Applied w/ Guy

3/28

Many problems we want to solve have a small parameter

$$F(u, \varepsilon) = 0$$

soln (for u), parameter $\varepsilon \ll 1$

F may be an integral, algebraic eqn, ODE, PDE. Suppose that it's "easy" (tractable) to solve $F(u,0)=0$

This class is about how to do this

$$F(u,\varepsilon) = \underbrace{F(u,0)}_{\text{"easy"}} + \boxed{\text{corrections}(\varepsilon)}$$

Could approximate $F(u,\varepsilon)=0$ using
- numerical methods
  - ↳ fixed parameter, get soln
- asymptotic analysis
  - ↳ soln. structure
  - ↳ analytic approximation for how the soln depends on parameters

Satellite Problems

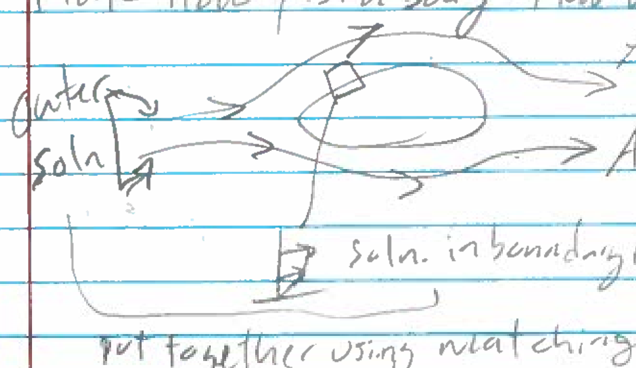
$$m\ddot{r} = -\frac{\gamma m M}{r^2}\hat{r}$$

Newton's 2^{nd} law + Newton's gravitation law + we can solve it

- \+ air resistance
- \+ earth not a sphere
- \+ effect of the moon/others
- \+ relativistic effects

corrections/perturbations

Fluid flow past a body w/ low viscosity



Assume 2-D, inviscid, irrotational

Amounts to solving $\Delta\phi = 0$

Ex. $\underbrace{\varepsilon \ddot{x}}_{\text{small inertia}} + \underbrace{\dot{x}}_{\text{friction}} + \underbrace{x}_{\text{spring}} = 0$

$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \to \varepsilon\lambda^2 + \lambda + 1 = 0$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$$

look at $\varepsilon \to 0$ (Taylor series of $\sqrt{\ }$ in λ)

$\sqrt{1-4\varepsilon} = \underline{1-2\varepsilon} - 2\varepsilon^2 - 4\varepsilon^3 + \dots$

$\lambda_1 \approx \frac{-1+(1-2\varepsilon)}{2\varepsilon} = -1 + \dots$ slow

$\lambda_2 \approx \frac{-1-(1-2\varepsilon)}{2\varepsilon} = -\frac{1}{\varepsilon} + 1 + \dots$ fast

Two time scales at work here: 1 and $\underbrace{\varepsilon}_{\text{rapid decay}}$

$\Rightarrow x \approx \underbrace{Ae^{-t}}_{\text{slow decay}} + \underbrace{Be^{-t/\varepsilon}}_{\text{fast decay}}$

Consider again $\varepsilon\lambda^2 + \lambda + 1 = 0$ (imagine intractable)

set $\varepsilon = 0 \to \lambda = -1 + ?$

Look for a solution of the form $\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots$

$\varepsilon(\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots)^2 + (\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots) + 1 = 0$

$(\lambda_0 + 1) + \varepsilon(\lambda_0^2 + \lambda_1) + \varepsilon^2(2\lambda_0\lambda_1 + \lambda_2) + \dots = 0$

At order 1 $\to \lambda_0 + 1 = 0$

$\lambda_0 = -1$ $\Rightarrow \lambda \approx -1 - \varepsilon - 2\varepsilon^2 + \dots$

At order $\varepsilon \to \lambda_0^2 + \lambda_1 = 0 \to \lambda_1 = -1$

At order $\varepsilon^2 \to 2\lambda_0\lambda_1 + \lambda_2 = 0 \to \lambda_2 = -2$

What about the other root $-1/\varepsilon$?

$\varepsilon\lambda^2 + \lambda + 1$ try $\frac{y}{\varepsilon} = \lambda$ try to catch the root that goes off to infinity by rescaling time/problem

$\varepsilon\left(\frac{y^2}{\varepsilon^2}\right) + \frac{y}{\varepsilon} + 1 = 0$

$y^2 + y + \varepsilon = 0$

set $\varepsilon = 0$, $y^2 + y = 0 \Rightarrow y = 0, -1 \to y = 0$ is the root that doesn't go to infinity

$y = -1$ goes off to infinity at a constant rate

What about the error terms for this root?

Try $y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots$ into $y^2 + y + \varepsilon = 0$

$(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots)^2 + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots) + \varepsilon = 0$

constant eqn: $y_0^2 + y_0 = 0 \rightarrow y_0 = -1$

order ε eqn: $2y_0 y_1 + y_1 + 1 = 0 \rightarrow y_1 = 1$

Asymptotic Expansions

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

$\longrightarrow$ these terms are getting smaller

Order symbols

Let $f(\varepsilon), g(\varepsilon)$

1. $f(\varepsilon) = O(g(\varepsilon))$ if $\exists$ constants C & ε_1 indep. of ε s.t.

$$|f| \leq C|g| \quad \forall\, 0 < \varepsilon < \varepsilon_1$$

f is "big oh" of g

2. $f = o(g)$ if $\forall \delta > 0\ \exists\ \varepsilon_1 > 0$ ind. of ε s.t. $|f| \leq \delta |g| \quad \forall\, 0 < \varepsilon <$

f is "little oh" of g

Stricter: If $\lim_{\varepsilon \to 0^+} \frac{f(\varepsilon)}{g(\varepsilon)} = 0$, then $f = o(g)$

If $\lim_{\varepsilon \to 0^+} \frac{f(\varepsilon)}{g(\varepsilon)} = C < \infty$, then $f = O(g)$

Note that if $f = o(g)$, then $f = O(g)$ but converse not true!

Can say f is "big oh sharp" of g $f = O_s(g)$ if $f = O(g)$ but $f \neq o(g)$

3. $f \sim g$ is asymptotically equivalent to g if $\lim_{\varepsilon \to 0^+} \frac{f}{g} = 1$

other notation

$f << g$ means $f = o(g)$

$f \lesssim g$ " $f = o(g)$

$f \precsim g$ " $f = O(g)$

HW assigned today: TAOM 2-4 F

4/1

A sequence of fns $\{\phi_n(\varepsilon)\}$ is an asymptotic sequence if
$\phi_{n+1} = o(\phi_n)$, i.e. $\phi_0 \gg \phi_1 \gg \phi_2 \gg \dots$
$\phi_0 \succ \phi_1 \succ \phi_2 \succ \dots$

If $\{\phi_n\}$ is an asymptotic sequence, then
$$u \sim u_0\phi_0(\varepsilon) + u_1\phi_1(\varepsilon) + u_2\phi_2(\varepsilon) + \dots + u_n\phi_n(\varepsilon)$$
is an n-term asymptotic expansion of u.
The ϕ fns. are called <u>scale fns</u> or <u>gauge fns</u>.

Ex. of asymptotic sequences

$\phi_n = \varepsilon^{\alpha_n} \quad \alpha_n < \alpha_{n+1} < \alpha_{n+2} < \dots$

$\phi_n = (\ln(\varepsilon))^{\alpha_n}$

$\phi_n = \varepsilon,\ \varepsilon/\log\varepsilon,\ \varepsilon^2,\ \varepsilon^2/\log\varepsilon$

Asymptotic expansions need not converge!
Approximate the Error fn.
$$Erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}\,ds$$

$$e^{-s^2} = \sum_{k=0}^{\infty}\frac{(-s^2)^k}{k!} \quad \text{~~uniform convergence on } x \in [-R, R]~~$$

$$Erf(x) = \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3} + \frac{x^5}{2\cdot 5} + \dots + \frac{(-1)^n}{(2n+1)n!}x^{2n+1} + \dots\right)$$
converges for all x

This is an asymptotic expansion as $x \to 0$
# of terms to get error $< 10^{-6}$?

x	# terms
0.5	5
1	9
2	18
4	51

gets really bad before it converges

this expansion sucks away from 0

Note that

$$\operatorname{Erf}(\infty) = 1 = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2} = \frac{2}{\sqrt{\pi}}\Big(\underbrace{\int_0^x e^{-s^2}}_{\operatorname{Erf}(x)} + \int_x^\infty e^{-s^2}\,dx\Big)$$

$$\operatorname{Erf}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-s^2}\,ds$$

$$\int_x^\infty \frac{\overbrace{(-2s)e^{-s^2}}^{dv}}{\underbrace{-2s}_{u}}\,ds = \left[-\frac{e^{-s^2}}{2s}\right]_x^\infty - \int_x^\infty \frac{e^{-s^2}}{2s^2}\,ds$$

$$= \frac{e^{-x^2}}{2x} - \int_x^\infty \frac{e^{-s^2}}{2s^2}\,ds \leftarrow \text{this integral looks smaller}$$

Repeating this many times yields

$$\int_x^\infty e^{-s^2}\,ds = \frac{e^{-x^2}}{2x}\Big(1 - \frac{1}{2x} + \frac{1\cdot 3}{(2x^2)^2} - \frac{1\cdot 3\cdot 5}{(2x^2)^3} + \dots \frac{(-1)^n(2n-1)!!}{(2x^2)^n}$$

$$+ \frac{(-1)^{n+1}(2n+1)!!}{2^{n+1}}\int_x^\infty \frac{e^{-s^2}}{s^{2n+2}}\,ds \leftarrow \text{error term blows up as } n\to\infty$$

but $\to 0$ as $x \to \infty$

$$\operatorname{Erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(2n-1)!!}{2^n}\frac{1}{x^{n+1}}$$

] not a convergent sum

each term $\to\infty$ as $n\to\infty$ for x fixed

each term $\to 0$, n fixed $x\to\infty$

Is this divergent series useful?

$x = 1/2$, best approx is 1-term & error ≈ 0.4

$x = 1$, best approx is 1-term & error ≈ 0.05, rel. error $\approx 6\%$

$x = 2$, best approx is 3-term w/ error $\approx 2\cdot 10^{-4}$

↳ a 1-term expansion has error $\approx 5\cdot 10^{-4}$

$x = 4$, 1 term exp. has error $\approx 5\cdot 10^{-10}$

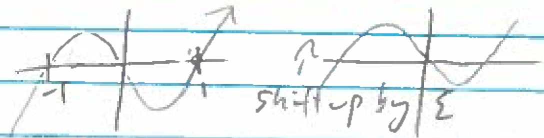
4/4

Algebraic Equs

Regular: $x^3 - x + \varepsilon = 0$ $\qquad \varepsilon \to 0 \Rightarrow x^3 - x = 0$

Singular: $\varepsilon x^3 - x + 1 = 0$ $\qquad \varepsilon \to 0 \Rightarrow \underbrace{-x + 1 = 0}$

missing 2 roots



$x^3 - x = 0$, $\quad x^3 - x + \varepsilon = 0$

Regular problem: let $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots)^3 - (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots) + \varepsilon = 0$

$O(1)$: $x_0^3 - x_0 = 0$, $x_0 = 0, \pm 1$

$O(\varepsilon)$: $3x_0^2 x_1 - x_1 + 1 = 0 \Rightarrow x_1 = \frac{1}{1 - 3x_0^2} = \underset{x_0=0}{1}, \underset{x_0=\pm1}{-\frac{1}{2}}$

$O(\varepsilon^2)$: ...

$\Rightarrow x = 0 + \varepsilon \cdot 1, \ +1 - \varepsilon/2, \ -1 - \varepsilon/2 \ + O(\varepsilon^2)$

$\Rightarrow x = \varepsilon + O(\varepsilon^2), \ \pm 1 - \frac{1}{2}\varepsilon + O(\varepsilon^2)$

Singular problem: let $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$

if did this, would get $x = 1 + \varepsilon + O(\varepsilon^2)$ (missing 2 roots)

Rescale:

$x = \varepsilon^\alpha y$ for some α TBD plug into $\varepsilon x^3 - x + 1 = 0$

$\Rightarrow \varepsilon^{3\alpha + 1} y^3 - \varepsilon^\alpha y + 1 = 0$ idea: pick α s.t. at least two terms match in size & others are higher order.

Match 1 & 2	match 1 & 3	match 2 & 3
$3\alpha + 1 = \alpha$	$3\alpha + 1 = 0$	$\alpha = 0$
$\alpha = -1/2$	$\alpha = -1/3$	(original scaling problem)

orders of all terms —— "

$\varepsilon^{-1/2}, \varepsilon^{-1/2}, \underbrace{\varepsilon^0}_{o(\varepsilon^{-1/2})}$ ✓ $\qquad \underbrace{\varepsilon^0}_{\text{not } o(\varepsilon^0)}, \varepsilon^{-1/3}, \varepsilon^0$ ✗ $\qquad \underbrace{\varepsilon^1}_{o(\varepsilon^0)}, \varepsilon^0, \varepsilon^0$ ✓

So set $x = \varepsilon^{-1/2} y$

$\varepsilon^{-1/2} y^3 - \varepsilon^{-1/2} y + 1 = 0$

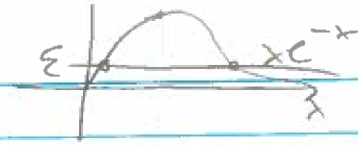
$y^3 - y + \varepsilon^{1/2} = 0$

expected, should check

Expand in powers of ε: $y = y_0 + \varepsilon^{1/2} y_1 + \varepsilon^1 y_2 + \varepsilon^{3/2} y_3 + \dots$

$y = \pm 1 - \frac{1}{2}\varepsilon^{1/2} + O(\varepsilon) \to x = \pm \frac{1}{\sqrt{\varepsilon}} - \frac{1}{2} + O(\varepsilon^{1/2})$

Ex w/ nonobvious gauge fns:



$$xe^{-x} = \varepsilon$$

As $\varepsilon \to 0$, one root $\to 0$ other $\to \infty$

lets take a ln to catch the wandering root!

Can use powers of ε, Taylor series
$x = \varepsilon + O(\varepsilon^2)$

$$\ln x - x = \ln \varepsilon$$

suggests leading order as $\varepsilon \to 0$ is $x = \ln(\varepsilon^{-1}) + o(\ln \varepsilon^{-1})$

Try $x = \ln(\varepsilon^{-1}) + g$, $g = o(\ln(\varepsilon^{-1}))$

$$\ln(\ln(\varepsilon^{-1}) + g) - \ln(\varepsilon^{-1}) - g = \ln(\varepsilon)$$

$$\ln(\ln(\varepsilon^{-1}) + \underbrace{g}_{o(\ln \varepsilon^{-1}) \text{ so ignore}}) = g \implies g = \ln(\ln(\varepsilon^{-1}))$$

or $\ln\left(\ln(\varepsilon^{-1})\left(1 + \frac{g}{\ln(\varepsilon^{-1})}\right)\right) = g$

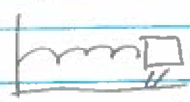
$$\ln(\ln(\varepsilon^{-1})) + \ln\Big(1 + \underbrace{\frac{g}{\ln(\varepsilon^{-1})}}_{\text{small}}\Big) = g$$

$$\implies g = \ln(\ln(\varepsilon^{-1})) + \underbrace{\frac{g}{\ln(\varepsilon^{-1})}}_{\text{small}} + \dots$$

$$\implies x = \ln(\varepsilon^{-1}) + \underbrace{\ln(\ln(\varepsilon^{-1})) + o(\ln(\ln \varepsilon^{-1}))}_{\text{not actually much smaller}}$$

4/6 **Nondimensionalization:**

What the small parameter



$$\underbrace{m\ddot{x}}_{\text{inertia}} + \underbrace{b\dot{x}}_{\text{friction}} + \underbrace{kx}_{\text{restoring spring force}} = 0$$

ignore inertia $b\dot{x} + kx = 0$ is this a good approx?

Incompressible Navier-Stokes eqns.

$$\rho(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}) = -\nabla p + \underbrace{\mu \nabla \vec{u}}_{\text{viscosity}}$$

incompressibility $\to \nabla \cdot \vec{u} = 0$ — μ is dynamic viscosity

ρ is density, $\vec{u}$ is velocity, p is pressure, ξ is position, t is time

$$\frac{Df}{dt} = f_t + (u \cdot \nabla) f = f_t + (u\partial_x + v\partial_y + w\partial_z) f$$

Nondim'lize $\rho(\vec{u}_t + \vec{u}\cdot\nabla\vec{u}) = -\nabla p + \mu\nabla\vec{u}$

$$\nabla\cdot\vec{u} = 0$$

Let $u = U\hat{u}(\hat{x},\hat{t})$ U char. velocity, $\hat{u}$ dim'less

$p = P\hat{p}$ P char. pressure, $\hat{p}$ dim'less

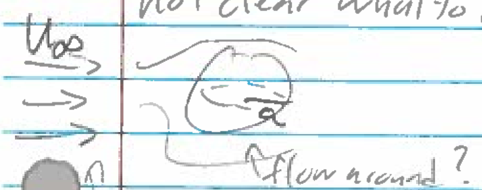
$x = L\hat{x}$ " "

$t = T\tau$ " "

$$\rho\left(\frac{U}{T}\hat{u}_\tau + \frac{U^2}{L}\hat{u}\cdot\hat{\nabla}\hat{u}\right) = -\frac{P}{L}\hat{\nabla}\hat{p} + \frac{\mu}{L^2}\hat{\nabla}\hat{u}$$

$$\frac{U}{L}\hat{\nabla}\cdot\hat{u} = 0$$

not clear what to cancel, need a particular problem to solve:



flow at infinity is U_∞

Pick $U = U_\infty$ (length/time)

$L = a$

$T = a/U_\infty$ (time taken to go char. length at char. velocity)

$= L/U$

with these choices, we have:

$$\rho\frac{U^2}{L}\left(\hat{u}_\tau + \hat{u}\cdot\hat{\nabla}\hat{u}\right) = -\frac{P}{L}\hat{\nabla}\hat{p} + \frac{\mu U}{L^2}\hat{\Delta}\hat{u}$$

$$\hat{\nabla}\cdot\hat{u} = 0$$

Can pick $P = \rho U^2$ or $P = \mu U/L$] no obvious choice, just pick 1st, why not?

inertial scale viscous scale

dropping ^ now

$$u_\tau + u\cdot\nabla u = -\nabla p + \left(\frac{\mu}{\rho U L}\right)\Delta u$$

$$\nabla\cdot u = 0$$

Define $Re = \rho U L/\mu$ is the Reynold's #

Re = "size of inertial forces"/"size of viscous forces"

Re large ($\gg 1$) jets

Re low ($\ll 1$) cells swimming

Supp. $Re \gg 1$

$$u_t + u\cdot\nabla u = -\nabla p + \frac{1}{Re}\Delta u$$

$$\nabla\cdot u = 0$$

$Re \to \infty$: $u_t + u\cdot\nabla u = -\nabla p$, $\nabla\cdot u = 0$] incompressible Euler eqns.

