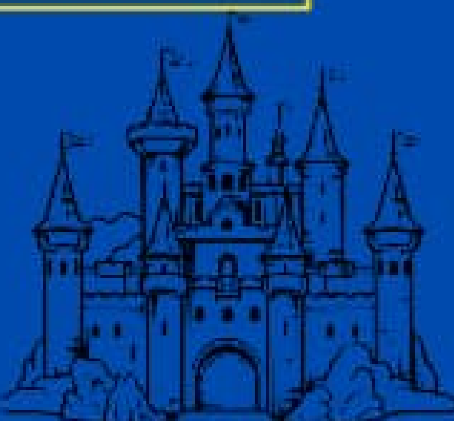
SOLUTION MANUAL
OPAL

BRIDGERTON EDITION

David Obot



OPAL TUTORIALS

G.E.T213 SOLUTION MANUAL

By God's grace, as you read this manual your knowledge about Engineering mathematics will improve magnificently and your understanding towards this course will be better!!! Give me the opportunity to guide you!!

Differentiation

y	dy/dx
x^n	nx^{n-1}
K	0
e^x	e^x
$\ln x$	$1/x$
a^x	$a^x \ln a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
Kx	K
e^{Kx}	Ke^{Kx}

Differentiation of Inverse Trigonometrical Function

$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1}x$	$\frac{1}{1+x^2}$ or $\frac{1}{x^2+1}$
$\cot^{-1}x$	$\frac{-1}{1+x^2}$
$\sec^{-1}x$	$\frac{1}{x\sqrt{x^2-1}}$
$\operatorname{cosec}^{-1}x$	$\frac{-1}{x\sqrt{x^2-1}}$
$\sinh^{-1}x$	$\frac{1}{\sqrt{x^2+1}}$
$\cosh^{-1}x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1}x$	$\frac{1}{1-x^2}$

Product Rule: If $y = uv$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule: If $y = \frac{u}{v}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Function of A function

(2)

For me, this is the most important topic in differentiation because this is the stepping stone or foundation to every other topics in differentiation.

(1.) $y = \sin(2x)$

$$\frac{dy}{dx} = 2\cos 2x$$

Step 1: Differentiate $2x$ which is 2

Step 2: Take $(2x)$ as u

Step 3: $\sin(2x)$ becomes $\sin u$

Step 4: Differentiate $\sin u$ which is $\cos u$

Step 5: multiply step 1 and step 4 which is

$2\cos u = 2\cos 2x$

(2.) $y = (3x^2+5)^4$

$$\frac{dy}{dx} = 4(3x^2+5)^{4-1} \times D(3x^2+5)$$

$$= 4(3x^2+5)^3(6x)$$

$$= 24x(3x^2+5)^3$$

(3) $y = \sin^3 x = (\sin x)^3$

If you understand no 2; this follows same pattern.

$$\frac{dy}{dx} = 3(\sin x)^{3-1} \times D(\sin x)$$
$$= 3(\sin x)^2 \cos x$$
$$= 3\sin^2 x \cos x \text{ or } 3\cos x \sin^2 x$$

(4.) $y = \cos^3(3x)$

This takes the same pattern as well but there is a slight difference.

do $y = (\cos 3x)^3$

$$\frac{dy}{dx} = 3(\cos 3x)^{3-1} \times D(\cos 3x)$$
$$= 3(\cos 3x)^2 \times -3\sin 3x$$
$$= -9\cos^2 3x \sin 3x$$

(5) $y = \ln(\sin 3x)$

$$\frac{dy}{dx} = \frac{1}{\sin 3x} \times D(\sin 3x)$$

$$\frac{dy}{dx} = \frac{1}{\sin 3x} \times 3\cos 3x = 3\cot 3x$$

$$\boxed{\frac{\cos 3x}{\sin 3x} = \cot 3x}$$

3

(6.) $y = \ln 5x^2$

$$\frac{dy}{dx} = \frac{1}{5x^2} \times 10x = \frac{2}{x}$$

You can notice; $\ln(fx)$ has one pattern which is $\frac{1}{f(x)} \cdot Df(x)$

(7.) $y = e^{4x}$

$$\frac{dy}{dx} = 4e^{4x}$$

The pattern is $e^{fx} \Rightarrow D(fx) \times e^{fx}$

(8) $y = e^{\sin x}$

$\frac{dy}{dx} = D(\sin x) \times e^{\sin x}$

$= \cos x \cdot e^{\sin x}$

(9.) $y = e^{3x^2}$

$\frac{dy}{dx} = 6xe^{3x^2}$

(10.) $y = 3\tan x$

4

$\frac{dy}{dx} = 3\sec^2 x$

(11.) $y = \operatorname{cosec}(x^2+1)$

$\frac{dy}{dx} = 2x\{-\operatorname{cosec}(x^2+1)\cot(x^2+1)\}$

$= -2x\operatorname{cosec}(x^2+1)\cot(x^2+1)$

Look at number 1 & see the differential of cosecx then start smiling. Life no suppose hard.

(12.) $y = \tan(5x+1)$

$\frac{dy}{dx} = 5\sec^2(5x+1)$

Provings

(1.) If $y = \ln\left[e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right]$, show that $\frac{dy}{dx} = \frac{x^2-1}{x^2-4}$

Recall $\ln ab = \ln a + \ln b$

Let $a = e^x$; $b = \left(\frac{x-2}{x+2}\right)^{3/4}$

$y = \ln\left[e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right] = \ln e^x + \ln\left(\frac{x-2}{x+2}\right)^{3/4}$

$y = \ln e^x + 3/4 \ln\left(\frac{x-2}{x+2}\right)$

$$\frac{dy}{dx} = \frac{1}{e^x} \cdot e^x + \frac{3}{4} \cdot \frac{1}{\left(\frac{x-2}{x+2}\right)} \cdot D\left(\frac{x-2}{x+2}\right)$$

$$\frac{dy}{dx} = 1 + \frac{3}{4}\left(\frac{x+2}{x-2}\right) D\left(\frac{x-2}{x+2}\right)$$

$$D\left(\frac{x-2}{x+2}\right) \Rightarrow u = x-2 \ ; \ v = x+2$$
$$du = 1 \ ; \ dv = 1$$
$$\Rightarrow \frac{vdu - udv}{v^2} = \frac{1(x+2) - 1(x-2)}{(x+2)^2} = \frac{x+2-x+2}{(x+2)^2} = \frac{4}{(x+2)^2}$$

$$\frac{dy}{dx} = 1 + \frac{3}{\cancel{4}}\left(\frac{\cancel{x+2}}{x-2}\right) \cdot \frac{\cancel{4}}{(x+2)^{\cancel{2}}}$$

$$\frac{dy}{dx} = 1 + \frac{3}{(x-2)(x+2)} = 1 + \frac{3}{(x^2-4)}$$

$$\frac{dy}{dx} = \frac{x^2-4+3}{x^2-4} = \frac{x^2-1}{x^2-4}$$

(2) If $xy + y^2 = 2$, show that $2y(x+y)dx^2 = (x+2y)^3 d^2y$

$$y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x+2y) = -y$$

$$\frac{dy}{dx} = \frac{-y}{x+2y}$$

$u = -y$

$du = -\frac{dy}{dx}$

$v = x + 2y$

$dv = 1 + 2\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} = \frac{vdu - udv}{v^2} = \frac{x+2y\left(-\frac{dy}{dx}\right) + y\left(1 + 2\frac{dy}{dx}\right)}{(x+2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-x\frac{dy}{dx}\quad x+2y\left(\frac{y}{x+2y}\right) + y + 2y\,dy/dx}{(x+2y)^2}$$

$$\frac{d^2y}{dx^2} = y + y + 2y\left(\frac{-y}{x+2y}\right) \div (x+2y)^2$$

$$\frac{d^2y}{dx^2}(x+2y)^2 = 2y - \frac{2y^2}{x+2y}$$

$$\frac{d^2y}{dx^2}(x+2y)^2 = \frac{2y(x+2y) - 2y^2}{(x+2y)}$$

$$\frac{d^2y}{dx^2}(x+2y)^3 = 2xy + 4y^2 - 2y^2$$

$$\frac{d^2y}{dx^2}(x+2y)^3 = 2xy + 2y^2$$

$$\frac{d^2y}{dx^2}(x+2y)^3 = 2y(x+y)$$

$$\therefore d^2y\,(x+2y)^3 = 2y(x+y)\,dx^2$$

3. If $Z = [v+\sqrt{1+v^2}]^p$ Show that $(1+v^2)\frac{d^2Z}{dv^2} + v\frac{dZ}{dv} - p^2Z = 0$

$$\frac{dZ}{dv} = p\left(v+\{1+v^2\}^{1/2}\right)^{p-1} \times 1 + \tfrac{1}{\cancel{2}}(1+v^2)^{-1/2}\cdot\cancel{2}v$$

$$= p\left(v+\sqrt{1+v^2}\right)^{p-1}\left(1+\frac{v}{\sqrt{1+v^2}}\right)$$

$$= p\left(v+\sqrt{1+v^2}\right)^{p-1}\left(\frac{\sqrt{1+v^2}+v}{\sqrt{1+v^2}}\right)$$

$$\frac{dZ}{dv} = \frac{p}{\sqrt{1+v^2}}\left(v+\sqrt{1+v^2}\right)^{p-1+1} = \frac{p\left(v+\sqrt{1+v^2}\right)^p}{\sqrt{1+v^2}}$$

$$\frac{d^2Z}{dv^2} \Rightarrow \quad u = p\left(v+\sqrt{1+v^2}\right)^p$$

$$du = p^2\left(v+\sqrt{1+v^2}\right)^{p-1}\left(1+\tfrac{1}{\cancel{2}}(1+v^2)^{-1/2}\cdot\cancel{2}v\right)$$

$$= p^2\left(v+\sqrt{1+v^2}\right)^{p-1}\left(1+\frac{v}{\sqrt{1+v^2}}\right)$$

$$du = \frac{p^2}{\sqrt{1+v^2}}\left(v+\sqrt{1+v^2}\right)^{p-1}\left(v+\sqrt{1+v^2}\right)'$$

$$du = \frac{p^2\left(v+\sqrt{1+v^2}\right)^p}{\sqrt{1+v^2}}$$

$$v = (1+v^2)^{1/2}$$

$$dv = 1/2\,(1+v^2)^{-1/2}\cdot 2v$$

$$= v\,(1+v^2)^{-1/2}$$

$$= \frac{v}{(1+v^2)^{1/2}} = \frac{v}{\sqrt{1+v^2}}$$

$$\frac{d^2Z}{dv^2} = \frac{v\,du - u\,dv}{v^2} = \frac{\cancel{\sqrt{1+v^2}}\cdot\frac{p^2\left(v+\sqrt{1+v^2}\right)^p}{\cancel{\sqrt{1+v^2}}} - \frac{p\left(v+\sqrt{1+v^2}\right)^p\cdot v}{\sqrt{1+v^2}}}{\left(\sqrt{1+v^2}\right)^2}$$

$$\frac{d^2Z}{dv^2}(1+v^2) = p^2\left(v+\sqrt{1+v^2}\right)^p - \frac{p\left(v+\sqrt{1+v^2}\right)^p\cdot v}{\sqrt{1+v^2}}$$

Recall $Z = \left(v+\sqrt{1+v^2}\right)^p$

$$\frac{dZ}{dv} = \frac{p\left(v+\sqrt{1+v^2}\right)^p}{\sqrt{1+v^2}}$$

substitute accordingly

$$\therefore \frac{d^2Z}{dv^2} = p^2Z - v\frac{dZ}{dv}$$

$$\therefore \frac{d^2Z}{dv^2} + v\frac{dZ}{dv} - p^2Z = 0$$

(4) If $x = \sin(p\sin^{-1}\theta)$, show that $(1-\theta^2)\frac{d^2x}{d\theta} - \theta\frac{dx}{d\theta}$
$+ p^2x = 0$

Before I do this guy, let's do a bit of differentiation of inverse of trig differentiation.

(i) $y = \sin^{-1}3x$

Recall that $D(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-(3x)^2}} \times D(3x) = \frac{3}{\sqrt{1-9x^2}}$$

(ii) $y = \tan^{-1}5x$

$$\frac{dy}{dx} = \frac{1}{1+(5x)^2} \cdot D(5x) = \frac{5}{1+25x^2}$$

(iii) $y = \sec^{-1}3x$

$$\frac{dy}{dx} = \frac{1}{|3x|\sqrt{(3x)^2-1}} \cdot D(3x) = \frac{3}{|3x|\sqrt{9x^2-1}}$$

So back to number 4.

$$x = \sin(p\sin^{-1}\theta)$$

$$\frac{dx}{d\theta} = \frac{p\cos(p\sin^{-1}\theta)}{\sqrt{1-\theta^2}}$$

$\frac{d^2x}{d^2\theta} \Rightarrow u = p\cos(p\sin^{-1}\theta)$

$$du = \frac{-p^2\sin(p\sin^{-1}\theta)}{\sqrt{1-\theta^2}}$$

$$v = \sqrt{1-\theta^2} = (1-\theta^2)^{1/2}$$

$$dv = \frac{1}{\cancel{2}}(1-\theta^2)^{-1/2} \cdot -\cancel{2}\theta = \frac{-\theta}{\sqrt{1-\theta^2}}$$

$$\frac{d^2x}{d\theta^2} = \frac{vdu - udv}{v^2} = \frac{\sqrt{1-\theta^2}\left(\frac{-P^2\sin(P\sin^{-1}\theta)}{\sqrt{1-\theta^2}}\right) - P\cos(P\sin^{-1}\theta)\frac{(-\theta)}{\sqrt{1-\theta^2}}}{\left(\sqrt{1-\theta^2}\right)^2}$$

Cross multiply.

$$\frac{d^2x}{d\theta^2}(1-\theta^2) = -P^2\sin(P\sin^{-1}\theta) + \frac{P\theta\cos(P\sin^{-1}\theta)}{\sqrt{1-\theta^2}}$$

Recall $x = \sin(P\sin^{-1}\theta)$

$$\frac{dx}{d\theta} = \frac{P\cos(P\sin^{-1}\theta)}{\sqrt{1-\theta^2}}$$

$$\frac{d^2x}{d\theta^2} = -P^2x + \theta\frac{dx}{d\theta}$$

$$\therefore \frac{d^2x}{d\theta^2} + P^2x - \theta\frac{dx}{d\theta} = 0$$

(5) If $x = 2\theta - \sin 2\theta$ \& $y = 1 - \cos 2\theta$, show that

$$\frac{dy}{dx} + \frac{d^2y}{dx^2} = \cot\theta - \frac{1}{4\sin^4\theta}$$

$$\frac{dx}{d\theta} = 2 - 2\cos 2\theta \; ; \; \frac{dy}{d\theta} = 2\sin 2\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 2\sin 2\theta \times \frac{1}{2-2\cos 2\theta}$$

$$\frac{dy}{dx} = \frac{\cancel{2}\sin 2\theta}{\cancel{2}(1-\cos 2\theta)} = \frac{\sin 2\theta}{1-\cos 2\theta}$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin 2\theta}{1-\cos 2\theta}\right) \times \frac{d\theta}{dx}$$

$$\frac{d}{d\theta}\left(\frac{\sin 2\theta}{1-\cos 2\theta}\right) \Rightarrow$$

$u = \sin 2\theta$

$v = 1 - \cos 2\theta$

$du = 2\cos 2\theta$

$dv = 2\sin 2\theta$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin 2\theta}{1-\cos 2\theta}\right) = \frac{v\,du - u\,dv}{v^2}$$

$$= \frac{2\cos 2\theta(1-\cos 2\theta) - \sin 2\theta(2\sin 2\theta)}{(1-\cos 2\theta)^2}$$

$$= \frac{2\cos 2\theta - 2\cos^2 2\theta - 2\sin^2 2\theta}{(1-\cos 2\theta)^2} = \frac{2\cos 2\theta - 2(\cos^2 2\theta + \sin^2 2\theta)}{(1-\cos 2\theta)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2\cos 2\theta - 2(\cos^2 2\theta + \sin^2 2\theta)}{(1-\cos 2\theta)^2} \times \frac{1}{2(1-\cos 2\theta)}$$

N/B: $\cos^2 2\theta + \sin^2 2\theta = 1$

$$\frac{d^2y}{dx^2} = \frac{2\cos 2\theta - 2(1)}{(1-\cos 2\theta)^2} \times \frac{1}{2(1-\cos 2\theta)}$$

$$= \frac{2(\cos 2\theta - 1)}{(1-\cos 2\theta)^2} \times \frac{1}{2(1-\cos 2\theta)}$$

$$\frac{d^2y}{dx^2} = \frac{-\cancel{2}\,\cancel{(1-\cos 2\theta)}}{(1-\cos 2\theta)^2} \times \frac{1}{\cancel{2}\,\cancel{(1-\cos 2\theta)}} = \frac{-1}{(1-\cos 2\theta)^2}$$

$$\therefore \frac{dy}{dx} + \frac{d^2y}{dx^2} = \frac{\sin 2\theta}{1-\cos 2\theta} - \frac{1}{(1-\cos 2\theta)^2}$$

But $\cos 2\theta = 1 - 2\sin^2\theta$;

$\sin 2\theta = 2\sin\theta\cos\theta$

$$= \frac{2\sin\theta\cos\theta}{1-(1-2\sin^2\theta)} - \frac{1}{(1-(1-2\sin^2\theta))^2}$$

$$= \frac{2\sin\theta\cos\theta}{1-1+2\sin^2\theta} - \frac{1}{(2\sin^2\theta)^2}$$

$$= \frac{\cancel{2}\,\cancel{\sin\theta}\cos\theta}{\cancel{2}\sin^{\cancel{2}}\theta} - \frac{1}{4\sin^4\theta}$$

$$= \cot\theta - \frac{1}{4\sin^4\theta}$$

(6.) If $a^2+2ab+3b^2=1$ prove that $(a+3b)^3\frac{d^2b}{da^2}+2(a^2+2ab+3b^2)=0$

$$2a+2b+2a\frac{db}{da}+6b\frac{db}{da}=0$$

$$\frac{db}{da}(2a+6b)=-2a-2b$$

$$\frac{db}{da}=\frac{2(-a-b)}{2(a+3b)}=\frac{-a-b}{a+3b}$$

$$\frac{d^2b}{da^2} \Rightarrow u=-a-b$$

$$du=-1-\frac{db}{da}$$

$$v=a+3b$$

$$dv=1+3\frac{db}{da}$$

$$\frac{d^2b}{da^2}=\frac{vdu-udv}{v^2}=\frac{(a+3b)\left(-1-\frac{db}{da}\right)-\left(1+3\frac{db}{da}\right)(-a-b)}{(a+3b)^2}$$

$$\frac{d^2b}{du^2}(a+3b)^2=(a+3b)\left(-1-\frac{db}{du}\right)-\left(1+3\frac{db}{da}\right)(-a-b)$$

$$\frac{d^2b}{da^2}(a+3b)^2=-a-a\frac{db}{da}-3b-3b\frac{db}{da}-\left[-a-b-3a\frac{db}{da}-3b\frac{db}{da}\right]$$

$$=-\cancel{a}-a\frac{db}{da}-\cancel{3}b-3b\frac{db}{da}+\cancel{a}+\cancel{b}+3a\frac{db}{da}+3b\frac{db}{da}$$

$$\frac{d^2b}{da^2}(a+3b)^2=-2b+2a\frac{db}{da}$$

Recall $\frac{db}{da}=\frac{-a-b}{a+3b}$

$$\frac{d^2b}{da^2}(a+3b)^2 = -2b + 2a\left(\frac{-a-b}{a+3b}\right)$$

$$\frac{d^2b}{da^2}(a+3b)^2 = \frac{-2b(a+3b)+2a(-a-b)}{(a+3b)}$$

$$\frac{d^2b}{da^2}(a+3b)^3 = -2ab - 6b^2 - 2a^2 - 2ab$$

$$\frac{d^2b}{da^2}(a+3b)^3 = -4ab - 6b^2 - 2a^2$$

$$\frac{d^2b}{da^2}(a+3b)^3 = -2(2ab+3b^2+a^2)$$

$$\frac{d^2b}{da^2}(a+3b)^3 + 2(2ab+3b^2+a^2) = 0$$

7.) If $y = (\sqrt{1-i^2})\arcsin i$, show that $(1-i^2)\frac{dy}{di}$

$= 1 - i^2 - iy.$

$$y = \sqrt{1-i^2}\sin^{-1}i = (1-i^2)^{1/2}\sin^{-1}i$$

$$u = (1-i^2)^{1/2}$$

$$du = \tfrac{1}{\cancel{2}}(1-i^2)^{-1/2} \cdot -\cancel{2}i = \frac{-i}{\sqrt{1-i^2}}$$

$$v = \sin^{-1}i$$

$$dv = \frac{1}{\sqrt{1-i^2}}$$

$$\frac{dy}{di} = u\,dv + v\,du = \cancel{(1-i^2)^{1/2}}\frac{1}{\cancel{\sqrt{1-i^2}}} + \sin^{-1}i\left(\frac{-i}{\sqrt{1-i^2}}\right)$$

$$\frac{dy}{di} = 1 - \frac{i\sin^{-1}i}{\sqrt{1-i^2}}$$

$$\frac{dy}{di} = \frac{\sqrt{1-i^2} - i\sin^{-1}i}{\sqrt{1-i^2}}$$

$$\frac{dy}{di}\sqrt{1-i^2} = \sqrt{1-i^2} - i\sin^{-1}i$$

Multiply both sides by $\sqrt{1-i^2}$

$$\frac{dy}{di}\left(\sqrt{1-i^2}\right)\left(\sqrt{1-i^2}\right) = \sqrt{1-i^2}\left(\sqrt{1-i^2} - i\sin^{-1}i\right)$$

$$\frac{dy}{di}(1-i^2) = 1-i^2 - i\sqrt{1-i^2}\sin^{-1}i$$

Recall $y = \sqrt{1-i^2}\sin^{-1}i$

$$\frac{dy}{di}(1-i^2) = 1-i^2 - iy$$

(8.) If $y = \frac{\sqrt{1+\cos x}}{\sqrt{1-\cos x}}$; Prove that $y' = \frac{-1}{1-\cos x}$

