```
Clear["*"];
    (*sphere/sphere collision resolver*)
p_1 = \{p_{1.x}, p_{1.y}, p_{1.z}\}; (*position*)
 u_1 = \{u_{1.x}, u_{1.y}, u_{1.z}\}; (*velocity*)
r<sub>1</sub>; (*radius*)
p_2 = \{p_{2.x}, p_{2.y}, p_{2.z}\};
u_2 = \{u_{2.x}, u_{2.y}, u_{2.z}\};
 Solve[EuclideanDistance[p_1 + u_1 t, p_2 + u_2 t] = r_1 + r_2, t]
    \left\{ \left\{ t \rightarrow \left( -2\,p_{1.\,x}\,u_{1.\,x} + 2\,p_{2.\,x}\,u_{1.\,x} + 2\,p_{1.\,x}\,u_{2.\,x} - 2\,p_{2.\,x}\,u_{2.\,x} - 2\,p_{1.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{2.\,y} + 2\,p_{2
                                                                       2 p_{1, y} u_{2, y} - 2 p_{2, y} u_{2, y} - 2 p_{1, z} u_{1, z} + 2 p_{2, z} u_{1, z} + 2 p_{1, z} u_{2, z} - 2 p_{2, z} u_{2, z} 
                                                                     \sqrt{\left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} - 
                                                                                                                                           2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                        4 \left( p_{1,x}^2 - 2 p_{1,x} p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2 p_{1,y} p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2 p_{1,z} p_{2,z} + p_{2,z}^2 - r_1^2 - 2 r_1 r_2 - 2 r_1 r_2 + r_2^2 r_1 r_2 - r_1^2 - r_1^2 r_1 r_2 - r_1^2 r_1 r_1 r_2 - r_1^2 r
                                                                                                                                         (u_{1,x}^2) (u_{1,x}^2 - 2u_{1,x}u_{2,x} + u_{2,x}^2 + u_{1,y}^2 - 2u_{1,y}u_{2,y} + u_{2,y}^2 + u_{1,z}^2 - 2u_{1,z}u_{2,z} + u_{2,z}^2))
                                                \left(2\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)\right)\right\}
               \left\{t \rightarrow \left(-2\,p_{1.\,x}\,u_{1.\,x} + 2\,p_{2.\,x}\,u_{1.\,x} + 2\,p_{1.\,x}\,u_{2.\,x} - 2\,p_{2.\,x}\,u_{2.\,x} - 2\,p_{1.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{2.\,y} + 2\,p_{2.\,y}\,
                                                                        2 p_{1. y} u_{2. y} - 2 p_{2. y} u_{2. y} - 2 p_{1. z} u_{1. z} + 2 p_{2. z} u_{1. z} + 2 p_{1. z} u_{2. z} - 2 p_{2. z} u_{2. z} +
                                                                       \sqrt{\left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} - 
                                                                                                                                           2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                         4 \left( p_{1,x}^2 - 2 p_{1,x} p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2 p_{1,y} p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2 p_{1,z} p_{2,z} + p_{2,z}^2 - r_1^2 - 2 r_1 r_2 - 2 r_1 r_2 + r_2^2 r_1 r_2 - r_1^2 - r_1^2 r_2 r_1 r_2 - r_1^2 r_1 r_2 r_1 r_2 - r_1^2 r_1 r_2 r
                                                                                                                                         (u_{1}^{2} + 2) (u_{1}^{2} + 2 u_{1} + u_{2} + u_{2} + u_{1}^{2} + u_{1}^{2} - 2 u_{1} + u_{2} + u_{
