```
In[1]:= Clear["*"];
        |n[1]:= (*sphere/sphere collision resolver*)
                                             p_1 = \{p_{1.x}, p_{1.y}, p_{1.z}\}; (*position*)
                                              u_1 = \{u_{1.x}, u_{1.y}, u_{1.z}\}; (*velocity*)
                                             r<sub>1</sub>; (*radius*)
                                             p_2 = \{p_{2.x}, p_{2.y}, p_{2.z}\};
                                             u_2 = \{u_{2.x}, u_{2.y}, u_{2.z}\};
                                              Solve[EuclideanDistance[p_1 + u_1 t, p_2 + u_2 t] = r_1 + r_2, t]
 \text{Out} \text{[7]= } \left\{ \left\{ t \rightarrow \left( -2\,p_{1\,.\,x}\,u_{1\,.\,x} + 2\,p_{2\,.\,x}\,u_{1\,.\,x} + 2\,p_{1\,.\,x}\,u_{2\,.\,x} - 2\,p_{2\,.\,x}\,u_{2\,.\,x} - 2\,p_{1\,.\,y}\,u_{1\,.\,y} + 2\,p_{2\,.\,y}\,u_{1\,.\,y} + 2\,p_{2\,.\,y}\,u_{1\,.\,y} + 2\,p_{2\,.\,y}\,u_{2\,.\,y} 
                                                                                                             2 p_{1, y} u_{2, y} - 2 p_{2, y} u_{2, y} - 2 p_{1, z} u_{1, z} + 2 p_{2, z} u_{1, z} + 2 p_{1, z} u_{2, z} - 2 p_{2, z} u_{2, z} 
                                                                                                            \sqrt{\left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} - 
                                                                                                                                                                           2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                                                            4 \left( p_{1,x}^2 - 2 p_{1,x} p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2 p_{1,y} p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2 p_{1,z} p_{2,z} + p_{2,z}^2 - r_1^2 - 2 r_1 r_2 - 2 r_1 r_2 + r_2^2 r_1 r_2 - r_1^2 - r_1^2 r_2 r_1 r_2 - r_1^2 r_1 r_2 r_1 r_2 - r_1^2 r_1 r_2 r
                                                                                                                                                                          (u_{1,x}^2) (u_{1,x}^2 - 2u_{1,x}u_{2,x} + u_{2,x}^2 + u_{1,y}^2 - 2u_{1,y}u_{2,y} + u_{2,y}^2 + u_{1,z}^2 - 2u_{1,z}u_{2,z} + u_{2,z}^2))
                                                                                         \left(2\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)\right)\right\}
                                                           \left\{t \rightarrow \left(-2\,p_{1\,.\,x}\,u_{1\,.\,x} + 2\,p_{2\,.\,x}\,u_{1\,.\,x} + 2\,p_{1\,.\,x}\,u_{2\,.\,x} - 2\,p_{2\,.\,x}\,u_{2\,.\,x} - 2\,p_{1\,.\,y}\,u_{1\,.\,y} + 2\,p_{2\,.\,y}\,u_{1\,.\,y} + 2\,p_{2\,.\,y}\,u_{1\,.\,y} + 2\,p_{2\,.\,y}\,u_{2\,.\,y} + 2\,p_{2\,.\,y}\,u_{2\,.\,
                                                                                                               2 p_{1. y} u_{2. y} - 2 p_{2. y} u_{2. y} - 2 p_{1. z} u_{1. z} + 2 p_{2. z} u_{1. z} + 2 p_{1. z} u_{2. z} - 2 p_{2. z} u_{2. z} +
                                                                                                             \sqrt{\left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} - 
                                                                                                                                                                           2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                                                             4 \left( p_{1,x}^2 - 2 p_{1,x} p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2 p_{1,y} p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2 p_{1,z} p_{2,z} + p_{2,z}^2 - r_1^2 - 2 r_1 r_2 - 2 r_1 r_2 + r_2^2 r_1 r_2 - r_1^2 - r_1^2 r_2 r_1 r_2 - r_1^2 r_1 r_2 r_1 r_2 - r_1^2 r_1 r_2 r_2 r_1 r
