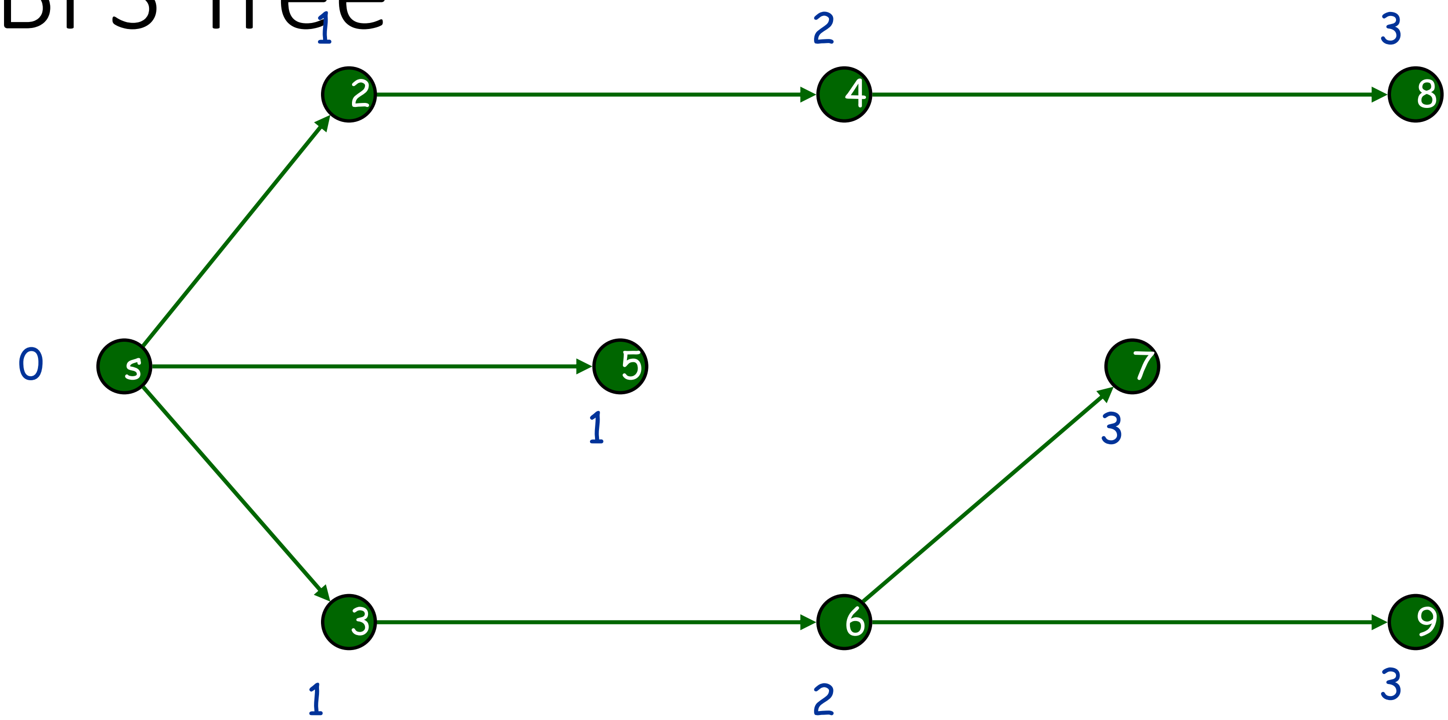
BFS Example



Undiscovered
Discovered