Supp. $Re << 1$

$\underline{\Delta u = 0}$, $\nabla \cdot u = 0$ overdetermined

with appropriate BC's, soln is unique

no way to enforce $\nabla \cdot u = 0$

since we chose pressure to scale like inertial force

not like viscous forces, so it is gone now

& was the only way to enforce $\nabla \cdot u = 0$.

Chose wrong scale for P!!

Go back. let $P = \frac{\mu U}{L}$

$\Rightarrow Re(u_t + u \cdot \nabla u) = -\nabla p + \Delta u$

$\nabla \cdot u = 0$

$Re \to 0$: $\left.\begin{aligned} -\nabla p + \Delta u &= 0 \\ \nabla \cdot u &= 0 \end{aligned}\right\}$ Stokes eqs.

4/8 Application w/ Regular Perturbation

micro-organism swimming

$Re \to 0$ $\quad$ $\Delta u - \nabla p = 0$

$\nabla \cdot u = 0$

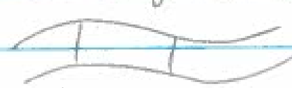
no inertia $\to$ can't "glide", very viscous medium,

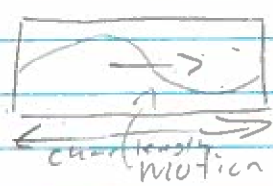
have to work the whole time to move, can't just kick off & glide

no "memory" or "history" of movement

Want some sort of undulating tail motion

Look at 2-D sheet





pick a body-centered coordinate system

$x = 0$, $y = A\sin(kx - \omega t)$

$\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ on the boundary

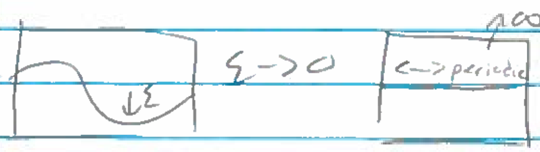
$\left.\begin{aligned} \dot{x} &= u = 0 \\ \dot{y} &= -\omega A\cos(kx - \omega t) = v \end{aligned}\right]$

use wavelength as characteristic length scale $\to$ $1/k$

want to look at limit as $A \to 0$

Nondimensionalize $L = k^{-1}$, $T = \omega^{-1}$

$$\left.\begin{aligned} u &= 0 \\ V &= -\varepsilon \cos(x-t), \quad \varepsilon = Ak \end{aligned}\right\} \text{boundary}$$



$$\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \text{horizontal flow} \\ \text{vertical flow} \end{pmatrix}$$

$$\mu \Delta u - \nabla p = 0$$
$$\nabla \cdot u = 0$$

$$\left.\begin{aligned} \text{Let } u &= \psi_y \\ v &= -\psi_x \end{aligned}\right] \ \psi \text{ is the stream function} \Big\} \text{ satisfy } \nabla\cdot u = 0 \text{ for free}$$

$$\nabla \cdot u = u_y + v_x = (\psi_y)_x + (-\psi_x)_y = 0$$

$$\left.\begin{aligned} \mu \Delta \psi_y - p_x &= 0 \\ \mu \Delta(-\psi_x) - p_y &= 0 \end{aligned}\right\} \to \begin{aligned} \frac{\partial}{\partial y}(\mu \Delta \psi_y) - p_{xy} &= 0 \\ \frac{\partial}{\partial x}(\mu \Delta -\psi_x) - p_{yx} &= 0 \end{aligned}$$

$$\Rightarrow \mu \Delta(\psi_{yy} + \psi_{xx}) = 0$$

PDE + BCs

$$\begin{cases} \Rightarrow \Delta^2 \psi = 0 & \leftarrow \text{biharmonic eqn.} \\ v = -\psi_x(x, \varepsilon \sin(x-t)) = -\varepsilon \cos(x-t) & \leftarrow \text{vertical flow on boundary} \\ u = \psi_y(x, \varepsilon \sin(x-t)) = 0 & \leftarrow \text{horizontal flow on boundary} \end{cases}$$

Solve this PDE & BC's & do some asymptotics!

Expand boundary conditions as $\varepsilon \to 0$

boundary

$$\begin{aligned} &\psi_x(x,0) + \varepsilon \psi_{xy}(x,0)\sin(x-t) + \varepsilon^2 \psi_{xyy}(x,0)\sin^2(x-t) + \ldots = \varepsilon \cos(x-t) \\ &\psi_y(x,0) + \varepsilon \psi_{yy}(x,0)\sin(x-t) + \varepsilon^2 \psi_{yyy}(x,0)\sin^2(x-t) + \ldots = 0 \end{aligned}$$

Expand ψ in powers of ε (on whole domain)

$$\psi = \varepsilon \psi_1 + \varepsilon^2 \psi_2 + \ldots$$

At $O(\varepsilon)$: $\psi_{1x}(x,0) = \cos(x-t)$

$\psi_{1y}(x,0) = 0$

At $\mathcal{O}(\varepsilon^2)$, $\psi_{2x}(x,0)+\psi_{1xy}(x,0)\sin(x-t)=0$
$\psi_{2y}(x,0)+\psi_{1yy}(x,0)\sin(x-t)=0$

$$\Delta^2\psi=\Delta(\psi_{xx}+\psi_{yy})=\psi_{xxxx}+2\psi_{xxyy}+\psi_{yyyy}=0$$

Fourier transform in x
$\varphi_n(y)=\hat{\psi}(k,y)$
$\Rightarrow k^4\varphi-2k^2\varphi''+\varphi^{(4)}=0$

Char eqn: $\lambda^4-2k^2\lambda^2+k^4=0$
$(\lambda^2-k^2)^2=0$
$\lambda=\pm k,\pm k$

solns will be linear combinations of
$(k\neq 0)$ $[e^{-ky}, ye^{-ky}, e^{ky}, ye^{ky}]$ or $\sinh(ky), y\sinh(ky), \cosh(ky), y\cosh(ky)$

$(k=0)$ $1, y, y^2, y^3$

Find ψ_1: $\begin{cases}\Delta^2\psi_1=0\\ \psi_{1x}(x,0)=\cos(x-t)\\ \psi_{1y}(x,0)=0\end{cases}$ ← note we have a single Fourier mode w/ $k=1$, so expect soln to be

$$\psi_1(x,y)=\left((A+By)e^{-y}+\underbrace{(C+Dy)e^{y}}_{0\text{ to have bounded derivatives/flow at }\infty}\right)\sin(x-t)$$

Maple to solve for A & B:

$$\psi_1(x,y)=(y+1)e^{-y}\sin(x-t)$$

ψ_1 & all derivs $\to 0$ as $y\to\infty$ no swimming since $\underline{u}=\begin{pmatrix}\psi_y\\-\psi_x\end{pmatrix}\to\begin{pmatrix}0\\0\end{pmatrix}$

Need 2nd order:

$$\begin{cases}\Delta^2\psi_2=0\\ \psi_{2x}(x,0)=-\psi_{1xy}(x,0)\sin(x-t)=0\\ \psi_{2y}(x,0)=\underbrace{-\psi_{1yy}(x,0)\sin^2(x-t)}_{\text{known by 1st order}}=\sin^2(x-t)\end{cases}$$

$$=\tfrac{1}{2}(\underset{k=0}{1}-\underset{k=2}{\cos(2(x-t))})$$

Expect soln $\Psi_2(x,y) = \left((A+By)e^{-2y}\right)\cos(2(x-t))$
$+ Cy + \underbrace{Dy^2 + Ey^3}_{=0}$

details on Maple worksheet

$$\Psi_2(x,y) = \frac{y}{2}\left(1 - e^{-2y}\cos(2(x-t))\right)$$

$\Psi_y \to \frac{1}{2}$ as $y \to \infty$ horizontal flow at infinity

Swimming speed is $U = \frac{\varepsilon^2}{2} + O(\varepsilon^3)$

4/11 Singular Perturbation Theory

Singular $\boxed{vs}$ Regular problems

Regular expansion: $u(x;\varepsilon) = \sum_k f_k(x)\,\phi_k(\varepsilon)$ ← gauge fns, ← coeff. fns

this expansion does not provide a uniform approx. to a singular problem.

Ex: damped linear oscillator - two limits

1. small damping
$$\ddot{u} + 2\varepsilon\dot{u} + u = 0$$
2. large damping
$$\varepsilon\ddot{u} + \dot{u} + u = 0$$
$\varepsilon \to 0$

(1) Small damping + $u(0)=1$, $\dot{u}(0)=0$ (BC's)

Try a regular expansion $u = u_0 + u_1\varepsilon + u_2\varepsilon^2 + \dots$

$$\begin{cases} (\ddot{u}_0 + \varepsilon\ddot{u}_1 + \dots) + 2\varepsilon(\dot{u}_0 + \varepsilon\dot{u}_1 + \dots) + (u_0 + \varepsilon u_1 + \dots) = 0 \\ u_0(0) + \varepsilon u_1(0) + \dots = 1, \quad \dot{u}_0(0) + \varepsilon\dot{u}_1(0) + \dots = 0 \end{cases}$$

$O(1)$: $\ddot{u}_0 + u_0 = 0$, $u_0(0) = 1$, $\dot{u}_0(0) = 0$

$O(\varepsilon)$: $\ddot{u}_1 + 2\dot{u}_0 + u_1 = 0$, $u_1(0) = 0$, $\dot{u}_1(0) = 0$

$O(1) \Rightarrow u_0(t) = \cos t$

$O(\varepsilon)$: $\ddot{u}_1 + u_1 = -2\sin t \Rightarrow u_1(t) = A\cos t + B\sin t - t\cos t$

$u_1(0) = 0$, $\dot{u}_1(0) = 0$ $\xrightarrow{IC}$ $u_1(t) = \sin t - t\cos t$

$\Rightarrow u = \cos t + \varepsilon(\sin t - t\sin t) + O(\varepsilon^2)$

We get $u = \cos t + \varepsilon(\sin t - \underbrace{t\cos t}_{\text{grows large for } t \text{ large}}) + O(\varepsilon^2)$

This expansion is not valid on long time scales $t = O(1/\varepsilon)$

OK for small time, bad for large time.

form of analytic soln: $u(t) = e^{-\varepsilon t}[A\cos((1-\varepsilon^2)^{1/2}t) + B\sin((1-\varepsilon^2)^{1/2}t)]$

expand amplitude for small time

$e^{-\varepsilon t} = 1 - \varepsilon t$ ← same linear growth

If goal is approx. for all time, we've failed w/ this method.

(2) Large Damping w/ $u(0)=0$, $\varepsilon\dot{u}(0)=1$

$\varepsilon\ddot{u} + \dot{u} + u = 0$

Want approximation for all $t > 0$.

Try expansion of the form $u = \frac{1}{\varepsilon}u_{-1} + u_0 + \varepsilon u_1 + \dots$

$$\begin{cases} \varepsilon(\frac{1}{\varepsilon}\ddot{u}_{-1} + \ddot{u}_0 + \dots) + \frac{1}{\varepsilon}\dot{u}_{-1} + \dot{u}_0 + \varepsilon\dot{u}_1 + \frac{1}{\varepsilon}u_{-1} + u_0 + \varepsilon u_1 = 0 \\ \frac{1}{\varepsilon}u_{-1}(0) + u_0(0) + \dots = 0 \\ (\frac{1}{\varepsilon}\dot{u}_{-1}(0) + \dot{u}_0(0) + \dots) = 1/\varepsilon \end{cases}$$

$O(\varepsilon^{-1})$: $\dot{u}_{-1} + u_{-1} = 0$, $u_{-1}(0)=0$, $\dot{u}_{-1}(0)=1$

$O(1)$: $\dot{u}_0 + u_0 = -\ddot{u}_{-1}$, $u_0(0)=0$, $\dot{u}_0(0)=0$

overspecified problem: 1st order w/ 2 BCs

Perhaps we're working on the "wrong" timescale

make change of variables: $t = \varepsilon\tau$

$v(\tau) = u(t)$

$$\frac{\partial u}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial u}{\partial \tau} = \frac{\partial \tau}{\partial t}\frac{\partial v}{\partial \tau} = \frac{1}{\varepsilon}\frac{\partial v}{\partial \tau}$$

$$\Rightarrow \begin{cases} \frac{\varepsilon v_{\tau\tau}}{\varepsilon^2} + \frac{v_\tau}{\varepsilon} + v = 0 \\ v(0) = 0 \\ \varepsilon\frac{1}{\varepsilon}v_\tau(0) = 1 \end{cases} \Rightarrow \begin{cases} v_{\tau\tau} + v_\tau + \varepsilon v = 0 \\ v(0) = 0 \\ v_\tau(0) = 1 \end{cases}$$

Now try (regular) expansion $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots$

plug in ⟷

$O(1)$: $\begin{cases} v_{0\tau\tau} + v_{0\tau} = 0 \\ v_0(0) = 0 \\ v_{0\tau}(0) = 1 \end{cases} \Rightarrow v_0(\tau) = 1 - e^{-\tau}$

$O(\varepsilon)$: $\begin{cases} v_{1\tau\tau} + v_{1\tau} = -v_0 = -1 + e^{-\tau} \\ v_1(0) = 0 \\ v_{1\tau}(0) = 0 \end{cases} \Rightarrow v_1(\tau) = (2-\tau) - (\tau+2)e^{-\tau}$

linear growth has problems on long time scale

This approx. is not valid for large time.

Both problems ① & ② have two time scales

① small damping problem

$u = \underbrace{Ae^{-\varepsilon t}}_{\text{amplitude changes slowly compared to period}} \cos\left((1-\varepsilon^2)^{1/2} t - \varphi\right)$ } secular problem

② large damping problem

$u = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$, $\lambda_1 = O(1)$, $\lambda_2 = O(\varepsilon^{-1})$

$u \approx \underbrace{Ae^{-t}}_{\approx \text{const}} + \underbrace{Be^{-t/\varepsilon}}_{\text{changing}}$ } layers

near $t=0$ ≈ const changing

(beyond $O(\varepsilon)$) changing $\approx 2e10$

this is a layer problem / boundary layer problem

two different regions of domain w/ diff. scalings necessary

4/13 ① low friction

$\ddot{x} + 2\varepsilon\dot{x} + x = 0$

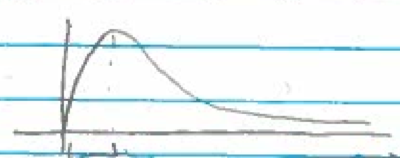
both involve two time scales $O(1)$, $O(\varepsilon)$

high friction ②

$\varepsilon\ddot{x} + \dot{x} + x = 0$

② $\begin{cases} u(0) = 0 \\ \varepsilon\dot{u} = 0 \end{cases}$ IC's $\Rightarrow$

$\Rightarrow u \approx Ae^{-t} + Be^{-t/\varepsilon}$



last time: $\begin{cases} \dot{u}_0 + u_0 = 0 \\ u_0(0) = 0 \\ \dot{u}_0(0) = 1 \end{cases}$ outer soln

fair to ignore $\varepsilon\ddot{u}$ on the long time scale

OK approx away from initial layer

$\Rightarrow$ don't enforce these IC's, they don't match the initial layer

Call this soln on the outside layer, the outer soln: $u_0(t) = Ae^{-t}$

Outer soln: $\dot{u}_0 + u_0 = 0$

$$u_0(t) = Ae^{-t}$$

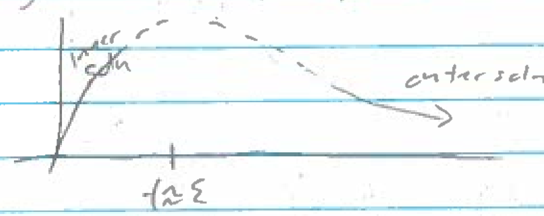
~~We will determine A by "matching" at the layer boundary~~

Last time we rescaled $t = \varepsilon\tau$ to look at the inner soln in the layer.

$$v(\tau) = u(t)$$

$$\begin{cases} v_0'' + v_0' = 0 \\ v_0(0) = 0, \ \dot{v}_0(0) = 1 \end{cases} \Rightarrow v_0(\tau) = 1 - e^{-\tau}$$

in original variables, $u_{in}(t) = 1 - e^{-t/\varepsilon} + O(\varepsilon)$



from within the layer, outer layer looks as if its at $\tau = \infty$

from outside the layer, inner layer looks as if its at $t = 0$

$$\lim_{t\to\infty} u_{in}(t) = \lim_{t\to 0^+} u_{out}(t) \quad \Big] \text{ matching condition}$$

$$\lim_{t\to\infty} (1 - e^{-t/\varepsilon}) = 1$$

$$\lim_{t\to 0^+} Ae^{-t} = A$$

to match, pick $A = 1$.

