LECTURE 4 (30/01/2025)

IN LECTURE 2 WE STUDIED THE EXPONENTIAL EULER METHOD

$$y_{n+1} = e^{\tau A} y_n + \tau \varphi_1(\tau A)\, g(y_n)$$

WHICH ACTUALLY FOR EXPLICIT TIME-DEPENDENT NONLINEARITY IS

$$y_{n+1} = e^{\tau A} y_n + \tau \varphi_1(\tau A)\, g(t_n, y_n)$$

$$\Downarrow$$

$$\begin{cases} y'(t) = A y(t) + g(t, y(t)) & t \in (0, t^*] \\ y(0) = y_0 \end{cases}$$

TODAY WE "SIMPLIFY" THE SETTING AND DEVELOP EXPONENTIAL INTEGRATOR FOR <u>LINEAR</u> SYSTEMS OF ODEs

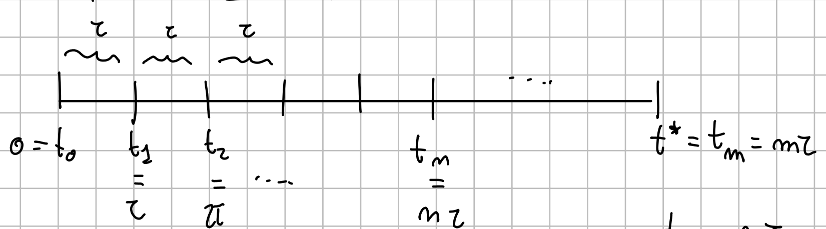
$$\begin{cases} y'(t) = A y(t) + g(t) = f(t, y(t)) & t \in (0, t^*] \\ y(0) = y_0 \end{cases} \qquad (*)$$

WHERE $y(t) \in \mathbb{C}^N$, $A \in \mathbb{C}^{N \times N}$ (STIFF PART), $g(t) \in \mathbb{C}^N$ IS THE TIME-DEPENDENT LINEAR PART (NONSTIFF) $\Rightarrow$ SOURCE TERM

TO BUILD EXPONENTIAL INTEGRATORS WE START FROM THE VARIATION-OF-CONSTANTS FORMULA FOR $(*)$

$$y(t) = e^{tA} y_0 + \int_0^t e^{(t-s)A} g(s)\, ds$$

WE INTRODUCE A TIME DISCRETIZATION



$t_n = n\tau$ WITH $\tau = \dfrac{t^*}{m}$

AND THE V.O.C. FORMULA AT TIME t_{n+1} STARTING FROM t_n GIVES

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^{\tau} e^{(\tau-s)A}\, \underline{g(t_n+s)}\, ds$$

TO GENERATE A TIME MARCHING METHOD WE APPROXIMATE g IN THE INTEGRAL. IN PARTICULAR WE DO $g(t_n+s) \approx g(t_n)$ IN $s \in [0,\tau]$ TO GET $(y(t_n) \approx y_n)$

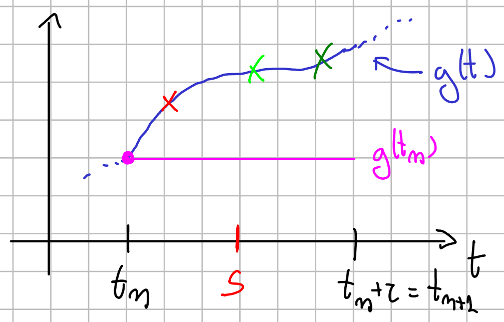
$$\varphi_1(x) = \int_0^1 e^{(1-\theta)x}\, d\theta$$

$$\begin{aligned} y_{n+1} &= e^{\tau A} y_n + \int_0^{\tau} e^{(\tau-s)A}\, ds\; g(t_n) \\ &= \boxed{e^{\tau A} y_n + \tau \varphi_1(\tau A)\, g(t_n)} \quad (\Delta) \\ &= y_n + \tau \varphi_1(\tau A)\,(A y_n + g(t_n)) \\ &= y_n + \tau\, \varphi_1(\tau A)\, f(t_n, y_n) \end{aligned}$$

$$x\varphi_1(x) = e^x - I$$

THIS BELONGS TO THE CLASS OF THE SO-CALLED EXPONENTIAL QUADRATURE RULES (HERE WITH A SINGLE COLLOCATION POINT).

NOT SURPRISINGLY, THIS IS A FIRST-ORDER METHOD (A-STABLE) SINCE IT IS ESSENTIALLY EXPONENTIAL EULER.