Top of queue
Finished

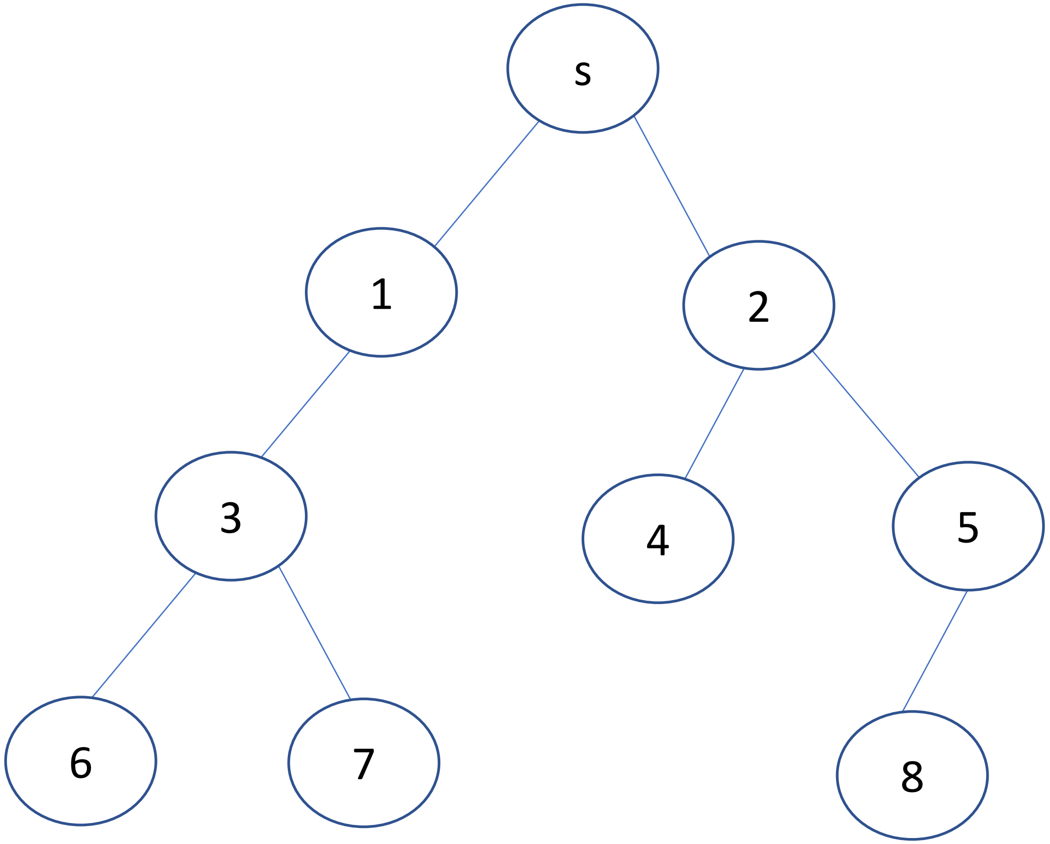
Queue:

