LECTURE 5 (3/02/2025)

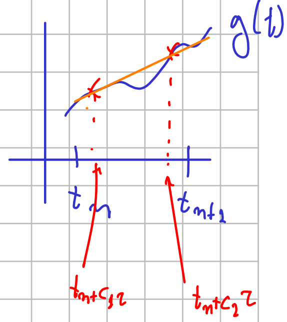
AS A RECALL DURING THE LAST LECTURE WE SAW EXPONENTIAL INTEGRATORS FOR LINEAR PROBLEMS

$$\begin{cases} y'(t) = Ay(t) + g(t) & t \in (0, t^*] \\ y(0) = y_0 \end{cases}$$

V.O.C. FORMULA ON A TIME GRID $t_n = n\tau$

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^{\tau} e^{(\tau - s)A} \underline{\underline{g(t_n + s)}}\, ds \quad *$$

$$\approx g(t_n + c_1 \tau) \qquad \approx \alpha_s g(t_n + c_1 \tau) + \beta_s g(t_n + c_2 \tau)$$



CAN WE DO SOMETHING SIMILAR IN THE SEMILINEAR CASE ?

TO FIX THE NOTATION

$$(SL) \quad \begin{cases} y'(t) = Ay(t) + g(y(t)) = f(y(t)) & t \in (0, t^*] \\ y(0) = y_0 \end{cases}$$

THE V.O.C. FORMULA FOR (SL) ON THE TIME GRID $t_n = n\tau$ IS

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^{\tau} e^{(\tau - s)A} \underline{\underline{g(y(t_n + s))}}\, ds \quad *$$

"IMPLICIT"

FOR EXPONENTIAL EULER WE APPROXIMATED $g(y(t_n + s)) \approx g(y(t_n))$

AND CALLING $y_n \approx y(t_n)$ WE GET SOMETHING "COMPUTABLE"

$$y_1 = e^{\tau A} \underline{\underline{y_0}} + \tau \varphi_1(\tau A)\, g(\underline{\underline{y_0}})$$

↳ INITIAL CONDITION

$$y_2 = e^{\tau A} \underline{\underline{y_1}} + \tau \varphi_1(\tau A)\, g(\underline{\underline{y_1}})$$

↳ FROM THE PREVIOUS ITERATION

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^{\tau} e^{(\tau-s)A} \underline{\underline{g(y(t_n+s))}}\, ds$$

$$\downarrow$$

$$\underline{\underline{g(y(t_n+c_2\tau))}}$$

IS SOMETHING THAT WE DON'T HAVE AVAILABLE, IN CONTRAST TO WHAT HAPPENS IN THE LINEAR CASE (EXPONENTIAL QUADRATURE RULES)

THE NATURAL THING TO DO IS TO <u>CREATE</u> AN APPROXIMATION OF THE NEEDED QUANTITY $\Rightarrow$ BUILDING OF AN INTERMEDIATE STAGE

$$Y_{n2} \approx y(t_n + c_2\tau) \qquad y_{n+1} \approx y(t_{n+1}) = y(t_n+\tau) \qquad y_n \approx y(t_n)$$

$\boxed{c_2 \in (0,1]}$

$$Y_{n2} = e^{c_2\tau A} y_n + c_2 \tau \varphi_1(\tau A) g(y_n)$$ $\rightarrow$ EXP. EULER FROM t_n TO $t_n + c_2\tau$

COMPARE WITH V.O.C.

$$y_{n+1} = e^{\tau A} y_n + \int_0^{\tau} e^{(\tau-s)A} g(Y_{n2})\, ds$$
$$= e^{\tau A} y_n + \tau\varphi_1(\tau A)\, g(Y_{n2})$$

$$\boxed{\begin{aligned} Y_{n2} &= e^{c_2\tau A} y_n + c_2 \tau \varphi_1(c_2\tau A)\, g(y_n) \\ y_{n+1} &= e^{\tau A} y_n + \tau \varphi_1(\tau A)\, g(Y_{n2}) \end{aligned}} \qquad (*)$$

THE TIME-MARCHING METHOD $(*)$ IS THE EQUIVALENT OF WHAT WE SAW LAST TIME IN THE LINEAR CASE (WITH ONE COLLOCATION POINT).

