Depth First Search (DFS) + Topological Sort

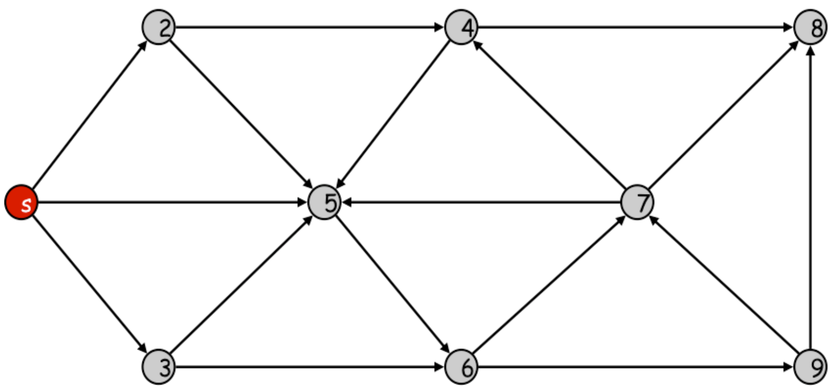
Instructor: Krishna Venkatasubramanian

CSC 212

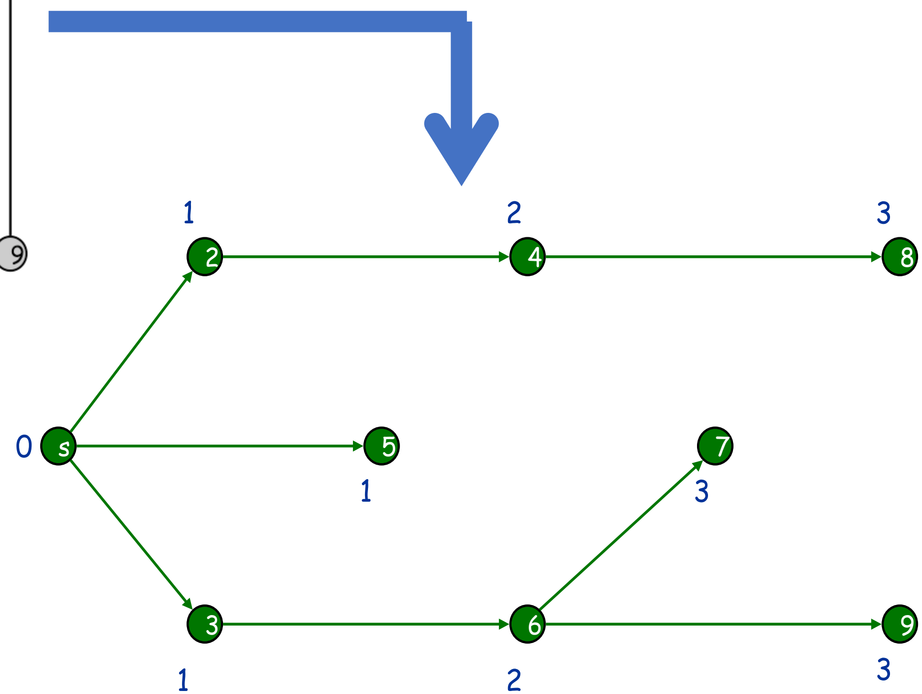
Based on slides from Prof. Carolina Ruiz at WPI

We Already Covered Breadth First Search(BFS)

- **Traverses the graph one level at a time**
  - **Visit all outgoing edges from a node before you go deeper**
- **Needs a queue**



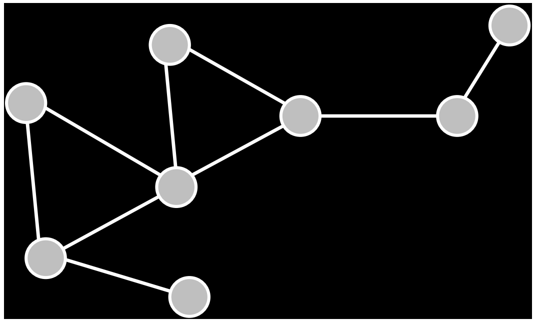
BFS creates a tree called BFS-Tree



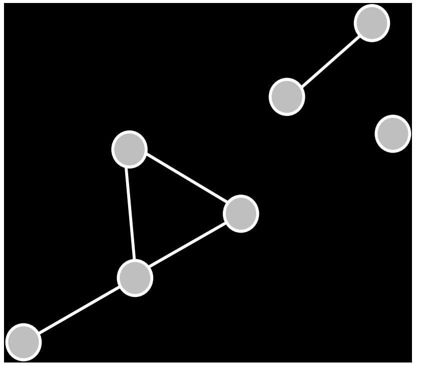
Time Complexity O(V+E)

Connected Undirected Graph

- **G = (V, E) is called connected iff**
  - From any node u, we can reach all other nodes



Connected graph



Un-Connected graph

- ***What is the time complexity to decide if G is a connected graph?***
  - **Take any node from G and apply BFS**
  - **If you reached all nodes → G is connected**
  - **Otherwise → G is not connected**

Time Complexity O(V+E)

Depth First Search (DFS)

- **Traverse the graph by going deeper whenever possible**

- **DFS uses a stack, hence can be implemented using recursion**

- **While traversing keep some useful information**
  - **u.color:**
    - White → u has not been visited yet
    - Gray → u is visited but its descendent are not completed yet
    - Black → u and its all descendent are visited
  - **u.startTime (u.d)**= the first time u is visited
  - **u.endTime (u.f)** = the last time u will be seen (after all descendent of u are processed)

DFS: Pseudocode

U is just discovered…make it gray

```
DFS-VISIT(u)

color[u] ← GRAY          ▸White vertex u has just been discovered.
time ← time +1
d[u]=time

for each v in Adj[u]     ▸Explore edge(u, v).
    do if color[v] = WHITE
        then π[v] ← u
             DFS-VISIT(v)

color[u]=BLACK           ▸ Blacken u; it is finished.

f [u] ▸ time ← time +1
```

Time is a global variable

U start time

If v is not seen before, recursively visit v

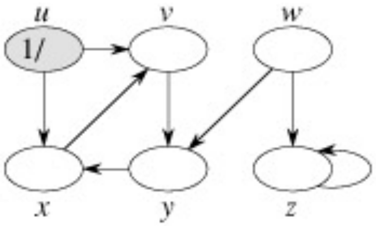
```
DFS (G)

for each vertex u in G.V
    color[u] = WHITE
    π(u) < −NIL
time  <- 0
for each vertex  u in G.V
    if  color[u] = WHITE
        DFS-VISIT(u)
```

<u>Notice:</u> We maintain 4 arrays during the traversal

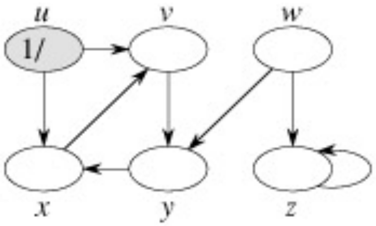
d[u] ➔First time u is seen
f[u] ➔Last time u is seen
color[u] ➔ {White, Gray, Black}
π[u] ➔ parent of node u

Depth First Search (DFS): Example

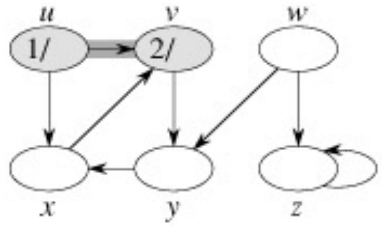


(a)

Depth First Search (DFS): Example

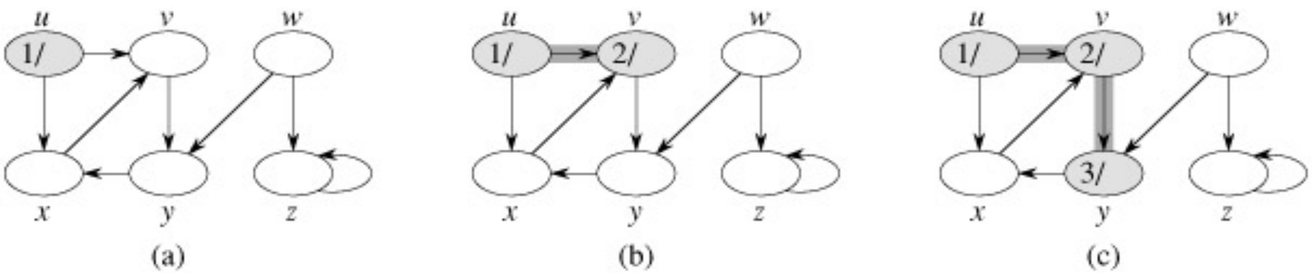


(a)

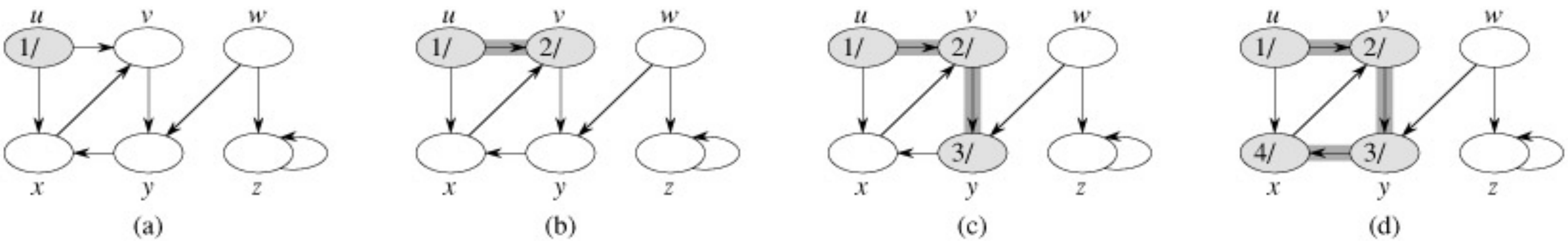


(b)

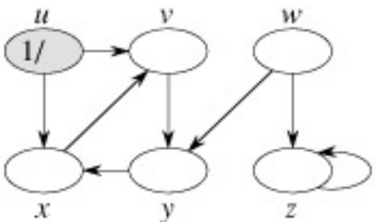
Depth First Search (DFS): Example



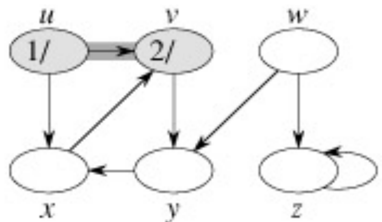
Depth First Search (DFS): Example



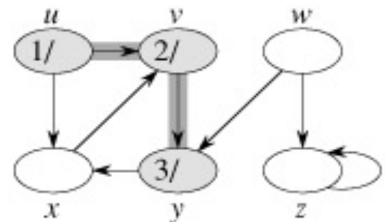
Depth First Search (DFS): Example



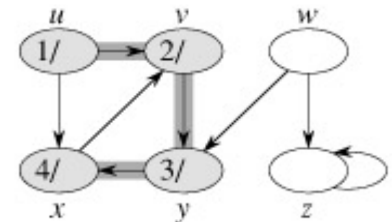
(a)



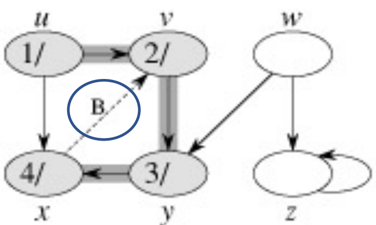
(b)



(c)

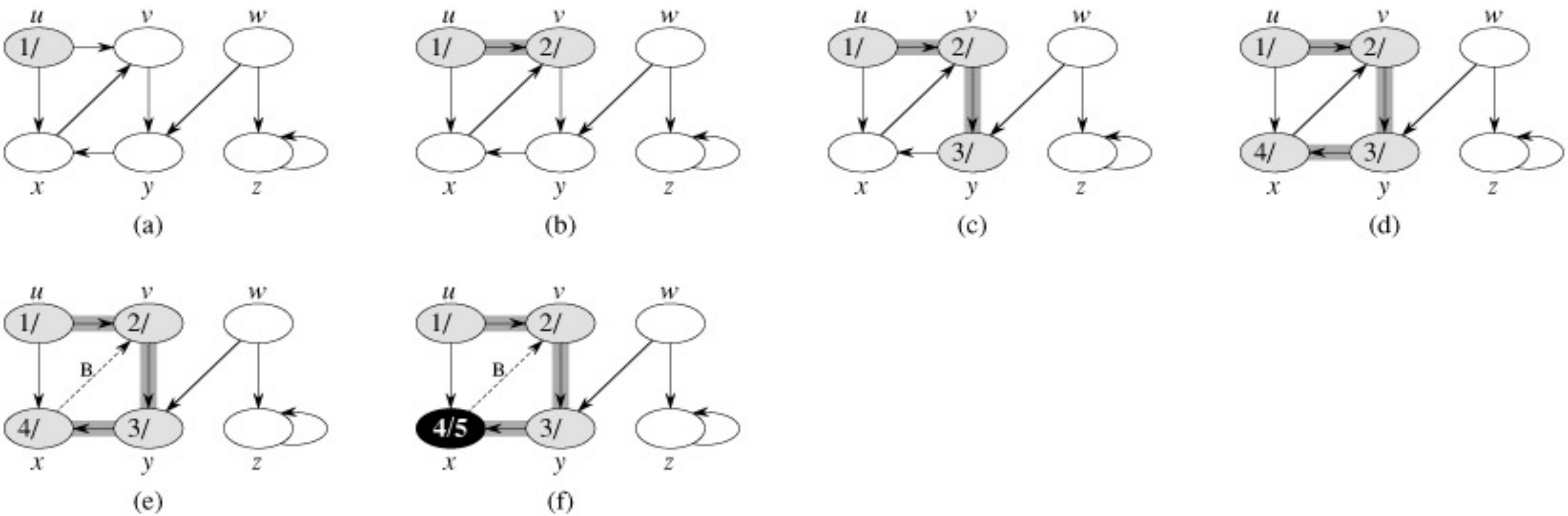


(d)

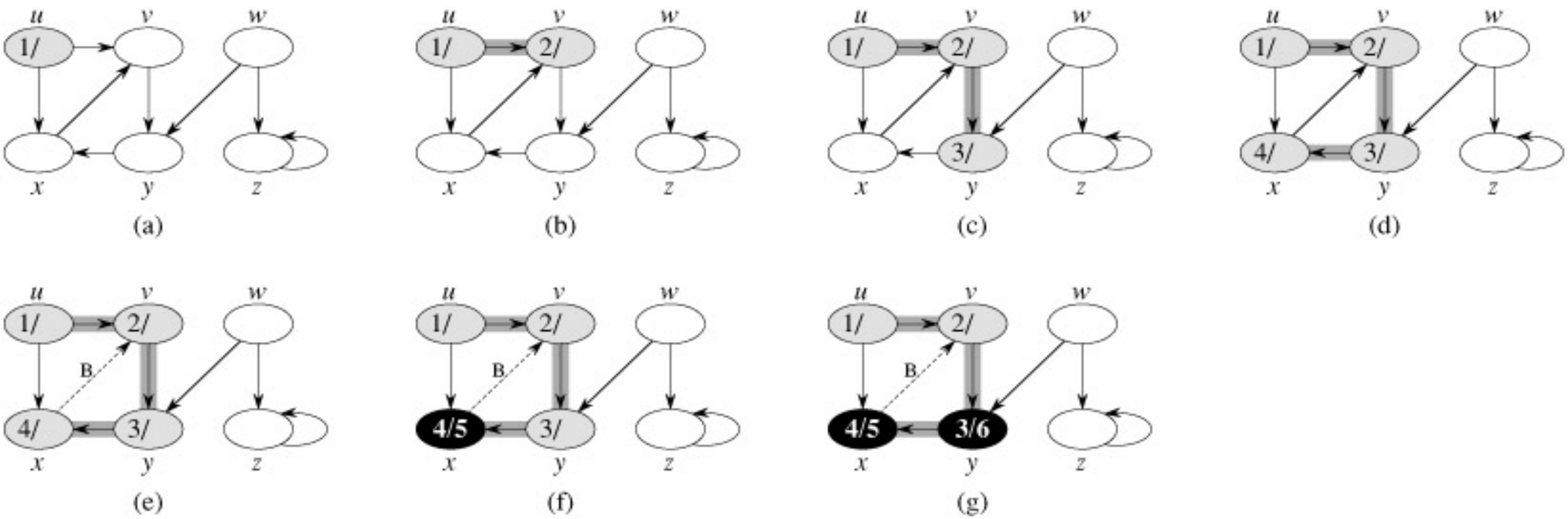


(e)

