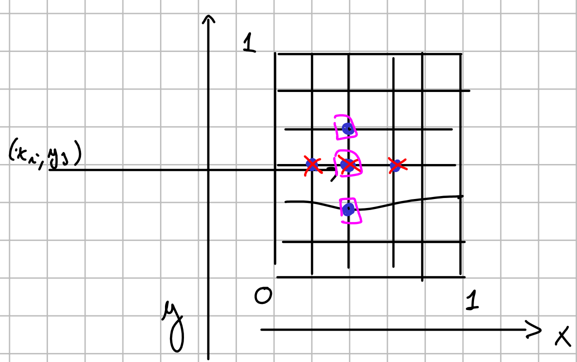
LECTURE 9 B (20/02/2025)

NOW WE FOCUS OUR ATTENTION TO EXPLOIT SOME FEATURES THAT THE SYSTEM OF ODEs MAY HAVE TO EFFICIENTLY COMPUTE THE NEEDED "ACTIONS OF" MATRIX FUNCTIONS. IN PARTICULAR WE WILL FOCUS ON THE KRONECKER-STRUCTURED CASE.

MOTIVATING EXAMPLE

WE WANT TO APPROXIMATE A ∇^{2D} LAPLACIAN WITH FINITE DIFFERENCES



$$(\partial_{xx} + \partial_{yy})\, u \Big|_{(x_i, y_j)}$$

$$\approx \frac{u(x_{i-1}, y_j) - 4u(x_i, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j-1}) + u(x_i, y_{j+1})}{h^2}$$

$$= \boxed{\frac{u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j)}{h^2}} + \boxed{\frac{u(x_i, y_{j-1}) - 2u(x_i, y_j) + u(x_i, y_{j+1})}{h^2}}$$

$\approx \partial_{xx}$ IN 1D

$\approx \partial_{yy}$ IN 1D

SO THE IDEA IS TO "PROPAGATE" THE 1D STENCILS FOR ALL THE 2D GRID TO OBTAIN THE DESIRED APPROXIMATION. WE CAN DO THIS BY EXPLOITING THE SO-CALLED KRONECKER PRODUCT

$$(\partial_{xx} + \partial_{yy}) \approx \underbrace{\underbrace{I^y}_{\in \mathbb{R}^{m_y \times m_y}} \otimes \underbrace{D2^x}_{\in \mathbb{R}^{m_x \times m_x}} + \underbrace{D2^y}_{\in \mathbb{R}^{m_y \times m_y}} \otimes \underbrace{I^x}_{\in \mathbb{R}^{m_x \times m_x}}}_{K} \in \mathbb{R}^{m_x m_y \times m_x m_y}$$

IDENTITY MATRICES

$\approx \partial_{xx}$ IN 1D

$\approx \partial_{yy}$ IN 1D

STRUCTURED GRIDS

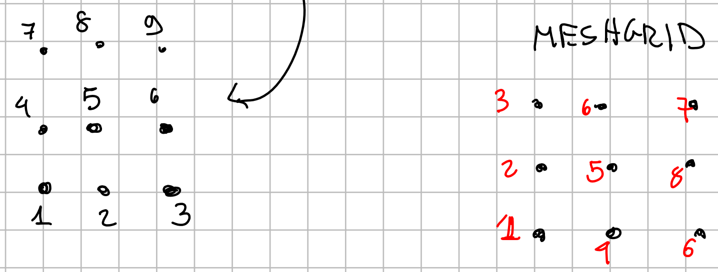
THE KRONECKER PRODUCT IS DEFINED AS

$$A \otimes B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix}$$

$$\underset{\mathbb{R}^{n_x n_y \times n_x n_y}}{K}\, (w) \to \in \mathbb{R}^{n_x n_y}$$

w CONTAINS $u(x_i, y_j)\ \forall i, j$

$\Rightarrow$ IF YOU WANT TO USE K, A CONSISTENT ORDERING OF THE GRID NODES IS GIVEN BY NDGRID OF MATLAB



NOT CONSISTENT WITH K

THE APPROACH WORKS IN THE SAME WAY FOR

$$\partial_x + \partial_y \approx I^y \otimes D_1^x + D_1^y \otimes I^x$$

AND IT GENERALIZES TO nD. IN 3D WE WOULD HAVE

$$\partial_{xx} + \partial_{yy} + \partial_{zz} \approx \boxed{I} \otimes \boxed{I} \otimes \boxed{D_2} + \boxed{I} \otimes \boxed{D_2} \otimes \boxed{I} + \boxed{D_2} \otimes \boxed{I} \otimes \boxed{I}$$

• X DIRECTION • Y DIRECTION • Z DIRECTION

WE FOCUS ON THE 2D CASE. FOR OUR STRUCTURED GRID (TENSOR-PRODUCT GRID). WE CONSIDER THE PDE

$$\begin{cases} \partial_t u(t,x,y) = \delta\, \Delta u(t,x,y) + \alpha\left(\partial_x u(t,x,y) + \partial_y u(t,x,y)\right) + \rho(x,y,u(t,x,y)) \\ u(0,x,y) = u_0(x,y) \\ + \text{ BDY CONDITIONS} \end{cases}$$

$(x,y) \in [a,b] \times [c,d]$

MUST WORK PROPERLY WITH THE KRONECKER STRUCTURE

ADVECTION-DIFFUSION-REACTION EQUATION

THEN, WE CAN DISCRETIZE IT AS

$$\begin{cases} u'(t) = K u(t) + \underline{g(u(t))} \\ u(0) = u_0 \end{cases}$$

WHERE

$$K = 1^y \otimes (\delta D_2^x + \alpha D_1^x) + (\delta D_2^y + \alpha D_1^y) \otimes 1^x$$