WE WILL NOT SEE THE PROOF BUT WE CAN SHOW THAT UNDER SUITABLE SMOOTHNESS ASSUMPION WE HAVE FOR INTEGRATOR $(*)$

- IF $c_2 \neq \frac{1}{2}$ $\Rightarrow$ FIRST-ORDER CONVERGENCE
- IF $c_2 = \frac{1}{2}$ $\Rightarrow$ SECOND-ORDER CONVERGENCE

STIFF ORDER CONDITIONS ARE NOT SATISFIED IN STRONG SENSE

IN THE "CONTINUOUS" CASE (E.G., $A = \partial_{xx} \oplus$ B.C.) WE MAY ENCOUNTER ORDER REDUCTION

$$\boxed{\begin{aligned} Y_{n2} &= e^{c_2\tau A} y_n + c_2 \tau \varphi_1(c_2\tau A)\, g(y_n) \\ y_{n+1} &= e^{\tau A} y_n + \tau \varphi_1(\tau A)\, g(Y_{n2}) \end{aligned}} \iff \begin{array}{c|cc} 0 & \times & \times \\ c_2 & c_2\varphi_1(c_2\cdot) & \times \\ \hline & 0 & \varphi_1(\cdot) \end{array}$$

BUTCHER TABLEAU

$c_2 \varphi_{1,2}(\cdot)$

THE IDEA IS TO EXPLOIT $g(y_n)$ ALSO IN THE FINAL APPROXIMATION. THIS TRANSLATES INTO USING A SCHEME OF THE FORM

$$Y_{n2} = e^{c_2\tau A} y_n + c_2 \tau \varphi_1(c_2\tau A)\, g(y_n)$$

$$y_{n+1} = e^{\tau A} y_n + \tau\big(b_1(\cdot)\, g(y_n) + b_2(\cdot)\, g(Y_{n2})\big)$$

$$\begin{array}{c|cc} 0 & & \\ c_2 & c_2\varphi_{1,2}(\cdot) & \\ \hline & b_1(\cdot) & b_2(\cdot) \end{array}$$

SUITABLE COEFFICIENTS (MATRIX FUNCTIONS) TO BE FOUND

AS FOR EXPLICIT RUNGE KUTTA METHODS THE b_1, b_2 COEFFICIENTS CAN BE FOUND BY LOOKING INTO THE ORDER CONDITIONS ($\Rightarrow$ WE COMPARE THE EXACT SOLUTION WITH THE NUMERICAL SOLUTION ASSUMING TO START FROM THE EXACT SOLUTION).

FOR SIMPLICITY OF NOTATION WE SET $h(t) \doteq g(y(t))$. THEN FOR THE EXACT SOLUTION (V.O.C. FORMULA) WE HAVE

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^\tau e^{(\tau-s)} h(t_n + s)\, ds \qquad (*)$$

BY TAYLOR EXPANSION WE HAVE

$$h(t_n + s) = h(t_n) + h'(t_n)\, s + \int_0^s h''(t_n+\sigma)(s-\sigma)\, d\sigma$$

THEREFORE, INSERTING THIS IN $(*)$ WE HAVE

$$y(t_{n+1}) = e^{\tau A} y(t_n) + \underbrace{\boxed{\int_0^\tau e^{(\tau-s)A} ds}}_{}\ h(t_n) + \boxed{\int_0^\tau e^{(\tau-s)A} s\, ds}\ h'(t_n)$$

$\tau\varphi_1(\tau A)$ $\quad$ $\tau^2\varphi_2(\tau A)$

$$+ \int_0^\tau e^{(\tau-s)A} \int_0^s h''(t_n+\sigma)(s-\sigma)\,d\sigma\,ds$$

$$= e^{\tau A} y(t_n) + \tau\varphi_1(\tau A)\,h(t_n) + \tau^2\varphi_2(\tau A) h'(t_n)$$

$$+ \int_0^\tau e^{(\tau-s)A} \int_0^s h''(t_n+\sigma)\underline{\underline{(s-\sigma)}}\,d\sigma\,ds \quad \rightsquigarrow \quad \leq C\tau^3$$

THIS ACCOUNTS FOR THE EXACT SOLUTION. FOR THE NUMERICAL SOLUTION STARTING FROM THE EXACT WE HAVE

$$y^*_{n+1} = e^{\tau A} y(t_n) + \tau\left(b_1\, g(y(t_n)) + b_2\, g(y(t_n+c_2\tau))\right)$$

$$= e^{\tau A} y(t_n) + \tau\left(b_1\, h(t_n) + b_2\, \underline{\underline{h(t_n+c_2\tau)}}\right)$$