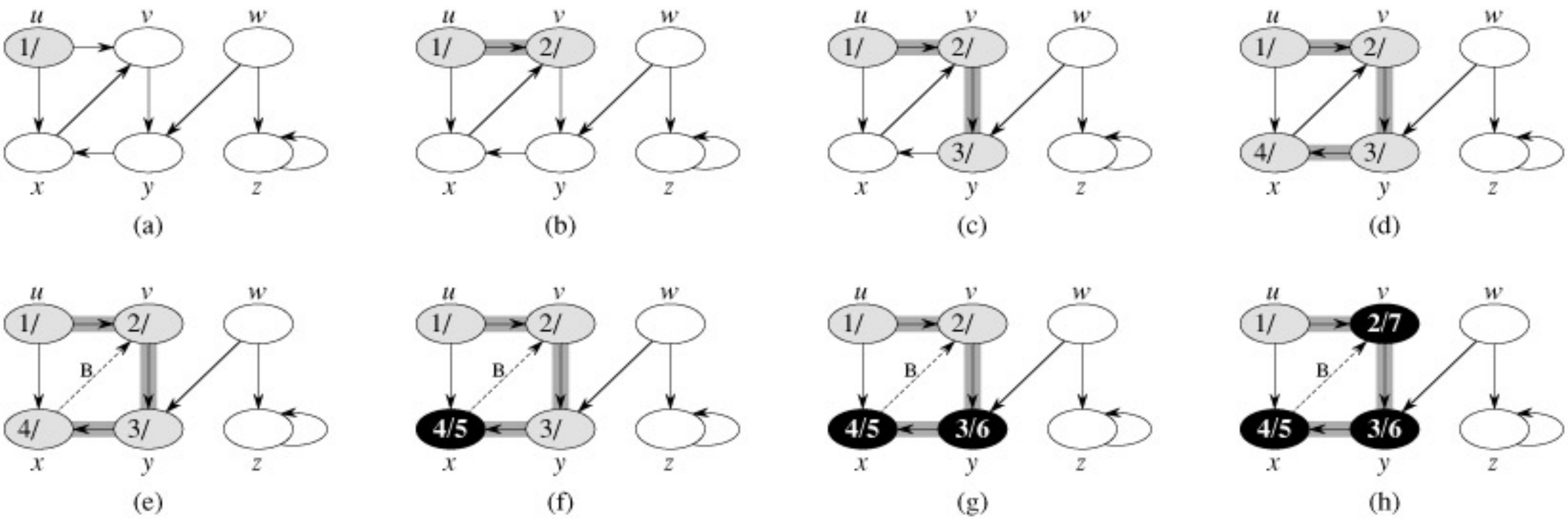
Depth First Search (DFS): Example



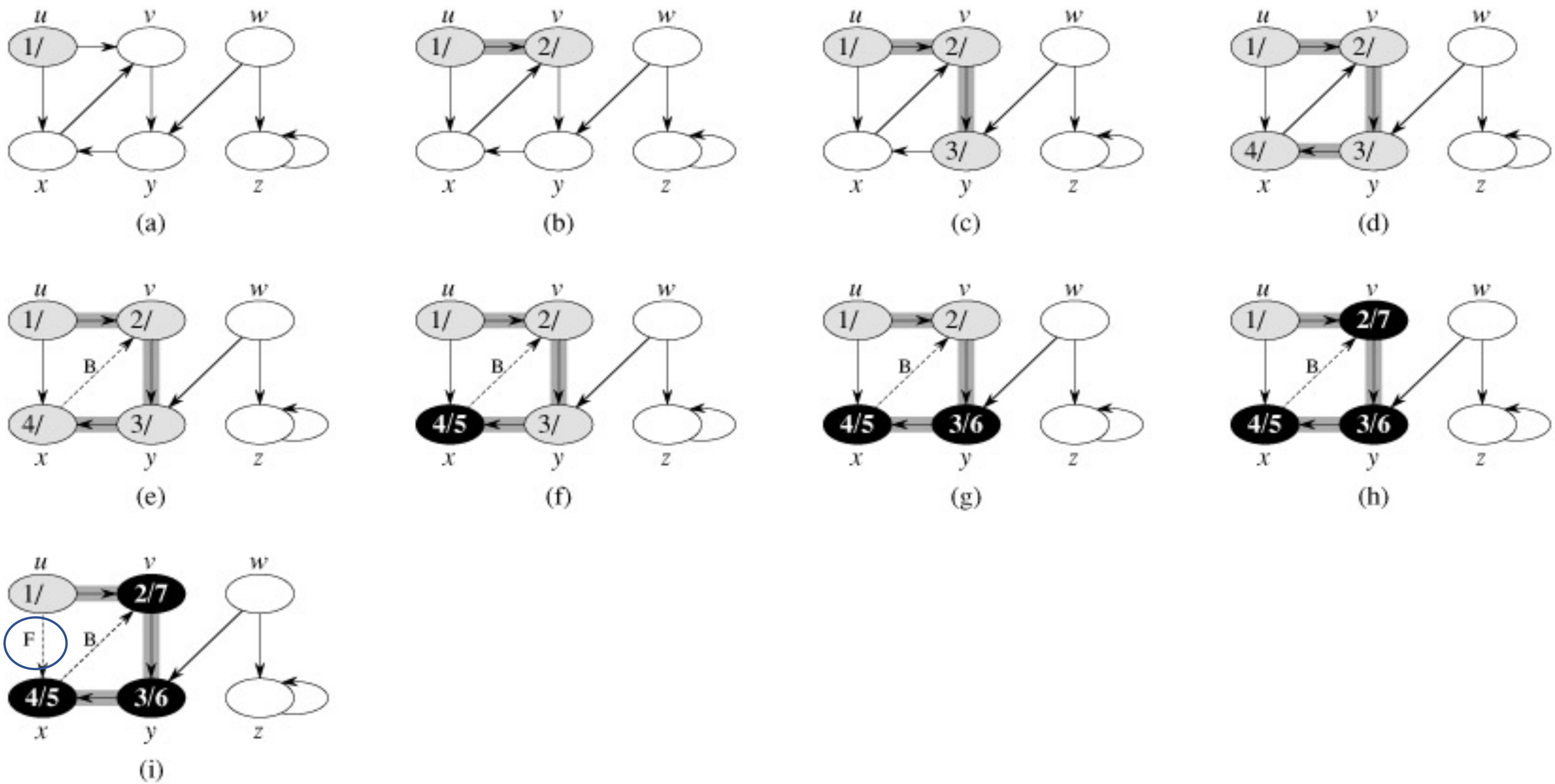
Depth First Search (DFS): Example



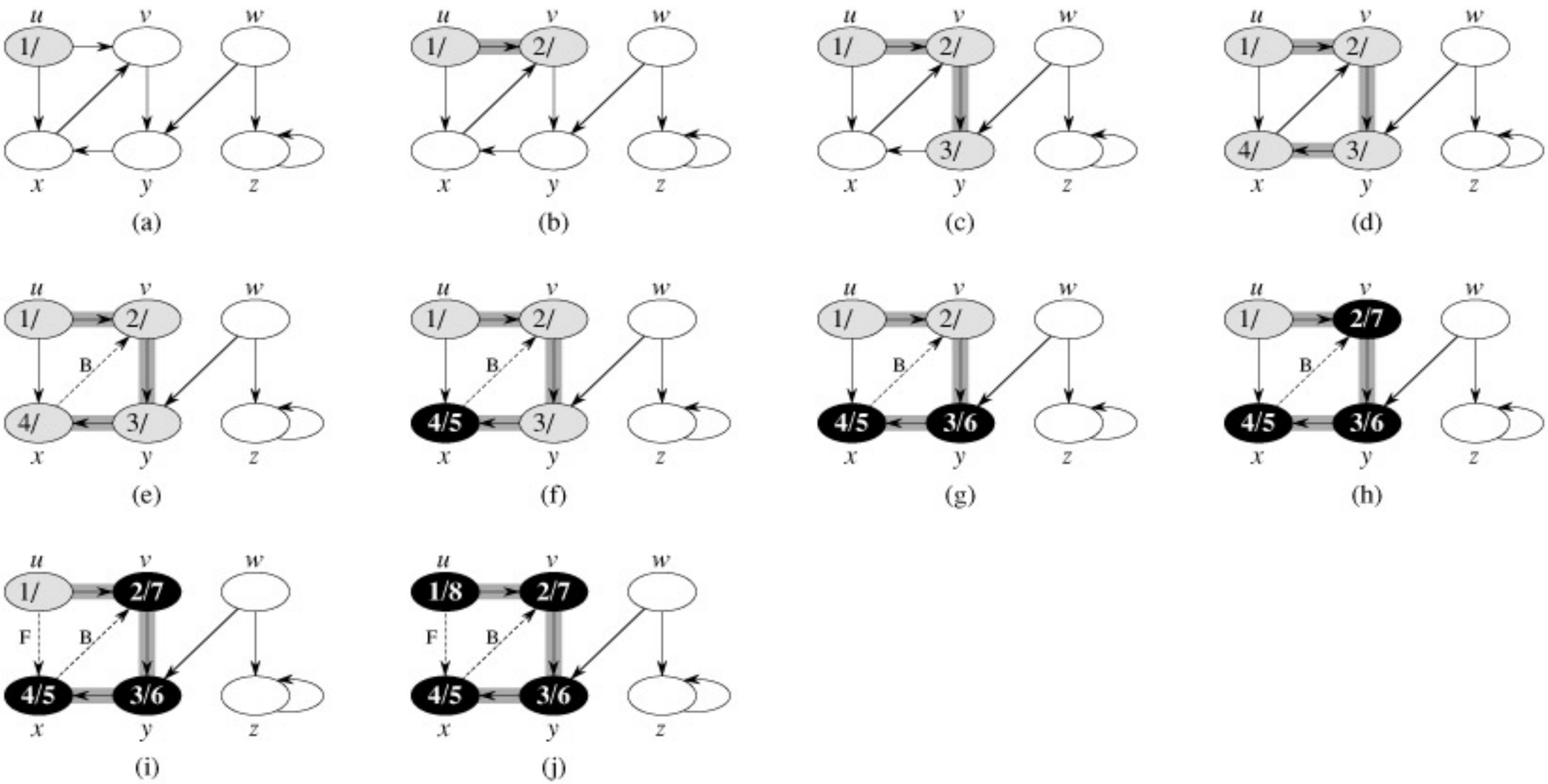
Depth First Search (DFS): Example



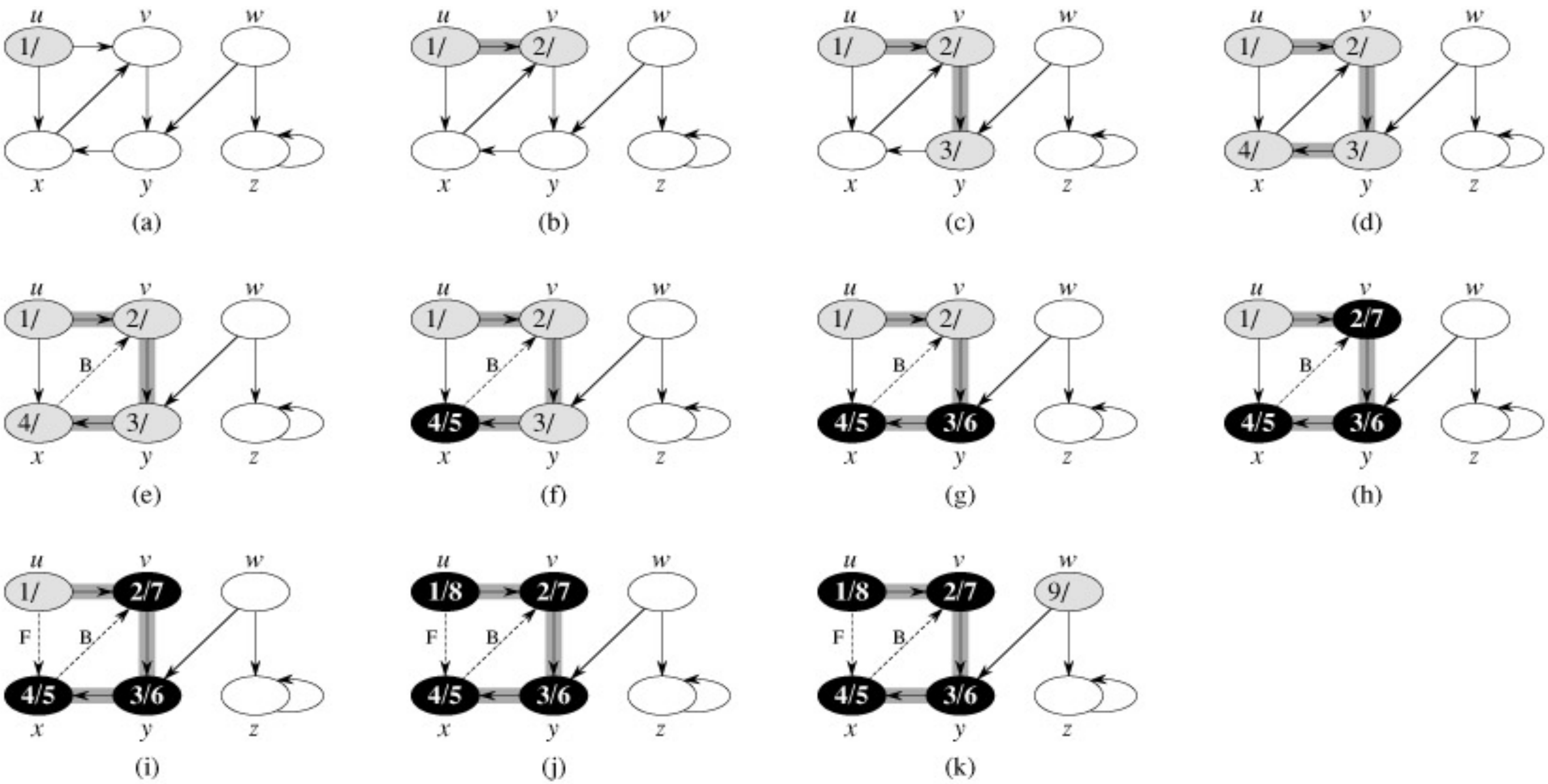
Depth First Search (DFS): Example



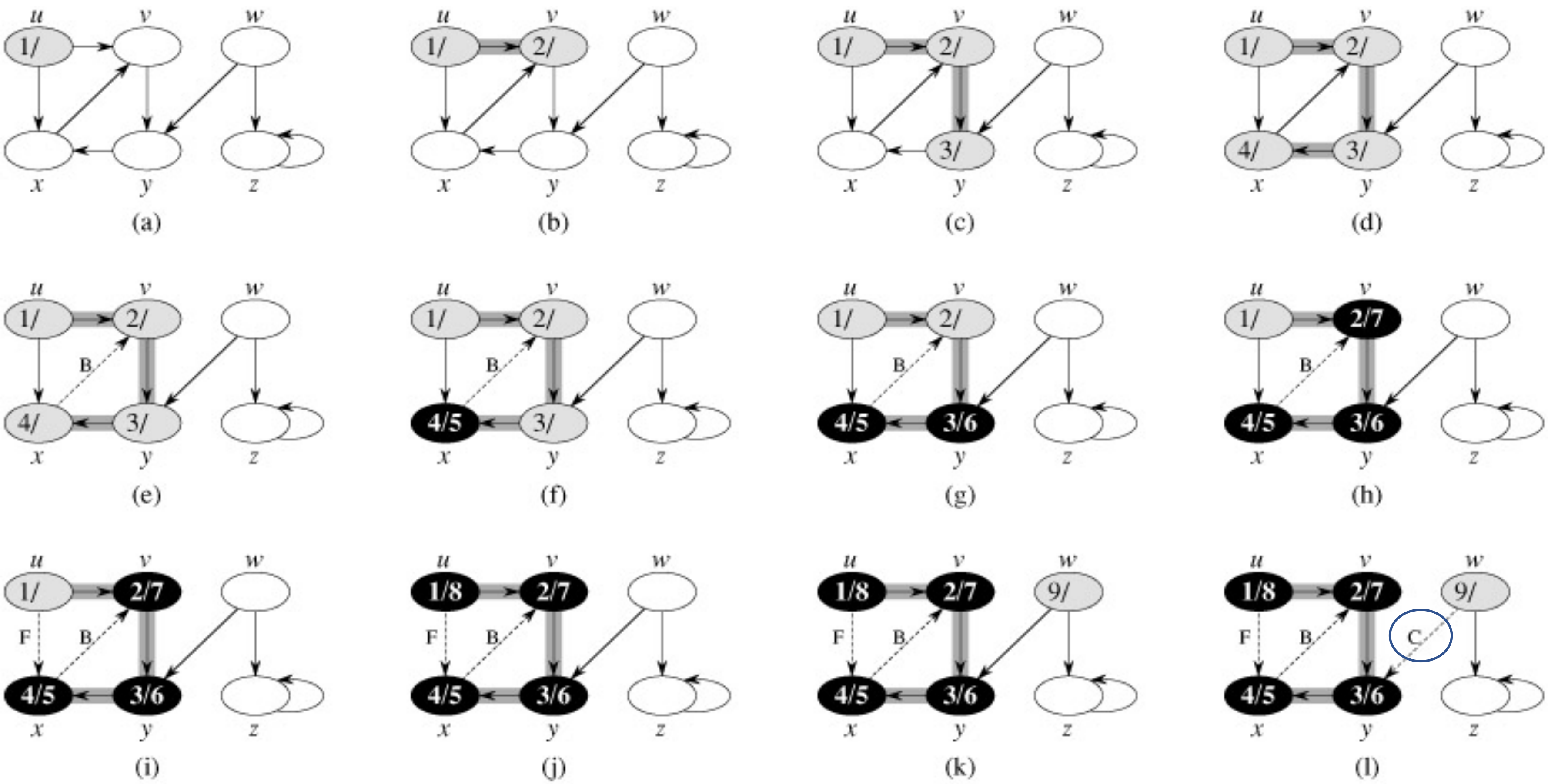
Depth First Search (DFS): Example



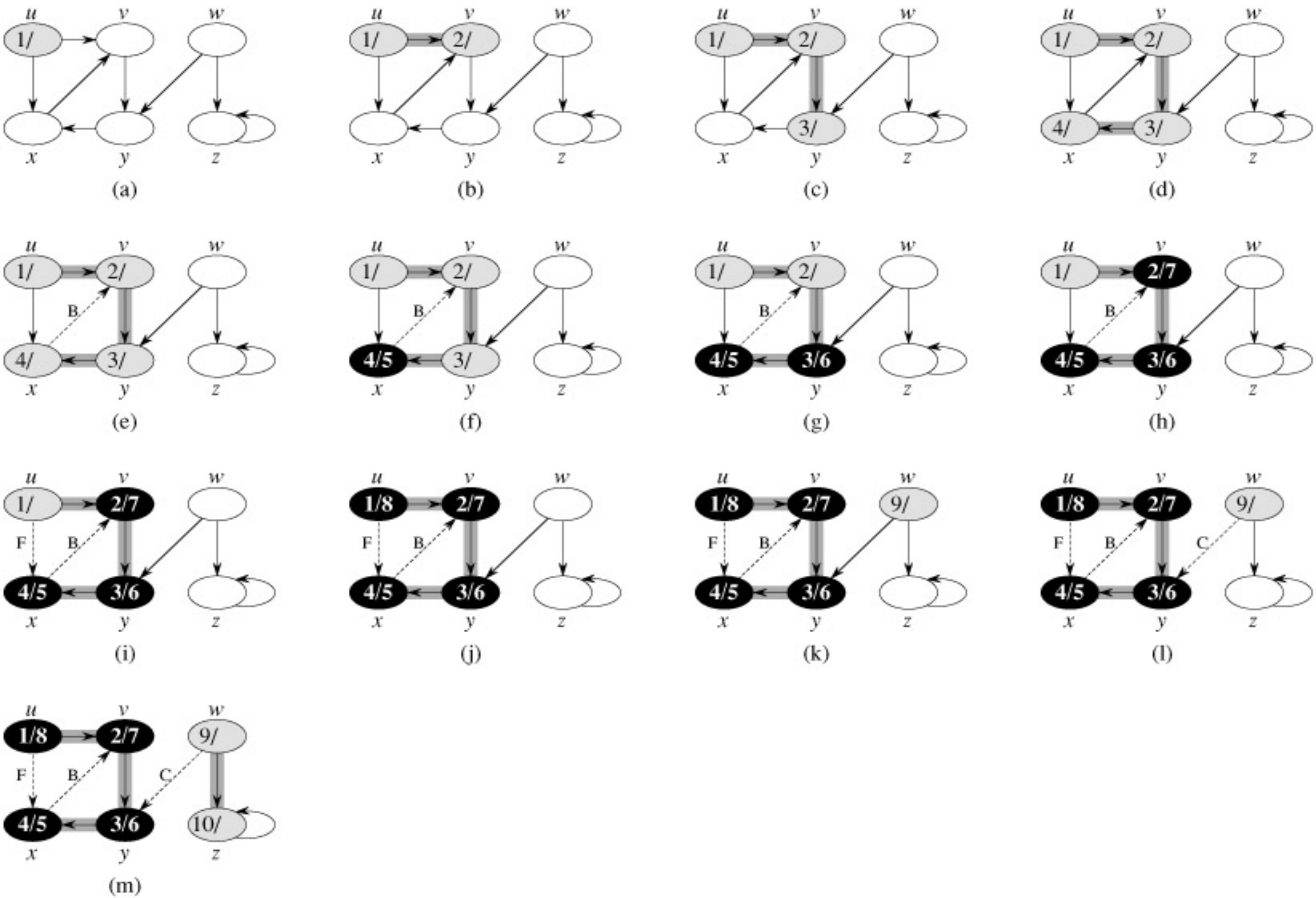
Depth First Search (DFS): Example



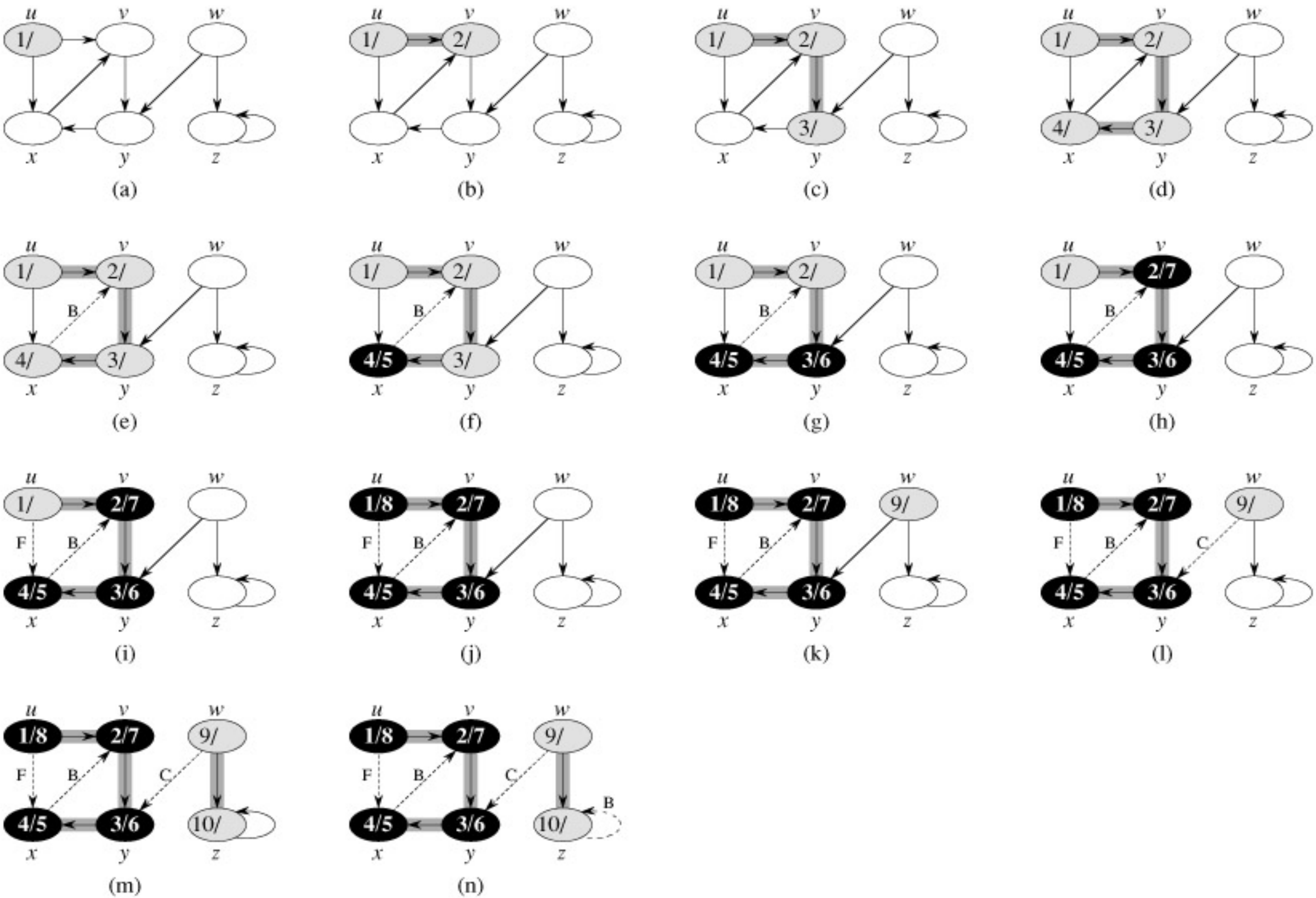
Depth First Search (DFS): Example



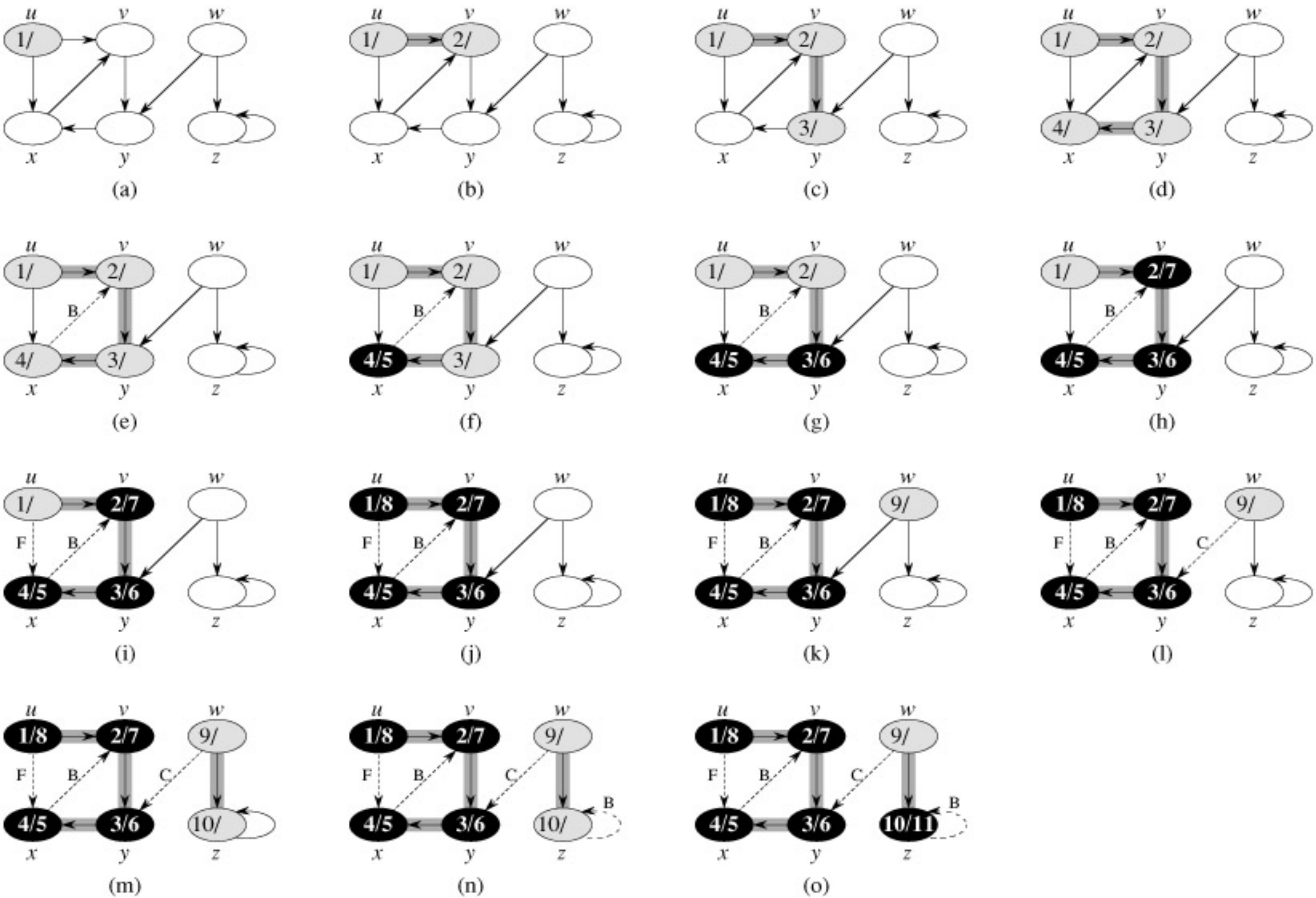
Depth First Search (DFS): Example



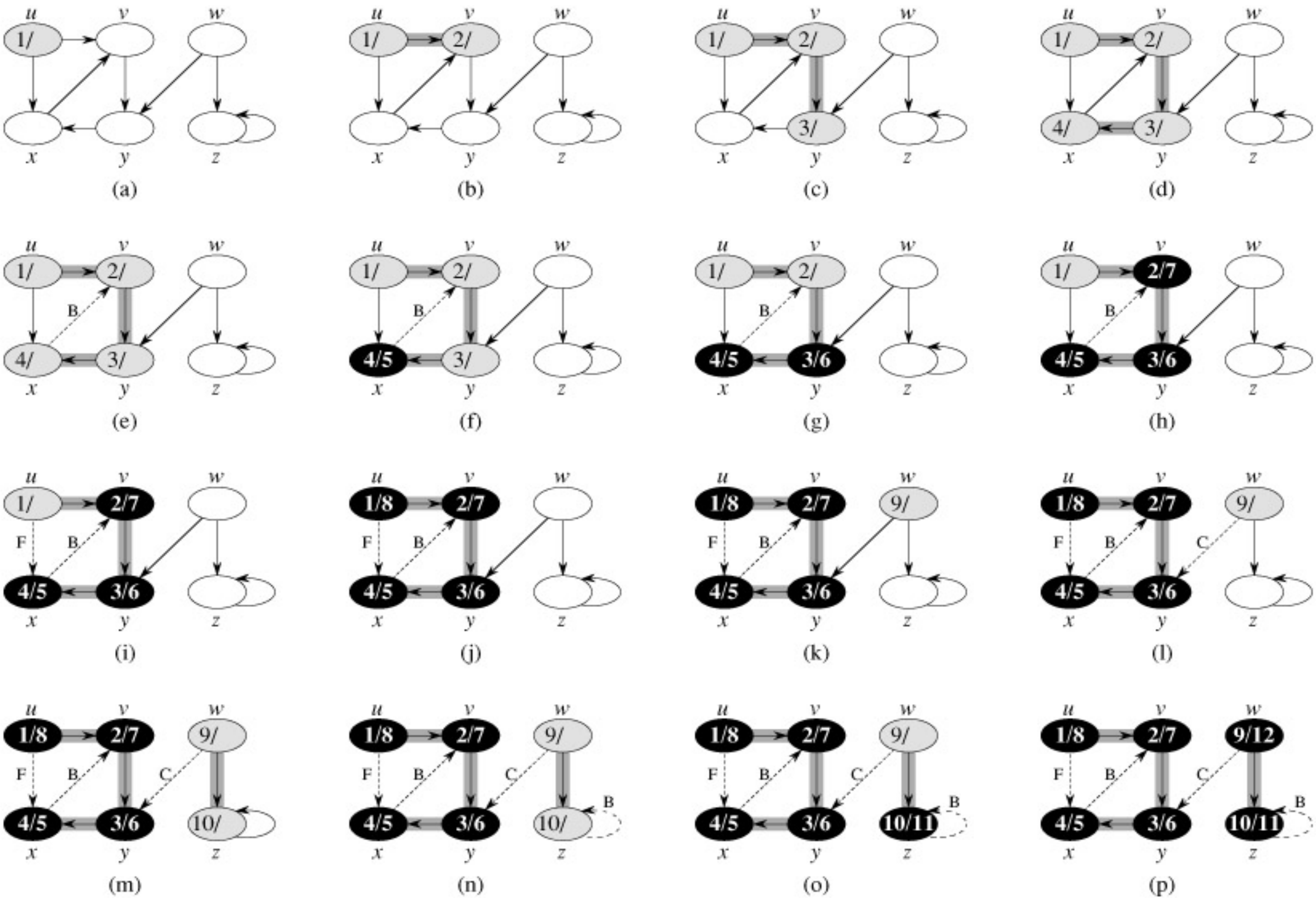
Depth First Search (DFS): Example



Depth First Search (DFS): Example

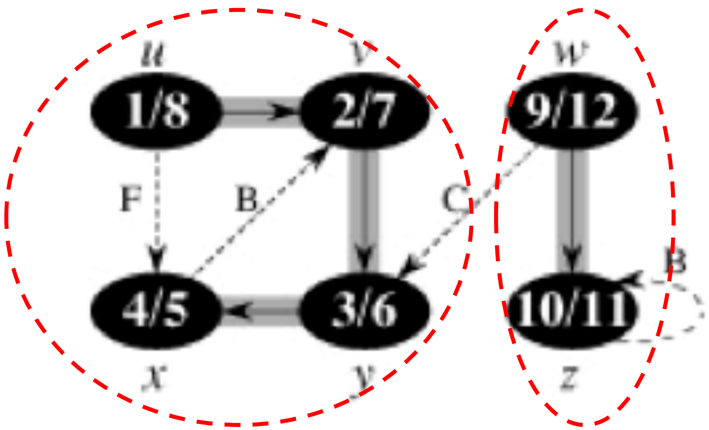


Depth First Search (DFS): Example



DFS: Forest

- The DFS may create **multiple disconnected trees** called **forest**



DFS: Time Complexity

```
DFS-VISIT(u)

1  color[u] ← GRAY        ▸White vertex u has just been discovered.
2  time ← time +1
3  d[u]=time

4  for each v in Adj[u]   ▸Explore edge(u, v).
5       do if color[v] = WHITE
6             then π[v] ← u
7                   DFS-VISIT(v)

8  color[u]=BLACK         ▸ Blacken u; it is finished.

9  f [u] ▸ time ← time +1