Define $u_{match} = \lim_{t\to\infty} u_{in}(t) = \lim_{t\to 0^+} u_{out}(t)$

Composite expansion

$$u_c(t) = u_{in}(t) + u_{out}(t) \underbrace{- u_{match}(t)}_{\text{get rid of double counting at layer}}$$

$$\Rightarrow u_c(t) = (1 - e^{-t/\varepsilon}) + e^{-t} - 1$$

$$u_c(t) = e^{-t} - e^{-t/\varepsilon}$$ is valid for all $t > 0$.

Example of an Initial layer

Enzyme Kinetics

$S \rightarrow P$ substrate to product w/ the help of an enzyme E

$s = [S]$, $p = [P]$

$\frac{dp}{dt} = \frac{V_{max} s}{K + s}$ ← large S, reaction is enzyme-limited

$$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \xrightarrow{k_2} P + E \quad , \quad e = [E], \; c = [C]$$

$\frac{ds}{dt} = k_{-1} c - k_1 e s$ $\qquad \frac{dc}{dt} = k_1 e s - k_{-1} c - k_2 c$

$\frac{de}{dt} = k_{-1} c + k_2 c - k_1 e s$ $\qquad \frac{dp}{dt} = k_2 c$

Michaelis-Menton assumed $S + E \rightleftharpoons C$ in equilibrium

Assumption: $k_1 s e = k_{-1} c$

~~use the fact that enzyme is either free or tied up in C,~~

so $\frac{de}{dt} + \frac{dc}{dt} = 0 \Rightarrow e + c = e_0$ is a conserved quantity

$$\Rightarrow c = \frac{s e_0}{\frac{k_{-1}}{k_1} + s} \Rightarrow \frac{dp}{dt} = k_2 c = \frac{k_2 e_0 s}{\frac{k_{-1}}{k_1} + s} = \frac{V s}{K + s}$$

Let's derive something similar via perturbation analysis

$\frac{ds}{dt} = (k_{-1} - k_1 s) c - k_1 e_0 s$ $\qquad$ using $c + e = e_0$

$\frac{dc}{dt} = k_1 e_0 s - (k_{-1} + k_2 + k_1 s) c$

nondimensionalize: $s = s_0 \sigma$, $c = e_0 x$, $t = \left(\frac{1}{k_1 e_0}\right) \tau$

$$(k_1 e_0 s_0) \frac{d\sigma}{d\tau} = (k_{-1} - k_1 s_0 \sigma) e_0 x - k_1 e_0 s_0 \sigma$$

$$\frac{d\sigma}{d\tau} = \left(\frac{k_{-1}}{k_1 s_0} + \sigma\right) x - \sigma$$

$$(k_1 e_0^2) \frac{dx}{d\tau} = k_1 e_0 s_0 \sigma - (k_{-1} + k_2 + k_1 s_0 \sigma) e_0 x$$

$$\left(\frac{e_0}{s_0}\right) \frac{dx}{d\tau} = \sigma - \left(\frac{k_{-1} + k_2}{k_1 s_0} + \sigma\right) x$$

typically $e_0 \ll s_0$

$\Rightarrow \frac{e_0}{s_0} = \varepsilon$

$\alpha = \frac{k_{-1}}{k_1 s_0}$

$K = \frac{k_{-1} + k_2}{k_1 s_0}$

Nondim'l Eqs become:

$$\frac{d\sigma}{d\tau} = (\alpha+\sigma)x - \sigma$$

$$\varepsilon \frac{dx}{d\tau} = \sigma - (K+\sigma)x$$

fast-slow system

make the approx. This is small ($\varepsilon \to 0$)

$$\Rightarrow x = \frac{\sigma}{K+\sigma} \quad \text{or} \quad c = \frac{e_0 s}{\left(\frac{k_{-1}+k_2}{k_1}\right)+s}$$

← slightly different

Suppose initial conditions $\sigma(0) = 1$, $x(0) = 0$

not satisfied initially

Expect an initial layer

$\varepsilon \frac{dx}{d\tau}$ is not small for short time (near zero)

Choose $T = \tau/\varepsilon$, $\Sigma(T) = \sigma(\tau)$, $X(T) = x(\tau)$

Inner layer

$$\Rightarrow \frac{1}{\varepsilon}\frac{d\Sigma}{dT} = \left[(\alpha+\Sigma)X - \Sigma\right]\cdot\varepsilon \to 0$$

$$\frac{dX}{dT} = \Sigma - (K+\Sigma)X$$

Use initial condition $\Sigma(0) = 1$, $X(0) = 0$

In the layer, $\Sigma = 1$

$$\frac{dX}{dT} = 1 - (K+1)X$$

$$\Rightarrow X(T) = \frac{1}{K+1}\left(1 - e^{-(K+1)T}\right)$$

outer soln: $x(\tau) = \frac{\sigma(\tau)}{K+\sigma(\tau)} \xrightarrow{\tau \to 0} \frac{1}{K+1}$

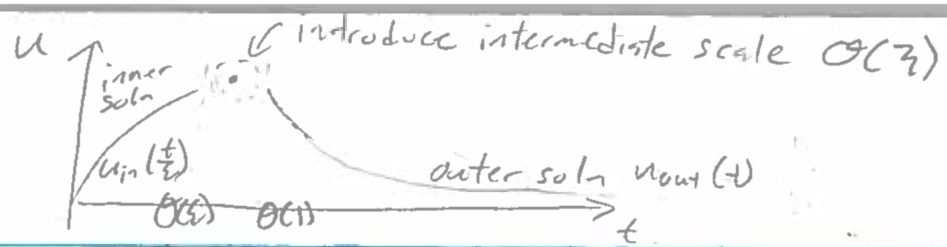
inner soln: $x(\tau/\varepsilon) = \frac{1}{K+1}\left(1 - e^{-(K+1)\tau/\varepsilon}\right)$

$$x(\tau/\varepsilon) \xrightarrow{\tau \to \infty} \frac{1}{K+1}$$

Composite expansion:

$$x(\tau) = \frac{\sigma}{K+\sigma} + \frac{1}{K+1}\left(1 - e^{-(K+1)\tau/\varepsilon}\right) - \frac{1}{K+1}$$

$$= \frac{\sigma}{K+\sigma} - \frac{1}{K+1} e^{-(K+1)\tau/\varepsilon}$$



match $\lim_{t \to \infty} u_{in}(t/\varepsilon) = \lim_{t \to 0^+} u_{out}(t)$.

this may not always work & usually doesn't at next order

matching via intermediate scale

introduce $t_\eta = t/\eta$ where $\varepsilon < \eta(\varepsilon) < 1$

e.g. $\sqrt{\varepsilon}$, some fn. of ε.

match $\varepsilon \to 0$ w/ t_η fixed

$$\lim_{\varepsilon \to 0} u_{in}\left(\frac{\eta}{\varepsilon} t_\eta\right) = \lim_{\varepsilon \to 0} u_{out}(\eta t_\eta)$$

$$u_{in}\left(\frac{\eta}{\varepsilon} t_\eta\right) - u_{out}(\eta t_\eta) = o(1)$$

$$\Rightarrow \text{at } \mathcal{O}(\varepsilon): \quad u_{in}(\eta/\varepsilon \, t_\eta) - u_{out}(\eta t_\eta) = o(\varepsilon)$$

4/18 – boundary layers

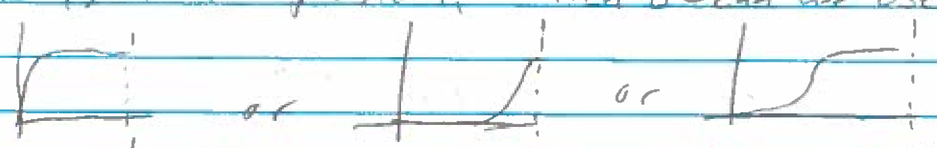
Ex: $\varepsilon u'' + (1+x)u' + u = 0$

$$u(0) = 0, \quad u(1) = 1$$

As $\varepsilon \to 0$, $(1+x)u_0' + u_0 = 0$ ← 1st order eq.

$u_0(0) = 0, \quad u_0(1) = 1$ ← 2 b.c.'s } no soln?

This is an outer problem, which BC can we use?



layer at left   layer at right   layer in middle   etc...

Introduce $X = \frac{x - x_0}{\varepsilon^\alpha}$ layer coordinate

$x = x_0 + \varepsilon^\alpha X$

$U(X) = u(x)$

inner   outer

$$\Rightarrow \frac{\varepsilon}{\varepsilon^{2\alpha}} \frac{d^2 U}{dX^2} + (1 + x_0 + \varepsilon^\alpha X)\frac{1}{\varepsilon^\alpha}\frac{dU}{dX} + U = 0$$

$$\Rightarrow \varepsilon \frac{d^2 U}{dX^2} + (1 + x_0 + \varepsilon^\alpha X)\varepsilon^\alpha \frac{dU}{dX} + \varepsilon^{2\alpha} U = 0$$

Find α so that at least two terms match in size & others are higher order

$$\varepsilon \frac{d^2U}{dX^2} + (1+x_0+\varepsilon^{\alpha}X)\,\varepsilon^{\alpha}\frac{dU}{dX} + \varepsilon^{2\alpha}U = 0$$

Pick $\alpha = 0$ gives us outer soln $\rightarrow$ already saw this

Other possibilities, $1 = \alpha \Rightarrow$ match 1 & 2

$1 = 2\alpha \Rightarrow$ match 1 & 3

For $\alpha = 1$, orders of terms are $\mathcal{O}(\varepsilon)$, $\mathcal{O}(\varepsilon)$, $\overbrace{\mathcal{O}(\varepsilon^2)}^{\text{higher or}}$

For $\alpha = 1/2$, orders are $\mathcal{O}(\varepsilon)$, $\underbrace{\mathcal{O}(\varepsilon^{1/2})}_{\text{lower order}}$, $\mathcal{O}(\varepsilon)$ X

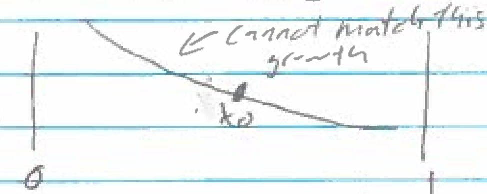
Inner problem:

$$\frac{d^2U}{dX^2} + (1+x_0+\varepsilon X)\frac{dU}{dX} + \varepsilon U = 0$$

As $\varepsilon \to 0$, leading order problem is

$$\ddot{u}_0 + (1+x_0)\dot{u}_0 = 0$$

$$\Rightarrow u_0 = A + Be^{-(1+x_0)X}$$



$$X = \frac{x - x_0}{\varepsilon}$$

grows exponentially on left as leave the layer, so impossible to match on left.

Can only leave the layer at the right, so layer is at $x_0 = 0$.

$\Rightarrow$ There is a boundary layer at $x_0 = 0$ of thickness ε

Construct a two-term composite expansion

$$\left\{ \begin{array}{l} \varepsilon u'' + (1+x)u' + u = 0 \\ u(0) = 0,\ u(1) = 1 \end{array} \right\} \quad u_{out} = u_0 + \varepsilon u_1 + \dots$$

outer soln:

$$\mathcal{O}(1): \left.\begin{array}{l} (1+x)u_0' + u_0 = 0 \\ u_0(1) = 1 \end{array}\right] \Rightarrow u_0(x) = \frac{2}{1+x}$$

$$\mathcal{O}(\varepsilon): \left.\begin{array}{l} (1+x)u_1' + u_1 = -u_0'' \\ u_1(1) = 0 \end{array}\right] \Rightarrow u_1(x) = \frac{2}{(1+x)^3} - \frac{1}{2(1+x)}$$

Outer soln: $u_{out} = \dfrac{2}{1+x} + \varepsilon\left(\dfrac{2}{(1+x)^3} - \dfrac{1}{2(1+x)}\right) + \mathcal{O}(\varepsilon^2)$

Inner soln: $X = \frac{x}{\varepsilon}$, $U(X) = u(x)$

$$\begin{cases} \ddot{U} + (1+\varepsilon X)\dot{U} + \varepsilon U = 0 \\ U(0) = 0 \end{cases}$$

Let $U = U_0 + \varepsilon U_1 + \ldots$

$O(1)$: $\left.\begin{matrix}\ddot{U}_0 + \dot{U}_0 = 0 \\ U_0(0) = 0\end{matrix}\right] \to U_0 = A_0(1-e^{-X})$

$O(\varepsilon)$: $\left.\begin{matrix}\ddot{U}_1 + \dot{U}_1 = -X\dot{U}_0 - U_0 \\ U_1(0) = 0\end{matrix}\right] \to U_1 = A_0(\frac{1}{2}X^2e^{-X} - X) + A_1(1-e^{-X})$

inner soln: $U = A_0(1-e^{-X}) + \varepsilon(A_0(\frac{1}{2}X^2e^{-X} - X) + A_1(1-e^{-X})) \ldots$

leading order match:

$$\lim_{x\to 0} \frac{2}{1+x} = 2 = A_0 = \lim_{X\to\infty} A_0(1-e^{-x/\varepsilon}) \Rightarrow A_0 = 2$$

can't just match at order ε since problem here as $X \to \infty$.
Need to use intermediate scale technique!!
match via intermediate scale.

introduce $x_\eta = x/\eta$, $\varepsilon < \eta(\varepsilon) < 1$

"simple" matching $\lim_{\varepsilon\to 0}\left(u_{out}(\eta x_\eta) - U\left(\frac{\eta x_\eta}{\varepsilon}\right)\right) = 0$

or $u_{out}(\eta x_\eta) - U\left(\frac{\eta x_\eta}{\varepsilon}\right) = o(1)$

for higher order, require $u_{out}(\eta x_\eta) - U(\eta x_\eta/\varepsilon) = o(\varepsilon)$

or $\lim_{\varepsilon\to 0^+} \frac{u_{out}(\eta x_\eta) - U(\eta x_\eta/\varepsilon)}{\varepsilon} = 0$

Change vars: $x = \eta x_\eta$, $X = \eta/\varepsilon \, x_\eta$

outer: $u_{out} = \frac{2}{1+\eta x_\eta} + \varepsilon\left(\frac{2}{(1+\eta x_\eta)^3} - \frac{1}{2(1+\eta x_\eta)}\right) + O(\varepsilon^2)$

inner: $U = 2(1-e^{-\eta/\varepsilon\, x_\eta}) + \varepsilon\left[\frac{\eta^2 x_\eta^2}{\varepsilon^2}e^{-\eta x/\varepsilon^2} - \frac{2\eta x_\eta}{\varepsilon} + A_1(1-e^{-\frac{\eta x_\eta}{\varepsilon}})\right] + O(\varepsilon^2)$

ignore $e^{-\frac{\eta x_\eta}{\varepsilon}}$ b/c they are transcendentally small

$u = 2 - 2\eta x_\eta + O(\eta^2) + \varepsilon(3/2) + O(\eta\varepsilon)$

$U = \underbrace{2 - 2\eta x_\eta}_{match} + \varepsilon A_1 + T.S.T \to$ match $A_1 = 3/2$.
provided $\eta^2 < \varepsilon$.

Finally, $u_{match} = 2 - 2x + \frac{3}{2}\varepsilon$

$$\Rightarrow u_{composite} = \frac{2}{1+x} + \varepsilon\left(\frac{2}{(1+x)^2} - \frac{1}{2(1+x)}\right) + 2(1-e^{-x/\varepsilon}) + \varepsilon\left(\frac{x^2}{\varepsilon^2}e^{-x/\varepsilon} - \frac{2x}{\varepsilon} + \frac{3}{2}(1-e^{-x/\varepsilon}\right) - \left(2 - 2x + \tfrac{3}{2}\varepsilon\right)$$

4/20

$\left(\lim_{x\to 0} u_{out} = \lim_{x\to\infty} u_{in}\right)$ equivalent to

$$\lim_{\varepsilon\to 0}\left(u_{out}(\eta x_\eta) - u_{in}(\eta x_\eta/\varepsilon)\right) = 0$$

Want $\lim_{\varepsilon\to 0}\left(\frac{u_{out} - u_{in}}{\varepsilon}\right) = 0$

$$u_{out} = \frac{2}{1+x} \Rightarrow u_{out} = \overbrace{2 - 2\eta x_\eta}^{O(1)} + \underbrace{O(\eta^2)}_{\text{Ignore } \eta^2} + \varepsilon\left(\tfrac{3}{2}\right) + O(\eta\varepsilon)$$

$$u_{in} = 2(1-e^{-x/\varepsilon}) \Rightarrow u_{in} = \underbrace{\overbrace{2}^{O(1)} - 2\eta x_\eta}_{\text{match w/ } O(1) \text{ in } u_{out}} + \underbrace{\overbrace{\varepsilon A_1}^{O(\varepsilon)}}_{\text{match w/ } O(\varepsilon) \text{ in } u_{out}} + \text{T.S.T}$$

$\Rightarrow A_1 = 3/2$

this all works provided $\eta^2 < \varepsilon$

leading order need $\varepsilon < \eta < 1$

at order $O(\varepsilon)$ need $\varepsilon < \eta < \sqrt{\varepsilon}$

☆ at $O(\varepsilon)$ only need to look at u_{in}

at $O(1)$ only need to look at u_{out}

between them $[O(\eta)]$ need to look at both

between $O(\sqrt{\varepsilon})$ & $O(1)$, see leading orders mingle but higher order terms distinct.

