# 207A - Applied Math

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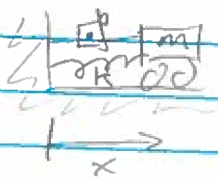
Oct 26 - Midterm 1, Nov. 16 - Midterm 2

207A - ODE's   207B - some ODE's & PDEs that can be solved

207C - Ways to approximate solns

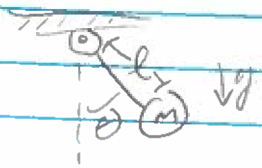
mass on a spring




$m\ddot{x} + b\dot{x} + kx = 0$   $x(0) = 0,\ \dot{x}(0) = 1$

↑ linear ordinary d.e., $2^{nd}$-order, solvable

pendulum




$ml^2\ddot{\theta} + mgl\sin\theta = 0$,   $\theta(0) = 0,\ \dot{\theta}(0) = 1$

↑ nonlinear $2^{nd}$ order ODE

1-D Equations: can turn any autonomous ($t$ doesn't appear explicitly) $2^{nd}$ order ODE into a pair of 1-D ODEs

$m\ddot{x} + b\dot{x} + kx = 0 \Rightarrow \dot{x} = v$

$\dot{v} = -\frac{b}{m}v - \frac{k}{m}x$

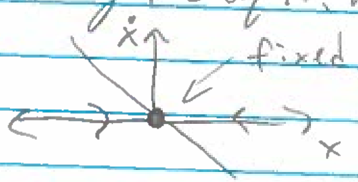
In general, $\ddot{x} = f(x, \dot{x}, t) \Rightarrow x_1 = \dot{x},\ x_2 = x,\ x_3 = t$

Then $\dot{x}_1 = f(x_1, x_2, x_3),\ \dot{x}_2 = x_1,\ \dot{x}_3 = 1$

1-D eqns: $\dot{x} = f(x)$

Ex: $\dot{x} = -ax,\ a > 0 \Leftrightarrow x = x_0 e^{-at},\ x_0 = x(0)$

w/o solving the eqn., how do we get intuition about what soln looks like?




f.p. stable since deriv. is (+) on left & (−) on R.

x = linspace(0, 12, 1001); makes 1001 equally spaced pts. in btwn 0 and 12.

9/28

Recall: 1-D ODES: $\dot{x} = f(x)$, ~~typically we can't solve this eqn.~~

Ex: $+\downarrow$ $\uparrow \gamma v$ $\downarrow mg$ bead falling in honey.

$$m\dot{v} = -\gamma v + mg \qquad (\dot{v} = f(v))$$

$$\dot{v} = g - \frac{\gamma}{m} v$$

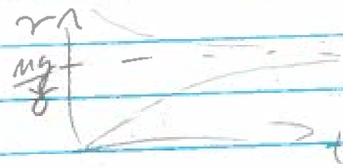
$$\dot{v} = -\frac{\gamma}{m}\left(-\frac{mg}{\gamma} + v\right)$$

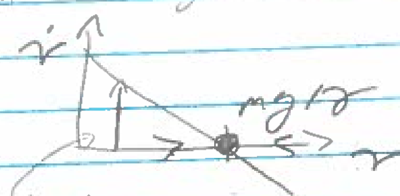
let $u = v - \frac{mg}{\gamma} \Rightarrow \dot{u} = -\frac{\gamma}{m} u$

$$\Rightarrow \quad u = u_0 e^{-(\gamma/m)t}$$

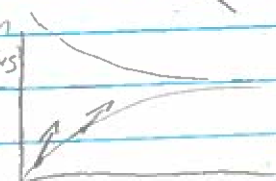
Tranform back: $v - \frac{mg}{\gamma} = \left(v_0 - \frac{mg}{\gamma}\right) e^{-\gamma t/m}$

thus as $t \to \infty$, $v \to mg/\gamma$ terminal velocity

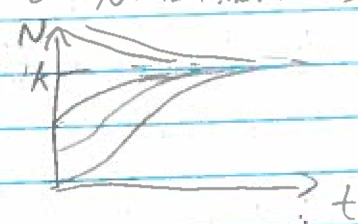



projection of trajectories slope on the v-axis



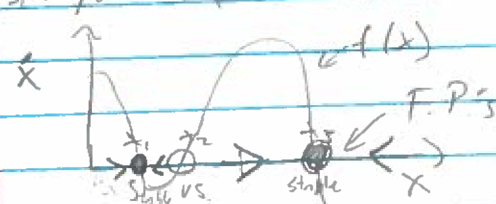
Ex: $\dot{N} = aN\left(1 - \frac{N}{K}\right)$, $a, k > 0$ N: # of individuals


K gives FP.

a scales time

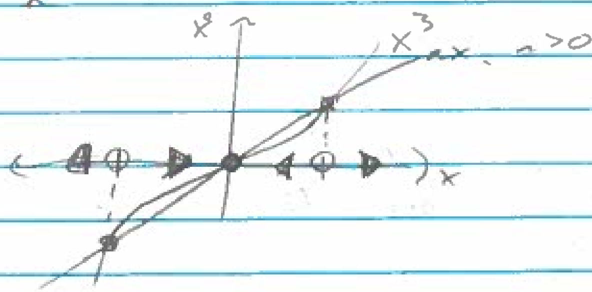
In general, $\dot{x} = f(x)$




Stability: small perturbations at the fixed pt. grows - unstable, decays - stable

What happens when $f(x)$ is hard to sketch?

Ex: $\dot{x} = -ax + x^3$




Ex: $\dot{x} = ax(1+bx)$ Where are the f.p.'s? Stability? System behavior?

$x^* = 0, -1/b$

Stability? $\varepsilon(t) = x - x^*$ small! So $\dot{x} = f(x)$

$$\Rightarrow \dot{\varepsilon} = f(\varepsilon + x^*) \sim \overset{=0}{\cancel{f(x^*)}} + \varepsilon \cdot \frac{df}{dx}\Big|_{x^*} + O(\varepsilon^2)$$

so $\dot{\varepsilon} \sim \frac{df}{dx}\Big|_{x^*} \cdot \varepsilon$

hence $\varepsilon = \varepsilon_0 \exp\left(\frac{df}{dx}\Big|_{x^*} t\right)$

Thus $\frac{df}{dx}\Big|_{x^*} > 0 \Rightarrow$ perturbations $\varepsilon$ grow $\Rightarrow x^*$ is unstable

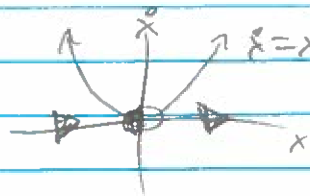
$\frac{df}{dx}\Big|_{x^*} < 0 \Rightarrow$ perturbations $\varepsilon$ decay $\Rightarrow x^*$ is stable.