```

```
DFS (G)

for each vertex u in G.V
    color[u] = WHITE
    π(u) < —NIL
time <- 0
for each vertex  u in G.V
     if color[u] = WHITE
         DFS-VISIT(u)
```

Each node is recursively called once ➔ O(V)

The For Loop (Step 4) visits the outgoing edges of each node ➔ Total O(E)

Total Complexity O(V + E)

Also written in a more precise form as O(|V| +|E|)

Analyzing The Collected Info.

- **The algorithm computes for each node u**
  - **u.startTime (u.d)**
  - **u.endTime (u.f)**

- **These intervals form a nested structure**

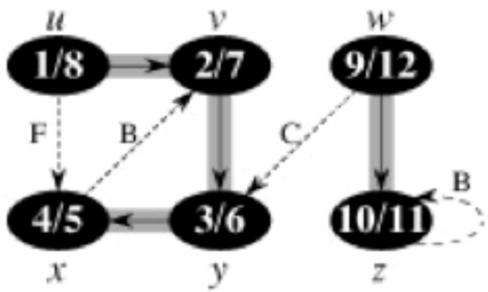
- **Parenthesis Theorem**
  - **If u has (u.d, u.f) , and v has (v.d, v.f), then either:**
  - **Case 1: u descendant of v in DFS**

    v.d < u.d < u.f < v.f
  - **Case 2: v descendant of u in DFS**

    u.d < v.d < v.f < u.f
  - **Case 3: u and v are disjoint (different trees in the forest)**

    (u.d, u.f) and (v.d, v.f) are not overlapping

Example



Based on the numbers (in the node) you can draw the DFS <u>trees (plural)</u>

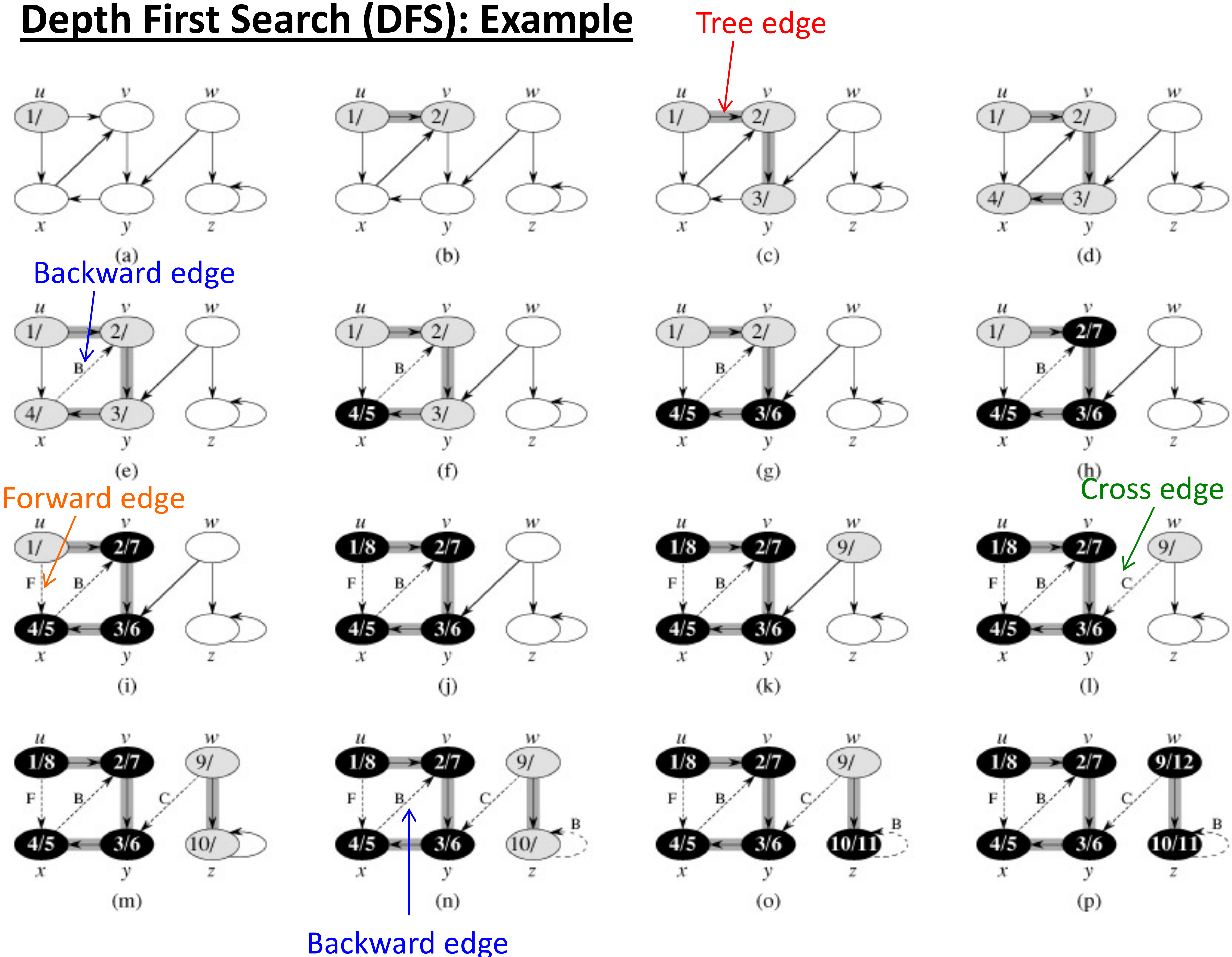
- v, y, x are all descendants of u
  - In other words: u is an ancestor to v, y, x

- w and u are disjoint

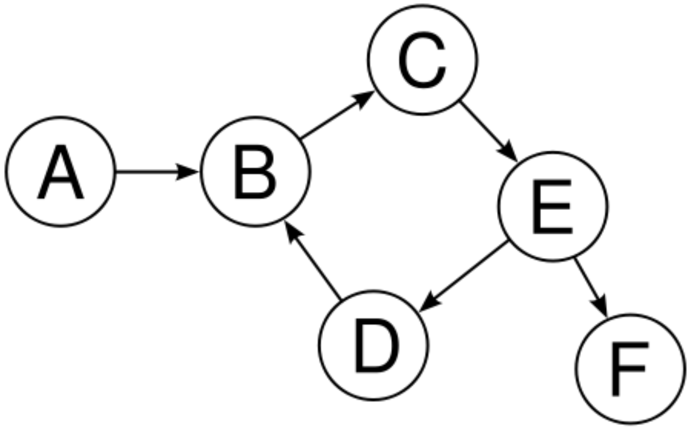
Classification of Graph Edges