BFS Tree

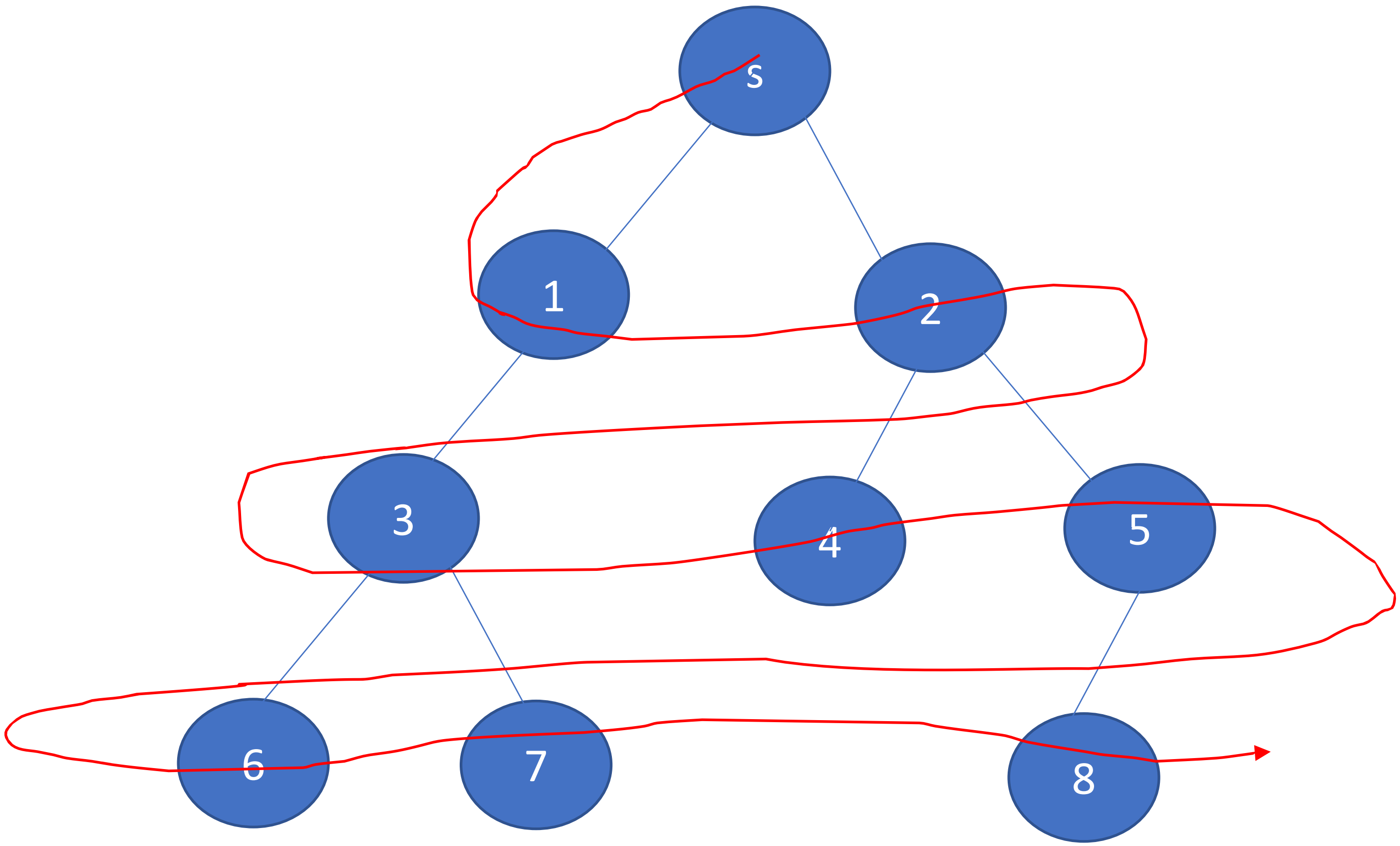


- Starting from s, we visited all (reachable) nodes
- BFS forms a tree rooted at s (<u>BFS Tree</u>)
- For each node *x* reachable from s ➔ we created a shortest path from s to *x*