If $\frac{df}{dx}\Big|_{x^*} = 0$, then higher order terms in approx. are NOT negligible compared to the 1st order term, so need a smarter method.

9/30 Linear stability – soln. approx near F.P.

What if $\frac{df}{dx}\Big|_{x^*} = 0$?

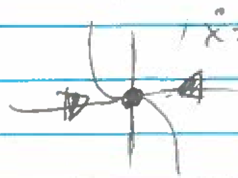
Eg.



$\dot{x} = x^2$ unstable (half stable, but def. of stable says all perturbs decay).

perturbations in the $+x$ direction grow

" " $-x$ direction decay



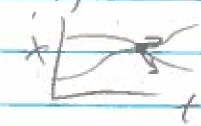
$\dot{x} = -x^3$ stable!

207B - John Hunter
207C - Bob Guy

# Existence & Uniqueness - $\dot{x} = f(x)$ When do soln's exist?

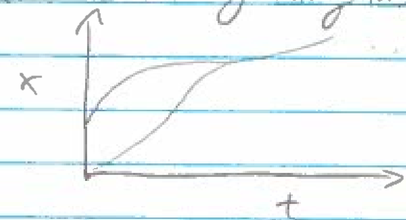
(1) Can trajectories cross? Why / why not? No
(2) Can trajectories converge in finite time? Sometimes
(3) Can oscillatory solns happen? Why/why not? No

(1) No! Trajectories cannot cross b/c $f(x)$ is single-valued

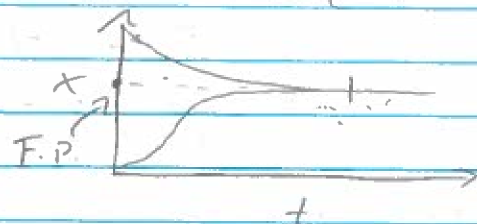


derivatives are different, but $\dot{x} = f(x)$ can only have one value!

(2) Yes, but something funny has to happen



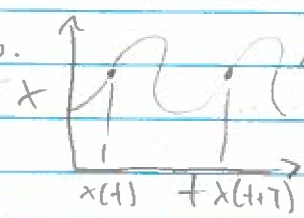
← this is not poss. by same reasoning as in (1)



only way trajectories converge is if they reach a f.p. in finite time.

This means that the linearization around the fixed pt. must break, since linearization predicts exponential decay to the f.p. which does not reach the f.p. in finite time!

(3) No.



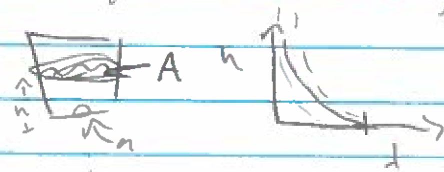
← $x(t)$ is coming from $\dot{x} = f(x)$

Can't get oscillatory soln's in 1-D.

Question (2) relates to uniqueness.

What is required in a physical system to have non-unique solns?

Non-uniqueness means the system loses history information after some p.



after water drains, no way to tell how much water there was originally!

Prof Walcott OH 10-12 Tuesday 2148

Eugene Shwartz OH 1-2 friday, 12-1 Monday

What funny thing happened in the bucket problem? Where did linearization fail?

Let's do the problem. Cons. of mass gives $-\dot{h}A = ra$

Cons. of energy gives $\rho A g h = \frac{1}{2}\rho a r^2$

$\Rightarrow \dot{h} = -k\sqrt{h}$, $k>0$

$\frac{\dot{h}}{\sqrt{h}} = -k \Rightarrow \sqrt{h} = -\frac{k}{2}t + \hat{a}$, $\hat{a} = \sqrt{h_0}$

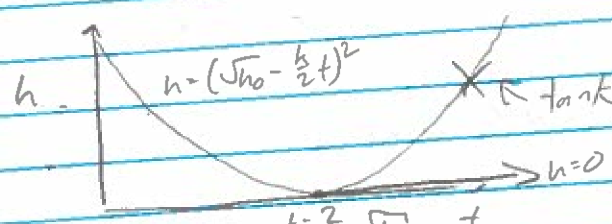
$\Rightarrow h = (\sqrt{h_0} - \frac{k}{2}t)^2$ ← one soln.

other soln is $h(t) = 0$.

ose

$=0$

soln




reach f.p. in finite time

linearization?

$f(h) = -k\sqrt{h}$, $f'(h) = -\frac{k}{2}\cdot\frac{1}{\sqrt{h}}$ @ 0 DNE

$\Rightarrow$ need $f(x)$ to not be $C^1$ to get nonuniqueness.

10/2 $\dot{x} = f(x)$ – Exist. & Uniqueness of soln's?

① Non-uniqueness — system reaches a fixed pt. in finite time

— linearization breaks down → $\frac{df}{dx}\big|_{x^*}$ DNE

② Existence – Can a soln blow up in finite time? yes.

Ex: $\dot{x} = 1+x^2 \Rightarrow \frac{dx}{1+x^2} = dt \Rightarrow \arctan x = t + c$

$x(0) = 0 \longrightarrow c = 0 \Rightarrow x = \tan t$

$-\frac{\pi}{2}$ $\frac{\pi}{2}$

Can make this soln blow up arbitrarily fast by adding small time constant $\dot{x} = \varepsilon(1+x^2)$

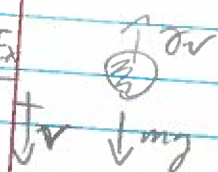
Thm (Existence & Uniqueness) Consider $\dot{x} = f(x)$. If $f(x)$ is $C^1$ ①, then a unique soln. exists for $-T < t < T$ ②, $T > 0$.

robo-fly

## Non-dimensionalization

1. How many parameters does a model have?
2. What is "big" & what is "small"?

Straightforward Ex




$m\dot{v} = mg - \gamma v$   3 "parameters" $m, g, \gamma$

$\dot{v} = g - \frac{\gamma}{m} v$   2 variables $v, t$

idea: let the problem dictate our choice of scale in the variables.

$T = t/\tau$   $\tau$ is the characteristic time scale, dictated by the eq.