N|B: $1+\cos x = 2\cos^2(x/2)$

$1-\cos x = 2\sin^2(x/2)$

$1-\cos 2x = 2\sin^2 x$

$$y = \sqrt{\frac{2\cos^2(x/2)}{2\sin^2(x/2)}} = \sqrt{\frac{\cos^2(x/2)}{\sin^2(x/2)}}$$

$$y = \frac{\cos(x/2)}{\sin(x/2)} = \cot x/2$$

$$\therefore \frac{dy}{dx} = -\operatorname{cosec}^2 x/2 \times 1/2$$

$$= -1/2\left(\frac{1}{\sin^2 x/2}\right)$$

$$= \frac{-1}{2\sin^2 x/2} = \frac{-1}{1-\cos x}$$

(9.) If $(x-y)^3 = A(x+y)$, Prove that $(2x+y)\frac{dy}{dx} = x+2y$.

$$A = \frac{(x-y)^3}{(x+y)}$$

$$3(x-y)^2\left(1-\frac{dy}{dx}\right) = A\left(1+\frac{dy}{dx}\right)$$

$$3(x-y)^2\left(1-\frac{dy}{dx}\right) = \frac{(x-y)^3}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$3\left(1-\frac{dy}{dx}\right) = \frac{(x-y)}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$3(x+y)\left(1-\frac{dy}{dx}\right) = (x-y)\left(1+\frac{dy}{dx}\right)$$

$$(3x+3y)\left(1-\frac{dy}{dx}\right) = (x-y)\left(1+\frac{dy}{dx}\right)$$

$$3x - 3x\frac{dy}{dx} + 3y - 3y\frac{dy}{dx} = x + x\frac{dy}{dx} - y - y\frac{dy}{dx}$$

$$-3x\frac{dy}{dx} - 3y\frac{dy}{dx} - x\frac{dy}{dx} + y\frac{dy}{dx} = -3x - 3y + x - y$$

$$-4x\frac{dy}{dx} - 2y\frac{dy}{dx} = -2x - 4y$$

$$-2\frac{dy}{dx}(2x+y) = -2(x+2y)$$

$$\frac{dy}{dx}(2x+y) = (x+2y)$$

(10.) If $y = \sin(m\arcsin x)$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

$$y = \sin(m\sin^{-1}x)$$

$$\frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}\cos(m\sin^{-1}x)$$

$$\frac{d^2y}{dx^2} \Rightarrow u = m\cos(m\sin^{-1}x)$$

$$du = \frac{-m^2}{\sqrt{1-x^2}}\sin(m\sin^{-1}x)$$

$$v = \sqrt{1-x^2}\ ;\ dv = \tfrac{1}{\cancel{2}}(1-x^2)^{-1/2}\cdot -\cancel{2}x$$

$$dv = \frac{-x}{\sqrt{1-x^2}}$$

$$\frac{d^2y}{dx^2} = \frac{vdu - udv}{v^2} = \frac{\cancel{\sqrt{1-x^2}}\left(-\frac{m^2\sin(m\sin^{-1}x)}{\cancel{\sqrt{1-x^2}}}\right) + \frac{x}{\sqrt{1-x^2}}\left(m\cos(m\sin^{-1}x)\right)}{\left(\sqrt{1-x^2}\right)^2}$$

$$\frac{d^2y}{dx^2}(1-x^2) = -m^2\sin(m\sin^{-1}x) + x\left[\frac{m\cos(m\sin^{-1}x)}{\sqrt{1-x^2}}\right]$$

Recall $y = \sin(m\sin^{-1}x)$

$$\frac{dy}{dx} = \frac{m\cos(m\sin^{-1}x)}{\sqrt{1-x^2}}$$

$$\therefore \frac{d^2y}{dx^2}(1-x^2) = -m^2y + x\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}(1-x^2) + m^2y - x\frac{dy}{dx} = 0$$

(11.) If $\sinh y = \frac{4\sinh x - 3}{4 + 3\sinh x}$, show that $\frac{dy}{dx} = \frac{5}{4 + 3\sinh x}$

$$\cosh y\frac{dy}{dx} = \frac{(4+3\sinh x)(4\cosh x) - (4\sinh x - 3)(3\cosh x)}{(4+3\sinh x)^2}$$

$$\cosh y\frac{dy}{dx} = \frac{16\cosh x + 12\sinh x\cosh x - [12\sinh x\cosh x - 9\cosh x}{(4+3\sinh x)^2}$$

$$\cosh y\frac{dy}{dx} = \frac{16\cosh x + 12\sinh x\cosh x - 12\sinh x\cosh x + 9\cosh x}{(4+3\sinh x)^2}$$

$$\cosh y\frac{dy}{dx} = \frac{25\cosh x}{(4+3\sinh x)^2}$$

Hmmmm!!! we didn't get it. Let's use a different pattern. I want to get rid of y.

$$\sinh y = \frac{4\sinh x - 3}{4 + 3\sinh x}$$

$$y = \sinh^{-1}\left(\frac{4\sinh x - 3}{4 + 3\sinh x}\right) \quad ; \text{Let } a = \left(\frac{4\sinh x - 3}{4 + 3\sinh x}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2+1}} \cdot D(u)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\left(\frac{4\sinh x - 3}{4+3\sinh x}\right)^2 + 1}} \cdot D\left(\frac{4\sinh x - 3}{4 + 3\sinh x}\right)$$

$$D\left(\frac{4\sinh x - 3}{4 + 3\sinh x}\right) \Rightarrow \begin{array}{l} u = 4\sinh x - 3 \\ du = 4\cosh x \\ v = 4 + 3\sinh x \\ dv = 3\cosh x \end{array}$$

$$= \frac{v\,du - u\,dv}{v^2}$$

$$= \frac{(4+3\sinh x)(4\cosh x) - (4\sinh x - 3)(3\cosh x)}{(4 + 3\sinh x)^2}$$

$$= \frac{16\cosh x + 12\sinh x\cosh x - [12\sinh x\cosh x - 9\cosh x]}{(4+3\sinh x)^2}$$

$$= \frac{16\cosh x + 12\sinh x\cosh x - 12\sinh x\cosh x + 9\cosh x}{(4+3\sinh x)^2}$$

$$= \frac{25\cosh x}{(4+3\sinh x)^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{(4\sinh x - 3)^2 + (4+3\sinh x)^2}{(4+3\sinh x)^2}}} \cdot \frac{25\cosh x}{(4+3\sinh x)^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{16\sinh^2 x - \cancel{12\sinh x} - \cancel{12\sinh x} + 9 + 16 + \cancel{12\sinh x} + \cancel{12\sinh x} + 9\sinh^2 x}{(4+3\sinh x)^2}}}$$

$$\times \frac{25\cosh x}{(4+3\sinh x)^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{25\sinh^2 x + 25}{(4+3\sinh x)^2}}} \cdot \frac{25\cosh x}{(4+3\sinh x)^2}$$

But $\cosh^2 x = 1 + \sinh^2 x$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\frac{25(\sinh^2 x + 1)}{(4+3\sinh x)^2}}} \cdot \frac{25\cosh x}{(4+3\sinh x)^2}$$

$$\frac{dy}{dx} = \frac{1}{\frac{\sqrt{25\cosh^2 x}}{(4+3\sinh x)}} \cdot \frac{25\cosh x}{(4+3\sinh x)^2}$$

$$\frac{dy}{dx} = \frac{\cancel{4+3\sinh x}}{\cancel{5}\cancel{\cosh x}} \cdot \frac{\cancel{25}\cancel{\cosh x}}{(4+3\sinh x)^{\cancel{2}}} = \frac{5}{(4+3\sinh x)}$$

You are blushing now bah!!! yesss!!
Don't mention!!

Continuation of Differentiation

Eqn of Normal: $y - y_1 = -\frac{1}{m}(x - x_1)$
Eqn of tangent: $y - y_1 = m(x - x_1)$
x_1 & y_1 are mostly given or solved for pertaining to the question. while x & y are constants. $\frac{dy}{dx} = m$.
Radius of Curvature: $R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$

Centre of curvature $\Rightarrow h = x_1 - R\sin\theta$

$$K = y_1 + R\cos\theta$$

$$\tan\theta = \frac{dy}{dx}$$

If $\frac{d^2y}{dx^2} = -ve$; then y = maximum (y_{max})

If $\frac{d^2y}{dx^2} = +ve$; then y = minimum (y_{min})

for P.O.I (Point of Inflexion); $\frac{d^2y}{dx^2} = 0$

Stationary point: $\frac{dy}{dx} = 0$

(12) Find the stationary values of y and the points of inflexion of $y = te^{-t}$

$$\frac{dy}{dt} = t(-e^{-t}) + e^{-t}(1)$$

$$\frac{dy}{dt} = -te^{-t} + e^{-t} = e^{-t}(1-t)$$

$$e^{-t}(1-t) = 0$$

$$e^{-t} = 0$$

$$1-t = 0$$

Since $e^{-t} \neq 0$ for any real t, we focus on $1-t=0$

$$\therefore t = 1$$

$$y = 1e^{-1} = \frac{1}{e}$$

Stationary values: $(1, 1/e)$

For P.O.I; $\frac{d^2y}{dt^2} = 0$

$$\frac{dy}{dt} = e^{-t}(1-t)$$

$$\frac{d^2y}{dt^2} = e^{-t}(-1) + (1-t)(-e^{-t})$$

$$\frac{d^2y}{dt^2} = -e^{-t} - e^{-t} + te^{-t}$$
$$= -2e^{-t} + te^{-t}$$
$$= e^{-t}(t-2)$$

$$e^{-t}(t-2) = 0$$

we focus on $t-2=0$ since $e^{-t} \neq 0$ for any real t.

$$t = 2$$

for P.O.I to occur there has to be a change of sign at $t=2$

Pick a point before $t=2$ (i.e $2-a$) & a point after $t=2$ (i.e $2+a$)

$$\frac{d^2y}{dt^2} = e^{-t}(2-a-2) = e^{-t}(-a) \Rightarrow -ve$$

$$\frac{d^2y}{dt^2} = e^{-t}(2+a-2) = e^{-t}(a) \Rightarrow +ve$$

Since there is a sign change, P.O.I occurs at $t=2$

$$y = 2e^{-2} = \frac{2}{e^2}$$
$$\Rightarrow \left(2, \tfrac{2}{e^2}\right)$$

(13) Find the radius of curvature at any point on the curve $y = a\log\sec(\theta/a)$

$$\frac{dy}{d\theta} = \frac{a}{\sec(\theta/a)} \times D\left(\sec\{\theta/a\}\right)$$

$$= \frac{\cancel{a}}{\cancel{\sec(\theta/a)}} \times \frac{1}{\cancel{a}}\cancel{\sec(\theta/a)}\tan(\theta/a)$$

$$\frac{dy}{d\theta} = \tan(\theta/a)$$

$$\frac{d^2y}{d\theta^2} = \frac{1}{a}\sec^2(\theta/a)$$

$$R = \frac{\left[1+\left(\frac{dy}{d\theta}\right)^2\right]^{3/2}}{\frac{d^2y}{d\theta^2}} = \frac{\left[1+\tan^2(\theta/a)\right]^{3/2}}{\frac{1}{a}\sec^2(\theta/a)}$$

But $1+\tan^2\theta = \sec^2\theta$

$$R = \frac{\left[\sec^2(\theta/a)\right]^{3/2}}{\frac{1}{a}\sec^2(\theta/a)} = \frac{a\sec^3(\theta/a)}{\sec^2(\theta/a)} = a\sec(\theta/a)$$

(14) Given that $r = 2\cos t + \cos 2t$ \$ $y = 2\sin t - \sin 2t$, find $\frac{dy}{dr}$ interms of t \$ Compute its value for $t = \frac{\pi}{2}$

$$\frac{dr}{dt} = -2\sin t - 2\sin 2t$$

$$\frac{dy}{dt} = 2\cos t - 2\cos 2t$$

$$\frac{dy}{dr} = \frac{dy}{dt} \times \frac{dt}{dr} = \frac{2\cos t - 2\cos 2t}{-2\sin t - 2\sin 2t} = \frac{\cancel{2}(\cos t - \cos 2t)}{-\cancel{2}(\sin t + \sin 2t)}$$

$$\frac{dy}{dr} = \frac{\cos \pi/2 - \cos 2(\pi/2)}{-(\sin \pi/2 + \sin 2(\pi/2))} = \frac{\cos 90 - \cos 180}{-(\sin 90 + \sin 180)}$$

$\pi/2 = 90$

$\pi = 180$

$$\therefore \frac{dy}{dr} = \frac{0-(-1)}{-(1+0)} = \frac{1}{-1} = -1$$

(15.) The parametric eqns of a fxn are $x = 2\cos^3\theta$ & $y = 2\sin^3\theta$. Find the eqn of the normal & radius of curvature at the point for which $\theta = \pi/4$

$$\frac{dx}{d\theta} = 6\cos^2\theta(-\sin\theta) = -6\cos^2\theta\sin\theta$$

$$\frac{dy}{d\theta} = 6\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{\cancel{6}\sin^{\cancel{2}}\theta\,\cancel{\cos\theta}}{-\cancel{6}\cos^{\cancel{2}}\theta\,\cancel{\sin\theta}} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

$$y_1 = 2(\sin\pi/4)^3 = 2(\sin 45)^3 = 2\left(\tfrac{1}{\sqrt{2}}\right)^3 = \tfrac{1}{\sqrt{2}}$$

$$x_1 = 2(\cos\pi/4)^3 = 2(\cos 45)^3 = 2\left(\tfrac{1}{\sqrt{2}}\right)^3 = \tfrac{1}{\sqrt{2}}$$

Eqn of normal: $y - y_1 = -\frac{1}{m}(x - x_1)$

Recall $m = -\tan\theta = -\tan 45 = -1$

$$y - \tfrac{1}{\sqrt{2}} = -\tfrac{1}{-1}\left(x - \tfrac{1}{\sqrt{2}}\right)$$

$$y - \tfrac{1}{\sqrt{2}} = x - \tfrac{1}{\sqrt{2}}$$

$$y - x = 0$$

~~Assuming that the eqn $\frac{1}{5}x\sin(\pi y) + y - x^2 = 2$~~

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}(-\tan\theta) \times \frac{d\theta}{dx}$$

$$= -\sec^2\theta \times \frac{1}{-6\cos^2\theta\sin\theta}$$

$$= \frac{-1}{\cos^2\theta} \times \frac{1}{-6\cos^2\theta\sin\theta} = \frac{1}{6\cos^4\theta\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{1}{6\sin 45 (\cos 45)^4} = \frac{1}{6(1/\sqrt{2})(1/4)}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2\sqrt{2}}{3}$$

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+1)^{3/2}}{\frac{2\sqrt{2}}{3}} = \frac{\cancel{2^{3/2}}}{\cancel{2\sqrt{2}}} \times 3 = 3 \text{ units}$$

(16.) Find the Eqn of tangent $ normal to the curve

$$t^{2/3} + y^{2/3} = 2 \text{ at } (1,1)$$

in this context $y_1 = 1$ $ $t_1 = 1$

$$\frac{2}{3}t^{2/3-1} + 2/_3\, y^{2/3-1}\frac{dy}{dt} = 0$$

$$\frac{2}{3}t^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dt} = 0$$

$$\frac{2}{3\sqrt[3]{t}} + \frac{2}{3\sqrt[3]{y}}\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{\frac{-2}{3\sqrt[3]{t}}}{\frac{2}{3\sqrt[3]{y}}}$$

$$\text{at } (1,1) \Rightarrow \frac{\frac{-2}{3\sqrt[3]{1}}}{\frac{2}{3\sqrt[3]{1}}} = \frac{-2}{3} \div 2/_3 = \frac{-2}{3} \times 3/_2 = -1$$

Tangent: $y - y_1 = m(t - t_1)$

$$\Rightarrow y - 1 = -1(t-1)$$

$$y - 1 = -t + 1$$

$$y = -t + 1 + 1$$

$$y = 2 - t$$

$$\text{Normal: } Y - y_1 = -\frac{1}{m}(t - t_1)$$

$$y - 1 = \frac{-1}{-1}(t - 1)$$

$$y - 1 = t - 1$$

$$y = t - 1 + 1$$

$$y = t$$

$$y - t = 0$$

(17.) If $y = \ln\left(\frac{1+x}{1-x}\right)$ find $y'(x)$

$$\frac{dy}{dx} = \frac{1}{\left(\frac{1+x}{1-x}\right)} \times D\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1-x}{1+x} \times D\left(\frac{1+x}{1-x}\right)$$

$$D\left(\frac{1+x}{1-x}\right) \Rightarrow \begin{array}{l} u = 1+x \\ du = 1 \\ v = 1-x \\ dv = -1 \end{array}$$

$$= \frac{v\,du - u\,dv}{v^2} = \frac{1(1-x) - (1+x)(-1)}{(1-x)^2}$$

$$= \frac{1 - x + 1 + x}{(1-x)^2} = \frac{2}{(1-x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{1-x}{1+x} \cdot \frac{2}{(1-x)^2} = \frac{2}{(1+x)(1-x)} = \frac{2}{(1-x^2)}$$

(18.) Assuming that the eqn $\frac{1}{5}x\sin(\pi y) + y - x^2 = 2\pi$ defines y implicitly as a function of x in some neighbourhood of the Point $(x, y) = (1, 1/2)$. Find y' at this point.

$$\frac{1}{5}x\left[\pi\cos(\pi y)\frac{dy}{dx}\right] + \frac{1}{5}\sin(\pi y) + \frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx}\left[\frac{\pi x}{5}\cos(\pi y)\right] + \frac{dy}{dx} = 2x - \frac{\sin \pi y}{5}$$

$$\frac{dy}{dx}\left[\frac{\pi x\cos(\pi y)}{5}+1\right]=2x-\frac{\sin(\pi y)}{5}$$

$$\frac{dy}{dx}=2x-\frac{\sin(\pi y)}{5}\div\left(\frac{\pi x\cos(\pi y)}{5}+1\right)$$

at $(1, 1/2)$; $\frac{dy}{dx}=2(1)-\frac{\sin(\pi/2)}{5}\div\left(\frac{\pi(1)\cos(\pi/2)}{5}+1\right)$

$$=2-1/5\div\left(\frac{\pi(0)}{5}+1\right)$$

$$=\frac{10-1}{5}\div(1)$$

$$=9/5$$

(15.) Find the stationary points of the graph of the given function $f(x)=(x-1)^3(x-5)$

$$y=(x-1)^3(x-5)$$

$$\frac{dy}{dx}=(x-1)^3(1)+(x-5)3(x-1)^2(1)$$

$$\frac{dy}{dx}=(x-1)^3+(3x-15)(x-1)^2$$

$$\frac{dy}{dx}=\cancel{x(x-1)-1(x-1)}$$

$$\frac{dy}{dx}=(x-1)^2[(x-1)+3x-15]$$

$$=(x-1)^2[4x-16]$$

$$=4(x-1)^2(x-4)$$

$$\frac{dy}{dx}=0$$

$(x-1)^2=0$; $x-1=0$; $x=1$

$4(x-4)=0$; $x=4$

$\therefore$ stationary point are at $x=1, 4$

(20.) Find the radius of curvature & the coordinates of the centre of curvature at the point on the curve whose eqn is $y = x^2 + 5\ln x - 24$ where $x = 4$

$$\frac{dy}{dx} = 2x + \frac{5}{x} = 2x + 5x^{-1}$$

at $x = 4$; $\frac{dy}{dx} = 2(4) + \frac{5}{4} = 9.25$

$$\frac{d^2y}{dx^2} = 2 + \left(-\frac{5}{x^2}\right) = 2 - \frac{5}{x^2}$$

$$\frac{d^2y}{dx^2} = 2 - \frac{5}{4^2} = 1.6875$$

$$R = \frac{\left[1 + (9.25)^2\right]^{3/2}}{1.6875} = 477.2$$

$$\frac{dy}{dx} = \tan\theta$$

$9.25 = \tan\theta$

$\theta = \tan^{-1}(9.25) = 83.82^\circ$

$y_1 = (4)^2 + 5\ln 4 - 24 = -1.0685$

$h = x_1 - R\sin\theta$

$= 4 - 477.2\sin 83.82$

$= -470.4$

$K = y_1 + R\cos\theta$

$K = -1.0685 + 477.2\cos 83.82$

$= 50.30$

$\therefore$ Centre of curvature $= (-470.4, 50.30)$

(21.) If $\left(\frac{e^{4r}\sin r}{r\cos 2r}\right) = y$; Prove that $\frac{d}{dr}\left(\frac{e^{4r}\sin r}{r\cos 2r}\right) =$