BFS on a TREE: Example

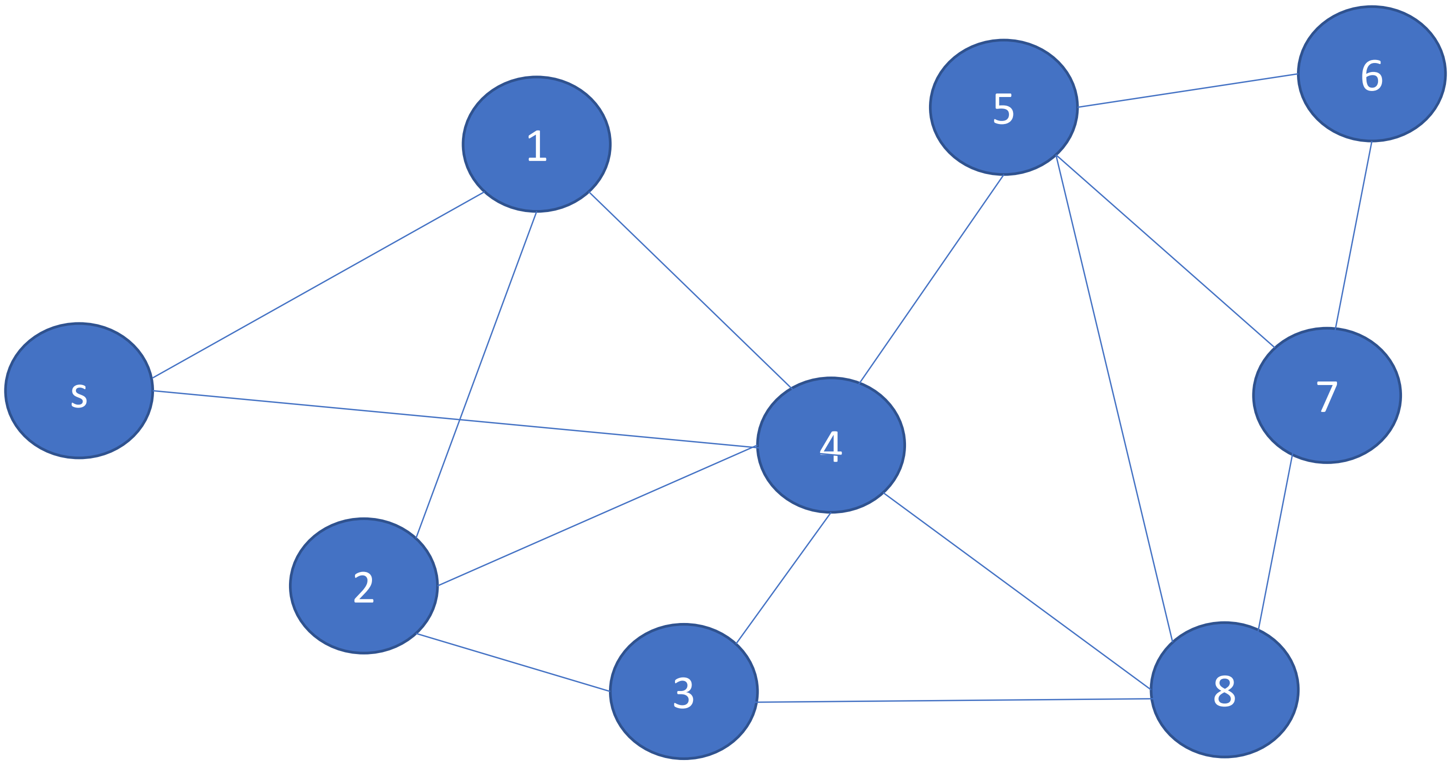


BFS on a TREE: Outcome

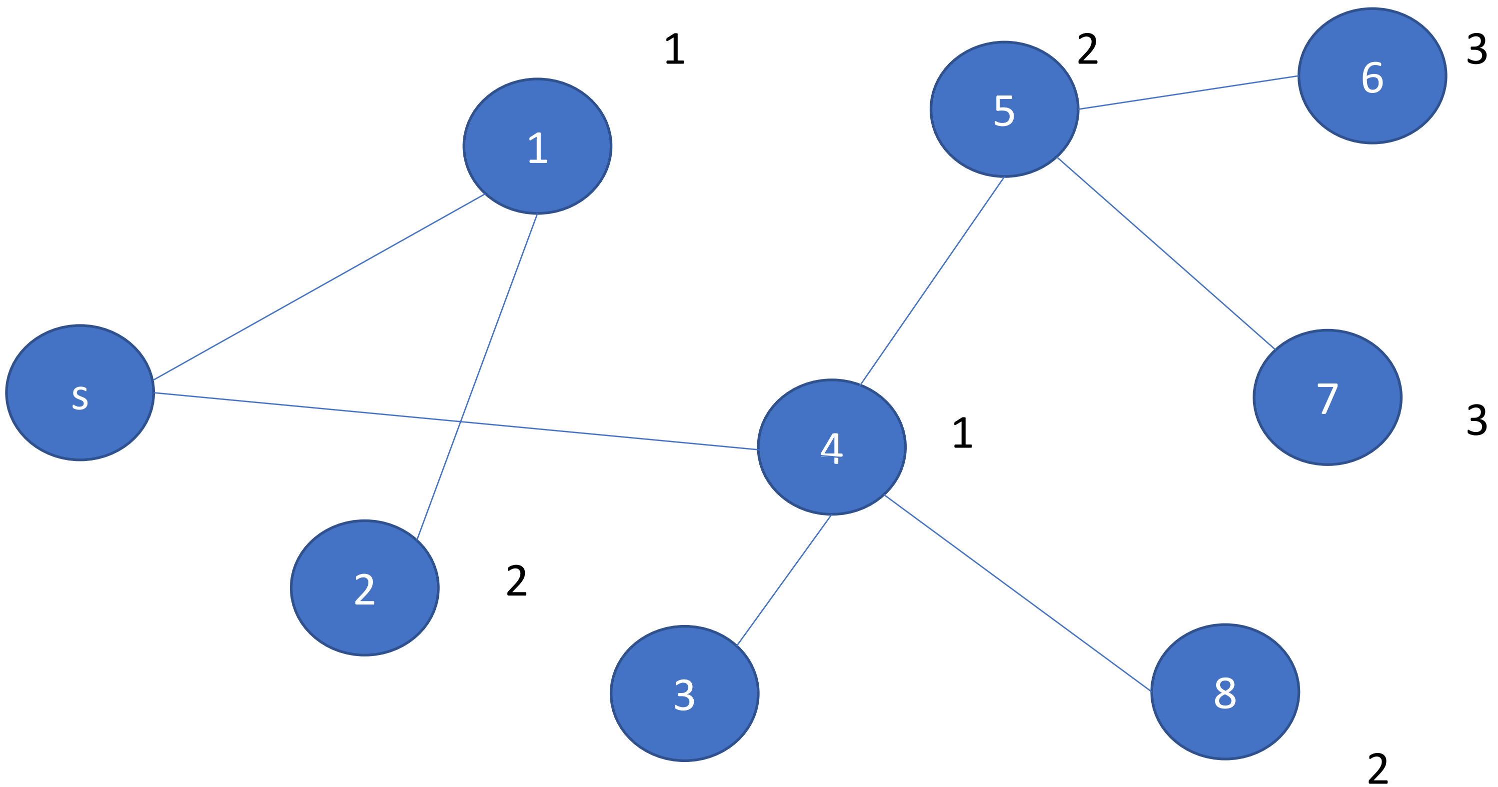


See why it is called Breadth First Search? It goes through the breadth of the graph first

Another BFS Example To Try



Another BFS Example: Outcome



Breadth-First Search

Example problems in which we use BFS

