eak solutions to differential equations. Hence there is a brief excursion into stribution theory at a fairly elementary level, and I have tried to bring together e mathematical and physical underpinnings in a cohesive way. To compensate ır the increased length, two chapters from the first edition (Similarity Methods ıd Finite Difference Methods) have been deleted from this edition. Further, :lected changes have been made throughout the book to enhance the exposi-ɔn, and many of the exercises have been replaced. The result is a book that is ightly more traditional than its predecessor. But I have maintained the empha-s on modeling, and there are examples from biology, chemistry and physics, ıd most of the areas of engineering.

Since the first edition of this book in 1987, the curriculum in mathematics ıs been undergoing fundamental changes, including technology-based courses the undergraduate level using calculators and symbolic programs, and, along ith this, a new emphasis on modeling. It is not clear at the present time where l of this is leading; yet I feel a strong sense that radical changes will ulti-ately bring more and more computer-based instruction to our profession. This :w edition, however, takes only a small step in this direction. I have included small number of Maple programs in various places to show students how oblems can be solved using symbolic packages; I have not repeated standard aple commands used to solve routine problems; these are available in the any Maple manuals on the market. In my own course I use Maple as the ısis of projects that I assign during the semester. Some students prefer to use athematica, or other packages, to do the projects. The text is in no way based ı any particular software package.

I would like to especially thank my colleague, Professor Glenn Ledder, who ıs taught this course on numerous occasions. I have benefitted from his hand-ıts, exams, and many important suggestions for improvement of the book. My ife Tess deserves special thanks for her constant encouragement and support.

I always remember the comment of my teacher, Professor Stefan Drobot, ho said that mathematics does not consist of only definitions, theorems, oofs, and rigor, but rather ideas and motivations as well. I hope this book cedes to this philosophy.

Suggestions for use of the text. Chapters 1 through 3 (scaling and dimen-ɔnal analysis, modeling, perturbation methods, and variational methods) can rm the basis of a one semester course that focuses on ordinary differential uations. By deleting some of the material on calculus of variations, one can ck up some of the basic ideas on stability and bifurcation in Chapter 7. A sec-ıd semester can focus on partial differential equations, which form the basis of e models in Chapters 4 through 6 (diffusion, eigenfunction expansions, trans-rm methods, Green's functions, wave propagation, and fluid mechanics). With fairly rapid pace the entire book can be covered in a year-course.

J. DAVID LOGAN

ncoln, Nebraska

1

DIMENSIONAL ANALYSIS AND SCALING

The techniques of dimensional analysis and scaling are basic in the theory and practice of mathematical modeling. In every physical setting a good grasp of the possible relationships and comparative magnitudes among the various dimensioned parameters nearly always leads to a better understanding of the problem and sometimes points the way toward approximations and solutions. In this chapter we briefly introduce some of the basic concepts from these two topics. Along with several examples, a statement and proof of the fundamental result in dimensional analysis, the Buckingham Pi theorem, is presented, and scaling is discussed in the context of reducing problems to dimensionless form. The notion of scaling also points the way toward a proper treatment of perturbation methods, especially boundary layer phenomena in singular perturbation theory.

1.1 DIMENSIONAL ANALYSIS

The Program of Applied Mathematics

There are many phases to the solution of a problem that arises in a physical context and that requires careful mathematical analysis. One way to view the attack on such a physical problem is as follows. When a problem arises in empirics, the first stage is to formulate a mathematical model of the situation. This step includes defining the relevant quantities and formulating a set of governing equations that describe the process involved in detail. We can regard the mathematical problem represented by these model equations as a pure mathematics problem. Its solution by some mathematical technique is the second stage of analysis. Once the solution is obtained, the third stage is to go back

and verify that the analytical results are consistent with the experimental observations in the original physical problem. If, indeed, there is consistency, and if the solution is predictive of other similar physical results, then we can conclude that the devised mathematical equations do, in fact, represent a realistic model.

It would be a limited view, in fact an incorrect one, to believe that applied mathematics consists only of developing techniques and algorithms to solve problems that arise in a physical or applied context. Applied mathematics deals with all these stages, not merely the formal solution as represented in stage two. It is true that an important aspect of applied mathematics consists of studying, investigating, and developing procedures that are useful in solving such mathematical problems: these include analytic and approximation techniques, numerical analysis, and methods for solving differential and integral equations. It is more the case, however, that applied mathematics deals with every phase of the problem. Formulating the model and understanding its origin in empirics are crucial steps. Because there is a constant interplay between the various stages, the scientist, engineer, or mathematician must understand each phase. For example, in the second stage the solution to a problem sometimes involves making approximations that lead to a simplification. The approximations often come from a careful examination of the physical reality, which in turn suggests what terms may be neglected, what quantities (if any) are small, and so on. Finally, inaccurate predictions may suggest refinements in the model that lead to even better descriptions of reality. All of this is the practice of applied mathematics; heuristic reasoning, manipulative skills, and physical insight are all essential elements.

In this chapter our aim is to focus on the first stage, or modeling process. We carry this out by formulating models for various physical systems while emphasizing the interdependence of mathematics and the physical world. Through study of the modeling process we gain insight into the equations themselves. For example, it is possible to study the diffusion equation, a partial differential equation of the form

$$u_t(x,t) - u_{xx}(x,t) = 0$$

without regard to its origin. We can investigate it mathematically by asking questions regarding the existence of solutions, methods of solution, and so on. Such an endeavor, however, is sterile from the point of view of applied mathematics; the origins and analysis are equally important. Indeed, physical insight forces us toward the right questions and at times leads us to the theorems and their proofs.

In addition to presenting some concrete examples of modeling, we also discuss two techniques that are useful in developing and interpreting the model equations. One technique is dimensional analysis, and the other is scaling. The former permits us to understand the dimensional (meaning length, time, mass, etc.) relationships of the quantities in the equations and the resulting implications of dimensional homogeneity. Scaling is a technique that helps us understand the magnitude of the terms that appear in the model equations by comparing the quantities to intrinsic reference quantities that appear naturally in the physical situation.

Dimensional Methods

One of the basic techniques that is useful in the initial or modeling stage of a problem is the analysis of the relevant quantities and how they must relate to each other in a dimensional way. Simply put, apples cannot equal oranges; equations must have a consistency to them that precludes every possible relationship among the variables. Stated still differently, equations must be dimensionally homogeneous. These simple observations form the basis of the subject known as *dimensional analysis*. The methods of dimensional analysis developed over the last century or so have led to important results in determining the nature of physical phenomena, even when the governing equations were not known. This has been especially true in continuum mechanics, out of which the general methods of dimensional analysis evolved.

The cornerstone result in dimensional analysis is known as the *Pi theorem*. Roughly, the Pi theorem states that if there is a physical law that gives a relation among a certain number of physical quantities, then there is an equivalent law that can be expressed as a relation among certain dimensionless quantities (often noted by $\pi_1, \pi_2, \ldots$, and hence the name). The Pi theorem appears to have been first stated by A. Vashy in 1892. Later, in 1914, E. Buckingham gave a proof of the Pi theorem for special cases, and now the theorem often carries his name. Birkhoff [1] can be consulted for a bibliography and history.

To communicate the flavor and power of this classic result, let us consider an example. This calculation was made by G. I. Taylor in the late 1940s to compute the yield of the first atomic explosion after viewing photographs of the spread of the fireball. In such an explosion a large amount of energy e is released in a short time (essentially instantaneously) in a region small enough to be considered a point. From the center of the explosion a strong shock wave spreads outward; the pressure behind it is on the order of hundreds of thousands of atmospheres, far greater than the ambient air pressure whose magnitude can be accordingly neglected in the early stages of the explosion. It is reasonable that there should exist a relation between the radius of the blast wave front r, time t, the initial air density ρ, and the energy released e. Hence we *assume* there is a physical law

$$g(t,r,\rho,e) = 0 \tag{1}$$

which provides a relationship among these quantities. The Pi theorem of Buckingham states that there is an equivalent physical law between the independent dimensionless quantities that can be formed from t, r, e, and ρ. We note that t has dimensions of time, r has dimensions of length, e has dimensions

off vertically upward at velocity v_f. If E_g is the Gurney energy [joules/kilogram (J/kg)] of the explosive, that is, the energy available in the explosive to do work, determine as far as possible, using dimensional analysis methods, the velocity of the flyer in terms of m_3, m_f, and E_g. The correct relation is

$$v_f = \sqrt{2E_g}\left(\frac{m_f}{m_e} + \frac{1}{3}\right)^{-1/2}$$

2.6 Using dimensional analysis methods prove Pythagoras' theorem, which states that in a right triangle the sum of the squares of the legs equals the square of the hypotenuse. (*Hint:* The area A of the right triangle is determined by its hypotenuse c and the smallest acute angle ϕ. Apply the same principle to the two similar right triangles found by dropping a perpendicular to the hypotenuse c.)

2.7 A pendulum executing small vibrations has period T, length l, and m is the mass of the bob. Can T depend only on l and m? If we assume T depends on l, m, and the acceleration g due to gravity, then show that $T = \text{constant} \cdot (L/g)^{1/2}$.

2.8 Suppose one wishes to determine the power P that must be applied to keep a ship of length l moving at a constant speed V. If it is the case, as seems reasonable, that P depends on the density of water ρ, the acceleration due to gravity g, and the viscosity of water ν (in length squared per unit time), as well as l and V, then show that

$$\frac{P}{\rho l^2 V^3} = f(\text{Fr}, \text{Re})$$

where Fr is the Froude number and Re is the Reynolds number defined by

$$\text{Fr} \equiv \frac{V}{\sqrt{lg}}, \qquad \text{Re} \equiv \frac{Vl}{\nu}$$

2.9 A spherical gas bubble with ratio of specific heats γ is surrounded by an infinite sea of liquid of density ρ_l. The bubble oscillates with growth and contraction periodically with small amplitude at a well-defined frequency ω. Assuming a physical law

$$f(P, R, \rho_l, \omega, \gamma) = 0$$

where P is the mean pressure inside the bubble and R is the mean radius,

show that the frequency must vary inversely with the mean radius R. The constant γ is dimensionless.

1.3 SCALING

Characteristic Scales

Another procedure useful in formulating a mathematical model of a physical situation is that of *scaling*. Roughly, scaling means selecting new, usually dimensionless variables and reformulating the problem in terms of those variables. Not only is the procedure useful but it often is a necessity, especially when comparisons of the magnitudes of various terms in an equation must be made in order to neglect small terms, for example. These ideas are of particular importance in the application of perturbation methods to a problem to identify small and large parameters.

For motivation let us suppose that time t is a variable in a given problem, measured in units of seconds. If the problem involved the motion of a glacier, clearly the unit of seconds is too fast because significant changes in the glacier could not be observed on the order of seconds. On the other hand, if the problem involved a nuclear reaction then the unit of seconds is too slow; all of the important action would be over before the first second ticked. Evidently, every problem has an *intrinsic time scale*, or *characteristic time* t_c, which is appropriate to the given problem. This is the shortest time for discernible changes to be observed in the physical quantities. For example, the characteristic time for glacier motion would be of the order of years, whereas the characteristic time for a nuclear reaction would be of the order of microseconds. Some problems may have multiple time scales; a chemical reaction, for example, may begin slowly and then rapidly go to completion. Figure 1.1 depicts such a reac-

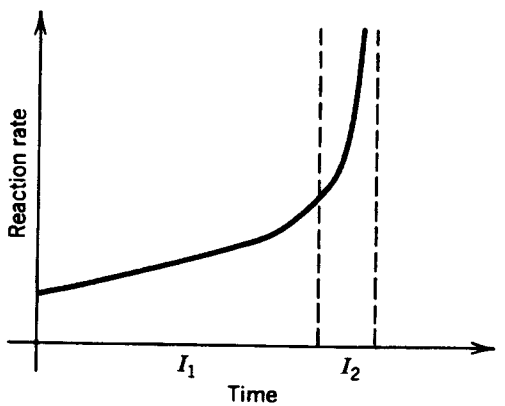

Figure 1.1. Physical process with a multiple time scale.

tion. Obviously, a time scale appropriate for the time interval I_1 would not be appropriate for the interval I_2 where rapid changes are occurring. This basic observation is at the heart of singular perturbation or boundary layer theory.

Once the characteristic time is known, then a new dimensionless variable $\bar{t}$ can be defined by

$$\bar{t} = \frac{t}{t_c}$$

If t_c is chosen correctly, then the *dimensionless time* $\bar{t}$ is neither too large nor too small, but rather of order unity. The question remains to determine the time scale t_c for a particular problem. The same question applies to lengths or to any other variables in the problem. The general rule is that the characteristic quantities are formed by taking combinations of the various dimensional constants in the problem and should be roughly the same order of magnitude of the quantity itself. This general principle is illustrated by the following examples.

A Chemical Reactor Problem

A basic problem in chemical engineering is to understand how the concentration of chemical species vary when undergoing a reaction in a chemical reactor. To illustrate some of the concepts of nondimensionalization and scaling we analyze a simple model of an isothermal, continuously stirred, batch reactor (see Fig. 1.2). The reactor has a fixed volume V, and a chemical **C** of fixed concentration c_i, given in mass per volume, enters the reactor through the feed at a constant flow rate q, given in volume per second; initially the concentration of the chemical in the reactor is c_0. When the chemical enters the reactor, the mixture is perfectly stirred while undergoing a chemical reaction, and then the mixture exits the reactor at the same flow rate q. At any time t, we denote the concentration of the chemical **C** in the reactor by $c = c(t)$. The reactant chemical is assumed to disappear, that is, it is consumed by reaction, with a rate $r = r(c)$, given in mass per unit volume, per unit second. We are thinking of a simple reaction of the form **C** $\rightarrow$ *products*. Usually reaction rates depend also on temperature, but here we assume that r depends only on the concentration c; this is what makes our problem isothermal. The *perfectly stirred* assumption

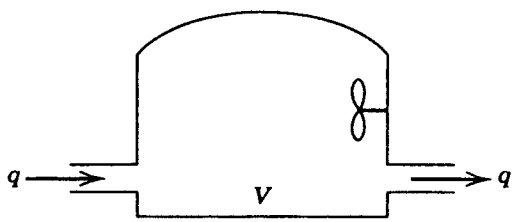

Figure 1.2. Batch chemical reactor.

is an idealization and implies that there are no concentration gradients in t reactor; otherwise, c would also depend on a spatial variable.