                                                                                                                                                                          (u_{1,x}^2) (u_{1,x}^2 - 2u_{1,x}u_{2,x} + u_{2,x}^2 + u_{1,y}^2 - 2u_{1,y}u_{2,y} + u_{2,y}^2 + u_{1,z}^2 - 2u_{1,z}u_{2,z} + u_{2,z}^2))
                                                                                         \left.\left(2\,\left(u_{1.\,x}^{2}\,-\,2\,\,u_{1.\,x}\,\,u_{2.\,x}\,+\,u_{2.\,x}^{2}\,+\,u_{1.\,y}^{2}\,-\,2\,\,u_{1.\,y}\,\,u_{2.\,y}\,+\,u_{2.\,y}^{2}\,+\,u_{1.\,z}^{2}\,-\,2\,\,u_{1.\,z}\,\,u_{2.\,z}\,+\,u_{2.\,z}^{2}\right)\,\right)\,\right\}\right\}
        _{\ln[8]:=} (*if overlapping pick nearest t<0, for collision detection check t>0 and t<1*)
                                             np_1 = p_1 + u_1 t; (*move out of collision*)
                                             np_2 = p_2 + u_2 t;
                                             nml = Normalize[np<sub>1</sub> - np<sub>2</sub>];(*collision plane normal*)
                                             m1; (*mass*)
                                             \mathbf{m}_2;
                                                                                                                velocity given
                                                                                                                                                                                                                                                                                           velocities received along collision plane normal*)
                                                 (*
                                             v_1 = u_1 - (u_1.nml) nml + (u_1.nml) nml ((m_1 - m_2) / (m_1 + m_2)) + (u_2.nml) nml (2 m_2 / (m_1 + m_2));
                                              v_2 = u_2 - (u_2 \cdot nm1) \cdot nm1 + (u_2 \cdot nm1) \cdot nm1 \cdot ((m_2 - m_1) / (m_1 + m_2)) + (u_1 \cdot nm1) \cdot nm1 \cdot (2m_1 / (m_1 + m_2));
                                             np_1 = np_1 + v_1 (1 - t); (*perform remaining dt with new velocities*)
                                             np_2 = np_2 + v_2 (1 - t);
ln[17] = v_1 = (u_1 \cdot nml) nml ((m_1 - m_2) / (m_1 + m_2)) + (u_2 \cdot nml) nml (2 m_2 / (m_1 + m_2));
                                             v_2 = (u_2.nml) nml ((m_2 - m_1) / (m_1 + m_2)) + (u_1.nml) nml (2 m_1 / (m_1 + m_2));
                                             m_1 u_1 + m_2 u_2 == m_1 v_1 + m_2 v_2;
```

```
In[20]:= (*sphere/plane collision resolver*)
       p_g = \{0, 0, 0\}; (*dot on plane*)
       n_q = Normalize[{0, 1, 0}];(*normal*)
       p = {0, 1.5, 0}; (*sphere position*)
       u = \{0, -1, 0\}; (*velocity*)
       r = 1; (*radius*)
       Remove[t];
       f = (p + ut - p_g) \cdot n_g - r = 0; (*signed distance to plane*)
       Plot[f, {t, 0, 1}]
       Solve[f, t]
       t = .5;
       (*pick nearest t<0, t>1 or t<0 collision is in future or past*)
       np = p + u t (*move sphere out of collision*)
       v = u - 2 (u.n_g) n_g (*reflect*)
       np = np + v (1 - t) (*second part of dt*)
\begin{array}{c} 1.0 \\ 0.6 \\ 0.4 \\ 0.2 \end{array} Out[27]= $DisplayFunction \begin{bmatrix} 0.6 \\ 0.4 \\ 0.2 \end{bmatrix}
Out[28]= \{ \{ t \rightarrow -1. (-0.5 + 1. \{0, 1.5, 0\}_g) \} \}
Out[30]= \{0., 1., 0.\}
Out[31]= \{0, 1, 0\}
Out[32]= \{0., 1.5, 0.\}
        (*simplified*)
In[34]:= Clear["*"];
```

```
In[1]:= (*sphere/sphere collision resolver*)
                                             p_1 = \{p_{1.x}, p_{1.y}, p_{1.z}\}; (*position*)
                                              u_1 = \{u_{1.x}, u_{1.y}, u_{1.z}\}; (*velocity*)
                                             r<sub>1</sub>; (*radius*)
                                             p_2 = \{p_{2.x}, p_{2.y}, p_{2.z}\};