                                                \left.\left(2\,\left(u_{1.\,x}^{2}\,-\,2\,\,u_{1.\,x}\,\,u_{2.\,x}\,+\,u_{2.\,x}^{2}\,+\,u_{1.\,y}^{2}\,-\,2\,\,u_{1.\,y}\,\,u_{2.\,y}\,+\,u_{2.\,y}^{2}\,+\,u_{1.\,z}^{2}\,-\,2\,\,u_{1.\,z}\,\,u_{2.\,z}\,+\,u_{2.\,z}^{2}\right)\,\right)\,\right\}\right\}
    (*if overlapping pick nearest t<0, for collision detection check t>0 and t<1*)
np_1 = p_1 + u_1 t; (*move out of collision*)
np_2 = p_2 + u_2 t;
nml = Normalize[np<sub>1</sub> - np<sub>2</sub>];(*collision plane normal*)
m1; (*mass*)
\mathbf{m}_2;
                                                                          velocity given
                                                                                                                                                                                                                                                                     velocities received along collision plane normal*)
    (*
v_1 = u_1 - (u_1.nml) nml + (u_1.nml) nml ((m_1 - m_2) / (m_1 + m_2)) + (u_2.nml) nml (2 m_2 / (m_1 + m_2));
 v_2 = u_2 - (u_2 \cdot nm1) nm1 + (u_2 \cdot nm1) nm1 ((m_2 - m_1) / (m_1 + m_2)) + (u_1 \cdot nm1) nm1 (2 m_1 / (m_1 + m_2));
np_1 = np_1 + v_1 (1 - t); (*perform remaining dt with new velocities*)
np_2 = np_2 + v_2 (1 - t);
 v_1 = (u_1.nml) nml ((m_1 - m_2) / (m_1 + m_2)) + (u_2.nml) nml (2 m_2 / (m_1 + m_2));
v_2 = (u_2.nml) nml ((m_2 - m_1) / (m_1 + m_2)) + (u_1.nml) nml (2 m_1 / (m_1 + m_2));
m_1 u_1 + m_2 u_2 == m_1 v_1 + m_2 v_2;
```

```
(*sphere/plane collision resolver*)
p_g = \{0, 0, 0\}; (*dot on plane*)
n_q = Normalize[{0, 1, 0}];(*normal*)
p = {0, 1.5, 0}; (*sphere position*)
u = \{0, -1, 0\}; (*velocity*)
r = 1; (*radius*)
Remove[t];
f = (p + ut - p_g) \cdot n_g - r = 0; (*signed distance to plane*)
Plot[f, {t, 0, 1}]
Solve[f, t]
t = .5;
(*pick nearest t<0, t>1 or t<0 collision is in future or past*)
np = p + u t (*move sphere out of collision*)
v = u - 2 (u.n_g) n_g (*reflect*)
np = np + v (1 - t) (*second part of dt*)
\begin{array}{c} 0.8 \\ 0.8 \\ 0.4 \\ 0.2 \end{array} $DisplayFunction \begin{bmatrix} 0.8 \\ 0.4 \\ 0.2 \end{bmatrix}
\left\{ \left. \left\{ \, t \, \rightarrow \, -\, 1 \, . \, \, \left( \, -\, 0 \, .\, 5 \, +\, 1 \, . \, \, \left\{ \, 0 \, ,\, \, 1 \, .\, 5 \, ,\, \, 0 \, \right\}_{g} \right) \, \right\} \right\}
{0., 1., 0.}
{0,1,0}
\{0., 1.5, 0.\}
(*simplified*)
Clear["*"];
```

```
(*sphere/sphere collision resolver*)
p_1 = \{p_{1.x}, p_{1.y}, p_{1.z}\}; (*position*)
 u_1 = \{u_{1.x}, u_{1.y}, u_{1.z}\}; (*velocity*)
r<sub>1</sub>; (*radius*)
p_2 = \{p_{2.x}, p_{2.y}, p_{2.z}\};
 u_2 = \{u_{2.x}, u_{2.y}, u_{2.z}\};
r2;
  Solve [EuclideanDistance [p_1 + u_1 t, p_2 + u_2 t] = r_1 + r_2, t]
  \left\{\left\{t \to \left(-2 \, p_{1.\,x} \, u_{1.\,x} + 2 \, p_{2.\,x} \, u_{1.\,x} + 2 \, p_{1.\,x} \, u_{2.\,x} - 2 \, p_{2.\,x} \, u_{2.\,x} - 2 \, p_{1.\,y} \, u_{1.\,y} + 2 \, p_{2.\,y} \, u_{1.\,y} + 2 \, p_{2.\,y} \, u_{1.\,y} + 2 \, p_{2.\,y} \, u_{2.\,x} + 2 \, p_{2.\,y} \, u_{2.\,x} + 2 \, p_{2.\,y} \, u_{2.\,x} + 2 \, p_{2.\,y} \, u_{2.\,y} + 2 \, p