THE QUESTION IS

DO WE GAIN SOMETHING IF WE CHANGE COLLOCATION POINT?

SPOILER ⇒ YES

TO FORMALIZE THIS, INSTEAD OF APPROXIMATING $g(t_n+s) \approx g(t_n)$

WE DO

$$g(t_n+s) \approx g(t_n+c_1\tau) \qquad c_1 \in [0,1]$$

IF YOU REPEAT THE CALCULATIONS ABOVE WE GET THE SCHEME

$$\boxed{y_{n+1} = e^{\tau A} y_n + \tau\varphi_1(\tau A)\, g(t_n+c_1\tau)}$$

TO STUDY THE CONVERGENCE OF THIS METHOD, WE PROCEED AS DONE FOR EXPONENTIAL EULER.

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^\tau e^{(\tau-s)A} g(t_n+s)\, ds$$

V.O.C.

BY TAYLOR EXPANSION OF g WE GET

$$g(t_n+s) = g(t_n) + g'(t_n)s + \int_0^s g''(t_n+\sigma)(s-\sigma)\,d\sigma$$

WE INSERT THIS IN THE V.O.C. TO OBTAIN

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \boxed{\int_0^\tau e^{(\tau-s)A} ds}\, g(t_n) + \boxed{\int_0^\tau e^{(\tau-s)A} s\, ds}\, g'(t_n)$$

$$+ \int_0^\tau e^{(\tau-s)A} \int_0^s g''(t_n+\sigma)(s-\sigma)\,d\sigma\, ds$$

$$\varphi_\ell(X) = \frac{1}{(\ell-1)!}\int_0^1 e^{(1-\theta)X}\theta^{\ell-1}\,d\theta$$

$$= e^{\tau A} y(t_n) + \tau\varphi_1(\tau A)\, g(t_n) + \tau^2\varphi_2(\tau A)\, g'(t_n)$$

$$+ \int_0^\tau e^{(\tau-s)A} \int_0^s g''(t_n+\sigma)(s-\sigma)\,d\sigma\, ds$$

THEN WE COMPARE THIS WITH OUR NUMERICAL SOLUTION

$$\boxed{y_{n+1} = e^{\tau A} y_n + \tau\varphi_1(\tau A)\, g(t_n+c_1\tau)} \quad \text{(EQ1)}$$

IF $c_1=0$ WE RECOVER WHAT DID FOR EXPEULER

$$y_{n+1} - y(t_{n+1}) = e^{\tau A}(y_n - y(t_n)) + \tau\varphi_1(\tau A)\left(g(t_n+c_1\tau) - g(t_n)\right) - \tau^2\varphi_2(\tau A)\, g'(t_m)$$

$$- \int_0^\tau e^{(\tau-s)A} \int_0^s g''(t_n+\sigma)(s-\sigma)\,d\sigma\, ds$$

EXPAND NOW $g(t_{n}+c_1\tau)$ IN TAYLOR TO GET

$$g(t_n+c_1\tau) = g(t_n) + g'(t_n)c_1\tau + \int_0^{c_1\tau} g''(t_n+\sigma)(c_1\tau-\sigma)\,d\sigma$$

$$\Leftrightarrow \quad g(t_n+c_1\tau) - g(t_n) = g'(t_n)c_1\tau + \underline{\int_0^{c_1\tau} g''(t_n+\sigma)(c_1\tau-\sigma)\,d\sigma}$$

SO WE HAVE

$$y(t_{n+1}) - y_{n+1} = e^{\tau A}(y(t_n)-y_n) + \underline{c_1\tau^2\varphi_1(\tau A)g'(t_n)} + c_1\tau\,\varphi_1(\tau A)\int_0^{c_1\tau} g''(t_n+\sigma)(c_1\tau-\sigma)\,d\sigma$$

$$\underline{-\tau^2\varphi_2(\tau A)\,g'(t_n)} - \int_0^{\tau} e^{(\tau-s)A}\int_0^{s} g''(t_n+\sigma)(s-\sigma)\,d\sigma\,ds$$

$$= e^{\tau A}(y(t_n)-y_n) + \boxed{\tau^2\left(\boxed{c_1\varphi_1(\tau A) - \varphi_2(\tau A)}\right)g'(t_n)} \le C\tau^2$$

$$+\, c_1\tau\,\varphi_1(\tau A)\int_0^{c_1\tau} g''(t_n+\sigma)(c_1\tau-\sigma)\,d\sigma - \int_0^{\tau} e^{(\tau-s)A}\int_0^{s} g''(t_n+\sigma)(s-\sigma)\,d\sigma\,ds$$

$$\le C\tau^3 \qquad\qquad \le C\tau^3$$

IF $\quad c_1\varphi_1(\tau A) - \varphi_2(\tau A) = \tau\,\eta(\tau A) \quad \Rightarrow \quad \bullet \le C\tau^3$