- **While doing DFS, we can label the edges**
  - **Tree edges:** **Included in the DFS trees**
  - **Forward edges:** **From ancestor to already-visited descendant**
  - **Backward edges:** **From descendant to already-visited ancestor**
  - **Cross edges:** **Any other edges**

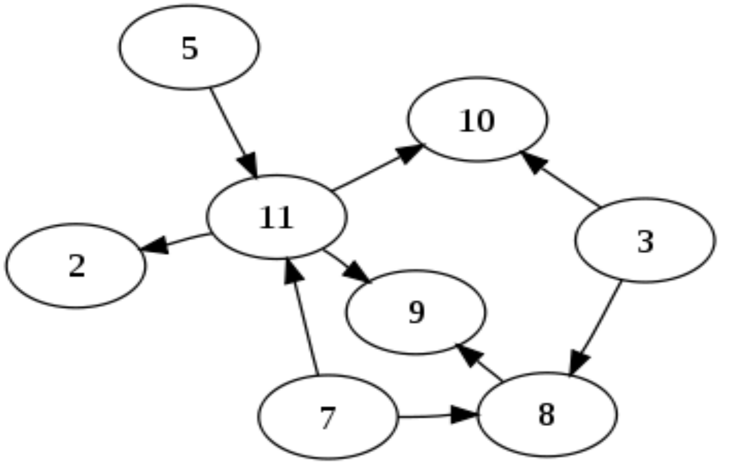
Depth First Search (DFS): Example



Cycles in Graphs



Cyclic graph (Graph containing cycles)



Acyclic graph (Graph without cycles)

Called: Directed Acyclic Graph DAG

- ***How to know if a graph has a cycle or not? What is the time complexity?***

Cycles in Graphs (Cont'd)

- ***Answer***
  - **Perform DFS on the graph**
  - **If there are backward edges ➔ there is a cycle**
  - **Otherwise ➔ there are no cycles**

Topological Sort

Topological Sort

- **Sorting technique over DAGs (Directed Acyclic Graphs)**
- **It creates a linear sequence (ordering) for the nodes such that:**
  - **If u has an outgoing edge to v → then u must finish before v starts**
- **Very common in ordering jobs or tasks**

Topological Sort Example

A job consists of 10 tasks with the following precedence rules:

Must start with 7, 5, 4 or 9.

Task 1 must follow 7.

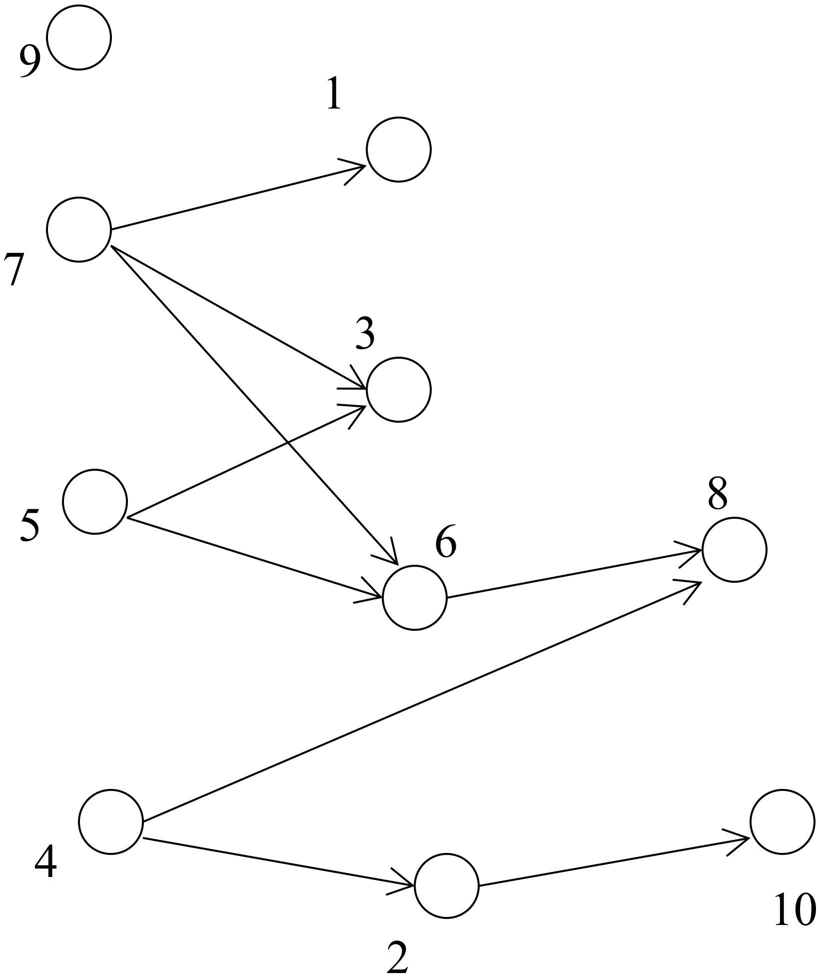
Tasks 3 & 6 must follow both 7 & 5.