                                              u_2 = \{u_{2.x}, u_{2.y}, u_{2.z}\};
                                             r2;
                                              Solve [EuclideanDistance [p_1 + u_1 t, p_2 + u_2 t] = r_1 + r_2, t]
      \text{Out}[7] = \left\{ \left\{ t \rightarrow \left( -2 \, p_{1. \, x} \, u_{1. \, x} + 2 \, p_{2. \, x} \, u_{1. \, x} + 2 \, p_{1. \, x} \, u_{2. \, x} - 2 \, p_{2. \, x} \, u_{2. \, x} - 2 \, p_{1. \, y} \, u_{1. \, y} + 2 \, p_{2. \, y} \, u_{1. \, y} + 2 \, p_{2. \, y} \, u_{1. \, y} + 2 \, p_{2. \, y} \, u_{2. \, x} \right\} \right\} \right\} = \left\{ \left\{ t \rightarrow \left( -2 \, p_{1. \, x} \, u_{1. \, x} + 2 \, p_{2. \, x} \, u_{1. \, x} + 2 \, p_{1. \, x} \, u_{2. \, x} - 2 \, p_{2. \, x} \, u_{2. \, x} - 2 \, p_{1. \, y} \, u_{1. \, y} + 2 \, p_{2. \, y} \, u_{1. \, y} + 2 \, p_{2. \, y} \, u_{2. \, x} \right\} \right\} \right\}
                                                                                                         2\;p_{1.\;y}\;u_{2.\;y}-2\;p_{2.\;y}\;u_{2.\;y}-2\;p_{1.\;z}\;u_{1.\;z}+2\;p_{2.\;z}\;u_{1.\;z}+2\;p_{1.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2\;p_{2.\;z}-2
                                                                                                         \sqrt{\left(2 p_{1,x} u_{1,x} - 2 p_{2,x} u_{1,x} - 2 p_{1,x} u_{2,x} + 2 p_{2,x} u_{2,x} + 2 p_{1,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{2,x} + 2 p_{2,x} u_{2,x} + 2 p_{2,y} u_{2,y} - 
                                                                                                                                                                 2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                                                      4 \left( p_{1.\,x}^2 - 2\,p_{1.\,x}\,p_{2.\,x} + p_{2.\,x}^2 + p_{1.\,y}^2 - 2\,p_{1.\,y}\,p_{2.\,y} + p_{2.\,y}^2 + p_{1.\,z}^2 - 2\,p_{1.\,z}\,p_{2.\,z} + p_{2.\,z}^2 - r_1^2 - 2\,r_1\,r_2 - 2\,r_1^2 + 2\,r_2^2\,p_{2.\,z}^2 + 2\,r_2^2\,p_{2.\,z}^
                                                                                                                                                                 r_{2}^{2}) \left( u_{1.\,x}^{2} - 2\,u_{1.\,x}\,u_{2.\,x} + u_{2.\,x}^{2} + u_{1.\,y}^{2} - 2\,u_{1.\,y}\,u_{2.\,y} + u_{2.\,y}^{2} + u_{1.\,z}^{2} - 2\,u_{1.\,z}\,u_{2.\,z} + u_{2.\,z}^{2} \right) \right) \right) / r_{2}^{2} 
                                                                                      \left(2\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)\right)\right\}
                                                       \left\{ \text{t} \rightarrow \left( \text{-2 p}_{\text{1. x}} \, \text{u}_{\text{1. x}} + \text{2 p}_{\text{2. x}} \, \text{u}_{\text{1. x}} + \text{2 p}_{\text{1. x}} \, \text{u}_{\text{2. x}} - \text{2 p}_{\text{2. x}} \, \text{u}_{\text{2. x}} - \text{2 p}_{\text{1. y}} \, \text{u}_{\text{1. y}} + \text{2 p}_{\text{2. y}} \, \text{u}_{\text{1. y}} + \text{2 p}_{\text{2. y}} \, \text{u}_{\text{1. y}} + \text{2 p}_{\text{2. y}} \, \text{u}_{\text{2. y}} + \text{2 p}_{\text{2. y}} \, \text{u}_{\text{2. y}} + \text{2 p}_{\text{2. y}} \, \text{u}_{\text{2. y}} \right\} \right\} = 0
                                                                                                         2 p_{1, y} u_{2, y} - 2 p_{2, y} u_{2, y} - 2 p_{1, z} u_{1, z} + 2 p_{2, z} u_{1, z} + 2 p_{1, z} u_{2, z} - 2 p_{2, z} u_{2, z} +
                                                                                                       \sqrt{\left(2 p_{1,x} u_{1,x} - 2 p_{2,x} u_{1,x} - 2 p_{1,x} u_{2,x} + 2 p_{2,x} u_{2,x} + 2 p_{1,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{2,x} + 2 p_{2,x} u_{2,x} + 2 p_{2,y} u_{2,y} - 