                                                                    2\ p_{1.\ y}\ u_{2.\ y}-2\ p_{2.\ y}\ u_{2.\ y}-2\ p_{1.\ z}\ u_{1.\ z}+2\ p_{2.\ z}\ u_{1.\ z}+2\ p_{1.\ z}\ u_{2.\ z}-2\ p_{2.\ z}\ u_{2.\ z}-2
                                                                    \sqrt{\left(2 p_{1,x} u_{1,x} - 2 p_{2,x} u_{1,x} - 2 p_{1,x} u_{2,x} + 2 p_{2,x} u_{2,x} + 2 p_{1,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{2,x} + 2 p_{2,y} u_{2,x} + 2 p_{2,y} u_{2,y} - 
                                                                                                                                      2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                      4 \left( p_{1.\,x}^2 - 2\,p_{1.\,x}\,p_{2.\,x} + p_{2.\,x}^2 + p_{1.\,y}^2 - 2\,p_{1.\,y}\,p_{2.\,y} + p_{2.\,y}^2 + p_{1.\,z}^2 - 2\,p_{1.\,z}\,p_{2.\,z} + p_{2.\,z}^2 - r_1^2 - 2\,r_1\,r_2 - 2\,r_1^2 + 2\,r_2^2\,p_{2.\,z}^2 + 2\,r_2^2\,p_{2.\,z}^
                                                                                                                                      (u_{1,x}^2) (u_{1,x}^2 - 2u_{1,x}u_{2,x} + u_{2,x}^2 + u_{1,y}^2 - 2u_{1,y}u_{2,y} + u_{2,y}^2 + u_{1,z}^2 - 2u_{1,z}u_{2,z} + u_{2,z}^2))
                                              \left(2\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)\right)\right\}
           \left\{ \text{t} \,\rightarrow\, \left( \text{-2}\,\, \text{p}_{\text{1.}\,\text{x}}\,\, \text{u}_{\text{1.}\,\text{x}} \,+\, 2\,\, \text{p}_{\text{2.}\,\text{x}}\,\, \text{u}_{\text{1.}\,\text{x}} \,+\, 2\,\, \text{p}_{\text{1.}\,\text{x}}\,\, \text{u}_{\text{2.}\,\text{x}} \,-\, 2\,\, \text{p}_{\text{2.}\,\text{x}}\,\, \text{u}_{\text{2.}\,\text{x}} \,-\, 2\,\, \text{p}_{\text{1.}\,\text{y}}\,\, \text{u}_{\text{1.}\,\text{y}} \,+\, 2\,\, \text{p}_{\text{2.}\,\text{y}}\,\, \text{u}_{\text{1.}\,\text{y}} \,+\, 2\,\, \text{p}_{\text{2.}\,\text{y}}\,\, \text{u}_{\text{1.}\,\text{y}} \,+\, 2\,\, \text{p}_{\text{2.}\,\text{y}}\,\, \text{u}_{\text{2.}\,\text{y}} \,+\, 2
                                                                    2 p_{1, y} u_{2, y} - 2 p_{2, y} u_{2, y} - 2 p_{1, z} u_{1, z} + 2 p_{2, z} u_{1, z} + 2 p_{1, z} u_{2, z} - 2 p_{2, z} u_{2, z} +
                                                                   \sqrt{\left(2 p_{1,x} u_{1,x} - 2 p_{2,x} u_{1,x} - 2 p_{1,x} u_{2,x} + 2 p_{2,x} u_{2,x} + 2 p_{1,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{2,x} + 2 p_{2,x} u_{2,x} + 2 p_{2,y} u_{2,y} - 
                                                                                                                                      2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                    4 \left( p_{1,x}^2 - 2 p_{1,x} p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2 p_{1,y} p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2 p_{1,z} p_{2,z} + p_{2,z}^2 - r_1^2 - 2 r_1 r_2 - 2 r_1 r_2 \right)
                                                                                                                                      (u_{1,x}^2) (u_{1,x}^2 - 2u_{1,x}u_{2,x} + u_{2,x}^2 + u_{1,y}^2 - 2u_{1,y}u_{2,y} + u_{2,y}^2 + u_{1,z}^2 - 2u_{1,z}u_{2,z} + u_{2,z}^2))