$y\left\{\cot r + 4 + 2\tan 2r - \frac{1}{r}\right\}$

This is logarithmic differentiation. When you see more than two functions in a question, solve such questions using logarithmic differentiation

$$\ln y = \ln e^{4r} + \ln \sin r - \left(\ln r + \ln \cos 2r\right)$$

$$\frac{1}{y}\frac{dy}{dr} = \frac{1}{e^{4r}} \cdot 4e^{4r} + \frac{1}{\sin r} \cdot \cos r - \left(\frac{1}{r} \cdot 1 + \frac{1}{\cos 2r} \cdot -2\sin 2r\right)$$

$$\frac{1}{y}\frac{dy}{dr} = 4 + \cot r - \left(\frac{1}{r} - 2\tan 2r\right)$$

$$\frac{1}{y}\frac{dy}{dr} = 4 + \cot r - \frac{1}{r} + 2\tan 2r$$

$$\frac{dy}{dr} = y\left\{4 + \cot r - \frac{1}{r} + 2\tan 2r\right\}$$

(22) Determine the Equation of the line normal to the graph of $y = \tan^{-1}(\ln x)$ at $x = e$

$$y_1 = \tan^{-1}(\ln e)$$

But $\ln e = 1$

$$y_1 = \tan^{-1}(1) = 45^\circ = \pi/4$$

$$x_1 = e$$

$$\frac{dy}{dx} = \frac{1}{1+(\ln e)^2} \times \frac{1}{e} = \frac{1}{1+1} * \frac{1}{e} = \frac{1}{2e}$$

$$m = \frac{dy}{dx}$$

Eqn to normal: $y - \pi/4 = \frac{-1}{1/2e}(x - e)$

$$y - \pi/4 = -2e(x - e)$$

(23.) Find the maximum & minimum points of $f(\theta) = 3\theta^4 - 8\theta^3 - 6\theta^2 + 24\theta$. Distinguish b/w them & sketch the graph of the function.

$$f'\theta = 12\theta^3 - 24\theta^2 - 12\theta + 24$$

$$f'(\theta) = 12(\theta^3 - 2\theta^2 - \theta + 2) = 0$$

Divide b.s by 12.

$$\theta^3 - 2\theta^2 - \theta + 2 = 0$$

factorize the above expression.

$$\begin{array}{r}
\theta^2 - \theta - 2 \\
\theta - 1 \overline{\big) \theta^3 - 2\theta^2 - \theta + 2} \\
-(\theta^3 - \theta^2) \\
\hline
-\theta^2 - \theta + 2 \\
-(-\theta^2 + \theta) \\
\hline
-2\theta + 2 \\
-(-2\theta + 2) \\
\hline
0
\end{array}$$

$$(\theta - 1)(\theta^2 - \theta - 2)$$

$$(\theta - 1)(\theta + 1)(\theta - 2) = 0$$

$$\theta = 1, -1, 2$$

$$f''(\theta) = 3\theta^2 - 4\theta - 1$$

at $\theta = 1$; $f''(\theta) = 3(1)^2 - 4(1) - 1$

$= 3 - 4 - 1$

$= -ve$

$$f_{max} = 3(1)^4 - 8(1)^3 - 6(1)^2 + 24(1)$$

$= 13$

at $\theta = -1$; $f''(\theta) = 3(-1)^2 - 4(-1) - 1$

$= 3(1) + 4 - 1$

$= +ve$

$$f_{min} = 3(-1)^4 - 8(-1)^3 - 6(-1)^2 + 24(-1)$$

$= 3(1) - 8(-1) - 6(1) + 24(-1)$

$= 3 + 8 - 6 - 24$

$= -19$

at $\theta = 2$; $f''(\theta) = 3(2)^2 - 4(2) - 1$

$= 12 - 8 - 1 = +ve$

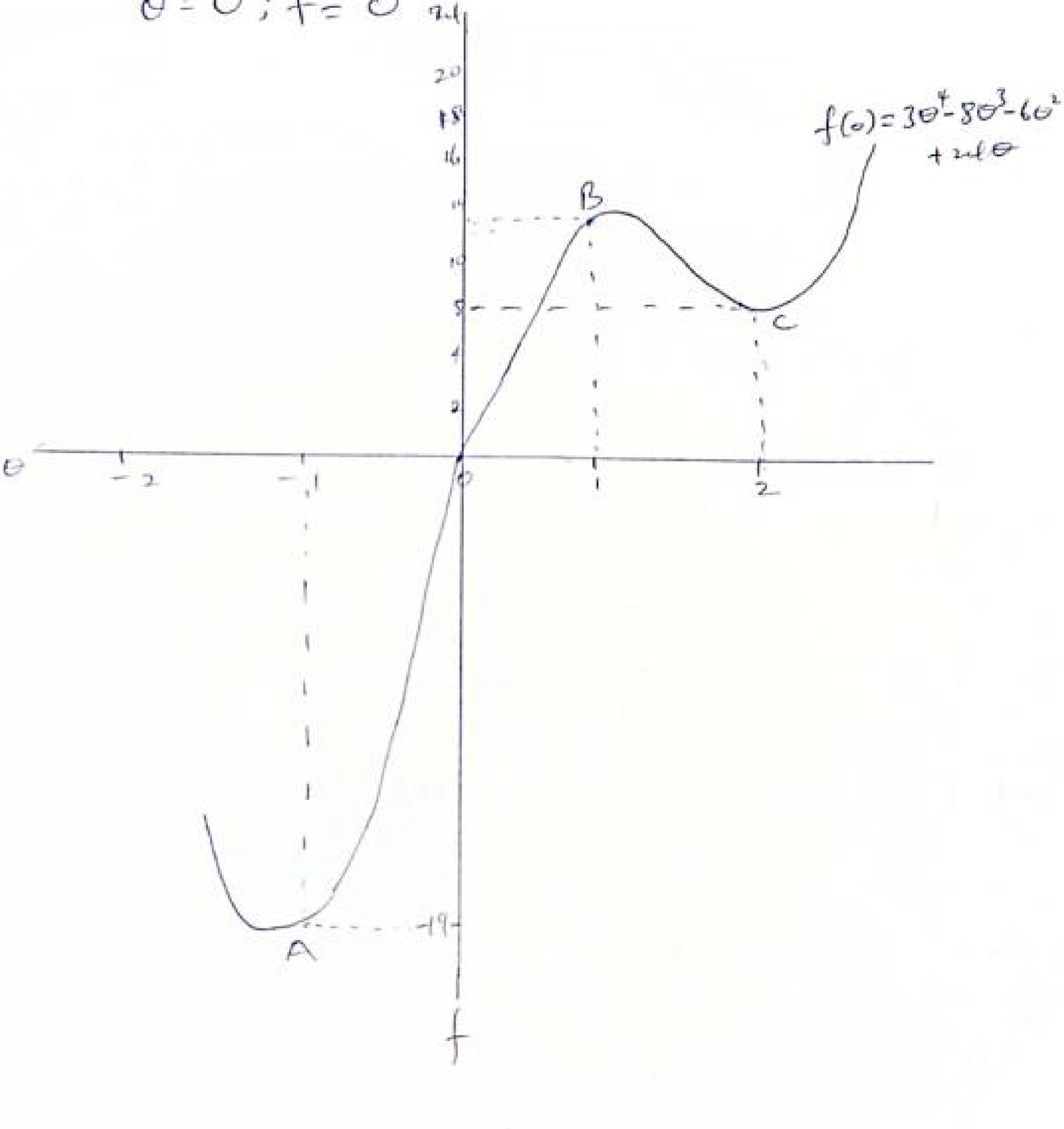
$$f_{min} = 3(2)^4 - 8(2)^3 - 6(2)^2 + 24(2)$$
$$= 48 - 64 - 24 + 48$$
$$= 96 - 88$$
$$= 8$$

at $\theta = -1$; $f_{min} = -19$

$\theta = 1$; $f_{max} = 13$

$\theta = 2$; $f_{min} = 8$

$\theta = 0$; $f = 0$



(24.) Prove that the centre of curvature (h, k) at the point $P(at^2, 2at)$ on the parabola $y^2 = 4ax$ has coordinates $h = 2a + 3at^2$, $K = -2at^3$.

~~$x = at$~~

Here, solving for R isn't necessary.

$$h = x - \frac{\frac{dy}{dx}\left(1 + \left(\frac{dy}{dx}\right)^2\right)}{\frac{d^2y}{dx^2}}$$

— this particular formula is the one without angle

$x = at^2$ | $y = 2at$

$\frac{dx}{dt} = 2at$ | $\frac{dy}{dt} = 2a$

$$\frac{dy}{dx} = \frac{2a}{2at} = 1/t = t^{-1}$$

$$\frac{d^2y}{dx^2} = -t^{-2} \times 1/2at = -\frac{1}{t^2}\left(\frac{1}{2at}\right) = \frac{-1}{2at^3}$$

$$h = x - \frac{\frac{1}{t}\left(1 + \left(\frac{1}{t}\right)^2\right)}{\frac{-1}{2at^3}}$$

$$h = at^2 - \frac{1/t\,(1 + 1/t^2)}{\frac{-1}{2at^3}}$$

$$h = at^2 - 1/\cancel{t}\,(1 + 1/t^2) \times \frac{-2at^{\cancel{3}}}{1}$$

$$h = at^2 + (1 + 1/t^2)(2at^2)$$

$$h = at^2 + (2at^2 + 2a)$$

$$h = 3at^2 + 2a$$

$$K = y_1 + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$ — Same with this one (without angle)

$$K = 2at + (1 + 1/t^2) \div \frac{-1}{2at^3}$$

$$K = 2at + (1 + 1/t^2)(-2at^3)$$

$$K = 2at - 2at^3 - 2at$$

$$K = -2at^3$$

$\therefore$ the coordinates $= (h, k) = (3at^2 + 2a, -2at^3)$

(25) Show that the numerical value of the radius of curvature (x_1, r_1) on the parabola $r^2 = 4ax$ is $\frac{-2(a+x_1)^{3/2}}{\sqrt{a}}$

$$r^2 = 4ax$$

$$r = \sqrt{4ax} = 2\sqrt{ax} = 2(ax)^{1/2}$$

$$\frac{dr}{dx} = 1/\cancel{2} \cdot \cancel{2}(ax)^{1/2-1} \times a = a(ax)^{-1/2}$$

$$\frac{dr}{dx} = a^1 \cdot a^{-1/2} \cdot x^{-1/2} = a^{1-1/2}x^{-1/2}$$

$$\frac{dr}{dx} = a^{1/2}x^{-1/2} = \frac{\sqrt{a}}{\sqrt{x}} = \sqrt{\frac{a}{x}}$$

$$\left(\frac{dr}{dx}\right)^2 = \left(\sqrt{\frac{a}{x}}\right)^2 = \frac{a}{x}$$

$$\frac{d^2r}{dx^2} = D\sqrt{\frac{a}{x}}$$

$$u = \sqrt{a};\ du = 0$$

$$v = \sqrt{x};\ dv = 1/2\, x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\therefore \frac{d^2r}{dx^2} = \frac{v\,du - u\,dv}{v^2} = \frac{\sqrt{x}(0) - \sqrt{a}\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x})^2}$$

$$\frac{d^2r}{dx^2} = \frac{0 - 2\sqrt{\frac{a}{x}}}{x} = \frac{-\sqrt{a}}{2x\sqrt{x}}$$

$$R = \frac{\left\{1 + \left(\frac{dr}{dr}\right)^2\right\}^{3/2}}{\frac{d^2r}{dx^2}} = \frac{\left\{1 + a/x\right\}^{3/2}}{\frac{-\sqrt{a}}{2x\sqrt{x}}}$$

$$R = \frac{2x\sqrt{x}\left\{\frac{x+a}{x}\right\}^{3/2}}{-\sqrt{a}} = \frac{2x\sqrt{x}\,\frac{(x+a)^{3/2}}{x^{3/2}}}{-\sqrt{a}} = \frac{-2(x+a)^{3/2}}{\sqrt{a}}$$

$$\therefore R = \frac{-2(a+x_1)^{3/2}}{\sqrt{a}}$$

(26) If $3ay^2 = x(x-a)^2$ with $a > 0$, proof that the radius of curvature at the point $(3a, 2a)$ is $\frac{50a}{3}$.

$$3ay^2 = x(x-a)^2$$

$$6ay\frac{dy}{dx} = x\left[2(x-a)\right] + (x-a)^2(1)$$

$$6ay\frac{dy}{dx} = 2x^2 - 2ax + x^2 - ax - ax + a^2$$

$$6ay\frac{dy}{dx} = 3x^2 - 4ax + a^2$$

$$\frac{dy}{dx} = \frac{3x^2 - 4ax + a^2}{6ay}$$

at $(3a, 2a)$; $$\frac{dy}{dx} = \frac{3(3a)^2 - 4a(3a) + a^2}{6a(2a)} = \frac{27a^2 - 12a^2 + a^2}{12a^2}$$

$$\frac{dy}{dx} = \frac{16\cancel{a^2}}{12\cancel{a^2}} = 4/3$$

$$\frac{d^2y}{dx^2} \Rightarrow u = 3x^2 - 4ax + a^2$$

$$du = 6x - 4a$$

$$v = 6ay$$

$$dv = 6a\,dy/dx = 6a(4/3) = 8a$$

$$\frac{d^2y}{dx^2} \Rightarrow \frac{v\,du - u\,dv}{v^2} = \frac{6ay(6x-4a) - (3x^2-4ax+a^2)(8a)}{(6ay)^2}$$

$$\frac{d^2y}{dx^2} = \frac{6a(2a)(6(3a)-4a) - (3(3a)^2-4a(3a)+a^2)(8a)}{36a^2y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12a^2(18a-4a) - 8a(27a^2-12a^2+a^2)}{36a^2(2a)^2}$$

$$= \frac{12a^2(14a) - 8a(16a^2)}{144a^4}$$

$$\frac{d^2y}{dx^2} = \frac{168a^3 - 128a^3}{144a^4} = \frac{40a^3}{144a^4} = \frac{5}{18a}$$

$$R = \frac{\left\{1+\left(\frac{4}{3}\right)^2\right\}^{3/2}}{\frac{5}{18a}} = \frac{\left(1+\frac{16}{9}\right)^{3/2}}{\frac{5}{18a}} = \frac{18a\left(\frac{25}{9}\right)^{3/2}}{5}$$

$$R = \frac{18a\left(\frac{125}{27}\right)}{5} = \frac{2}{3}a(125)\times\frac{1}{5} = \frac{50a}{3}$$

(27.) If $x = 2\theta - \sin 2\theta$ and $y = 1 - \cos 2\theta$, show that $\frac{dy}{dx} + \frac{d^2y}{dx^2} = \cot\theta - \frac{1}{4\sin^4\theta}$. If R is the radius of curvature at any point on the curve show that $R^2 = 8y$

Check reference i.e (no 5)

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Recall $\frac{dy}{dx} = \frac{\sin 2\theta}{1-\cos 2\theta}$

$$\frac{d^2y}{dx^2} = \frac{-1}{(1-\cos 2\theta)^2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{\sin^2 2\theta}{(1-\cos 2\theta)^2}$$

$$R^2 = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2 \times 2}}{\left(\frac{d^2y}{dx^2}\right)^2}$$

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[\frac{-1}{(1-\cos 2\theta)^2}\right]^2 = \frac{1}{(1-\cos 2\theta)^4}$$

$$\therefore R^2 = \left[1 + \frac{\sin^2 2\theta}{(1-\cos 2\theta)^2}\right]^3 \div \frac{1}{(1-\cos 2\theta)^4}$$

$$R^2 = \frac{\left((1-\cos 2\theta)^2 + \sin^2 2\theta\right)^3}{(1-\cos 2\theta)^6} \times \frac{(1-\cos 2\theta)^4}{1}$$

$$R^2 = \frac{\left[1(1-\cos 2\theta) - \cos 2\theta(1-\cos 2\theta) + \sin^2 2\theta\right]^3}{(1-\cos 2\theta)^2}$$

$$R^2 = \frac{\left[1-\cos 2\theta - \cos 2\theta + \cos^2 2\theta + \sin^2 2\theta\right]^3}{(1-\cos 2\theta)^2}$$

$$R^2 = \frac{(1-2\cos 2\theta + 1)^3}{(1-\cos 2\theta)^2}$$

$$R^2 = (2-2\cos 2\theta)^3 \div (1-\cos 2\theta)^2$$

$$R^2 = 2^3(1-\cos 2\theta)^3 \div (1-\cos 2\theta)^2$$

$$R^2 = 8\left(\frac{(1-\cos 2\theta)^{3}}{\cancel{(1-\cos 2\theta)^2}}\right) = 8(1-\cos 2\theta)$$

Recall $y = (1-\cos 2\theta)$

$$\therefore R^2 = 8y$$

(28.) Find the stationary values of y and the points of inflexion of $y = x^3 - 6x^2 + 9x + 6$

for stationary point, $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = 3x^2 - 12x + 9$$

$$3x^2 - 12x + 9 = 0$$
$$x^2 - 4x + 3 = 0$$
$$x^2 - 3x - x + 3 = 0$$
$$x(x-3) - 1(x-3) = 0$$
$$(x-3)(x-1) = 0$$
$$x = 3, 1$$

Point of inflexion: $\frac{d^2y}{dx^2} = 2x - 4 = 0$

$$2x = 4$$
$$x = 2$$

Test for change of signs; POI changes from +ve to -ve & vice versa. Take a point just before $x=2$; $x = 2-a$ & a point just after $x=2$; $x = 2+a$

at $x = 2-a$; $\frac{d^2y}{dx^2} = 6(2-a) - 12$
$= 12 - 6a - 12$
$= -6a$ (negative)

at $x = 2+a$; $\frac{d^2y}{dx^2} = 6(2+a) - 12$
$= 12 + 6a - 12$
$= 6a$ (Positive)

$\therefore$ POI $= 2$

at $x = 3$; $y = 3^3 - 6(3)^2 + 9(3) + 6 = 6$
$x = 1$; $y = (1)^3 - 6(1)^2 + 9(1) + 6 = 10$

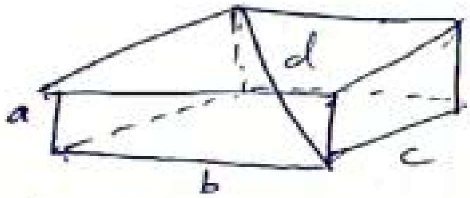
$\therefore$ stationary values $= (3, 6)$ & $(1, 10)$

~~Differentiate~~ Partial Differentiation

Small Increment

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

(8.) A rectangular box has sides measuring as 30mm, 40mm & 60mm. If these measurements are liable to be in error by ±0.5mm, ±0.8mm & ±1.0mm respectively. Calculate the length of the diagonal of the box & maximum possible error. (10 mks)



$$d^2 = a^2+b^2+c^2$$
$$d = \sqrt{a^2+b^2+c^2}$$

$$\delta d = \frac{\partial d}{\partial a}\delta a + \frac{\partial d}{\partial b}\delta b + \frac{\partial d}{\partial c}\delta c$$

$\delta a = \pm 0.5mm$; $\delta b = \pm 0.8mm$, $\delta c = \pm 1.0mm$

$a = 30mm$; $b = 40mm$; $c = 60mm$

$$d = (a^2+b^2+c^2)^{1/2}$$

$$\frac{\partial d}{\partial a} = \tfrac{1}{2}(a^2+b^2+c^2)^{1/2-1} \times 2a^2$$
$$= \frac{1}{\cancel{2}}(a^2+b^2+c^2)^{-1/2} \times \cancel{2}a$$
$$= (a^2+b^2+c^2)^{-1/2}a$$

$$\frac{\partial d}{\partial a} = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{30}{\sqrt{30^2+40^2+60^2}} = \frac{3}{\sqrt{61}}$$

$$\frac{\partial d}{\partial b} = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{40}{\sqrt{30^2+40^2+60^2}} = \frac{4}{\sqrt{61}}$$

$$\frac{\partial d}{\partial c} = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{60}{\sqrt{30^2+40^2+60^2}} = \frac{6}{\sqrt{61}}$$

$$\delta d = \frac{3}{\sqrt{61}} \times (\pm 0.5) + \frac{4}{\sqrt{61}}(\pm 0.8) \pm \frac{6}{\sqrt{61}}(\pm 1.0)$$

$$\delta d = \pm 1.360 mm$$

$$d = \sqrt{30^2 + 40^2 + 60^2} = 78.10 mm$$

(9.) If $Y = \frac{ws^3}{d^4}$, find percentage increase in y when w increases by 2%, s decreases by 3% & d increases by 1%. (10 mks)

$$y = ws^3 d^{-4}$$

$$\delta y = \frac{\partial y}{\partial w}\delta w + \frac{\partial y}{\partial s}\delta s + \frac{\partial y}{\partial d}\delta d$$

$$\frac{\partial y}{\partial w} = s^3 d^{-4} ; \frac{\partial y}{\partial s} = 3ws^2 d^{-4} ; \frac{\partial y}{\partial d} = -4ws^3 d^{-5}$$

$$\delta w = \frac{2w}{100} ; \delta s = \frac{-3s}{100} ; \delta d = \frac{1d}{100}$$

$$\delta y = \frac{s^3}{d^4}\left(\frac{2w}{100}\right) - \left(\frac{3ws^2}{d^4}\right)\left(\frac{3s}{100}\right) - \left(\frac{4ws^3}{d^5}\right)\left(\frac{d}{100}\right)$$

$$\delta y = \frac{ws^3}{d^4}\left(\frac{2}{100}\right) - \frac{ws^3}{d^4}\left(\frac{9}{100}\right) - \frac{ws^3}{d^4}\left(\frac{4}{100}\right)$$

$$\delta y = \frac{ws^3}{d^4}\left[\frac{2}{100} - \frac{9}{100} - \frac{4}{100}\right] \Rightarrow \frac{ws^3}{d^4}\left[\frac{-11}{100}\right]$$

Recall $\frac{ws^3}{d^4} = y$

$\therefore$ Y decreases by 11%

Rate of Change

$$V = f(r, h)$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial v}{\partial h}\cdot\frac{dh}{dt}$$