8 must follow 6 & 4.

2 must follow 4.

10 must follow 2.

Make a directed graph and then do DFS.



Tasks shown as a directed graph.

Topological Sort using DFS

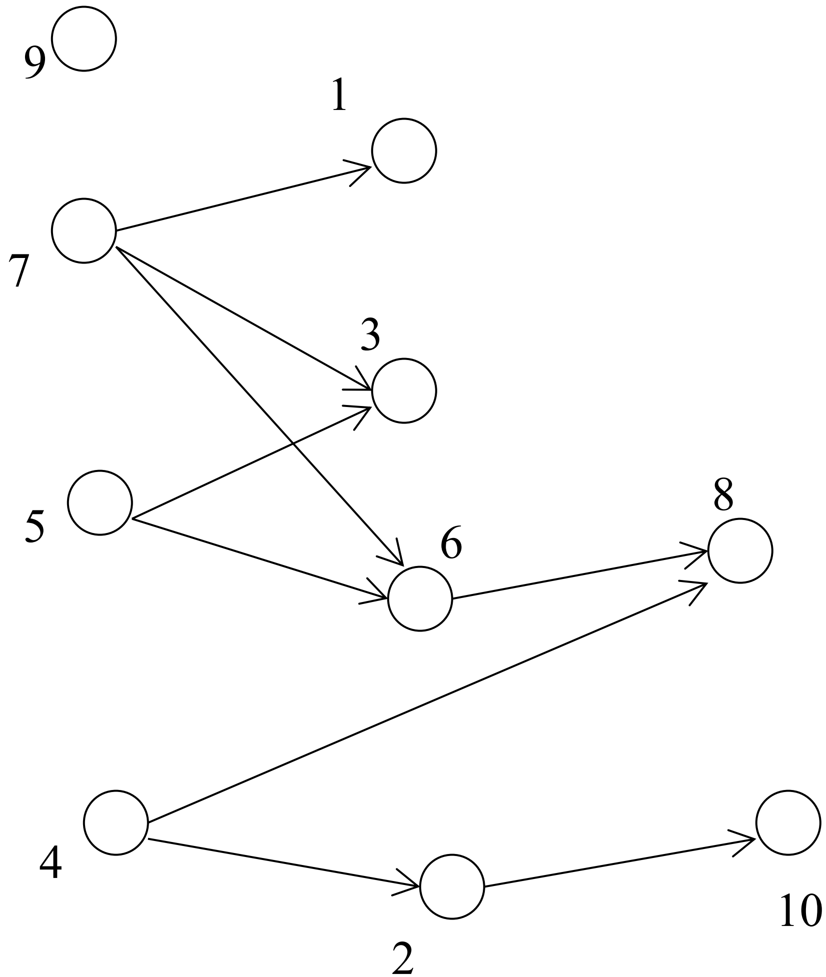
- To create a topological sort from a DAG