BY TAYLOR EXPANSION WE GET

$$h(t_n+c_2\tau) = h(t_n) + c_2\tau h'(t_n) + \int_0^{c_2\tau} h''(t_n+\sigma)(c_2\tau-\sigma)\,d\sigma$$

AND THEREFORE

$$y^*_{n+1} = e^{\tau A} y(t_n) + \tau(b_1+b_2)\,h(t_n) + \tau^2 c_2 b_2\, h'(t_n)$$

$$+ \tau b_2 \int_0^{c_2\tau} h''(t_n+\sigma)(c_2\tau-\sigma)\,d\sigma$$

FINALLY WE COMPARE $y(t_{n+1})$ AND y^*_{n+1}

$$y^*_{n+1} - y(t_{n+1}) = \tau\,\boxed{\left(b_1+b_2-\varphi_1(\tau A)\right)}\,h(t_n) + \tau^2\,\boxed{\left(c_2 b_2 - \varphi_2(\tau A)\right)}\,h'(t_n)$$

$$+ \tau b_2 \int_0^{c_2\tau} h''(t_n+\sigma)(c_2\tau-\sigma)\,d\sigma - \int_0^\tau e^{(\tau-s)A}\int_0^s h''(t_n+\sigma)(s-\sigma)\,d\sigma\,ds$$

$\leq C\tau^3$ $\qquad\qquad$ $\leq C\tau^3$

SINCE WE AIM FOR A SECOND-ORDER METHOD OUR LOCAL ERROR SHOULD BE $\leq C\tau^3$. THIS CAN BE ACHIEVED IF $\blacksquare = 0$, THAT IS

$$\boxed{\begin{aligned} b_1 + b_2 &= \varphi_1(\tau A) \\ c_2 b_2 &= \varphi_2(\tau A) \end{aligned}}$$

THEY CAN BE EASILY SOLVED AS

$$\boxed{\begin{aligned} b_2 &= \frac{1}{c_2}\varphi_2(\tau A) \\ b_1 &= \varphi_1(\tau A) - \frac{1}{c_2}\varphi_2(\tau A) \end{aligned}} \qquad c_2 \neq 0$$

THEREFORE THE FINAL FORM OF OUR SCHEME IS

$$(**) \quad \boxed{\begin{aligned} Y_{n2} &= e^{c_2 \tau A} y_n + c_2 \tau \varphi_1(c_2 \tau A)\, g(y_n) \\ y_{n+1} &= e^{\tau A} y_n + \tau\left(\left(\varphi_1(\tau A) - \frac{1}{c_2}\varphi_2(\tau A)\right) g(y_n) + \frac{1}{c_2}\varphi_2(\tau A)\, g(Y_{n2})\right) \end{aligned}}$$

OR IN BUTCHER TABLEAU FORM

$$\begin{array}{c|cc} 0 & & \\ c_2 & c_2\varphi_{1,2} & \\ \hline & \varphi_1(\tau A) - \frac{1}{c_2}\varphi_2(\tau A) & \frac{1}{c_2}\varphi_2(\tau A) \end{array} \qquad \left(\text{COMPARE WITH} \rightarrow \begin{array}{c|cc} 0 & & \\ c_2 & c_2\varphi_{1,2} & \\ \hline & 0 & \varphi_1 \end{array}\right)$$

WE CAN PROVE THAT INTEGRATOR (**) IS SECOND-ORDER CONVERGENT (UNDER SUITABLE SMOOTHNESS/BOUNDENNESS ASSUMPIONS) FOR ANY CHOICE OF c_2 $(\neq 0)$. → THIS INTEGRATOR SATISFIES ALL THE STIFF ORDER CONDITIONS IN STRONG SENSE

ALL THESE KINDS OF INTEGRATORS BELONG TO THE CLASS OF THE SO-CALLED EXPONENTIAL RUNGE-KUTTA METHODS (OF COLLOCATION TYPE).

LABORATORY

$$\begin{cases} \partial_t y(t,x) = \delta\, \partial_{xx} y(t,x) + \dfrac{1}{1+y(t,x)^2} & t \in (0,t^*] \quad x \in (0,1) \\ y(t,0) = y(t,1) = 0 \\ y(0,x) = y_0(x) = 4x(1-x) \end{cases}$$

SEMIDISCRETIZATION IN SPACE $\Rightarrow$ SECOND-ORDER CENTERED FINITE DIFF.

WITH INTERIOR / INNER NODES