LECTURE 6 (10/02/25)

UP TO NOW WE CONSIDERED EXPONENTIAL INTEGRATORS FOR

- LINEAR PROBLEMS $\Rightarrow$ EXPONENTIAL QUADRATURE RULES
- SEMILINEAR PROBLEMS $\Rightarrow$ EXPONENTIAL RUNGE-KUTTA METHODS

$\Rightarrow$ EXPONENTIAL MULTISTEP METHODS (WE DON'T CONSIDER THEM HERE)

AS A NATURAL QUESTION WE CAN ASK

CAN WE DO SOMETHING FOR FULLY NONLINEAR PROBLEMS?

$\Rightarrow$ YES, DEVELOPING THE SO-CALLED EXPONENTIAL ROSENBROCK METHODS

EXPONENTIAL ROSENBROCK METHODS

UP TO NOW WE ASSUMED TO HAVE AT HAND A "NATURAL" SEPARATION AMONG A LINEAR PART (STIFF) AND A NONLINEAR PART (NON STIFF)

$$\begin{cases} y'(t) = Ay(t) + g(y(t)), & t \in (0, t^*] \\ y(0) = y_0 \end{cases}$$

V.O.C. $y(t_{n+1}) = e^{\tau A} y(t_n) + \int_0^{\tau} e^{(\tau - s)A} g(y(t_n + s))\,ds$

BUT MORE IN GENERAL WE COULD HAVE

$$\begin{cases} y'(t) = f(y(t)), & t \in (0, t^*] \\ y(0) = y_0 \end{cases} \qquad (\Delta)$$

$y(t) \in \mathbb{R}^N$

TO OBTAIN A LINEAR PART A NONLINEAR PART IN (Δ) WE PERFOR A LINEARIZATION ALONG THE NUMERICAL SOLUTION y_n (WHICH IS STILL TO BE DETERMINED). THIS MEANS THAT WE CONSIDER

$$\begin{cases} y'(t) = \underbrace{\underset{A}{\underline{\underline{J_n}}}\, y(t)}_{\text{LINEAR PART}} + \underbrace{g_n(y(t))}_{\text{NONLINEAR PART}} \\ y(0) = y_0 \end{cases} \qquad (\square)$$

WHERE $J_n \doteq \left.\dfrac{\partial f}{\partial y}\right|_{y_n}$ $\qquad g_n \doteq f(y(t)) - \overset{A}{\underline{\underline{J_n}}}\, y(t)$.

THE JACOBIAN OF f (EVALUATED AT y_n)

NON LINEAR REMAINDER

- NOTICE THAT (Δ) AND $(\square)$ ARE EQUIVALENT $\Rightarrow$ NO APPROXIMATION IS PERFORMED AT THIS LEVEL (EXACT LINEARIZATION)
- THE JACOBIAN J_n MAKES SENSE SINCE THIS IS WHAT YOU WOULD OBTAIN BY TAYLOR EXPANDING THE RIGHT HAND SIDE OF THE SYSTEM

$$f(y(t)) = f(y_n) + \underline{\underline{J_n}}\,(y(t) - y_n) + \dots$$

SINCE $(\square)$ IS A SEMILINEAR SYSTEM WE CAN APPLY THE EXPONENTIAL EULER METHOD, OBTAINING

$$\boxed{\begin{aligned} y_{n+1} &= e^{\tau J_n} y_n + \tau \varphi_1(\tau J_n)\, g_n(y_n) \\ &= y_n + \tau \varphi_1(\tau J_n)\, f(y_n) \end{aligned}} \qquad (\circ)$$

THIS IS THE SO-CALLED EXPONENTIAL ROSENBROCK EULER METHOD.

$$f(y(t)) = A y(t) + g(y(t))$$

$$J_n = A + \left.\frac{\partial g}{\partial y}\right|_{y_n}$$

$$y_{n+1} = y_n + \tau \varphi_1(\tau \underline{A})\, f(y_n) \qquad\qquad y_{n+1} = y_n + \tau \varphi_1(\tau \underline{\underline{J_n}})\, f(y_n)$$

MATRIX FUNCTION CONSTANT AT EACH TIME STEP

MATRIX FUNCTION THAT CHANGES AT EACH TIME STEP

FIRST-ORDER ACCURATE

SECOND-ORDER ACCURATE

FOR THE CONVERGENCE PROOF OF EXPONENTIAL ROSENBROCK EULER WE ASSUME THAT $f(y(t)) = Ay(t) + g(y(t))$. AS USUAL WE ASSUME THAT ALL OCCURING DERIVATIVES ARE BOUNDED.