                                                                                                                                                                 2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                                                    4 \left( p_{1,x}^2 - 2 p_{1,x} p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2 p_{1,y} p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2 p_{1,z} p_{2,z} + p_{2,z}^2 - r_1^2 - 2 r_1 r_2 - 2 r_1 r_2 \right)
                                                                                                                                                                 (u_{1,x}^2) (u_{1,x}^2 - 2u_{1,x}u_{2,x} + u_{2,x}^2 + u_{1,y}^2 - 2u_{1,y}u_{2,y} + u_{2,y}^2 + u_{1,z}^2 - 2u_{1,z}u_{2,z} + u_{2,z}^2))
                                                                                      \left\{2\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)\right\}\right\}
           ln[8]:= EuclideanDistance[p_1 + u_1 t, p_2 + u_2 t]
      Out[8]= \sqrt{\left(\text{Abs}[p_{1.x} - p_{2.x} + t u_{1.x} - t u_{2.x}]^2 + \right)}
                                                                          Abs \left[ p_{1.y} - p_{2.y} + t u_{1.y} - t u_{2.y} \right]^{2} + Abs \left[ p_{1.z} - p_{2.z} + t u_{1.z} - t u_{2.z} \right]^{2}
        \ln[9] = \sqrt{\text{Total}[(p_1 + u_1 t - (p_2 + u_2 t))^2]} = r_1 + r_2
   \text{Out}[9] = \sqrt{\left( \left( p_{1.\,x} - p_{2.\,x} + t \, u_{1.\,x} - t \, u_{2.\,x} \right)^2 + \left( p_{1.\,y} - p_{2.\,y} + t \, u_{1.\,y} - t \, u_{2.\,y} \right)^2 + \left( p_{1.\,z} - p_{2.\,z} + t \, u_{1.\,z} - t \, u_{2.\,z} \right)^2 \right)} = 0
   \ln[10] := \left( \sqrt{\text{Total} \left[ (p_1 + u_1 t - (p_2 + u_2 t))^2 \right]} \right)^2 == (r_1 + r_2)^2
 \text{Out} [\text{10}] = \left( p_{1.\,x} - p_{2.\,x} + \text{t} \, u_{1.\,x} - \text{t} \, u_{2.\,x} \right)^2 + \left( p_{1.\,y} - p_{2.\,y} + \text{t} \, u_{1.\,y} - \text{t} \, u_{2.\,y} \right)^2 + \\ \left( p_{1.\,z} - p_{2.\,z} + \text{t} \, u_{1.\,z} - \text{t} \, u_{2.\,z} \right)^2 = \\ \left( r_1 + r_2 \right)^2 + \left( r_1 + r_2 \right)^2 + \left( r_2 + r_2 \right)^2 + \\ \left( r_1 + r_2 \right)^2 + \left( r_2 + r_2 \right)^2 + \\ \left( r_1 + r_2 \right)^2 + \left( r_2 + r_2 \right)^2 + \\ \left( r_1 + r_2 \right)^2 + \\ \left( r_2 + r_2 \right)^2 + \\ \left( r_1 + r_2 \right)^2 + \\ \left( r_2 + r_2 \right)^2 + \\ \left( r_1 + r_2 \right)^2 + \\ \left( r_2 + r_2 \right)^2 + \\ \left( r_1 + r_2 \right)^2 + \\ \left( r_2 + r_2 \right)^2 + \\ \left( r_1 + r_2 \right)^2 + \\ \left( r_1
   ln[11] = Total[(p_1 + u_1 t - (p_2 + u_2 t))^2] = (r_1 + r_2)^2
Out[11] = (p_{1.x} - p_{2.x} + t u_{1.x} - t u_{2.x})^2 + (p_{1.y} - p_{2.y} + t u_{1.y} - t u_{2.y})^2 + (p_{1.z} - p_{2.z} + t u_{1.z} - t u_{2.z})^2 = (r_1 + r_2)^2
   ln[12] = \Delta p = p_1 - p_2
Out[12]= \{p_{1.x} - p_{2.x}, p_{1.y} - p_{2.y}, p_{1.z} - p_{2.z}\}
```

```
ln[13] = Collect [Expand [Total [(\Delta p + u_1 t - u_2 t)^2] - (r_1 + r_2)^2 = 0], t]
Out[13] = p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_
                                                                                 r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.x} + 2 p_{2.y} u_{2.y} - 2 p_{2.y} u_
                                                                                                                     2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                 t^{2}\left(u_{1,x}^{2}-2\,u_{1,x}\,u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2\,u_{1,y}\,u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2\,u_{1,z}\,u_{2,z}+u_{2,z}^{2}\right) = 0
      log[14] = Collect[Expand[Total[(\Delta p + (u_1 - u_2) t)^2] - (r_1 + r_2)^2 = 0], t]
Out[14]= p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_1