$V = v/u$   $u$ is the characteristic velocity scale, dictated by the eqn.

$$\frac{dv}{dt} = \frac{d}{dT}(uV)\frac{dT}{dt} = \frac{u}{\tau}\frac{dV}{dT}$$

$$\frac{u}{\tau}\frac{dV}{dT} = g - \frac{\gamma}{m} uV$$

$$\frac{dV}{dT} = \frac{\tau}{u} g - \frac{\gamma\tau}{m} V$$

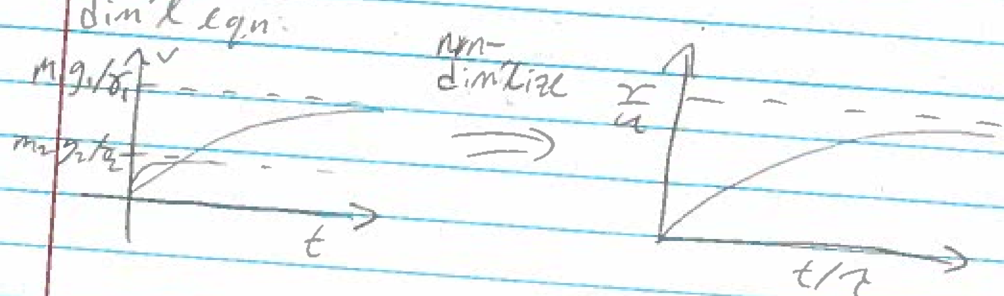
set $\tau = m/\gamma$ $\Rightarrow$ $\frac{dV}{dT} = \frac{mg}{\gamma}\cdot\frac{1}{u} - V$

set $u = mg/\gamma$ $\rightarrow$ $\boxed{\frac{dV}{dT} = 1 - V}$ ← non-dim'lized eqn.

(2) is answered now b/c we can compare $V$ w/ 1 for size

(1) is answered: no parameters!

dim'l eqn:



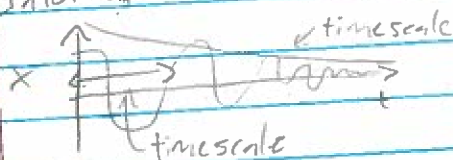
Non-straightforward Ex:

∃ scale that is unclear:


$m\ddot{x}+b\dot{x}+kx=0$ (i)

$x(0)=x_0,\ \dot{x}(0)=0$

Intuition:




Choice of timescale depends on what we are interested in asking questions about.

Variables: $x, t$

let $X = x/\ell$ ← length scale, dictated by eq.

$T = t/\tau$

length/time$^2$

$\frac{dx}{dt} = \frac{d}{dT}(\ell X)\frac{dT}{dt} = \frac{\ell}{\tau}\frac{dX}{dT}$ || $\frac{d^2x}{dt^2} = \frac{\ell}{\tau^2}\frac{d^2X}{dT^2}$

length/time — clear b/c of units

(i) becomes $\frac{\ell}{\tau^2}\frac{d^2X}{dT^2}+\frac{b}{m}\frac{\ell}{\tau}\frac{dX}{dT}+\frac{k}{m}\ell X=0$

If we divide by $\ell$, it disappears, so the eqn. is not dictating the length scale.

If we think physically, we can see $\ell$ is dictated by the initial condition

$$\frac{d^2X}{dT^2}+\frac{b}{m}\tau\frac{dX}{dT}+\frac{k}{m}\tau^2X=0$$

Two choices for $\tau$:

① $\tau = m/b$ ← timescale of the dampening

② $\tau = \sqrt{m/k}$ ← time scale of oscillations

① $\frac{d^2X}{dT^2}+\frac{dX}{dT}+\left(\frac{km}{b^2}\right)X=0$ 1 parameter in eqn. $p=\frac{km}{b^2}$

HW assigned W $^{10/7}$, due next W (10/14)

10/5 Recall: 

$m\ddot{x} + b\dot{x} + kx = 0$, $x(0) = x_0$, $\dot{x}(0) = 0$

$X = x/\ell$, $T = t/\tau$

$\ddot{X} + \frac{b}{m}\tau\dot{X} + \frac{k}{m}\tau^2 X = 0$   2 choices of timescale

pick $\tau = \sqrt{m/k}$ arbitrarily

$\Rightarrow \ddot{X} + p\dot{X} + X = 0$, $p = \frac{b}{\sqrt{mk}}$

guess: $X(t) = e^{rt} \Rightarrow r^2 + pr + 1 = 0 \Rightarrow r_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - 1}$

soln: $X(t) = a_1 e^{r_1 t} + a_2 e^{r_2 t}$ ← changing $p$ changes soln

What is our length scale $\ell$? Look at IC!

In this particular case, it makes sense to pick $\ell = x_0$.

Then, $\ddot{X} + p\dot{X} + X = 0$, $X(0) = 1$, $\dot{X}(0) = 0$

Back to $\dot{x} = f(x)$:

Suppose $\dot{x} = f(x;\alpha)$. What can happen as $\alpha$ is varied?

Ex: Gene switch: $\dot{p} = -kp + A\frac{p^2}{p^2+B}$, $p$ is concentration of some protein

① Nondimensionalize ② How does the system depend on parameters

$T = t/\tau$, $P = p/\rho$

$\frac{dp}{dt} = \frac{\rho}{\tau}\frac{dP}{dT} = -k\rho P + A\frac{\rho^2 P^2}{B + \rho^2 P^2}$

$\Rightarrow \frac{dP}{dT} = -k\tau P + \frac{A}{\rho}\tau\frac{P^2}{\frac{B}{\rho^2} + P^2}$, choose $\tau = 1/k$

$\Rightarrow \frac{dP}{dT} = -P + \frac{A}{\rho k}\frac{P^2}{P^2 + B/\rho^2}$   2 choices for $\rho$, we pick $\rho = \sqrt{B}$.

$\Rightarrow \frac{dP}{dT} = -P + \alpha\frac{P^2}{P^2+1}$, $\alpha = A/(k\sqrt{B})$