THEOREM 3 (CONVERGENCE OF EXPONENTIAL ROSENBROCK EULER)

CONSIDER PROBLEM (Δ) AND INTEGRATOR (o). ASSUME g IS SUFFICIENTLY OFTEN DIFFERENTIABLE. THEN

$$\| y(t_n) - y_n \| \le C \tau^2$$

HERE $0 \le t_n \le t^*$ AND THE CONSTANT C MAY DEPEND ON THE FINAL TIME t^* BUT ON n.

PROOF

FROM THE V.O.C. FORMULA WE GET (SEE $(\square)$)

$$y(t_{n+1}) = e^{\tau J_n} y(t_n) + \int_0^\tau e^{(\tau-s)J_n} g_n(y(t_n+s))ds$$

FOR SHORTHAND NOTATION WE SET $h_n(t) \doteq g_n(y(t))$ SO THAT

$$y(t_{n+1}) = e^{\tau J_n} y(t_n) + \int_0^\tau e^{(\tau-s)J_n} h_n(t_n+s)ds.$$

BY TAYLOR EXPANDING h_n WE GET

$$h_n(t_n+s) = h_n(t_n) + h_n'(t_n)s + \int_0^s h''(t_n+\sigma)(s-\sigma)d\sigma.$$

HENCE BY PLUGGING IT IN THE V.O.C. FORMULA WE GET

$$y(t_{n+1}) = e^{\tau J_n} y(t_n) + \tau\varphi_1(\tau J_n) h_n(t_n) + \tau^2 \varphi_2(\tau J_n) h_n'(t_n)$$
$$+ \int_0^\tau e^{(\tau-s)J_n} \int_0^s h''(t_n+\sigma)(s-\sigma)d\sigma ds$$

WE COMPARE THIS WITH THE NUMERICAL SOLUTION

$$y_{n+1} = e^{\tau J_n} y_n + \tau\varphi_1(\tau J_n) g_n(y_n)$$

$$y(t_{n+1}) - y_{n+1} = e^{\tau J_n}\left(y(t_n) - y_n\right) + \tau\varphi_1(\tau J_n)\left(h_n(t_n) - g_n(y_n)\right)$$
$$+\tau^2 \varphi_2(\tau J_n)\, h_n'(t_n) + \int_0^\tau e^{(\tau-s)J_n}\int_0^s h''(t_n+\sigma)(s-\sigma)\,d\sigma\, ds$$

FOR THE PART $\bullet$

$$g_n(y(t)) = f(y(t)) - J_n\, y(t)$$

$$h_n(t_n) - g_n(y_n) = g_n(y(t_n)) - g_n(y_n)$$
$$\overset{!}{=} \left(f(y(t_n)) - J_n\, y(t_n)\right) - \left(f(y_n) - J_n y_n\right)$$
$$\overset{!}{=} \left(A y(t_n) + g(y(t_n)) - J_n\, y(t_n)\right) -$$
$$\left(A y_n + g(y_n) - J_n y_n\right)$$

SINCE $f(y(t)) = Ay(t) + g(y(t)) \Rightarrow J_n = \left.\dfrac{\partial f}{\partial y}\right|_{y_n} = A + \dfrac{\partial g}{\partial y}(y_n)$