BY TAYLOR EXPANSION OF THE φ FUNCTIONS WE SEE

$$\varphi_1(\tau A) = \sum_{k=0}^{+\infty}\frac{(\tau A)^k}{(k+1)!} = I + \frac{\tau A}{2} + \dots \qquad \varphi_2(\tau A) = \sum_{k=0}^{+\infty}\frac{(\tau A)^k}{(k+2)!} = \frac{I}{2} + \frac{\tau A}{6} + \dots$$

$$c_1\varphi_1(\tau A) - \varphi_2(\tau A) = \left(c_1 - \frac{1}{2}\right)I + \tau\left(\frac{c_1}{2} - \frac{1}{6}\right)A + \dots$$

IF $c_1 = \frac{1}{2}$ $\Rightarrow$ WE HAVE A GUESS THAT WE OBTAIN $\tau\eta(\tau A)$

HENCE WE LOOK FOR AN EXPRESSION FOR $\frac{1}{2}\varphi_1(\tau A) - \varphi_2(\tau A)$

THE φ FUNCTIONS SATISFY THE RECURSIVE FORMULA

$$x\,\varphi_{\ell+1}(x) = \varphi_\ell(x) - \varphi_\ell(0)$$

HENCE

$$\tau A \varphi_3(\tau A) = \varphi_2(\tau A) - \frac{1}{2} I \qquad \tau A \varphi_2(\tau A) = \varphi_1(\tau A) - I$$

$$\frac{1}{2}\tau A \varphi_2(\tau A) = \frac{1}{2}\varphi_1(\tau A) - \frac{1}{2} I$$

$$\ominus$$

$$\tau A\left(\frac{1}{2}\varphi_2(\tau A) - \varphi_3(\tau A)\right) = \frac{1}{2}\varphi_1(\tau A) - \varphi_2(\tau A) \iff \frac{1}{2}\varphi_1(\tau A) - \varphi_2(\tau A) = \underbrace{\tau A\left(\frac{1}{2}\varphi_2(\tau A) - \varphi_3(\tau A)\right)}_{\eta(\tau A)}$$

IF $c_1 = \frac{1}{2}$ (AND ONLY IF) WE HAVE CHANCES TO GET HIGHER ORDER. IN FACT,

$$y(t_{n+1}) - y_{n+1} = e^{\tau A}(y(t_n) - y_n) + \underbrace{\tau^3\left(\frac{1}{2}\varphi_2(\tau A) - \varphi_3(\tau A)\right) A g'(t_n)}_{\text{IN NORM } \le C\tau^3}$$

$$\underbrace{+ c_1\tau\varphi_1(\tau A)\int_0^{c_1\tau} g''(t_n+\sigma)(c_1\tau - \sigma)d\sigma}_{\le C\tau^3} \underbrace{- \int_0^{\tau} e^{(\tau-s)A}\int_0^s g''(t_n+\sigma)(s-\sigma)d\sigma ds}_{\le C\tau^3}$$

AS FOR EXPONENTIAL EULER WE CALL

$$\varepsilon_{n+1} \doteq y(t_{n+1}) - y_{n+1} \qquad \delta_{n+1} = \bullet \qquad \text{TO OBTAIN}$$

$$\varepsilon_{n+1} = e^{\tau A}\varepsilon_n + \delta_{n+1} \iff \varepsilon_n = \sum_{j=0}^{n-1} e^{j\tau A}\delta_{n-j} \qquad (\varepsilon_0 = 0)$$

THEN WE PASS TO THE NORMS TO GET

$$\|\varepsilon_n\| \le \sum_{j=0}^{n-1} \underbrace{\|e^{j\tau A}\|}_{\le C} \|\delta_{n-j}\| \le C\sum_{j=0}^{n-1}\|\delta_{n-j}\| \le C\sum_{j=0}^{n-1}\tau^3$$

$$\le C\tau^3 \sum_{j=0}^{n-1} 1 \quad \le n$$

$$\le C\tau^3 \frac{t^*}{\tau} \le C\tau^2$$

HENCE $\boxed{\|y(t_{n+1}) - y_n\| \le C\tau^2}$

AND WE GOT A SECOND-ORDER METHOD

THEOREM 2 (CONVERGENCE OF EXPONENTIAL QUADRATURE RULE WITH SINGLE POINT)