(10.) The total Surface area S of a cone of base radius r & Perpendicular height h is given by $S = \pi r^2 + \pi r\sqrt{r^2+h^2}$. If r & h are each increasing at the rate of 0.25cm/s, find the rate at which S is increasing at the instant when r = 3cm & h = 4cm. (10 mks)

$$S = \pi r^2 + \pi r\,(r^2+h^2)^{1/2}$$

Le

$$\frac{dS}{dt} = \frac{\partial s}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial s}{\partial h}\cdot\frac{dh}{dt}$$

I used product rule here.

$$\frac{\partial s}{\partial r} = 2\pi r + \pi r\left[\tfrac{1}{2}(r^2+h^2)^{1/2-1}\times 2r\right] + (r^2+h^2)^{1/2}(\pi)$$

$$= 2\pi r + \pi r\left[\frac{r}{(r^2+h^2)^{1/2}}\right] + \pi\sqrt{r^2+h^2}$$

$$= 2\pi r + \frac{\pi r^2}{\sqrt{r^2+h^2}} + \pi\sqrt{r^2+h^2}$$

$$\frac{\partial s}{\partial r} = 2\pi(3) + \frac{\pi(3)^2}{\sqrt{3^2+4^2}} + \pi\sqrt{3^2+4^2}$$

$$\frac{\partial s}{\partial r} = 6\pi + \frac{9\pi}{5} + 5\pi = 11\pi + \frac{9\pi}{5}$$

$$\frac{\partial s}{\partial r} = \frac{64\pi}{5}$$

$$\frac{\partial s}{\partial h} = \tfrac{1}{\cancel{2}}\pi r(r^2+h^2)^{-1/2}\times \cancel{2}h + 0\cdot(r^2+h^2)^{1/2}$$

$$= \frac{\pi r h}{\sqrt{r^2+h^2}} = \frac{\pi(3)(4)}{5} = \frac{12\pi}{5}$$

$$\frac{ds}{dt} = \frac{64\pi}{5}(0.25) + \frac{12\pi}{5}(0.25) = 3.8\pi$$

$$\therefore \frac{ds}{dt} = 11.94\text{cm}^2/\text{s}$$

(11.) The base radius of a cone, r, is decreasing at the rate of 0.1cm/s while the perpendicular height h is increasing at the rate of 0.2cm/s. Find the rate at which the volume, V is changing when r=2cm & h=3cm.
(5 mks)

$$\frac{dr}{dt} = -0.1\text{cm/s}; \quad \frac{dh}{dt} = 0.2\text{cm/s}$$

$$V = \pi r^2 h \; ; \; \frac{\partial v}{\partial r} = 2\pi r h ; \; \frac{\partial v}{\partial h} = \pi r^2$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial v}{\partial h} \cdot \frac{dh}{dt}$$

$$\frac{dv}{dt} = (2\pi r h)(-0.1) + \pi r^2 (0.2)$$

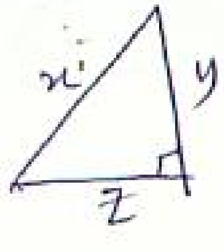
where $r=2, h=3$

$$\frac{dv}{dt} = 2\pi(2)(3)(-0.1) + \pi(2)^2(0.2)$$

$$= -1.2\pi + 0.8\pi$$

$$= -0.4\pi = -1.257\text{cm}^3/\text{s}$$

(12.) In the right-angled triangle shown, x is increasing at 3cm/s. Calculate at 2cm/s while y is decreasing the rate at which z is changing when x=5cm & y=3.



$$z = \sqrt{x^2 - y^2}$$

$$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2 - y^2)^{-1/2} \times 2x$$

$$= \frac{x}{\sqrt{x^2 - y^2}} = \frac{5}{4}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{x^2 - y^2}} = \frac{-3}{\sqrt{5^2 - 3^2}} = \frac{-3}{4}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{5}{4}(2) + \left(\frac{-3}{4}\right)(-3) = \frac{5}{2} + \frac{9}{4} = 4.75\text{cm}$$

Lagrange multiplier

for $f(x, y, z)$

$$\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

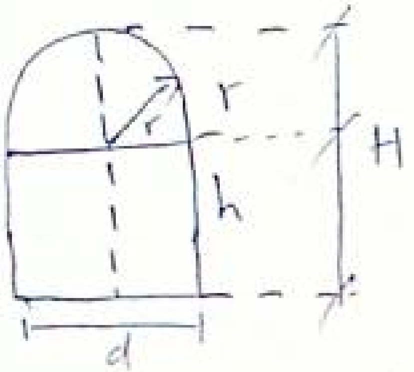
$$\frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$$\frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

with $\phi(x, y, z) = 0$ as constraint.

Constraint always has value.

(13.) A hot water storage tank is a vertical cylinder surmounted by a hemispherical top of the same diameter. The tank is designed to hold $500\,m^3$ of liquid. Determine the total height & the diameter of the tank if the surface heat loss is minimum.



Area is subjected to volume (constraint)

$H = h + r;\ d = 2r$

$A = 2\pi r h + 3\pi r^2$

The surface area of the hemisphere is $2\pi r^2$, the area of the base of tank is πr^2 & the area of the cylindrical side is $2\pi r h$, giving a total of $3\pi r^2 + 2\pi r h$.

$$V = \pi r^2 h + \frac{2}{3}\pi r^3 = 500 \quad \text{— (a)}$$

$$\frac{\partial A}{\partial r} + \lambda \frac{\partial V}{\partial r} = 0 \quad \text{— (i)}$$

$$\frac{\partial A}{\partial h} + \lambda \frac{\partial V}{\partial h} = 0 \quad \text{— (ii)}$$

(14) Find the stationary Points of the function $U = x^2 + y^2$ subject to the constraint $x^2 + y^2 + 2x - 2y + 1 = 0$

$$\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \quad \text{--- (i)}$$

$$\frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \quad \text{--- (ii)}$$

$U = x^2 + y^2$

$\frac{\partial u}{\partial x} = 2x ; \frac{\partial u}{\partial y} = 2y$

$\phi(x, y) = x^2 + y^2 + 2x - 2y + 1 = 0$

$\frac{\partial \phi}{\partial x} = 2x + 2 ; \frac{\partial \phi}{\partial y} = 2y - 2$

from (1): $2x + \lambda(2x + 2) = 0$

from (2): $2y + \lambda(2y - 2) = 0$

$2[x + \lambda(x+1)] = 0$ --- (i)

$2[y + \lambda(y-1)] = 0$ --- (ii)

Divide (i) by (ii)

$$\frac{x}{y} = \frac{-\lambda(x+1)}{-\lambda(y-1)}$$

$x(y-1) = y(x+1)$

$xy - x = xy + y$

$xy - xy = x = y$

$\therefore y = -x$

Substituting $y = -x$ into ϕ

$x^2 + (-x)^2 + 2x - 2(-x) + 1 = 0$

$x^2 + x^2 + 2x + 2x + 1 = 0$

$2x^2 + 4x + 1 = 0$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(2)(1)}}{2(2)} = \frac{-4 \pm \sqrt{8}}{4} = -\frac{4}{4} \pm \frac{2\sqrt{2}}{4}$$

$x = -1 \pm \frac{\sqrt{2}}{2}$

$$A = 2\pi r h + 3\pi r^2$$

$$\frac{\partial A}{\partial r} = 2\pi h + 6\pi r$$

$$\frac{\partial A}{\partial h} = 2\pi r$$

$$\frac{\partial V}{\partial r} = 2\pi r h + 2\pi r^2$$

$$\frac{\partial V}{\partial h} = \pi r^2$$

(1) becomes: $2\pi h + 6\pi r + \lambda(2\pi r h + 2\pi r^2) = 0$

(2) becomes: $2\pi r + \lambda(\pi r^2) = 0$

$$\lambda = \frac{-2\pi r}{\pi r^2} = \frac{-2}{r}$$

Put $\lambda = \frac{-2}{r}$ into (1)

$$2\pi h + 6\pi r + \left(\frac{-2}{r}\right)\left[2\pi r h + 2\pi r^2\right] = 0$$

$r(2\pi h + 6\pi r) - 2(2\pi r h + 2\pi r^2) = 0 \rightarrow$ I multiplied both sides by r.

$$2\pi r h + 6\pi r^2 - 4\pi r h - 4\pi r^2 = 0$$

$$2\pi r^2 - 2\pi r h = 0$$

$$2\pi r(r - h) = 0$$

$$r - h = 0$$

$$r = h$$

Put $r = h$ into eqn (4): $V = \pi(h^2)h + \frac{2}{3}\pi(h)^3 = 500$

$$\pi h^3 + \frac{2}{3}\pi h^3 = 500$$

multiply b.s by 3

$$3\pi h^3 + 2\pi h^3 = 1500$$

$$5\pi h^3 = 1500$$

$$\pi h^3 = 300$$

$$h^3 = \frac{300}{\pi}$$

$$h = \sqrt[3]{\frac{300}{\pi}} = 4.57m$$

$$\therefore r = h = 4.57m$$

$$H = (4.57 + 4.57)m = 9.14m$$

$$d = 2r = 2(4.57) = 9.14m$$

$\therefore y = 1 \mp \frac{\sqrt{2}}{2}$

Implicit Function / General Partial Diff

(15)

$$\frac{dy}{dx} = -\frac{\partial z}{dx} \div \frac{\partial z}{\partial y}$$

(15.) By means of Partial differentiation, determine $\frac{dy}{dx}$ in each of the following: (a) $xy + 2y - x = 4$
(b) $x^3y^2 - 2x^2y + 3xy^2 - 8xy = 5$ (5 mks)

Let $z = xy + 2y - x = 4$

$z = xy + 2y - x - 4 = 0$

$\frac{\partial z}{\partial x} = y - 1$; $\frac{\partial z}{\partial y} = x + 2$

$\therefore \frac{dy}{dx} = -\frac{(y-1)}{x+2} = \frac{1-y}{x+2}$

(b.) $z = x^3y^2 - 2x^2y + 3xy^2 - 8xy - 5 = 0$

$\frac{\partial z}{\partial x} = 3x^2y^2 - 4xy + 3y^2 - 8y$

$\frac{\partial z}{\partial y} = 2x^3y - 2x^2 + 6xy - 8x$

$\frac{dy}{dx} = \frac{-(3x^2y^2 - 4xy + 3y^2 - 8y)}{2x^3y - 2x^2 + 6xy - 8x} = \frac{4xy + 8y - 3x^2y^2 - 3}{2x^3y - 2x^2 + 6xy - 8}$

(1

$\frac{dy}{dx}$ = No do pass yourself o!

(16.) Find the first & second partial differential coefficients of

$z = \cos(2x + 3y)$

$z = \cos(2x + 3y)$

$\frac{\partial z}{\partial x} = -2\sin(2x + 3y)$

$\frac{\partial^2 z}{\partial x^2} = -4\cos(2x + 3y)$

$\frac{\partial z}{\partial y} = -3\sin(2x + y)$

$\frac{\partial^2 z}{\partial y^2} = -9\cos(2x + y)$

$\frac{\partial^2 z}{\partial x \partial y} = -6\cos(2x + 3y)$

$$\frac{\partial^2 z}{\partial y \partial x} = -6\cos(2x+3y)$$

(17) To find $\frac{\partial^2 z}{\partial y \partial x}$; go to $\frac{\partial z}{\partial y}$ & differentiate x

To find $\frac{\partial^2 z}{\partial x \partial y}$; go to $\frac{\partial z}{\partial x}$ & differentiate y.

(17)

Taylor's Theorem

(17) Using Taylor's theorem, show that; If $z = f(x,y)$ then $\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial x}{\partial y}\delta y$

Let $h = \delta x$, $k = \delta y$

$$z + \delta z = z + \left\{h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right\} + \frac{1}{2!}\left\{h^2\frac{\partial^2 z}{\partial x^2} + 2hk\frac{\partial^2 z}{\partial y \partial x} + k^2\frac{\partial^2 z}{\partial y^2}\right\} + \cdots$$

Subtracting z from each side:

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \frac{1}{2!}\left\{\frac{\partial^2 z}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 z}{\partial y \partial x}(\delta x \delta y) + \frac{\partial^2 z}{\partial y^2}(\delta y)^2\right\}$$

Since δx & δy are small, the expression in the brackets is of the next order of smallness & can be discarded for our purposes. Therefore, the result: If $z = f(x,y)$ then $\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$

Minimum & Maximum

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$$

If $\frac{\partial^2 z}{\partial x^2}$ & $\frac{\partial^2 z}{\partial y^2}$ are both positive, it is minimum & vice versa. If the expression above is greater than zeros it is either minimum or maximum.

Saddle Point: $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < 0$

To solve for stationary point, $\frac{\partial z}{\partial x} = 0$ & $\frac{\partial z}{\partial y} = 0$ to find x, y

(18.) Determine the position & nature of the stationary points of the function $z = 5xy - 4x^2 - y^2 - 2x - y + 5$

$$\frac{\partial z}{\partial x} = 5y - 8x - 2$$

$$\frac{\partial z}{\partial y} = 5x - 2y - 1$$

$5y - 8x = 2$ (i)

$5x - 2y = 1$ (ii)

Solving simultaneously: $-8x + 5y = 2$

$5x - 2y = 1$

Using Elimination method: $2(-8x + 5y = 2) \Rightarrow -16x + 10y = 4$

$5(5x - 2y = 1) \Rightarrow 25x - 10y = 5$

$9x = 9$

$x = 1$

from (ii): $5(1) - 2y = 1$

$5 - 2y = 1$

$-2y = -4$; $y = 2$

$$\frac{\partial^2 z}{\partial x^2} = -8 \; ; \; \frac{\partial^2 z}{\partial y^2} = -2 \; ; \; \frac{\partial^2 z}{\partial x \partial y} = 5$$

Testing if it's minimum, maximum or saddle point

$\Longrightarrow (-8)(-2) - (5)^2 = -9$

since $-9 < 0$; it is saddle point.

(19.) Determine the position & nature of the stationary point of the stationary point of the function

$$z = 2x^2y^2 + 4xy^2 - 4y^3 + 16y + 5$$

$$\frac{\partial z}{\partial x} = 4xy^2 + 4y^2 \; ; \; \frac{\partial z}{\partial y} = 4x^2y + 8xy - 12y^2 + 16$$

$$\frac{\partial^2 z}{\partial x^2} = 4y^2 \; ; \; \frac{\partial^2 z}{\partial y^2} = 4x^2 + 8x - 24y$$

Where $\frac{\partial z}{\partial x}$ \$ $\frac{\partial z}{\partial y} = 0$ to find x, y

$4xy^2 + 4y^2 = 0$ — (i)

$4x^2y + 8xy - 12y^2 + 16 = 0$ — (ii)

from (i); $4xy^2 = -4y^2$

$x = -4/4 = -1$

from (ii); $4(-1)^2y + 8(-1)y - 12y^2 + 16 = 0$

$4y - 8y - 12y^2 + 16 = 0$

$-4y - 12y^2 + 16 = 0$

multiply b.s by -1

$12y^2 + 4y - 16 = 0$

Divide through by 4

$3y^2 + y - 4 = 0$

$3y^2 + 4y - 3y - 4 = 0$

$y(3y+4) - 1(3y+4) = 0$

$(3y+4)(y-1) = 0$

$3y = -4$; $y = 1$

$y = -4/3$; $y = 1$

The points: $(-1, 1)$ \$ $(-1, -4/3)$

$\frac{\partial^2 z}{\partial x \partial y} = 8xy + 8y$

$\frac{\partial^2 z}{\partial x \partial y}$ at $(-1,1) = 8(-1)(1) + 8(1) = 0$

$\frac{\partial^2 z}{\partial x \partial y}$ at $(-1, -4/3) = 8(-1)(\frac{-4}{3}) + 8(-4/3) = 0$

$= 0$

at $(-1,1)$; $\frac{\partial^2 z}{\partial x^2} = 4(1)^2 = 4$

at $(-1,1)$; $\frac{\partial^2 z}{\partial y^2} = 4(-1)^2 + 8(-1) - 24(1)$

at $(-1, -4/3)$; $\partial^2 z/\partial x^2 = 4(-4/3)^2 = 64/9$

$$\frac{\partial^2 z}{\partial y^2} = 4(-1)^2 + 8(-1) - 24(-4/3)$$
$$= 28$$

Recall: $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$

$f(1,1) \Rightarrow (4)(-28) - (0) < 0$

$\therefore$ saddle point

$(-1, -4/3) \Rightarrow \left(\frac{64}{9}\right)(28) - (0)^2 > 0$

$\therefore$ minimum point because both $\left(\frac{\partial^2 z}{\partial x^2}\right)$ & $\left(\frac{\partial^2 z}{\partial y^2}\right)$ are positive.

$\therefore Z_{min} = 2(-1)^2(-4/3)^2 + 4(-1)\left(\frac{-4}{3}\right)^2 - 4\left(\frac{-4}{3}\right)^3 + 16\left(\frac{-4}{3}\right) + 5$

$Z_{min} = \frac{-278}{27}$ //

(20.) Given that $f(x,r) = 2x^4 + 2r^4 - 8xr + 12$, identify any critical points & any local extrema.

$$\frac{\partial f}{\partial x} = 8x^3 - 8r$$

$$\frac{\partial f}{\partial r} = 8r^3 - 8x$$

$$\frac{\partial^2 f}{\partial x^2} = 24x^2 ; \frac{\partial^2 f}{\partial r^2} = 24r^2$$

$$\frac{\partial^2 f}{\partial x \partial r} = -8 ; \cancel{\frac{\partial^2 f}{\partial r \partial x} = -8}$$

To find x & r: $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial r} = 0$

$\frac{\partial f}{\partial x} = 0 : 8x^3 - 8r = 0$

$8x^3 = 8r$

$x^3 = r$

$\frac{\partial f}{\partial r} = 0 : 8r^3 - 8x = 0$

$8(x^3)^3 - 8x = 0$

$8x^9 - 8x = 0$

$$8x^2 = 8x$$

$$8x(x^2 - 1) = 0$$

$$8x = 0 \; ; \; x^2 - 1 = 0$$

$$x = 0 \; ; \; x^2 = 1 \; ; \; x = 1$$

$$\therefore x = 0, 1$$

at $x = 0$; $r = 0^3 = 0$

at $x = 1$; $r = 1^3 = 1$

$$(x, r) = (0, 0) \; \& \; (1, 1)$$

at $0, 0$; $\dfrac{\partial^2 f}{\partial r^2} = 24(0)^2 = 0$

$$\frac{\partial^2 f}{\partial x^2} = 24(0)^2 = 0$$

at $(1, 1)$; $\dfrac{\partial^2 f}{\partial r^2} = 24(1)^2 = 24$

$$\frac{\partial^2 f}{\partial x^2} = 24(1)^2 = 24$$

at $(0, 0)$; $(0)(0) - (-8)^2 < 0$

$\therefore$ the critical point at $(0, 0)$ gives a saddle point.

at $(1, 1)$; $(24)(24) - (-8)^2$

$$576 - 64 > 0$$

$\therefore$ the critical point at $(1,)$ gives a minimum.

$$\therefore f_{min} = 2(1)^4 + 2(1)^4 - 8(1)(1) + 12$$

$$= 8$$

Change of variable

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial R} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial R} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial R}$$

21.) If $Z = x^4 + 2x^2y + y^3$ \$ $x = r\cos\theta$ \$ $y = r\sin\theta$. Find $\frac{\partial z}{\partial r}$ in their simplest form.

$$\frac{\partial z}{\partial x} = 4x^3 + 4xy \; ; \; \frac{\partial z}{\partial y} = 2x^2 + 3y^2$$

$$\frac{\partial x}{\partial r} = \cos\theta \; ; \; \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = (4x^3 + 4xy)\cos\theta + (2x^2 + 3y^2)\sin\theta$$

$$= 4x\cos\theta(x^2 + y) + \sin\theta(2x^2 + 3y^2)$$

Inverse functions (Jacobian Fxn \$ its Properties)

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{array}{c c} \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} & \begin{array}{l} \leftarrow x \\ \leftarrow y \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ u & v \end{array} & \end{array}$$

To find $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.