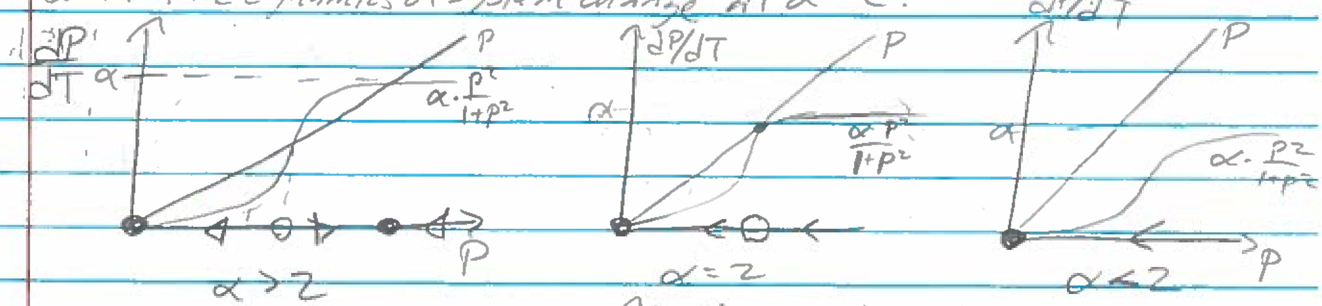
(2) Look for fixed pts

$P^* = \alpha P^{*2}/(1 + P^{*2}) \Rightarrow P^{*3} - \alpha P^{*2} + P^* = 0$

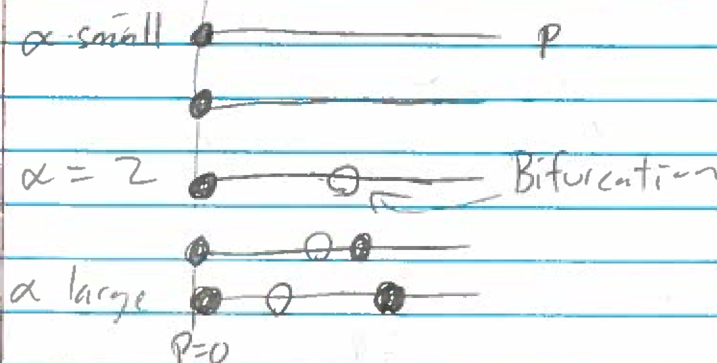
$P_1^* = 0$, $P^{*2} - \alpha P^* + 1 = 0 \Rightarrow P^*_{2,3} = \frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - 1}$

As we change $\alpha$ from small ($<2$) to large ($>2$), transition from $1 \to 3$ fixed points.

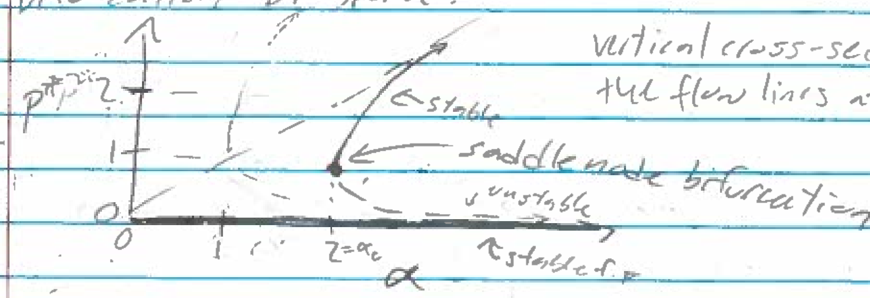
Qualitative dynamics of system change at $\alpha = 2$.




↑double root = tangent curves




Bifurcation Diagram:

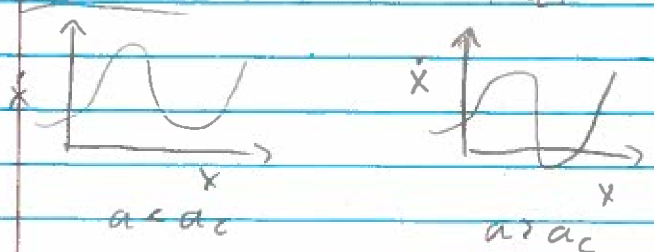



vertical cross-sections are the flow lines at a given $\alpha$.

6/7

Idea of "normal form" is that the dynamics of "all" systems undergoing a (saddle node) bifurcation are identical near the bifurcation point. (*)

Motivation - SN bifurcation at $a_c$




If (*) is true, let's see it in action!

Consider again $\dot{p} = -p + \alpha \frac{p^2}{p^2+1} = g(p)$

Must be "near" bifurcation, so let $p = 1+\varepsilon$, $\alpha = 2+\delta$

$$\Rightarrow \dot{\varepsilon} = -1-\varepsilon+(2+\delta)\,g(1+\varepsilon)$$

$$\dot{\varepsilon} = -1-\varepsilon+(2+\delta)\left(\tfrac{1}{2}+\tfrac{1}{2}\varepsilon+c\varepsilon^2+\mathcal{O}(\varepsilon^3)\right)$$

$$\dot{\varepsilon} = 2c\varepsilon^2+\tfrac{\delta}{2}+\tfrac{1}{2}\delta\varepsilon+c\delta\varepsilon^2+\mathcal{O}(\varepsilon^3)$$

Let $\delta \sim \varepsilon^2 \Rightarrow \dot{\varepsilon} = 2c\left(\frac{\delta}{4c}+\varepsilon^2+\mathcal{O}(\varepsilon^3)\right)$

"parameter" — satisfies

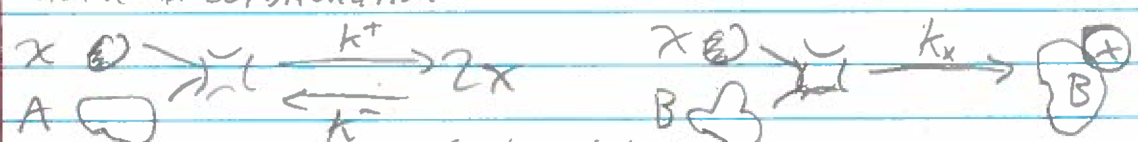
Normal form of saddle node $\dot{x} \sim a + x^2$

Our qualification that "all" dynamics of systems in SN bifs. follow the normal form depends on Taylor expansion $\sim x^2$ near bif. pt.

$\dot{x} = \alpha + x^4$ would work

The point of the normal form is that it tells us how most (saddle node) bifurcations behave.

Another kind of bifurcation




~Simple model of a prion disease.