To obtain a mathematical model of this problem we look for a physical pri ciple that may apply. A common principle that is fundamental to all flow prc lems is mass balance. That is, the time rate of change of the mass of the chen cal inside the reactor must equal the rate mass flows in (qc_i), minus the r that mass flows out (qc), plus the rate that mass is consumed by the reacti (Vr). At any given time the mass of the chemical in the reactor is Vc. Thus, symbols, the mass balance equation is

$$\frac{d}{dt}(Vc(t)) = qc_i - qc(t) - Vr(c(t))$$

To fix the idea, let us take the reaction rate r to be proportional to c, that $r = kc$, where k is the rate constant (having dimensions of inverse time). Th the mathematical model is given by the initial value problem

$$\frac{dc}{dt} = -\frac{q}{V}(c_i - c) - kc, \qquad t > 0$$

$$c(0) = c_0$$

To nondimensionalize the problem we choose dimensionless independ and dependent variables. This means we must select a characteristic time an characteristic concentration by which to measure the real time and concent tion. These characteristic values are formed from the constants in the proble c_i, c_0, V, q, k. Generally, we measure the dependent variable relative to so maximum value in the problem, or any other value that represents the order magnitude of that quantity. There are two constant concentrations, c_i and and either one of them is a suitable concentration scale. Therefore, we def a *dimensionless* concentration C by

$$C = \frac{c}{c_i}$$

Thus all concentrations in the problem will be measured relative to c_i, the conc tration of the feed. To select the time scale we observe that there are two quantit with dimensions of time that can be formed from the constants in the problem, V and k^{-1}. The former is based on the flow rate, and the latter is based on the re tion rate. The choice of either one will lead to a correct dimensionless proble For definiteness, we choose V/q and define a dimensionless time by

$$\tau = \frac{t}{V/q}$$

Now we may reformulate the model in dimensionless form. By the chain rule we have

$$\frac{dc}{dt} = \frac{c_i}{V/q}\frac{dC}{d\tau}$$

Therefore the model becomes

$$\beta\frac{dC}{d\tau} = \beta(1 - C) - C, \qquad \tau > 0 \tag{3}$$

$$C(0) = \gamma \tag{4}$$

where β and γ are dimensionless constants given by

$$\beta = \frac{q/V}{k}, \quad \gamma = \frac{c_0}{c_i}$$

So the problem has been reduced to dimensionless form, and any results that are obtained are free of any specific set of units that we select. Moreover, the number of parameters has been decreased, and the problem is simpler.

As we observed above, the choice of the scales is somewhat arbitrary and the result of nondimensionalization is not unique. However, in some problems it is important to choose the *correct* scales so that each scale is representative of the magnitude of the dimensioned variable. Sometimes we use the term "scaling" to refer to the process of making a rational, rather than arbitrary, choice of dimensionless variables for the nondimensionalization. For example, suppose we choose the time scale differently from above and set

$$\bar{t} = \frac{t}{k^{-1}}$$

Thus, time is being measured relative to the reaction rate. Then it is easy to check that the model becomes

$$\frac{dC}{d\bar{t}} = \beta(1 - C) - C, \qquad \bar{t} > 0 \tag{5}$$

$$C(0) = \gamma \tag{6}$$

Notice that the dimensionless constants appear in a different position in the equations (3) and (5). Both are correct nondimensionalizations. However, let us put another condition on the problem and assume that the chemical reaction is very fast, or k is very large. Specifically, we assume that k is large compared to q/V. This assumption forces the dimensionless parameter β to be small. Notice

that β measures the ratio of the two time scales. Suppose further that we want to make the problem easier, and obtain an approximation, by deleting small terms in the equation (this is a very common strategy in equations that we cannot solve). If we ignore the terms involving β in (3) then we just get $C(\tau) = 0$, which is uninteresting and does not satisfy the initial condition; if we ignore the terms involving β in (5) then we get

$$\frac{dC}{d\bar{t}} = -C, \qquad \bar{t} > 0; \qquad C(0) = \gamma \tag{7}$$

which gives the approximate solution $C(\bar{t}) = \gamma \exp(-\bar{t})$. This approximation is believable; the reaction dominates and quickly depletes the chemical. So (5) is the correctly scaled problem; under the assumption of fast chemistry, the terms in the equation involving the flow rate are small. The small parameter β appears where it should be when we scale time by the reaction rate. Because k is large, k^{-1} is small, and therefore we might expect our approximation to be valid when the real time t is small.

Consequently, if approximations are to be made by deleting small terms, it is important how we nondimensionalize the problem. Proper scaling is learned through experience and careful analysis. One helpful scaling strategy is to nondimensionalize the problem using a generic time scale t_c that is chosen later to make the coefficients of the terms in the equation reflect their size or reflect which terms balance in the process.

The Projectile Problem

The following example, as first pointed out by Lin and Segel [2], is a good illustration of the importance of choosing correct scales in a problem, particularly when it is desired to make a simplification by neglecting small quantities. Terms in an equation that appear small are not always as they seem, and proper scaling is essential in determining the orders of magnitude of the terms.

In this example the problem of describing the motion of a projectile thrust vertically upward from the surface of the earth is analyzed in detail. The steps are similar to those taken in the preceding example, namely, the development of a mathematical model and then the determination of appropriate scales that are used to cast the problem in dimensionless form.

At time $t = 0$ on the surface of the earth, with radius R and mass M, an object of mass m is given a vertical upward velocity of magnitude V. To be determined is the height h above the earth's surface that the mass reached at time t (see Fig. 1.3). The forces on the object are the force due to gravity and the force due to air resistance. In an initial effort to formulate a governing equation we assume the force due to air resistance is negligible in the particular physical situation we are considering. In general, as a first approximation it is common to neglect what are believed to be small effects, since in that case the

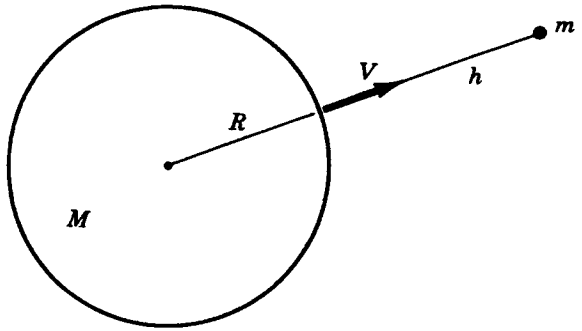

Figure 1.3.

equations are more tractable for analysis. Should the analytic results compare unfavorably with experiment or should a more detailed description be required, then additional effects can be included.

In the present problem the governing equation or mathematical model comes from a physical law. Newton's universal gravitational law, which states that the force between the two objects is proportional to the product of the masses and inversely proportional to the square of the distance between them, where the mass of each object can be regarded as concentrated at its center. Therefore, since by Newton's second law the force on m must equal the mass of m times its acceleration,

$$m\frac{d^2h}{dt^2} = -G\frac{Mm}{(h+R)^2}$$

where G is the proportionality constant in the universal gravitational law. When $h = 0$, that is, at the earth's surface, the gravitational force must equal $-mg$, and so

$$\frac{GM}{R^2} = g$$

where g is the acceleration due to gravity. Thus

$$\frac{d^2h}{dt^2} = -\frac{R^2g}{(h+R)^2} \tag{8}$$

with initial conditions

$$h(0) = 0, \qquad \frac{dh}{dt}(0) = V \tag{9}$$

The initial value problem (8) and (9) represents the mathematical model for the problem.

At this point we can undertake a dimensional analysis of the problem and gain considerable insight without actually attempting a solution. From our model the relevant dimensional quantities are t, h, R, V, and g having dimensions

$$[t] = \text{time } (T) \qquad [V] = \text{velocity } (LT^{-1})$$
$$[h] = \text{length } (L) \qquad [g] = \text{acceleration } (LT^{-2})$$
$$[R] = \text{length } (L)$$

We are using T (time) and L (length) as fundamental units. Following the procedure described in Section 1.2, if π is a dimensionless combination of t, h, R, V, and g, then

$$\begin{aligned}[\pi] &= [t^{\alpha_1}h^{\alpha_2}R^{\alpha_3}V^{\alpha_4}g^{\alpha_5}] \\ &= T^{\alpha_1-\alpha_4-2\alpha_5}L^{\alpha_2+\alpha_3+\alpha_4+\alpha_5} \\ &= 1\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_1 - \alpha_4 - 2\alpha_5 &= 0 \\ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 &= 0\end{aligned} \tag{10}$$

This system has rank two and so there are three independent dimensionless variables. Either by inspection or solving (10) we find that these quantities are

$$\pi_1 = \frac{h}{R}, \qquad \pi_2 = \frac{t}{R/V}, \qquad \pi_3 = V/\sqrt{gR} \tag{11}$$

By the Pi theorem, if there is a physical law relating t, h, R, V, and g (and we assume there must be, since in theory we could solve (8) and (9) to obtain that law), then there is an equivalent law that can be expressed as

$$\frac{h}{R} = f\left(\frac{t}{R/V}, \frac{V}{\sqrt{gR}}\right) \tag{12}$$

for some function $f(\pi_2, \pi_3)$.

Actually there is considerable information in (12). For example suppose we are interested in finding the time t_{max} that is required for the object to reach its maximum height for a given velocity V. Then differentiating (12) with respect

to t and setting $h'(t)$ equal to zero gives

$$\frac{\partial f}{\partial \pi_2}\left(\frac{t_{\max}}{R/V}, \frac{V}{\sqrt{gR}}\right) = 0$$

or

$$\frac{t_{\max}}{R/V} = F\left(\frac{V}{\sqrt{gR}}\right) \tag{13}$$

for some function F. Remarkably, with little analysis beyond dimensional reasoning, we have found that the time to maximum height depends only on the single quantity $V/\sqrt{gR}$. The value in knowing this kind of information lies in the efficiency of (13); a *single* graph of $t_{\max}/(R/V)$ versus $V/\sqrt{Rg}$ contains all of the data of the graphs of $t_{\max}$ versus V for *all* choices of g and R. For example, an experimenter making measurements on different planets of $t_{\max}$ versus V would not need a separate plot of data for each planet.

The next step in the analysis is to choose characteristic time and length scales and recast the problem represented by (8) and (9) into dimensionless form. For the present problem this is more subtle than it originally appears. The general method requires us to select a new dimensionless dependent variable $\bar{h}$ and independent variable $\bar{t}$ by

$$\bar{t} = \frac{t}{t_c}, \qquad \bar{h} = \frac{h}{h_c} \tag{14}$$

where t_c is an intrinsic time scale and h_c is an intrinsic length scale; the values of t_c and h_c should be chosen by taking combinations of the constants in the problem, which in this case are R, V, and g. This problem presents several choices. For a length scale h_c we could take either R or V^2/g. Possible time scales are R/V, $\sqrt{R/g}$, and V/g. Which choice is the most appropriate? Actually, Equations (14) represent a legitimate transformation of variables for any choice of t_c and h_c; after the change of variables an equivalent problem would result. From a scaling viewpoint, however, one particular choice will be advantageous. The three choices

$$\bar{t} = \frac{t}{R/V}, \qquad \bar{h} = \frac{h}{R} \tag{15}$$

$$\bar{t} = \frac{t}{\sqrt{R/g}}, \qquad \bar{h} = \frac{h}{R} \tag{16}$$

and

$$\bar{t} = \frac{t}{Vg^{-1}}, \qquad \bar{h} = \frac{h}{V^2 g^{-1}} \tag{17}$$

lead to the following three dimensionless problems, which are equivalent to (8) and (9)

$$\epsilon \frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}, \qquad \bar{h}(0) = 0, \qquad \frac{d\bar{h}}{d\bar{t}}(0) = 1 \tag{18}$$

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}, \qquad \bar{h}(0) = 0, \qquad \frac{d\bar{h}}{d\bar{t}}(0) = \sqrt{\epsilon} \tag{19}$$

and

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\epsilon\bar{h})^2}, \qquad \bar{h}(0) = 0, \qquad \frac{d\bar{h}}{d\bar{t}}(0) = 1 \tag{20}$$

respectively, where ϵ is a dimensionless parameter defined by

$$\epsilon = \frac{V^2}{gR}$$

To illustrate how difficulties may arise in selecting an incorrect scaling let us modify our original problem by examining the situation when ϵ is known to be a small quantity; that is, V^2 is much smaller than gR. Then one may be tempted, in order to make an approximation, to delete the terms involving ϵ in the scaled problem. Problem (18) would then become

$$(1+\bar{h})^{-2} = 0, \qquad \bar{h}(0) = 0, \qquad \frac{d\bar{h}}{d\bar{t}}(0) = 1$$

which has no solution, and problem (19) would become

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}, \qquad \bar{h}(0) = 0, \qquad \frac{d\bar{h}}{d\bar{t}}(0) = 0$$

which has no physically valid solution. In the latter case note that the graph of $\bar{h}(\bar{t})$ would pass though the origin with zero slope and be concave down-

ward, thereby making $\bar{h}$ negative. Therefore it appears that terms involving small parameters cannot be neglected. This is indeed unfortunate, since this kind of technique is a common practice in making approximations in applied problems. What went wrong was that (15) and (16) represent *incorrect* scalings; in that case, terms that appear small may in fact not be small. For example, in the term $\epsilon d^2\bar{h}/d\bar{t}^2$ the parameter ϵ may be small but $d^2\bar{h}/d\bar{t}^2$ may be large, and hence the term may not be negligible compared to other terms in the equation.

If, on the other hand, the term $\epsilon\bar{h}$ is neglected in (20), then $d^2\bar{h}/d\bar{t}^2 = -1$, or $\bar{h} = \bar{t} - \bar{t}^2/2$, after applying the initial conditions. Therefore

$$h = -\tfrac{1}{2}gt^2 + Vt$$

and we have obtained an approximate solution that is consistent with our experience with falling bodies. In this case we are able to neglect the small term and obtain a valid approximation because the scaling was correct. That (17) gives the correct time and length scales can be argued physically. If V is small, then the body will be acted on by a constant gravitational field; hence, launched with speed V, it will uniformly decelerate and reach its maximum height in V/g units of time, which is the characteristic time. It will travel a distance of about (V/g) times its average velocity $\frac{1}{2}(V + 0)$, or $V^2/2g$. Hence V^2/g is a good selection for the length scale.

In general, if a correct scaling is chosen, then terms in the equations that appear small are indeed small and may be safely neglected. In fact, one goal of scaling is to select intrinsic, characteristic reference quantities so that each term in the dimensional equation transforms into a term that the dimensionless coefficient in the transformed term represents the order of magnitude or approximate size of that term. Pictorially

$$\begin{bmatrix}\text{Dimensional}\\ \text{term}\end{bmatrix} \rightarrow \begin{bmatrix}\text{coefficient}\\ \text{representing}\\ \text{the order of}\\ \text{magnitude}\\ \text{of the term}\end{bmatrix} \cdot \begin{bmatrix}\text{dimensionless}\\ \text{factor}\\ \text{of order unity}\end{bmatrix}$$

By *order of unity* we mean a term that is neither extremely large nor small. In the next section this notion is made precise.