1) $\begin{matrix} u & v \\ x & y \end{matrix}$; $\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y} / J$

2) $\begin{matrix} u & v \\ x & y \end{matrix}$; $\frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y} / J$

3) $\begin{matrix} u & v \\ x & y \end{matrix}$; $\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x} / J$

4.) $\begin{matrix} u & v \\ x & y \end{matrix}$; $\frac{\partial y}{\partial v} = \frac{\partial u}{\partial x} / J$

N/B: u, x \$ v, y are in the same column. Once you leave the column, it becomes negative

22.) ~~Determine~~ If $Z = \cos 2x \sin 3y$, $u = e^x(1+y^2)$ \$ $v = 2ye^{-x}$

Determine $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}$ \$ $\frac{\partial y}{\partial v}$; $\frac{\partial z}{\partial u}$ \$ $\frac{\partial z}{\partial v}$ (15mks)

$$u = e^x(1+y^2)$$

$\frac{\partial u}{\partial x} = e^x(0) + (1+y^2)e^x$ → I used product rule.

$$\frac{\partial u}{\partial x} = e^x(1+y^2)$$

$$\therefore \frac{\partial x}{\partial u} = \frac{1}{e^x(1+y^2)} *$$

wisdom wan wound you abi. No try this nonsense for Exams hall abeg!! This is where Jacobian comes in. Men mount!!!

$$Z = \cos 2x \sin 3y \qquad V = 2ye^{-x}$$

$$\frac{\partial u}{\partial y} = 2ye^x \; ; \; \frac{\partial z}{\partial x} = -2\sin 2x \sin 3y \qquad \frac{\partial v}{\partial x} = -2ye^{-x}$$

$$\frac{\partial z}{\partial y} = 3\cos 2x \cos 3y \qquad \frac{\partial v}{\partial y} = 2e^{-x}$$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x(1+y^2) & -2ye^{-x} \\ 2ye^x & 2e^{-x} \end{vmatrix}$$

$$J = 2e^{-x}[e^x(1+y^2)] - (2ye^x)(-2ye^{-x})$$

$$J = 2(1+y^2) + 4y^2$$

$$J = 2 + 2y^2 + 4y^2$$

$$J = 2 + 6y^2 = 2(1+3y^2)$$

$$\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y} / J = \frac{2e^{-x}}{2(1+3y^2)} = \frac{e^{-x}}{1+3y^2}$$

$$\frac{\partial x}{\partial v} = \frac{-\partial u}{\partial y} / J = \frac{-2ye^x}{2(1+3y^2)} = \frac{-ye^x}{(1+3y^2)}$$

$$\frac{\partial y}{\partial u} = \frac{-\partial v}{\partial x} / J = \frac{-(-2ye^{-x})}{2(1+3y^2)} = \frac{\cancel{2}ye^{-x}}{\cancel{2}(1+3y^2)} = \frac{ye^{-x}}{(1+3y^2)}$$

$$\frac{\partial y}{\partial v} = \frac{\partial u}{\partial x} / J = \frac{e^x(1+y^2)}{2(1+3y^2)}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

Let's substitute accordingly

$$\left(2\sin 2x\sin 3y\right)\left(\frac{-e^{-x}}{1+3y^2}\right)+\left(3\cos 2x\cos 3y\right)\left(\frac{ye^{-x}}{1+3y^2}\right)$$

$$= \frac{-2e^{-x}\sin 2x\sin 3y+3ye^{-x}\cos 2x\cos 3y}{(1+3y^2)}$$

$$= \frac{e^{-x}(3y\cos 2x\cos 3y-2\sin 2x\sin 3y)}{(1+3y^2)}$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v}=\left(-2\sin 2x\sin 3y\right)\left(\frac{-ye^{x}}{1+3y^2}\right)+3\cos 2x\cos 3y\left(\frac{e^{x}(1+y^2)}{2(1+3y^2)}\right)$$

$$\frac{\partial z}{\partial v}=\frac{(2\sin 2x\sin 3y)ye^{x}}{1+3y^2}+\frac{3\cos 2x\cos 3y\,(e^{x}+y^2e^{x})}{2(1+3y^2)}$$

$$\frac{\partial z}{\partial v}=\frac{2(2\sin 2x\sin 3y)ye^{x}+3\cos 2x\cos 3y(e^{x}(1+y^2))}{2(1+3y^2)}$$

$$\frac{\partial z}{\partial v}=\frac{e^{x}[4y\sin 2x\sin 3y+3\cos 2x\cos 3y(1+y^2)]}{2(1+3y^2)}$$

(23.) If $z=2x^2+3xy+4y^2$ & $u=x^2+y^2$ & $v=x+2y$; determine $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ & $\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}$

$\frac{\partial v}{\partial x}=1;\frac{\partial v}{\partial y}=2$ $\frac{\partial u}{\partial x}=2x$; $\frac{\partial u}{\partial y}=2y$; $\frac{\partial z}{\partial x}=4x+3y$; $\frac{\partial z}{\partial y}=3x+8y$

$$J=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}2x & 1\\ 2y & 2\end{vmatrix}=4x-2y$$

$$\frac{\partial x}{\partial u}=\frac{\partial v}{\partial y}/J=\frac{2}{4x-2y}=\frac{2}{2(2x-y)}=\frac{1}{2x-y}$$

$$\frac{\partial x}{\partial v}=-\frac{\partial u}{\partial y}/J=\frac{-\cancel{2}y}{\cancel{2}(2x-y)}=\frac{-y}{2x-y}$$

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x} / J = \frac{-1}{4x-2y}$$

$$\frac{\partial y}{\partial v} = \frac{\partial u}{\partial x} / J = \frac{2x}{2(2x-y)} = \frac{x}{2x-y}$$

(b.) $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= (4x+3y)\left(\frac{1}{2x-y}\right) + (3x+8y)\left(\frac{-1}{4x-2y}\right)$$

$$= \frac{4x+3y}{2x-y} - \frac{3x+8y}{4x-2y}$$

$$= \frac{2(4x+3y) - (3x+8y)}{2(2x-y)}$$

$$= \frac{8x+6y-3x-8y}{2(2x-y)} = \frac{5x-2y}{4x-2y}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = (4x+3y)\left(\frac{-y}{2x-y}\right) + (3x+8y)\left(\frac{x}{2x-y}\right)$$

$$\frac{\partial z}{\partial v} = \frac{-4xy-3y^2}{2x-y} + \frac{3x^2+8xy}{2x-y} = \frac{-4xy-3y^2+3x^2+8xy}{2x-y}$$

$$\frac{\partial z}{\partial v} = \frac{4xy+3x^2-3y^2}{2x-y}$$

Vectors

(29.) Scalar Product: $A \cdot B = |A||B| \cos\theta$

$$\cos\theta = \frac{A \cdot B}{|A||B|}$$

If $A = ai + bj + cK$

$$|A| = \sqrt{a^2+b^2+c^2}$$

Vector Product: $|A \times B| = |A||B| \sin\theta$

Scalar Tripple Product: $a \cdot (b \times c) = (a \times b) \cdot c$

$$(a \cdot b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

This is used to find volume & workdone.

Vector Calculus

$$\text{Grad } F,\ \nabla F = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} K$$

$$\text{Divergence } F;\ \nabla \cdot F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$\text{Curl } F;\ \nabla \times F = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

(29) Compute $\nabla \times F$ & verify that $\nabla \cdot (\nabla \times F) = 0$

$$F = xi + yj + 2zK$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 2z \end{vmatrix}$$

$i\left(\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z} y\right) - j\left(\frac{\partial}{\partial x} 2z - x\frac{\partial}{\partial z}\right) + k\left(\frac{\partial}{\partial x} y - x\frac{\partial}{\partial y}\right)$

$i(2z\,\partial/\partial y - y\,\partial/\partial z) - j\left(2z\,\partial/\partial x - x\frac{\partial}{\partial z}\right) + k\left(y\,\partial/\partial x - x\,\partial/\partial y\right)$

$i(0-0) - j(0-0) + k(0-0) = 0$

Since $\nabla \times F = 0$

$\therefore \nabla \cdot (\nabla \times F) = 0$

(30.) Compute $\nabla(\phi + \psi)$ at the point $(-2,1,6)$ if

$\psi = 18xyz + e^x$ & $\phi = x - y + 2z^2$

$\phi + \psi = 18xyz + e^x + x - y + 2z^2$

$\nabla(\phi+\psi) = \frac{\partial}{\partial x}(\phi+\psi)i + \frac{\partial}{\partial y}(\phi+\psi)j + \frac{\partial}{\partial z}(\phi+\psi)k$

$= \frac{\partial}{\partial x}(18xyz + e^x + x - y + 2z^2)i + \frac{\partial}{\partial y}(18xyz + e^x + x - y + 2z^2)j$

$+ \frac{\partial}{\partial z}(18xyz + e^x + x - y + 2z^2)k$

$= (18yz + e^x + 1 - 0 + 0)i + (18xz + 0 + 0 - 1 + 0)j$

$+ (18xy + 0 + 0 - 0 + 4z)k$

$= (18yz + e^x + 1)i + (18xz - 1)j + (18xy + 4z)k$

(31.) The force in an electrostatic field $f(x, y, z)$ has the ~~dir~~ direction of the gradient f. Find ∇f & it value at $P : (4, -3)$ where $f = (x-1)^2 - (y+1)^2$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

$$\nabla f = [2(x-1)] i + [-2(y+1)] j + 0$$

$$\nabla f = (2x-2) i + (-2y-2) j$$

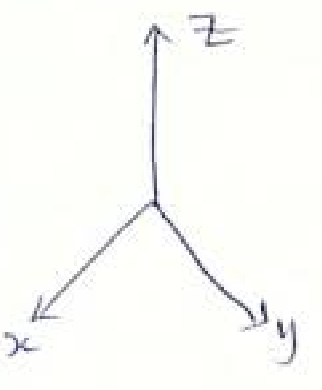
$$\nabla f \text{ at } (4,-3) \Rightarrow [2(4)-2] i + [-2(-3)-2] j$$

$$\Rightarrow (8-2) i + (6-2) j$$

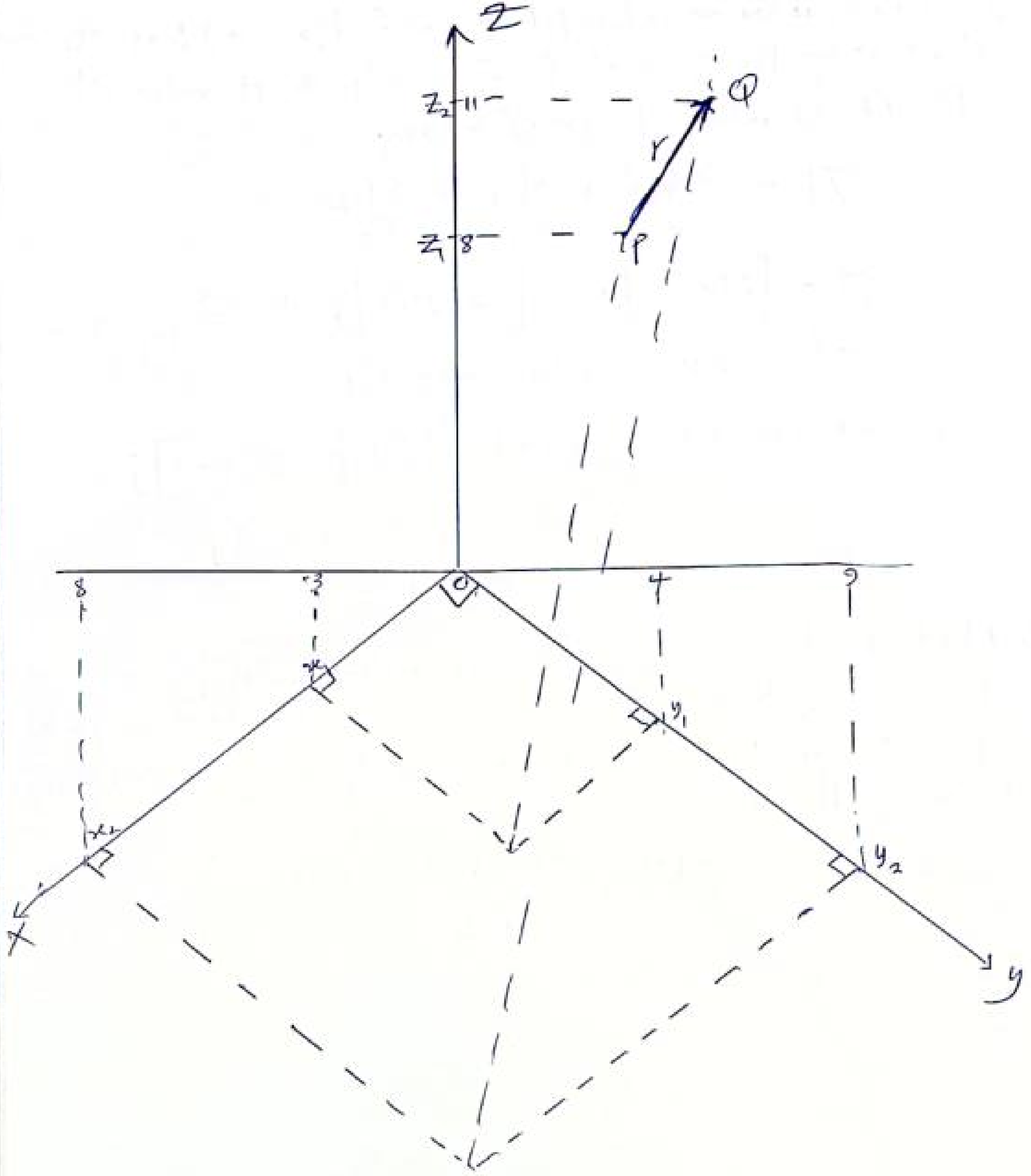
$$6i + 4j$$

(32.) Let r be a given vector with initial point $P : (3, 4, 8)$ & terminal point $Q : (8, 9, 11)$, Using graphical representation. Show each Point in space by a position Vector of the point.

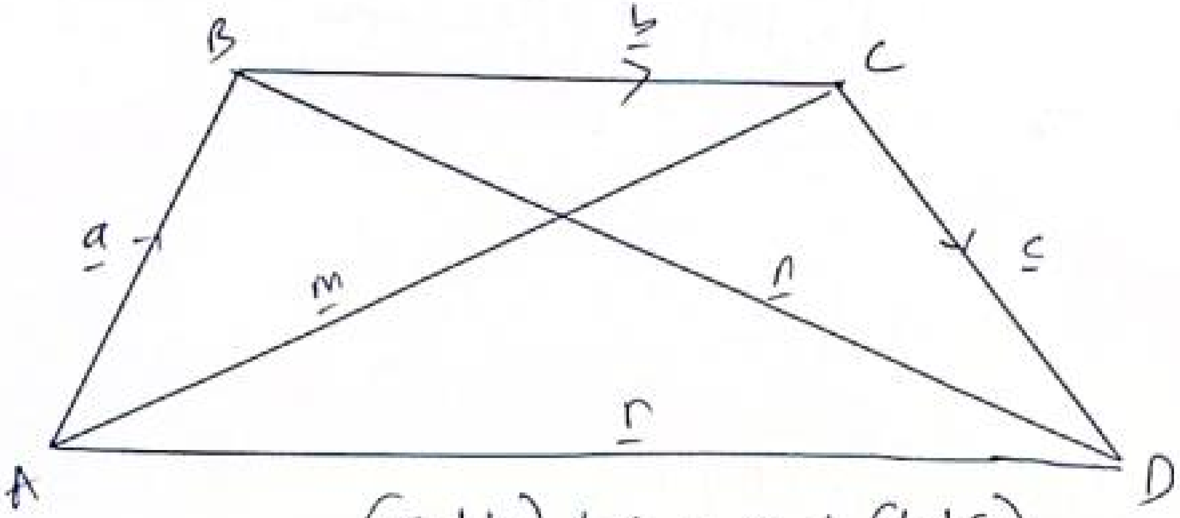
Components of vectors (x, y, z)



See solution next page →



(33.) Associativity of vector addition is given as $(\underline{a}+\underline{b})+\underline{c} =$ $\underline{a}+(\underline{b}+\underline{c})$, Using graphical representation, Prove that th can be verified geometrically.



$$(\underline{a}+\underline{b})+\underline{c} = \underline{a}+(\underline{b}+\underline{c})$$

$$\underline{a}+\underline{b} = \underline{m}$$

$$\underline{b}+\underline{c} = \underline{n}$$

$$\underline{m}+\underline{n} = \underline{r}$$

(34.) Find a unit normal vector V of the cone revolution, $2x^2+3y^2+z^2$ at the point $P:(3,1,0)$

$$\phi = 2x^2+3y^2+z^2$$

$$\nabla\phi = \frac{\partial v}{\partial x}i + \frac{\partial v}{\partial y}j + \frac{\partial v}{\partial z}k$$

$$= 4xi + 6yj + 2zk$$

$$\nabla\phi_{(3,1,0)} = 4(3)i + 6(1)j + 2(0)k$$

$$= 12i + 6j + 0$$

$$\hat{v} = \frac{\nabla v}{|\nabla v|} = \frac{12i+6j+0}{\sqrt{12^2+6^2}} = \frac{12i+6j+0}{\sqrt{180}}$$

$$\hat{v} = \frac{12i}{6\sqrt{5}} + \frac{6j}{6\sqrt{5}} + 0$$

$$\hat{v} = \frac{2}{\sqrt{5}}i + \frac{j}{\sqrt{5}}$$

$\therefore$ The unit normal vector is: $V = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$ //

Determinant

(35.) Given $a_1x + b_1y + d_1 = 0$

$a_2x + b_2y + d_2 = 0$

$$x = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}$$

$$y = \frac{a_1d_2 - a_2d_1}{a_1b_2 - a_2b_1}$$

Solve the Equations: $5x + 2y + 19 = 0$

$3x + 4y + 17 = 0$

The Key to the method is: $\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta_0}$

$\cancel{\Delta_0 \neq}$ N/B: To find Δ_0, omit the constant term.

To find Δ_1, omit the x-terms

To find Δ_2, omit the y-terms.

$$\Delta_0 = \begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix} = (5\times4) - (3\times2)$$

$$20 - 6$$

$$= 14$$

$$\Delta_1 = \begin{vmatrix} 2 & 19 \\ 4 & 17 \end{vmatrix} = (2\times17) - (19\times4)$$

$$= 34 - 76$$

$$= -42$$

$$\Delta_2 = \begin{vmatrix} 5 & 19 \\ 3 & 17 \end{vmatrix} \Rightarrow (5\times 17) - (3\times 19)$$
$$= 85 - 57$$
$$= 28$$

$$\therefore \frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{1}{\Delta_0} \Rightarrow \frac{x}{-42} = \frac{-y}{28} = \frac{1}{14}$$

$$14x = -42\times 1$$
$$x = \frac{-42}{14} = -3$$
$$-14y = 28\times 1$$
$$y = \frac{28}{-14} = -2$$

(36.) Find the value of x from the Eqns:

$$2x + 3y - z - 4 = 0$$
$$3x + y + 2z - 13 = 0$$
$$x + 2y - 5z + 11 = 0$$

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0}$$

To find Δ_0, omit the constant

To find Δ_1, omit the x-terms

To find Δ_2, omit the y-terms

To find Δ_3, omit the z-terms

$$\Delta_0 = \begin{vmatrix} 2 & 3 & -1 \\ 3 & 1 & 2 \\ 1 & 2 & -5 \end{vmatrix}$$

$$2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$2\left[(1\times -5) - (2\times 2)\right] - 3\left[(3\times -5) - (2\times 1)\right] - 1\left[(3\times 2) - (1\times 1)\right]$$

$$2\left[-5 - 4\right] - 3\left[-15 - 2\right] - 1\left[6 - 1\right]$$

$2(-9) - 3(-17) - 1(5)$

$-18 + 51 - 5$

$= 28$

$$\Delta_1 = \begin{vmatrix} 3 & -1 & -4 \\ 1 & 2 & -13 \\ 2 & -5 & 11 \end{vmatrix}$$

$$3\begin{vmatrix} 2 & -13 \\ -5 & 11 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -13 \\ 2 & 11 \end{vmatrix} - 4\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix}$$

$3[(2\times 11) - (-13\times -5)] + 1[1\times 11 - (2\times -13)] - 4[1\times -5 - (2\times 2)]$

$3[22 - (65)] + 1[11 - (-26)] - 4[-5 - 4]$

$3(-43) + 1(37) - 4(-9)$

$-129 + 37 + 36$

$= -56$

$\frac{x}{\Delta_1} = \frac{-1}{\Delta_0} \Rightarrow \frac{x}{-56} = \frac{-1}{28}$

$28x = 56$

$x = 2$

Just incase you are asked to find y $ z

$$\Delta_2 = \begin{vmatrix} 2 & -1 & -4 \\ 3 & 2 & -13 \\ 1 & -5 & 11 \end{vmatrix}$$

$$2\begin{vmatrix} 2 & -13 \\ -5 & 11 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -13 \\ 1 & 11 \end{vmatrix} - 4\begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix}$$

$2[2\times 11 - (-5\times -13)] + 1[3\times 11 - (1\times -13)] - 4[3\times -5 - (2\times 1)]$

$2[22 - 65] + 1[33 - (-13)] - 4[-15 - 2]$

$2(-43)+1(46)-4(-17)$

$-86+46+68$

$=28$

$$\frac{-y}{\Delta_2} = \frac{-1}{\Delta_0} \Rightarrow \frac{-y}{28} = \frac{-1}{28}$$

$$-28y = -28$$

$$y = 1$$

$$\Delta_3 = \begin{vmatrix} 2 & 3 & -4 \\ 3 & 1 & -13 \\ 1 & 2 & 11 \end{vmatrix}$$

$$2\begin{vmatrix} 1 & -13 \\ 2 & 11 \end{vmatrix} - 3\begin{vmatrix} 3 & -13 \\ 1 & 11 \end{vmatrix} - 4\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$2[1\times 11-(2\times -13)]-3[3\times 11-(1\times -13)]-4[3\times 2-(1\times 1)]$$

$$2[11-(-26)]-3[33-(-13)]-4[6-1]$$

$$2(37)-3(46)-4(5)$$

$$74-138-20$$

$$=-84$$

$$\frac{z}{\Delta_3} = \frac{-1}{\Delta_0} \Rightarrow \frac{z}{-84} = \frac{-1}{28}$$

$$28z = 84$$

$$z = \frac{84}{28} = 3$$

Try this: $\begin{cases} 2x - 2y - z - 3 = 0 \\ 4x + 5y - 2z + 3 = 0 \\ 3x + 4y - 3z + 7 = 0 \end{cases}$

$x = 2;\ y = -1,\ z = 3$

N|B: for an eqn to be consistent, $\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0$

I will be solving the last two questions using GAUSSIAN ELIMINATION METHOD

In this particular method, the augmented matrix must be changed to $\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix}$

Let's ride in!!

$$2x + 3y - z = 4$$
$$3x + y + 2z = 13$$
$$x + 2y - 5z = -11$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 3 & 1 & 2 & 13 \\ 1 & 2 & -5 & -11 \end{array}\right] \begin{array}{l} \longrightarrow R_1 \text{ (Row 1)} \\ \text{---} R_2 \text{ (Row 2)} \\ \longrightarrow R_3 \text{ (Row 3)} \end{array}$$

Now there are many operations you can do which consist of swapping rows, addition, subtraction, multiplication & even division.

step 1: Swap R_1 with R_3.

