Athemath Spring 2022 Admissions Quiz

Athemath Staff

Due January 17th, 2021

§1 Instructions

For all of the problems below, **proof-based solutions are encouraged**. We would like you to explain all of your steps, instead of just giving an answer. If you don't have experience with proofs, just try to explain your answer as much as you can. LaTeX submissions and *neat*, dark handwriting submissions are both allowed.

We also encourage you to try the entire test. While later problems will generally be harder, they also play to different strengths and you may find one particularly easy.

Please do not use computer programs, Google, WolframAlpha, GeoGebra, etc. to help you find solutions. Additionally, please do not discuss this quiz with anyone else until after the application deadline has passed. **If you find the test difficult, that's because it's designed to be.** If you get stuck, take a walk, try a different problem, or try a strategy you dismissed at first. And remember that you don't have to solve all—or even a majority—of the problems to get in. Historically, the average admitted student solves around two problems.

Ask for clarifications by emailing Serena at serena.an@athemath.org. Submit your completed solutions to the application Google Form by **January 17th, 11:59PM Eastern**. As a reminder, only students of underrepresented genders can apply. Have fun!

§ 2 The Problems!

Problem 1

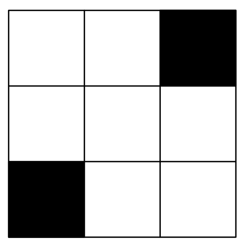
For which real numbers m is $4^{\log_2 m}$ not equal to m^2?

Problem 2

Let $S(n)$ denote the sum of the digits in the base-10 representation of n for any natural number n. Find the numbers n between 1 and 100 inclusive that have the property $S(S(n)+1) < S(S(n))$.

Problem 3

Jessie has seven small square tiles of side length 1 meter, each painted a different color, and a large empty tile of side length 3 meters. The large tile is divided into nine square areas of side length 1 meter each, but two diagonally opposite areas are already filled, leaving seven blank areas.



In how many distinguishable ways can Jessie use her seven small tiles to fill the large tile? (Two fillings are *distinguishable* is one cannot be rotated or flipped over to be identical to the other.)

Problem 4

Let $ABCD$ and $AEFG$ be two non-coinciding unit squares. Prove that it is impossible for the points B, C, E, and F to all lie on one circle.[1]

Problem 5

Find the smallest $k > 1$ such that for all polynomials P with all integer coefficients,

$$P(x^k) \equiv P(x) \pmod{210}$$

for all $x \in \mathbb{N}$, the set of natural numbers.

Remember to prove that no smaller number can work.

[1] For an accessible introduction to cyclic quadrilaterals, see chapter 1 of *Euclidean Geometry in Mathematical Olympiads* by Evan Chen.