Population Models

We end this section with an example from biology where we formulate and scale a simple model of population dynamics in a closed ecosystem. Such models play an important role in understanding the factors that contribute to population growth. The simplest model is to assume that the rate of population changes

is proportional to the number of individuals present; this is the classic Malthus model. To be precise, let $p = p(t)$ denote the population at time t. Then the Malthus model is that dp/dt is proportional to p, or

$$\frac{dp}{dt} = rp$$

where r is the constant *growth rate*, given in inverse-time units. Easily this model predicts that the population will grow exponentially for all time, that is, $p = p_0 \exp(rt)$, where p_0 is the initial population. Many books on an impending world population explosion have been written with the Malthusian model as a premise. Clearly, however, as a population grows competition for food, living space, and natural resources will limit the growth. Therefore one might think of modifying the Malthus model to include a competition term. The simplest approach is to notice that if there are p individuals in the system, then the number of encounters, which is a measure of competition, can be modeled by p^2; therefore we subtract a term proportional to p^2 from the growth rate to obtain the model

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right), \quad p(0) = p_0 \tag{21}$$

Here K is the *carrying capacity*, a constant that is interpreted as the number of individuals that the ecosystem can sustain. This model is called the *logistics model*; as the population grows, the negative p^2 term will kick in and limit the growth. Later we shall show that over a long time the population will approach the carrying capacity K.

To reduce (21) to dimensionless form we need to select new dimensionless variables (both independent and dependent). The time scale and population scale are formed from the constants in the problem, r, K, and p_0. Of these, only r contains the dimensions of time, and therefore we scale time by $1/r$ giving a new, dimensionless time τ defined by

$$\tau = rt$$

There are two choices for the population scale, K or p_0. Either will do, and so we select K to obtain a dimensionless population P given by

$$P = \frac{p}{K}$$

Thus, the interpretation is that we are measuring population in the problem relative to the carrying capacity. Using these variables it is straightforward to

obtain the

$$\frac{dP}{d\tau} = P(1 - P), \qquad P(0) = \alpha \tag{22}$$

where $\alpha \equiv p_0/K$ is a dimensionless constant. The scaled model (22) has only one constant (α), a significant simplification over (21) where there are three constants (r, K, p_0). The constant α represents a scaled, initial population.

The initial value problem (22) can be solved by separating variables to obtain

$$P(\tau) = \frac{\alpha}{\alpha + (1 - \alpha)e^{-\tau}} \tag{23}$$

Clearly

$$\lim_{\tau \to \infty} P(\tau) = 1$$

It follows that, confirming our earlier statement, the limiting population p is equal to the carrying capacity K. We observe that there are two *equilibrium populations* [constant solutions of (21)], $p = K$ and $p = 0$. The population $p = K$ is an *attractor*; that is, regardless of the initial population, the population $p(t)$ tends to the value K as time gets large.

A large number of interesting population models is contained in the book by Murray [3].

Exercises

3.1 Let $u = u(t)$, $0 \le t \le b$ be a given continuous function. If $M = max|u(t)|$, then u can be scaled by M to obtain the dimensionless dependent variable $U = u/M$. A time scale can be taken as $t_c = M/\max|u'(t)|$, the ratio of the maximum value of the function to the maximum slope. Find M and t_c for the following functions:

(a) $u(t) = A \sin \omega t$, $t > 0$.

(b) $u(t) = Ae^{-\lambda t}$, $t > 0$.

(c) $u(t) = Ate^{-\lambda t}$, $0 \le t \le 2/\lambda$.

3.2 Consider the function $u(t) = 1 + e^{-t/\epsilon}$ on the interval $0 \le t \le 1$, where ϵ is a small number. Use Exercise 3.1 to determine a time scale. Is this time scale appropriate for the entire interval [0, 1]? (Sketch a graph of $u(t)$ when $\epsilon = 0.05$). Explain why two time scales might be required for a process described by $u(t)$.

3.3 In a classic work modeling the outbreak of the spruce budworm in Canada's balsam fir forests, researchers proposed that the budworm population $n = n(t)$ was governed by the law

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right) - P(n)$$

where r and K are the growth rate and carrying capacity, respectively, and $P(n)$ is a *predation* term (think of birds eating the budworms) given by

$$P(n) = \frac{bn^2}{a^2 + n^2}$$

where a and b are positive constants.

(a) Determine the dimensions of the constants a and b.

(b) Graph the predation rate $P(n)/b$ for $a = 1, 5, 10$ and make a qualitative statement about the effect that the parameter a has on the model.

(c) Select dimensionless variables $N = n/a$ and $\tau = t/(a/b)$ and reduce the differential equation to dimensionless form (introduce constants $q \equiv K/a$ and $s \equiv ar/b$).

(d) Researchers are often interested in equilibrium populations (constant solutions of the differential equation). Working with the dimensionless model, and ignoring the zero population, show that there is always at least one equilibrium population, and show that there may be two or three, depending on the values of the parameters. Find the equilibrium populations when $q = 12$ and $s = 0.25$, and when $q = 35$ and $s = 0.4$.

(e) In the case $q = 35$ and $s = 0.4$ use a numerical differential equations solver to graph the population curve when the initial population is given by $N(0) = 40$, 25, 2, and 0.05, respectively.

3.4 A uniform bar of mass m and length l is connected by a pin joint to a vertical shaft that rotates with angular velocity ω (see Fig. 1.4). The governing differential equation for the angle θ of deflection is

$$\frac{d^2\theta}{dt^2} + \left(\frac{3g}{2l} - \omega^2\right) \sin \theta = 0$$

where g is the constant acceleration due to gravity.

(a) Determine two time scales for the motion and put the differential equation in dimensionless form using each of the time scales (do not scale θ, which is already dimensionless).

(b) Which of the two time scales is appropriate when ω is small? When ω is large?

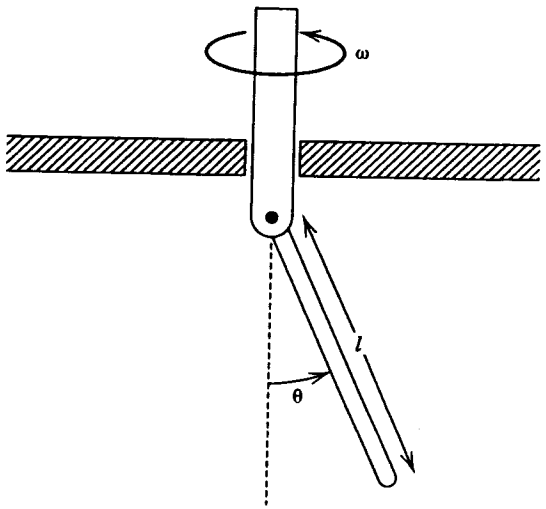

Figure 1.4.

3.5 Verify (23).

3.6 A rocket blasts off from the earth's surface. During the initial phase of flight fuel is burned at the maximum possible rate α, and the exhaust gas is expelled downward with velocity β relative to the velocity of the rocket. The motion is governed by the following set of equations:

$$m'(t) = -\alpha, \qquad m(0) = M$$

$$v'(t) = \frac{\alpha\beta}{m(t)} - \frac{g}{(1 + x(t)/R)^2}, \qquad v(0) = 0$$

$$x'(t) = v(t), \qquad x(0) = 0$$

where $m(t)$ is the mass of the rocket, $v(t)$ is the upward velocity, $x(t)$ is the height above the earth's surface, M is the initial mass, g is the gravitational constant, and R is the radius of the earth. Reformulate the problem in terms of dimensionless variables using appropriate scales for m, x, v, t. (Scale m and x by obvious choices; then choose the time scale and velocity scale to ensure that the terms in the v equation and x equation are of the same order. Assume that the acceleration is due primarily to fuel burning and that the gravitational force is relatively small.)

3.7 In the chemical reactor problem assume that the chemical reaction rate is given by $r = kc^2$, where k is a positive rate constant. What are the dimensions of k? Define dimensionless variables and reformulate the problem in dimensionless form, and then solve the problem to determine the concentration.

3.8 The temperature $T(t)$ of a chemical sample in a furnace at time t is governed by the initial value problem

$$\frac{dT}{dt} = qe^{-\theta/T} - k(T - T_f), T(0) = T_0$$

where T_0 is the initial temperature of the sample, T_f is the temperature in the furnace, and q, k, and θ are positive constants.

(a) What must be the dimensions of the constants q, k, θ?

(b) Reduce the problem to dimensionless form (i.e., all variables and constants you introduce should be dimensionless) using T_f as the temperature scale and choosing a time scale to be one appropriate to the case when the heat loss term is large compared to the heat generated by the reaction.

3.9 A ball of mass m is tossed upward with initial velocity V. Assuming the force caused by air resistance is proportional to the square of the velocity of the ball and the gravitational field is constant, formulate an initial value problem for the height of the ball at any time t. Choose characteristic length and time scales and recast the problem in dimensionless form.

3.10 A pendulum of length l with a bob of mass m executes angular displacements $\theta = \theta(t)$ from its attachment point, with $\theta = 0$ when the pendulum is vertically downward. Use Newton's law to derive the equation of motion for the pendulum (note that the acceleration of the mass is d^2s/dt^2 where $s = l\theta$ is the length of the circular arc traveled by the bob; the force is $-mg$ vertically downward, and only the component tangential to the arc of oscillation affects the motion). The answer is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

If the bob is released from a small angle θ_0 at time $t = 0$, formulate a dimensionless initial value problem describing the motion.

REFERENCES

1. G. Birkhoff, *Hydrodynamics: A Study in Logic, Fact, and Similitude*, Princeton University Press, Princeton, NJ (1950).

2. C. C. Lin and L. A. Segel, *Mathematics Applied to Deterministic Problems in the Natural Sciences*, Macmillan, New York (1974) [reprinted by SIAM (Soc. Industr. Appl. Math.), Philadelphia (1988)].

3. J. D. Murray, *Mathematical Biology*, Springer-Verlag, New York (1989).

Now we may reformulate the model in dimensionless form. By the chain rule we have

$$\frac{dc}{dt} = \frac{c_i}{V/q}\frac{dC}{d\tau}$$

Therefore the model becomes

$$\beta\frac{dC}{d\tau} = \beta(1 - C) - C, \qquad \tau > 0 \tag{3}$$

$$C(0) = \gamma \tag{4}$$

where β and γ are dimensionless constants given by

$$\beta = \frac{q/V}{k}, \quad \gamma = \frac{c_0}{c_i}$$

So the problem has been reduced to dimensionless form, and any results that are obtained are free of any specific set of units that we select. Moreover, the number of parameters has been decreased, and the problem is simpler.

As we observed above, the choice of the scales is somewhat arbitrary and the result of nondimensionalization is not unique. However, in some problems it is important to choose the *correct* scales so that each scale is representative of the magnitude of the dimensioned variable. Sometimes we use the term "scaling" to refer to the process of making a rational, rather than arbitrary, choice of dimensionless variables for the nondimensionalization. For example, suppose we choose the time scale differently from above and set

$$\bar{t} = \frac{t}{k^{-1}}$$

Thus, time is being measured relative to the reaction rate. Then it is easy to check that the model becomes

$$\frac{dC}{d\bar{t}} = \beta(1 - C) - C, \qquad \bar{t} > 0 \tag{5}$$

$$C(0) = \gamma \tag{6}$$

Notice that the dimensionless constants appear in a different position in the equations (3) and (5). Both are correct nondimensionalizations. However, let us put another condition on the problem and assume that the chemical reaction is very fast, or k is very large. Specifically, we assume that k is large compared to q/V. This assumption forces the dimensionless parameter β to be small. Notice

that β measures the ratio of the two time scales. Suppose further that we want to make the problem easier, and obtain an approximation, by deleting small terms in the equation (this is a very common strategy in equations that we cannot solve). If we ignore the terms involving β in (3) then we just get $C(\tau) = 0$, which is uninteresting and does not satisfy the initial condition; if we ignore the terms involving β in (5) then we get

$$\frac{dC}{d\bar{t}} = -C, \qquad \bar{t} > 0; \qquad C(0) = \gamma \tag{7}$$

which gives the approximate solution $C(\bar{t}) = \gamma \exp(-\bar{t})$. This approximation is believable; the reaction dominates and quickly depletes the chemical. So (5) is the correctly scaled problem; under the assumption of fast chemistry, the terms in the equation involving the flow rate are small. The small parameter β appears where it should be when we scale time by the reaction rate. Because k is large, k^{-1} is small, and therefore we might expect our approximation to be valid when the real time t is small.

Consequently, if approximations are to be made by deleting small terms, it is important how we nondimensionalize the problem. Proper scaling is learned through experience and careful analysis. One helpful scaling strategy is to nondimensionalize the problem using a generic time scale t_c that is chosen later to make the coefficients of the terms in the equation reflect their size or reflect which terms balance in the process.

The Projectile Problem

The following example, as first pointed out by Lin and Segel [2], is a good illustration of the importance of choosing correct scales in a problem, particularly when it is desired to make a simplification by neglecting small quantities. Terms in an equation that appear small are not always as they seem, and proper scaling is essential in determining the orders of magnitude of the terms.

In this example the problem of describing the motion of a projectile thrust vertically upward from the surface of the earth is analyzed in detail. The steps are similar to those taken in the preceding example, namely, the development of a mathematical model and then the determination of appropriate scales that are used to cast the problem in dimensionless form.

At time $t = 0$ on the surface of the earth, with radius R and mass M, an object of mass m is given a vertical upward velocity of magnitude V. To be determined is the height h above the earth's surface that the mass reached at time t (see Fig. 1.3). The forces on the object are the force due to gravity and the force due to air resistance. In an initial effort to formulate a governing equation we assume the force due to air resistance is negligible in the particular physical situation we are considering. In general, as a first approximation it is common to neglect what are believed to be small effects, since in that case the

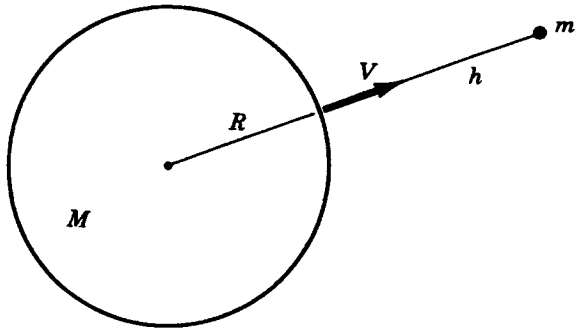

Figure 1.3.

equations are more tractable for analysis. Should the analytic results compare unfavorably with experiment or should a more detailed description be required, then additional effects can be included.

In the present problem the governing equation or mathematical model comes from a physical law. Newton's universal gravitational law, which states that the force between the two objects is proportional to the product of the masses and inversely proportional to the square of the distance between them, where the mass of each object can be regarded as concentrated at its center. Therefore, since by Newton's second law the force on m must equal the mass of m times its acceleration,

$$m\frac{d^2h}{dt^2} = -G\frac{Mm}{(h+R)^2}$$

where G is the proportionality constant in the universal gravitational law. When $h = 0$, that is, at the earth's surface, the gravitational force must equal $-mg$, and so

$$\frac{GM}{R^2} = g$$

where g is the acceleration due to gravity. Thus

$$\frac{d^2h}{dt^2} = -\frac{R^2g}{(h+R)^2} \tag{8}$$

with initial conditions

$$h(0) = 0, \qquad \frac{dh}{dt}(0) = V \tag{9}$$

The initial value problem (8) and (9) represents the mathematical model for the problem.