At leading order, $u_{match} = 2$

At $O(\varepsilon)$, $u_{match} = 2 - 2x + 3/2\,\varepsilon$

Unknowns in u_{in} or u_{out}, dep. on IC/BC's.
To get unknowns to match at $o(\varepsilon)$,

Set $$\lim_{\varepsilon \to 0} \frac{u_{out}(\xi t_2) - u_{in}(\xi t_2/\varepsilon)}{\varepsilon} = 0$$

Then $$u_{match} = \lim_{\varepsilon \to 0} \frac{u_{out}(\xi t_2)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{u_{in}(\xi t_2/\varepsilon)}{\varepsilon}$$

TA Training w/ Andres [illegible]
Rm 315

How [illegible] [illegible] – [illegible]
[illegible]
[illegible]

4/20 - 207C

Flow around an object at low Reynolds #

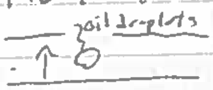
1851 - Stokes computed the force on a "slowly moving" sphere

$$\rho \underline{u} \cdot \nabla \underline{u} = \mu \Delta \underline{u} - \nabla p$$
$$\nabla \cdot u = 0$$
] Navier-Stokes dimensional steady state

could scale → Re. Stokes analyzed Re = 0 limit.

$$\underline{F} = 6 \pi a \mu \underline{u}$$ a = radius of sphere, μ = viscosity, u = velocity of sphere

1907 - Millikan used this to measure the charge of an electron



need to balance gravity, $\underbrace{\text{friction}}_{\text{Stokes}}$, & $\underbrace{\text{electrostatic force}}_{\text{proportional to \# of electrons}}$

Brownian Motion

Stokes-Einstein Relation - diffusion constant

diffusion of small particle

$$D = \frac{K_b T}{6 \pi \mu a}$$ ← balance of thermal forces & friction

---

Stokes wants to look at cylinder moving through fluid
↳ 2D there is no solution — "Stokes paradox"

1889 - Whitehead - tried to derive a correction to Stokes' formula for $\varepsilon = Re \neq 0$

$$\varepsilon \underline{u} \cdot \nabla \underline{u} = \Delta \underline{u} - \nabla p$$
$$\nabla \cdot \underline{u} = 0$$
$$\underline{u}(r=a) = 0$$
$$\underline{u}(\infty) = 1 \underline{e}_x \quad \leftarrow x \text{ direction}$$

$u = u_0 + \varepsilon u_1$ (u_0: Stokes; u_1: no soln for u_1)

"Whitehead paradox"

1910 - Oseen: recognized the problem - able to get correction $O(\varepsilon)$ in 3D

$$\varepsilon \underline{U} \cdot \nabla \underline{u} = \Delta \underline{u} - \nabla p$$

↳ Why?

→ Lamb able to get leading order in 2D

4/22

$$\underbrace{\rho \underline{u}\cdot\nabla\underline{u}}_{\frac{\rho U^2}{L}} - \underbrace{\mu\Delta\underline{u} - \nabla p}_{\frac{\mu U}{L^2}} = 0$$

$\frac{\rho U^2}{L} << \frac{\mu U}{L^2}$ want at low Re

For sphere $U =$ speed at ∞

$L = a =$ sphere radius

$\frac{\rho U_\infty a}{\mu} << 1$ → how about other scales?

$$\underbrace{\left(\frac{\rho U_\infty a}{\mu}\right)\left(\frac{L}{a}\right) = Re\left(\frac{L}{a}\right) \overset{\text{want}}{<<} 1}$$

breaks down for

$\frac{L}{a} = O(Re^{-1})$

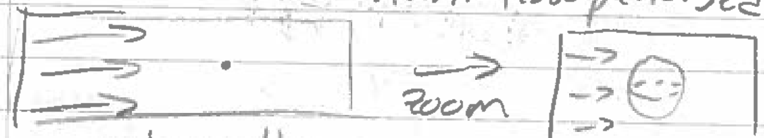
$L \approx \frac{a}{Re}$

Oseen eqn

$$\left.\begin{aligned} \rho\, \underline{U}_\infty\cdot\nabla u &= \mu\Delta\underline{u} - \nabla p \\ \nabla\cdot u &= 0 \end{aligned}\right\}$$ gets $O(\varepsilon)$ correction in 3D

We've considered zero Re perturbed by small Re.

Consider now constant flow perturbed by small object



outer problem

inner problem

$$\varepsilon\underline{u}\cdot\nabla\underline{u} = \Delta\underline{u} - \nabla p$$

rescale $\left.\begin{aligned} \underline{y} &= \varepsilon\underline{x} \\ \underline{V}(\underline{y}) &= \underline{u}(\underline{x}) \end{aligned}\right\}$ outer problem

$$\underline{v}\cdot\nabla\underline{v} = \Delta\underline{v} - \nabla\hat{p}$$

$$\underline{v} = \underline{U} + \varepsilon v_1 \quad \text{w/ } \underline{U} = \text{flow at } \infty$$

→ $\underline{U}\cdot\nabla v_1 = \Delta\underline{v}_1 - \nabla\hat{p}_1$ ← Oseen's eqn.

this outer problem holds all the way to the sphere (lucky) mathematical / structure model, not physical rep.

Lagerstrom Model for Low Re Flow

$$\underbrace{\frac{d^2U}{dR^2} + \frac{K}{R}\frac{dU}{dR}}_{} + \underbrace{\varepsilon U \frac{dU}{dR}}_{\text{like convection}} = 0$$

$$\underbrace{U(1) = 0}_{\text{no flow at sphere}}, \quad \underbrace{U(\infty) = 1}_{\text{constant flow at } \infty}$$

(Analog of force is $\left.\frac{dU}{dR}\right|_{R=1}$)

model of spherically symmetric laplacian in $K+1$ dimensions

(in 2D, $\Delta = \frac{\partial^2}{dr^2} + \frac{1}{r}\frac{\partial}{\partial r} \leftarrow K=1$

in 3D, $\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} \leftarrow K=2$)

Analyze $K=1$ problem — see Stokes's problem & its resolution

$$U'' + \frac{1}{R}U' + \varepsilon U U' = 0$$

$$U(1) = 0, \quad U(\infty) = 1$$

leading order problem: $U_0'' + \frac{1}{R}U_0' = 0$

$$\Rightarrow \frac{1}{R}\frac{d}{dR}\left(R\frac{dU_0}{dR}\right) = 0$$

$$\Rightarrow R\frac{dU_0}{dR} = A$$

$$U_0 = A\ln(R) + B$$

$U_0 \to \infty$ as $R \to \infty$, so cannot satisfy $U(\infty) = 1$

Stokes' Paradox!

$$\underset{A/R^2}{U''} + \underset{A/R^2}{\frac{1}{R}U'} + \underbrace{\underset{\varepsilon(A\ln R + B)\frac{A}{R}}{\varepsilon U U'}}_{} = 0$$

dominates $\frac{A}{R^2}$ for large R

cannot neglect at big R!

Expansion of $U = U_0 + \varepsilon U_1 + O(\varepsilon^2)$ does not approximate the soln. everywhere

We actually started on the inner problem!

Apply inner BC: $U(1) = 0 \Rightarrow B = 0$

inner layer leading order soln $\Rightarrow U_0 = A\ln(R)$.

Rescale to get the outer problem

$r = \varepsilon R$ or $R = r/\varepsilon$ (when $R = O(\varepsilon^{-1})$ want $r = O(1)$)

$u(r) = U(R)$

$$\Rightarrow \varepsilon^2 \frac{d^2u}{dr^2} + \frac{\varepsilon^2}{r}\frac{du}{dr} + \varepsilon^2 u \frac{du}{dr} = 0$$

$u(\infty) = 1$, $u(\varepsilon) = 0$ ← don't apply since u-terℓ.

This is the free stream velocity field that we are perturbing w/ the small object. but on outer layer we ignore the object

$\Rightarrow U_0 = 1$ soln (obviously true).

Simple matching doesn't work here, so introduce intermediate scale

$r_\eta = r/\eta$ w/ $\varepsilon \prec \eta(\varepsilon) < 1$

$r = r_\eta \eta$, $R = \frac{\eta r_\eta}{\varepsilon}$

$$u_0 - U_0 = 1 - A\ln\left(\frac{\eta r_\eta}{\varepsilon}\right)$$

$$= 1 - A(\ln(\eta) + \ln(r_\eta) - \ln(\varepsilon))$$

$$= 1 - A(-\ln\varepsilon)\Bigg(\underbrace{1 - \frac{\ln(\eta) + \ln(r_\eta)}{\ln(\varepsilon)}}_{\to 1 \text{ as } \varepsilon \to 0}\Bigg)$$

Pick $A = -\frac{1}{\ln\varepsilon}$, then $\lim_{\varepsilon \to 0} u_0 - U_0 = 0$

as long as η is the right order.

when does $\frac{\ln(\eta)}{\ln(\varepsilon)} \to 0$?

need $\eta \to 0$ slower than ε^α $\forall \alpha > 0$

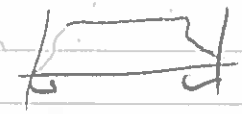
choose $\eta = \frac{1}{|\ln\varepsilon|}$. works

Hence $U(R) = \frac{\ln(R)}{-\ln(\varepsilon)}$

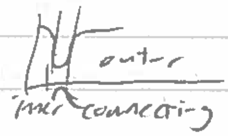
$\Rightarrow \frac{dU}{dR}(1) = \frac{-1}{\ln(\varepsilon)}$ ← doesn't blow up physically since $\varepsilon \neq 0$ just small & this exp. is OK sized.

4/25 - Many other types of layers

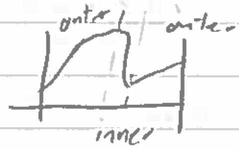
Sec 2.3 – multiple layers
– at different places



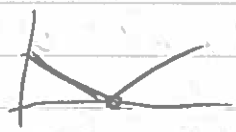
– within layers



Sec 2.4 – interior layers



Sec 2.5 – corner layer



no rapid change in soln, rapid change in derivative

Toy Problem: $\frac{\varepsilon}{2} y'' + yy' - y = 0$

$y(0) = A,\ y(1) = B$

Clearly singular, probably a layer problem – where?
Depends on scales of A & B

outer soln:

$O(1)$: $y_0(y_0' - 1) = 0$

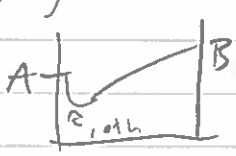
eg. $y_0 = A$, $y_0 = x + A$, $y_0 = x - 1 + B$

Inner eq: $\bar{x} = \frac{x - x_0}{\varepsilon} \Rightarrow \frac{1}{2} Y'' + YY' + \varepsilon Y = 0$

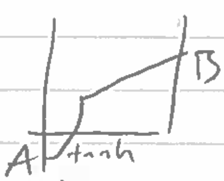
$\Rightarrow Y_0 = C\tanh(C(\bar{x}+D))$ or $Y_0 = C\coth(C(\bar{x}+D))$

How do we stitch these together? Dep. on A & B

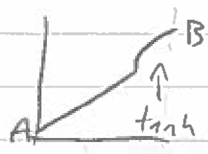
layers at left: layer at right:



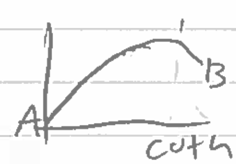
$0 < |A| < B - 1$



$A > B - 1 > 0$



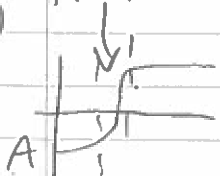
$A + 1 < 1$
$|B| < A + 1$



$B < A + 1 < 0$

$-1 < A + B < 1,\ B > A + 1$

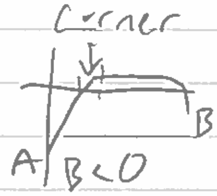
interior



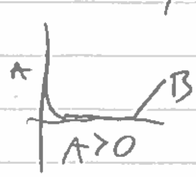
$A > 0,\ B < 0$

two boundary layers

corner

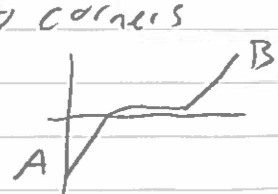


$-1 < A < 0$



$0 < B < 1$

two corners



$0 < -A < 1 - B < 1$

Seemingly easy problems can have wide variety of solns, no real rhyme or reason, have to just try things!

Linear Layer Problems

$$\begin{cases} \varepsilon u'' + a(x)u' + b(x)u = 0 \\ u(0) = \alpha, \quad u(1) = \beta \end{cases}$$ (a, b linear fns)

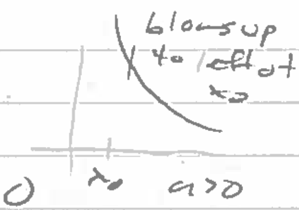
Supp. $a(x)$ does not change sign & bd. away from zero

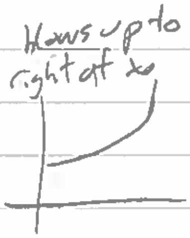
$$\bar{X} = \frac{x - x_0}{\varepsilon}$$

leading order layer eqn:

$$U_0'' + a(x_0)U' = 0$$

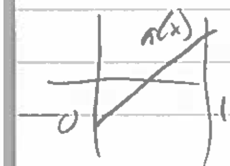
$$\Rightarrow U_0 = \frac{A}{a(x_0)} + Be^{-a(x_0)\bar{X}}$$





Supp. $a(x) > 0$, then $\exists$ boundary layer at $x_0 = 0$ (to prevent blow-up on left of x_0 layer location)

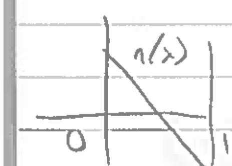
Similarly, if $a(x) < 0$, $\exists$ boundary layer at $x_0 = 1$.



Suppose $a(x)$ switches sign & $a'(x) > 0$

↳ can't have layer at $x_0 = 0$ or 1 $\Rightarrow$ no boundary layer

↳ must have an interior layer or corner layer.



Supp. $a(x)$ switches sign & $a'(x) < 0$

then $\exists$ layer at $x_0 = 0$ & $x_0 = 1$

$\Rightarrow$ double boundary layers

Corner Layer Example

$$\begin{cases} \varepsilon u'' + x u' - u = 0 \\ u(-1) = 1, \quad u(1) = 2 \end{cases}$$

By previous analysis, know interior layer, no boundary layers

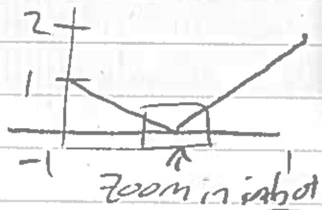
outer problem: $x u_0' - u_0 = 0$

$u_0 = Ax$ ← should apply both left & right boundaries, so get 2 solns w/ 2 constants

At $x=-1$, pick $A=-1 \Rightarrow u = -x$

$x = 1$, pick $A = 2 \Rightarrow u = 2x$

☆ Just try mashing them together, if they miss we need a rapid transient interior layer, if they hit then we smooth the cusp w/ a corner layer



outer soln:

$$u = \begin{cases} -x, & -1 \le x < 0 \\ 2x, & 0 < x \le 1 \end{cases}$$

suggests have corner layer at $x=0$, $u=0$

Scale both u & x:

$x = x_0 + \varepsilon^\alpha \bar{x}$, $u = u_0 + \varepsilon^\gamma U$

$\bar{X} = \frac{x - x_0}{\varepsilon^\alpha}$ $\quad U = \frac{u - u_0}{\varepsilon^\gamma}$

in our example, $x_0 = 0$, $u_0 = 0$

$$\Rightarrow \varepsilon^{1-2\alpha+\gamma} U'' + \varepsilon^\gamma \bar{x} U' - \varepsilon^\gamma U = 0$$

Generally, we pick α first, then γ has more freedom to take care of whatever problems arise in problem — determine at matching

Pick $\alpha = 1/2$: $U'' + \bar{X} U' - U = 0$ (looks like original)

Sometimes this ODE will not be as readily solved, but it doesn't matter if we don't really want a closed form soln.

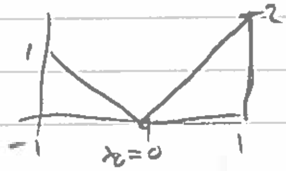
Might be sufficient to show existence of a soln to the ODE and thus prove that there exists a corner layer.

4/27

$$\varepsilon u'' + x u' - u = 0, \quad u(-1)=1, \; u(1)=2$$

outer soln:

$$u_0 = \begin{cases} -x & -1 \le x < 0 \\ 2x & 0 \le x \le 1 \end{cases}$$



Corner layer problems → outer soln continuous but not differentiable!

Rescale to get corner layer:

constant, not soln. → $\left.\begin{aligned} x &= x_0 + \varepsilon^{\alpha} X \\ u &= u_0 + \varepsilon^{\gamma} U \end{aligned}\right\} \alpha = 1/2, \; \gamma = ?$ (in this ex, $x_0 = 0$, $u_0 = 0$)

Inner problem $\underline{U'' + XU' - U} = 0$

guess $U = X$ → reduction of order

closed form not approx. → $U = AX + BX\left[\exp\left(-\frac{X^2}{2}\right) + \int_0^X \exp\left(-\frac{s^2}{2}\right) ds\right]$

Determine A & B by matching with both outer solns at leading order

↳ but if matching at higher order, A, B might depend on ε . . .