  1- Final linked list is empty

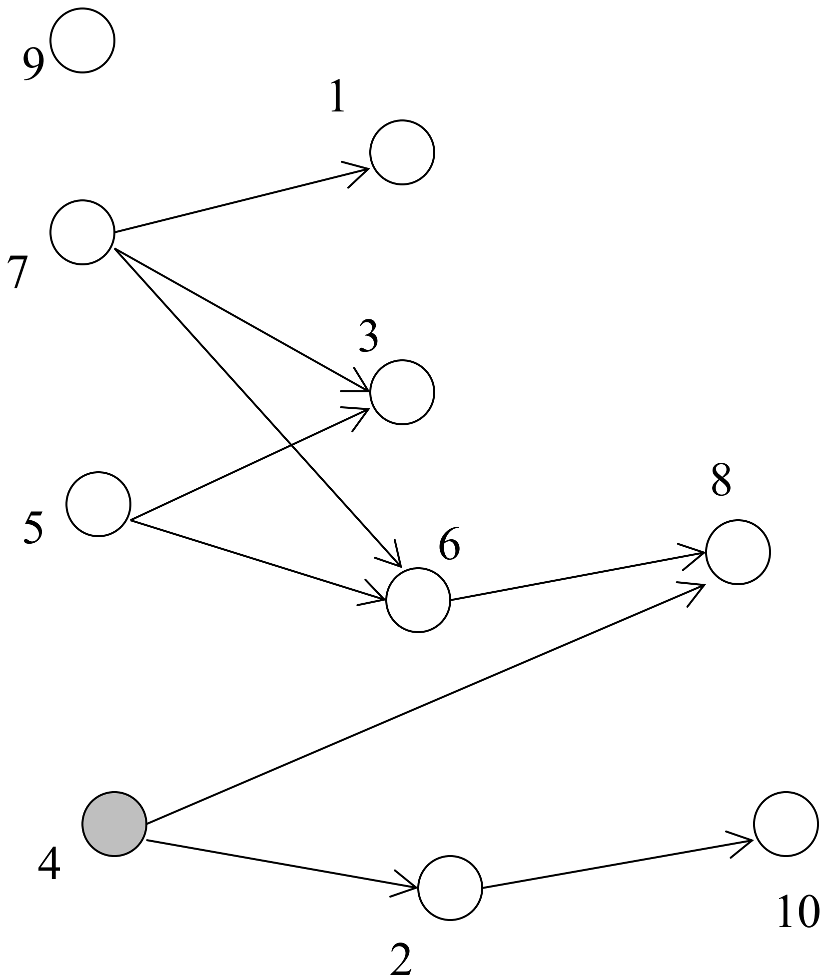
  2- Run DFS

  3- When a node becomes black (finishes) insert it to the top of a linked list

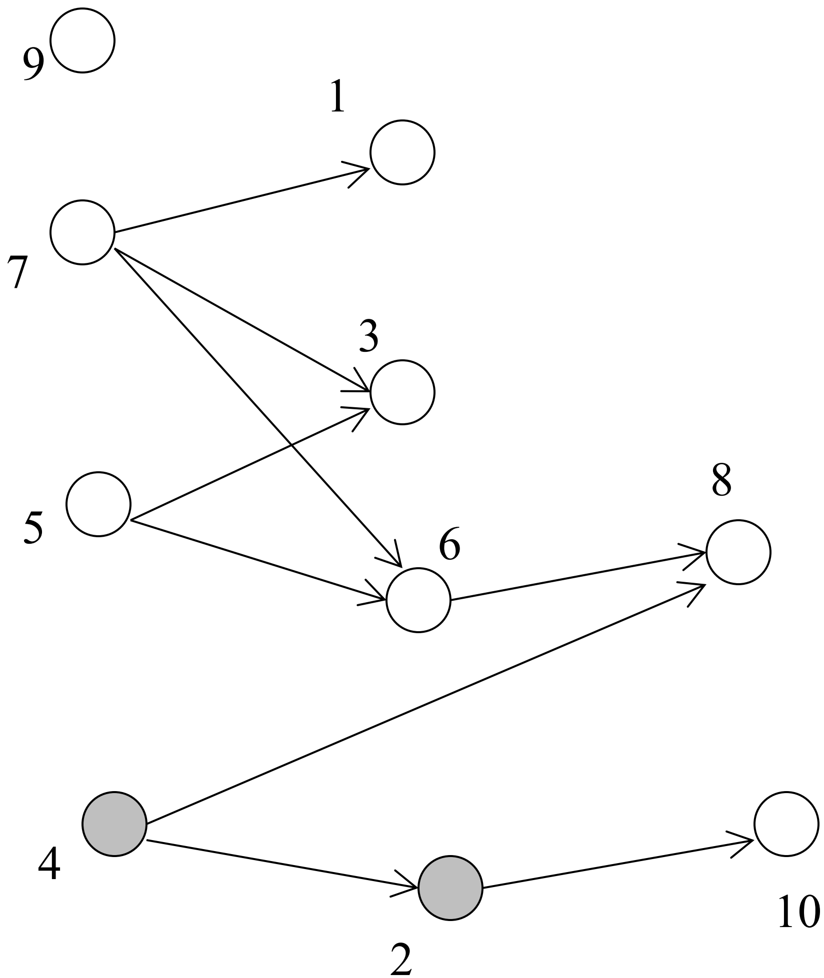
Example



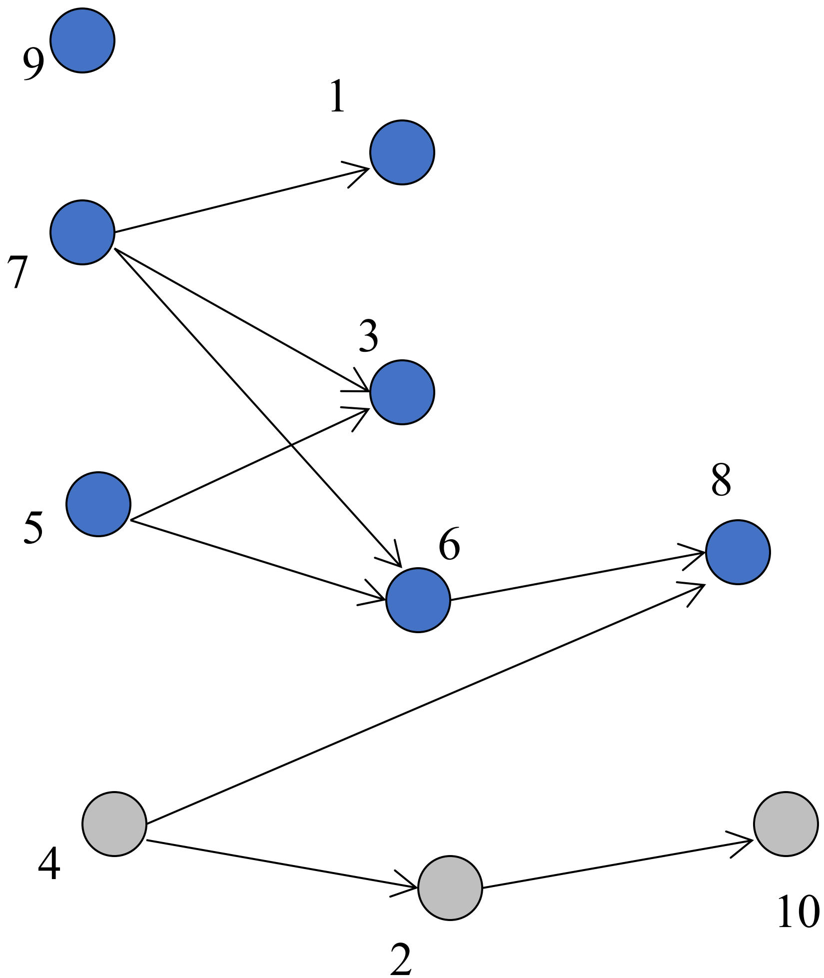
Example



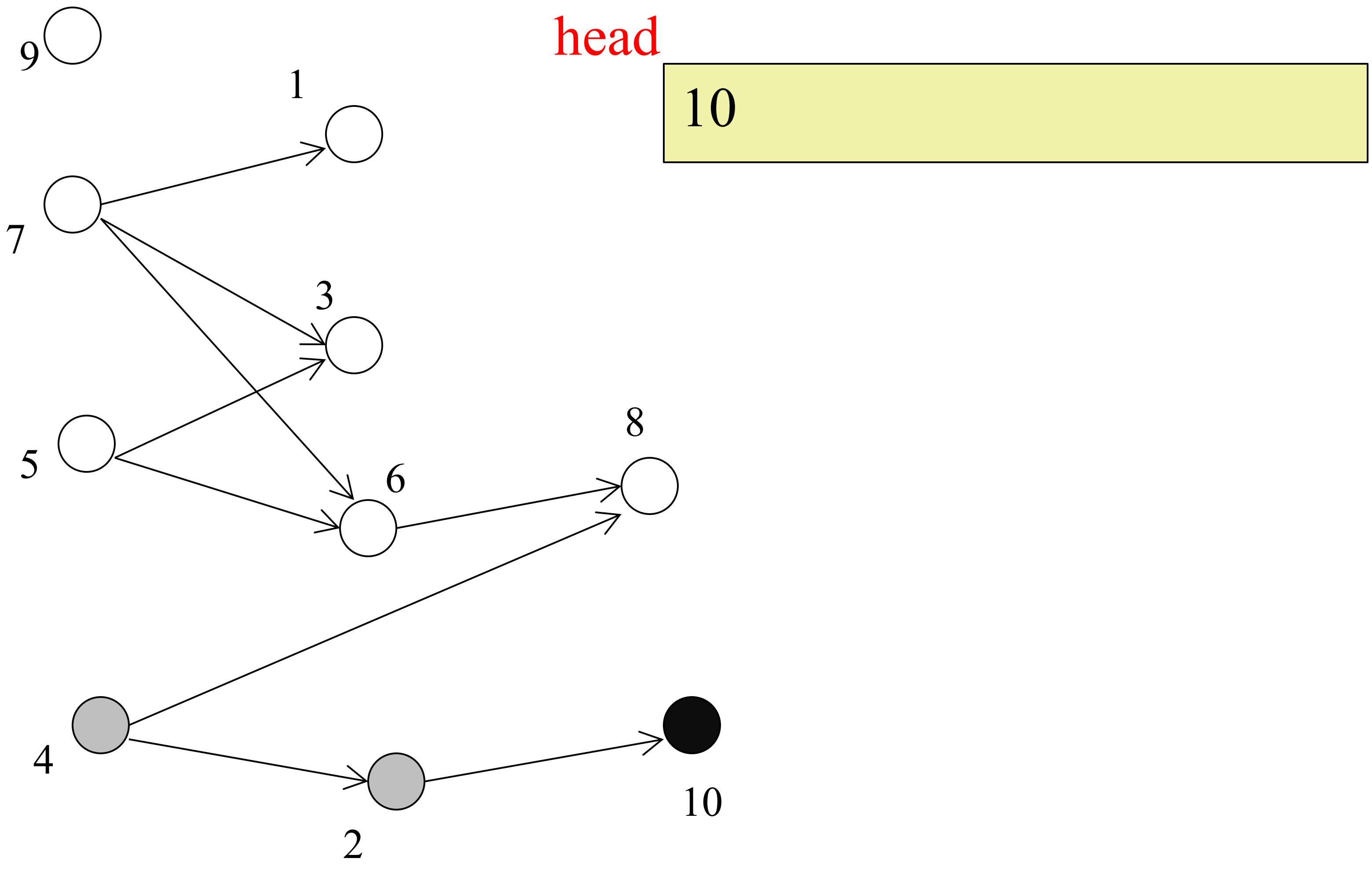
Example



Example

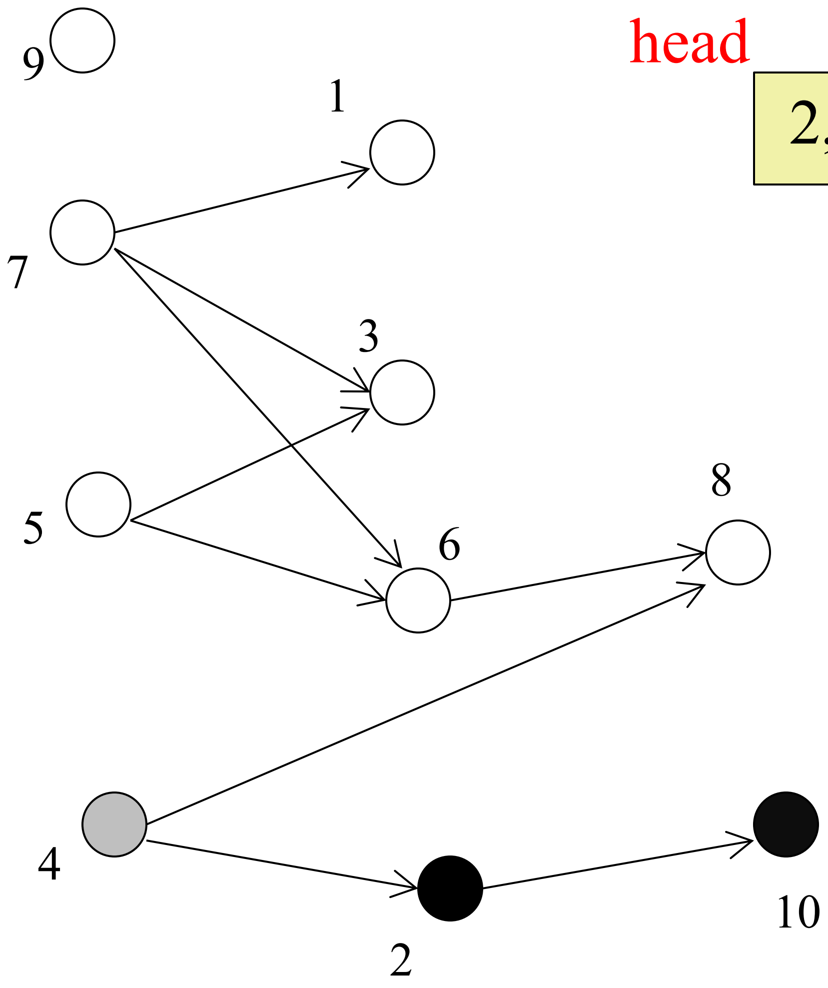


Example



Note, you need a stack to run the DFS algorithm, that’s not shown here. We only show the linked-list used for topological sort.

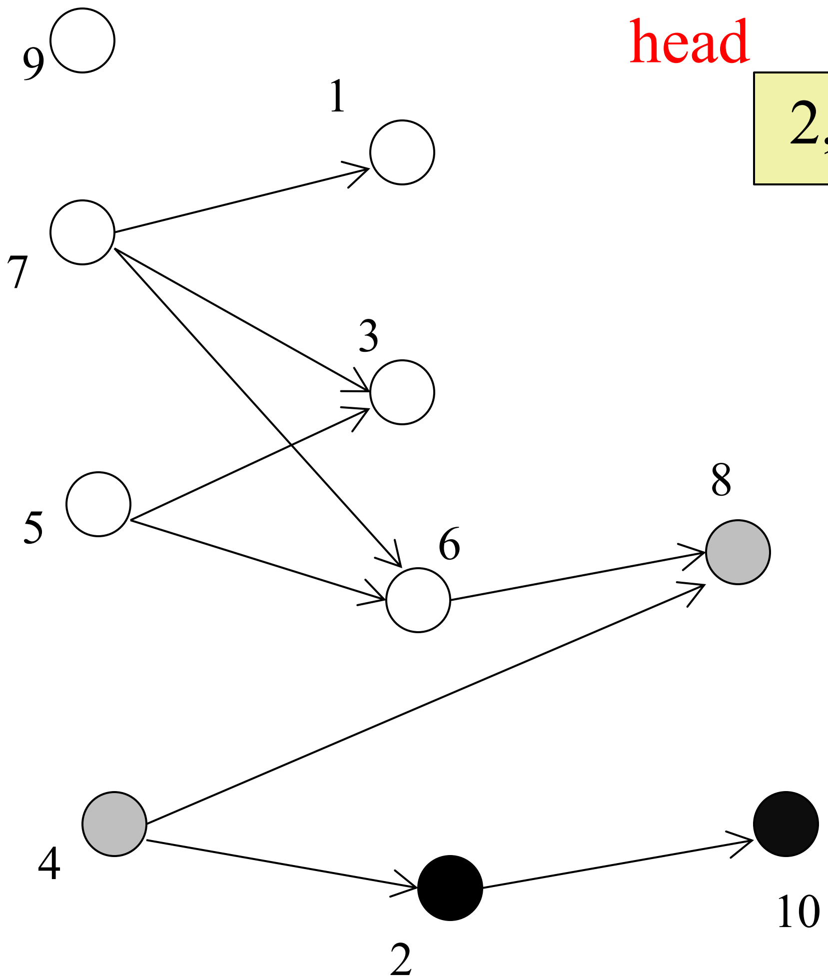
Example



head

2, 10

Example



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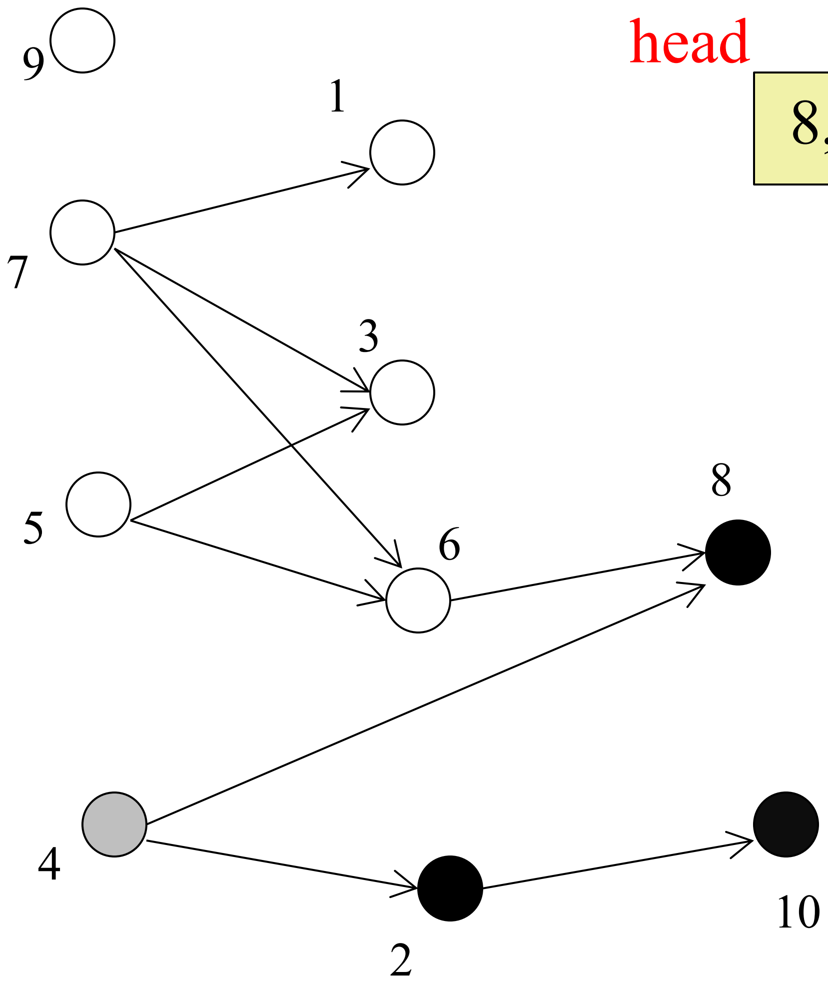
2, 10

Example



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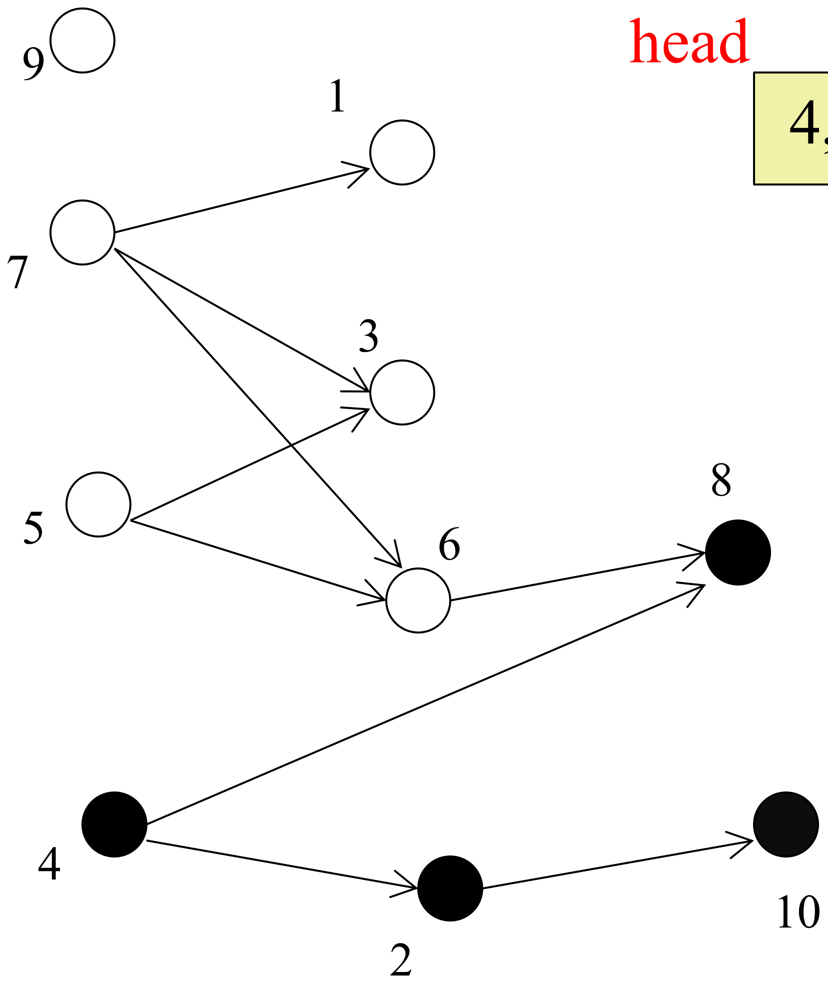
8, 2, 10

Example



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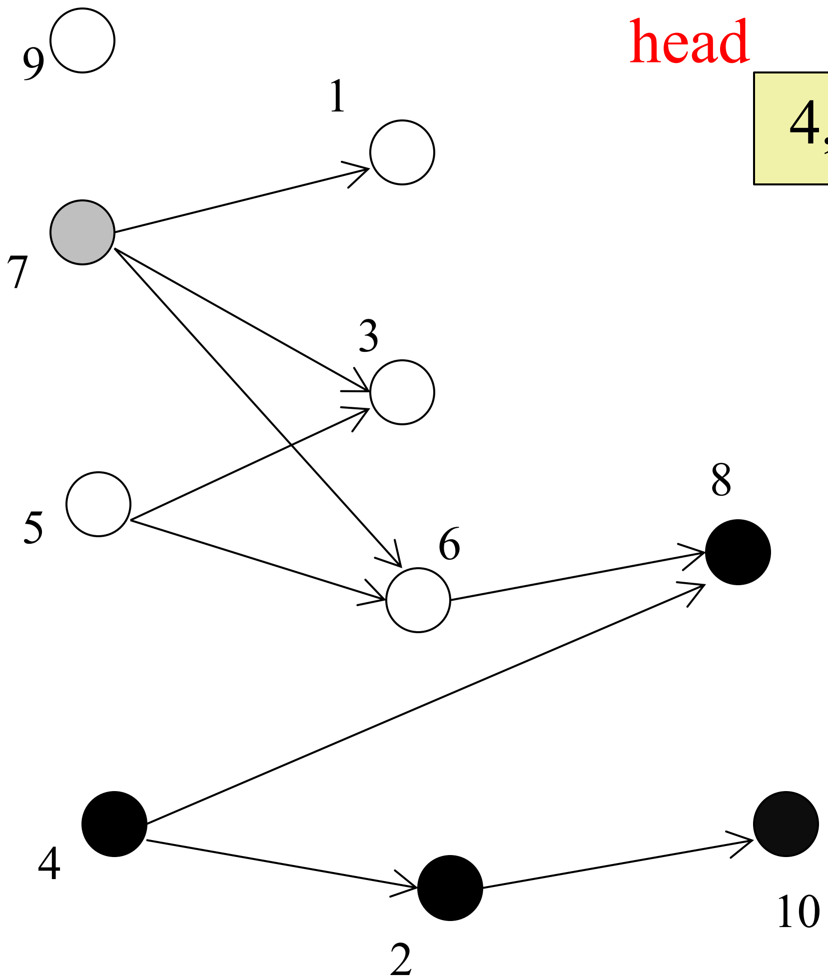
4, 8, 2, 10

Example



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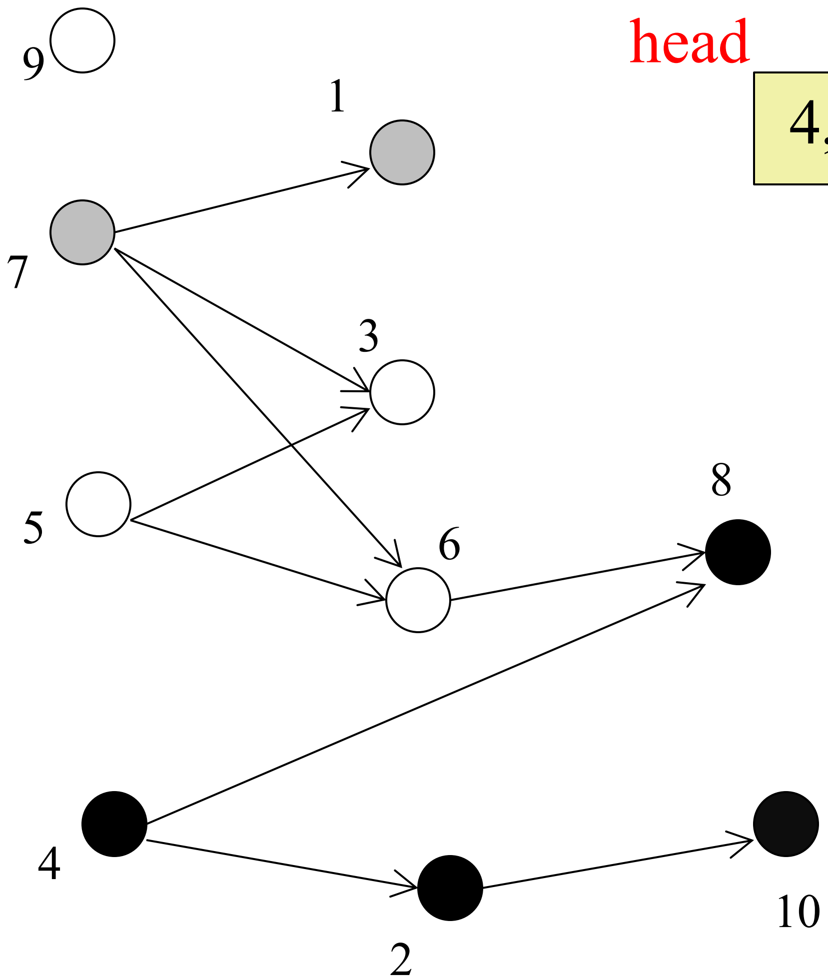
4, 8, 2, 10