- **Find if node *x* is reachable from node *y***
  - ***Start from node y and do BFS***
- **Find the shortest path from node x to node y**
  - ***Start from node x and perform BFS***
- **Search for a value v in the graph**
  - ***Start from any node and perform BFS***

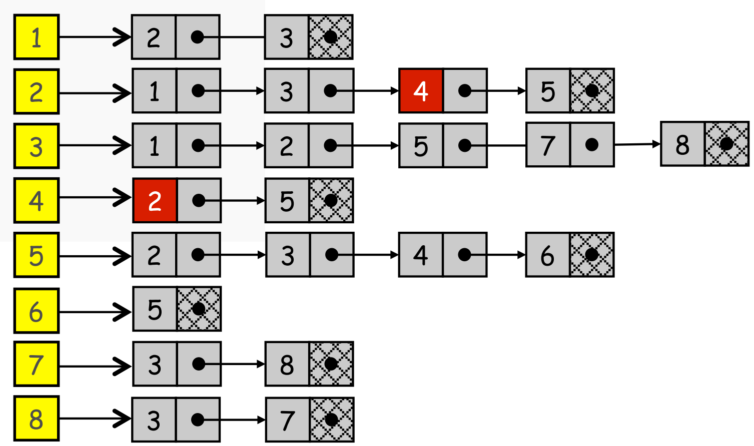
Always keep in mind whether we talk about undirected graph or directed graph

