Fall 2024 Admissions Problem Set

Athemath Staff

Due August 31st, 2024

§1 Instructions

For all problems, **proof-based solutions are encouraged**. We would like you to explain all of your steps, instead of just giving an answer. If you don't have experience with proofs, just try to explain your answer as much as you can. LaTeX submissions and *neat*, dark handwriting submissions are both allowed.

We also encourage you to try all the problems. While later problems will generally be harder, they also play to different strengths and you may find one particularly easy.

Please do not use computer programs, Google, WolframAlpha, GeoGebra, etc. to help you find solutions. Additionally, please do not discuss this quiz with anyone else until after the deadline has passed. **If you find the problems difficult, that's because it's designed to be.** If you get stuck, take a break, try a different problem, or try a strategy you dismissed at first. And remember that you don't have to solve all of the problems to get in. Historically, the average admitted student solved around two or three problems.

Ask for clarifications by emailing Vivian at vivian.loh@athemath.org. Submit your completed solutions to the application form by **August 31st, 11:59pm Eastern**. However, we will evaluate applications as they come in, so the sooner you apply, the sooner you'll receive your results. As a reminder, only students of underrepresented genders can apply. Have fun!

§ 2 The Problems!

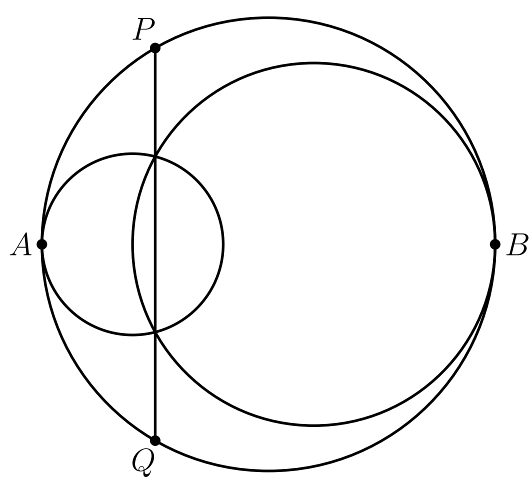
Problem 1

Find the number of 4 digit numbers that satisfy these 2 conditions:

1. divisible by 99
2. the last two digits form a two-digit perfect square

Problem 2

Circle Ω has diameter $AB = 5$. Circles ω_1 and ω_2 have radii 1 and 2 respectively, and are internally tangent to Ω at A and B respectively. The common chord of ω_1 and ω_2 intersects Ω at P and Q. What is the length of PQ?



Problem 3

Chelsea writes down the sequence of integers from 1 to 24. For each integer $0 \leq n \leq 7$, Charlie randomly permutes the $3n+1$, $3n+2$, $3n+3$ in Chelsea's sequence (not necessarily changing the order of these three numbers), and writes them directly under Chelsea's sequence. Then, Chandra forms her own sequence with 24 terms by multiplying each term in Chelsea's sequence with the corresponding term in Charlie's sequence. What is the expected sum of all the numbers in Chandra's sequence?

Note: *"Expected" means the average value of a variable.*

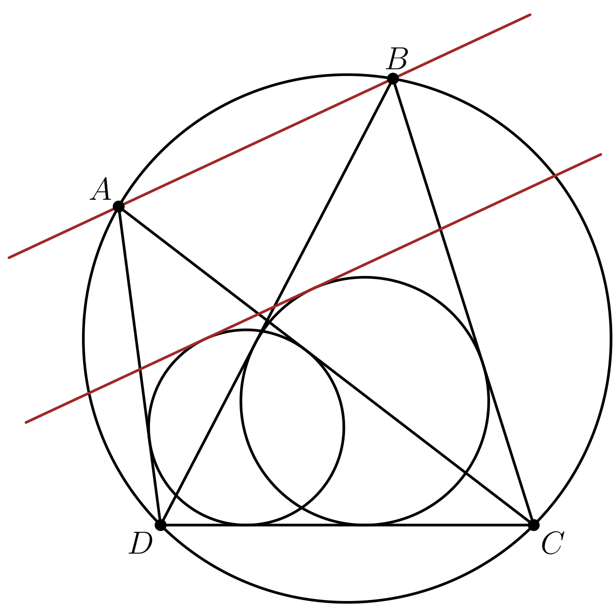
Problem 4

For a given positive integer m, you have a round table of $2m$ seats, and m couples attending this party (all $2m$ people are distinct). Given an odd integer $1 < n < m$, you want to seat everyone such that the two people in each couple are exactly n seats apart (as in there are $n-1$ seats between them). How many distinct ways are there to achieve this?

(Rotations and reflections of an arrangement are considered the same.)

Problem 5

Let $ABCD$ be a cyclic quadrilateral, and ω_A, ω_B the incircles of $\triangle ACD$ and $\triangle BCD$. Show that the common external tangent of ω_A and ω_B which is not CD is parallel to AB.



Problem 6

Alice and Bob play a game. They start off with a pile of N rocks and alternate turns. On each player's turn, they may take away a number of rocks that satisfies one of the following restrictions:

- is divisible by 17
- is in $S = \{1, 2, 3, 5, 7, 11, 13\}$

The player who takes the last rock wins. The value of N is a random integer from 1 to 289, inclusive. If Alice goes second, what is the probability that she has a winning strategy?

Bonus: If we redefine winning the game as making a move that leaves no legal moves for the other player, then show that for any other subset $S \in \{1, 2, 3, .., 16\}$, the probability of winning is the same!