Example



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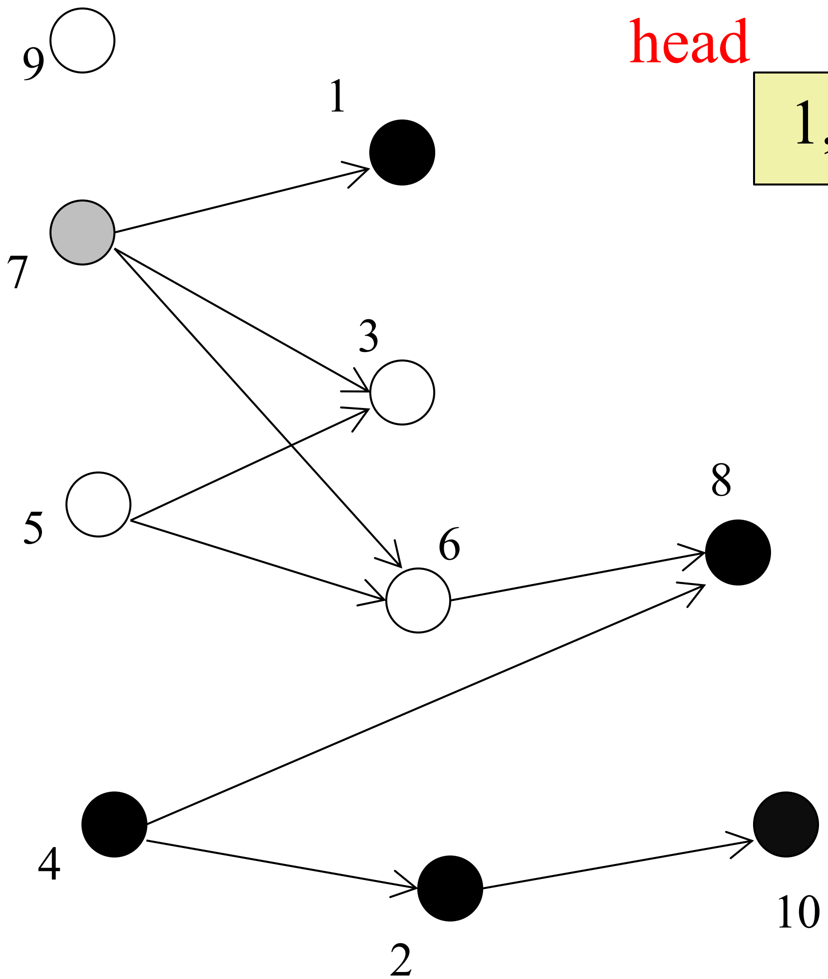
4, 8, 2, 10

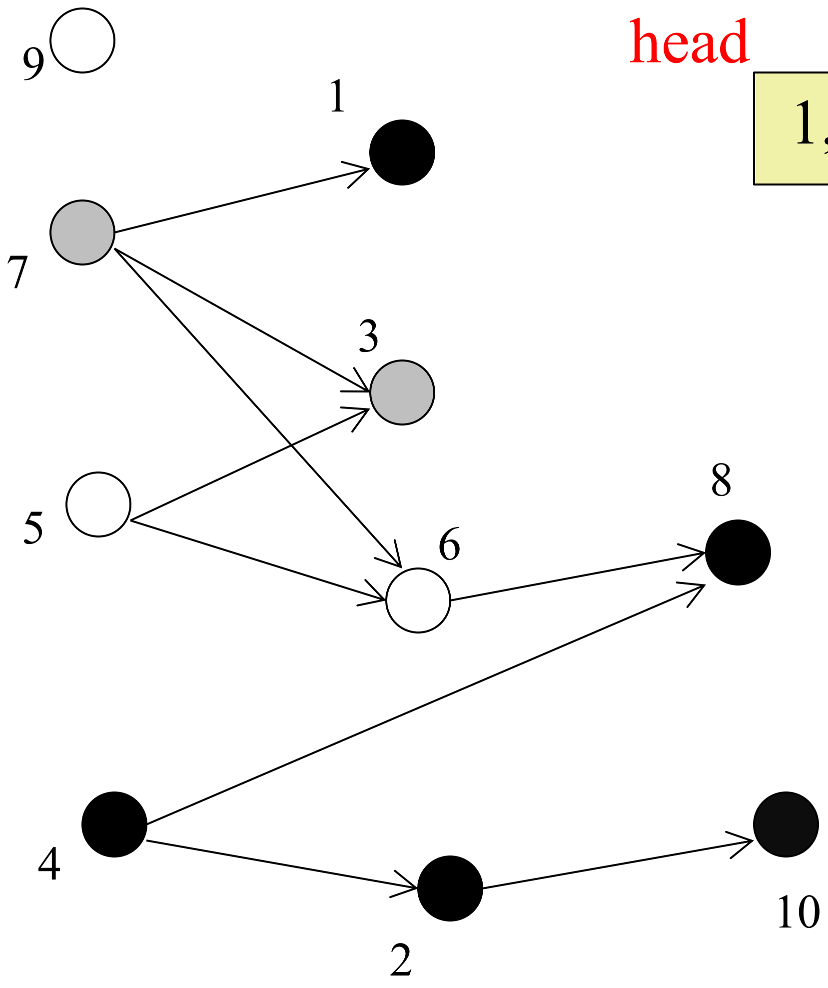
Example



head

1, 4, 8, 2, 10

Example



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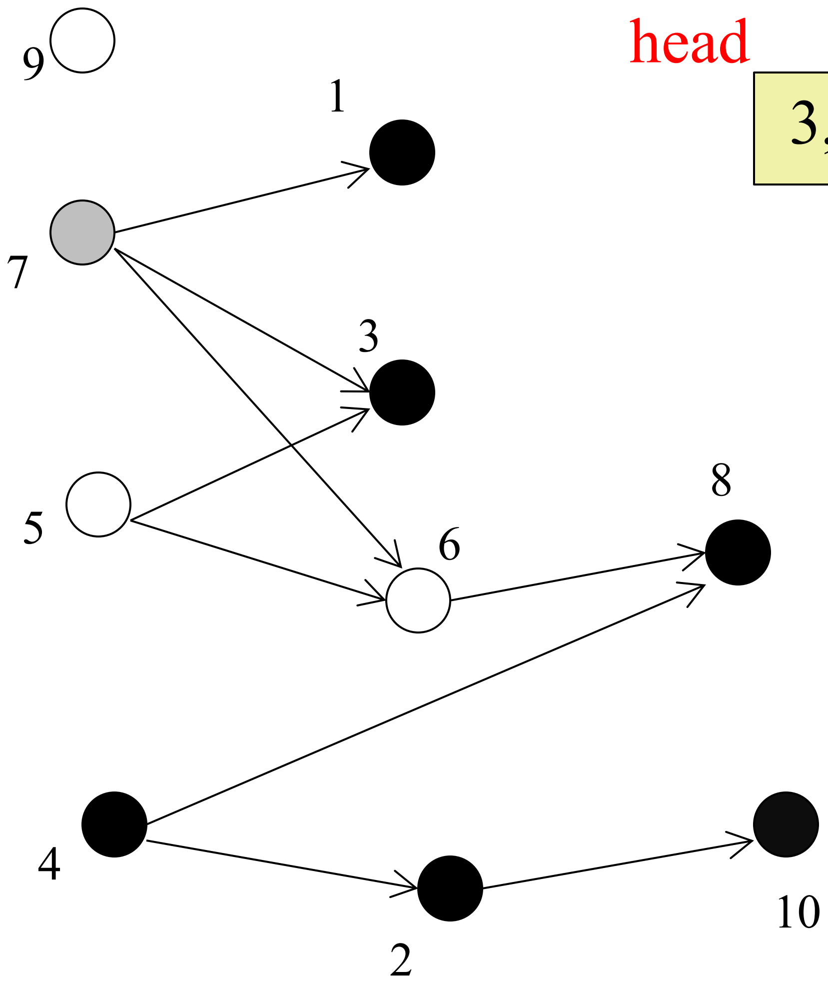
1, 4, 8, 2, 10