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & -11 \\ 3 & 1 & 2 & 13 \\ 2 & 3 & -1 & 4 \end{array}\right]$$

New row 2

Step 2: $NR_2 = 3R_1 - R_2$
$NR_3 = 2R_1 - R_3$
$= \left[\begin{array}{ccc|c} 1 & 2 & -5 & -11 \\ 0 & 5 & -17 & -46 \\ 0 & 1 & -9 & -26 \end{array}\right]$.

Working: NR_2; for column one, row 2: $3(1) - 3 = 0$
for column two, row 2: $3(2) - 5 = 1$

for column three, row 2; $3(-5)-2=-17$
for column four, row 2; $3(-11)-13=-46$

NR_3; for column one, row 3; $2(1)-2=0$
" " two, " ; $2(2)-3=1$
" " three, " ; $2(-5)-(4)=-9$
" " four, " ; $2(-11)-4=-26$

Now As you can see it is remaining for us to make the second column, third row, zero.

$$NR_3 = 5R_3 - R_2 = \left[\begin{array}{ccc|c} 1 & 2 & -5 & -11 \\ 0 & 5 & -17 & -46 \\ 0 & 0 & -28 & -84 \end{array}\right]$$

$$x + 2y - 5z = -11 \quad \text{--- (i)}$$
$$5y - 17z = -46 \quad \text{--- (ii)}$$
$$-28z = -84 \quad \text{--- (iii)}$$

from (iii); $z = \frac{-84}{-28} = 3$

from (ii); $5y - 17(3) = -46$
$5y - 51 = -46$
$5y = -46 + 51$
$5y = 5$
$y = 5/5 = 1$

from (i); $x + 2(1) - 5(3) = -11$
$x + 2 - 15 = -11$
$x - 13 = -11$
$x = 2$

$\therefore$ $x = 2$, $y = 1$ & $z = 3$

There is another Method called INVERSE METHOD

$$\text{Assuming } A = \begin{pmatrix} 2 & 3 & -1 \\ 3 & 1 & 2 \\ 1 & 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 13 \\ -11 \end{pmatrix}$$

Step 1: Find the determinant of A

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 3 & 1 & 2 \\ 1 & 2 & -5 \end{vmatrix}$$

$$2\begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 2(-5(1) - 2(2)) - 3(-5(3) - 2(1)) - 1(3(2) - 1(1))$$

$$= 2(-9) - 3(-17) - 1(5)$$

$$= -18 + 51 - 5$$

$$= 28$$

Step 2: Find the cofactors of the Elements

Cofactors of any one element is its minor together with its place sign.

$$\text{i.e } C = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$A_{11} = \begin{vmatrix} 1 & 2 \\ 2 & -5 \end{vmatrix} = -9 \; ; \; A_{12} = -\begin{vmatrix} 3 & 2 \\ 1 & -5 \end{vmatrix} = +17 \; ; \; A_{13} = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}$$

$(-5(1) - 2(2))$ $\quad -(-5(3) - 2(1))$ $\quad 3(2) - 1(1)$

$$A_{13} = 5 \; ; \; A_{21} = -\begin{vmatrix} 3 & -1 \\ 2 & -5 \end{vmatrix} = +13 \; ; \; A_{22} = \begin{vmatrix} 2 & -1 \\ 1 & -5 \end{vmatrix} = -9$$

$-(3(-5) - 2(-1))$ $\quad 2(-5) - 1(-1)$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -1 \quad A_{31} = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 7 \quad A_{32} = -\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = -7$$

$-(2(2) - 1(3))$ $\quad 3(2) - 1(-1)$ $\quad -(2(2) - 3(-1))$

$A_{33} = \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = -7$

$2(1) - 3(3)$

$$\therefore C = \begin{pmatrix} -9 & 17 & 5 \\ 13 & -9 & -1 \\ 7 & -7 & -7 \end{pmatrix}$$

Step 3: Write the transpose of C, C^T in which we write rows as columns & columns as rows.

$$C^T = \begin{pmatrix} -9 & 13 & 7 \\ 17 & -9 & -7 \\ 5 & -1 & -7 \end{pmatrix}$$

Step 4: $$\begin{pmatrix} -9 & 13 & 7 \\ 17 & -9 & -7 \\ 5 & -1 & -7 \end{pmatrix} \begin{pmatrix} 4 \\ 13 \\ -11 \end{pmatrix} = \begin{pmatrix} -36 & 169 & -77 \\ 68 & -117 & 77 \\ 20 & -13 & 77 \end{pmatrix}$$

Step 5: Find A^{-1} which is $\frac{1}{|A|} \times C^T$

$$= \frac{1}{28} \begin{pmatrix} 56 \\ 28 \\ 84 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$\therefore x = 2, y = 1$ & $z = 3$

(37.) Solve
$$4x_1 + 5x_2 + x_3 = 2$$
$$x_1 - 2x_2 - 3x_3 = 7$$
$$3x_1 - x_2 - 2x_3 = 1$$

I will so use Gaussian Elimination method & Inverse method. Let's go!!!

Oya O! which one will I start with like this? OK Gaussian Elimination method first.

$$\left[\begin{array}{ccc|c} 4 & 5 & 1 & 2 \\ 1 & -2 & -3 & 7 \\ 3 & -1 & -2 & 1 \end{array}\right]$$

Now the goal of this is to reduce this augmented matrix to this:

$$\begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix}$$

$$\begin{aligned} NR_2 &= 4R_2 - R_1 \\ NR_3 &= 3R_2 - R_3 \end{aligned} = \left[\begin{array}{ccc|c} 4 & 5 & 1 & 2 \\ 0 & -13 & -13 & 26 \\ 0 & -5 & -7 & 20 \end{array}\right]$$

$$NR_2 = \frac{R_2}{-13} = \left[\begin{array}{ccc|c} 4 & 5 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & -5 & -7 & 20 \end{array}\right]$$

$$NR_3 = 5R_2 + R_3 = \left[\begin{array}{ccc|c} 4 & 5 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & -2 & 10 \end{array}\right]$$

$$4x_1 + 5x_2 + x_3 = 2 \quad \text{— i}$$

$$x_2 + x_3 = -2 \quad \text{— ii}$$

$$-2x_3 = 10 \quad \text{— iii}$$

from iii, $x_3 = \frac{10}{-2} = -5$

from ii, $x_2 - 5 = -2$

$x_2 = -2 + 5 = 3$

from i, $4x_1 + 5(3) - 5 = 2$

$4x_1 + 15 - 5 = 2$

$4x_1 + 10 = 2$

$4x_1 = 2 - 10 = -8$

$x_1 = -8/4 = -2$

$x_1 = -2,\ x_2 = 3,\ x_3 = -5$

Using Inverse method:

$$\text{Let } A = \begin{pmatrix} 4 & 5 & 1 \\ 1 & -2 & -3 \\ 3 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$$

$$|A| = 4\begin{vmatrix} -2 & -3 \\ -1 & -2 \end{vmatrix} - 5\begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}$$

$$= 4[-2(-2) - (-1)(-3)] - 5[1(-2) - 3(-3)] + 1[1(-1) - 3(-2)]$$

$$= 4(\ \ -3) - 5(-2+9) + 1(-1+6)$$

$$= 4(\ \ ) - 5(7) + 1(5)$$

$$= 4 - 35 + 5$$

$$= -26$$

Let's solve for the cofactor

$$C = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$A_{11} = \begin{vmatrix} -2 & -3 \\ -1 & -2 \end{vmatrix} = 1 \ ; \ A_{12} = -\begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} = -7$$

$$A_{13} = \begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix} = 5 \ ; \ A_{21} = -\begin{vmatrix} 5 & 1 \\ -1 & -2 \end{vmatrix} = 9 \ ; \ A_{22} = \begin{vmatrix} 4 & 1 \\ 3 & -2 \end{vmatrix} = -11$$

$A_{23} = -\begin{vmatrix} 4 & 5 \\ 3 & -1 \end{vmatrix} = 19$; $A_{31} = \begin{vmatrix} 5 & 1 \\ -2 & -3 \end{vmatrix} = -13$; $A_{32} = -\begin{vmatrix} 4 & 1 \\ 1 & -3 \end{vmatrix} = 13$

$A_{33} = \begin{vmatrix} 4 & 5 \\ 1 & -2 \end{vmatrix} = -13$

$$\therefore C = \begin{pmatrix} 1 & -7 & 5 \\ 9 & -11 & 19 \\ -13 & 13 & -13 \end{pmatrix} ; \; C^T = \begin{pmatrix} 1 & 9 & -13 \\ -7 & -11 & 13 \\ 5 & 19 & -13 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} \times C^T = \frac{1}{-26} \begin{pmatrix} 1 & 9 & -13 \\ -7 & -11 & 13 \\ 5 & 19 & -13 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix} = \frac{1}{-26} \begin{pmatrix} 1 & 9 & -13 \\ -7 & -11 & 13 \\ 5 & 19 & -13 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$$

$$= -\frac{1}{26} \begin{pmatrix} 2 & 63 & -13 \\ -14 & -77 & 13 \\ 10 & 133 & -13 \end{pmatrix}$$

$$= -\frac{1}{26} \begin{pmatrix} 52 \\ -78 \\ 130 \end{pmatrix} = -\begin{pmatrix} 2 \\ -3 \\ 5 \end{pmatrix}$$

$\therefore x_1 = -2$; $x_2 = 3$ \$ $x_3 = -5$

You feel me?

Let me drop one last Alpha in this topic before I get out!

(38) Solve $x_1 + 3x_2 - 2x_3 + x_4 = -1$

$$2x_1 - 2x_2 + x_3 - 2x_4 = 1$$

$$x_1 + x_2 - 3x_3 + x_4 = 6$$

$$3x_1 - x_2 + 2x_3 - x_4 = 3$$

You can't solve the above using Inverse method, the only method Efficient for this is Gaussian Elimination method. And the Augmented matrix should be reduced to

$$\left|\begin{matrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{matrix}\right| \begin{matrix} \text{— } R_1 \\ \text{— } R_2 \\ \text{— } R_3 \\ \text{— } R_4 \end{matrix} \qquad \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 2 & -2 & 1 & -2 & 1 \\ 1 & 1 & -3 & 1 & 6 \\ 3 & -1 & 2 & -1 & 3 \end{array}\right]$$

$$NR_2 = 2R_1 - R_2$$
$$NR_3 = R_1 - R_3 \qquad = \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & 8 & -5 & 4 & -3 \\ 0 & 2 & 1 & 0 & -7 \\ 0 & 10 & -8 & 4 & -6 \end{array}\right]$$
$$NR_4 = 3R_1 - R_4$$

~~$NR_3 = 2R_1 - 3R_3$~~

$$NR_3 = 8R_3 - 2R_2 = \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & 8 & -5 & 4 & -3 \\ 0 & 0 & 18 & -8 & -50 \\ 0 & 10 & -8 & 4 & -6 \end{array}\right]$$

$$NR_4 = \frac{R_4}{2} = \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & 8 & -5 & 4 & -3 \\ 0 & 0 & 18 & -8 & -50 \\ 0 & 5 & -4 & 2 & -3 \end{array}\right]$$

$$NR_4 = 8R_4 - 5R_2 = \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & 8 & -5 & 4 & -3 \\ 0 & 0 & 18 & -8 & -50 \\ 0 & 0 & -7 & -4 & -9 \end{array}\right]$$

$$NR_4 = -7R_3 - 18R_4 = \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & -1 \\ 0 & 8 & -5 & 4 & -3 \\ 0 & 0 & 18 & -8 & -50 \\ 0 & 0 & 0 & 128 & 512 \end{array}\right]$$

$$x_1 + 3x_2 - 2x_3 + x_4 = -1 \quad \text{— (i)}$$

$$8x_2 - 5x_3 + 4x_4 = -3 \quad \text{— (ii)}$$

$$18x_3 - 8x_4 = -50 \quad \text{— (iii)}$$

$$128x_4 = 512 \quad \text{— (iv)}$$

$$x_4 = \frac{512}{128} = 4$$

from (iii); $18x_3 - 8(4) = -50$

$$18x_3 - 32 = -50$$

$$18x_3 = -50 + 32$$

$$x_3 = -1$$

from (ii); $8x_2 - 5(-1) + 4(4) = -3$

$$8x_2 + 5 + 16 = -3$$

$$8x_2 + 21 = -3$$

$$8x_2 = -3 - 21$$

$$x_2 = -\frac{24}{8} = -3$$

from (i); $x_1 + 3(-3) - 2(-1) + 4 = -1$

$$x_1 - 9 + 2 + 4 = -1$$

$$x_1 - 3 = -1$$

$$x_1 = -1 + 3 = 2$$

$$x_1 = 2,\ x_2 = -3,\ x_3 = -1,\ x_4 = 4$$

You can try: $x_1 + 2x_2 - x_3 + 3x_4 = 9$

$$2x_1 - x_2 + 3x_3 + 2x_4 = 23$$

$$3x_1 + 3x_2 + x_3 + x_4 = 5$$

$$4x_1 + 5x_2 - 2x_3 + 2x_4 = -2$$

$x_1 = 1$; $x_2 = -2$, $x_3 = 3$ \& $x_4 = 5$

InTeGrAtIoN

A	$\int A\,dx$
$\frac{1}{x}$	$\ln x + c$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\sec^2 x$	$\tan x + c$
$\sinh x$	$\cosh x + c$
$\cosh x$	$\sinh x + c$
$\sec x \tan x$	$\sec x + c$
K	$Kx + c$
x^n	$\frac{x^{n+1}}{n+1} + c$
a^x	$\frac{a^x}{\ln a}$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$
$\frac{-1}{\sqrt{1-x^2}}$	$\cos^{-1} x + c$
$\frac{1}{x^2+1}$	$\tan^{-1} x + c$

$\frac{-1}{x^2+1}$	$\cot^{-1}x + c$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1}x + c$
$\frac{-1}{x\sqrt{x^2-1}}$	$\mathrm{cosec}^{-1}x + c$
$\frac{1}{\sqrt{x^2+1}}$	$\sinh^{-1}x + c$
$\frac{1}{1-x^2}$	$\tanh^{-1}x + c$
$\frac{1}{\sqrt{x^2-1}}$	$\cosh^{-1}x + c$

Trigonometrical Identities

$$\left.\begin{aligned}\sin^2 x &= \frac{1}{2}(1-\cos 2x)\\ \cos^2 x &= \frac{1}{2}(1+\cos 2x)\end{aligned}\right\}$$ utilize for integration since

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 1 - 2\sin^2 x$$

$$= 2\cos^2 x - 1$$

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$1 + \tan^2 x = \sec^2 x$$

$$\sin^2 2x + \cos^2 2x = 1$$

$$1 + \cos x = 2\cos^2(x/2)$$

$$1 - \cos x = 2\sin^2(x/2)$$

$$1 - \cos 2x = 2\sin^2 x$$

$$\operatorname{cosec}^2 x = 1 + \cot^2 x$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Functions of a linear function of x.

(1.) $\int (5x-4)^7 \, dx$

$$\frac{(5x-4)^8}{8 \times D(5x-4)} = \frac{(5x-4)^8}{8 \times 5} = \frac{(5x-4)^8}{40} + c$$

(2.) $\int e^{5x} \, dx \Rightarrow \dfrac{e^{5x}}{De^{5x}} = \dfrac{e^{5x}}{5} + c$

(3.) $\int \sin(3x+5) \, dx \Rightarrow \dfrac{-\cos(3x+5)}{D(3x+5)} = \dfrac{-\cos(3x+5)}{3} + c$

(4.) $\int \sinh 8x \, dx \Rightarrow \dfrac{\cosh 8x}{D(8x)} = \dfrac{\cosh 8x}{8} + c$

(5.) $\int \frac{1}{4x+5}\,dx \Rightarrow \frac{\ln(4x+5)}{4} + c$

(6.) $\int 3^{4x}\,dx \Rightarrow \frac{3^{4x}}{4\ln 3} + c$

Integrals of the form $\int \frac{f'(x)}{f(x)}$

(7.) $\int \frac{(2x+3)}{x^2+3x-4}\,dx = \ln(x^2+3x-4) + c$

(8.) $\int \frac{3x^2}{x^3-9}\,dx \Rightarrow \ln(x^3-9) + c$

(9.) $\int \tan x\,dx = \int \frac{\sin x}{\cos x} = -\int \frac{-\sin x}{\cos x}\,dx$

$= -\ln \cos x + c$

(10) $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln \sin x + c$

Now, if you notice in this one you will observe that the numerator is the differential of the denominator.

Integrals In The Form $\int f(x) f'(x)\, dx$

(11.) $\int \tan x \sec^2 x \, dx \Rightarrow \frac{(\tan x)^2}{2} + c$

$= \frac{\tan^2 x}{2} + c$

(12) $\int \sin x \cos x \, dx = \frac{(\sin x)^2}{2} + c \Rightarrow \frac{\sin^2 x}{2} + c$

(13) $\int \frac{\ln x}{x} \, dx = \int \ln x \cdot \frac{1}{x} \, dx = \frac{(\ln x)^2}{2} + c$

(14.) $\int (x^2+7x-4)(2x+7) \, dx = \frac{(x^2+7x-4)^2}{2} + c$

Now if you notice, the differential of the function is just beside ~~the~~ it. In summary,

$$\int f(x) f'(x) dx \Rightarrow \frac{[f(x)]^2}{2} + c$$

$$\int \frac{f'(x)}{f(x)} dx \Rightarrow \ln f(x) + c$$

Integration By Parts.

$$\int u\,dv = uv - \int v\,du.$$

Determination of "u"

Rules: If one factor is a log function, i.e ln or log, that must be taken as u.

— If there is no log function, but a power of x, that becomes u.

— If there is neither a log function nor a Power of x, then the exponential function is taken as u.

N/B: It is not like Product rule in differentiation that you can take any function as u. BE CAREFUL

In Summary: $\left.\begin{array}{l}\log x / \ln x \\ x^n \\ e^{kx}\end{array}\right\}$ The hierachy in determining "u"

(Eg.) Evaluate $\int x^4 \cos x\,dx$ [very Likely an Exams Question]

Short cut:

$+\ x^4$	$\cos x$
$-\ 4x^3$	$\sin x$
$+\ 12x^2$	$-\cos x$
$-\ 24x$	$-\sin x$
$+\ 24$	$\cos x$
$-\ 0$	$\sin x$

$x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24x\cos x + 24\sin x + C$

Long method: $u = x^4$

$du = 4x^3$

$dv = \cos x$

$v = \sin x$

$$\int x^4 \cos x \, dx \Rightarrow x^4 \sin x - \int \sin x \, (4x^3) \, dx$$

$$x^4 \sin x - 4 \int x^3 \sin x \, dx$$

$$\int x^3 \sin x \Rightarrow u = x^3 \; ; \; du = 3x^2$$
$$dv = \sin x \; ; \; v = -\cos x$$

$$-x^3 \cos x - \int -\cos x \, (3x^2) \, dx$$

$$= -x^3 \cos x + 3 \int x^2 \cos x \, dx$$

$$\int x^2 \cos x \, dx \Rightarrow u = x^2 \; ; \; dv = \cos x$$
$$du = 2x \; ; \; v = \sin x$$

$$= x^2 \sin x - \int \sin x \, (2x) \, dx$$

$$= x^2 \sin x - 2 \int x \sin x \, dx$$

$$\int x \sin x \, dx \Rightarrow u = x \; ; \; du = 1$$
$$dv = \sin x \; ; \; \cancel{d} v = -\cos x$$

$$-x \cos x - \int -\cos x \, (1) \, dx$$

$$= -x \cos x + \sin x$$

$$\therefore \int x^4 \cos x \, dx = x^4 \sin x - 4 \left[-x^3 \cos x + 3 \left\{ x^2 \sin x - 2 \left(-x \cos x + \sin x \right) \right\} \right]$$

$$= x^4 \sin x - 4 \left[-x^3 \cos x + 3 \left\{ x^2 \sin x + 2x \cos x - 2 \sin x \right\} \right.$$

1

$$= x^4\sin x - 4\left[-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x\right] + c$$

$$= x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24x\cos x + 24\sin x + c$$

Also note: $\int e^{ax}\cos bx = \dfrac{e^{ax}(b\sin bx + a\cos bx)}{a^2+b^2} + c$

$$\int e^{ax}\sin bx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + c$$

(40) Determine $I = \int e^{5x}\sin 3x\, dx$

$$a = 5x,\ b = 3x$$

$$= \frac{e^{5x}(5\sin 3x - 3\cos 3x)}{5^2+3^2} + c$$

$$= \frac{e^{5x}(5\sin 3x - 3\cos 3x)}{34} + c$$

The above method is like a checker. If you want full marks, use this:

$$u = e^{5x}\ ;\ du = 5e^{5x}$$

$$dv = \sin 3x\ ;\ v = -\frac{\cos 3x}{3}$$

$$I = -\frac{e^{5x}\cos 3x}{3} - \int -\frac{\cos 3x}{3}(5e^{5x})\,dx$$

$$I = -\frac{e^{5x}\cos 3x}{3} + \frac{5}{3}\int \cos 3x\, e^{5x}\,dx$$

$$\int \cos 3x\, e^{5x}\, dx \Rightarrow u = e^{5x};\ du = 5e^{5x}$$

$$dv = \cos 3x;\ v = \frac{\sin 3x}{3}$$

$$\frac{\sin 3x\, e^{5x}}{3} - \int \frac{\sin 3x}{3} \cdot 5e^{5x}\, dx$$

$$\frac{\sin 3x\, e^{5x}}{3} - \frac{5}{3}\int \sin 3x\, e^{5x}\, dx$$

$$\text{But } I = \int e^{5x} \sin 3x\, dx$$

$$\therefore I = -\frac{e^{5x}\cos 3x}{3} + \frac{5}{3}\left[\frac{\sin 3x\, e^{5x}}{3} - \frac{5}{3} I\right]$$

$$\cancel{I + 5}\ I = -\frac{e^{5x}\cos 3x}{3} + \frac{5\sin 3x\, e^{5x}}{9} - \frac{25}{9} I$$

$$I + \frac{25}{9} I = -\frac{e^{5x}\cos 3x}{3} + \frac{5\sin 3x\, e^{5x}}{9}$$

$$\frac{34 I}{9} = \frac{e^{5x}}{3}\left[\frac{5\sin 3x}{3} - \cos 3x\right]$$

$$I = \frac{9e^{5x}}{34 \cdot 3}\left[\frac{5\sin 3x}{3} - \cos 3x\right]$$

$$I = \frac{3e^{5x}}{34}\left[\frac{5\sin 3x}{3} - \cos 3x\right] + c$$

If you are writing the exams with ur village PPL, $\frac{\cancel{3}}{\cancel{3}4} = \frac{1}{4}$

Be careful!!!

