Athemath Spring 2025 Admissions Quiz

Athemath Staff

Due January 25th, 2025

§1 Instructions

For all problems, **proof-based solutions are encouraged**. We would like you to explain all of your steps instead of just giving an answer. If you don't have experience with proofs, just try to explain your answer as much as you can. LaTeX submissions and *neat*, dark handwriting submissions are both allowed.

We also encourage you to try all the problems. While later problems will generally be harder, they also play to different strengths and you may find one particularly easy.

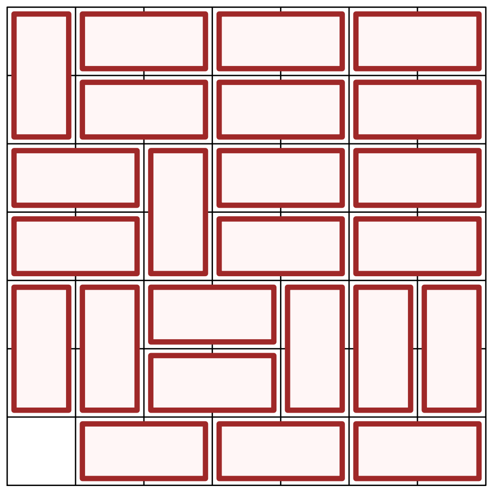
Please do not use computer programs, Google, WolframAlpha, etc. to help you find solutions (however, GeoGebra is allowed). Additionally, please do not discuss this quiz with anyone else until after the application deadline has passed. If you find the test difficult, that's because it's designed to be! If you get stuck, take a walk, try a different problem, or try a strategy you dismissed at first. And remember that you don't have to solve all of the problems to get in. Historically, the average admitted student solved around two or three problems (admissions problem sets have varied in difficulty over the years though).

Ask for clarifications by emailing contact@athemath.org. Submit your completed solutions to the application form by the end of the day on **Saturday, January 25th**. As a reminder, only students of underrepresented genders can apply. Have fun!

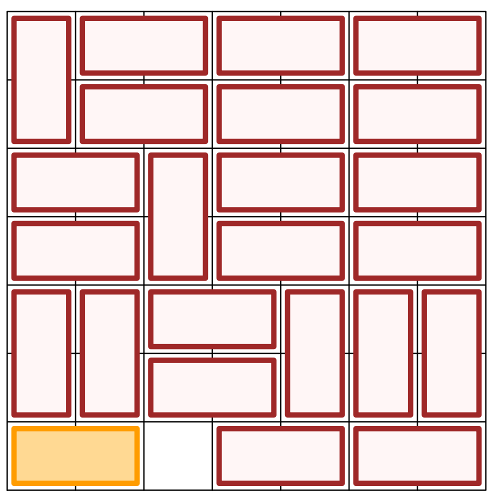
§ 2 The Problems!

Problem 1

You can slide 1×2 dominoes on this grid so that each of them occupies two grid squares, none of them overlap each other, and none of them stick out of the grid. There is always exactly one unoccupied grid square. Try to achieve as many possible positions for this unoccupied square as you can by sliding dominoes.



For example, one way to make the first move is:



Bonus: If you think you have found all the possible unoccupied squares, can you prove that there are no more?

Problem 2

For any two positive integers a and b, let $f(a,b)$ be the number of ordered integer pairs (x,y) (not necessarily positive) such that $ax+by=xy$. Find $f(2^{14},3^{34})$.

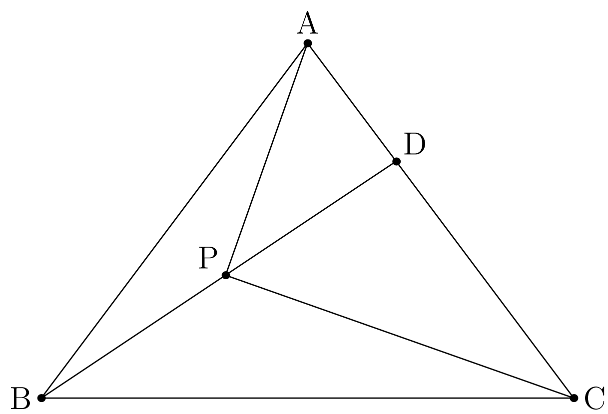
Problem 3

Consider a set S consisting of all positive integers from 1 to 8, inclusive. Find the smallest number k such that one can find k subsets of S, each of size 4, such that every two-element subset of S is be a subset of at least one of these k subsets.

Bonus: Can you find the value of k if S is the set of all positive integers from 1 to $4n$, for each positive integer n?

Problem 4

Let $\triangle ABC$ be an isosceles acute triangle with $AB=AC$. Let P be in a point in $\triangle ABC$ so that $\angle APC = 90°$ and $\angle ABP = \angle BCP$. Let D be the intersection of BP and AC. If $AD = 108$, find the value of the length of CD.



Hint: *Reinterpret the condition regarding $\angle APC$ as a circle and link it to the line BD!*

Problem 5

Fix $k=1434$ and let S be the set of ordered positive integer pairs (m,n) such that $\gcd(m,n)=1$. Additionally, both m and n are not larger than k. If for any two positive integers a,b we define $f(a,b)$ as $\left\lfloor \dfrac{k}{\max(a,b)} \right\rfloor$, find the sum of $f(m,n)$ for all pairs (m,n) in S.

Hint: *Experiment with small values of k.*