                                              \left\{2\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)\right\}\right\}
EuclideanDistance [p_1 + u_1 t, p_2 + u_2 t]
  \sqrt{\left(\text{Abs}[p_{1.x} - p_{2.x} + t u_{1.x} - t u_{2.x}]^2 + \right)}
                                 Abs \left[ p_{1.y} - p_{2.y} + t u_{1.y} - t u_{2.y} \right]^{2} + Abs \left[ p_{1.z} - p_{2.z} + t u_{1.z} - t u_{2.z} \right]^{2}
 \sqrt{\text{Total}[(p_1 + u_1 t - (p_2 + u_2 t))^2]} == r_1 + r_2
 \sqrt{\left(\left.\left(p_{1.\,x}-p_{2.\,x}+t\,u_{1.\,x}-t\,u_{2.\,x}\right)^{\,2}+\left(p_{1.\,y}-p_{2.\,y}+t\,u_{1.\,y}-t\,u_{2.\,y}\right)^{\,2}+\left.\left(p_{1.\,z}-p_{2.\,z}+t\,u_{1.\,z}-t\,u_{2.\,z}\right)^{\,2}\right)}=0
    \left(\sqrt{\text{Total}[(p_1 + u_1 t - (p_2 + u_2 t))^2]}\right)^2 == (r_1 + r_2)^2
   \left( p_{1.\,x} - p_{2.\,x} + t\,u_{1.\,x} - t\,u_{2.\,x} \right)^{\,2} + \, \left( p_{1.\,y} - p_{2.\,y} + t\,u_{1.\,y} - t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{1.\,z} - p_{2.\,z} + t\,u_{1.\,z} - t\,u_{2.\,z} \right)^{\,2} = \, \left( r_1 + r_2 \right)^{\,2} + \, \left( p_{1.\,y} - p_{2.\,y} + t\,u_{1.\,y} - t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{1.\,z} - p_{2.\,z} + t\,u_{1.\,z} - t\,u_{2.\,z} \right)^{\,2} = \, \left( p_{1.\,y} - p_{2.\,y} + t\,u_{1.\,y} - t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{1.\,z} - p_{2.\,z} + t\,u_{1.\,z} - t\,u_{2.\,z} \right)^{\,2} = \, \left( p_{1.\,y} - p_{2.\,y} + t\,u_{1.\,y} - t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{1.\,y} - p_{2.\,z} + t\,u_{2.\,z} \right)^{\,2} = \, \left( p_{1.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2} + \, \left( p_{2.\,y} - p_{2.\,y} + t\,u_{2.\,y} \right)^{\,2
 Total [(p_1 + u_1 t - (p_2 + u_2 t))^2] = (r_1 + r_2)^2
     (p_{1,x} - p_{2,x} + t u_{1,x} - t u_{2,x})^2 + (p_{1,y} - p_{2,y} + t u_{1,y} - t u_{2,y})^2 + (p_{1,z} - p_{2,z} + t u_{1,z} - t u_{2,z})^2 = (r_1 + r_2)^2 
\Delta p = p_1 - p_2
  \{p_{1.x} - p_{2.x}, p_{1.y} - p_{2.y}, p_{1.z} - p_{2.z}\}
```

```
Collect Expand Total \left[ \left( \Delta p + u_1 t - u_2 t \right)^2 \right] - \left( r_1 + r_2 \right)^2 = 0, t
p_{1.x}^2 - 2p_{1.x}p_{2.x} + p_{2.x}^2 + p_{1.y}^2 - 2p_{1.y}p_{2.y} + p_{2.y}^2 + p_{1.z}^2 - 2p_{1.z}p_{2.z} + p_{2.z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_1^
                        r_{2}^{2} + t \left(2\,p_{1.\,x}\,u_{1.\,x} - 2\,p_{2.\,x}\,u_{1.\,x} - 2\,p_{1.\,x}\,u_{2.\,x} + 2\,p_{2.\,x}\,u_{2.\,x} + 2\,p_{1.\,y}\,u_{1.\,y} - 2\,p_{2.\,y}\,u_{1.\,y} - 2\,p_{2.\,y}\,u_{1.\,y} - 2\,p_{2.\,y}\,u_{2.\,x} + 2\,p_{2.\,x}\,u_{2.\,x} + 2\,p_{2.\,x}\,u_{2.\,x} + 2\,p_{2.\,x}\,u_{2.\,x} + 2\,p_{2.\,y}\,u_{2.\,x} + 2\,p_{2.\,y}\,u_{2.\,x} + 2\,p_{2.\,y}\,u_{2.\,y} + 2\,
                                                           2 p_{1.y} u_{2.y} + 2 p_{2.y} u_{2.y} + 2 p_{1.z} u_{1.z} - 2 p_{2.z} u_{1.z} - 2 p_{1.z} u_{2.z} + 2 p_{2.z} u_{2.z}
                        t^{2}\left(u_{1,x}^{2}-2\,u_{1,x}\,u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2\,u_{1,y}\,u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2\,u_{1,z}\,u_{2,z}+u_{2,z}^{2}\right) = 0