(41.) Determine $I = \int \frac{\sin^{-1}x}{\sqrt{1-x^2}}\, dx$

$$I = \int \sin^{-1}x \cdot \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \sin^{-1}x \cdot d(\sin^{-1}x)$$

$$= \frac{(\sin^{-1}x)^2}{2} + c$$

or

Let $u = \sin^{-1}x$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \; ; \; dx = \sqrt{1-x^2}\, du$$

$$I = \int \frac{u}{\cancel{\sqrt{1-x^2}}} \cdot \cancel{\sqrt{1-x^2}}\, du = \int u\, du$$

$$I = \frac{u^2}{2} + c$$

$$I = \frac{(\sin^{-1}x)^2}{2} + c$$

(42.) Determine $\int \frac{-5}{9\sqrt[4]{x^2}}\, dx = \frac{-5}{9} \int \frac{1}{\sqrt[4]{x^2}}\, dx$

$$= \frac{-5}{9} \int \frac{1}{4(x^{1/2})}\, dx = \frac{-5}{9} \int \frac{1}{x^2}\, dx$$

This is wrong o!

$$= \frac{-5}{9} \int x^{-2}\, dx$$

$$= \frac{-5}{9} \left[\frac{x^{-2+1}}{-2+1} \right] + c$$

$$= \frac{-5}{9} \left[\frac{x^{-1}}{-1} \right] + c$$

Jesus!!! opal calm down now!

$$= \frac{-5}{9}\int \frac{1}{(x^{2/4})}\,dx = \frac{-5}{9}\int \frac{1}{x^{1/2}}\,dx$$

$$= \frac{-5}{9}\int x^{-1/2}\,dx = \frac{-5}{9}\left[\frac{x^{-1/2+1}}{-1/2+1}\right] + c$$

$$= \frac{-5}{9}\left[\frac{x^{1/2}}{1/2}\right] + c$$

$$= \frac{-5}{9}\left[2x^{1/2}\right] + c$$

$$= \frac{-10x^{1/2}}{9} + c$$

$$= \frac{-10\sqrt{x}}{9} + c \quad //$$

Integration of trigometrical functions.

(43.) Determine $\int_0^{\pi/2} \sin^9 x \cos^3 x\,dx$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x;\ dx = \frac{du}{\cos x}$$

$$\int_0^{\pi/2} (\sin^9 x \cos^3 x)\,\frac{du}{\cos x} = \int_0^{\pi/2} \sin^9 x \cos^2 x\,du$$

$$\int_0^{\pi/2} \sin^9 x\,(1-\sin^2 x)\,du$$

$$\int_0^{\pi/2} u^9(1-u^2)\,du$$

$$\int_0^{\pi/2} (u^9-u^{11})\,du$$

$$= \left[\frac{u^{10}}{10} - \frac{u^{12}}{12}\right]_0^{\pi}$$

$$\left[\frac{\sin^{10} x}{10} - \frac{\sin^{12} x}{12}\right]_0^{\pi/2}$$

$\pi/2 = 90$; N|B: π is 180 in trigonometry

$$\frac{\sin^{10} 90}{10} - \frac{\sin^{12} 90}{12} - \left(\frac{\sin^{10} 0}{10} - \frac{\sin^{12} 0}{12}\right)$$

$\sin 90 = 1$ \& $\sin 0 = 0$

$$\frac{(1)^{10}}{10} - \frac{(1)^{12}}{12} - (0-0)$$

$$\frac{1}{10} - \frac{1}{12}$$

$$\frac{6-5}{60} = 1/60$$

Products of sines & Cosines.

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

(44.) Prove that $\int_0^{\pi/w} \sin wt\cos 2wt\,dt = \frac{2}{3w}$

Let $A = wt$
$B = 2wt$

Using: $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
$\sin A\cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$

$$\therefore \sin wt\cos 2wt = \frac{1}{2}\left[\sin(wt+2wt) + \sin(wt-2wt)\right]$$
$$= \frac{1}{2}\left[\sin 3wt - \sin wt\right]$$
$$\therefore \int \sin wt\cos 2wt = \frac{1}{2}\int(\sin 3wt - \sin wt)\,dt$$
$$= \frac{1}{2}\left[\frac{-\cos 3wt}{3w} + \frac{\cos wt}{w}\right] + c$$
$$= -\frac{\cos 3wt}{6w} + \frac{\cos wt}{2w} + c$$
$$\therefore \int_0^{\pi/w} \sin wt\cos 2wt\,dt = \left[-\frac{\cos 3wt}{6w} + \frac{\cos wt}{2w}\right]_0^{\pi/w}$$
$$\frac{-\cos 3w(\pi/w)}{6w} + \frac{\cos w(\pi/w)}{2} - \left[\frac{-\cos 3w(0)}{6w} + \frac{\cos w(0)}{2w}\right],$$

$$\frac{-\cos 3\pi}{6w} + \frac{\cos \pi}{2w} - \left[\frac{-\cos 0}{6w} + \frac{\cos 0}{2w}\right]$$

N/B: $\cos n\pi = -1$ for $n =$ odd nos

$\cos = 1$

$$\frac{-(-1)}{6w} + \frac{(-1)}{2w} - \left(\frac{-1}{6w} + \frac{1}{2w}\right)$$

$$1/6w - 1/2w + 1/6w - 1/2w$$

$$= \frac{2}{6w} - \frac{2}{2w}$$

$$= \frac{1}{3w} - \frac{1}{w}$$

$$\frac{1-3}{3w} = \frac{-2}{3w}$$

(45.) Determine $\int \cos 6x \cos 4x \, dx$

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$\cos A \cos B = 1/2 \left[\cos(A+B) + \cos(A-B)\right]$$

$A = 6x$; $B = 4x$

$$\therefore \int \cos 6x \cos 4x \, dx = 1/2 \int \left(\cos(6x+4x) + \cos(6x-4x)\right) dx$$

$$= 1/2 \int (\cos 10x + \cos 2x) \, dx$$

$$= 1/2 \left(\frac{\sin 10x}{10}\right) + 1/2 \left(\frac{\sin 2x}{2}\right) + c$$

$$= \frac{\sin 10x}{20} + \frac{\sin 2x}{4} + c$$

Partial Fraction Integration

(46.) Determine $I = \int \frac{x^3+x+1}{x^4+x^2}\,dx$

$$\frac{x^3+x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$\frac{x^3+x+1}{x^2(x^2+1)} = \frac{Ax(x^2+1)+B(x^2+1)+(Cx+D)x^2}{x^2(x^2+1)}$$

$$x^3+x+1 = Ax^3+Ax+Bx^2+B+Cx^3+Dx^2$$

x^3: $A+C=1$ — (i)

x^2: $B+D=0$ — (ii)

x: $A=1$ — (iii)

K: $B=1$ — (iv)

from (ii); $1+D=0$

$D=-1$

from (i); $1+C=1$

$C=1-1=0$

$$\int \frac{x^3+x+1}{x^2(x^2+1)} = \int \frac{1}{x}dx + \int \frac{1}{x^2}dx + \int \frac{0x-1}{x^2+1}dx$$

$$= \int \frac{1}{x}dx + \int x^{-2}dx - \int \frac{1}{x^2+1}dx$$

$$= \ln x + \frac{x^{-2+1}}{-2+1} - \tan^{-1} x$$

$$= \ln x - x^{-1} - \tan^{-1} x$$

$$= \ln x - \frac{1}{x} - \tan^{-1} x + c$$

(47.) solve $\int \left(\theta^2 - 2\theta + 5 + \frac{3\theta - 1}{\theta^2 + 3\theta - 4}\right) d\theta$

$$\int \theta^2 d\theta - \int 2\theta d\theta + \int 5 d\theta + \int \frac{3\theta - 1}{\theta^2 + 3\theta - 4} d\theta$$

$$\int \frac{3\theta - 1}{\theta^2 + 3\theta - 4} d\theta = \frac{3\theta - 1}{(\theta + 4)(\theta - 1)} = \frac{A}{\theta + 4} + \frac{B}{\theta - 1}$$

$$3\theta - 1 = A(\theta - 1) + B(\theta + 4)$$

Let $\theta = 1$

$$3(1) - 1 = A(1 - 1) + B(1 + 4)$$

$$2 = 5B \text{ ; } B = \frac{2}{5}$$

Let $\theta = -4$

$$3(-4) - 1 = A(-4 - 1) + B(-4 + 4)$$

$$-13 = -5A$$

$$A = \frac{13}{5}$$

$$\therefore \frac{3\theta - 1}{(\theta + 4)(\theta - 1)} = \frac{13}{5(\theta + 4)} + \frac{2}{5(\theta - 1)}$$

$$\therefore \int \theta^2 d\theta - \int 2\theta\, d\theta + \int 5\, d\theta + \int \frac{3\theta - 1}{\theta^2 + 3\theta - 4}\, d\theta$$

$$= \frac{\theta^3}{3} - \frac{2\theta^2}{2} + 5\theta + \frac{13}{5}\ln(\theta+4) + \frac{2}{5}\ln(\theta-1) + c$$

(48) show that $\int \frac{\beta^3 + 3\beta^2 - \beta + 1}{\beta^2 + 2\beta - 3}\, d\beta = \frac{\beta^2}{2} + \beta + \ln\left(\frac{\beta - 1}{\beta + 3}\right) + c$

$$\begin{array}{r|l} & \beta + 1 \\ \hline \beta^2 + 2\beta - 3 & \beta^3 + 3\beta^2 - \beta + 1 \\ & -(\beta^3 + 2\beta^2 - 3\beta) \\ \hline & \beta^2 + 2\beta + 1 \\ & -(\beta^2 + 2\beta - 3) \\ \hline & 4 \end{array}$$

$$\beta + 1 + \frac{4}{\beta^2 + 2\beta - 3}$$

$$\frac{4}{\beta^2 + 2\beta - 3} = \frac{4}{(\beta+3)(\beta-1)} = \frac{A}{\beta+3} + \frac{B}{\beta-1}$$

$$4 = A(\beta - 1) + B(\beta + 3)$$

when $\beta = 1$

$$4 = A(1-1) + B(1+3)$$

$$4 = 4B;\ B = 1$$

when $\beta = -3$

$$4 = A(-3-1) + B(-3+3)$$

$$4 = -4A;\ A = -1$$

$$\int \frac{\beta^3+3\beta^2-\beta+1}{\beta^2+2\beta-3}\,d\beta = \int \beta + \int 1 + \int \frac{-1}{\beta+3} + \int \frac{1}{\beta-1}$$

$$= \frac{\beta^2}{2} + \beta - \ln(\beta+3) + \ln(\beta-1) + c$$

$$= \frac{\beta^2}{2} + \beta + \ln\left(\frac{\beta-1}{\beta+3}\right) + c$$

(49.) Determine $\int \frac{2z^2-6z-7}{(2z+3)(z^2+1)}\,dz$

$$\frac{2z^2-6z-7}{(2z+3)(z^2+1)} = \frac{A}{2z+3} + \frac{B\cancel{z}+C}{z^2+1}$$

$$2z^2-6z-7 = A(z^2+1) + (B\cancel{z}+C)(2z+3)$$

$$2z^2-6z-7 = Az^2 + A + 2Bz^2 + 3Bz + 2Cz + 3C$$

z^2: $A + 2B = 2$ — (I)

z: $3B + 2C = -6$ — (II)

k: $A + 3C = -7$ — (III)

~~$2B = 2 - A$ from (I)~~

~~$B = \frac{2-A}{2}$~~

from (I); $A = 2 - 2B$

from (III); $2 - 2B + 3C = -7$

$$3C - 2B = -9$$

$$3C = 2B - 9$$

$$C = \frac{2B-9}{3}$$

from (II); $3B + 2\left(\frac{2B-9}{3}\right) = -6$

$$3B + \frac{4B-18}{3} = -6$$

$$\frac{9B + 4B - 18}{3} = -6$$

$$\frac{13B-18}{3} = -6$$

$$13B - 18 = -18$$

$$13B = 0$$

$$B = 0$$

$$C = \frac{2(0)-9}{3}$$

$$C = -9/3 = -3$$

$$A = 2 - 2(0) = 2$$

$$A = 2,\ B = 0;\ C = 0$$

$$= \frac{2}{2z+3} + \frac{0z-3}{z^2+1}$$

$$\therefore \int \frac{2z^2-6z-7}{(2z+3)(z^2+1)} = \int \frac{2dz}{2z+3} - 3\int \frac{1}{z^2+1}\,dz$$

$$= \ln(2z+3) - 3\tan^{-1}z + c$$

Standard Integral

(1) $$\int \frac{dz}{z^2-A^2} = \frac{1}{2A}\ln\left\{\frac{z-A}{z+A}\right\} + c$$

(2) $$\int \frac{dz}{A^2-z^2} = \frac{1}{2A}\ln\left\{\frac{A+z}{A-z}\right\} + c$$

(3) $\displaystyle\int \frac{dz}{A^2+z^2} = \frac{1}{A}\tan^{-1}\left\{\frac{z}{A}\right\} + c$

(4.) $\displaystyle\int \frac{dz}{\sqrt{A^2-z^2}} = \sin^{-1}\left\{\frac{z}{A}\right\} + c$

(5.) $\displaystyle\int \frac{dz}{\sqrt{z^2+A^2}} = \sinh^{-1}\left\{\frac{z}{A}\right\} + c$

(6.) $\displaystyle\int \frac{dz}{\sqrt{z^2-A^2}} = \cosh^{-1}\left\{\frac{z}{A}\right\} + c$

(7.) $\displaystyle\int \sqrt{A^2-z^2}\, dz = \frac{A^2}{2}\left\{\sin^{-1}\left(\frac{z}{A}\right) + \frac{z\sqrt{A^2-z^2}}{A^2}\right\} + c$

(8.) $\displaystyle\int \sqrt{z^2+A^2} = \frac{A^2}{2}\left\{\sinh^{-1}\left(\frac{z}{A}\right) + \frac{z\sqrt{z^2+A^2}}{A^2}\right\} + c$

(9.) $\displaystyle\int \sqrt{z^2-A^2}\, dz = \frac{A^2}{2}\left\{\frac{z\sqrt{z^2-A^2}}{A^2} - \cosh^{-1}\left(\frac{z}{A}\right)\right\} + c$

Integrals of the form

$\displaystyle\int \frac{1}{a+b\sin^2 x+c\cos^2 x}\, dx$; $t = \tan x$; $\dfrac{dt}{dx} = 1+t^2$

$dx = \dfrac{dt}{1+t^2}$

$\sin x = \dfrac{t}{\sqrt{1+t^2}}$; $\cos x = \dfrac{1}{\sqrt{1+t^2}}$

Integrals of the form

$$\int \frac{dx}{a + b\sin x + c\cos x} \quad ; t = \tan x/2 \;;\; \sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2} \;;\; dx = \frac{2dt}{1+t^2}$$

(50.) Determine $I = \int \frac{d\theta}{3\sin\theta + 4\cos\theta}$

This is under the form: $\int \frac{dx}{a + b\sin x + c\cos x}$

$$3\sin\theta + 4\cos\theta = 3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right)$$

$$= \frac{6t}{1+t^2} + \frac{4-4t^2}{1+t^2}$$

$$= \frac{6t+4-4t^2}{1+t^2}$$

$$\therefore \int \frac{d\theta}{3\sin\theta + 4\cos\theta} = \int \frac{1+t^2}{6t+4-4t^2} \cdot \frac{2dt}{1+t^2}$$

$$= \int \frac{1}{\cancel{2}(2+3t-2t^2)} dt \cdot \cancel{2}$$

$$= \int \frac{1}{2+3t-2t^2} dt$$

$$= \frac{1}{2} \int \frac{1}{1+\frac{3t}{2}-t^2} dt$$

(

$$1+\frac{3}{2}t-t^2 = 1-\left(t^2-\frac{3}{2}t+\left(\frac{3}{4}\right)^2\right)+\left(\frac{3}{4}\right)^2$$

$$= 1-(t-3/4)^2+9/16$$

$$= \frac{25}{16}-(t-3/4)^2$$

$$= \left(\frac{5}{4}\right)^2-(t-3/4)^2$$

which is under the formula (2)

$$\int\frac{dz}{A^2-z^2} = \frac{1}{2A}\ln\left\{\frac{A+z}{A-z}\right\}+c$$

where $A = 5/4$ \$ $Z = t-3/4$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot 5/4}\ln\left\{\frac{5/4+t-3/4}{5/4-(t-3/4)}\right\}+c$$

$$= \frac{1}{5}\ln\left\{\frac{2/4+t}{8/4-t}\right\}+c$$

$$= 1/5\ln\left\{\frac{1/2+t}{4/2-t}\right\}+c$$

$$= 1/5\ln\left\{\frac{1+2t}{4-2t}\right\}+c$$

Recall $t = \tan\frac{x}{2}$

$$= \frac{1}{5}\ln\left\{\frac{1+2\tan x/2}{4-2\tan x/2}\right\}+c$$

(51.) Determine $\int_1^4 \frac{dr}{\sqrt{(r+2)(4-r)}}$

$$\int_1^4 \frac{dr}{\sqrt{8+2r-r^2}}$$

$8+2r-r^2 \Rightarrow 8-(r^2-2r+1^2)+1^2 \rightarrow$ factorizing using completing the square

$9-(r-1)^2$

$3^2-(r-1)^2 \rightarrow$ You don see say e de under the category $\int \frac{dx}{\sqrt{A^2-Z^2}}$

$A=3;\ Z=r-1$

Recall $\int \frac{dz}{\sqrt{A^2-Z^2}} = \sin^{-1}\left(\frac{Z}{A}\right)+c$

$= \sin^{-1}\left(\frac{r-1}{3}\right)+c$

(52.) Determine $\int \sqrt{x^2+4x+13}\,dx$

Omo you go determine taya. Infact, you have to be determined to determine. SHARP!!! Word!!!