AND SO

$$= \left(g(y(t_n)) - \frac{\partial g}{\partial y}(y_n)\, y(t_n)\right) -$$
$$\left(g(y_n) - \frac{\partial g}{\partial y}(y_n)\, y_n\right)$$
$$= \left(g(y(t_n)) - g(y_n)\right) - \frac{\partial g}{\partial y}(y_n)\left(y(t_n) - y_n\right)$$

SINCE g IS LIPSCHITZ (AS BDD DERIVATIVES) WE HAVE

$$\|h_n(t_n) - g_n(y_n)\| \le C\,\|y(t_n) - y_n\|$$

$$g_n(y) = f(y) - J_n y$$
$$\overset{!}{=} Ay + g(y) - J_n y$$

FOR THE PART $\bullet$

$$h_n'(t_n) = \left.\frac{\partial g_n}{\partial y}\right|_{y(t_n)} y'(t_n) = \left(\frac{\partial g}{\partial y}(y(t_n)) - \frac{\partial g}{\partial y}(y_n)\right) y'(t_n)$$

$$\Rightarrow \|h_n'(t_n)\| \le C\,\|y(t_n) - y_n\|$$

FOR THE TERM $\bullet$

$$\left\| \int_0^{\tau} e^{(\tau-s)J_n} \int_0^{s} \ell''(t_n+\sigma)(s-\sigma)\,d\sigma\,ds \right\| \le c\,\tau^3$$

SO FOR THE RECURSION WE HAVE $\varepsilon_n \doteq y(t_n) - y_n$ AND

$\delta_{n+1} = \bullet + \bullet + \bullet$

$$\varepsilon_{n+1} = e^{\tau J_n}\varepsilon_n + \delta_{n+1} \iff \varepsilon_n = \sum_{j=1}^{n}\left(\prod_{k=1}^{n-j} e^{\tau J_{n-k}}\right)\delta_j$$

FINALLY USING THE PREVIOUS BOUNDS (AND $\|e^{\tau J_{n-k}}\| \le c$)

WE GET

$$\|\varepsilon_n\| \le \sum_{j=1}^{n}\|\delta_j\| \le c\tau\sum_{j=0}^{n-1}\|\varepsilon_j\| + c\tau^2\sum_{j=0}^{n-1}\|\varepsilon_j\| + \overbrace{c\tau^3\sum_{j=0}^{n-1}1}^{\le c\tau^2}$$

AND WE CONCLUDE WITH THE DISCRETE GRONWALL LEMMA (SEE LECTURE 2)

$$\|\varepsilon_n\| \le c\,\tau^2 \iff \|y(t_n) - y_n\| \le c\tau^2$$

$\square$

AS FOR STANDARD EXPONENTIAL RUNGE KUTTA METHODS YOU CAN GO HIGHER ORDER BY SUITABLY DESIGNING INTERMEDIATE STAGES (WE WILL NOT DISCUSS FURTHER)

NON-AUTONOMOUS CASE

FOR EXPONENTIAL EULER WE HAVE

$$y'(t) = Ay(t) + g(y(t)) \Rightarrow y_{n+1} = y_n + \tau\varphi_1(\tau A)f(y_n)$$

$$y'(t) = Ay(t) + g(t, y(t)) \Rightarrow y_{n+1} = y_n + \tau\varphi_1(\tau A)f(t_n, y_n)$$

FOR EXPONENTIAL ROSENBROCK EULER

$$y'(t) = \underline{f(y(t))} \quad \Rightarrow \quad y_{n+1} = y_n + \tau \varphi_1(\tau \underline{J_n}) f(y_n) \quad \cancel{}$$

$$y'(t) = \underline{f(t, y(t))} \quad \overset{?}{\Rightarrow} \quad y_{n+1} = y_n + \tau \varphi_1(\tau J_n) f(t_n, y_n)$$

NOT CORRECT

BECAUSE WE SHOULD LINEARIZE ALSO WRT t

THE PROCEDURE TO OBTAIN THE "CORRECT" INTEGRATOR IS :