Collect \left[ \text{Expand} \left[ \text{Total} \left[ (\Delta p + (u_1 - u_2) t)^2 \right] - (r_1 + r_2)^2 = 0 \right], t \right]
p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_2^2 - r_1^2 - r_2^2 - r_2^
                      r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} + 2 p_{2.y} u_
                                                           2 p_{1.y} u_{2.y} + 2 p_{2.y} u_{2.y} + 2 p_{1.z} u_{1.z} - 2 p_{2.z} u_{1.z} - 2 p_{1.z} u_{2.z} + 2 p_{2.z} u_{2.z} + 
                        t^{2}\left(u_{1}^{2} + 2u_{1} + u_{2} + u_{2}^{2} + u_{1}^{2} + u_{1}^{2} + u_{1}^{2} + u_{2}^{2} + u_{1}^{2} + u_{2}^{2} + u_{2}^{2} + u_{1}^{2} + u_{2}^{2} + u_{2}^{2} \right) = 0
\Delta \mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2
   \{u_{1.x} - u_{2.x}, u_{1.y} - u_{2.y}, u_{1.z} - u_{2.z}\}
Collect \left[ \text{Expand} \left[ \text{Total} \left[ (\Delta p + \Delta u t)^2 \right] - (r_1 + r_2)^2 = 0 \right], t \right]
p_{1.x}^2 - 2p_{1.x}p_{2.x} + p_{2.x}^2 + p_{1.y}^2 - 2p_{1.y}p_{2.y} + p_{2.y}^2 + p_{1.z}^2 - 2p_{1.z}p_{2.z} + p_{2.z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_1^
                      r_2^2 + t \left( 2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} \right)
                                                           2 p_{1.y} u_{2.y} + 2 p_{2.y} u_{2.y} + 2 p_{1.z} u_{1.z} - 2 p_{2.z} u_{1.z} - 2 p_{1.z} u_{2.z} + 2 p_{2.z} u_{2.z} + 1 + 2 p_{2.z} u_{2.z} + 2 p_{2.z} u_{2.z}
                        t^{2}\left(u_{1,x}^{2}-2\,u_{1,x}\,u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2\,u_{1,y}\,u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2\,u_{1,z}\,u_{2,z}+u_{2,z}^{2}\right)=0
Collect Expand Total [\Delta p^2 + 2 \Delta p \Delta u t + \Delta u^2 t^2] - (r_1 + r_2)^2 = 0], t
p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2
                      r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} + 2 p_{2.y} u_
                                                           2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z} + 2 p_{2,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                        \mathsf{t}^2 \left( \mathsf{u}^2_{1.\,x} - 2\,\mathsf{u}_{1.\,x}\,\mathsf{u}_{2.\,x} + \mathsf{u}^2_{2.\,x} + \mathsf{u}^2_{1.\,y} - 2\,\mathsf{u}_{1.\,y}\,\mathsf{u}_{2.\,y} + \mathsf{u}^2_{2.\,y} + \mathsf{u}^2_{1.\,z} - 2\,\mathsf{u}_{1.\,z}\,\mathsf{u}_{2.\,z} + \mathsf{u}^2_{2.\,z} \right) \, = \, 0
Collect \left[ \text{Total} \left[ \Delta p^2 + 2 \Delta p \Delta u \, t + \Delta u^2 \, t^2 \right] - (r_1 + r_2)^2 = 0, t \right]
    (p_{1.x} - p_{2.x})^2 + (p_{1.y} - p_{2.y})^2 + (p_{1.z} - p_{2.z})^2 - (r_1 + r_2)^2 +
                         \text{t} \, \left( 2 \, \left( p_{1.\,x} - p_{2.\,x} \right) \, \left( u_{1.\,x} - u_{2.\,x} \right) \, + \, 2 \, \left( p_{1.\,y} - p_{2.\,y} \right) \, \left( u_{1.\,y} - u_{2.\,y} \right) \, + \, 2 \, \left( p_{1.\,z} - p_{2.\,z} \right) \, \left( u_{1.\,z} - u_{2.\,z} \right) \right) \, + \, 2 \, \left( p_{2.\,x} - p_{2.\,z} \right) \, \left( u_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - p_{2.\,z} \right) \, \left( u_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - p_{2.\,z} \right) \, \left( u_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z} \right) \, + \, 2 \, \left( p_{2.\,x} - u_{2.\,z}