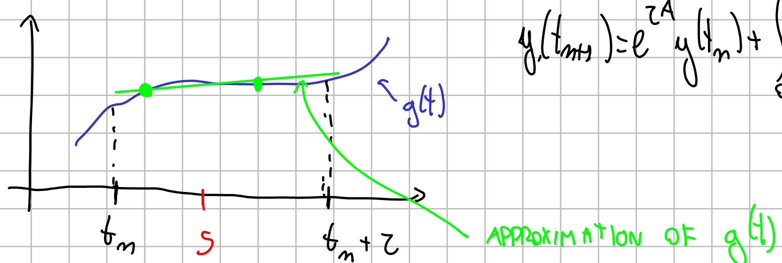
CONSIDER THE LINEAR PROBLEM (*) AND THE TIME-MARCHING INTEGRATOR (EQ1). THEN, ASSUMING g SUFFICIENTLY OFTEN DIFFERENTIABLE WITH BOUNDED DERIVATIVES WE HAVE:

- IF $c_1 \neq \frac{1}{2}$ $\quad \|y_n - y(t_n)\| \leq C\tau$
- IF $c_1 = \frac{1}{2}$ $\quad \|y_n - y(t_n)\| \leq C\tau^2$

HERE $0 \leq t_n \leq t^*$ AND THE CONSTANT C MAY DEPEND ON THE FINAL TIME t^* BUT NOT ON n.

PROOF: LOOK AT THE CALCULATIONS ABOVE $\square$

INTEGRATORS WITH TWO COLLOCATION POINTS



$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^\tau e^{(\tau-s)A} g(t_n+s)\,ds$$

ESSENTIALLY WE APPROXIMATE g WITH A LINEAR FUNCTION PASSING THROUGH TWO POINTS

$$g(t_n+s) \approx \alpha_s\, g(t_n+c_1\tau) + \beta_s\, g(t_n+c_2\tau)$$

WHERE $c_1, c_2 \in [0,1]$ AND α_s AND β_s ARE GIVEN BY INTERPOLATION CONDITIONS.

$\to$ NON-CONFLUENT NODES $\qquad$ LINEAR POLYNOMIALS

IF YOU DO THE CALCULATIONS YOU GET

$$y_{n+1} = e^{\tau A} y_n + \tau\left(\frac{c_2}{c_2-c_1}\varphi_1(\tau A) - \frac{1}{c_2-c_1}\varphi_2(\tau A)\right) g(t_n+c_1\tau)$$
$$+\ \tau\left(-\frac{c_1}{c_2-c_1}\varphi_1(\tau A) + \frac{1}{c_2-c_1}\varphi_2(\tau A)\right) g(t_n+c_2\tau) \qquad \text{(EQ2)}$$

SIMILARLY TO THE PREVIOUS CASE, ASSUMING g SUFFICIENTLY OFTEN DIFFERENTIABLE (WITH BDD DERIVATIVES) THE INTEGRATOR (EQ2) IS SECOND ORDER ACCURATE. IF WE IMPOSE ADDITIONAL CONDITIONS ON c_1, c_2 WE MAY GET HIGHER ORDER. FOR INSTANCE IF

$$\boxed{\frac{1}{3} - \frac{1}{2}(c_1+c_2) + c_1 c_2 = 0}$$

→ E.G. GAUSS-RADAU POINTS

THE INTEGRATOR MAY BE THIRD-ORDER ACCURATE

REMARK

EXPONENTIAL QUADRATURE RULES WITH SUFFICIENTLY HIGH NUMBER OF QUADRATURE POINTS ARE EXACT WHEN g IS A POLYNOMIAL.

LAB PART

$$\begin{cases} \partial_t y(t,x) = \delta\, \partial_{xx} y(t,x) + g(t) & t \in (0, t^*] \quad x \in [0,1] \\ y(t,0) = y(t,1) = 0 \\ y(0,x) = y_0(x) \end{cases}$$

1. $g(t)$ S.T. THE EXACT SOLUTION IS $y(t,x) = e^t x(1-x)$
2. " " $y(t,x) = e^t \sin(2\pi x)$