Breadth-First Search: Algorithm

Start at node v

```
procedure BFS(G,v):
    create a queue Q
    enqueue v onto Q
    mark v
    while Q is not empty:
        t ← Q.dequeue()
        if t is what we are looking for:
            return t
        for all edges e in G.adjacentEdges(t) do
            u ← G.adjacentVertex(t,e)
            if u is not marked:
                mark u
                enqueue u onto Q
    return none
```

Can do any processing on t



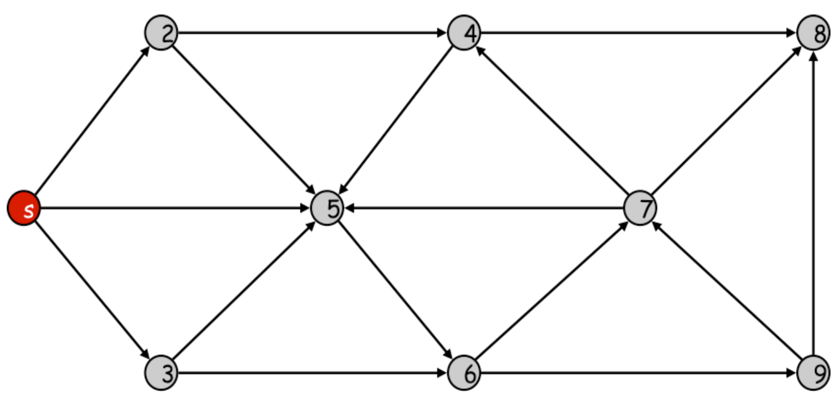
Breadth-First Search: Analysis

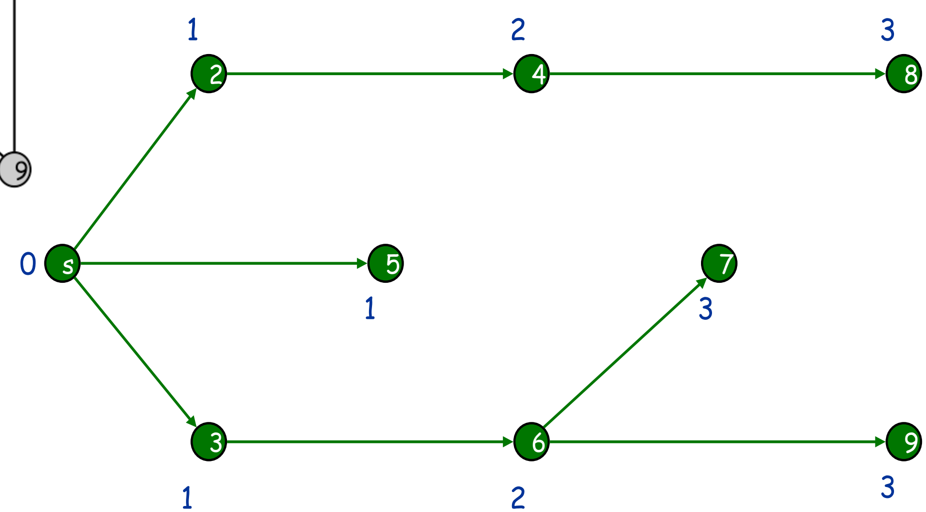
The above implementation of BFS runs in O(V + E) time if the graph is given by its adjacency list.

- Proof
  - Each node will be queued only once → O(V)
  - For each node, we visit all its out edges → O(E)
    - In fact each edge (u,v) is visited twice once from u's side and once form v's side
  - Total is O(V + E)

BFS & Shortest Path

- **BFS can be used to compute the shortest path (minimum number of edges) from source s and any reachable nodes v**
  - **Maintain a counter for each node**
  - **When a node *x* is first visited from parent *y* ➔ x.counter = y.counter + 1**






That's all Folks!
Any Question?