↳ Need intermediate scale to match: $x_\eta = \frac{x - x_0}{\eta} = \frac{x}{\eta}$

$\Rightarrow x = \eta x_\eta, \; X = \eta x_\eta / \varepsilon^{1/2}$ $\quad \boxed{\varepsilon^{1/2} \ll \eta \ll 1}$

outer soln:

soln, not constant! → $u_0 = \begin{cases} -\eta x_\eta & x_\eta < 0 \\ 2\eta x_\eta & x_\eta > 0 \end{cases}$

inner problem: $u = \underbrace{u_0}_{\text{constant, not asym. soln}} + \varepsilon^{\gamma} U = \varepsilon^{\gamma} U$

location of corner

$$\Rightarrow \varepsilon^{\gamma} U = \frac{A \varepsilon^{\gamma} \eta x_\eta}{\varepsilon^{1/2}} + \frac{B \varepsilon^{\gamma} \eta x_\eta}{\varepsilon^{1/2}} \left[\underbrace{\exp\left(\frac{-\eta^2 x_\eta^2}{2\varepsilon}\right)}_{\frac{\eta^2}{\varepsilon} \to \infty \text{ since } \varepsilon^{1/2} \ll \eta. \;\; \text{T.S.T}} + \underbrace{\int_0^{\frac{\eta x_\eta}{\varepsilon^{1/2}} \to \pm\infty} \exp\left(-\frac{s^2}{2}\right) ds}_{\to \pm\sqrt{\pi/2}}\right]$$

$$= A \varepsilon^{\gamma - 1/2} \eta x_\eta + B \varepsilon^{\gamma - 1/2} \eta x_\eta \left(\pm \sqrt{\pi/2}\right)$$

In order to match, need $\gamma = 1/2$.

$$= \eta x_\eta \left(A \pm B\sqrt{\pi/2}\right) \quad (+ \text{ when } x_\eta > 0, \; - \text{ when } x_\eta < 0)$$

$$\begin{aligned} A - B\sqrt{\pi/2} &= -1 \\ A + B\sqrt{\pi/2} &= 2 \end{aligned} \Rightarrow A = \frac{1}{2}, \; B = 3/\sqrt{2\pi}$$

Hence the composite soln is

for $x<0$

$$u = \cancel{-x} + \frac{\varepsilon^{1/2}x}{\varepsilon^{1/2}}\left(\frac{1}{2} - \frac{3}{\sqrt{2\pi}}\left(\exp\left(-\frac{x^2}{2\varepsilon}\right) + \int_0^{x/\varepsilon^{1/2}} e^{-s^2/2}\,ds\right) - \cancel{(-x)}\right)$$

Same thing happens for $x>0$

$$\Rightarrow \quad u = x\left[\frac{1}{2} - \frac{3}{\sqrt{2\pi}}\left(e^{-x^2/2\varepsilon} + \int_0^{x/\varepsilon^{1/2}} e^{-s^2/2}\,ds\right)\right]$$

Relaxation oscillations

(oscillation between fast & slow timescales) – Van der Pol oscillator

– Fitzhugh-Nagumo ← model of excitable media for understanding electrical propagation in nerves & cardiac tissue.

Van der pol eqn:

looks similar to damped oscillator $\left\{ \quad u'' + \underbrace{\mu(u^2-1)u'} + u = 0 \right.$, $\mu > 0$

for $|u|<1$ like negative friction coeff ⇒ powers oscillator

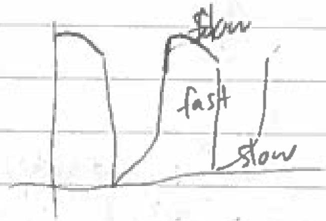
for $|u|>1$ like friction that grows w/ displacement

↳ loss of energy

↳ put energy in at small displacement

We wave hands and expect a limit cycle.

Interested in case of large μ.



fast-slow split!

$$\frac{d^2u}{d\tau^2} + \mu(u^2-1)\frac{du}{d\tau} + u = 0$$

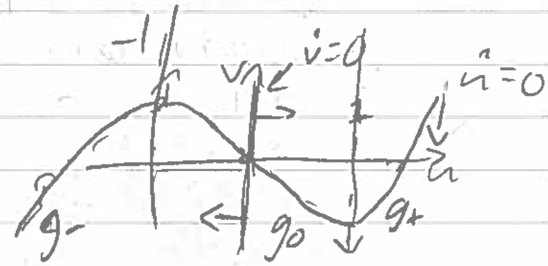
rescale $\tau = t\mu$

$$\Rightarrow \underbrace{\frac{1}{\mu^2}}u'' + (u^2-1)u' + u = 0$$

$\varepsilon = 1/\mu^2$ small param

$$\Rightarrow \frac{d}{dt}\left(\varepsilon u' + \left(\tfrac{1}{3}u^3 - u\right)\right) + u = 0 \Rightarrow \text{let } v = \varepsilon u' + \left(\tfrac{1}{3}u^3 - u\right)$$

$\Rightarrow \frac{dv}{dt} + u = 0$



$$\Rightarrow \begin{cases} \varepsilon \frac{du}{dt} = v + u - \frac{1}{3}u^3 \\ \frac{dv}{dt} = -u \end{cases}$$

single eq. at $(u^*, v^*) = (0,0)$
unstable

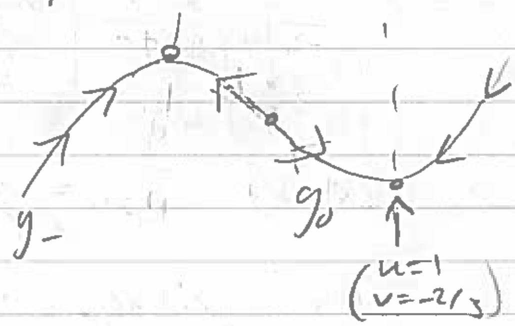
expect limit cycle

Leading order outer soln $\varepsilon \to 0$

$\frac{dv}{dt} = -u$, $v = \frac{1}{3}u^3 - u$ (not invertible generally)

can invert u on different parts of u-space

$u = g_{+/-/0}(v)$



What happens when move down nullcline to $(u=1, v=-2/3)$?

Check the innerlayer eqns:

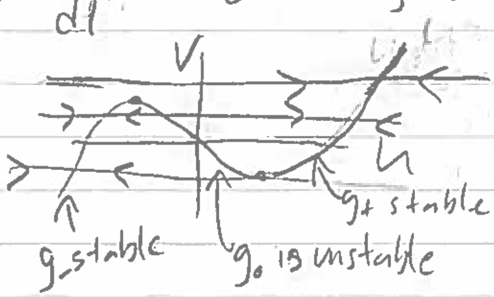
$T = \frac{t - t_0}{\varepsilon}$ or $t = t_0 + \varepsilon T$

$$\left.\begin{aligned} &\Rightarrow \frac{dU}{dT} = V + U - \tfrac{1}{3}U^3 \\ &\frac{1}{\varepsilon}\frac{dV}{dT} = -U \Rightarrow \frac{dV}{dT} = -\varepsilon U \end{aligned}\right]$$

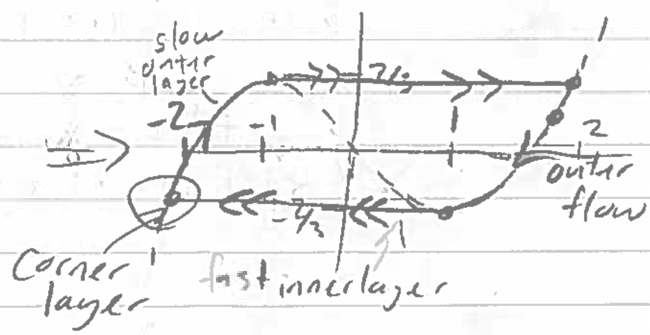
leading order soln

$\frac{dV}{dt} = 0 \to V$ is constant

$\Rightarrow \frac{dU}{dT} = V_0 + U - \frac{1}{3}U^3$



$\Rightarrow$



innerlayer connects g_+ w/ g_- through rapid transients

Can calculate the period of this limit cycle up to leading order by looking only at outer soln.

$$v = \frac{1}{3}u^3 - u, \quad \frac{dv}{dt} = -u \;\Rightarrow\; \frac{dv}{dt} = (u^2-1)\frac{du}{dt} = -u$$

$$\Rightarrow \left(\frac{1}{u} - u\right)\frac{du}{dt} = 1$$ ← integrate on g_+ from $u=2$ to $u=1$.

$$\int_2^1 \frac{1}{u} - u \, du = \int_0^{T_{per}/2} dt$$

$$\Rightarrow T_{per}/2 = 3/2 - \ln 2 \Rightarrow T_{per} = 3 - 2\ln 2 \approx 1.61$$

Numerically:

ε	T_{per}	% off	leading order	next order
10^{-2}	1.9155	18%	1.61	1.939
10^{-3}	1.68		1.61	1.683

next order: $T_{per} = 3 - 2\ln 2 + 3\alpha\,\varepsilon^{2/3}$

$\alpha = 2.338$ ← a root Airy fn.

The inner layer is thickness $O(\varepsilon)$, so why does T_{per} have a correction $O(\varepsilon^{2/3})$?

↳ The corner layer has thickness $O(\varepsilon^{2/3})$

Corner layer is thicker than the inner layer

↳ book sketches simple analysis of this in Sec 6.5

↳ After corner at $u=1$, $v=-2/3$, $t=t_0$

↳ rescale $u = 1 + \varepsilon^{\gamma}\hat{u}$, $v = -2/3 + \varepsilon^{\beta}\tilde{v}$, $t = t_0 + \varepsilon^{\alpha}\hat{t}$.

pick γ, β, α to balance terms in eqn & get new scaling

↳ get $\beta = \alpha$ from $\frac{dv}{dt}$ eqn

↳ get $\gamma = 1/3$, $\alpha = 2/3$ from $\varepsilon\frac{du}{dt}\ldots$ eqn.

5/2

Method of strained time

↳ looking for periodic soln.

Duffing's Eqn: $\begin{cases} \ddot{u} + u + \varepsilon u^3 = 0 \\ u(0) = 1, \ \dot{u}(0) = 0 \end{cases}$

Vander Pol – $\ddot{u} + \varepsilon(u^2 - 1)\dot{u} + u = 0$

Regular expansions for both problems $u = u_0 + \varepsilon u_1 + \dots$

$\Rightarrow \ddot{u}_0 + u_0 = 0$, $u_0(0) = 1$, $\dot{u}_0(0) = 0$ (Duffing)

$\ddot{u}_0 + u_0 = 0$ (VanderPol) expect limit cycle

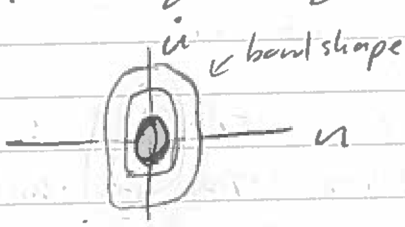
Duffing's is a conservative system

$\dot{u}\ddot{u} + (u + \varepsilon u^3)\dot{u} = 0$

$$\frac{d}{dt}\left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2 + \frac{\varepsilon}{4}u^4\right) = 0$$

↳ trajectories are level curves of this energy fn.

$E(u, \dot{u}) = \frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2 + \frac{\varepsilon}{4}u^4$



Duffing's should also yield periodic soln.

Duffing's eq. $\ddot{u} + u + \varepsilon u^3 = 0$, $u(0) = 1$, $\dot{u}(0) = 0$

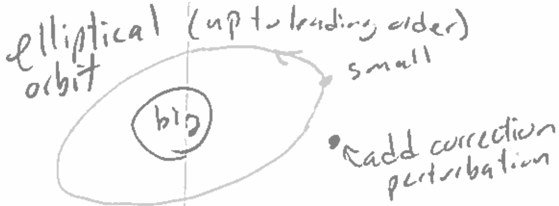
Try $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots$

$O(1)$: $\left.\begin{array}{l} \ddot{u}_0 + u_0 = 0 \\ u_0(0) = 1 \\ \dot{u}_0(0) = 0 \end{array}\right\} \rightarrow u_0 = \cos t = \dfrac{e^{it} + e^{-it}}{2}$

$O(\varepsilon)$: $\begin{cases} \ddot{u}_1 + u_1 = -u_0^3 = -\cos^3 t = -\left(\dfrac{e^{it} + e^{-it}}{2}\right)^3 = -\left[\dfrac{e^{3it} + e^{-3it}}{8} + \dfrac{3}{8}(e^{it} + e^{-it})\right] \\ u_1(0) = 0 \\ \dot{u}_1(0) = 0 \end{cases}$

$= -\frac{1}{4}(\cos 3t + \underbrace{3\cos t}_{\text{resonance}})$

not periodic! OK for short time $\longrightarrow$ yields $t\cos t$, $t\sin t$



comes from Latin "seculum" - century, of the current age

comes from calculation of planetary motion

These terms are called <u>secular terms</u>

↳ solns w/ these are okay for a short period of time very bad on secular time scales

Regular perturbation did not account for the fact that we are looking for a periodic soln.

→ We expect that the perturbation will alter the period

Our attempt: $\ddot{u}_1 + u_1 = F(u_0)$

always expect fns. of natural frequency, higher harmonics, & growth term

Fixing the period! Not allowing for perturbation to change.

Eg. $\cos((1+\varepsilon)t) = \cos t - \underbrace{\varepsilon t \sin t} + O(\varepsilon^2)$

This secular term came from a bad approach!

Method of Poincaré-Lindsted / Strained time

$$\tau = \omega(\varepsilon) t = (\omega_0 + \varepsilon\omega_1 + \dots)t$$

$$u(t) = v(\tau) = v_0(\tau) + \varepsilon v_1(\tau) + \dots$$

↳ Use this method to find the limit cycle in Van der Pol:

$$\ddot{u} + \varepsilon(u^2-1)\dot{u} + u = 0$$

$$\frac{d}{dt} = \frac{d\tau}{dt} \cdot \frac{d}{d\tau} = \omega \frac{d}{d\tau}$$

$$\Rightarrow \omega^2 v'' + \varepsilon(v^2-1)\omega v' + v = 0$$

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \dots$$

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots$$

↳ plug into eqn & collect terms of similar order

$O(1)$: $v_0'' + v_0 = 0$

$$\Rightarrow v_0 =$$

$O(\varepsilon)$: $v_1'' + v_1 = -v_0'(v_0^2-1) - 2\underline{\omega_1} v_0''$

$O(\varepsilon^2)$: $v_2'' + v_2 = -(1+2\underline{\omega_2})v_0'' - ((v_0^2-1)\underline{\omega_1} + 2v_0v_1)v_0' - 2\underline{\omega_1} v_1'' - (v_0^2-1)v_1'$

We can choose ω_1, ω_2 to deal w/ secular terms (eliminate them)

5/4 Strained Time – Poincaré-Lindsted

↑ specific for time periodic problems

Recall: $\ddot{u} + \varepsilon(u^2-1)\dot{u} + u = 0$ ← expect periodic soln.

if $u = u_0(t) + \varepsilon u_1(t) + \ldots$ then nonperiod solution at $O(\varepsilon)$ due to secular terms

Scale: $\tau = \omega t = (\omega_0 + \varepsilon\omega_1 + \ldots)t + (1 + \varepsilon\omega_1 + \ldots)t$

$v(\tau) = u(t) = v_0(\tau) + \varepsilon v_1(\tau) + \ldots$

↳ $\omega^2 v'' + \varepsilon(v^2-1)\omega v' + v = 0$

↳ use maple to expand & collect powers of ε

$O(1)$: $v_0'' + v_0 = 0$

Three equiv. ways to express this general soln:

$$v_0 = \begin{cases} A\sin\tau + B\cos\tau \\ A\cos(\tau+\phi) \\ Ae^{i\tau} + \bar{A}e^{-i\tau} \end{cases}$$

← A complex, most convenient rep. for this problem.

$O(\varepsilon)$: $v_1'' + v_1 = -v_0'(v_0^2-1) - 2\omega_1 v_0''$

Looking for a limit cycle, phase is irrelevant, so let A be real.

$\Rightarrow v_0 = \frac{A_0}{2}(e^{i\tau} + e^{-i\tau}) = A_0\cos(\tau)$ amplitude not determined at leading order

$O(\varepsilon)$:

$$v_1'' + v_1 = \frac{-iA_0^3}{8}e^{3i\tau} + \underbrace{\left(\omega_1 A_0 + i\left(\frac{A_0}{2} - \frac{A_0^3}{8}\right)\right)}_{\text{need this to be zero to kill secular terms \& give periodic soln.}}\underbrace{e^{i\tau}}_{\text{secular terms come from here}} + \text{complex conjugates}$$

$\Rightarrow \omega_1 A_0 = 0$

$\frac{A_0}{2} - \frac{A_0^3}{8} = 0 \to 0 = \frac{A_0}{8}(4 - A_0^2) = \underset{\text{expt}}{0}, \underbrace{2, -2}_{\text{difference is just phase}} \Rightarrow \omega_1 = 0, \underbrace{A_0 = 2}_{\text{gives leading-order soln}}$

So now have $v_0 = 2\cos\tau \to u = 2\cos(\omega t) + \mathcal{O}(\varepsilon)$
$\omega = 1 + \mathcal{O}(\varepsilon^2)$

$\mathcal{O}(\varepsilon)$: $v_1'' + v_1 = -i(e^{3i\tau} - e^{-3i\tau})$

$\Rightarrow v_1 = \frac{i}{8}(e^{3i\tau} - e^{-3i\tau}) + \frac{A_1}{2}(e^{i\tau} + e^{-i\tau})$

$v_1 = -\frac{1}{4}\sin(3\tau) + A_1\cos(\tau)$ ← set A_1 at 2nd order

$\mathcal{O}(\varepsilon^2) \Longrightarrow A_1 = 0.\ \omega_2 = -\frac{1}{16}$

$\Rightarrow$ know amplitude thru $\mathcal{O}(\varepsilon)$, freq. through $\mathcal{O}(\varepsilon^2)$.