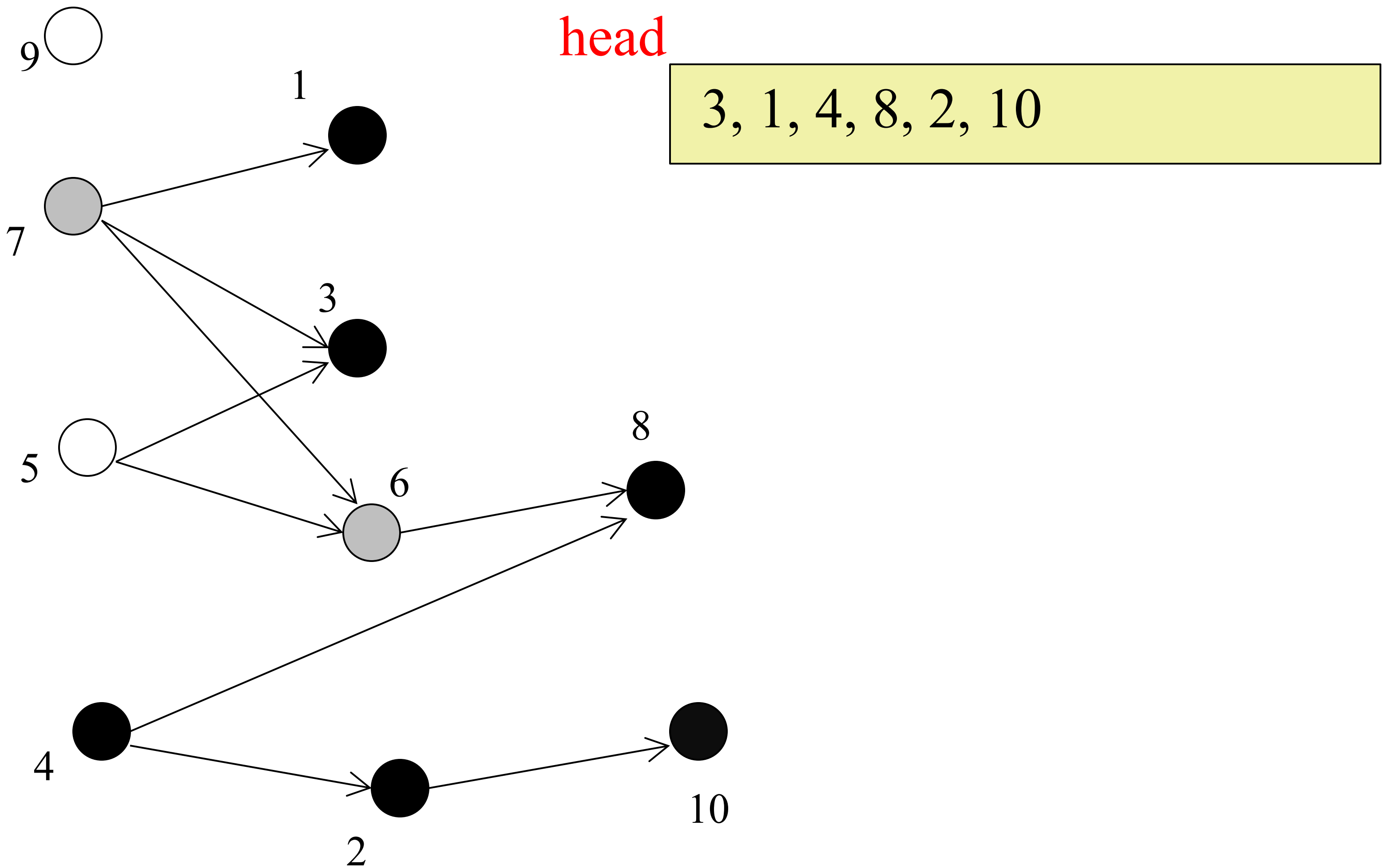
Example



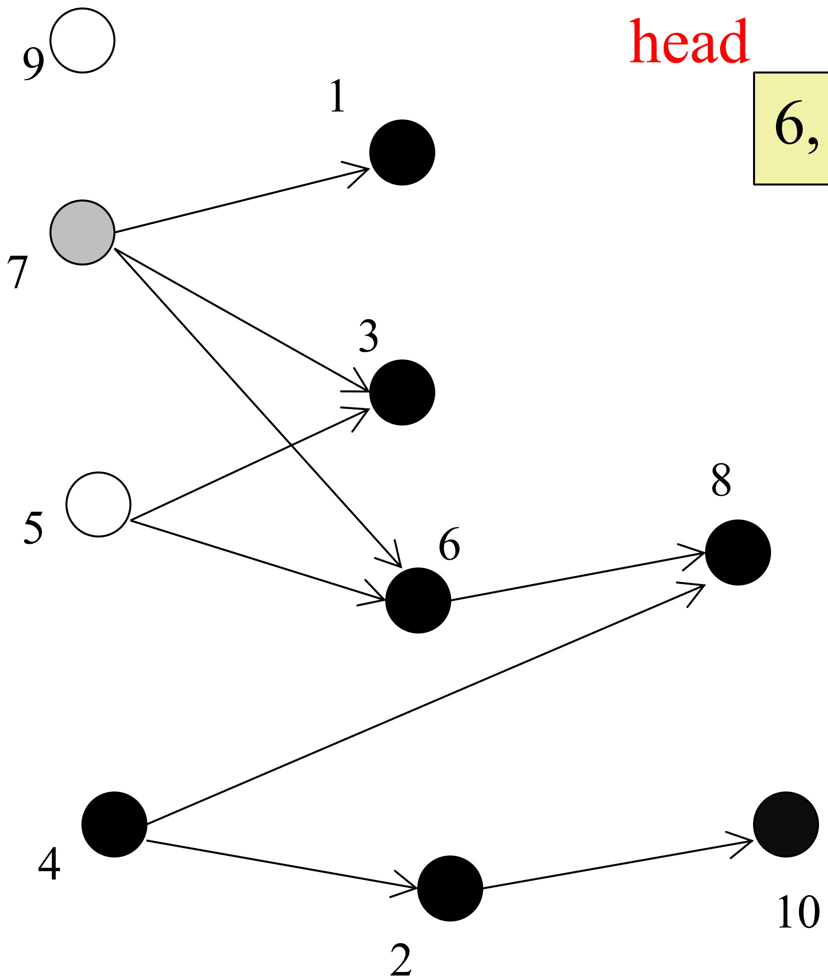
head

3, 1, 4, 8, 2, 10

Example



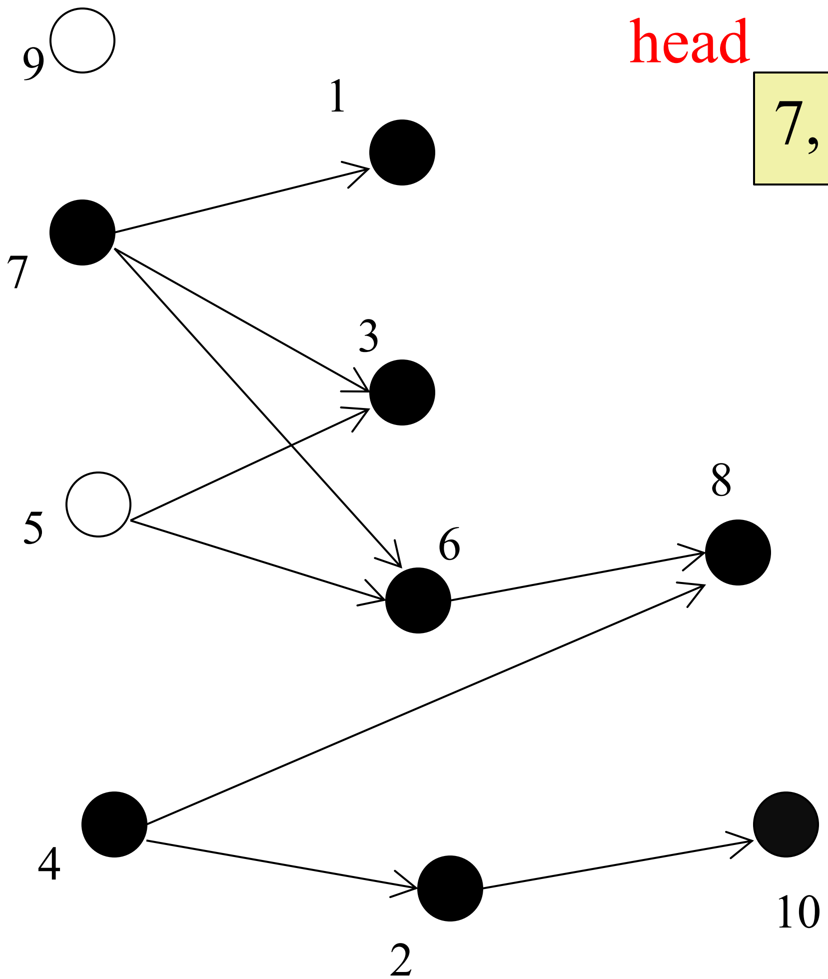
Example



head

6, 3, 1, 4, 8, 2, 10

Example



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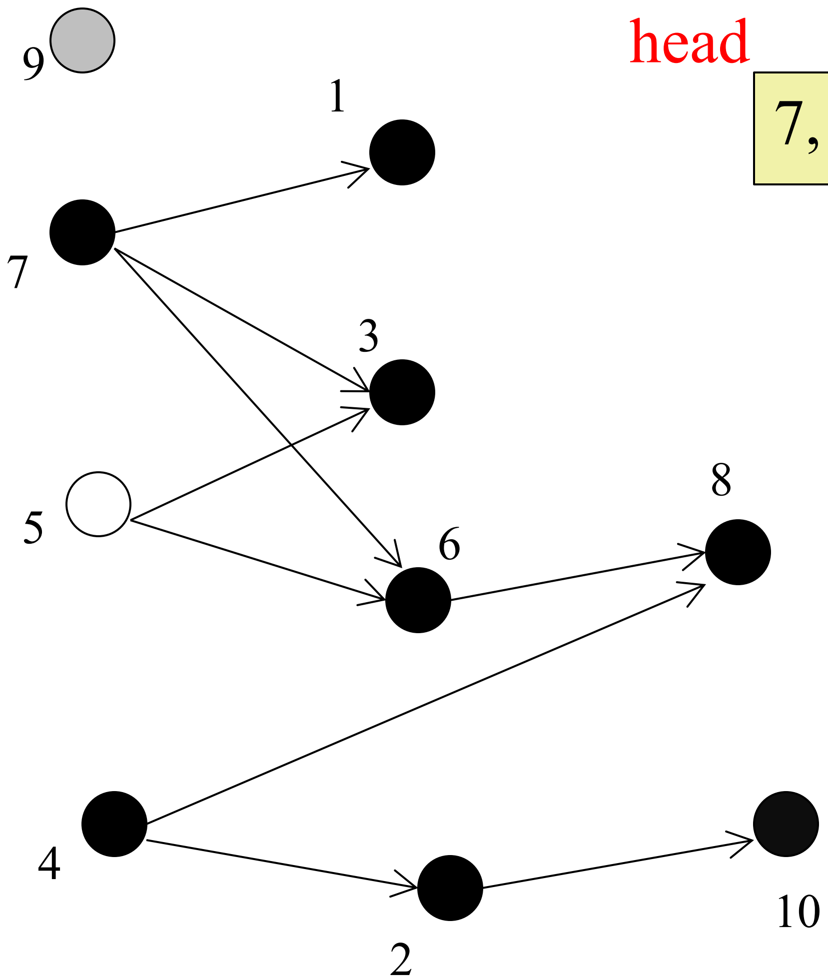
7, 6, 3, 1, 4, 8, 2, 10

Example



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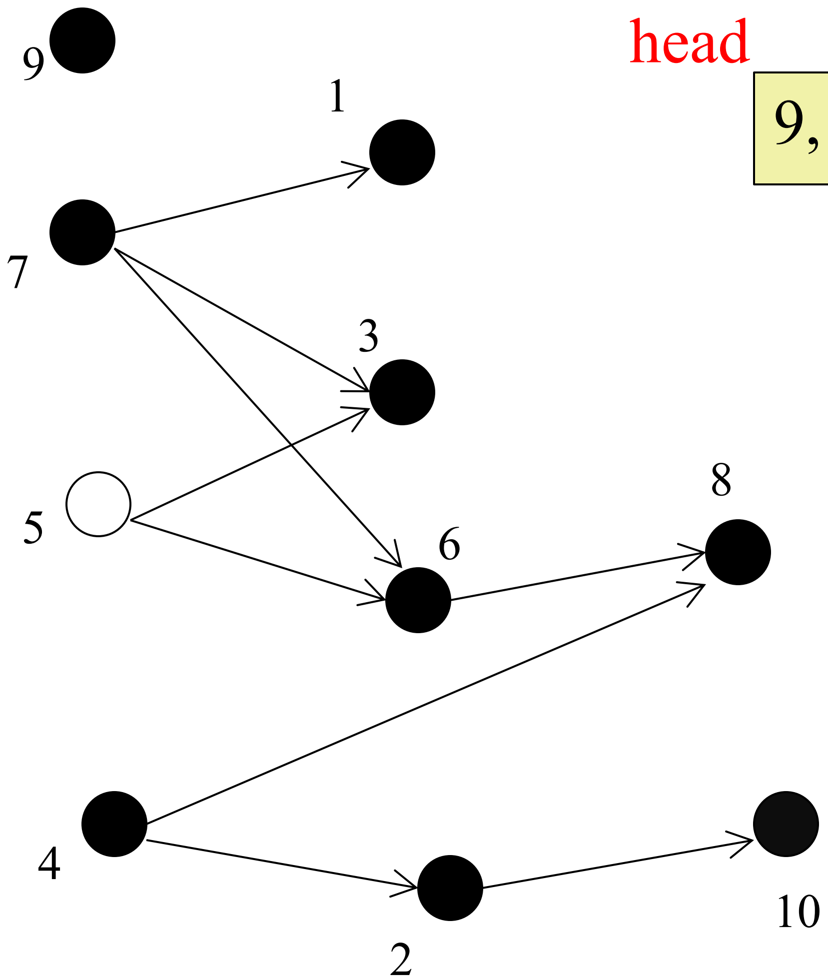
7, 6, 3, 1, 4, 8, 2, 10

Example



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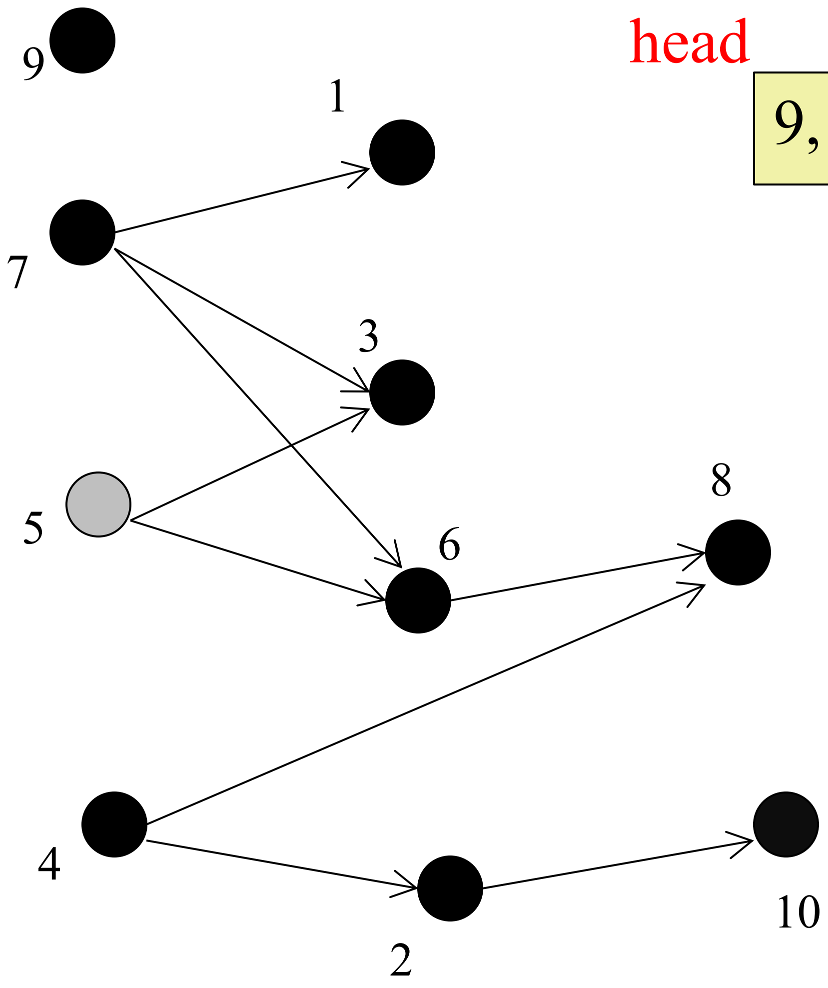
9, 7, 6, 3, 1, 4, 8, 2, 10

Example



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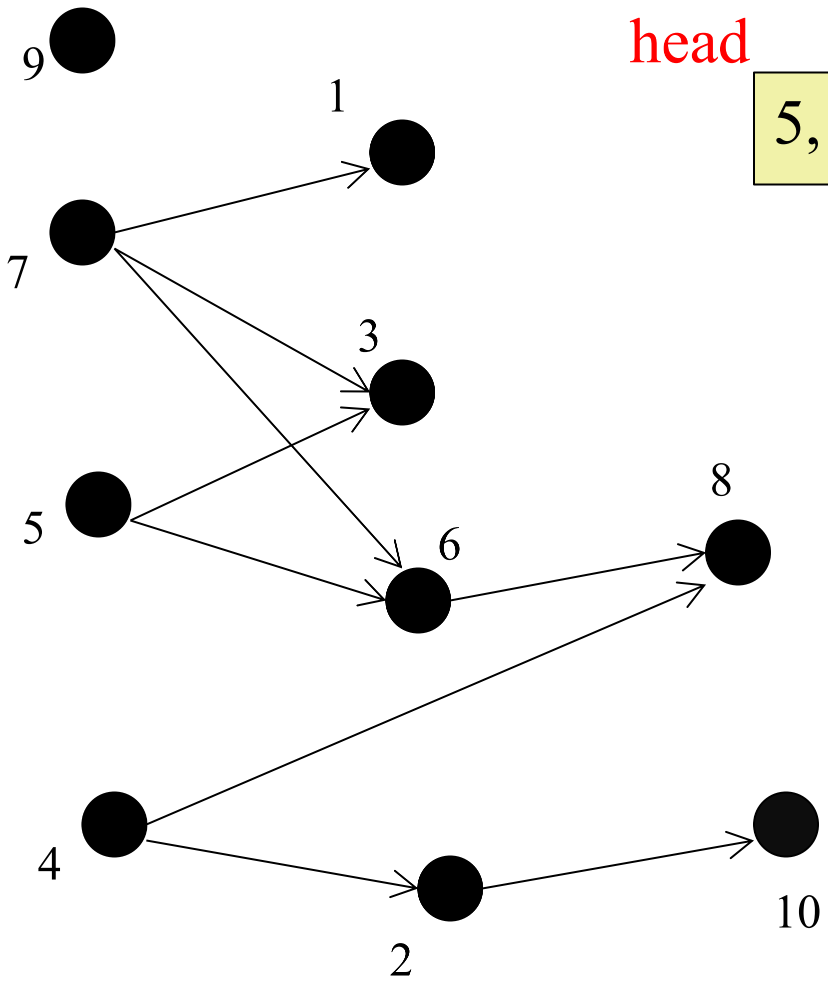
9, 7, 6, 3, 1, 4, 8, 2, 10

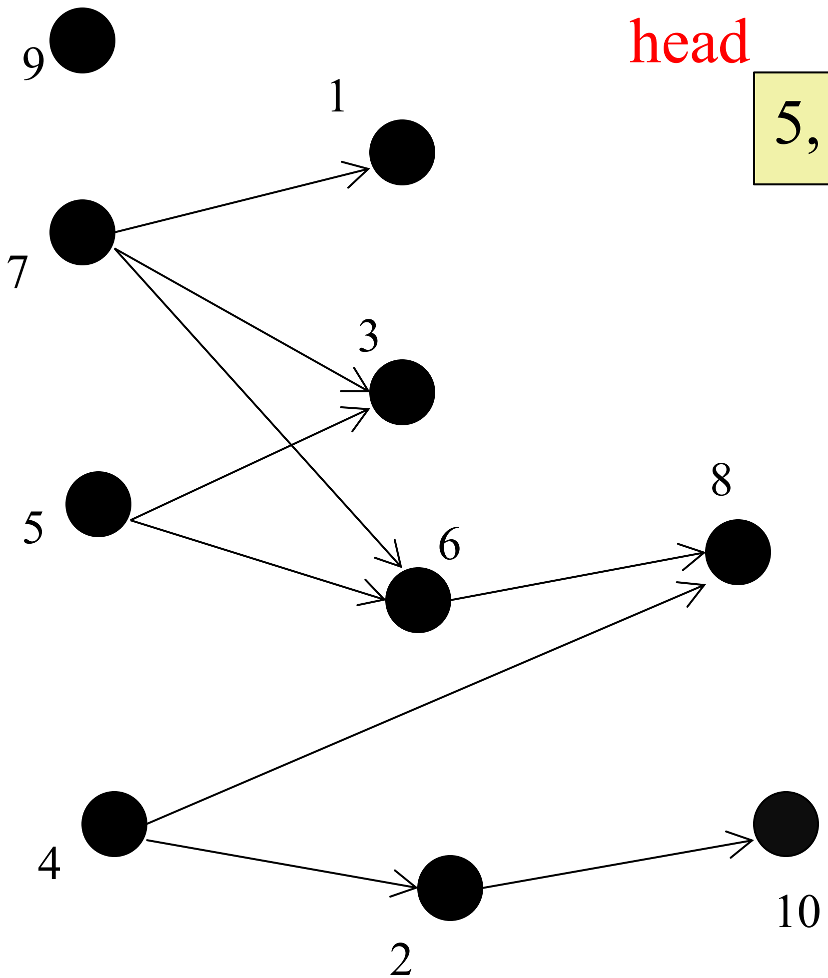
Example



head

5, 9, 7, 6, 3, 1, 4, 8, 2, 10

Topological Sort: Summary



head

5, 9, 7, 6, 3, 1, 4, 8, 2, 10

The final order or jobs

Time complexity = DFS complexity O(V + E)

There can be many orders that meet the requirements

Quiz 5 next Tuesday

- Quiz 5 will be held next Nov 26
- The quiz will primarily focus on Graph algorithms covered BFS and DFS
- My office hours at 1pm today is cancelled.
- Assignment 2 due Tuesday (Nov 26) midnight


That's all Folks!
Any Question ?