At this point we can undertake a dimensional analysis of the problem and gain considerable insight without actually attempting a solution. From our model the relevant dimensional quantities are t, h, R, V, and g having dimensions

$$[t] = \text{time } (T) \qquad [V] = \text{velocity } (LT^{-1})$$
$$[h] = \text{length } (L) \qquad [g] = \text{acceleration } (LT^{-2})$$
$$[R] = \text{length } (L)$$

We are using T (time) and L (length) as fundamental units. Following the procedure described in Section 1.2, if π is a dimensionless combination of t, h, R, V, and g, then

$$\begin{aligned}[\pi] &= [t^{\alpha_1}h^{\alpha_2}R^{\alpha_3}V^{\alpha_4}g^{\alpha_5}] \\ &= T^{\alpha_1-\alpha_4-2\alpha_5}L^{\alpha_2+\alpha_3+\alpha_4+\alpha_5} \\ &= 1\end{aligned}$$

Therefore,

$$\begin{aligned}\alpha_1 - \alpha_4 - 2\alpha_5 &= 0 \\ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 &= 0\end{aligned} \tag{10}$$

This system has rank two and so there are three independent dimensionless variables. Either by inspection or solving (10) we find that these quantities are

$$\pi_1 = \frac{h}{R}, \qquad \pi_2 = \frac{t}{R/V}, \qquad \pi_3 = V/\sqrt{gR} \tag{11}$$

By the Pi theorem, if there is a physical law relating t, h, R, V, and g (and we assume there must be, since in theory we could solve (8) and (9) to obtain that law), then there is an equivalent law that can be expressed as

$$\frac{h}{R} = f\left(\frac{t}{R/V}, \frac{V}{\sqrt{gR}}\right) \tag{12}$$

for some function $f(\pi_2, \pi_3)$.

Actually there is considerable information in (12). For example suppose we are interested in finding the time t_{max} that is required for the object to reach its maximum height for a given velocity V. Then differentiating (12) with respect

to t and setting $h'(t)$ equal to zero gives

$$\frac{\partial f}{\partial \pi_2}\left(\frac{t_{\max}}{R/V}, \frac{V}{\sqrt{gR}}\right) = 0$$

or

$$\frac{t_{\max}}{R/V} = F\left(\frac{V}{\sqrt{gR}}\right) \tag{13}$$

for some function F. Remarkably, with little analysis beyond dimensional reasoning, we have found that the time to maximum height depends only on the single quantity $V/\sqrt{gR}$. The value in knowing this kind of information lies in the efficiency of (13); a *single* graph of $t_{\max}/(R/V)$ versus $V/\sqrt{Rg}$ contains all of the data of the graphs of $t_{\max}$ versus V for *all* choices of g and R. For example, an experimenter making measurements on different planets of $t_{\max}$ versus V would not need a separate plot of data for each planet.

The next step in the analysis is to choose characteristic time and length scales and recast the problem represented by (8) and (9) into dimensionless form. For the present problem this is more subtle than it originally appears. The general method requires us to select a new dimensionless dependent variable $\bar{h}$ and independent variable $\bar{t}$ by

$$\bar{t} = \frac{t}{t_c}, \qquad \bar{h} = \frac{h}{h_c} \tag{14}$$

where t_c is an intrinsic time scale and h_c is an intrinsic length scale; the values of t_c and h_c should be chosen by taking combinations of the constants in the problem, which in this case are R, V, and g. This problem presents several choices. For a length scale h_c we could take either R or V^2/g. Possible time scales are R/V, $\sqrt{R/g}$, and V/g. Which choice is the most appropriate? Actually, Equations (14) represent a legitimate transformation of variables for any choice of t_c and h_c; after the change of variables an equivalent problem would result. From a scaling viewpoint, however, one particular choice will be advantageous. The three choices

$$\bar{t} = \frac{t}{R/V}, \qquad \bar{h} = \frac{h}{R} \tag{15}$$

$$\bar{t} = \frac{t}{\sqrt{R/g}}, \qquad \bar{h} = \frac{h}{R} \tag{16}$$

and

$$\bar{t} = \frac{t}{Vg^{-1}}, \qquad \bar{h} = \frac{h}{V^2 g^{-1}} \tag{17}$$

lead to the following three dimensionless problems, which are equivalent to (8) and (9)

$$\epsilon \frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}, \qquad \bar{h}(0) = 0, \qquad \frac{d\bar{h}}{d\bar{t}}(0) = 1 \tag{18}$$

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}, \qquad \bar{h}(0) = 0, \qquad \frac{d\bar{h}}{d\bar{t}}(0) = \sqrt{\epsilon} \tag{19}$$

and

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\epsilon\bar{h})^2}, \qquad \bar{h}(0) = 0, \qquad \frac{d\bar{h}}{d\bar{t}}(0) = 1 \tag{20}$$

respectively, where ϵ is a dimensionless parameter defined by

$$\epsilon = \frac{V^2}{gR}$$

To illustrate how difficulties may arise in selecting an incorrect scaling let us modify our original problem by examining the situation when ϵ is known to be a small quantity; that is, V^2 is much smaller than gR. Then one may be tempted, in order to make an approximation, to delete the terms involving ϵ in the scaled problem. Problem (18) would then become

$$(1+\bar{h})^{-2} = 0, \qquad \bar{h}(0) = 0, \qquad \frac{d\bar{h}}{d\bar{t}}(0) = 1$$

which has no solution, and problem (19) would become

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{(1+\bar{h})^2}, \qquad \bar{h}(0) = 0, \qquad \frac{d\bar{h}}{d\bar{t}}(0) = 0$$

which has no physically valid solution. In the latter case note that the graph of $\bar{h}(\bar{t})$ would pass though the origin with zero slope and be concave down-

ward, thereby making $\bar{h}$ negative. Therefore it appears that terms involving small parameters cannot be neglected. This is indeed unfortunate, since this kind of technique is a common practice in making approximations in applied problems. What went wrong was that (15) and (16) represent *incorrect* scalings; in that case, terms that appear small may in fact not be small. For example, in the term $\epsilon d^2\bar{h}/d\bar{t}^2$ the parameter ϵ may be small but $d^2\bar{h}/d\bar{t}^2$ may be large, and hence the term may not be negligible compared to other terms in the equation.

If, on the other hand, the term $\epsilon\bar{h}$ is neglected in (20), then $d^2\bar{h}/d\bar{t}^2 = -1$, or $\bar{h} = \bar{t} - \bar{t}^2/2$, after applying the initial conditions. Therefore

$$h = -\tfrac{1}{2}gt^2 + Vt$$

and we have obtained an approximate solution that is consistent with our experience with falling bodies. In this case we are able to neglect the small term and obtain a valid approximation because the scaling was correct. That (17) gives the correct time and length scales can be argued physically. If V is small, then the body will be acted on by a constant gravitational field; hence, launched with speed V, it will uniformly decelerate and reach its maximum height in V/g units of time, which is the characteristic time. It will travel a distance of about (V/g) times its average velocity $\frac{1}{2}(V + 0)$, or $V^2/2g$. Hence V^2/g is a good selection for the length scale.

In general, if a correct scaling is chosen, then terms in the equations that appear small are indeed small and may be safely neglected. In fact, one goal of scaling is to select intrinsic, characteristic reference quantities so that each term in the dimensional equation transforms into a term that the dimensionless coefficient in the transformed term represents the order of magnitude or approximate size of that term. Pictorially

$$\begin{bmatrix}\text{Dimensional}\\ \text{term}\end{bmatrix} \rightarrow \begin{bmatrix}\text{coefficient}\\ \text{representing}\\ \text{the order of}\\ \text{magnitude}\\ \text{of the term}\end{bmatrix} \cdot \begin{bmatrix}\text{dimensionless}\\ \text{factor}\\ \text{of order unity}\end{bmatrix}$$

By *order of unity* we mean a term that is neither extremely large nor small. In the next section this notion is made precise.

Population Models

We end this section with an example from biology where we formulate and scale a simple model of population dynamics in a closed ecosystem. Such models play an important role in understanding the factors that contribute to population growth. The simplest model is to assume that the rate of population changes

is proportional to the number of individuals present; this is the classic Malthus model. To be precise, let $p = p(t)$ denote the population at time t. Then the Malthus model is that dp/dt is proportional to p, or

$$\frac{dp}{dt} = rp$$

where r is the constant *growth rate*, given in inverse-time units. Easily this model predicts that the population will grow exponentially for all time, that is, $p = p_0 \exp(rt)$, where p_0 is the initial population. Many books on an impending world population explosion have been written with the Malthusian model as a premise. Clearly, however, as a population grows competition for food, living space, and natural resources will limit the growth. Therefore one might think of modifying the Malthus model to include a competition term. The simplest approach is to notice that if there are p individuals in the system, then the number of encounters, which is a measure of competition, can be modeled by p^2; therefore we subtract a term proportional to p^2 from the growth rate to obtain the model

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right), \quad p(0) = p_0 \tag{21}$$

Here K is the *carrying capacity*, a constant that is interpreted as the number of individuals that the ecosystem can sustain. This model is called the *logistics model*; as the population grows, the negative p^2 term will kick in and limit the growth. Later we shall show that over a long time the population will approach the carrying capacity K.

To reduce (21) to dimensionless form we need to select new dimensionless variables (both independent and dependent). The time scale and population scale are formed from the constants in the problem, r, K, and p_0. Of these, only r contains the dimensions of time, and therefore we scale time by $1/r$ giving a new, dimensionless time τ defined by

$$\tau = rt$$

There are two choices for the population scale, K or p_0. Either will do, and so we select K to obtain a dimensionless population P given by

$$P = \frac{p}{K}$$

Thus, the interpretation is that we are measuring population in the problem relative to the carrying capacity. Using these variables it is straightforward to

obtain the

$$\frac{dP}{d\tau} = P(1 - P), \qquad P(0) = \alpha \tag{22}$$

where $\alpha \equiv p_0/K$ is a dimensionless constant. The scaled model (22) has only one constant (α), a significant simplification over (21) where there are three constants (r, K, p_0). The constant α represents a scaled, initial population.

The initial value problem (22) can be solved by separating variables to obtain

$$P(\tau) = \frac{\alpha}{\alpha + (1 - \alpha)e^{-\tau}} \tag{23}$$

Clearly

$$\lim_{\tau \to \infty} P(\tau) = 1$$

It follows that, confirming our earlier statement, the limiting population p is equal to the carrying capacity K. We observe that there are two *equilibrium populations* [constant solutions of (21)], $p = K$ and $p = 0$. The population $p = K$ is an *attractor*; that is, regardless of the initial population, the population $p(t)$ tends to the value K as time gets large.

A large number of interesting population models is contained in the book by Murray [3].

Exercises

3.1 Let $u = u(t)$, $0 \le t \le b$ be a given continuous function. If $M = max|u(t)|$, then u can be scaled by M to obtain the dimensionless dependent variable $U = u/M$. A time scale can be taken as $t_c = M/\max|u'(t)|$, the ratio of the maximum value of the function to the maximum slope. Find M and t_c for the following functions:

(a) $u(t) = A \sin \omega t$, $t > 0$.

(b) $u(t) = Ae^{-\lambda t}$, $t > 0$.

(c) $u(t) = Ate^{-\lambda t}$, $0 \le t \le 2/\lambda$.

3.2 Consider the function $u(t) = 1 + e^{-t/\epsilon}$ on the interval $0 \le t \le 1$, where ϵ is a small number. Use Exercise 3.1 to determine a time scale. Is this time scale appropriate for the entire interval [0, 1]? (Sketch a graph of $u(t)$ when $\epsilon = 0.05$). Explain why two time scales might be required for a process described by $u(t)$.

3.3 In a classic work modeling the outbreak of the spruce budworm in Canada's balsam fir forests, researchers proposed that the budworm population $n = n(t)$ was governed by the law

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right) - P(n)$$

where r and K are the growth rate and carrying capacity, respectively, and $P(n)$ is a *predation* term (think of birds eating the budworms) given by

$$P(n) = \frac{bn^2}{a^2 + n^2}$$

where a and b are positive constants.

(a) Determine the dimensions of the constants a and b.

(b) Graph the predation rate $P(n)/b$ for $a = 1, 5, 10$ and make a qualitative statement about the effect that the parameter a has on the model.

(c) Select dimensionless variables $N = n/a$ and $\tau = t/(a/b)$ and reduce the differential equation to dimensionless form (introduce constants $q \equiv K/a$ and $s \equiv ar/b$).

(d) Researchers are often interested in equilibrium populations (constant solutions of the differential equation). Working with the dimensionless model, and ignoring the zero population, show that there is always at least one equilibrium population, and show that there may be two or three, depending on the values of the parameters. Find the equilibrium populations when $q = 12$ and $s = 0.25$, and when $q = 35$ and $s = 0.4$.

(e) In the case $q = 35$ and $s = 0.4$ use a numerical differential equations solver to graph the population curve when the initial population is given by $N(0) = 40$, 25, 2, and 0.05, respectively.

3.4 A uniform bar of mass m and length l is connected by a pin joint to a vertical shaft that rotates with angular velocity ω (see Fig. 1.4). The governing differential equation for the angle θ of deflection is

$$\frac{d^2\theta}{dt^2} + \left(\frac{3g}{2l} - \omega^2\right) \sin \theta = 0$$

where g is the constant acceleration due to gravity.

(a) Determine two time scales for the motion and put the differential equation in dimensionless form using each of the time scales (do not scale θ, which is already dimensionless).

(b) Which of the two time scales is appropriate when ω is small? When ω is large?

4

PARTIAL DIFFERENTIAL EQUATION MODELS

The reader has encountered many physical problems that are modeled by ordinary differential equations and from elementary courses has been exposed to some of the basic solution techniques for such equations. In this chapter we expand our view and begin to examine partial differential equations and some of the standard methods for solving them, including eigenfunction expansions and transform methods. We focus here on diffusion problems and equilibrium problems, saving wave propagation problems to Chapter 6, where we study mathematical models of continuous media.

4.1 PARTIAL DIFFERENTIAL EQUATIONS

Definitions

Partial differential equations are one of the fundamental areas of applied analysis, and it is hard to imagine any area of scientific endeavor where the impact of this subject is not felt. The basic issues in the subject traditionally deal with (i) existence and uniqueness of solutions, (ii) stability of solutions to small perturbations, and (iii) methods for constructing solutions. Here we focus on the latter with the advisory that only rarely is it possible to solve a partial differential equation in closed form, that is, find a formula for its solution. For special linear problems Fourier analysis and integral transform methods often lead to an infinite series or integral representation of the solution. But for the majority of problems approximation techniques or numerical methods are more in order. In this chapter we present some of the basic methods and results while maintaining contact with the origins in empirical science. Our aim is to present partial

differential equations in a broad context from both application and solutions points of view.