In our model $x \ll A, B$ so A and B are constants $(A_0, B_0)$.

$$\frac{dx}{dt} = xA_0k^+ - x^2k^- - xB_0k_x$$

nondimlize $\bar{X} = X/c \quad T = t/\tau \Rightarrow \frac{c}{\tau}\frac{d\bar{X}}{dT} = c\bar{X}(A_0k^+ - B_0k_x) - c^2k^-\bar{X}^2$

$$\Rightarrow \frac{d\bar{X}}{dT} = A_0k^+\tau\bar{X} - k^-\tau c\bar{X}^2 - k_xB_0\tau\bar{X}$$

let $\tau = 1/A_0k^+$

$$\frac{d\bar{X}}{dT} = \bar{X} - \frac{k^-c}{A_0k^+}\bar{X}^2 - \frac{k_xB_0}{A_0k^+}\bar{X} \qquad c = \frac{A_0k^+}{k^-}$$

$$\Rightarrow \frac{d\bar{X}}{dT} = \alpha\bar{X} - \bar{X}^2, \text{ where } \alpha = \frac{A_0k^+ - k_xB_0}{A_0k^+}$$

$\alpha$ can be pos. or neg.

draw bif. diagram for $\frac{d\bar{X}}{dT} = \alpha\bar{X} - \bar{X}^2$

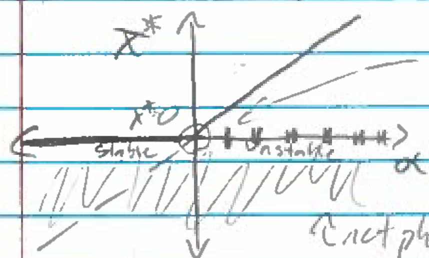
$\bar{X}^*(\alpha)$: $0 = \alpha\bar{X}^* - \bar{X}^{*2} \Rightarrow \alpha\bar{X}^* = \bar{X}^{*2} \Rightarrow \bar{X}^* = 0, \bar{X}^* = \alpha.$

determine stability: $f(\bar{X}) = \alpha\bar{X} - \bar{X}^2$

$f'(\bar{X}^*) = \alpha - 2\bar{X}^* = \begin{cases} \alpha & \bar{X}^* = 0 \\ -\alpha & \bar{X}^* = \alpha \end{cases}$

So for $\alpha < 0$, $\bar{X}^* = 0$ stable, $\bar{X}^* = \alpha$ unstable

for $\alpha > 0$, $\bar{X}^* = 0$ unstable, $\bar{X}^* = \alpha$ stable



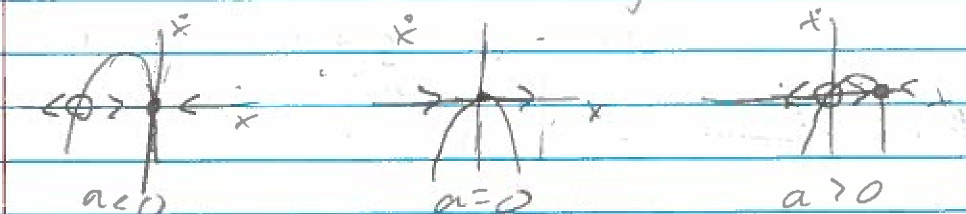

Transcritical bifurcation

it turns out that $\dot{x} = ax - x^2$ is the normal form.

Note: This is a special case

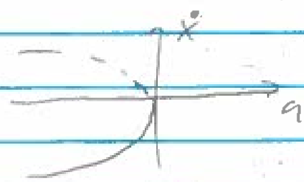
$\dot{x} = ax - x^2 \leftarrow$ transcritical at 0

$\dot{x} = ax - x^2 + \varepsilon \leftarrow$ arb. small yields a saddle node bif.



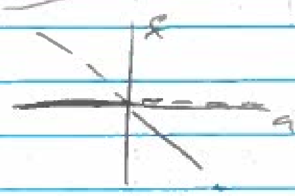

10/9 Saddle Node

$\dot{x} = a + x^2$



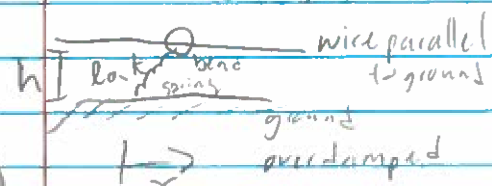

Transcritical

$\dot{x} = ax + x^2$




* $\dot{x} = ax + x^2 + \varepsilon$ is not transcrit.

3rd kind of bif:

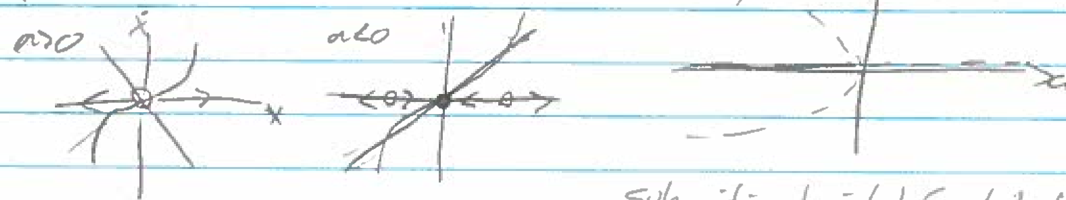




pitchfork bifurcation $\Rightarrow \dot{x} = ax - x^3$

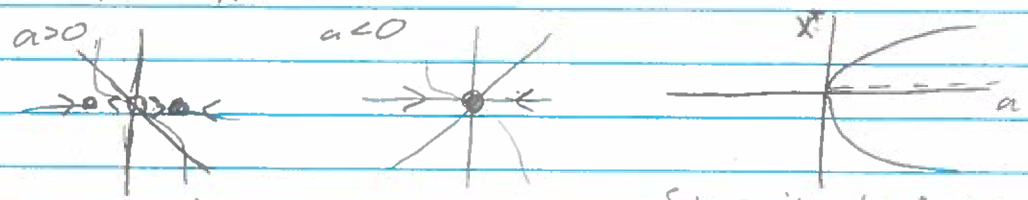
know normal form must be cubic b/c we need 3 roots, & can't have

$\dot{x} = x^3 + ax$




subcritical pitchfork bif.

$\dot{x} = -x^3 + ax$



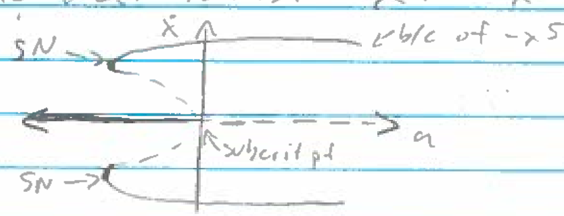

Supercritical p.f. bif.

As with the transcritical, pitchfork bifurcations require something sp

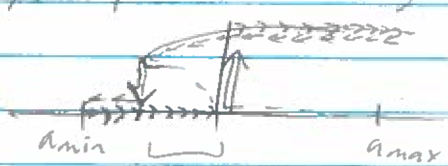
$\dot{x} = \pm x^3 + ax + \varepsilon$ does not have a pitchfork bif!