$\Rightarrow u = 2\cos\omega t - \frac{\varepsilon}{4}\sin(3\omega t) + \mathcal{O}(\varepsilon^2)$

$\omega = 1 - \frac{\varepsilon^2}{16} + \mathcal{O}(\varepsilon^3)$

Strained Time ~ Eigenvalue Problem

5/9 $(A + \varepsilon B)\vec{v} = \lambda\vec{v}$

Assume $\vec{v} = \vec{v}_0 + \varepsilon\vec{v}_1 + \dots$

$\lambda = \lambda_0 + \varepsilon\lambda_1 + \dots$

$(A + \varepsilon B)(\vec{v}_0 + \varepsilon\vec{v}_1 + \dots) = (\lambda_0 + \varepsilon\lambda_1 + \dots)(v_0 + \varepsilon v_1 + \dots)$

$\mathcal{O}(1)$: $Av_0 = \lambda_0 v_0$ or $(A - \lambda_0 I)v_0 = 0$

$\mathcal{O}(\varepsilon)$: $Av_1 + Bv_0 = \lambda_0 v_1 + \lambda_1 v_0$

or $(A - \lambda_0 I)v_1 = (\lambda_1 I - B)v_0$

Assume λ_0 has algebraic & geometric multiplicity one

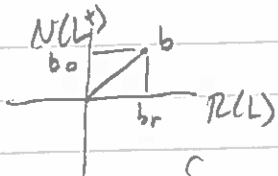
$L = (A - \lambda_0 I)$ is singular, $\vec{v}_0$ spans the nullspace.

$L\vec{v}_1 = (\lambda_1 I - B)\vec{v}_0$ ← can only solve if this is in range of L.
Pick λ_1 to force RHS into range of L.
so there is a solution for v_1

How do we force $(\lambda_1 I - B)\vec{v}_0$ to be in range of L?

$R(L)^\perp = N(L^*)$ by analysis, orthog. compl. of range is nullspace of the adjoint.

$\Rightarrow R(L) \oplus N(L^*) = \mathbb{R}^n$



$b = b_r + b_0$

$b_r = b - b_0 \leftarrow$ project off piece of b in $N(L^*)$

Supp. $N(L^*)$ is spanned by w_0 (if $L = L^*$, $v_0 = w_0$)

(since $\dim N(L^*) = \dim N(L) = 1$).

$Lv_1 = \lambda_1 v_0 - Bv_0$

$\underbrace{w_0^* \cdot Lv_1}_{0} = \lambda_1 w_0^* v_0 - w_0^* Bv_0$

$0 = \;//$

$\Rightarrow \lambda_1 = \dfrac{w_0^* B v_0}{w_0^* v_0} = \dfrac{\langle Bv_0, w_0\rangle}{\langle v_0, w_0\rangle}$

Now that we have λ_1, can solve

$Lv_1 = (\lambda_1 I - B)v_0$ for v_1. & so on.

Can extend this same idea for nonlinear EVPs

$A\vec{v} + \varepsilon \vec{F}(\vec{v}) = \lambda \vec{v}$

$v = v_0 + \varepsilon v_1 + \ldots, \quad \lambda = \lambda_0 + \varepsilon \lambda_1 + \ldots$

$O(1)$: $Av_0 = \lambda_0 v_0. \rightarrow (A - \lambda_0 I)v_0 = Lv_0 = 0$

$O(\varepsilon)$: $Av_1 + F(v_0) = \lambda_0 v_1 + \lambda_1 v_0.$

$\Rightarrow (A - \lambda_0 I)v_1 = \lambda_1 v_0 - F(v_0)$

pick s.t. rhs. in ran L.

& do same deal.

$\rightsquigarrow$ useful for stuff like

$u_{xx} + f(u) = \lambda u, \; u(0) = u(1) = 0$

$f(0) = 0$, $u \equiv 0$, $\lambda = 0$ is a solution $\rightarrow$ perturb this zero soln. to find nontrivial solns.

Van der Pol:

$$\frac{\partial^2 u}{\partial t^2} + \varepsilon(u^2-1)\frac{du}{dt} + u = 0$$

$O(1)$: $\left(\frac{\partial^2}{\partial t^2}+1\right)u_0 = 0$ ← eigenfns are $e^{i\omega t}$ periodic in time.

leading order eigenvalue is $(i\omega_0)^2 = -1$.

perturb eigenfns & eigenvalues:

$$u = u_0 + \varepsilon u_1 + \dots$$

$$\omega = \omega_0 + \varepsilon\omega_1 + \dots$$

Method of Multiple Scales

Damped linear oscillator: $\ddot{u} + 2\varepsilon\dot{u} + u = 0$

regular perturbation generates secular terms at $O(\varepsilon)$.

solns are not periodic, can't use strained time method

exact soln:

$$u = Ae^{-\varepsilon t}\cos\left(\sqrt{1-\varepsilon^2}(t-t_0)\right)$$

two timescales always present

$T_1 = \frac{1}{\varepsilon}$ slow decay of amplitude

$T_2 = \frac{1}{\sqrt{1-\varepsilon^2}}$ fast (relative to this) period of oscillation

Method: Introduce slow time variable $\tau = \varepsilon t$

Assume that the soln. can be written as $u(t) = v(t,\tau)$

expand: $u(t) = v_0(t,\tau) + \varepsilon v_1(t,\tau) + \dots$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\cdot\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau}$$

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial \tau}\right)^2 = \frac{\partial^2}{\partial t^2} + 2\varepsilon\frac{\partial^2}{\partial t\partial\tau} + \varepsilon^2\frac{\partial^2}{\partial\tau^2}$$

Ex: $\ddot{u} + \varepsilon\dot{u}^3 + u = 0$, $u(0)=1$, $\dot{u}(0)=0$

expect to get decaying oscillations since

$$\dot{u}\ddot{u} + \varepsilon\dot{u}^4 + \dot{u}u = 0$$

$$\frac{1}{2}\frac{d}{dt}\left(\underbrace{\dot{u}^2+u^2}_{\text{square distance to origin in } (u,\dot{u}) \text{ plane decreases to } 0}\right) = -\varepsilon\dot{u}^4$$

$\ddot{u} + \varepsilon \dot{u}^3 + u = 0, \quad u(0) = 1, \dot{u}(0) = 0$

introduce $u(t) = v_0(t,\tau) + \varepsilon v_1(t,\tau)$

$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}, \quad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial^2}{\partial \tau^2}$

$$\Rightarrow \left(\frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial^2}{\partial \tau^2}\right)(v_0 + \varepsilon v_1 + \dots) + \varepsilon\left[\left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau}\right)(v_0 + \varepsilon v_1 + \dots)\right]^3$$
$$+ (v_0 + \varepsilon v_1 + \dots) = 0$$

$O(1)$: $\frac{\partial^2}{\partial t^2} v_0 + v_0 = 0$ → harmonic oscillator expected

$O(\varepsilon)$: $\frac{\partial^2}{\partial t^2} v_1 + 2\frac{\partial^2}{\partial t \partial \tau} v_0 + \left(\frac{\partial v_0}{\partial t}\right)^3 + v_1 = 0$

$$\frac{\partial^2}{\partial t^2} v_1 + v_1 = \underbrace{-2\frac{\partial^2}{\partial t \partial \tau} v_0 - \left(\frac{\partial v_0}{\partial t}\right)^3}$$

determine τ dependence of $O(1)$ by forcing solvability of RHS.

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leading order soln: $v_0(t,\tau) = A(\tau)e^{it} + \bar{A}(\tau)e^{-it}$

$O(\varepsilon)$: $\frac{\partial^2}{\partial t^2} v_1 + v_1 = -2(iA'e^{it} - i\bar{A}'e^{-it}) - (iAe^{it} - i\bar{A}e^{-it})^3$

$$= -2(iA'e^{it} - i\bar{A}'e^{-it}) - (-iA^3e^{3it} + i\bar{A}^3e^{-3it} + 3iA^2\bar{A}e^{it} - 3iA\bar{A}^2e^{-it})$$

$$= iA^3e^{3it} - (2iA' + 3iA^2\bar{A})\underbrace{e^{it}}_{\text{gives secular terms, undesirable}} + \text{c.c.}$$

choose A to eliminate secular terms

$2A' + 3A^2\bar{A} = 0$ ← use polar coordinates

$A(\tau) = r(\tau)e^{i\theta(\tau)}$

$\hookrightarrow 2(r'e^{i\theta} + i\theta' r e^{i\theta}) + 3r^2e^{2i\theta} \cdot re^{-i\theta} = 0$

$2r' + ir\theta' + 3r^3 = 0 + 0i$

$\Rightarrow \left.\begin{array}{l} r\theta' = 0 \\ 2r' + 3r^3 = 0 \end{array}\right\}$ → solve to get $\theta = \theta_0$, $r = \frac{1}{\sqrt{3\tau + C}}$

leading order soln: $u = v_0 = \frac{1}{\sqrt{3\tau + C}}\left(e^{i(t+\theta_0)} + e^{-i(t+\theta_0)}\right)$

$$= \frac{2}{\sqrt{3\tau + C}} \cos(t + \theta_0)$$

$$v_0 = \frac{2}{\sqrt{3\tau + c}} \cos(t+\theta_0)$$

Now solve IVP: $u(0)=1,\ \dot u(0)=0$

$\Rightarrow v_0(0,0)=1,\quad \frac{\partial}{\partial t} v_0 + \varepsilon \frac{\partial}{\partial \tau} v_0 \Big|_{(0,0)} = 0$

at $O(1)$: $\quad \frac{\partial}{\partial t} v_0 \Big|_{(0,0)} = 0$

$$\frac{\partial v_0}{\partial t}\Big|_{(0,0)} = \frac{-2}{\sqrt{3\tau+c}}\sin(t+\theta_0)\Big|_{(0,0)} = \frac{-2}{\sqrt{c}}\sin(\theta_0) = 0 \Rightarrow \theta_0 = 0$$

$$v_0(0,0) = \frac{2}{\sqrt{c}}\cos(\theta_0) = 1 \Rightarrow c = 4$$

$$\Rightarrow u = v_0 = \frac{2}{\sqrt{3\varepsilon t + 4}}\cos(t)$$

Parametric Resonance

$$\ddot u + (\underbrace{\lambda + \varepsilon\cos t}_{\text{small variations in the frequency}})u = 0 \qquad \leftarrow \text{Mathieu's eqn}$$

we could scale by the period of the oscillator

$$u'' + (1 + \tilde\varepsilon\cos(kt))u = 0 \qquad k = 1/\sqrt{\lambda}$$

pendulum & change length periodically in time

Can we get resonance?

From swinging, expect $k=2$? twice the natural frequency

Try regular expansion

$u = u_0 + \varepsilon u_1 + \dots$

$O(1)$: $\quad \ddot u_0 + \lambda u_0 = 0 \to u_0 = Ae^{i\sqrt{\lambda}t} + \bar A e^{-i\sqrt{\lambda}t}$

$O(\varepsilon)$: $\quad \ddot u_1 + \lambda u_1 = -\cos t\, u_0 = -\frac{1}{2}\left(Ae^{it(1+\sqrt{\lambda})} + \bar A e^{-it(1+\sqrt{\lambda})} + \bar A e^{it(1-\sqrt{\lambda})} + Ae^{-it(1-\sqrt{\lambda})}\right)$

$\Rightarrow 1 \pm \sqrt{\lambda} = \pm\sqrt{\lambda}$ leads to secular terms.

$1+\sqrt{\lambda} = \sqrt{\lambda}$ no soln $\to 1-\sqrt{\lambda} = -\sqrt{\lambda}$

$1-\sqrt{\lambda} = +\sqrt{\lambda} \to \lambda = \frac{1}{4} \searrow 1+\sqrt{\lambda} = -\sqrt{\lambda}$

$\Rightarrow \lambda = 1/4$ gives secular terms at $O(\varepsilon)$.

$\Rightarrow k=2$ gives resonance (for $\varepsilon=0$)

(simple analysis suggests this)

Other frequencies lead to resonance?
At higher orders in ε, get secular terms for

$$\lambda = \frac{n^2}{4}, \quad n = 1, 2, \ldots$$

What about resonance for ε near 0, ie. λ near $1/4$?

Let $\lambda = \frac{1}{4} + \varepsilon \lambda_1$

$\hookrightarrow \ddot{u} + \left(\frac{1}{4} + \varepsilon\lambda_1 + \varepsilon \cos t\right) u = 0$

regular expansion will lead to same analysis, so instead lets look for multiple scales:

introduce slow time scale $\tau = \varepsilon t$

$u = v(t, \tau) \qquad \frac{d}{dt} = \frac{\partial^2}{\partial t^2} + 2\varepsilon \frac{\partial^2}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial^2}{\partial \tau^2}$

$O(1)$: $\frac{\partial^2 v_0}{\partial t^2} + \frac{1}{4} v_0 = 0 \quad \Rightarrow v_0 = A(\tau) e^{it/2} + \bar{A}(t) e^{-it/2}$

$O(\varepsilon)$: $\frac{\partial^2 v_1}{\partial t^2} + \frac{1}{4} v_1 = -2 \frac{\partial^2 v_0}{\partial t \partial \tau} - (\lambda_1 + \cos t) v_0$

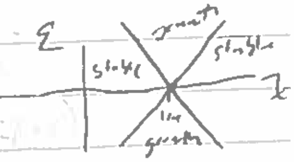
$$\Rightarrow \frac{\partial^2 v_1}{\partial t^2} + \frac{1}{4} v_1 = -2\left(\frac{i}{2} A' e^{it/2} - \frac{i}{2} \bar{A}' e^{-it/2}\right) - \left(\lambda_1 + \frac{e^{it} + e^{-it}}{2}\right)\left(A e^{it/2} + \bar{A} e^{-it/2}\right)$$

$$= -i\left(A' e^{it/2} - \bar{A}' e^{-it/2}\right) - \lambda_1 A e^{it/2} - \lambda_1 \bar{A} e^{-it/2} - \frac{A}{2} e^{3it/2} - \frac{\bar{A}}{2} e^{-3it/2} - \frac{\bar{A}}{2} e^{it/2} - \frac{A}{2} e^{-it/2}$$

$$= -\underbrace{\left(iA' + \lambda_1 A + \frac{\bar{A}}{2}\right)}_{} e^{it/2} + \frac{A}{2} e^{3it/2} + \text{complex conjugates}$$

pick λ_1 to eliminate secular terms

$\hookrightarrow iA' + \lambda_1 A + \frac{1}{2}\bar{A} = 0 \quad \Rightarrow$ let $A = \alpha + i\beta$



$$\begin{cases} -\beta' + \lambda_1 \alpha + \frac{1}{2}\alpha = 0 \\ \alpha' + \lambda_1 \beta - \frac{1}{2}\beta = 0 \end{cases}$$

$\Rightarrow \alpha'' + (\lambda_1 - \frac{1}{2})\beta' = 0$

$\Rightarrow \alpha'' + (\lambda_1 - \frac{1}{2})(\lambda_1 + \frac{1}{2})\alpha = 0$

$\Rightarrow \alpha'' + (\lambda_1^2 - \frac{1}{4})\alpha = 0$

$\alpha(\tau) = K \exp\left(\pm\sqrt{\frac{1}{4} - \lambda_1^2}\, \tau\right) \leftarrow$ slow growth for $|\lambda_1| < \frac{1}{2}$ (in τ) (resonance effect)

Method of Averaging

essentially equivalent to method of multiple scales
average the equations on the fast timescale to get
dynamics on the slow time scale...

Write system in "standard form"

$$\frac{d\underline{x}}{dt} = \varepsilon \underline{f}(\underline{x}, t, \varepsilon)$$, f is 2π-periodic fn. of time

averaging thm:

$\exists$ change of variables $x = y + \varepsilon \underline{U}(y, t, \varepsilon)$

so that $$\frac{dy}{dt} = \overbrace{\varepsilon f_0(y)}^{\text{average behavior}} + \overbrace{\varepsilon^2 \underline{F}(y, t, \varepsilon)}^{\text{corrections}}$$

where $$\underline{f_0}(y) = \frac{1}{2\pi}\int_0^{2\pi} \underline{f}(\underline{x}, t, 0)\, dt$$ leading order slow dynamics

is this soln. structurally stable?

Ex: $\ddot{u} + \varepsilon \dot{u}^3 + u = 0 \qquad u(0) = 1,\ \dot{u}(0) = 0$

$\dot{u} = v$

$\dot{v} = -\varepsilon v^3 - u$

for $\varepsilon = 0 \to u = a\cos(t+\phi)$

$v = -a\sin(t+\phi)$

change coordinates to a, ϕ

$$\Rightarrow a'\cos(t+\phi) - a(1+\phi')\sin(t+\phi) = -a\sin(t+\phi)$$

$$-a'\sin(t+\phi) - a(1+\phi')\cos(t+\phi) = -a\cos(t+\phi) + \varepsilon a^3 \sin(t+\phi)$$

$$\Rightarrow a' = -\varepsilon a^3 \sin^4(t+\phi)$$

$$\phi' = -\varepsilon a^2 \cos(t+\phi)\sin^3(t+\phi)$$

standard form for averaging

invoke averaging theorem!

$$\left.\begin{aligned} \bar{a}' &= -\varepsilon \tfrac{3}{8}\bar{a}^3 \\ \bar{\phi}' &= 0 \end{aligned}\right\} \to \bar{a} = \frac{2}{\sqrt{3\varepsilon t + C}}, \quad \bar{\phi} = \phi_0$$

$$\Rightarrow u = \frac{2}{\sqrt{3\varepsilon t + C}}\cos(t + \phi_0)$$ ← same soln as multiple scale analysis

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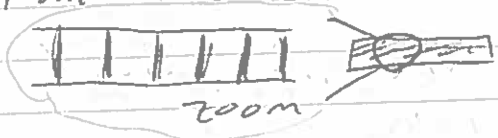
Homogenization (ch. 5)

material w/ microstructure

laminate: 

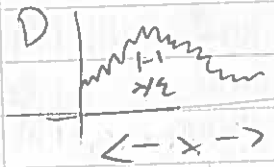
flow through a porous medium – flow through sand, bone, fibre
pores about 1mm

want to describe macroscale of the material – average out the microscale variations

Ex: $\frac{d}{dx}\left(\underbrace{D(x, x/\varepsilon)} \frac{du}{dx}\right) = f(x)$ + BC's

material properties described by diffusion coefficient

$D(x, x/\varepsilon)$ macroscale variation, microscale variation



Naive Approach – replace D w/ average

$$\langle D \rangle = \lim_{y\to\infty} \frac{1}{y}\int_0^y D(x,s)\,ds$$

$y = x/\varepsilon$ microvariable.

