A-stabity linear stability

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

$y : \mathbb{R} \to \mathbb{C}^m$

$y_{m+1}(t) := t$

$$\begin{cases} y'(t) = \lambda y(t) \\ y(t_0) = y_0 \end{cases}$$

$\lambda \in \mathbb{C}$

$y : \mathbb{R} \to \mathbb{R}$

$$y(t) = e^{(t-t_0)\lambda} y_0$$

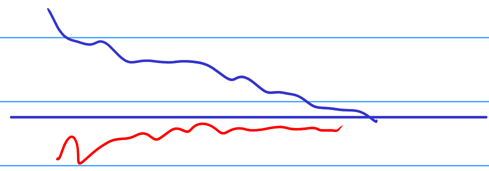
if $Re(\lambda) < 0$, $\boxed{\lim_{t \to \infty} y(t) = 0}$

time step k FIXED

FORWARD EULER

$$y_{n+1} = y_n + k \lambda y_n \qquad y_n \approx y(t_n)$$

$$= (1 + k\lambda) y_n$$



$$y_n = (1 + k\lambda)^n y_0$$

I would like $\lim_{n \to \infty} y_n = 0$

$$|1 + k\lambda|^2 < 1^2$$

$$(1 + k Re(\lambda))^2 + (k Im(\lambda))^2 < 1$$

$$\cancel{1} + k^{\cancel{2}} Re^2(\lambda) + 2\cancel{k} Re(\lambda) + k^{\cancel{2}} Im^2(\lambda) < \cancel{1}$$

$$k < \frac{-2 Re(\lambda)}{|\lambda|^2} \iff \lim_{n \to \infty} y_n = 0$$

BACKWARD EULER

$$y_{n+1} = y_n + k\lambda y_{n+1}$$

$$y_{n+1} = \left(\frac{1}{1-k\lambda}\right) y_n$$

$$y_n = \left(\frac{1}{1-k\lambda}\right)^n y_0$$

$$\left|\frac{1}{1-k\lambda}\right| < 1 \qquad |1-k\lambda| > 1$$

$$|\underbrace{1-k\,\mathrm{Re}(\lambda)}_{>1} - ik\,\mathrm{Im}(\lambda)| > 1$$

TRAPEZOIDAL RULE

$$y_{n+1} = y_n + \frac{k}{2}\lambda y_n + \frac{k}{2}\lambda y_{n+1}$$

$$y_{n+1} = \left(\frac{1+\frac{k}{2}\lambda}{1-\frac{k}{2}\lambda}\right) y_n$$

$$y_n = \left(\frac{1+\frac{k}{2}\lambda}{1-\frac{k}{2}\lambda}\right)^n y_0$$

$$\left|1-\frac{k}{2}\lambda\right| > \left|1+\frac{k}{2}\lambda\right|$$ ALWAYS TRUE

$$y'(t) = \lambda y(t)$$

DEF REGION OF ABSOLUTE STABILITY

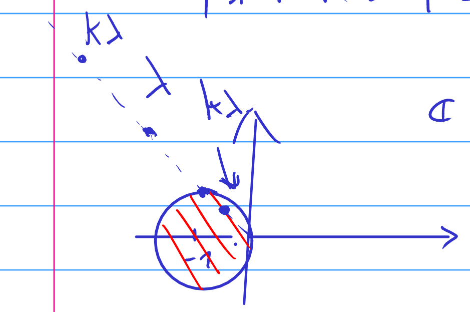
set of numbers $\underset{\in}{\overset{\mathbb{C}}{}} z = k\lambda$ such that

$$\lim_{n\to\infty} y_n = 0$$

FORWARD EULER

$|1 + k\lambda| < 1$ $\qquad$ $|1+z| < 1$

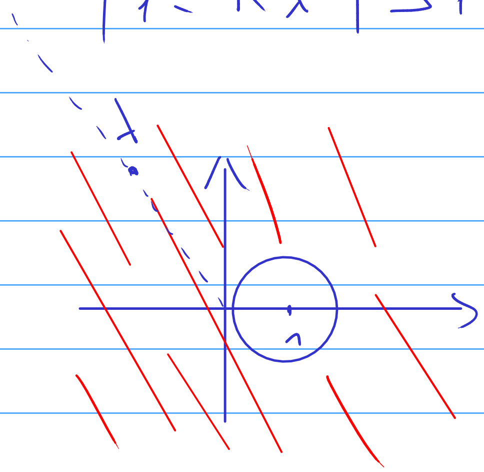
$|z-(-1)| < 1$



BACKWARD EULER

$|1 - k\lambda| > 1$ $\qquad$ $|1-z| > 1$

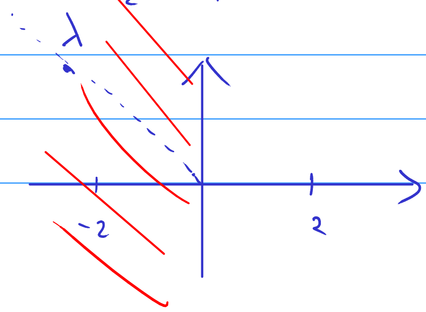
$|z-1| > 1$



TRAPEZOIDAL RULE

$\left|1 - \frac{k}{2}\lambda\right| > \left|1 + \frac{k}{2}\lambda\right|$ $\qquad$ $\left|1 - \frac{z}{2}\right| > \left|1 + \frac{z}{2}\right|$