Hysteresis

Ex: Typically w/ a subcritical pitchfork there would be stabilizing higher-order terms. $\dot{x} = -x^5 + x^3 + ax$




Thought experiment → change a periodically



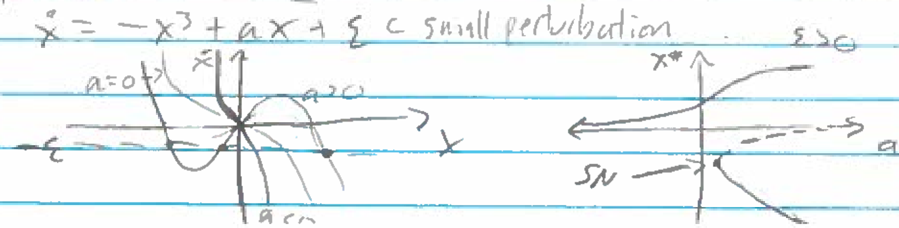

this region exhibits hysteresis – system state depends on memory of how it got there

the memory of how the system got to a state in this region will never fade! (as long as we don't change a anymore when it's there)

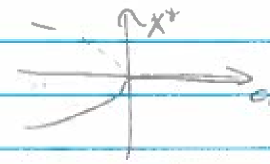
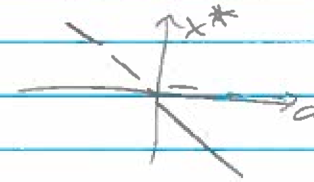
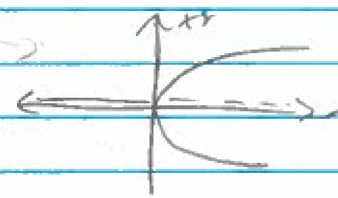
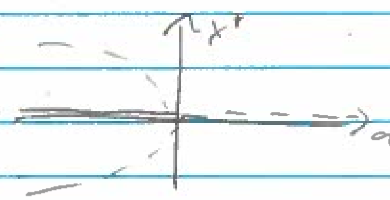
→ Return to the idea that transcritical & pitchfork bf's are special.

Imperfect bifurcations:

$\dot{x} = -x^3 + ax + \varepsilon$ ← small perturbation



## Recall: (§3 in Strogatz)

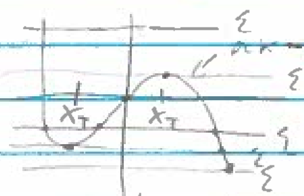
| Bif. name | normal form | plot |
|---|---|---|
| saddle node | $\dot{x} = a + x^2$ |  |
| transcritical | $\dot{x} = ax + x^2$ |  |
| pitchfork - supercrit | $\dot{x} = ax - x^3$ |  |
| - subcrit | $\dot{x} = ax + x^3$ |  |

## Imperfect bif

$\dot{x} = ax - x^3 + \varepsilon$ — attempt to model stuff we think we can neglect

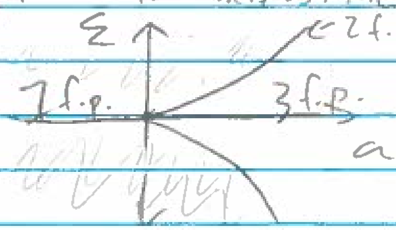
let's look for fixed points!

$0 = ax - x^3 + \varepsilon$ ← 3 fixed points




There are some choices of $a, \varepsilon$ that give 1 fixed pt (and some give 2 or 3)

picture to understand how this system depends on 2 params - $a$ & $\varepsilon$.




Stability diagram ↑

$a - 3x^2 = 0$ ← where could have max/min of $ax - x^3$

$x_T = \pm\sqrt{a/3}$

$\Rightarrow ax_T - x_T^3 = -\varepsilon$

$-(a - x_T^2)x_T = \varepsilon$

$-(a - \frac{a}{3})(\pm\sqrt{\frac{a}{3}}) = \varepsilon \Rightarrow \boxed{\varepsilon = \pm\left(\frac{2a}{3}\right)\sqrt{\frac{a}{3}}}$

2 f.p. here ↓

§5 in Strogatz

$\dot{\vec{x}} = \vec{f}(\vec{x})$

- Look for fixed points $\vec{f}(\vec{x}^*) = \vec{0}$

could be a hard problem, but typically solvable (numerically at least)

- stability? linear stability analysis → approx's soln near f.p.

Let $\dot{\vec{x}} = \vec{f}(\vec{x})$ Assume we've found $\vec{x}^*$. Stability?

We have

$\dot{x}_1 = f_1(x_1, x_2, \dots)$

$\dot{x}_2 = f_2(x_1, x_2, \dots)$

$\vdots$

So let $\xi_1 = x_1 - x_1^*$, $\xi_2 = x_2 - x_2^*$, ...

Then $\dot{\xi}_1 \sim \left.\frac{\partial f_1}{\partial x_1}\right|_{\vec{x}^*} \xi_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{\vec{x}^*} \xi_2 + \dots$

$\dot{\xi}_2 \sim \left.\frac{\partial f_2}{\partial x_1}\right|_{\vec{x}^*} \xi_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{\vec{x}^*} \xi_2 + \dots$

$\vdots$

So $\dot{\vec{\xi}} = \left.\vec{\vec{J}}\right|_{\vec{x}^*} \cdot \vec{\xi} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \dots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \dots \end{bmatrix}$

Jacobian matrix of partial derivatives

Linear stability requires us to solve $\dot{\vec{x}} = \vec{\vec{M}} \vec{x}$, $\vec{\vec{M}}$ $n \times n$

Recall: eigenvalues & eigenvectors $\vec{\vec{M}} \vec{s} = \lambda \vec{s}$

to find $\vec{s}$, solve $(\vec{\vec{M}} - \lambda \vec{\vec{I}})\vec{s} = 0$ (assume $\vec{s} \neq 0$)

$(\vec{\vec{M}} - \lambda \vec{\vec{I}})$ does not have an inverse (otherwise $\text{inv}(M - \lambda I)\vec{s} = 0 \Rightarrow \vec{s} = 0$)

$\Rightarrow \det(M - \lambda I) = 0$, which gives us $n^{th}$ poly eqn in $\lambda$.

In general, $\lambda_1, \dots, \lambda_n$ solve this, each has distinct $\vec{s}_1, \dots, \vec{s}_n$

Since $\dot{x} = mx \rightarrow x = e^{mt} x(0)$ it would be nice if

$\dot{\vec{x}} = \vec{\vec{M}} \vec{x} \rightarrow \vec{x} = \vec{x}(0) [e^{\vec{\vec{M}} t}]$.

What is $e^{Mt}$?