Many physical problems lead to ordinary differential equations, and several have been encountered in the first three chapters or in previous experience. In general an ordinary differential equation of second order is an equation of the form

$$F(t, y, \dot{y}, \ddot{y}) = 0$$

where t ranges over some interval I. A solution is a twice continuously differentiable function $y(t)$ of the single independent variable t that reduces the above equation to an identity on I, that is,

$$F(t, y(t), \dot{y}(t), \ddot{y}(t)) = 0, \qquad t \in I$$

We can regard y as a state function for the system and the differential equation as an equation that governs the evolution of the state y in time.

Many physical processes, however, cannot be modeled by ordinary differential equations because the state of the system depends on more than one independent variable. For example, the state u of a given physical system may depend on time t and a location x. Hence the system evolves in both space and time. For example, the temperature u in a bar of length l depends on the location x in the bar and the time t from when the initial conditions were applied. We shall see later that $u = u(x, t)$ has to satisfy a partial differential equation, the so-called heat equation

$$u_t - ku_{xx} = 0, \qquad t > 0, \quad 0 < x < l$$

where the constant k is the diffusivity of the bar.

In general, a *second-order partial differential equation* in two independent variables is an equation of the form

$$G(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0 \tag{1}$$

where (x, t) lies in some domain D in R^2. By a *solution* we mean a twice continuously differentiable function $u = u(x, t)$ on D, which when substituted into (1), reduces it to an identity for (x, t) in D. We assume that u is twice continuously differentiable so that it makes sense to calculate the second-order derivatives and substitute them into (1). A solution of (1) may be represented graphically as a smooth surface in three dimensional xtu space lying above the domain D in the xt plane as shown in Fig. 4.1. Here we regard x as a position or spatial coordinate and t as time. The domain D in R^2 where the problem is defined is referred to as a space-time domain and problems that include time as an independent variable are called *evolution problems*. When two spatial coordi-

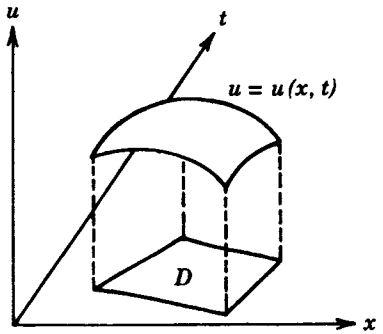

Figure 4.1. Solution surface $u = u(x, t)$ in xtu space.

nates, say x and y, are the independent variables we refer to the problem as an *equilibrium* or *steady-state problem.*

A partial differential equation of type (1) has infinitely many solutions. Just as the general solution of an ordinary differential equation depends on arbitrary constants, the general solution of a partial differential equation depends on arbitrary functions.

Example 1.1 Consider the simple partial differential equation

$$u_{tx} = tx$$

Integrating with respect to x gives

$$u_t = \tfrac{1}{2}tx^2 + f(t)$$

where f is an arbitrary function. Integrating with respect to t gives the general solution

$$u = \tfrac{1}{4}t^2x^2 + g(t) + h(x)$$

where h is an arbitrary function and $g(t) = \int f(t)dt$ is also arbitrary. Thus the general solution depends on two arbitrary functions; any choice of g and h will yield a solution.

For ordinary differential equations initial or boundary conditions fix the arbitrary constants of integration and thus often pick out a unique solution. Partial differential equations are usually accompanied by initial or boundary conditions that again select out one of the many solutions of the partial differential equation. A condition given at $t = 0$ along some segment of the x axis is called an *initial condition.* A condition given along any other curve in the xt plane is

called a *boundary condition.* Initial or boundary conditions may involve specifying values of u, its derivatives, or combinations of both along the given curves in the xt plane. Partial differential equations with auxiliary conditions are called *boundary value problems.*

Example 1.2 Heat flow in a bar of length l is governed by the partial differential equation

$$u_t - ku_{xx} = 0, \qquad t > 0, \quad 0 < x < l$$

where k is a physical constant and $u = u(x,t)$ is the temperature in the bar at location x at time t. An auxiliary condition of the form

$$u(x, 0) = f(x), \qquad 0 < x < l$$

is an initial condition, since it is given at $t = 0$. We regard $f(x)$ as the initial temperature distribution in the bar. Conditions of the form

$$u(0, t) = h(t), \qquad u(l, t) = g(t), \qquad t > 0$$

are boundary conditions, and $h(t)$ and $g(t)$ represent the given temperatures held at the boundaries $x = 0$ and $x = l$ for $t > 0$, respectively. These functions are represented in Fig. 4.2. Physically, we infer that there is a unique solution to the partial differential equation subject to these initial and boundary conditions; graphically the surface representing the solution would have f, g, and h as its boundaries.

Frequently of interest is a so-called *snapshot* of the solution frozen in time, or in other words, a graph of $u(x, t_0)$ for some fixed t_0. Figure 4.2 indicates such

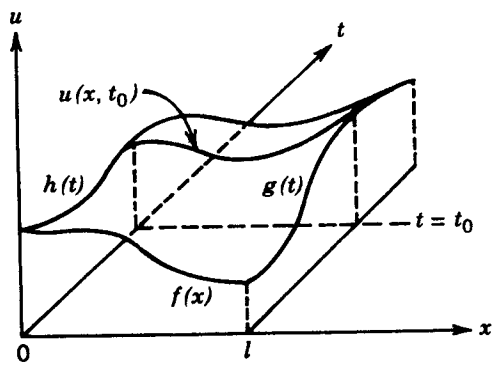

Figure 4.2. Temperature surface $u = u(x, t)$ bounded by initial temperature distribution $f(x)$ and boundary temperatures $h(t)$ and $g(t)$ at $x = 0$ and $x = l$. A time snapshot $u(x, t_0)$ is shown at time t_0.

a snapshot or time cross section of the solution surface. One way to view a solution $u(x,t)$ geometrically is to graph a sequence of snapshots $u(x,t_1), u(x,t_2), \ldots$, for $t_1 < t_2 < \ldots$. For example, Fig. 4.3 depicts a sequence of temperature profiles in a bar of length l whose ends are held at zero degrees and whose initial temperature profile is $f(x) = x(l-x)$.

The general solution of a partial differential equation is frequently difficult to find. Thus for partial differential equations we seldom solve a boundary value problem by determining the general solution and then finding the arbitrary functions from the initial and boundary data. This is in sharp contrast to ordinary differential equations where the general solution is found and the arbitrary constants are evaluated from the initial or boundary conditions.

Generalizations of the partial differential equation (1) can be made in various directions. Higher-order derivatives, several independent variables, and several unknown functions (governed by several equations) are all possibilities. As we shall see, there are fundamentally three types of partial differential equations—those that govern diffusion processes, those that govern wave propagation, and those that model equilibrium processes. These types are called *parabolic*, *hyperbolic*, and *elliptic*, respectively.

Linearity versus Nonlinearity

In the next few paragraphs we introduce the ideas of linearity and nonlinearity. The separation of partial differential equations and ordinary differential equations into these two distinct classes is a significant one. Besides being generally easier to solve, linear equations have a linear algebraic structure to their solution set, that is, the sum of two solutions of a linear homogeneous equation is again a solution, as is a constant multiple of a solution. Such is not the case for nonlinear equations. The sum of two solutions or a constant multiple of a solution may not be a solution. For nonlinear equations solutions do not add or superimpose. Superposition for linear equations often allows one to construct a variety of solutions that can meet diverse boundary or initial requirements.

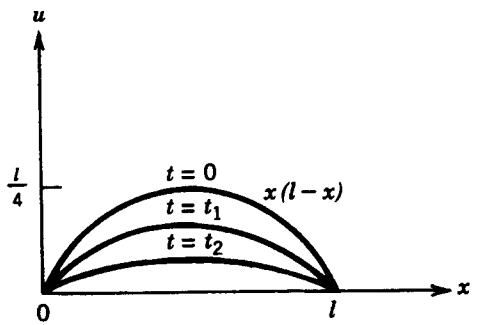

Figure 4.3. Temperature profiles or snapshots at $t = 0, t_1, t_2$.

In fact, this observation is the basis of the Fourier method or method of eigenfunction expansions for linear equations. Linear equations are also susceptible to transform methods for finding solutions. These include methods based on Laplace transforms and Fourier transforms. In summary, there is a profound difference between these two classes of problems (see, e.g., Logan [3] for a treatment of nonlinear equations).

To formulate the concepts more precisely we regard the partial differential equation (1) as defining a differential operator L acting on the unknown function $u(x,t)$ and we write (1) as

$$Lu(x,t) = f(x,t), \qquad (x,t) \in D$$

or suppressing the independent variables,

$$Lu = f, \qquad (x,t) \in D \tag{2}$$

In (2) all terms involving u are put on the left in the term Lu, and f is a known function. If $f = 0$ on D, then (2) is *homogeneous*; if f is not identically zero, then (2) is *nonhomogeneous*. The heat equation $u_t - ku_{xx} = 0$ can be written $Lu = 0$, where L is the partial differential operator $\partial/\partial t - k\partial^2/\partial x^2$, and it is clearly homogeneous. The partial differential equation $uu_t + 2txu - \sin tx = 0$ can be written $Lu = \sin tx$, where L is the differential operator defined by $Lu = uu_t + 2txu$. This equation is nonhomogeneous. The definition of linearity depends on the operator L in (2). We say that (2) is a *linear equation* if L has the properties

(i) $L(u+w) = Lu + Lw$.
(ii) $L(cu) = cLu$.

where u and w are functions and c is a constant. If (2) is not linear, then it is *nonlinear*.

Example 1.3 The heat equation is linear since

$$\begin{aligned} L(u+w) &= (u+w)_t - k(u+w)_{xx} \\ &= u_t + w_t - ku_{xx} - kw_{xx} \\ &= Lu + Lw \end{aligned}$$

and

$$\begin{aligned} L(cu) &= (cu)_t - k(cu)_{xx} \\ &= cu_t - cku_{xx} \\ &= cLu \end{aligned}$$

Example 1.4 The differential equation $uu_t + 2txu = \sin tx$ is nonlinear since

$$\begin{aligned} L(u+w) &= (u+w)(u+w)_t + 2tx(u+w) \\ &= uu_t + wu_t + ww_t + uw_t + 2txu + 2txw \end{aligned}$$

but

$$Lu + Lw = uu_t + 2txu + ww_t + 2txw$$

Note that the nonhomogeneous term $\sin tx$ does not affect linearity or nonlinearity.

It is clear that $Lu = f$ is linear if Lu is first-degree in u and its derivatives; that is, no products involving u and its derivatives occur. Hence the most general *linear equation of second order* is of the form

$$a(x,t)u_{tt} + b(x,t)u_{xt} + c(x,t)u_{xx} + d(x,t)u_t + e(x,t)u_x + g(x,t)u = f(x,t), \quad (x,y) \in D \tag{3}$$

where the functions a, b, c, d, e, g, and f are given continuous functions on D. If any of the coefficients $a, \ldots, g$ depend on u, we say that the equation is *quasi linear*.

Referring to an earlier remark, we say that (3) is *hyperbolic*, *parabolic*, or *elliptic* on a domain D if $b(x,t)^2 - 4a(x,t)c(x,t)$ is positive, zero, or negative, respectively, on that domain. For example, the heat equation $u_t - ku_{xx} = 0$ has $b^2 - 4ac = 0$ and is parabolic on all of R^2.

Superposition

If $Lu = 0$ is a linear homogeneous equation and u_1 and u_2 are two solutions, then it obviously follows that $u_1 + u_2$ is a solution, since $L(u_1 + u_2) = Lu_1 + Lu_2 = 0 + 0 = 0$. Also cu_1 is a solution, since $L(cu_1) = cLu_1 = c \cdot 0 = 0$. A simple induction argument shows that if $u_1, \ldots u_n$ are solutions of $Lu = 0$ and $c_1, \ldots, c_n$ are constants, then the finite sum or linear combination $c_1u_1 + \cdots + c_nu_n$ is also a solution; this is the *superposition principle* for linear equations. If certain convergence properties hold, the superposition principle can be extended to infinite sums $c_1u_1 + c_2u_2 + \cdots$.

Another form of a superposition principle is a continuous version of the one just cited. In this case let $u(x,t,\alpha)$ be a family of solutions on D, where α is a real number ranging over some interval Γ. That is, suppose $u(x,t,\alpha)$ is a solution of $Lu = 0$ for *each* value of $\alpha \in \Gamma$. Then *formally*† we may superimpose these solutions by forming

†Generally, a *formal* calculation in mathematics is one that lacks complete rigor but that can usually be justified under certain circumstances.

$$u(x,t) = \int_\Gamma c(\alpha)u(x,t,\alpha)\,d\alpha$$

where $c(\alpha)$ is a function representing a continuum of coefficients, the analog of $c_1, c_2, \ldots, c_n$. If we can write

$$\begin{aligned} Lu &= L\int_T c(\alpha)u(x,t,\alpha)\,d\alpha \\ &= \int_\Gamma c(\alpha)Lu(x,t,\alpha)\,d\alpha \\ &= \int_\Gamma c(\alpha)\cdot 0\,d\alpha = 0 \end{aligned}$$

then u is also a solution. This sequence of steps, and hence the superposition principle, depends on the validity of pulling the differential operator L inside the integral sign. For such a superposition principle to hold, these formal steps would need careful analytical verification.

Example 1.5 Consider the heat equation

$$u_t - ku_{xx} = 0, \qquad t > 0, \quad x \in R^1 \tag{4}$$

It is straightforward to verify that

$$u(x,t;\alpha) = \frac{1}{\sqrt{4\pi kt}}\exp\left(-\frac{(x-\alpha)^2}{4kt}\right), \qquad t > 0, \quad x \in R^1$$

is a solution of (4) for any $\alpha \in R^1$. This solution is called the *fundamental solution*. We may formally superimpose these solutions to obtain

$$u(x,t) = \int_{-\infty}^{\infty} c(\alpha)\,\frac{1}{\sqrt{4\pi kt}}\exp\left(-\frac{(x-\alpha)^2}{4kt}\right)d\alpha$$

where $c(\alpha)$ is some function. It can be shown that if $c(\alpha)$ is continuous and bounded, then differentiation under the integral sign can be justified and $u(x,t)$ is therefore a solution (see John [2]).

Exercises

1.1 By direct integration find the general solution of the following partial differential equations in terms of arbitrary functions.

(a) $u_x = 3xt + 4$, where $u = u(x, t)$.

(b) $u_{xx} = 6xy$, where $u = u(x, y)$.

(c) $u_{xy} + (1/x)u_y = y/x^2$, where $u = u(x, y)$. (Let $v = u_y$ and solve the resulting first-order equation for v by treating y as a parameter.)

(d) $u_{yx} + u_x = 1$, where $u = u(x, y)$.

1.2 By introducing polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, find the general solution of the equation

$$yu_x - xu_y = 0$$

Solution: $u = f(x^2 + y^2)$, where f is an arbitrary function.