This is not the correct approach!

Need to use method of multiple scales/averaging

$D(x, x/\varepsilon) = D(x,y)$

assume $0 < D_m(x) \le D(x,y) \le D_M(x)$

$y = x/\varepsilon$ is the "fast" variable

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}$$

$$u(x) = v(x,y) = v_0(x,y) + \varepsilon v_1(x,y) + \varepsilon^2 v_2(x,y) + \dots$$

$$\Rightarrow \left(\partial_x + \tfrac{1}{\varepsilon}\partial_y\right)\left(D(x,y)\left(\partial_x + \tfrac{1}{\varepsilon}\partial_y\right)(v_0 + \varepsilon v_1 + \dots)\right) = f(x)$$

$O(\varepsilon^{-2})$: $\partial_y(D(x,y)\cdot\partial_y v_0) = 0$

$O(\varepsilon^{-1})$: $\partial_y(D(x,y)\,\partial_y v_1) + \partial_x(D(x,y)\,\partial_y v_0) + \partial_y(D(x,y)\,\partial_x v_0) =$

$O(1)$: $\partial_y(D\partial_y v_2) + \partial_x(D\partial_y v_1) + \partial_y(D\partial_x v_1) + \partial_x(D\partial_x v_0) = f(x)$

Solve eqns:

$\mathcal{O}(\varepsilon^{-2})$: $\partial_y (D \partial_y v_0) = 0$

$\cancel{D}\, \partial_y v_0 = c_0(x)/D$ $\qquad D_m < D < D_M$

$$v_0(x,y) = c_1(x) + c_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)}$$

Note

$$\frac{y-y_0}{D_m(x)} = \int_{y_0}^{y} \frac{ds}{D_m(x)} \geq \int_{y_0}^{y} \frac{ds}{D(x,s)} \geq \int_{y_0}^{y} \frac{ds}{D_M(x)} = \frac{y-y_0}{D_M(x)}$$

bound integrand from above

Hence there is unbounded linear growth (by sandwiching) in the fast variable y (secular term) so need to force $c_0(x) = 0$

Then $v_0(x,y) = c_1(x)$ ← independent of microscale

$\mathcal{O}(\varepsilon^{-1})$: $\partial_y (D \partial_y v_1) = -\partial_x \cancel{(D \partial_y v_0)}^{0} - \partial_y (D \partial_x v_0)$

$$\Rightarrow D\partial_y v_1 + D\partial_x v_0 = b_0(x)$$

$$v_1 = b_1(x) + \underbrace{b_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)} - y\,\partial_x v_0}_{\text{both grow linearly in } y\text{. So balance! for large } y}$$

Take

$$\lim_{y\to\infty} \frac{1}{y}\left(b_0 \int_{y_0}^{y} \frac{ds}{D(x,s)} - y\,\partial_x v_0\right) = 0$$

$$\Rightarrow \partial_x v_0(x) = b_0(x) \cdot \lim_{y\to\infty} \frac{1}{y}\int_{y_0}^{y} \frac{ds}{D(x,s)}$$] average of 1/fast variation

(*) $\partial_x v_0(x) = b_0(x) \langle D^{-1} \rangle$

$\mathcal{O}(1)$: $\partial_y (D \partial_y v_2) = \underbrace{-\partial_x (D \partial_y v_1) - \partial_x (D \partial_x v_0)} - \partial_y (D \partial_x v_1) + f(x)$

$$\partial_y (D \partial_y v_2) = -b_0'(x) + f(x) - \partial_y (D \partial_x v_1)$$

$$D \partial_y v_2 = a_0(x) + y\,(f(x) - b_0'(x)) - D\,\partial_x v_1$$

$$v_2 = a_1(x) + \underbrace{a_0(x)\int_{y_0}^{y} \frac{ds}{D(x,s)} - \int_{y_0}^{y} \partial_x v_1\, ds}_{\text{linear in } y} + \underbrace{\int_{y_0}^{y} \frac{s(f(x) - b_0'(x))}{D(x,s)}\, ds}_{\text{quadratic in } y}$$

Need to balance linear growth terms, also balance quadratic terms

Quadratic growth elimination:

$$f(x) - b_0'(x) = 0 \Rightarrow b_0'(x) = f(x)$$

combine this w/ (*)

$$\frac{\partial_x v_0}{\langle D^{-1} \rangle} = b_0(x)$$

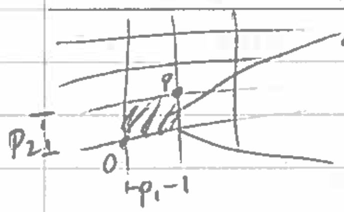
$$\Rightarrow \partial_x \left(\frac{\partial_x v_0}{\langle D^{-1} \rangle} \right) = f(x)$$

$$\Rightarrow \partial_x (D^* \partial_x v_0) = f(x) \leftarrow \text{homogenized eqn.}$$

where $D^* = \dfrac{1}{\lim\limits_{y \to \infty} \frac{1}{y} \int_{y_0}^{y} \frac{ds}{D(x,s)}}$ ← harmonic average

for 2 #'s $\dfrac{2}{\frac{1}{x} + \frac{1}{y}} = \dfrac{2xy}{x+y}$

Multi-dimensional Problem w/ periodic microstructure



assume microstructure periodic on this lattice

$D(\underline{y}) = D(\underline{y} + \underline{p})$ ← a 2-tensor, i.e. a matrix
2×2 in 2D, 3×3 in 3D

use Ω_0 for periodic cell.

required to be pos. definite

$$\nabla \cdot (D(x/\varepsilon) \nabla u) = f(x)$$

we will encounter $\nabla \cdot (D \nabla v) = g(y)$ on Ω_0 (periodic)
↳ v periodic, $D\nabla v$ periodic

need to enforce solvability: integrate over Ω_0

$$\underbrace{\int_{\partial \Omega_0} D \nabla v \cdot n \, dS}_{0 \text{ by periodicity}} = \int_{\Omega_0} g(y) \, dV$$

necessary condition for solvability: $\int_{\Omega_0} g(y) \, dV = 0$.

w/ this condition met, soln. is unique up to addition of a vector from the nullspace of $\nabla \cdot (D \nabla)$, i.e. a constant.

1

Use method of multiple scales $\rightarrow$ $y = \frac{x}{\varepsilon}$, $\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$

$u = v(x, y) = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots$

problem becomes:

$$\left(\nabla_x + \tfrac{1}{\varepsilon}\nabla_y\right)\cdot\left(D(y)\left(\nabla_x + \tfrac{1}{\varepsilon}\nabla_y\right)\left(v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \ldots\right)\right) = f(x)$$

$\mathcal{O}(\varepsilon^{-2})$: $\nabla_y \cdot (D \nabla_y v_0) = 0$

$$\Rightarrow v_0(x) = c(x).$$ (constant w.r.t. microstructure

$\mathcal{O}(\varepsilon^{-1})$: $\underbrace{\nabla_y \cdot (D \nabla_y}_{\text{only dep. on } y} v_1) = -\nabla_y \cdot (D \nabla_x c(x))$

$$= -\nabla_y D \cdot \nabla_x c(x)$$

Expect soln of the form: $v_1(x, y) = \underline{a}(y) \cdot \nabla_x c(x) + b(x)$

plug 'er in:

$$\nabla_y \cdot (D \nabla_y \underline{a} \cdot \nabla_x c(x)) = -\nabla_y D \cdot \nabla_x c(x)$$

$$\nabla_y \cdot (D \nabla_y \underline{a}) \cdot \underbrace{\nabla_x c(x)} = -\nabla_y D \cdot \underbrace{\nabla_x c(x)}$$

comes from homogenized leading order, shouldn't be problem-dependent so factor out

$$\Rightarrow \nabla_y \cdot (D \nabla_y \underline{a}) = -\nabla_y D$$

PDE on the microstructure to determine $\underline{a}$.

$\mathcal{O}(1)$: $\underbrace{\nabla_y \cdot (D \nabla_y v_2) + \nabla_y \cdot (D \nabla_x v_1)} + \nabla_x \cdot (D \nabla_y v_1) + \nabla_x \cdot (D \nabla_x v_0) = f($

Integrate over periodic cell Ω_0. these become 0 by div. thm + periodici

$$\int_{\Omega_0} \nabla_x \cdot (D(\nabla_y v_1 + \nabla_x v_0)) = \int_{\Omega_0} \nabla_x \cdot (D(\nabla_y a + \nabla_x v_0) + D \nabla v_0)$$

$$= \int_{\Omega_0} \nabla_x \cdot (\underbrace{D(I + \nabla_y \underline{a})} \nabla_x v_0) = \int_{\Omega_0} f(x)\, dx$$

all the y dependence live here

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Finish homogenization - multidimension periodic microstructure

$$\nabla \cdot (D(x/\varepsilon)\nabla u) = f(x)$$

We get:
$$\begin{cases} \nabla_x \cdot (D^* \nabla u_0) = f(x) \\ D^* = \frac{1}{|\Omega_0|}\int_{\Omega} D(y)(I + \nabla_y \underline{a})\,dy \\ \nabla_y (D \nabla_y \underline{a}) = -\nabla_y \cdot D \text{ on } \Omega_0. \end{cases}$$

WKB - Ch 4 - only applies to linear problem

$$\varepsilon u'' - q(x) u = f(x), \quad q(x) > 0$$
$$u(0) = a, \quad u(1) = b$$

Singular perturbation:
For $\varepsilon = 0$, we get the soln $u = \frac{-f(x)}{q(x)}$

In general, can't satisfy either boundary condition, so its singular

Let's look for layers at both boundaries.

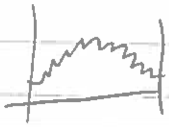
At $x=0$, $X = \frac{x}{\varepsilon^{1/2}}$ or $x = \sqrt{\varepsilon}\, X$
At $x=1$, $X = \frac{x-1}{\varepsilon^{1/2}}$ or $x = 1 + \sqrt{\varepsilon}\, X$
} layer coords.

At leading order,
$x=0$: $U'' - q(0) U = f(0)$ / $x=1$: $U'' - q(1) U = f(1)$
since $q > 0$
solns $e^{\pm\sqrt{q(0)}\,X}$ → exp. layers at both boundaries

What about $\varepsilon u'' + q(x) u = f(x)$, $q(x) > 0$?
$u(0) = A$, $u(1) = B$

Again look for boundary layers
$x=0 \Rightarrow U'' + q(0) U = f(0)$ → oscillatory solution
$$U = \frac{f(0)}{q(0)} + C_1 \exp(iX\sqrt{q(0)}) + C_2 \exp(-iX\sqrt{q(0)})$$

- doesn't look like a boundary layer problem anymore
- fundamentally changed the structure



$u'' + q(0)u = f(0)$ → look locally at x_0 on interior?

$q_0 = q(x_0)$, $f_0 = f(x_0)$

→ not a layer problem

– rapid ε-scale oscillations throughout domain

~~This is a multiple scales problem~~ → change to $y = \sqrt{\varepsilon}\, x = \delta$ >

$v'' + q(\delta x)v = f(\delta x)$

↳ oscillation w/ slow change in frequency & forcing

↳ see section 3.3 for method of multiscales analysis.

– slow time scale isn't obvious

WKB method – Wentzel, Kramers, Brillouin

↳ in this case above, easier than MMS

↳ popularized to approx. solns to Schrodinger's eqn, other app

$$\underbrace{-\frac{\hbar^2}{2m}}_{\text{small parameter}}\frac{\partial^2\psi}{\partial x^2} + (\underbrace{V(x)}_{\text{potential}} - \underbrace{E}_{\text{particle energy}})\psi = 0$$

$$\Rightarrow \varepsilon^2 = \frac{\hbar^2}{2m} \qquad \varepsilon^2\frac{\partial^2\psi}{\partial x^2} - q(x)\psi = 0$$

5/23 WKB method

$\underbrace{\varepsilon u'' - q(x)u = f(x)}_{\text{boundary layers expected}}$ & $\underbrace{\varepsilon u'' + q(x)u = 0}_{\text{fast oscillations → multiple scale prob}}$, $q(x) >$

Steady-State Schrodinger: $\varepsilon^2\frac{d^2\psi}{dx^2} - \underbrace{q(x)}_{\text{potential}}\psi = 0$

What if q is constant?

Trivial to solve → linear combinations of exponentials

All WKB problems assume a soln. of the form

$$u(x,\varepsilon) \sim \underbrace{A(x)}_{\text{amplitude – will expand in powers of } \alpha,\ A(x)\sim A_0(x)+\alpha A_1(x)}\exp\left(\frac{\theta(x)}{\alpha(\varepsilon)}\right) \text{ phase, can be complex}$$

Other notation:

$$u \sim \underbrace{\exp\left(\frac{\theta(x)}{\alpha(\varepsilon)} + \ln(A(x))\right)}_{\text{expand argument in powers of } \varepsilon \text{ \& include } \frac{1}{\alpha} \text{ term.}}$$

$$u \sim \exp\left(\frac{S_0(x)}{\alpha(\varepsilon)} + S_1(x) + \alpha(\varepsilon)S_2(x) + \alpha^2(\varepsilon)S_3(x) + \dots\right)$$

$$= \exp\left(\frac{1}{\alpha(\varepsilon)}\sum_{n=0}^{\infty} S_n(x)\alpha^n(\varepsilon)\right) \leftarrow \text{starting point for WKB}$$

linear problem, fast variation, looks like a WKB problem

Ex: $\varepsilon^2 u'' - q(x) u = 0$, $q(x) \neq 0$ cont's

Assume $u \sim \exp\left(\frac{S_0}{\alpha} + S_1 + \alpha S_2 + \dots\right)$

$u' \sim \left(\frac{S_0'}{\alpha} + S_1' + \alpha S_2' + \dots\right) \exp\left(\frac{S_0}{\alpha} + S_1 + \alpha S_2 + \dots\right)$

$u'' \sim \left[\left(\frac{S_0'}{\alpha} + S_1' + \alpha S_2' + \dots\right)^2 + \left(\frac{S_0''}{\alpha} + S_1'' + \alpha S_2'' + \dots\right)\right] \exp\left(\frac{S_0}{\alpha} + S_1 + \alpha S_2 + \dots\right)$

Plug u, u'' into problem, cancel exp terms, collect powers of $\frac{1}{\alpha}$

$\varepsilon^2\left[\left(\frac{S_0'}{\alpha} + S_1' + \alpha S_2' + \dots\right)^2 + \left(\frac{S_0''}{\alpha} + S_1'' + S_2''\alpha + \dots\right)\right] - q(x) = 0$

$\varepsilon^2\left[\frac{(S_0')^2}{\alpha^2} + \frac{1}{\alpha}\left(2S_0'S_1' + S_0''\right) + \left((S_1')^2 + 2S_0'S_2' + S_1''\right) + \dots\right] - q(x) = 0$

must balance leading orders - so match $\frac{\varepsilon^2 S_0'^2}{\alpha^2} = q(x)$

$\Rightarrow \alpha^2 = \varepsilon^2 \Rightarrow \alpha = \varepsilon \Rightarrow (S_0')^2 = q(x)$

$O(1)$: $(S_0')^2 - q(x) = 0$ ← nonlinear for S_0

$O(\varepsilon)$: $2S_0'S_1' + S_0'' = 0$ ← linear in S_1

$O(\varepsilon^2)$: $2S_2'S_0' + S_1'' + (S_1')^2 = 0$ ← linear in S_2

next order equations will be linear in next S_n

Typically need both S_0 & S_1 for leading order soln.

↳ leading order phase fn. → gives leading order amplitude.