$\vec{\vec{M}} [\vec{s}_1 \; \vec{s}_2 \dots \vec{s}_n] = [\lambda_1 s_1 \; \lambda_2 s_2 \dots \lambda_n s_n]$

columns are eigenvectors $\vec{\vec{M}} \vec{\vec{S}} = \vec{\vec{S}} \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{bmatrix}$

matrix w/ cols. of eigenvectors

$\vec{\vec{M}} = \vec{\vec{S}} \vec{\vec{\Lambda}} \vec{\vec{S}}^{-1}$

$e^{mt} = 1 + mt + \frac{1}{2}m^2t^2 + \dots$

let $e^{\vec{M}t} = \vec{I} + \vec{M}t + \frac{1}{2}\vec{M}^2t^2 + \dots$

$$\vec{M}^N = (\vec{S}\vec{\Lambda}\vec{S}^{-1})(\vec{S}\vec{\Lambda}\vec{S}^{-1}) \dots (S\Lambda S)$$

$$= S\Lambda^N S^{-1} = S\begin{pmatrix} \lambda_1^N & & 0 \\ & \ddots & \\ 0 & & \lambda_n^N \end{pmatrix} S^{-1}$$

$$\Rightarrow e^{\vec{M}t} = S\left(I + \Lambda t + \frac{1}{2}\Lambda^2t^2 + \frac{1}{3!}\Lambda^3t^3 + \dots\right)S^{-1}$$

$$= S\begin{pmatrix} 1+\lambda_1t+\frac{1}{2}\lambda_1^2t^2+\dots & & & 0 \\ & 1+\lambda_2t+\frac{1}{2}\lambda_2^2t^2+\dots & & \\ & & \ddots & \\ 0 & & & 1+\lambda_nt+\frac{1}{2}\lambda_n^2t^2+\dots \end{pmatrix} S^{-1}$$

$$= \vec{S}\begin{pmatrix} e^{\lambda_1t} & & & 0 \\ & e^{\lambda_2t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_nt} \end{pmatrix} S^{-1}$$

So for $\dot{\vec{x}} = \vec{M}\vec{x}$, $\vec{x}_0$, our general soln is

$$\vec{x} = e^{\vec{M}t} \cdot \vec{x}_0, \text{ where } e^{\vec{M}t} = \vec{S}\begin{bmatrix} e^{\lambda_1t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_nt} \end{bmatrix}\vec{S}^{-1}$$

Back to linear stability

$\dot{\vec{\varepsilon}} = \vec{J}|_{\vec{x}^*} \cdot \vec{\varepsilon}$. If $Re(\lambda_i) > 0$, then $\vec{x}^*$ is unstable.

If initial perturbation is along $\vec{S}_i$, then the soln. goes like $\vec{S}_i e^{\lambda_i t}$ & since $\lambda_i > 0$, this blows up $\Rightarrow$ unstable.

1. What about zero eigenvalues? (be worried - is our lin. stability valid?)
2. What about imaginary eigenvalues?

$\vec{\varepsilon} = c_1e^{\lambda_1t} + c_2e^{\lambda_2t} + \dots$ suppose $\lambda_1, \lambda_2$ are complex conjugates

$\lambda_1 = a + bi \Rightarrow c_1e^{\lambda_1t} = c_1e^{at}e^{bit} = c_1e^{at}(\cos bt + i\sin bt)$

★ Linear stability analysis gives more than just stability

- If $\lambda_{i,j}$ im. $\Rightarrow$ soln has oscillations

midterm monday 10/26
3 long problems

10/19 Recall: $\frac{dS}{dt} = 1 - S - aSI$ $\quad S + I + R = 1$

$\frac{dI}{dt} = aSI - 2I$

$\vec{x} = \begin{bmatrix} S \\ I \end{bmatrix}, \; \vec{x}_1^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \vec{x}_2^* = \begin{bmatrix} 2/a \\ \frac{a-2}{2a} \end{bmatrix}$

$$\vec{\vec{J}} = \begin{pmatrix} -1-aI & -aS \\ aI & aS-2 \end{pmatrix}$$

$\vec{\vec{J}}\Big|_{\vec{x}_1^*} = \begin{bmatrix} -1 & -a \\ 0 & a-2 \end{bmatrix}, \quad \vec{\vec{J}}\Big|_{\vec{x}_2^*} = \begin{bmatrix} -a/2 & -2 \\ \frac{a}{2}-1 & 0 \end{bmatrix}$

$\hookrightarrow \lambda_1 = -1, \; \lambda_2 = a-2$

to find eigenvectors $(\vec{\vec{M}} - \lambda_1 \vec{\vec{I}})\vec{S}_1 = \vec{0}$

$\begin{bmatrix} 0 & -a \\ 0 & a-1 \end{bmatrix} \vec{S}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{S}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1-a & -a \\ 0 & 0 \end{bmatrix} \vec{S}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow (1-a)x - a = 0$

$x - ax - a = 0$

$x = \frac{a}{1-a}$

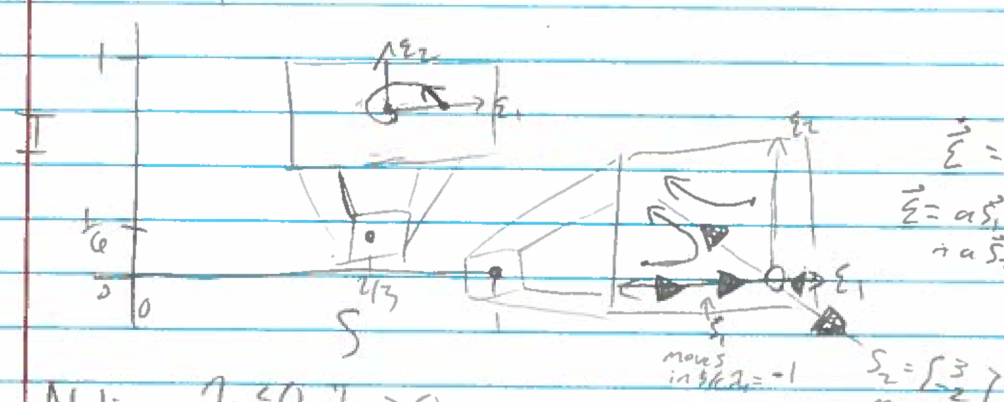
$\Rightarrow \vec{S}_2 = \begin{bmatrix} +a \\ 1-a \end{bmatrix}$

So for $\vec{x}_1^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda_1 = -1 \quad \vec{S}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = a-2 \quad \vec{S}_2 = \begin{bmatrix} -a \\ 1-a \end{bmatrix}$

for $\vec{x}_2^* = \begin{bmatrix} 2/a \\ \frac{a-2}{2a} \end{bmatrix}$, $\lambda_{1,2} = \left(-\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - 4(a-2)}\right)\frac{1}{2}$

Let $a = 3$, sketch $\vec{x}(t)$.