1.3 Determine regions where the following equations are hyperbolic, elliptic, or parabolic.

(a) $tu_{tt} + u_{xx} = 0$ **(c)** $u_{tt} + (1 + x^2)u_x - u_t = e^t$

(b) $u_{tt} - u_{xx} = 0$ **(d)** $u_{xx} + u_{yy} = f(x, y)$

1.4 Determine whether the following equations are linear or nonlinear.

(a) $u_t u_{tt} + 3tu = 0$ **(b)** $u_{tt} - u_{xx} = 0$

(c) $\exp(t)u_{tx} - x^2u = \cos t$ **(d)** $u_{tx} + u^2 = \sin x$

1.5 Find all solutions of the equation $u_t = u_{xx}$ of the form $u(x, t) = U(z)$, where $z = x/\sqrt{t}$.

1.6 Sketch snapshots of the fundamental solution of the heat equation

$$u(x, t; 0) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right)$$

for different times t. What do the temperature profiles look like as t approaches zero? Comment on the differences one would observe in the profiles for large k and for small k.

4.2 DIFFUSION EQUATIONS

Conservation Laws

Many of the fundamental equations occurring in the natural and physical sciences are obtained from *conservation laws*. Conservation laws are just balance laws, equations expressing the fact that some quantity is balanced throughout a process.

In thermodynamics, for example, the first law states that the change in internal energy in a given system is equal to, or is balanced by, the total heat added to the system plus the work done on the system. Thus the first law of thermodynamics is really an energy balance law, or conservation law. As another example, consider a fluid flowing in some region of space that consists of chemical species undergoing chemical reaction. For a given chemical species, the time rate of change of the total amount of that species in the region must equal the rate at which the species flows into the region, minus the rate at which the species flows out, plus the rate at which the species is created, or consumed, by the chemical reactions. This is a verbal statement of a conservation law for the amount of the given chemical species. Similar balance or conservation laws occur in all branches of science. In the biosciences, for example, the rate of change of a given animal population in a certain region must equal the birth rate, minus the death rate, plus the migration rate into or out of the region.

Mathematically, conservation laws usually translate into differential equations, which are then regarded as the *governing equations* or *equations of motion* of the process. These equations dictate how the process evolves in time. Here we are interested in processes governed by partial differential equations. We now formulate the basic one-dimensional conservation law, out of which will evolve some of the basic models and concepts.

Let us consider a quantity $u = u(x, t)$ that depends on a single spatial variable $x \in R$ and time $t > 0$. We assume that u is a density or concentration measured in an amount per unit volume, where the amount may refer to population, mass, energy, or any quantity. By definition, u varies in only one spatial direction, the direction denoted by x. We imagine further that the quantity is distributed in a tube of cross-sectional area A (see Fig. 4.4). Again, by assumption, u is constant in any cross section of the tube, and the variation is only in the x direction. Now consider an arbitrary segment of the tube denoted by the interval $I = [a, b]$. The total amount of the quantity u inside I at time t is

$$\text{Total amount of quantity in } I = \int_a^b u(x, t)A\,dx$$

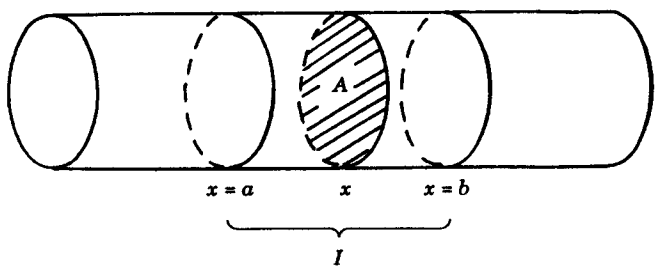

Figure 4.4. Cylindrical tube of cross-sectional area A showing a cross section at x and a finite section I: $a \le x \le b$.

Now assume that there is motion of the quantity in the tube in the axial direction. We define the *flux* of u at x at time t to be the scalar function $\phi(x,t)$; that is, $\phi(x,t)$ is the amount of the quantity u flowing through the cross section at x at time t, per unit area, per unit time. Thus the dimensions of ϕ are $[\phi]$ = amount/(area · time), where the bracket notation denotes *dimensions of*. By convention, we take ϕ to be positive if the flow at x is in the positive x direction, and ϕ is negative at x if the flow is in the negative x direction. Therefore, at time t the net rate that the quantity is flowing into the interval I is the rate that it is flowing in at $x = a$ minus the rate that it is flowing out at $x = b$. That is,

$$\text{Net rate that the quantity flows into } I = A\phi(a,t) - A\phi(b,t)$$

Finally, the quantity u may be created or destroyed inside I by some external or internal source (e.g., by a chemical reaction if u were a species concentration, or by birth or death if u were a population density). We denote this *source function*, which is a local function acting at each x, by $f(x,t,u)$ and its dimensions are given by $[f]$ = amount/(volume · time). Consequently, f is the rate that u is created (or destroyed) at x at time t, per unit volume. Note that the source function f may depend on u itself, as well as space and time. If f is positive, we say that it is a *source*, and if f is negative, we say that it is a *sink*. Now, given f, we may calculate the total rate that u is created in I by integration. We have

$$\text{Rate that quantity is produced in } I \text{ by sources} = \int_a^b f(x,t,u(x,t))A\,dx$$

The fundamental conservation law may now be formulated for the quantity u: For any interval I, we have

$$\begin{aligned}\text{Time rate of change of the total amount in } I \\ = \text{net rate that the quantity flows into } I \\ + \text{rate that the quantity is produced in } I\end{aligned}$$

In terms of the mathematical symbols and expressions that we introduced above, we have, after canceling the constant cross-sectional area A,

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b f(x,t,u)\,dx \qquad (1)$$

In summary, (1) states that the rate that u changes in I must equal the net rate at which u flows into I plus the rate that u is produced in I by sources. Equation

(1) is called a *conservation law in integral form*, and it holds even if u, ϕ, or f are not smooth (continuously differentiable) functions. The latter remark is important when we consider in subsequent chapters physical processes giving rise to shock waves, or discontinuous solutions.

If some restrictive conditions are place on the triad u, ϕ, and f, (1) may be transformed into a single partial differential equation. Two results from elementary integration theory are required to make this transformation: (i) the fundamental theorem of calculus, and (ii) the result on differentiating an integral with respect to a parameter in the integrand. Precisely,

(i) $\int_a^b \phi_x(x,t)\,dx = \phi(b,t) - \phi(a,t)$.
(ii) $d/dt \int_a^b u(x,t)\,dx = \int_a^b u_t(x,t)\,dx$.

These two results are valid if ϕ and u are continuously differentiable functions on R^2. Of course, (i) and (ii) remain correct under less stringent conditions, but our assumption of smoothness is all that is required in the subsequent discussion. Therefore, assuming smoothness of u and ϕ, as well as continuity of f, equations (i) and (ii) imply that the conservation law (1) may be written

$$\int_a^b [u_t(x,t) + \phi_x(x,t) - f(x,t,u)]\,dx = 0 \qquad \text{for all intervals } I = [a,b] \qquad (2)$$

Because the integrand is a continuous function of x, and because (2) holds for all intervals of integration I, it follows that the integrand must vanish identically; that is,

$$u_t + \phi_x = f(x,t,u), \qquad x \in R, \quad t > 0 \qquad (3)$$

Equation (3) is a partial differential equation relating the density $u = u(x,t)$ and the flux $\phi = \phi(x,t)$. Both are regarded as unknowns, whereas the source function f is assumed to be given. Equation (3) is called a *conservation law in differential form*, in contrast to the integral form (1). The ϕ_x term is called the *flux term* since it arises from the movement, or transport, of u through the cross section at x. The source term f is sometimes called a *reaction term* (especially in chemical contexts) or a *growth* or *interaction* term (in biological contexts). To be consistent with standard usage of terminology, (3) is frequently called a conservation law when the source term f is absent; in that case we should call (3) a conservation law with sources. We shall always make clear to what context we are referring. Finally, we have defined the flux ϕ as a function of x and t; it may happen that this dependence on space and time may occur through dependence on u or its derivatives. For example, a physical assumption may require us to posit $\phi(x,t) = \phi(x,t,u(x,t))$, where the flux is dependent on u itself.

Conservation in Higher Dimensions

It is straightforward to formulate conservation laws in higher dimensions. In this section we limit the discussion to three-dimensional Euclidean space R^3. For notation, we let $x = (x_1, x_2, x_3)$ denote a point in R^3, and we assume that $u = u(x, t)$ is a scalar density function representing the amount per unit volume of some quantity of interest distributed throughout some domain in R^3. In this domain let V be an arbitrary region, and assume that V has a smooth boundary, which is denoted by ∂V. It follows that, similar to the one-dimensional case, the total amount of the quantity in V is given by the volume integral

$$\text{Total amount in } V = \int_V u(x,t)\, dx$$

where $dx = dx_1\, dx_2\, dx_3$ represents a volume element in R^3. We prefer to write the volume integral over V with a single integral sign rather than the usual triple integral. Now, we know that the time rate of change of the total amount in V must be balanced by the rate that the quantity is produced in V by sources, plus the net rate that the quantity flows through the boundary of V. We let $f(x, t, u)$ denote the source term, so that the rate that the quantity is produced in V is given by

$$\text{Rate that } u \text{ is produced by sources} = \int_V f(x,t,u)\, dx$$

In three dimensions the flow can be in any direction, and therefore the flux is given by a vector $\boldsymbol{\phi}(x, t)$. If $\mathbf{n}(x)$ denotes the outward unit normal vector to the region V (see Fig. 4.5), the net outward flux of the quantity u through the boundary ∂V is given by the surface integral

$$\text{Net outward flux through } \partial V = \int_{\partial V} \boldsymbol{\phi}(x,t) \cdot \mathbf{n}(x)\, dS$$

where dS denotes a surface element on ∂V. Finally, therefore, the conservation law, or balance law for u, is given by

$$\frac{d}{dt} \int_V u\, dx = -\int_{\partial V} \boldsymbol{\phi} \cdot \mathbf{n}\, dS + \int_V f\, dx \tag{4}$$

The minus sign on the flux term occurs because outward flux decreases the rate that u changes in V.

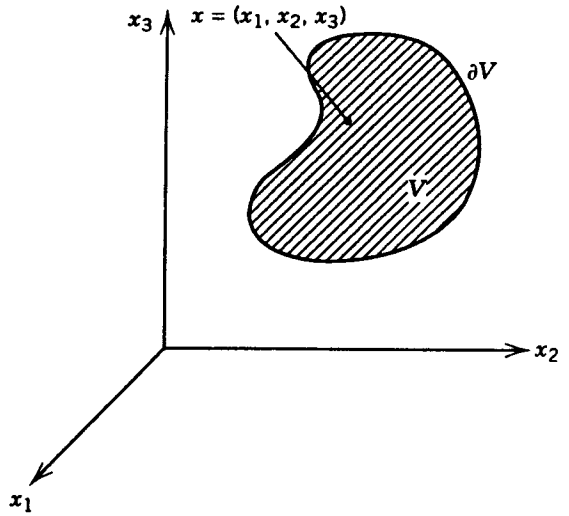

Figure 4.5. Volume V with boundary ∂V showing a surface element dS with outward normal $\mathbf{n}$ and flux vector $\boldsymbol{\phi}$.

The integral form of the conservation law (4) can be reformulated as a local condition, that is, partial differential equation (PDE), provided that u and $\boldsymbol{\phi}$ are sufficiently smooth functions. In this case the surface integral can be written as a volume integral over V using the *divergence theorem* (the divergence theorem is the fundamental theorem of calculus in three dimensions). The divergence theorem is embodied in the expression

$$\int_V \operatorname{div} \boldsymbol{\phi}\, dx = \int_{\partial V} \boldsymbol{\phi} \cdot \mathbf{n}\, dS \tag{5}$$

where div is the divergence operator. Using (5) and bringing the derivative under the integral on the left side of (4) yields

$$\int_V u_t\, dx = -\int_V \operatorname{div} \boldsymbol{\phi}\, dx + \int_V f\, dx$$

Using the arbitrariness of V then gives the differential form of the balance law as

$$u_t + \operatorname{div} \boldsymbol{\phi} = f(x,t,u), \qquad x \in V, \quad t > 0 \tag{6}$$

Equation (6) is the three-dimensional version of equation (3), the conservation law in one dimension.

Constitutive Equations

Because equation (3) [or (6)] is a single PDE for two unknown quantities (the density u and the flux ϕ), intuition indicates that another equation is required in order to have a well-determined system. This additional equation is often an equation that is based on an assumption about the physical properties of the medium, which, in turn, is based on empirical reasoning. Equations expressing these assumptions are called *constitutive relations* or *equations of state*. Thus constitutive equations are on a different level from the basic conservation law; the latter is a fundamental law of nature connecting the density u to the flux ϕ, whereas a constitutive relation is often an approximate equation whose origin is in empirics.

Example 2.1 (Diffusion Equation). At the outset assume that no sources are present ($f = 0$) and the process is governed by the basic conservation law in one dimension

$$u_t + \phi_x = 0, \qquad x \in R, \qquad t > 0 \tag{7}$$

In many physical problems it is observed that the amount of the substance represented by the density u that flows through a cross section at x at time t is proportional to the density gradient u_x, [i.e., $\phi(x,t) \propto u_x(x,t)$]. If $u_x > 0$, then $\phi < 0$ (the substance flows to the left), and if $u_x < 0$, then $\phi > 0$ (the substance flows to the right). Figure 4.6 illustrates the situation. For example, by the second law of thermodynamics, heat behaves in this manner; heat flows from hotter regions to colder regions, and the steeper the temperature distribution curve, the more rapid the flow of heat. As another example, if u represents a concentration of insects, one might observe that insects move from high concentrations to low concentrations with a rate proportional to the concentration

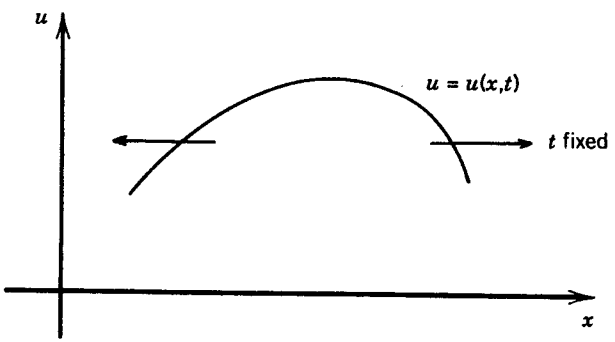

Figure 4.6. Time snapshot of the density distribution $u(x, t)$ illustrating Fick's law. The arrows indicate the direction of the flow, from higher concentrations to lower concentrations. The flow is said to be *down the gradient.*

gradient. Therefore, we assume the basic constitutive law

$$\phi(x,t) = -Du_x(x,t) \tag{8}$$

which is known as *Fick's law*. The positive proportionality constant D, called the *diffusion constant*, has dimensions given by $[D] = \text{length}^2/\text{time}$. Fick's law accurately describes the behavior of many physical and biological systems.