$|z-2| > |z-(-2)|$



DEF A method is A-stable if its absolute stability region contains $\mathbb{C}^- = \{z : Re(z) < 0\}$

$$\begin{cases} y'(t) = \lambda y(t) + b \\ y(t_0) = y_0 \end{cases}$$

$$y(t) = e^{(t-t_0)\lambda} y_0 + e^{(t-t_0)\lambda} \frac{b}{\lambda} - \frac{b}{\lambda}$$

$$y(t_0) = y_0 + \frac{b}{\lambda} - \frac{b}{\lambda} = y_0$$

$$y'(t) = \lambda e^{(t-t_0)\lambda} y_0 + \cancel{\lambda} e^{(t-t_0)\lambda} \frac{b}{\cancel{\lambda}} =$$

$$\lambda y(t) + b$$

$Re(\lambda) < 0$ $$\lim_{t \to \infty} y(t) = -\frac{b}{\lambda}$$

$$y(t) = e^{(t-t_0)\lambda} y_0 + (t-t_0) \boxed{\left(\frac{e^{(t-t_0)\lambda} - 1}{(t-t_0)\lambda}\right)} b$$

$$= e^{(t-t_0)\lambda} y_0 + (t-t_0)\, \varphi_1((t-t_0)\lambda)\, b$$

$$\varphi_1(z) = \begin{cases} \frac{e^z - 1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

$y(t)$ for $\lambda = 0$ is $y_0 + (t-t_0) b$

FORWARD EULER

$$y_{n+1} = y_n + k(\lambda y_n + b)$$

$$y_n = y_{n-1} + k(\lambda y_{n-1} + b)$$

$$y_m = (1+k\lambda)^m y_0 + \frac{(1+k\lambda)^m - 1}{\lambda} b$$

$$\lim_{m \to \infty} y_m = -\frac{b}{\lambda} \iff |1+k\lambda| < 1$$

NO EXPLICIT RUNGE-KUTTA METHOD IS A-STAB

NO EXPLICIT MULTISTEP METHOD IS A-STABLE

SOME IMPLICIT METHODS ARE A-STABLE

$$y'(t) = f(y(t))$$

$$y_{m+1} = y_m + f(y_{m+1})$$

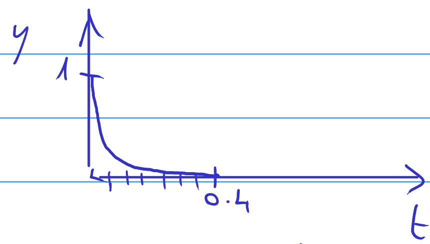
WE WILL PRESENT EXPLICIT METHODS A-STABLE

EXAMPLE
$$\begin{cases} y'(t) = -100\, y(t) \\ y(0) = 1 \end{cases} \qquad \lambda = -100$$

FORWARD EULER $\qquad k < \frac{1}{50} = 0.02$

$$y(t) = e^{-100t}$$

$$y(0.4) < 10^{-17} \qquad \varepsilon \approx 10^{-16}$$



21 time steps

$$1 - 10^{-17} = 1$$

$$\begin{cases} y'(t) = \begin{bmatrix} -100 & 0 \\ 0 & -1 \end{bmatrix} y(t) \\ y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{cases}$$

$y_1(t) = e^{-100t}$ ←
$y_2(t) = e^{-t}$ ←

$\| y(40) \|_\infty < 10^{-17}$

FORWARD EULER $\quad K < \frac{1}{50} = 0.02$

2001 time steps

$y'(t) = A y(t)$

if $AV = V\Lambda \quad \Lambda$ diagonal

$\underbrace{V^{-1} y'(t)} = \Lambda \underbrace{V^{-1} y(t)}$

$z'(t) = \Lambda z(t) \qquad K < -2 \frac{Re(\lambda)}{|\lambda|^2}$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_d \end{bmatrix}$$

→ $y'(t) = A y(t) + b$ $\qquad \boxed{y(t) = \exp(tA) y_0 + t \varphi_1(tA) b}$

$z'(t) = \Lambda z(t) + V^{-1} b$

$y'(t) = f(y(t)) \approx f(y_m) + J_{f_m}(y(t) - y_m)$

$= \underbrace{J_{f_m}} y(t) + \underbrace{(f(y_m) - J_{f_m} y_m)}$!

DEF

$y'(t) = f(y(t))$ is $\boxed{\text{STIFF}}$ around t_n when

1) J_{f_n} has at least two eigenvalues

λ_1, λ_2 $\quad Re(\lambda_1) < 0 \quad Re(\lambda_2) < 0$

2) $Re(\lambda_1) << Re(\lambda_2)$

Example POCA ACCURATEZZA RICHIESTA

EULERO ESPLICITO

$k = 0.1$ $\quad \rightarrow \quad \mathcal{O}(k)$

$k = 0.0001$

EULERO IMPLICITO

$k = 0.1$

$$\exp(A) = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

$AV = V\Lambda \qquad A = V\Lambda V^{-1} \qquad A^2 = V\Lambda \cancel{V^{-1}} \cancel{V} \Lambda V^{-1} = V\Lambda^2 V^{-1}$

$$\exp(A) = V \exp(\Lambda) V^{-1}$$

$$\exp(\Lambda) = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_d} \end{bmatrix}$$

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} \qquad x \ldots$$

$\cancel{1} + \cancel{z} + \frac{z^{\cancel{2}\,1}}{2} + \frac{z^{\cancel{3}\,2}}{3!} + \ldots$

$$\varphi_1(z) = \begin{cases} \frac{e^z - 1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

$1 + \frac{z}{2} + \frac{z^2}{3!} + \frac{z^3}{4!} + \ldots$

$$\varphi_1(z) = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}$$

ENTIRE FUNCTION

$$\varphi_1(A) = \sum_{i=0}^{\infty} \frac{A^i}{(i+1)!}$$

$$\varphi_1(z) = \frac{e^z - 1}{z}$$

$$z\varphi_1(z) = e^z - 1$$