                        t^{2} \left( \left( u_{1.\,x} - u_{2.\,x} \right)^{2} + \left( u_{1.\,y} - u_{2.\,y} \right)^{2} + \left( u_{1.\,z} - u_{2.\,z} \right)^{2} \right) \; = \; 0
Total[\Delta p^2]
    (p_{1,x} - p_{2,x})^2 + (p_{1,y} - p_{2,y})^2 + (p_{1,z} - p_{2,z})^2
             \text{Expand} \left[ \text{Total} \left[ \Delta p^2 \right] - (r_1 + r_2)^2 + \text{t} \left( 2 \left( p_{1, \ x} - p_{2, \ x} \right) \left( u_{1, \ x} - u_{2, \ x} \right) + 2 \left( p_{1, \ y} - p_{2, \ y} \right) \left( u_{1, \ y} - u_{2, \ y} \right) + 2 \left( p_{1, \ y} - p_{2, \ y} \right) \right) \right] 
                                                                                  2(p_{1,z} - p_{2,z})(u_{1,z} - u_{2,z}) +
                                                t^{2} \left( \left( u_{1, x} - u_{2, x} \right)^{2} + \left( u_{1, y} - u_{2, y} \right)^{2} + \left( u_{1, z} - u_{2, z} \right)^{2} \right) = 0 \right], t
p_{1.x}^2 - 2p_{1.x}p_{2.x} + p_{2.x}^2 + p_{1.y}^2 - 2p_{1.y}p_{2.y} + p_{2.y}^2 + p_{1.z}^2 - 2p_{1.z}p_{2.z} + p_{2.z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_1^
                        r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} - 2 p_{2.y} u_
                                                             2 p_{1.y} u_{2.y} + 2 p_{2.y} u_{2.y} + 2 p_{1.z} u_{1.z} - 2 p_{2.z} u_{1.z} - 2 p_{1.z} u_{2.z} + 2 p_{2.z} u_{2.z}
                        2 Δρ.Δu t
   2 t \left( (p_{1.x} - p_{2.x}) (u_{1.x} - u_{2.x}) + (p_{1.y} - p_{2.y}) (u_{1.y} - u_{2.y}) + (p_{1.z} - p_{2.z}) (u_{1.z} - u_{2.z}) \right)
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Collect [Expand]
                                   Total \left[ \Delta p^2 \right] - (r_1 + r_2)^2 + 2 \Delta p. \Delta u t + t^2 \left( (u_1 \cdot x - u_2 \cdot x)^2 + (u_1 \cdot y - u_2 \cdot y)^2 + (u_1 \cdot z - u_2 \cdot z)^2 \right) = 0 \right], t
p_{1.x}^2 - 2p_{1.x}p_{2.x} + p_{2.x}^2 + p_{1.y}^2 - 2p_{1.y}p_{2.y} + p_{2.y}^2 + p_{1.z}^2 - 2p_{1.z}p_{2.z} + p_{2.z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_1^
                                   r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_
                                                                                       2 p_{1.y} u_{2.y} + 2 p_{2.y} u_{2.y} + 2 p_{1.z} u_{1.z} - 2 p_{2.z} u_{1.z} - 2 p_{1.z} u_{2.z} + 2 p_{2.z} u_{2.z}
                                   t^{2}\left(u_{1,x}^{2}-2\,u_{1,x}\,u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2\,u_{1,y}\,u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2\,u_{1,z}\,u_{2,z}+u_{2,z}^{2}\right) = 0
  Total \left[\Delta u^2\right]
      (u_{1,x} - u_{2,x})^2 + (u_{1,y} - u_{2,y})^2 + (u_{1,z} - u_{2,z})^2
Collect \left[ \text{Expand} \left[ \text{Total} \left[ \Delta p^2 \right] - (r_1 + r_2)^2 + 2 \Delta p \cdot \Delta u \, t + t^2 \, \text{Total} \left[ \Delta u^2 \right] = 0 \right], t \right]
p_{1,x}^2 - 2p_{1,x}p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2p_{1,y}p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2p_{1,z}p_{2,z} + p_{2,z}^2 - r_1^2 - 2r_1r_2 - r_2^2 - r_1^2 - r_2^2 - r_1^2 - r_2^2 - r_2^