                                                                                r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} + 2 p_{2.y} u_
                                                                                                                     2 p_{1.y} u_{2.y} + 2 p_{2.y} u_{2.y} + 2 p_{1.z} u_{1.z} - 2 p_{2.z} u_{1.z} - 2 p_{1.z} u_{2.z} + 2 p_{2.z} u_{2.z} + 
                                                                                 t^{2}\left(u_{1}^{2} + 2u_{1} + u_{2} + u_{2}^{2} + u_{1}^{2} + u_{1}^{2} + u_{1}^{2} + u_{2}^{2} + u_{1}^{2} + u_{2}^{2} + u_{2}^{2} + u_{1}^{2} + u_{2}^{2} + u_{2}^{2} \right) = 0
    ln[15]:= \Delta u = u_1 - u_2
Out[15]= \{u_{1.x} - u_{2.x}, u_{1.y} - u_{2.y}, u_{1.z} - u_{2.z}\}
    ln[16]:= Collect \left[ Expand \left[ Total \left[ (\Delta p + \Delta u t)^{2} \right] - (r_{1} + r_{2})^{2} = 0 \right], t \right]
Out[16] = p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_
                                                                                r_2^2 + t \left( 2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} \right)
                                                                                                                     2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                 t^{2}\left(u_{1,x}^{2}-2\,u_{1,x}\,u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2\,u_{1,y}\,u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2\,u_{1,z}\,u_{2,z}+u_{2,z}^{2}\right) = 0
      log[17] = Collect[Expand[Total[\Delta p^2 + 2 \Delta p \Delta u t + \Delta u^2 t^2] - (r_1 + r_2)^2 = 0], t]
Out[17]= p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_1
                                                                                r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} + 2 p_{2.y} u_
                                                                                                                     2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z} + 2 p_{2,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                 \mathsf{t}^2 \left( \mathsf{u}^2_{1.\,x} - 2\,\mathsf{u}_{1.\,x}\,\mathsf{u}_{2.\,x} + \mathsf{u}^2_{2.\,x} + \mathsf{u}^2_{1.\,y} - 2\,\mathsf{u}_{1.\,y}\,\mathsf{u}_{2.\,y} + \mathsf{u}^2_{2.\,y} + \mathsf{u}^2_{1.\,z} - 2\,\mathsf{u}_{1.\,z}\,\mathsf{u}_{2.\,z} + \mathsf{u}^2_{2.\,z} \right) \, = \, 0
      log[18] := Collect[Total[\Delta p^2 + 2 \Delta p \Delta u t + \Delta u^2 t^2] - (r_1 + r_2)^2 == 0, t]
Out[18]= (p_{1,x} - p_{2,x})^2 + (p_{1,y} - p_{2,y})^2 + (p_{1,z} - p_{2,z})^2 - (r_1 + r_2)^2 +
                                                                                    t \left( 2 \left( p_{1.\,x} - p_{2.\,x} \right) \right. \left. \left( u_{1.\,x} - u_{2.\,x} \right) \right. \\ + \left. 2 \left( p_{1.\,y} - p_{2.\,y} \right) \right. \left( u_{1.\,y} - u_{2.\,y} \right) \\ + \left. 2 \left( p_{1.\,z} - p_{2.\,z} \right) \right. \left( u_{1.\,z} - u_{2.\,z} \right) \right) \\ + \left. 2 \left( p_{2.\,x} - p_{2.\,z} \right) \right. \left( u_{2.\,x} - u_{2.\,z} \right) \\ + \left. 2 \left( p_{2.\,x} - p_{2.\,z} \right) \right. \left( u_{2.\,x} - u_{2.\,z} \right) \\ + \left. 2 \left( p_{2.\,x} - p_{2.\,z} \right) \right. \left( u_{2.\,x} - u_{2.\,z} \right) \\ + \left. 2 \left( p_{2.\,x} - p_{2.\,z} \right) \right. \left( u_{2.\,x} - u_{2.\,z} \right) \\ + \left. 2 \left( p_{2.\,x} - p_{2.\,z} \right) \right. \left( u_{2.\,x} - u_{2.\,z} \right) \\ + \left. 2 \left( p_{2.\,x} - p_{2.\,z} \right) \right. \left( u_{2.\,x} - u_{2.\,z} \right) \\ + \left. 2 \left( p_{2.\,x} - p_{2.\,z} \right) \right. \\ \left( u_{2.\,x} - u_{2.\,z} \right) \\ + \left. 2 \left( p_{2.\,x} - u_{2.\,z} \right) \right. \\ \left( u_{2.\,x} - u_{2.\,z} \right) \\ + \left. 2 \left( p_{2.\,x} - u_{2.\,z} \right) \right. \\ \left( u_{2.\,x} - u_{2.\,z} \right) \\ + \left. 2 \left( p_{2.\,x} - u_{2.\,z} \right) \right. \\ \left( u_{2.\,x} - u_{2.\,z} \right) \\ + \left. 2 \left( p_{2.\,x} - u_{2.\,z} \right) \right. \\ \left( u_{2.\,x} - u_{2.\,z} \right) \\ \left( u_{2.