$x^2+4x+13 \Rightarrow \underbrace{x^2+4x+(2)^2}+13-(2)^2$

$(x+2)^2+9$

$(x+2)^2+3^2$

$A=3;\ Z=x+2$

Now this falls under $\int \sqrt{Z^2+A^2}\,dx$

$\frac{3^2}{2}\left\{\sinh^{-1}\left(\frac{x+2}{3}\right)+\frac{(x+2)\sqrt{x^2+4x+13}}{3^2}\right\}+c$

$\frac{9}{2}\left\{\sinh^{-1}\left(\frac{x+2}{3}\right)+\frac{(x+2)\sqrt{x^2+4x+13}}{9}\right\}+c$

$\ldots \frac{x}{2} + c \}$

(54) Determine $\int \frac{1}{2\sin^2 x + 4\cos^2 x}\, dx$

$$2\sin^2 x + 4\cos^2 x = 2\left(\frac{t^2}{1+t^2}\right) + 4\left(\frac{1}{1+t^2}\right)$$

$$= \frac{2t^2}{1+t^2} + \frac{4}{1+t^2} = \frac{2t^2+4}{1+t^2}$$

$$\therefore \int \frac{1}{2\sin^2 x + 4\cos^2 x}\, dx = \int \frac{\cancel{1+t^2}}{2t^2+4} \cdot \frac{dt}{\cancel{1+t^2}}$$

$$= \frac{1}{2} \int \frac{dt}{t^2+2}$$

$$Z = t;\ A^2 = 2;\ A = \sqrt{2}$$

$$\int \frac{dz}{A^2+Z^2} = \frac{1}{A} \tan^{-1}\left(\frac{Z}{A}\right) + c$$

$$\frac{1}{2} \cdot \frac{1}{\sqrt{2}} \tan^{-1}\left\{\frac{t}{\sqrt{2}}\right\} + c$$

Recall $t = \tan x$

$$= \frac{1}{2\sqrt{2}} \tan^{-1}\left\{\frac{\tan x}{\sqrt{2}}\right\} + c$$

(53) Determine $I = \int = \dfrac{dt}{\sqrt{5t^2+10t-16}}$

$$= \frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{t^2+2t-\frac{16}{5}}}$$

$$t^2+2t-\frac{16}{5} = t^2+2t+1^2-\frac{16}{5}-1^2$$

$$= (t+1)^2 - \frac{21}{5}$$

$$= (t+1)^2 - \left(\sqrt{\frac{21}{5}}\right)^2$$

$$Z = t+1 \; ; \; A = \sqrt{\frac{21}{5}}$$

$$\cancel{= \frac{1}{\sqrt{5}} \cosh^{-1} \left\{ \frac{\sqrt{21/5}}{t+1} \right\} + c}$$

$$= \frac{1}{\sqrt{5}} \cosh^{-1} \left\{ \frac{t+1}{\sqrt{21/5}} \right\} + c$$

Reduction formulae

N/B: $\tan^n x$ & $\cot^n x$ do not use Integrating by part

$\sin^n x = \sin^{n-1}x \sin x$ ∴
taking $u = \sin^{n-1}x$
$dv = \sin x$

$\cos^n x = \cos^{n-1}x \cdot \cos x$

$\sec^n x = \sec^{n-2}x \cdot \sec x$

$\tan^n x = \tan^{n-2}x \tan^2 x$
$= \tan^{n-2}x(\sec^2 x - 1)$

$\csc^n x = \csc^{n-2}x \csc^2 x$

(55.) Show that the reduction formula of $\int_0^{1/2}(1-2x)^n e^x dx = 2nI_{n-1} - 1$. Hence Evaluate $\int_0^{1/2}(1-2)^3 e^x dx$

See it's been long they asked Students to prove in reduction formula in Exams. But I am solving this just in case.

$$\int_0^{1/2}(1-2x)^n e^x dx$$

$u = (1-2x)^n$
$du = n(1-2x)^{n-1}(-2)$
$= -2n(1-2x)^{n-1}$
$dv = e^x$; $v = e^x$

$\Rightarrow uv - \int v du$

$e^x(1-2x)^n + 2n\int e^x (1-2x)^{n-1} dx$

$e^x(1-2x)^n + 2nI_{n-1}$

$\int_0^{1/2}(1-2x)^n e^x \Rightarrow \left[(1-2(1/2))^n e^{1/2} - (1-2(0))^n e^0\right] + 2nI_{n-1}$

$$\int_0^{1/2} (1-2x)^n e^x = \left[(1-1)^n e^{1/2} - (1-0)^n e^0\right] + 2n I_{n-1}$$

$$= \left[0 - (1)^n 1\right] + 2n I_{n-1}$$

1 raise to power of anything will always give me 1. So $\Rightarrow \left[0 - 1(1)\right] + 2n I_{n-1}$

$$= -1 + 2n I_{n-1}$$

$$\therefore \int_0^{1/2} (1-2x)^n e^x = 2n I_{n-1} - 1$$

Evaluating $\int_0^{1/2} (1-2x)^3 e^x dx = 2(3) I_{3-1} - 1$

$$= 6 I_2 - 1$$

$$I_2 = 2(2) I_{2-1} - 1$$

$$= 4 I_1 - 1$$

$$I_1 = 2(1) I_{1-1} - 1$$

$$= 2 I_0 - 1$$

$$I_0 = \left[(1-1)^0 e^{1/2} - (1-0)^0 e^0\right] + 2(0) I_{0-1} = -1$$

$$\therefore \int_0^{1/2} (1-2x)^3 e^x dx = 6\left[4(2(-1) - 1) - 1\right] - 1$$

$$= 6\left[4(-2-1) - 1\right] - 1$$

$$6\left[4(-3) - 1\right] - 1$$

$$6\left[-13\right] - 1$$

$$= -79$$

(56) If the reduction formula of $\int (a^2+1)^{n-1} da$

$$= \frac{a(a^2+1)^n}{2n} - \frac{2n+1}{2n} \int (a^2+1)^n da$$

Hence Evaluate $\int \frac{1}{(a^2+1)^2} da$

$$\int \frac{1}{(a^2+1)^2} = \int (a^2+1)^{n-1}$$

$$\int (a^2+1)^{-2} = \int (a^2+1)^{n-1}$$

$$-2 = n-1 \; ; \; n=-1$$

$$\int (a^2+1)^{-2} da = \frac{a(a^2+1)^{-1}}{2(-1)} - 2\frac{(-1)+1}{2(-1)} \int (a^2+1)^{-1} da$$

$$= \frac{a(a^2+1)^{-1}}{-2} + \frac{(-1)}{2} \int (a^2+1)^{-1} da$$

$$= \frac{-a}{2(a^2+1)} - \frac{1}{2} \tan^{-1} a + c$$

(57) If the reduction formula of $\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{(2n-2)a^2(x^2+a^2)^{n-1}}$

$$+ \frac{2n-3}{(2n-3)a^2} \int \frac{dx}{(x^2+a^2)^{n-1}} \text{ for } n \geq 2$$

Hence, evaluate $\int (x^2+4)^{-3} dx$

Let find a \$ n

comparing: $\int \frac{dx}{(x^2+a^2)^n} = \int (x^2+4)^{-3} dx$

$$\int \frac{dx}{(x^2+a^2)^n} = \int \frac{dx}{(x^2+2^2)^3}$$

$$a=2 \; ; \; n=3$$

$$\int \frac{dx}{(x^2+4)^3} dx = \frac{x}{(2(3)-2)2^2(x^2+2^2)^{3-1}} + \frac{2(3)-3}{(2(3)-3)2^2} \int \frac{dx}{(x^2+2^2)^{3-1}}$$

$$\frac{x}{(6-2)2^2(x^2+2^2)^2} + \frac{6-3}{(6-3)2^2} \int \frac{dx}{(x^2+2^2)^2}$$

Now the wahala is solving $\int \frac{dx}{(x^2+2^2)^2}$ abi

Remember that, fast!

Let $x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$$= \int \frac{2\sec^2\theta \, d\theta}{((2\tan\theta)^2+4)^2} = \int \frac{2\sec^2\theta \, d\theta}{16\sec^4\theta}$$

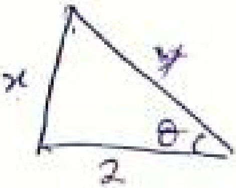
$$= \frac{1}{8}\int \cos^2\theta \, d\theta = \frac{1}{16}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{16}\left[\theta + \frac{\sin 2\theta}{2}\right] = \frac{1}{16}\left[\theta + \frac{2\sin\theta\cos\theta}{2}\right]$$

$x = 2\tan\theta; \ \frac{x}{2} = \tan\theta$

$\theta = \tan^{-1}\left(\frac{x}{2}\right)$

but since $\tan\theta = \frac{x}{2} \Rightarrow$



$y = \sqrt{x^2+2^2} = \sqrt{x^2+4}$

$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

$\cos\theta = \frac{2}{\sqrt{x^2+4}}$

Substitute accordingly

$$\frac{1}{16}\left[\tan^{-1}\left(\frac{x}{2}\right) + \frac{2 \cdot \frac{x}{\sqrt{x^2+4}} \cdot \frac{2}{\sqrt{x^2+4}}}{2}\right] + C$$

$$\frac{1}{16}\left[\tan^{-1}\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right] + C$$

$$\therefore \int \frac{dx}{(x^2+4)^3}dx = \frac{x}{(4)(4)(x^2+2^2)^2} + \frac{3}{(3)(2^2)}\left[\frac{1}{16}\left\{\tan^{-1}\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right\}\right]$$

$$= \frac{x}{16(x^2+2^2)^2} + \frac{3}{64}\left[\tan^{-1}\left(\frac{x}{2}\right) + \frac{2x}{x^2+4}\right] + c$$

(58.) If the reduction formula of $\int (a^2+1)^{n-1}da$
$= \frac{a(a^2+1)^n}{2n} - \frac{2n+1}{2n}\int (a^2+1)^n da$. Evaluate
$\int \frac{1}{(a^2+1)^3}da$

$$\int (a^2+1)^{-3}da = \int (a^2+1)^{n-1}$$

$$-3 = n-1$$
$$n = -3+1 = -2$$

$$= \frac{a(a^2+1)^{-2}}{2(-2)} - \frac{2(-2)+1}{2(-2)}\int (a^2+1)^{-2}da$$

$$= \frac{a(a^2+1)^{-2}}{-4} - \left[\frac{-3}{-4}\int (a^2+1)^{-2}da\right]$$

$$\frac{a(a^2+1)^{-2}}{-4} - \frac{3}{4}\int (a^2+1)^{-2}da$$

$$\int (a^2+1)^{-2}da = \int \frac{da}{(a^2+1)^2}$$

Let $x = \tan\theta$
$dx = \sec^2\theta\, d\theta$

$$= \int \frac{\sec^2\theta \, d\theta}{(\tan^2\theta + 1)^2} = \int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta - 1 + 1)^2}$$

$\int 2\, dy$

$$= \int \frac{\sec^2\theta \, d\theta}{(\sec^2\theta)^2} = \int \frac{\sec^2\theta}{\sec^4\theta} d\theta$$

$\int 2\, dx$

$2(x)$

$$= \int \frac{1}{\sec^2\theta} d\theta = \int \cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] = \cancel{\frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right.}$$

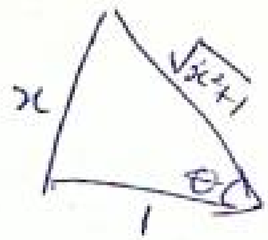
$$= \cancel{\frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right.}$$

but $\sin 2\theta = 2\sin\theta\cos\theta$

$$\frac{\theta}{2} + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{4}2}$$

Using reference right triangle.

$\theta = \tan^{-1}x$; $\tan\theta = \frac{x}{1}$:



$$\sin\theta = \frac{x}{\sqrt{x^2+1}}$$

$$\cos\theta = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{1}{2}\tan^{-1}x + \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}} \times \frac{1}{\sqrt{x^2+1}}\right)$$

$$\frac{1}{2}\tan^{-1}x + \frac{1 \cdot x}{2(x^2+1)} = \frac{1}{2}\left(\tan^{-1}x + \frac{x}{x^2+1}\right)$$

$$\therefore \int (a^2+1)^{-3}\,da = \frac{a(a^2+1)^{-2}}{4} - \frac{3}{2}\left[\frac{1}{2}\left(\tan^{-1}x + \frac{x}{x^2+1}\right)\right]$$

$$= \frac{a}{4(a^2+1)^2} - \frac{3}{4}\left[\tan^{-1}x + \frac{x}{x^2+1}\right] + c$$

Multiple Integral

(54.) $$\int_8^9\int_0^3\int_0^1 (i^2+j^2-k^2)\,di\,dj\,dk$$

$$\int_8^9\int_0^3\left[\frac{i^3}{3}+ij^2-ik^2\right]_0^1 dj\,dk$$

$$\int_8^9\int_0^3\left[\tfrac{1}{3}+j^2-k^2-\left(\frac{0}{3}+0\cdot j^2-0\cdot k^2\right)\right]dj\,dk$$

$$\int_8^9\int_0^3\left[\tfrac{1}{3}+j^2-k^2-0\right]dj\,dk$$

$$\int_8^9\int_0^3\left(\tfrac{1}{3}+j^2-k^2\right)dj\,dk$$

$$\int_8^9\left[\tfrac{1}{3}j+\tfrac{j^3}{3}-jk^2\right]_0^3 dk$$

$$\int_8^9\left[\tfrac{3}{3}+\tfrac{3^3}{3}-3k^2-\left(\tfrac{0}{3}+\tfrac{0^3}{3}-0\cdot k^2\right)\right]dk$$

$$\int_8^9 (10-3k^2)\,dk = \left[10k-\frac{3k^3}{3}\right]_8^9$$

$$\left[10k - k^3\right]_8^9 = 10(9) - (9)^3 - \left[10(8) - (8)^3\right]$$

$$= 90 - 729 - (80 - 512)$$

$$= -639 - (-432)$$

$$= -207$$

(59.) If $\int (\ln\theta)^n d\theta = \theta(\ln\theta)^n - nI_{n-1}$, evaluate $\int(\ln\theta)^4 d\theta$

$$= \theta(\ln\theta)^4 - 4I_{4-1}$$
$$= \theta(\ln\theta)^4 - 4I_3$$
$$I_3 = \theta(\ln\theta)^3 - 3I_2.$$
$$I_2 = \theta(\ln\theta)^2 - 2I_{2-1}$$
$$= \theta(\ln\theta)^2 - 2I_1$$
$$I_1 = \theta(\ln\theta)^1 - 1I_{1-1}$$
$$= \theta(\ln\theta) - I_0$$
$$I_0 = \int(\ln\theta)^0 d\theta = \int d\theta = \theta$$

$$I_4 = \theta(\ln\theta)^4 - 4\left[\theta(\ln\theta)^3 - 3\{\theta(\ln\theta)^2 - 2(\theta(\ln\theta) - \theta)\}\right]$$
$$= \theta(\ln\theta)^4 - 4\left[\theta(\ln\theta)^3 - 3\{\theta(\ln\theta)^2 - 2\theta\ln\theta - 2\theta\}\right]$$
$$= \theta(\ln\theta)^4 - 4\left[\theta(\ln\theta)^3 - 3\theta(\ln\theta)^2 - 6\theta\ln\theta - 6\theta\right]$$
$$= \theta(\ln\theta)^4 - 4\theta(\ln\theta)^3 + 12\theta(\ln\theta)^2 + 24\theta\ln\theta + 24\theta + c$$
$$= \theta\left((\ln\theta)^4 - 4(\ln\theta)^3 + 12(\ln\theta)^2 + 24\ln\theta + 24\right) + c$$

(62) Evaluate $p = \int_0^1 \int_0^{2-2y} (4x+5y)\,dx\,dy$

$$\int_0^1 \left(\frac{4x^2}{2} + 5xy\right)_0^{2-2y} dy$$

$$\int_0^1 \left[\frac{4(2-2y)^2}{2} + 5(2-2y)y - 0\right] dy$$

$$\int_0^1 \left[2(2-2y)^2 + 10y - 10y^2\right] dy$$

$$\int_0^1 \left[2(4-4y-4y+4y^2) + 10y - 10y^2\right] dy$$

$$\int_0^1 \left[8 - 8y - 8y + 8y^2 + 10y - 10y^2\right] dy$$

$$\int_0^1 \left[-2y^2 - 6y + 8\right] dy$$

$$\left[\frac{-2y^3}{3} - \frac{6y^2}{2} + 8y\right]_0^1$$

$$= \left[\frac{-2y^3}{3} - 3y^2 + 8y\right]_0^1$$

$$-\frac{2(1)^3}{3} - 3(1)^2 + 8(1)$$

$$= -2/3 - 3 + 8 = 5 - 2/3 = 13/3$$

Applications of Integration

Parametric Eqns: $A = \int_{\alpha}^{\beta} y\,dx$

Given y & x respectively.

Mean values: $M = \frac{1}{b-a}\int_{a}^{b} y\,dx$

$$M = \frac{A}{b-a}$$

A stands for area

Root mean square; RMS = $\sqrt{\text{mean value of } Y^2 \text{ b/w the given boundaries}}$

(62.) Find the mean value of $Y = \frac{5}{2-x-3x^2}$ b/w $x = -1/3$ & $x = 1/3$

$$M = \frac{1}{1/3-(-1/3)}\int_{-1/3}^{1/3} \frac{5}{2-x-3x^2}\,dx.$$

$$\int \frac{5}{2-x-3x^2}\,dx = 5\int \frac{1}{3\left(\frac{2}{3}-\frac{1}{3}x-x^2\right)}$$

$$\frac{2}{3}-\frac{1}{3}x-x^2 = \frac{2}{3}-\left(x^2+\frac{1}{3}x+\left(\frac{1}{6}\right)^2\right)+\left(\frac{1}{6}\right)^2$$

$$= \left(\frac{5}{6}\right)^2-\left(x+\frac{1}{6}\right)^2$$

$A = \frac{5}{6}$; $z = (x + 1/6)$

$$\frac{5}{3}\int \frac{dz}{A^2 - z^2} = \frac{5}{3}\cdot\frac{1}{2A} \ln\left\{\frac{A+z}{A-z}\right\}$$

$$\frac{5}{3}\cdot\frac{1}{2\left(\frac{5}{6}\right)} \ln\left\{\frac{5/6 + x + 1/6}{5/6 - x - 1/6}\right\}$$

$$\frac{5}{3}\times\frac{3}{5} \ln\left\{\frac{5+6x+1}{5-6x-1}\right\}$$

$$\frac{1}{1/3+1/3} \ln\left\{\frac{6+6x}{4-6x}\right\}$$

$$M = \frac{3}{2} \ln\left\{\frac{6(1+x)}{6(2/3 - x)}\right\}$$

$$m = \left[3/2 \ln\left\{\frac{1+x}{2/3 - x}\right\}\right]_{-1/3}^{1/3} + c$$

$$m = \frac{3}{2}\left[\ln\left\{\frac{1+1/3}{2/3 - 1/3}\right\} - \ln\left\{\frac{1-1/3}{2/3+1/3}\right\}\right]$$

$$m = \frac{3}{2}\left[\ln\left\{\frac{4/3}{1/3}\right\} - \ln\left\{\frac{2/3}{3/3}\right\}\right]$$

$$m = 3/2\left[\ln(4) - \ln(2/3)\right]$$

$$= 3/2 \ln\left(4/2/3\right) = 3/2 \ln 6$$

(63.) Find the rms value of $y = 400\sin 200\pi t$ b/w $t=0$ & $t = 1/100$

$Rms^2 = M$ of y^2 b/w $t=0$ to 0.01

$$\frac{1}{0.01 - 0}\int (400\sin 200\pi t)^2 dt$$

$$\frac{1}{0.01}\int 160000 \sin^2 200\pi t \, dt$$

$$100\int 160000\sin^2 2\omega\pi t\,dt$$

$$16\times10^6\int \sin^2 2\omega\pi t\,dt$$

$$16\times10^6\left[\tfrac{1}{2}\int(1-\cos4\omega\pi t)\,dt\right]$$

$$16\times10^6\left[\tfrac{1}{2}\left(t-\frac{\sin4\omega\pi t}{4\omega\pi}\right)\right.$$

$$8\times10^6\left[t-\frac{\sin4\omega\pi t}{4\omega\pi}\right]_0^{0.01}$$

$$8\times10^6\left[0.01-\frac{\sin4\omega\pi(0.01)}{4\omega\pi}-\left(0-\frac{\sin0}{4\omega\pi}\right)\right]$$

$$8\times10^6\left[0.01-0\right)$$

$$=80000$$

$$\text{rms}^2=80000$$

$$\text{rms}=\sqrt{80000}=282.84$$

(64.) A curve has parametric eqns $x=at^2, y=2at$. Find the area bounded by the curve, the x-axis & the ordinates at $t=1$ & $t=2$.

$$A=\int_a^b y\,dx$$

$$A=\int_a^b 2at$$

$$x=at^2;\ \frac{dx}{dt}=2at$$

$$dx=2at\,dt$$

... that $\int \sec^n x\,dx = \frac{}{n-1}$

$$A = \int_1^2 2at(2at)\,dt = \int_1^2 4a^2t^2\,dt$$

$$A = 4a^2\left[\frac{t^3}{3}\right]_1^2$$

$$A = 4a^2\left[\frac{2^3}{3} - \frac{1^3}{3}\right] = 4a^2\left[\frac{8}{3} - \frac{1}{3}\right]$$

$$A = \frac{28a^2}{3}$$

65) Find the area under the curve $y = 4\sin\frac{\beta}{2}$ b/w $\beta = \pi/3$ & $\beta = \pi$.

$$A = \int_a^b y\,dx = \int_{\pi/3}^{\pi} 4\sin\pi/2$$

$$= \left[-8\cos\beta/2\right]_{\pi/3}^{\pi}$$

$$= -8\cos\pi/2 - \left(-8\cos\frac{\pi/3}{2}\right)$$

$$= -8\cos\pi/2 + 8\cos\pi/6$$

$$= -8\cos 90 + 8\cos 30$$

$$= 8 \times \frac{\sqrt{3}}{2}$$

$$= 4\sqrt{3} \text{ sq. unit.}$$

Line Integral

If $F = P(x,y)\hat{i} + \varphi(x,y)\hat{j}$ is a vector field & C is Parameterized as $r(t) = x(t), y(t)$, then the line integral is : $\int_C F \cdot dr$ #

Since $dr = \frac{dx}{dt} dt\,\hat{i} + \frac{dy}{dt} dt\,\hat{j}$

$$\int_a^b \left[P(x(t), y(t)) \frac{dx}{dt} + \varphi(x(t), y(t)) \frac{dy}{dt} \right] dt$$

66) Evalute $\int_C F(x,y)\, ds$ where $f(x,y) = (3x^2+y)i + (5x-y)j$ and C is the Portion of the Curve $Y = 2x^2$ b/w $A(2,8)$ & $B(3,18)$.

$Y = 2x^2$; $x = t$ where $t \in (2,3)$

$\frac{dy}{d\cancel{x}} = 4x$; $\frac{dx}{dt} = 1$; $dx = dt$

$\frac{dy}{\cancel{dt}} = 4x\,d\cancel{x} = 4t\,dt$

The field components are:

$F_1 = 3x^2 + y = 3t^2 + 2t^2 = 5t^2$

$F_2 = 5x - Y = 5t - 2t^2$

$$F(t) = (5t^2, 5t-2t^2)$$

$$F \cdot dr = (5t^2, 5t-2t^2) \cdot (dt, 4t\,dt)$$

$$= 5t^2\,dt + (5t-2t^2)(4t\,dt)$$

$$= 5t^2\,dt + (20t^2-8t^3)\,dt$$

$$= (25t^2-8t^3)\,dt$$

$$\oint_c F \cdot ds = \int_2^3 (25t^2-8t^3)\,dt$$

$$\int = \frac{25t^3}{3} - \frac{8t^4}{4} = \left[\frac{25t^3}{3} - 2t^4\right]_2^3$$

$$\frac{25(3)^3}{3} - 2(3)^4 - \left(\frac{25(2)^3}{3} - 2(2)^4\right)$$

$$= 225 - 162 - \left(\frac{200}{3} - 32\right)$$

$$= 63 + 32 - \frac{200}{3}$$

$$= 95 - \frac{200}{3} = \frac{85}{3}$$