Equations (7) and (8) give a pair of PDEs for the two unknowns u and ϕ. They combine easily to form a single second-order linear PDE for the unknown density $u = u(x,t)$ given by

$$u_t - Du_{xx} = 0 \tag{9}$$

Equation (9), called the *diffusion equation*, governs conservative processes when the flux is specified by Fick's law. It may not be clear to the novice at this time why (9) should be termed the *diffusion equation*; suffice it to note for the moment that Fick's law seems to imply that a substance leaks (diffuses) into adjacent regions because of concentration gradients. We refer to the Du_{xx} term in (9) as the *diffusion term.*

We remark that the diffusion constant D defines a characteristic time (or time scale) T for the process. If L is a length scale (e.g., the length of the container), the quantity

$$T = \frac{L^2}{D}$$

is the only constant in the process with dimensions of time, and T gives a measure of the time required for discernible changes in concentration to occur.

Example 2.2 (Classical Heat Equation). In the case where u is an energy density (i.e., a quantity with dimensions of energy per unit volume), the diffusion equation describes the diffusion of energy in a one-dimensional medium. For example, in a homogeneous medium whose *density* is ρ and whose *specific heat* (at constant volume) is C, the energy density is given by

$$u(x,t) = C\rho T(x,t)$$

where T is the *temperature*. Recall that $[C] = \text{energy}/(\text{mass} \cdot \text{degree})$ and $[\rho] = \text{mass}/\text{volume}$. The conservation law may therefore be written

$$C\rho T_t + \phi_x = 0$$

In heat conduction, Fick's law has the form

$$\phi = -KT_x(x,t) \tag{10}$$

where K is the *thermal conductivity* [measured in energy/(mass · time · degree)]. In the context of heat conduction, Fick's law is called *Fourier's law of heat conduction*. Therefore, it follows that the temperature T satisfies the PDE

$$C\rho T_t - KT_{xx} = 0$$

or

$$T_t - kT_{xx} = 0, \qquad k = \frac{K}{C\rho} \tag{11}$$

Equation (11) is called the *heat equation*, and the constant k, which plays the role of the diffusion constant, is called the *diffusivity*. Thus the diffusivity k in heat flow problems is the analog of the diffusion constant, and the heat equation is just the diffusion equation. In this discussion we have assumed that the physical parameters C, K, and ρ of the medium are constant; however, these quantities could depend on the temperature T, which is an origin of nonlinearity in the problem.

Example 2.3 (Reaction-Diffusion Equation). If sources are present ($f \neq 0$), the conservation law

$$u_t + \phi_x = f(x,t,u)$$

and Fick's law (8) combine to give

$$u_t - Du_{xx} = f(x,t,u)$$

which is called a *reaction-diffusion equation*. Reaction-diffusion equations may be nonlinear if the source term f (or, as it is also called, the reaction term) is nonlinear in u. These equations are of great interest in nonlinear analysis and applications, particularly in combustion processes and in biological systematics. The following example introduces Fisher's equation, which is an important reaction-diffusion equation.

Example 2.4 (Fisher Equation). In studies of elementary population dynamics it is often proposed that a population is governed by the logistics law, which states that the rate of change of a population $u = u(t)$ is given by

$$\frac{du}{dt} = ru\left(1 - \frac{u}{K}\right)$$

where $r > 0$ is the *growth rate* and $K > 0$ is the *carrying capacity*.

Now suppose that the population u is a population density (population per unit volume) and depends on a spatial variable x as well as time t [i.e., $u = u(x,t)$]. Then, as in the preceding discussion, a conservation law may be formulated as

$$u_t + \phi_x = ru\left(1 - \frac{u}{K}\right)$$

where $f = f(u) = ru(1 - u/K)$ is the assumed local source term given by the logistics growth law, and ϕ is the population flux. Assuming Fick's law for the flux, we have

$$u_t - Du_{xx} = ru\left(1 - \frac{u}{K}\right) \tag{12}$$

The reaction-diffusion equation (12) is known as *Fisher's equation*, after R. A. Fisher, who studied the equation in the context of investigating the distribution of an advantageous gene in a given population.

The Heat Equation

In the context of heat conduction, we now discuss the types of auxiliary conditions that lead to a well-defined problem that has a physically meaningful solution. Consider a bar of length l and constant cross-sectional area A. By Example 2.2 the temperature $u = u(x,t)$ (we shall use u as the dependent variable rather than T) must satisfy the one-dimensional heat equation

$$u_t - ku_{xx} = 0, \qquad 0 < x < l, \quad t > 0 \tag{13}$$

where k is the diffusivity.

An auxiliary condition on the temperature u at time $t = 0$ is called an *initial condition* and is of the form

$$u(x,0) = f(x), \qquad 0 < x < l \tag{14}$$

where $f(x)$ is the given initial temperature distribution. Conditions prescribed at $x = 0$ and $x = l$ are called *boundary conditions*. If the temperature u at the ends of the bar are prescribed, then the boundary conditions take the form

$$u(0,t) = g(t), \qquad u(l,t) = h(t), \qquad t > 0 \tag{15}$$

where g and h are prescribed functions. Other boundary conditions are possible. It is easy to imagine physically that one end of the bar, say at $x = 0$, is insulated

so that no heat can pass through; this means by Fourier's heat law that the flux at $x = 0$ is zero or

$$u_x(0,t) = 0, \qquad t > 0 \tag{16}$$

Condition (16) is called the *insulated end* boundary condition. Or one may prescribe the flux at an end as a given function of t, for example,

$$-Ku_x(0,t) = \phi(t), \qquad t > 0 \tag{17}$$

If the conductivity K depends on u, then condition (17) will be nonlinear. Another type of boundary condition is given by Newton's law of cooling,

$$-Ku_x(0,t) = \alpha(u(0,t) - \psi(t)), \qquad t > 0$$

which requires the flux to be proportional to the difference between the temperature at the end and the temperature ψ of the environment.

If an initial condition is given and a boundary condition is prescribed at one end (say at $x = 0$) of a very long rod, then the problem can be considered on an infinite medium $x \geq 0$ on which the heat conduction equation holds. For example the boundary value problem

$$\begin{aligned} u_t - ku_{xx} &= 0, & t > 0, 0 < x < \infty \\ u(x,0) &= 0, & 0 < x < \infty \\ u(0,t) &= 1 - \cos t, & t > 0 \end{aligned}$$

governs heat flow in an infinite medium $x \geq 0$ of constant diffusivity k, initially at zero degrees, subject to the maintenance of the end at $x = 0$ at $1 - \cos t$ degrees. Here we expect the problem to model heat conduction in a long bar for a long enough time so that any condition at the far end would not affect the temperature distribution in the portion of the rod that we are interested in studying. As a practical example heat flow in an infinite medium arises from the study of the underground temperature variations, given the temperature changes at ground level on the earth. It seems clear that problems on the infinite interval $-\infty < x < \infty$ are also relevant in special physical situations. In this case only an initial condition can be prescribed. In order to restrict the class of solutions to those that are physically meaningful a condition is sometimes prescribed at infinity; for example, u is bounded at infinity or $\lim_{x \to +\infty} u(x,t) = 0$, $t > 0$.

In summary, by a *boundary value problem* for the heat conduction equation we mean the problem of solving some form of the heat equation subject to initial or boundary conditions. It may be clear from the physical context which auxiliary conditions should be prescribed to obtain a unique solution; in some cases it is not clear. A large body of mathematical literature is devoted

to proving the existence and uniqueness of solutions to various kinds of partial differential equations subject to sundry auxiliary data.

Existence-uniqueness theorems are useful even in the most applied contexts. For example, before embarking on a large numerical calculation of the solution to a heat conduction problem it is helpful to know that a unique solution to the problem exists. An example of such an existence-uniqueness theorem is the following:

Theorem 2.1 *The boundary value problem*

$$\begin{aligned} u_t - ku_{xx} &= 0, & 0 < x < l, \quad 0 < t < T \\ u(x,0) &= f(x), & 0 < x < l \\ u(0,t) &= g(t), & u(l,t) = h(t), \quad 0 < t < T \end{aligned} \tag{18}$$

where $f \in C[0,l]$ and $g, h \in C[0,T]$ has a unique solution $u(x,t)$ on the rectangle R: $0 \leq x \leq l$, $0 \leq t \leq T$, for any $T > 0$.

For this theorem we present the uniqueness proof to show an example of the *energy method.* By way of contradiction assume solutions are not unique and that there are two distinct solutions $u_1(x,t)$ and $u_2(x,t)$ to (18). Then their difference $w(x,t) \equiv u_1(x,t) - u_2(x,t)$ must satisfy the boundary value problem

$$w_t - kw_{xx} = 0, \qquad 0 < x < l, 0 < t < T \tag{19}$$

$$w(x,0) = 0, \qquad 0 < x < l \tag{20}$$

$$w(0,t) = w(l,t) = 0, \qquad 0 < t < T \tag{21}$$

If we show $w(x,t) \equiv 0$ on R, then $u_1(x,t) = u_2(x,t)$ on R, which is a contradiction. To this end define the *energy integral*

$$E(t) = \int_0^l w^2(x,t)\, dx \tag{22}$$

Clearly $E(t) \geq 0$ and $E(0) = 0$. Furthermore

$$E'(t) = \int_0^l 2ww_t\, dx = 2k \int_0^l ww_{xx}\, dx = 2kww_x\Big|_0^l - 2k \int_0^l w_x^2\, dx$$

where the last equality was obtained using integration by parts. From (21) the boundary term vanishes and

$$E'(t) = -2k \int_0^l w_x^2(x,t)\,dx \le 0$$

Thus $E(t)$ is nonincreasing, which along with the facts that $E(t) \ge 0$ and $E(0) = 0$, implies $E(t) = 0$. Therefore the integrand in (22) must vanish identically in $0 \le x \le l$, $0 \le t \le T$, since w is continuous in both its arguments. Thus $w \equiv 0$ on R and the uniqueness part of the theorem is proved. Existence will be established later by actually exhibiting a solution.

In addition to existence and uniqueness questions the notion of continuous dependence of the solution on the initial or boundary data is important. This concept is called *stability*. From a physical viewpoint it is reasonable that small changes in the initial or boundary temperatures should not lead to large changes in the overall temperature distribution. The mathematical model should reflect this stability in that small changes in the auxiliary data should lead to only small changes in the solution. Stated differently, the solution should be stable under small perturbations of the initial or boundary data.

If a given boundary value problem satisfies the three conditions—(i) there is a solution, (ii) the solution is unique, and (iii) the solution is stable—then the problem is said to be *well posed*. (We remark, parenthetically, that although stability seems desirable, many important physical processes are unstable, and so *ill-posed problems* are frequently studied in mathematics as well.)

Example 2.5 (Hadamard's Example) Consider the partial differential equation

$$u_{tt} + u_{xx} = 0, \qquad t > 0, \quad x \in R^1 \tag{23}$$

subject to the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad x \in R^1 \tag{24}$$

Note that (23) is elliptic and not parabolic like the heat equation. The solution of this problem is clearly the zero solution $u(x,t) \equiv 0$ for $t \ge 0$, $x \in R^1$. Now let us change (24) to

$$u(x,0) = 0, \qquad u_t(x,0) = 10^{-4} \sin 10^4 x \tag{25}$$

which represents a very small change in the initial data. The solution of (23) subject to (25) is

$$u(x,t) = 10^{-8} \sin(10^4 x) \sinh(10^4 t) \tag{26}$$

For large values of t, the function $\sinh(10^4 t)$ behaves like $\exp(10^4 t)$. Therefore

the solution (26) grows exponentially with t. Hence for the initial value problem (23)–(24) an arbitrarily small change in the initial data leads to an arbitrarily large solution, and the problem is not well posed.

In numerical calculations stability is essential. For example, suppose a numerical scheme is devised to propagate the initial and boundary conditions. Those conditions can never be represented exactly in the computer, as small errors will exist because of roundoff or truncation of the data. If the problem itself is unstable, then these small errors in the data may be propagated in the numerical scheme in such a way that the calculation becomes meaningless.

Exercises

2.1 Using the energy method as in the uniqueness proof presented in Theorem 2.1 prove that solutions to the boundary value problem

$$\begin{aligned} &u_t - ku_{xx} = 0, && 0 < x < l, \quad 0 < t < T \\ &u(x,0) = f(x), && 0 < x < l \\ &u_x(0,t) = 0, && u_x(l,t) = g(t), \qquad 0 < t < T \end{aligned}$$

are unique for any $T > 0$.

2.2 A homogeneous (constant ρ, C, and K) bar has cross-sectional area $A(x)$, $0 < x < l$, and there is only a small variation of $A(x)$ with x, so that the assumption of constant temperature in any cross section remains valid. There are no sources and the flux is given by $-Ku_x(x,t)$. From a conservation law obtain a partial differential equation for the temperature $u(x,t)$, which reflects the area variation of the bar.

2.3 Transform the boundary value problem

$$\begin{aligned} &u_t - u_{xx} = 0, && 0 < x < l, \quad t > 0 \\ &u(x,0) = f(x), && 0 < x < l \\ &u(0,t) = g(t), && u(l,t) = h(t), \qquad t > 0 \end{aligned}$$

to a problem with homogeneous boundary conditions.

2.4 Consider the boundary value problem

$$\begin{aligned} &u_t - ku_{xx} = q(x), && 0 < x < l, && t > 0 \\ &u_x(0,t) = A, && u_x(l,t) = B, && t > 0 \\ &u(x,0) = f(x), && 0 < x < l \end{aligned}$$

Interpret this problem in a physical context and find a condition under

6

WAVE PHENOMENA IN CONTINUOUS SYSTEMS

Two of the fundamental processes in nature are diffusion and wave propagation. In the last two chapters we studied the equation of heat conduction, a parabolic partial differential equation that is the prototype of equations governing linear diffusion processes. In the present chapter we investigate wave phenomena and obtain equations that govern the propagation of waves first in simple model settings and then in continuum mechanics. The evolution equations governing such phenomena are hyperbolic and are fundamentally different from their parabolic counterparts in diffusion and from elliptic equations that govern equilibrium states.

6.1 WAVE PROPAGATION

Waves

By a *wave* is meant an identifiable signal or disturbance in a medium that is propagated in time, carrying energy with it. A few familiar examples are electromagnetic waves, waves on the surface of water, sound waves, and stress waves in solids, as occur in earthquakes. Material or matter is not necessarily convected with the wave; it is the disturbance, that carries energy, that is propagated. In this section we investigate several model equations that occur in wave phenomena and point out basic features that are encountered in the study of the propagation of waves.

