LECTURE 3 (27/01/25)

LAST TIME WE SAW

- EXPONENTIAL EULER METHOD (FIRST ORDER, EXPONENTIAL INTEGRATOR)
- BACKWARD FORWARD EULER METHOD (FIRST ORDER, IMEX METHOD)

$$\Rightarrow \begin{cases} y'(t) = Ay(t) + g(y(t)) = f(y(t)) & t \in (0, t^*] \\ y(0) = y_0 \end{cases}$$

(EXP EUL.)

$$y_{n+1} = y_n + \tau \varphi_1(\tau A) f(y_n) = e^{\tau A} y_n + \tau \varphi_1(\tau A) g(y_n) \quad ||$$

(IMEX)

$$(I - \tau A) y_{n+1} = y_n + \tau g(y_n) \iff y_{n+1} = (I - \tau A)^{-1} (y_n + \tau g(y_n))$$

OUR NUMERICAL EXAMPLE IS A DIFFUSION EQUATION (ONE DIMENSIONAL)

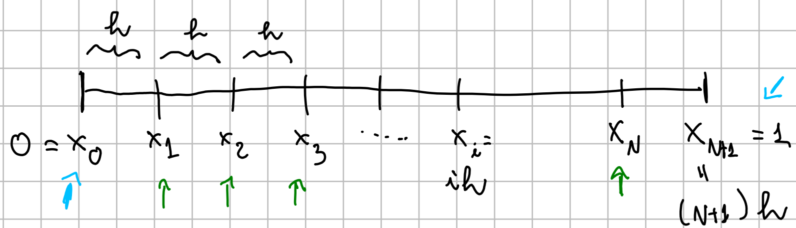
$$\begin{cases} \partial_t y(t,x) = \overset{>0}{\delta} \partial_{xx} y(t,x) + \dfrac{1}{1 + (y(t,x))^2}, & x \in (0,1), \ t \in (0, t^*] \\ y(0,x) = y_0(x) \\ y(t,0) = y(t,1) = 0 \end{cases}$$

$\Rightarrow$ HOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS

METHOD OF LINES $\Rightarrow$ WE SEMIDISCRETIZE IN SPACE

WE EMPLOY STANDARD SECOND-ORDER CENTERED FINITE DIFFERENCES

$x \in [0,1]$



$\Rightarrow$ $N+2$ EQUISPACED NODES OF SIZE $h = \dfrac{1}{N+1}$

FOR THE INNER NODES $x_i = ih$ FOR $i = 1, \dots, N$

$$\partial_{xx} y(t, x_i) \approx \frac{y_{i-1}(t) - 2y_i(t) + y_{i+1}(t)}{h^2}$$

$$= \quad " \quad\quad " \quad + \mathcal{O}(h^2)$$

WHERE $y(t, x_i) = y_i(t)$

$$\partial_{xx} y(t, x_2) = \frac{y_1(t) - 2y_2(t) + y_3(t)}{h^2} \qquad \dots \qquad \partial_{xx} y(t, x_i) \quad i = 2, \dots, N-1$$

FROM BDY COND. $0 =$

$$\partial_{xx} y(t, x_1) = \frac{y_0(t) - 2y_1(t) + y_2(t)}{h^2} = \frac{-2y_1(t) + y_2(t)}{h^2}$$

$= 0$ FROM BDY COND.

$$\partial_{xx} y(t, x_N) = \frac{y_{N-1}(t) - 2y_N(t) + y_{N+1}(t)}{h^2} = \frac{-2y_N(t) + y_{N-1}(t)}{h^2}$$

$$\mathbb{R}^N \ni y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_N(t) \end{bmatrix} \rightarrow \text{INNER NODES}$$

$$D_2 = \frac{1}{h^2} \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & \cdots & & 0 & 0 & 1 & -2 \end{bmatrix} \approx \partial_{xx} \oplus \text{HOM. DIRICHLET BDY CONDITIONS}$$

$$A = \delta D_2 \approx \delta \partial_{xx} \oplus \text{BDY COND.}$$

WE GET THE SYSTEM OF ODES (N EQUATIONS)

$$\begin{cases} y'(t) = A y(t) + \dfrac{1}{1 + y^2(t)} & t \in (0, t^*] \\ y(0) = y_0 \end{cases}$$

WHICH ACCOUNTS FOR THE INNER NODES AND $y_0 = 0$, $y_{N+1} = 0$.

THIS IS A STIFF SYSTEM BECAUSE OF $\mathbf{A}$.

FOR EXPONENTIAL EULER $\Rightarrow$ $\varphi_1(\tau A)\,w$

FOR BACKWARD FORWARD EULER $\Rightarrow$ $(I-\tau A)^{-1}w$

TO COMPUTE THESE MATRIX FUNCTIONS WE EMPLOY DIAGONALIZATION

$$\boxed{AV = VD}$$

WHERE $V \in \mathbb{R}^{N\times N}$ IS THE MATRIX CONTAINING THE EIGENVECTORS AND $D \in \mathbb{R}^{N\times N}$ IS THE DIAGONAL MATRIX CONTAINING THE EIGENVALUES

$$\boxed{A = VDV^{-1}}$$

AND THEN

$$\varphi_1\left(\tau\begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N\end{bmatrix}\right) = \begin{bmatrix}\varphi_1(\tau\lambda_1) & & & 0\\ & \varphi_1(\tau\lambda_2) & & \\ & & \ddots & \\ 0 & & & \varphi_1(\tau\lambda_N)\end{bmatrix}$$

$$\varphi_1(\tau A)\,w = V\,\varphi_1(\tau D)\,V^{-1}w$$

$$(I-\tau A)^{-1}w = V\,(I-\tau D)^{-1}\,V^{-1}w$$

REMARK: PAY ATTENTION TO THE POSSIBLE ILL-CONDITIONED DIAGONALIZATION

$$\text{COND}(V) \gg 1$$

$\hookrightarrow$ ADVECTION-DIFFUSION MATRIX

IN OUR CASE A IS SYMMETRIC $\Rightarrow$ $\text{COND}(V) = 1$ $\Rightarrow$ DIAGONALIZATION IS FINE.

$$\varphi_1(\tau A)\,w = V\,\varphi_1(\tau D)\,\underbrace{V^{-1}w}_{\text{LINEAR SYSTEM}}$$

BUT A IS SYMMETRIC $\Rightarrow V^{-1} = V^T$

$$\varphi_1(\tau A)\,w = V\varphi_1(\tau D)\,V^T w$$

$\Downarrow$

NO LINEAR SYSTEM HAS TO BE SOLVED