                                r_2^2 + t \left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} + 2 p_{2.y} u_
                                                                                     2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z} + 
                                    \mathsf{t}^2 \, \left( \mathsf{u}^2_{1.\,\, \mathsf{x}} \, - \, 2 \, \mathsf{u}_{1.\,\, \mathsf{x}} \, \mathsf{u}_{2.\,\, \mathsf{x}} \, + \, \mathsf{u}^2_{2.\,\, \mathsf{x}} \, + \, \mathsf{u}^2_{1.\,\, \mathsf{y}} \, - \, 2 \, \mathsf{u}_{1.\,\, \mathsf{y}} \, \mathsf{u}_{2.\,\, \mathsf{y}} \, + \, \mathsf{u}^2_{2.\,\, \mathsf{y}} \, + \, \mathsf{u}^2_{1.\,\, \mathsf{z}} \, - \, 2 \, \mathsf{u}_{1.\,\, \mathsf{z}} \, \mathsf{u}_{2.\,\, \mathsf{z}} \, + \, \mathsf{u}^2_{2.\,\, \mathsf{z}} \, \right) \, = \, 0 
  a = Total[\Delta u^2];
b = 2 \Delta p. \Delta u;
c = Total[\Delta p^{2}] - (r_{1} + r_{2})^{2};
Collect [Expand[at^2 + bt + c = 0], t]
p_{1,\,x}^2-2\,p_{1,\,x}\,p_{2,\,x}+p_{2,\,x}^2+p_{1,\,y}^2-2\,p_{1,\,y}\,p_{2,\,y}+p_{2,\,y}^2+p_{1,\,z}^2-2\,p_{1,\,z}\,p_{2,\,z}+p_{2,\,z}^2-r_1^2-2\,r_1\,r_2-r_2^2-r_1^2-r_2^2-r_1^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-r_2^2-
                                   r_2^2 + t \left(2 p_{1,x} u_{1,x} - 2 p_{2,x} u_{1,x} - 2 p_{1,x} u_{2,x} + 2 p_{2,x} u_{2,x} + 2 p_{1,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{1,y} - 2 p_{2,y} u_{2,x} + 2 p_{2,x} u_{2,x} + 2 p_{2,y} u_{2,y} - 2 p_{2,y} u_
                                                                                       2 p_{1.y} u_{2.y} + 2 p_{2.y} u_{2.y} + 2 p_{1.z} u_{1.z} - 2 p_{2.z} u_{1.z} - 2 p_{1.z} u_{2.z} + 2 p_{2.z} u_{2.z} + + 2 p_{2.z
                                   t^{2}\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)=0
    Solve [EuclideanDistance [p_1 + u_1 t, p_2 + u_2 t] == r_1 + r_2, t] == Solve [a t^2 + b t + c == 0, t]
    Solve[at^2+bt+c=0,t]
True
    \left\{\left\{t \to \left(-2\,p_{1.\,x}\,u_{1.\,x} + 2\,p_{2.\,x}\,u_{1.\,x} + 2\,p_{1.\,x}\,u_{2.\,x} - 2\,p_{2.\,x}\,u_{2.\,x} - 2\,p_{1.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{1.\,y} + 2\,p_{2.\,y}\,u_{2.\,y} + 2\,p_{2.\,y
                                                                                                       2 p_{1. y} u_{2. y} - 2 p_{2. y} u_{2. y} - 2 p_{1. z} u_{1. z} + 2 p_{2. z} u_{1. z} + 2 p_{1. z} u_{2. z} - 2 p_{2. z} u_{2. z} -
                                                                                                    \sqrt{\left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 2 p_{2.y} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{2.y} u_{2.y} - 
                                                                                                                                                                                                         2 p_{1,y} u_{2,y} + 2 p_{2,y} u_{2,y} + 2 p_{1,z} u_{1,z} - 2 p_{2,z} u_{1,z} - 2 p_{1,z} u_{2,z} + 2 p_{2,z} u_{2,z}