A simple mathematical model of a wave is the function

$$u(x,t) = f(x - ct) \tag{1}$$

which represents an undistorted right-traveling wave moving at constant velocity c. The coordinate x represents position, t time, and u the strength of the disturbance. At $t = 0$ the wave profile is $u = f(x)$ and at $t > 0$ units of time the disturbance has moved to the right ct units of length (see Fig. 6.1). Of particular importance is that the wave profile described by (1) moves without distortion. Not all waves have this property; it is characteristic of linear waves, or wave profiles that are solutions to linear partial differential equations. On the other hand, waves that distort and break are characteristic of nonlinear processes. To find a partial differential equation that governs a process described by (1) we compute u_t and u_x to get

$$u_t = -cf'(x - ct), \qquad u_x = f'(x - ct)$$

Hence

$$u_t + cu_x = 0 \tag{2}$$

Equation (2) is a first-order linear partial differential equation that, in the sense just described, is the simplest wave equation. It is called the *advection equation* and its general solution is (1), where f is an arbitrary function. The name comes from the fact that (2) describes what would happen if dye were squirted into a stream moving by at velocity c; the color would be advected downstream without distortion. Similarly, a traveling wave of the form $u = f(x+ct)$ is a left-moving wave and is a solution of the partial differential equation $u_t - cu_x = 0$.

Other waves of interest in many calculations are periodic, or sinusoidal waves. These traveling waves are represented by expressions of the form

$$u = A\cos(kx - \omega t) \tag{3}$$

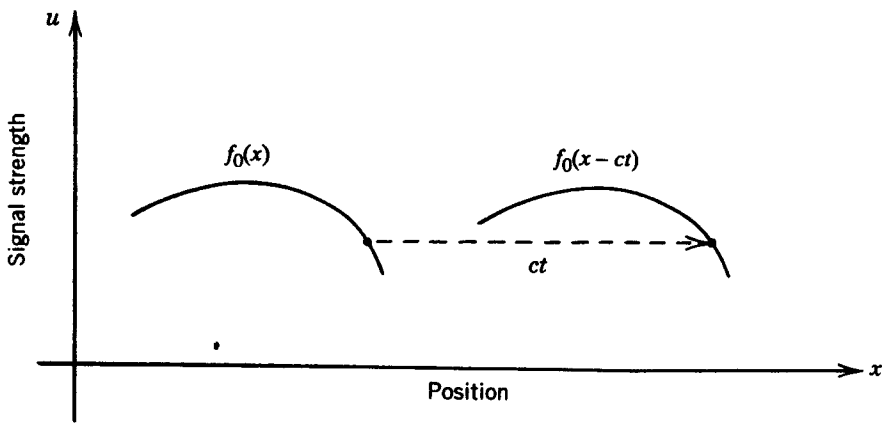

Figure 6.1. Right-traveling wave.

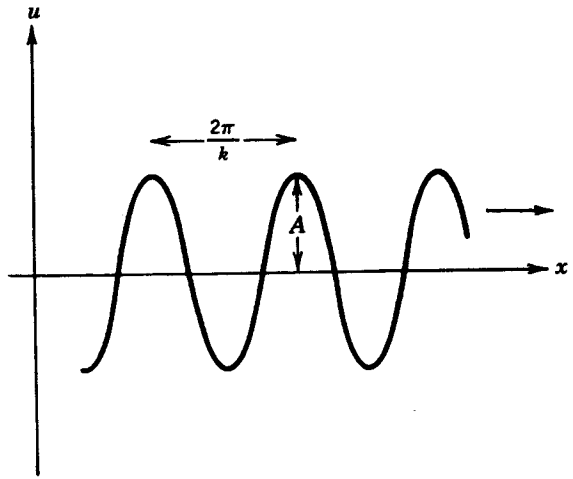

Figure 6.2. Periodic wave.

(see Fig. 6.2). The positive number A is the *amplitude*, k is the *wavenumber* (the number of oscillations in 2π units of space observed at a fixed time), and ω the *angular frequency* (the number of oscillations in 2π units of time observed at a fixed location x). The number $\lambda = 2\pi/k$ is the *wavelength* and $P = 2\pi/\omega$ is the *time period*. The wavelength measures the distance between successive crests and the time period is the time for an observer located at a fixed position x to see a repeat pattern. If we write (3) as

$$u = A\cos k\left(x - \frac{\omega}{k}t\right)$$

then we note that (3) represents a traveling wave moving to the right with velocity $c = \omega/k$. This number is called the *phase velocity*, and it is the speed one would have to move to remain at the same point on the traveling wave. For calculations the complex exponential form

$$u = \exp(i(kx - \omega t)) \tag{4}$$

is often used rather than (3). Computations involving differentiations are easier with (4) and afterward, making use of Euler's formula $\exp(i\theta) = \cos\theta + i\sin\theta$, the real or imaginary part may be taken to recover real solutions. Again, waves of the type (3) [or (4)] are characteristic of linear processes and linear equations.

Not all waves propagate in such a way that their profile remains unchanged or undistorted; surface waves on the ocean are obvious examples. Less familiar perhaps but just as common, are stress or pressure waves that propagate in solids or gases. To fix the idea and to indicate how nonlinearity affects the shape

of a wave, let us consider a stress wave propagating in a metallic bar, caused, for example, by a force on one end of the bar. The distortion of a wave profile results from the property of most materials to transmit signals at a speed that increases with increasing pressure. Therefore a stress wave that is propagating in a medium will gradually distort and steepen until it propagates as a discontinuous disturbance, or shock wave. Figure 6.3 shows various snapshots of a stress wave propagating into a material. The wave steepens as time increases because signals or disturbances travel faster when the pressure is higher. Thus the point A moves to the right faster than the point B. The shock that forms in pressure is accompanied by discontinuous jumps in the other flow parameters, such as density, particle velocity, temperature, energy, and entropy. Here we are considering finite amplitude waves rather than small amplitude waves. The latter can propagate as linear waves.

Physically, a shock wave is not strictly a discontinuity but rather an extremely thin region where the change in the state is steep. The width of the shock is small, usually of the order of a few mean free paths, or average distance to a collision, of the molecules. In a shock there are two competing effects that cause this thinness, the nonlinearity of the material that is causing the shock to form and the dissipative effects (e.g., viscosity) that are tending to smear the wave out. Usually these two effects just cancel and the front assumes a shape that does not change in time.

The same mechanism that causes pressure waves to steepen into shocks, that is, an increase in signal transmission speed at higher pressures, causes *release waves* or *rarefaction waves* to form that lower the pressure. Figure 6.4 shows a spreading rarefaction. Again, point A at higher pressure moves to the right faster than point B, thereby flattening the wave.

So far we have mentioned two types of waves, those that propagate undistorted at constant velocity and those that distort because the speed of propagation depends on the amplitude of the wave. There is a third phenomenon that is relevant in some problems, namely that of *dispersion*. In this case the speed of propagation depends on the wavelength of the particular wave. So, for

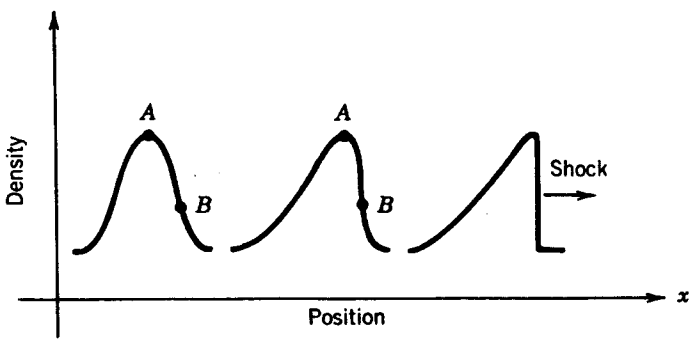

Figure 6.3. Distortion of a wave into a shock.

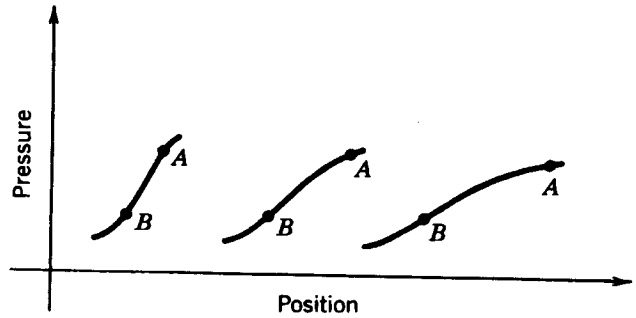

Figure 6.4. Snapshots of a rarefaction wave.

example, longer waves can travel faster than shorter ones. Thus an observer of a wave at fixed location x_0 may see a different temporal wave pattern from another observer at fixed location x_1. Dispersive wave propagation arises from both linear and nonlinear equations.

Linear Waves

In the preceding paragraphs we introduced the simplest wave equation, the advection equation,

$$u_t + cu_x = 0, \qquad x \in R^1, \qquad t > 0 \tag{5}$$

whose general solution is a right-traveling wave

$$u = f(x - ct) \tag{6}$$

propagating at constant velocity c, where f is an arbitrary function. If we impose the initial condition

$$u(x, 0) = \phi(x), \qquad x \in R^1 \tag{7}$$

then it follows that $f(x) = \phi(x)$ and so the solution to the initial value problem (5) and (7) is

$$u(x, t) = \phi(x - ct) \tag{8}$$

The straight lines $x - ct$ = constant play a special role in this problem; along those lines the initial values are propagated with constant value. We may interpret them as lines in space-time along which the signals are carried (see Fig. 6.5). Furthermore, along these lines the partial differential equation (5) reduces to the ordinary differential equation $du/dt = 0$. That is to say, if C is the curve

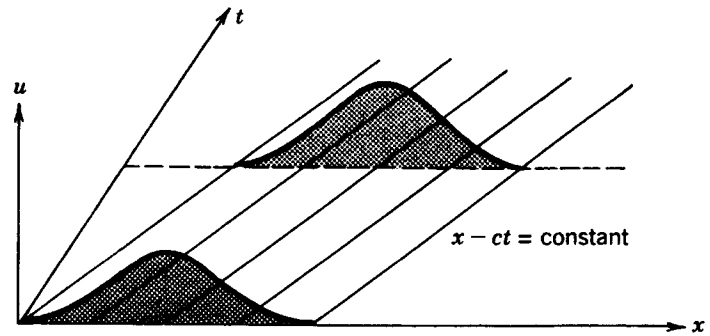

Figure 6.5. Solution to $u_t + cu_x = 0$, $c > 0$.

$x = ct + k$ for some constant k, then the directional derivative of u along that curve is

$$\frac{du}{dt}(x(t), t) = u_x(x(t), t)\frac{dx}{dt} + u_t(x(t), t)$$

$$= u_x(x(t), t)c + u_t(x(t), t)$$

which is the left side of (5) evaluated along C. The family of straight lines $x - ct = k$ (k constant) is called the family of *characteristic curves* for this problem. Note that their speed c is the reciprocal of their slope in the xt coordinate system.

Now let us complicate the partial differential equation (5) by replacing the constant c by a function of the independent variables t and x and consider the initial value problem

$$\begin{aligned} u_t + c(x, t)u_x &= 0, \qquad x \in R^1, \qquad t > 0 \\ u(x, 0) &= \phi(x), \qquad x \in R^1 \end{aligned} \tag{9}$$

where $c(x, t)$ is a given function. Let C be the family of curves defined by the differential equation

$$\frac{dx}{dt} = c(x, t) \tag{10}$$

Then along a member of C

$$\frac{du}{dt} = u_x\frac{dx}{dt} + u_t = u_x c(x, t) + u_t = 0$$

Hence u is constant on each member of C. The curves C defined by (10) are the *characteristic curves.*

Example 1.1 Consider the initial value problem

$$u_t + 2tu_x = 0, \qquad x \in R^1, t > 0$$
$$u(x,0) = \exp(-x^2), \qquad x \in R^1$$

The characteristic curves are defined by the differential equation $dx/dt = 2t$ that yields the family of parabolas

$$x = t^2 + k, \qquad k \text{ constant}$$

Knowing that u is constant on these characteristic curves allows us to find a solution to the initial value problem. Let (x,t) be an arbitrary point with $t > 0$. The characteristic curve through (x,t) passes through $(\xi,0)$ and has equation $x = t^2 + \xi$ (see Fig. 6.6). Since u is constant on this curve

$$u(x,t) = \exp(-\xi^2) = \exp(-(x - t^2)^2)$$

which is the unique solution to the initial value problem. The speed of the signal at (x,t) is $2t$, which is dependent on t. In general equation (9) propagates signals at the speed $c(x,t)$. In the present example the wave speeds up as time increases, but it retains its initial shape.

By following the same reasoning as in Example 1.1 we can write the solution to the initial value problem (9) as $u(x,t) = \phi(k)$, where $a(x,t) = k$ define the characteristic curves given by (10). The partial differential equation (9) itself has general solution $u(x,t) = f(a(x,t))$, where f is an arbitrary function (see Exercise 1.1).

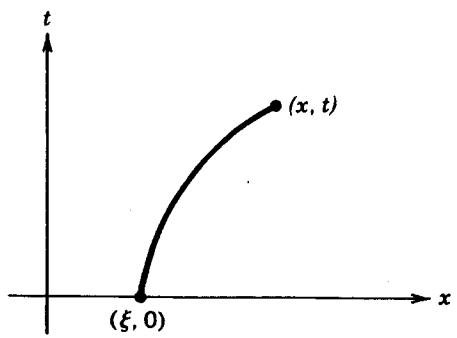

Figure 6.6. Characteristic $x = t^2 + k$.

Nonlinear Waves

In the last section we examined the two simple model wave equations $u_t + cu_x = 0$ and $u_t + c(x,t)u_x = 0$ that are both first-order and linear. Now we study the same type of equation when a nonlinearity is introduced. In particular we consider

$$u_t + c(u)u_x = 0, \qquad x \in R^1, \qquad t > 0 \tag{11}$$

where $c'(u) > 0$, with initial condition

$$u(x,0) = \phi(x), \qquad x \in R^1 \tag{12}$$

Using the guidance of the earlier examples we define the *characteristic curves* by the differential equation

$$\frac{dx}{dt} = c(u) \tag{13}$$

Then along a particular such curve $x = x(t)$ we have

$$\frac{du}{dt}(x(t),t) = u_x(x(t),t)c(u(x(t),t)) + u_t(x(t),t) = 0$$

Therefore u is constant along the characteristics, and the characteristics are straight lines since

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\,c(u(x(t),t)) = c'(u)\,\frac{du}{dt} = 0$$

In the nonlinear case, however, the speed of the characteristics as defined by (13) depends on the value u of the solution at a given point. To find the equation of the characteristic C through (x,t) we note that its speed is

$$\frac{dx}{dt} = c(u(\xi,0)) = c(\phi(\xi))$$

(see Fig. 6.7). This results from applying (13) at $(\xi,0)$. Hence, after integrating,

$$x = c(\phi(\xi))t + \xi \tag{14}$$

gives the equation of the desired characteristic C. Equation (14) defines $\xi = \xi(x,t)$ implicitly as a function of x and t, and the solution $u(x,t)$ of the initial