- FIRST REWRITE THE SYSTEM IN AUTONOMOUS WAY
- APPLY THE EXPONENTIAL ROSENBROCK EULER
- RETRIEVE THE INTEGRATOR IN TERMS OF THE ORIGINAL VARIABLES

$$\mathbb{R}^{N+1} \ni Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} t \\ y(t) \end{bmatrix} \Rightarrow Y'(t) = \begin{bmatrix} 1 \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 \\ f(Y_1(t), Y_2(t)) \end{bmatrix} = F(Y(t))$$

$$\underbrace{\frac{dF}{dY}}_{\in \mathbb{R}^{(N+1)\times(N+1)}} = \begin{bmatrix} \underbrace{0}_{\in \mathbb{R}} & \underbrace{0}_{\in \mathbb{R}^{1\times N}} \\ \underbrace{\frac{\partial f}{\partial t}}_{\in \mathbb{R}^N} & \underbrace{\frac{\partial f}{\partial y}}_{\in \mathbb{R}^{N\times N}} \end{bmatrix}$$

$$\left.\frac{dF}{dY}\right|_{Y_n} = \begin{bmatrix} 0 & 0 \\ \underbrace{\left.\frac{\partial f}{\partial t}\right|_{(t_n, y_n)}}_{=: V_n} & \underbrace{\left.\frac{\partial f}{\partial y}\right|_{(t_n, y_n)}}_{=: J_n} \end{bmatrix}$$

$$\varphi_1\left(\tau \left.\frac{dF}{dY}\right|_{Y_n}\right) = \begin{bmatrix} 1 & 0 \\ \tau \varphi_2(\tau J_n) V_n & \varphi_1(\tau J_n) \end{bmatrix}$$

NEXT LECTURE (AUGMENTED MATRIX)

$$\Rightarrow Y_{n+1} = Y_n + \tau \varphi_1\left(\tau \left.\frac{dF}{dY}\right|_{Y_n}\right) F(Y_n)$$

LOOK AT THE SECOND ROW YOU GET

$$\boxed{y_{n+1} = y_n + \tau \varphi_1(\tau J_n) f(t_n, y_n) + \underline{\tau^2 \varphi_2(\tau J_n) v_n}}$$

THIS IS THE EXPONENTIAL ROSENBROCK EULER METHOD FOR NON AUTONOMOUS SYSTEMS.

STANDARD ROSENBROCK METHODS

THEY ARE ESSENTIALLY IMEX SCHEMES IN WHICH YOU EXPLOIT THE JACOBIAN.

FOR SEMILINEAR SYSTEMS $y'(t) = Ay(t) + g(y(t))$

WE STUDIED BF EULER

$$(I - \tau A) y^{n+1} = y^n + g(y^n)$$

A ROSENBROCK METHOD FOR $y'(t) = f(y(t))$ IS

$$\boxed{\begin{aligned} \left(I - \frac{\tau}{2} J_n\right) k_1 &= \tau f(y_n) \\ y_{n+1} &= y_n + k_1 \end{aligned}}$$

$$J_n = \left.\frac{\partial f}{\partial y}\right|_{y_n}$$

$\Rightarrow$ SECOND-ORDER, A-STABLE

LABORATORY

$$\begin{cases} \partial_t y(t,x) = \delta \partial_{xx} y(t,x) + \dfrac{1}{1 + (y(t,x))^2} \\ y_0 = 4x(1-x) \\ \text{HOM. DIR. B.C.} \end{cases}$$

$\Rightarrow$ IN SPACE SECOND ORDER CENTERED F.D.

TO CHECK THE JACOBIAN $\Rightarrow$

$$\underbrace{\left.\frac{dF}{dy}\right|_{\bar{y}}}_{F(y)} V \approx \frac{F(\bar{y} + \varepsilon V) - F(\bar{y})}{\varepsilon} \qquad O(\varepsilon)$$

COMPLEX STEP $\longleftarrow$

$$\left.\frac{dF}{dy}\right|_{\bar{y}} V \approx \frac{\mathrm{Im}\left(F(\bar{y} + i\varepsilon V)\right)}{\varepsilon} \qquad O(\varepsilon^2)$$