                                                                                 t^{2} \left( (u_{1.x} - u_{2.x})^{2} + (u_{1.y} - u_{2.y})^{2} + (u_{1.z} - u_{2.z})^{2} \right) = 0
      ln[19]:= Total \Delta p^2
Out[19]= (p_{1,x} - p_{2,x})^2 + (p_{1,y} - p_{2,y})^2 + (p_{1,z} - p_{2,z})^2
      In[20]:= Collect[
                                                                      \text{Expand} \left[ \text{Total} \left[ \Delta p^2 \right] - (r_1 + r_2)^2 + \text{t} \left( 2 \left( p_{1, \ x} - p_{2, \ x} \right) \left( u_{1, \ x} - u_{2, \ x} \right) + 2 \left( p_{1, \ y} - p_{2, \ y} \right) \left( u_{1, \ y} - u_{2, \ y} \right) + 2 \left( p_{1, \ y} - p_{2, \ y} \right) \right) \right] 
                                                                                                                                             2(p_{1,z} - p_{2,z})(u_{1,z} - u_{2,z}) +
                                                                                                          t^{2} \left( \left( u_{1, x} - u_{2, x} \right)^{2} + \left( u_{1, y} - u_{2, y} \right)^{2} + \left( u_{1, z} - u_{2, z} \right)^{2} \right) = 0 \right], t
Out[20] = p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_
                                                                                 r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} + 2 p_{2.y} u_
                                                                                                                       2 p_{1.y} u_{2.y} + 2 p_{2.y} u_{2.y} + 2 p_{1.z} u_{1.z} - 2 p_{2.z} u_{1.z} - 2 p_{1.z} u_{2.z} + 2 p_{2.z} u_{2.z}
                                                                                 t^{2}\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)=0
    In[21]:= 2 Δp. Δu t
Out[21] = 2t \left( (p_{1,x} - p_{2,x}) (u_{1,x} - u_{2,x}) + (p_{1,y} - p_{2,y}) (u_{1,y} - u_{2,y}) + (p_{1,z} - p_{2,z}) (u_{1,z} - u_{2,z}) \right)
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In[22]:= Collect [Expand]
                                                                                                                    Total \left[ \Delta p^2 \right] - (r_1 + r_2)^2 + 2 \Delta p. \Delta u t + t^2 \left( (u_1 \cdot x - u_2 \cdot x)^2 + (u_1 \cdot y - u_2 \cdot y)^2 + (u_1 \cdot z - u_2 \cdot z)^2 \right) = 0 \right], t
Out[22] = p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_
                                                                                                                    r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_
                                                                                                                                                                         2\;p_{1.\;y}\;u_{2.\;y}+2\;p_{2.\;y}\;u_{2.\;y}+2\;p_{1.\;z}\;u_{1.\;z}-2\;p_{2.\;z}\;u_{1.\;z}-2\;p_{1.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}\Big)\;+
                                                                                                                    t^{2}\left(u_{1,x}^{2}-2\,u_{1,x}\,u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2\,u_{1,y}\,u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2\,u_{1,z}\,u_{2,z}+u_{2,z}^{2}\right) = 0
      ln[23]:= Total \Delta u^2
Out[23]= (u_{1,x} - u_{2,x})^2 + (u_{1,y} - u_{2,y})^2 + (u_{1,z} - u_{2,z})^2
      \ln[24] = \text{Collect}\left[\text{Expand}\left[\text{Total}\left[\Delta p^2\right] - (r_1 + r_2)^2 + 2\Delta p.\Delta u t + t^2 \text{Total}\left[\Delta u^2\right] = 0\right], t
