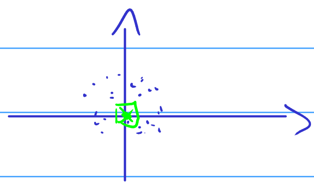
LECTURE 9A

A with minimal polynomial $\mu(x)$ of degree ν
($\mu(A) = 0$, it divides the characteristic polynomial)
Let $P_{\nu-1}(x)$ the interpolating polynomial of $f(x)$ in the Hermite sense at the roots of $\mu(x)$.
Then $f(A) = P_{\nu-1}(A)$.
It can be used as a definition.

It can be unfeasible to compute the eigenvalues of A.

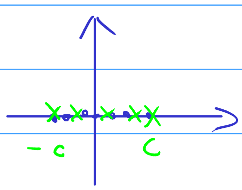
Truncated Taylor series is an interpolation of e^x at $m+1$ zeros.

$$A \to A - \mu I \quad , \quad \mu = \frac{\operatorname{trace}(A)}{N} \simeq \text{average eigenv.}$$



$L_{m,c}$ interpolation in $[-c, c]$ (Gershgorin, ...)

$A \simeq \partial_{xx}$



Krylov method $e^{A} v$ $\quad A \in \mathbb{R}^{N \times N}$

$K_m(A, v)$ Krylov space $= \langle \{ v, Av, A^2 v, \dots, A^{m-1} v \} \rangle$

$$\boxed{m \ll N}$$

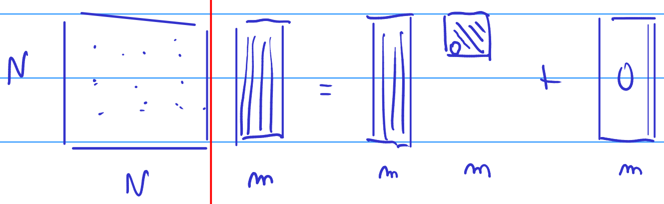
$$K_m(A, v) = \langle \{ v_1, v_2, \dots, v_m \} \rangle \qquad v_1 = \frac{v}{\|v\|_2}$$

$$V_m = [v_1, v_2, \dots, v_m] \in \mathbb{R}^{N \times m}$$

$$V_m^T V_m = I_m \qquad V_m V_m^T = \, ?$$

Arnoldi factorization

$$A V_m = V_m \underbrace{H_m}_{\text{Hessenberg}} + h_{m+1,m} \, v_{m+1} \, e_m^T \qquad H_m \in \mathbb{R}^{m \times m}$$



A is sparse $\left(nnz(A) = \mathcal{O}(N) \right)$

$$= V_{m+1} \bar{H}_m \qquad \bar{H}_m \in \mathbb{R}^{m+1 \times m}$$

$$V_m^T A V_m = H_m \qquad \text{“}A \text{ is similar to } H_m\text{”}$$

Cost: $\mathcal{O}(m^2) + \underbrace{\mathcal{O}(m N)}_{\text{matrix-vector products}}$ Av costs $\mathcal{O}(N)$ because A is sparse

(Arnoldi factorization is used in linear system solution with iterative methods GMRES, FOM, ...)

$$A V_m \approx V_m H_m$$

$$(\lambda I_N - A) V_m \approx V_m (\lambda I_m - H_m)$$

$$V_m (\lambda I_m - H_m)^{-1} e_1 \approx (\lambda I_N - A)^{-1} V_m e_1 = (\lambda I_N - A)^{-1} v_1$$

field of values

$$\lambda \notin \mathcal{F}(A) = \{ x^* A x \;:\; x \in \mathbb{C}^N,\ \|x\|_2 = 1 \} \supset \mathcal{F}(H_m)$$

$$e^{A} v = \frac{1}{2\pi i} \int_\Gamma e^{\lambda} (\lambda I_N - A)^{-1} v \, d\lambda \qquad \Gamma \text{ exterior to } \mathcal{F}(A)$$

Cauchy definition of $e^{A} v$ (contour integral formulas rational)

$$\approx \frac{1}{2\pi i} \int_\Gamma e^{\lambda} V_m (\lambda I_m - H_m)^{-1} e_1 \|v\| \, d\lambda$$

$$= \|v\| V_m e^{H_m} e_1$$

$$\boxed{e^{A} v \approx \|v\| V_m \boxed{e^{H_m}} e_1}$$

Krylov approx.

Taylor, Padé, diagonalization, ...

$h_{j+1,j} \neq 0 \Rightarrow$ the geometric multiplicity of each eigenvalue is 1.

minimal polynomial $\equiv$ characteristic polynomial

$$e^{H_m} = p_{m-1}(H_m)$$

$p_{m-1}(x)$ interpolates e^{x} at the eigenvalues of H_m

Lemma: $\quad p_j(A) v_1 = V_m p_j(H_m) e_1 \qquad j \leq m-1$

BY INDUCTION

PROOF: $j=0$ $\quad v_1 = V_m e_1$ ✓

Suppose true for $j \leq m-2$.

$V_m V_m^T w = w$ for $w \in K_m(A, v)$

$\Big(w = \alpha_1 v_1 + \dots + \alpha_m v_m$

$V_m^T w = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \qquad V_m \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = w \Big)$

$$\underbrace{A^{j+1} v_1}_{\in K_m} = V_m V_m^T A^{j+1} v_1 = V_m V_m^T A \underbrace{A^j v_1}_{\in K_m} =$$

$$= V_m \underbrace{V_m^T A V_m}_{H_m} V_m^T A^j v_1 =$$

$$= V_m H_m \underbrace{V_m^T A^j v_1}_{H_m^j e_1} = V_m H_m H_m^j e_1$$

$$= V_m H_m^{j+1} e_1 = A^{j+1} v_1 .$$

Theorem.

$$e^A v \approx \|v\| V_m e^{H_m} e_1 = p_{m-1}(A) v$$

PROOF:

$$\|v\| V_m e^{H_m} e_1 = \|v\| V_m p_{m-1}(H_m) e_1 =$$

$$\|v\| p_{m-1}(A) v_1 = p_{m-1}(A) v \quad \square$$

Krylov with incomplete orthogonalization: O(m)+O(mN)

rational Krylov $K_m(A, v) = \tau \{ v, (A-\lambda I)^{-1} v, (A-\lambda I)^{-2} v, \dots \}$

If I have to solve linear systems to apply exp. int. to solve $y'(t) = f(t, y(t))$
then I can use implicit methods to solve it.

$$(I - \tau A)^{-1} y_{m+1} = \dots$$