Recall: Multiple scales required two terms, Boundary layer problems involved two scales

$\exp\left(\frac{1}{\alpha}S_0 + S_1 + \alpha S_2 + \dots\right) = e^{S_1} e^{\frac{1}{\alpha}S_0} \exp(\alpha S_2 + \dots) = \underbrace{e^{S_1(x)}}_{\text{"slow" variation in amplitude}} \underbrace{e^{\frac{1}{\alpha}S_0(x)}}_{\text{"fast" variation in phase}} (1 + O(\alpha))$

Solve:

$O(1)$: $S_0 = \int \pm\sqrt{q(x)}\,dx$ → want both (+/-) looking for 2 solns

$O(\varepsilon)$: $S_1' = \frac{-S_0''}{2S_0'} = -\frac{1}{2}\frac{d}{dx}\left(\ln(S_0')\right) \Rightarrow S_1 = -\frac{1}{2}\ln(S_0') + C$

$= -\frac{1}{4}\ln(S_0'^2) + C$

$= -\frac{1}{4}\ln(q(x)) + C$

$\Rightarrow$ solns of the form:

$u_0 = \exp\left(\pm\frac{1}{\varepsilon}\int_a^x \sqrt{q(s)}\,ds + \ln\left(q^{-1/4}\right) + C\right)$

$= \frac{C_{+/-}}{\sqrt[4]{q(x)}} \exp\left(\pm\frac{1}{\varepsilon}\int_a^x \sqrt{q(s)}\,ds\right)$

$q(x) > 0$
$u(0) = 0$
Ex: Eigenvalue Problem: $u'' + \lambda q(x) u = 0$, $u(\pi) = 0$
Find approx. for λ-eigenvalue
~~Know for constant $q>0$ have $u_n = \sin(n\pi x)$, $\lambda = n^2\pi^2$~~

S-L theory: Discrete set of eigenfns/values → orthogonal eigenfns. in the weight inner product $\langle u, v\rangle_q = \int_0^\pi u(x)v(x)q(x)dx$

λ grows big → use $\frac{1}{\lambda}$ as "small" parameter in WKB for large

Let $\lambda = \frac{1}{\varepsilon^2}$: $\varepsilon^2 u'' + q(x)u = 0$ ← WKB problem

Use our derived soln: (w/ $q \to -q$)

$$u_0 = \frac{C_{+/-}}{\sqrt[4]{-q(x)}} \exp\left(\pm\frac{1}{\varepsilon}\int_0^x \sqrt{-q(s)}\,ds\right)$$

$$= A\frac{1}{\sqrt[4]{-q(x)}}\sin\left(\sqrt{\lambda}\int_0^x \sqrt{-q(s)}\,ds\right) + B\frac{1}{\sqrt[4]{-q(x)}}\cos\left(\sqrt{\lambda}\int_0^x \sqrt{-q(s)}\right.$$

Apply $u_0(0) = 0 \Rightarrow B = 0$

$$u_0(\pi) = 0 \Rightarrow \sin\left(\sqrt{\lambda}\int_0^\pi \sqrt{-q(s)}\,ds\right) = 0$$

$$\Rightarrow \sqrt{\lambda}\int_0^\pi \sqrt{-q(s)}\,ds = \pm n\pi \Rightarrow \lambda_n \sim \frac{n^2\pi^2}{\left(\int_0^\pi \sqrt{-q(s)}\,ds\right)^2}$$ for n large

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$u'' + \lambda q(x) u = 0$ — use WKB to estimate eigenvalues λ

$u(0) = u(\pi) = 0$

$\varepsilon u'' + q(x) u = 0$ — $\varepsilon^2 = 1/\lambda$

$$\text{WKB} \Rightarrow \lambda_n \sim \frac{n^2\pi^2}{\left(\int_0^\pi \sqrt{q(s)}\,ds\right)^2}$$

$n \to \infty$ but suprisingly good for n not that big

ex: $q(x) = (x+\pi)^4$

% error	8.1	2.6	1.3	0.77	0.51
n	1	2	3	4	5

Conditions for Validity of WKB expansion

$$u(x) \sim \exp\left(\sum_{n=0}^{\infty} \alpha^{n-1} S_n(x)\right)$$ ← series usually diverges

would like terms in the series to be getting smaller

$$\ldots \ll \alpha^2 S_3 \ll \alpha S_2 \ll S_1 \ll \frac{1}{\alpha}S_0 \Rightarrow \alpha\frac{S_{n+1}(x)}{S_n(x)} = o(1),\ \alpha = o($$

Requirement ① $\frac{S_{n+1}(x)}{S_n(x)}$ uniformly bounded.

This guarantees $\alpha \frac{S_{n+1}}{S_n} = o(1)$, but still not enough b/c of exp

What if we truncate the expansion at $n=N$? Need ignored stuff to be negligible.

Requirement ② $\alpha^N S_{N+1} \ll 1$.

Relative error of truncated expansion:

$$\text{rel error} = \frac{u(x) - \exp\left(\sum_{n=0}^{N} \alpha^{n-1} S_n(x)\right)}{u(x)} \sim 1 - \frac{\exp\left(\sum_{n=0}^{N} \alpha^{n-1} S_n(x)\right)}{\exp\left(\sum_{n=0}^{\infty} \alpha^{n-1} S_n(x)\right)}$$

$$= 1 - \exp\left(-\sum_{n=N+1}^{\infty} \alpha^{n-1} S_n(x)\right) = 1 - \underbrace{\exp\left(-\alpha^N S_{N+1}(x)\right)}_{\text{② makes this small}} \underbrace{\exp\left(+\sum_{n=1}^{\infty} \alpha^n \frac{S_{N+1+n}}{S_{N+1}}\right)}_{O(1)\ \text{① bounds this}}$$

$O(\alpha^N)$

$$\sim \alpha^N S_{N+1}(x).$$

Use WKB to find an asymptotic expansion to Airy fns for large x

$u'' - xu = 0$ is the Airy eqn.

has 2 solns $\to$ Airy functions

no small parameter! Why not use WKB for

$\varepsilon^2 u'' - \underbrace{x}_{q(x)} u = 0$, then set $\varepsilon = 1$.

$$u \sim \exp\left(\sum_{n=0}^{\infty} \alpha^{n-1} S_n(x)\right)$$

$$\begin{cases} O(1): (S_0')^2 = x \\ O(\varepsilon): 2S_0' S_1' + S_0'' = 0 \\ O(\varepsilon^2): 2S_2' S_0' + S_1'' + (S_1')^2 = 0 \end{cases}$$

$\to S_0' = \pm\sqrt{x} \Rightarrow S_0 = \pm \frac{2}{3} x^{3/2}$

$O(\varepsilon) \to S_1' = -\frac{1}{2} \frac{S_0''}{S_0'} = -\frac{1}{2}\frac{d}{dx}(\ln S_0') \to S_1 = -\frac{1}{2}\ln S_0'$

$= -\frac{1}{4}\ln (S_0')^2$

$= -\frac{1}{4}\ln x$

$\frac{S_1(x)}{S_0(x)} \to 0$ for large x ✓

but $S_1(x) \not\to 0$ for large x. X

$O(\varepsilon^2)$: $S_2 = \pm \frac{5}{48} x^{-3/2}$. $\frac{S_2}{S_1} \to 0$ for large x & $\varepsilon S_2 \to 0$ as $x \to \infty$

$$u \sim C_\pm \exp\left(\pm\frac{2}{3}x^{3/2} - \frac{1}{4}\ln x \pm \frac{5}{48}x^{-3/2}\right) = C_\pm x^{-\frac{1}{4}} \exp\left(\pm\frac{2}{3}x^{3/2} \pm \frac{5}{48}x^{-3/2}\right)$$

for large x

$$\Rightarrow u \sim \underbrace{C_{+/-} x^{-1/4} \exp\left(\pm \tfrac{2}{3} x^{3/2}\right)\left(1 \pm \tfrac{5}{48} x^{-3/2}\right)}$$

~~so two term expansion is valid.~~

For linear problems, WKB can capture both layer problems & multiple

Boundary Layer Example (old):

$$\varepsilon u'' + (1+x)u' + u = 0, \quad u(0)=0, \; u(1)=1$$

Leading order composite expansion: $u_c = \frac{2}{1+x} - 2e^{-x/\varepsilon}$

Apply WKB: $u = \exp\left(\frac{S_0}{\alpha} + S_1 + \alpha S_2 + \dots\right)$

$$\varepsilon\left(\left(\frac{S_0'}{\alpha} + S_1' + \dots\right)^2 + \left(\frac{S_0''}{\alpha} + S_1'' + \dots\right)\right) + (1+x)\left(\frac{S_0'}{\alpha} + S_1' + \dots\right) + 1 = 0$$

$$\varepsilon\left(\frac{S_0'^2}{\alpha^2} + \frac{2S_1'S_0' + S_0''}{\alpha} + \dots\right) + (1+x)\left(\frac{S_0'}{\alpha} + S_1' + \dots\right) + 1 = 0$$

2 choices of balance

$\alpha = \varepsilon$ → leaves 1 ✓

$\alpha = \sqrt{\varepsilon}$ → leaves $\frac{1}{\sqrt{\varepsilon}}$ ✗ not higher order

$$O(\varepsilon^{-1}): (S_0')^2 + (1+x)S_0' = 0$$

$$O(1): 2S_1'S_0' + S_0'' + (1+x)S_1' + 1 = 0$$

$$\Rightarrow S_0'(S_0' + 1 + x) = 0$$

$\hookrightarrow S_0' = 0$ or $S_0' = -(1+x)$

outer soln since S_0 constant can absorb ε.

$O(1) \Rightarrow (1+x)S_1' + 1 = 0$

$\Rightarrow S_1 = -\ln(1+x) + c_1$

outer soln

$\Rightarrow A\exp(-\ln(1+x))$

$\Rightarrow \frac{A}{1+x}$

inner soln

$\hookrightarrow$ ~~$-(1+x)S_1' = 0$~~

~~S_1 is constant~~

$S_0 = -(x + x^2/2)$

$\Rightarrow B\exp\left(\frac{-(x+x^2/2)}{\varepsilon}\right)$

$$\Rightarrow u = \frac{A}{1+x} + B\exp\left(-\frac{(x+x^2/2)}{\varepsilon}\right)$$

$u(0) = 0 = A + B$, $u(1) = 1 = \frac{A}{2} + \text{TST} \Rightarrow A = 2$, $B = -2$

$$u = \frac{2}{1+x} - \underbrace{2\exp\left(-\tfrac{1}{\varepsilon}(x + x^2/2)\right)} = \frac{2}{1+x} - 2\exp\left(-\tfrac{x}{\varepsilon}\right) + O(\varepsilon)$$

only relevant for x near ε

but there $x^2/\varepsilon = O(\varepsilon)$ so negligible.

$u'' + \omega(\varepsilon t)u = 0$ → m.m.s – tricky, see book
→ WKB after var. change

Turning Points

$$\varepsilon^2 u'' - q(x) = 0 \quad w/\ q(x) \neq 0$$ ← saw WKB analysis for this

Points where $q(x) = 0$ are called turning points

Classical particle in 1-D moving under some potential

$$m\frac{d^2x}{dt^2} + V'(x) = 0$$ where V is the potential energy

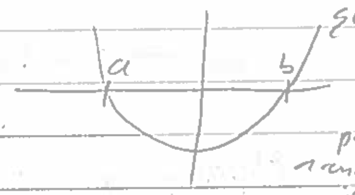
e.g. $V(x) = \frac{1}{2}x^2 \Rightarrow F = -V' = -x$.

Integrate

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + V(x) = E$$ ← energy of particle

Quantum

$$\underbrace{-\frac{\hbar^2}{2m}}_{\varepsilon^2}\frac{d^2\Psi}{dx^2} + \underbrace{(V(x) - E)}_{+q(x)}\Psi = 0$$



Classical particle "turns around" at $x = a, b$ as it rolls in this potential well

Quantum particle is not confined between a & b

$\varepsilon u'' - q(x)u = 0$ $\quad q(x) < 0$ for $a < x < b$

$u(\pm\infty) = 0$ $\quad q(x) > 0$ o/w

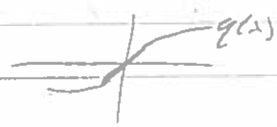
Single Turning Point Problem

$$\varepsilon^2 u'' - q(x)u = 0$$

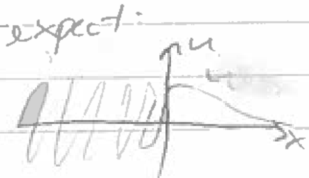
$q(x) > 0$ for $x > 0$

$q(x) < 0$ for $x < 0$

$q'(0) = p \neq 0$



expect:



Apply WKB on each side of $x = 0$, approximations valid away from $x = 0$

$$u_R = q(x)^{-1/4}\left(A_R \exp\left(-\frac{1}{\varepsilon}\int_0^x \sqrt{q(s)}\,ds\right) + B_R \exp\left(\frac{1}{\varepsilon}\int_0^x \sqrt{q(s)}\,ds\right)\right) \text{ for } x > 0$$

away from $x = 0$

$$u_L = q(x)^{-1/4}\left(A_L \exp\left(\frac{1}{\varepsilon}\int_x^0 \sqrt{q(s)}\,ds\right) + B_L \exp\left(-\frac{1}{\varepsilon}\int_x^0 \sqrt{q(s)}\,ds\right)\right) \text{ for } x < 0$$

Four constants w/ only 2 conditions → not independent constants
how to connect them together? Use a layer at $x = 0$.

Introduce a new scaling near $x = 0$

$$X = \frac{x}{\varepsilon^\beta} \quad U(X) = u(x) \quad , \quad x = \varepsilon^\beta X$$

derivs wrt. $\bar{X}$ not x

Layer eqn: $\frac{\varepsilon^2}{\varepsilon^{2\beta}} \bar{U}'' - q(\varepsilon^\beta \bar{X})\bar{U} = 0$

use balance argument to pick β

~~expand $q(\varepsilon^\beta X)$ in the limit as $\varepsilon \to 0$~~

$$\Rightarrow \frac{\varepsilon^2}{\varepsilon^{2\beta}} \bar{U}'' - (\cancelto{0}{q(0)} + \varepsilon^\beta \bar{X} q'(0) + \dots) V = 0$$

$$\Rightarrow \frac{\varepsilon^2}{\varepsilon^{2\beta}} = \varepsilon^\beta \Rightarrow \boxed{\beta = \varepsilon^{2/3}}$$

leading order layer eqn $O(\varepsilon^{2/3})$

$$\bar{U}'' - p\bar{X}\bar{U} = 0$$

$$\bar{X} = p^{-1/3} y \Rightarrow V(y) = \bar{U}(\bar{X})$$

$$\Rightarrow V'' - yV = 0 \rightarrow \text{Airy's eqn. (solns } A_i, B_i)$$

$$\Rightarrow \bar{U}(\bar{X}) = A\, A_i(p^{1/3}\bar{X}) + B\, B_i(p^{1/3}\bar{X})$$

u_L | $\bar{U}$ | u_R

$-c\varepsilon^{2/3}$ 0 $c\varepsilon^{2/3}$

Apply matching via intermediate scale at both places

↳ produces 4 constraints on the constants

matching at right: $A_R = \frac{1}{2\sqrt{\pi}} p^{1/6} \varepsilon^{1/6} A$

$B_R = \frac{1}{2\sqrt{\pi}} p^{1/6} \varepsilon^{1/6} B$

matching at left: $A_L = \frac{1}{2\sqrt{\pi}} p^{1/6} \varepsilon^{1/6} (A + iB)$

$B_L = \frac{1}{2\sqrt{\pi}} p^{1/6} \varepsilon^{1/6} (B + iA)$

$\Rightarrow$ $A_L = A_R + \frac{iB_R}{2}$

$B_L = iA_R + \frac{B_R}{2}$

connection formulas

6/1 – 207C

Last class of Year!

Turning point

$\varepsilon^2 u'' - q(x)u = 0$, $q(x) > 0, x > 0$

$q(x) < 0, x < 0$

$q'(0) = p \neq 0$, $q(0) = 0$.

WKB | layer | WKB

A_L, B_L 0 A_R, B_R

matching via intermediate scale

2 WKB solns, 1 layer → 6 unknown constants

Matching → 4 conditions on unknowns – "Connection Formulas"

$A_L = A_R + \frac{i}{2} B_R$, $B_L = i A_R + \frac{1}{2} B_R$

After some algebra:

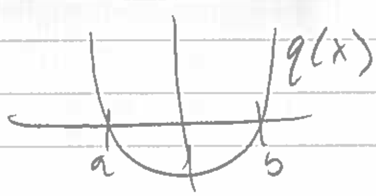
$$u(x) = \begin{cases} |q(x)|^{-1/4}\left[2A_R\cos\left(\frac{1}{\varepsilon}\theta(x) - \frac{\pi}{4}\right) + B_R\cos\left(\frac{1}{\varepsilon}\theta(x) + \frac{\pi}{4}\right)\right], & x < 0 \\ q(x)^{-1/4}\left[A_R\exp\left(-\frac{K(x)}{\varepsilon}\right) + B_R\exp\left(\frac{K(x)}{\varepsilon}\right)\right], & x > 0 \end{cases}$$

w/ $\theta(x) = \int_x^0 \sqrt{|q(s)|}\,ds$, $K(x) = \int_0^x \sqrt{q(s)}\,ds$

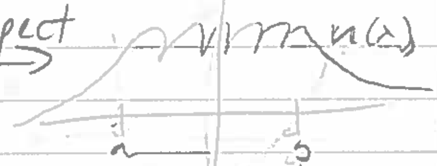
e.g.

in case $B_R = 0$

(force $u(\infty) = 0$)



expect



Two Turning Points – WKB Quantization

If apply above $u(\infty) = 0 \Rightarrow B_R = 0$ → bootstrap our soln here

1) Solve $\varepsilon^2 v'' - Q(x)v = 0$, $Q(x) < 0$, $x < b$

$Q(x) > 0$, $x > b$

Let $Q(x) = q(x-b) = q(y)$ → $y = x-b$ $\quad Q(\infty) = 0$

$v(x) = u(x-b) = u(y)$ → solves previous problem $\Rightarrow B_R = 0$

$\Rightarrow v(x) = |Q(x)|^{-1/4}\, 2A\cos\left(\frac{1}{\varepsilon}\int_x^b \sqrt{|Q(t)|}\,dt - \pi/4\right)$, $x < b$

2) Solve $\varepsilon^2 v'' - Q(x)v = 0$, $Q(x) < 0$ $x > a$

$Q(x) > 0$ $x < a$, $Q(-\infty) = 0$

$Q(x) = q(a-x) = q(y)$, $v(x) = u(a-x) = u(y)$

$\Rightarrow v(x) = |Q(x)|^{-1/4}\, 2A\cos\left(-\frac{1}{\varepsilon}\int_x^a \sqrt{|Q(t)|}\,dt + \pi/4\right)$

Require: $\frac{1}{\varepsilon}\int_x^b \sqrt{|Q(t)|}\,dt - \pi/4 = \frac{1}{\varepsilon}\int_x^a \sqrt{|Q(t)|}\,dt + \pi/4 + n\pi$, $n \in \mathbb{Z}$ to match these solns.

Matching v. require: $\frac{1}{\varepsilon}\left(\int_a^b \sqrt{|Q(t)|}\,dt\right) = \frac{\pi}{2} + n\pi$

$n \in \mathbb{N}$

This is a constraint on the potential $Q(t)$ to have nontrivial soln

Let $q(x) = V(x) - E$

$-\varepsilon^2 u'' + V(x)u = Eu$ (particle in potential)

$u(\pm\infty) = 0$

Nontrivial solns require

$$\frac{1}{\varepsilon}\int_a^b \sqrt{E - V(x)}\,dx = \left(n + \tfrac{1}{2}\right)\pi, \quad n = 0, 1, 2, \ldots$$