Out[24]= p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_1
                                                                                                                 r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} - 2 p_{2.y} u_
                                                                                                                                                                       2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z} + 2 p_{2,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                                     \mathsf{t}^2 \, \left( u_{1-\mathsf{x}}^2 \, - 2 \, u_{1-\mathsf{x}} \, u_{2-\mathsf{x}} + u_{2-\mathsf{x}}^2 + u_{1-\mathsf{y}}^2 - 2 \, u_{1-\mathsf{y}} \, u_{2-\mathsf{y}} + u_{2-\mathsf{y}}^2 + u_{1-\mathsf{z}}^2 - 2 \, u_{1-\mathsf{z}} \, u_{2-\mathsf{z}} + u_{2-\mathsf{z}}^2 \right) \, = \, 0 \, \mathsf{d} \, \mathsf{d}
        ln[29]:= a = Total[\Delta u^2];
                                                                                b = 2 \Delta p. \Delta u;
                                                                                c = Total[\Delta p^2] - (r_1 + r_2)^2;
                                                                                Collect [Expand[at^2+bt+c=0],t]
                                                                                  Solve[at^2+bt+c=0,t]
Out[32] = p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_
                                                                                                                 r_{2}^{2} + t (2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} - 2 p_{2.
                                                                                                                                                                       2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                                     t^2 \left( u_{1.\,x}^2 - 2\,u_{1.\,x}\,u_{2.\,x} + u_{2.\,x}^2 + u_{1.\,y}^2 - 2\,u_{1.\,y}\,u_{2.\,y} + u_{2.\,y}^2 + u_{1.\,z}^2 - 2\,u_{1.\,z}\,u_{2.\,z} + u_{2.\,z}^2 \right) \, = \, 0 
\text{Out}[33] = \left\{ \left\{ t \rightarrow \left( -2 \; p_{1 \ldots x} \; u_{1 \ldots x} + 2 \; p_{2 \ldots x} \; u_{1 \ldots x} + 2 \; p_{1 \ldots x} \; u_{2 \ldots x} - 2 \; p_{2 \ldots x} \; u_{2 \ldots x} - 2 \; p_{1 \ldots y} \; u_{1 \ldots y} + 2 \; p_{2 \ldots y} \; u_{1 \ldots y} + 2 \; p_{2 \ldots y} \; u_{1 \ldots y} + 2 \; p_{2 \ldots y} \; u_{2 \ldots x} - 2 \; p_{2 \ldots x} \; u_{2 \ldots x} - 2 \; p_{2 \ldots x} \; u_{2 \ldots x} - 2 \; p_{2 \ldots x} \; u_{2 \ldots x} - 2 \; p_{2 \ldots y} \; u_{2 \ldots y} + 2 \; p_{2 \ldots y} \; u_{2 \ldots x} \right\} \right\}
                                                                                                                                                                                       2 p_{1, y} u_{2, y} - 2 p_{2, y} u_{2, y} - 2 p_{1, z} u_{1, z} + 2 p_{2, z} u_{1, z} + 2 p_{1, z} u_{2, z} - 2 p_{2, z} u_{2, z} 
                                                                                                                                                                                       \sqrt{\left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} - 
                                                                                                                                                                                                                                                                                              2\;p_{1.\;y}\;u_{2.\;y}\;+\;2\;p_{2.\;y}\;u_{2.\;y}\;+\;2\;p_{1.\;z}\;u_{1.\;z}\;-\;2\;p_{2.\;z}\;u_{1.\;z}\;-\;2\;p_{1.\;z}\;u_{2.\;z}\;+\;2\;p_{2.\;z}\;u_{2.\;z}\;\right)^{2}\;-