                                                                                                                                                         4 \left( p_{1}^{2} \times -2 p_{1,x} p_{2,x} + p_{2}^{2} \times + p_{1}^{2} \times -2 p_{1,y} p_{2,y} + p_{2}^{2} \times + p_{1}^{2} \times -2 p_{1,z} p_{2,z} + p_{2}^{2} \times -r_{1}^{2} -2 r_{1} r_{2} - 2 r_{1} r_{2} - r_{1}^{2} -2 r_{1} r_{2} -2 r_{1} r
                                                                                                                                                                                                         (u_{1,x}^2) (u_{1,x}^2 - 2u_{1,x}u_{2,x} + u_{2,x}^2 + u_{1,y}^2 - 2u_{1,y}u_{2,y} + u_{2,y}^2 + u_{1,z}^2 - 2u_{1,z}u_{2,z} + u_{2,z}^2))
                                                                      \left(2\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)\right)\right\}
                     \left\{t \to \left(-2 \, p_{1, \, x} \, u_{1, \, x} + 2 \, p_{2, \, x} \, u_{1, \, x} + 2 \, p_{1, \, x} \, u_{2, \, x} - 2 \, p_{2, \, x} \, u_{2, \, x} - 2 \, p_{1, \, y} \, u_{1, \, y} + 2 \, p_{2, \, y} \, u_{1, \, y} + 2 \, p_{2, \, y} \, u_{1, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, y} + 2 \, p_{2, \, y} \, u_{2, \, 
                                                                                                       2\;p_{1.\;y}\;u_{2.\;y}-2\;p_{2.\;y}\;u_{2.\;y}-2\;p_{1.\;z}\;u_{1.\;z}+2\;p_{2.\;z}\;u_{1.\;z}+2\;p_{1.\;z}\;u_{2.\;z}-2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}\;u_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+2\;p_{2.\;z}+
                                                                                                       \sqrt{\left(2 p_{1.x} u_{1.x} - 2 p_{2.x} u_{1.x} - 2 p_{1.x} u_{2.x} + 2 p_{2.x} u_{2.x} + 2 p_{1.y} u_{1.y} - 2 p_{2.y} u_{1.y} - 
                                                                                                                                                                                                         2\;p_{1.\;y}\;u_{2.\;y}\;+\;2\;p_{2.\;y}\;u_{2.\;y}\;+\;2\;p_{1.\;z}\;u_{1.\;z}\;-\;2\;p_{2.\;z}\;u_{1.\;z}\;-\;2\;p_{1.\;z}\;u_{2.\;z}\;+\;2\;p_{2.\;z}\;u_{2.\;z}\;\rangle^{2}\;-
                                                                                                                                                         4 \left( p_{1,x}^2 - 2 p_{1,x} p_{2,x} + p_{2,x}^2 + p_{1,y}^2 - 2 p_{1,y} p_{2,y} + p_{2,y}^2 + p_{1,z}^2 - 2 p_{1,z} p_{2,z} + p_{2,z}^2 - r_1^2 - 2 r_1 r_2 - 2 r_1 r_2 + r_2^2 r_1 r_2 - r_1^2 - r_1^2 r_2 r_1 r_2 - r_1^2 r_1 r_2 r_1 r_2 - r_1^2 r_1 r_2 r
                                                                                                                                                                                                         r_{2}^{2} \left(u_{1,x}^{2} - 2u_{1,x}u_{2,x} + u_{2,x}^{2} + u_{1,y}^{2} - 2u_{1,y}u_{2,y} + u_{2,y}^{2} + u_{1,z}^{2} - 2u_{1,z}u_{2,z} + u_{2,z}^{2}\right)\right)
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 $\left(2\left(u_{1,x}^{2}-2u_{1,x}u_{2,x}+u_{2,x}^{2}+u_{1,y}^{2}-2u_{1,y}u_{2,y}+u_{2,y}^{2}+u_{1,z}^{2}-2u_{1,z}u_{2,z}+u_{2,z}^{2}\right)\right)\right\}$