                                                                                                                                                                                                                                            4 \left( p_{1,x}^2 - 2 p_{1,x} p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2 p_{1,y} p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2 p_{1,z} p_{2,z} + p_{2,z}^2 - r_1^2 - 2 r_1 r_2 - 2 r_1 r_2 + r_2^2 r_1 r_2 - r_1^2 - r_1^2 r_2 r_1 r_2 - r_1^2 r_1 r_2 r_1 r_2 - r_1^2 r_1 r_2 r_2 r_1 r
                                                                                                                                                                                                                                                                                           (u_{1,x}^2) (u_{1,x}^2 - 2u_{1,x}u_{2,x} + u_{2,x}^2 + u_{1,y}^2 - 2u_{1,y}u_{2,y} + u_{2,y}^2 + u_{1,z}^2 - 2u_{1,z}u_{2,z} + u_{2,z}^2))
                                                                                                                                                       \left(2\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)\right)\right\}
                                                                                                    \left\{t \to \left(-2\,p_{1.\,x}\,u_{1.\,x} + 2\,p_{2.\,x}\,u_{1.\,x} + 2\,p_{1.\,x}\,u_{2.\,x} - 2\,p_{2.\,x}\,u_{2.\,x} - 2\,p_{1.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{2.\,y} + 2\,p_{2.\,y}\,
                                                                                                                                                                                         2 p_{1. y} u_{2. y} - 2 p_{2. y} u_{2. y} - 2 p_{1. z} u_{1. z} + 2 p_{2. z} u_{1. z} + 2 p_{1. z} u_{2. z} - 2 p_{2. z} u_{2. z} +
                                                                                                                                                                                         \sqrt{\left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 
                                                                                                                                                                                                                                                                                              2p_{1,y}u_{2,y} + 2p_{2,y}u_{2,y} + 2p_{1,z}u_{1,z} - 2p_{2,z}u_{1,z} - 2p_{1,z}u_{2,z} + 2p_{2,z}u_{2,z}
                                                                                                                                                                                                                                            4 \left( p_{1,x}^2 - 2 p_{1,x} p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2 p_{1,y} p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2 p_{1,z} p_{2,z} + p_{2,z}^2 - r_1^2 - 2 r_1 r_2 - 2 r_1 r_2 + r_2^2 r_1 r_2 - r_1^2 - r_1^2 r_2 r_1 r_2 - r_1^2 r_1 r_2 r_1 r_2 - r_1^2 r_1 r_2 r
                                                                                                                                                                                                                                                                                              (u_{1}^{2}) (u_{1}^{2}) (u_{1}^{2}) (u_{1}^{2}) (u_{1}^{2}) (u_{2}^{2}) (u_{1}^{2}) (u_{2}^{2}) (u_{1}^{2}) (u_{2}^{2}) (u_{1}^{2}) (u_{2}^{2}) 
                                                                                                                                                       \left.\left(2\left(u_{1.\,x}^{2}-2\,u_{1.\,x}\,u_{2.\,x}+u_{2.\,x}^{2}+u_{1.\,y}^{2}-2\,u_{1.\,y}\,u_{2.\,y}+u_{2.\,y}^{2}+u_{1.\,z}^{2}-2\,u_{1.\,z}\,u_{2.\,z}+u_{2.\,z}^{2}\right)\right)\right\}\right\}
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