$\vec{\varepsilon} = \vec{\vec{J}}\Big|_{\vec{x}^*} \vec{\varepsilon}$

$\vec{\varepsilon} = a\vec{S}_1 e^{\lambda_1 t} + a\vec{S}_2 e^{\lambda_2 t}$

Notice: $\lambda_1 < 0, \lambda_2 > 0 \Rightarrow$ unstable

$a=3$ still.

Look at $\vec{x}_2^*$. $\lambda_{1,2} = \left(-\frac{3}{2} \pm \sqrt{(\frac{3}{2})^2 - 4}\right)\cdot\frac{1}{2} = -\frac{3}{4} \pm ib$

So the soln $\vec{x} = a\vec{s}_1 e^{\lambda_1 t} + b\vec{s}_2 e^{\lambda_2 t}$

$e^{-\frac{3}{4}t}\cdot\overrightarrow{(\text{sines \& cosines of } t)}$

Know we get spiraling into the f.p.

What direction do we spiral?

<u>Recall:</u> $\vec{\dot{\varepsilon}} = \overset{=}{J}|_{x_2^*}\vec{\varepsilon}$   $\overset{=}{J}|_{x_2^*} = \begin{bmatrix} -3/2 & -2 \\ 1/2 & 0 \end{bmatrix}$

$\Rightarrow \vec{\dot{\varepsilon}} = \begin{bmatrix} -3/2 & -2 \\ 1/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$ gives direction of spiral

$\lambda$  $\boxed{\text{* if } Re = 0,\ Im \neq 0 \Rightarrow \text{center}}$

| Re | Im | |
|---|---|---|
| +, − | 0 | − saddle |
| +, + | 0 | − unstable node |
| −, − | 0 | − stable node |
| + + | ≠0 | − unstable spiral |
| − − | ≠0 | − stable spiral |

complex conjugates guarantees same sign

10/21 Goal: Qualitative understanding of soln. $\vec{x}(t)$ everywhere.

Idea: Graphical stability analysis in 2-D

<u>Null clines</u>

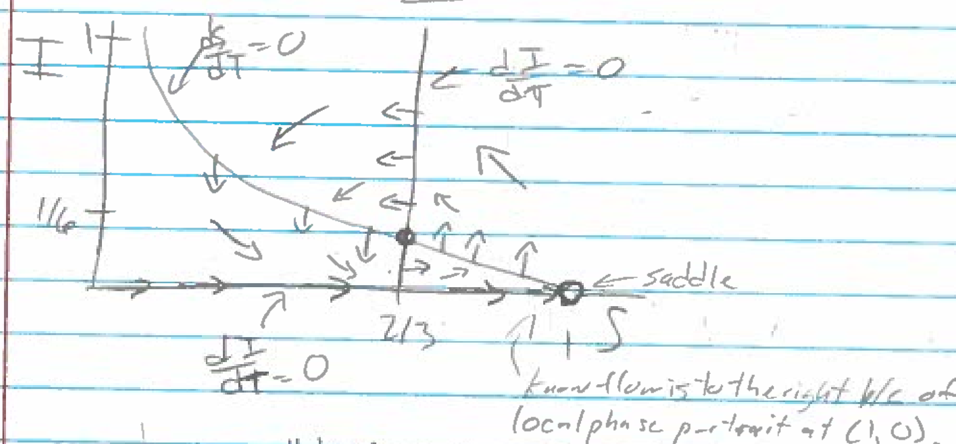
10/21

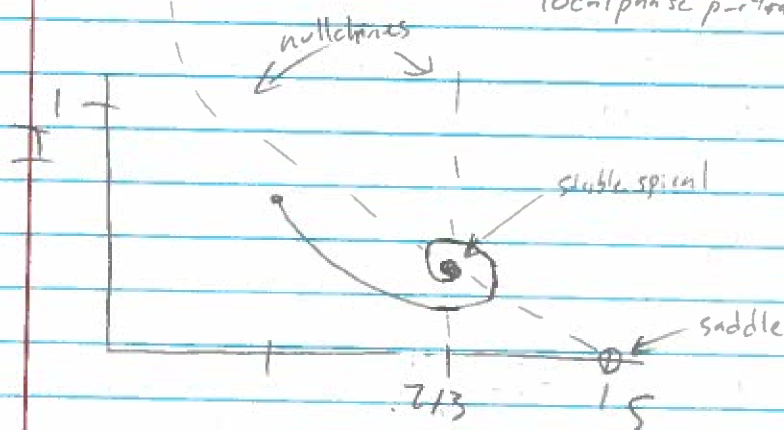
# Nullclines!

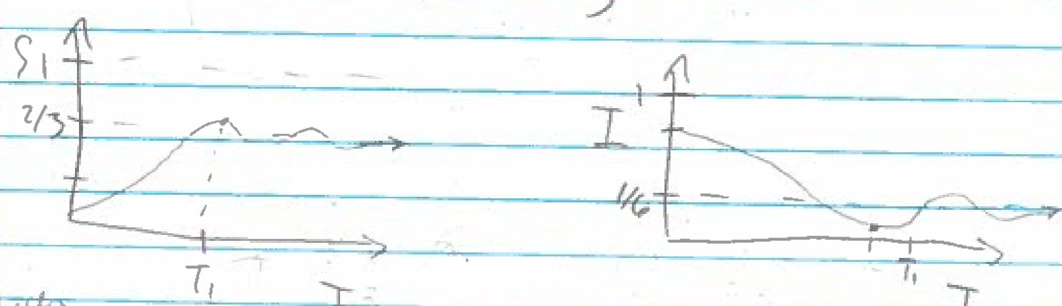
$$\frac{dS}{dT} = 1 - S - aSI \quad , \quad \frac{dI}{dT} = aSI - 2I$$

$a=3$: $\frac{dI}{dT} = 0 \Rightarrow I = 0$ or $S = \frac{2}{3}$ nullclines

$\frac{dS}{dT} = 0 \Rightarrow I = \frac{1-S}{3S}$




☆ Together, linear stability & nullcline graphical stability are a set of tools to take on phase plane analysis.

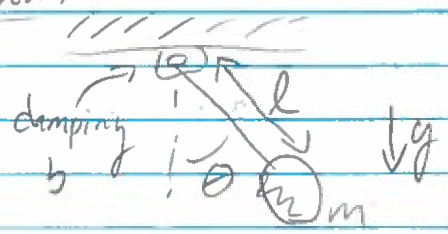
1. Find fixed pts → use nullcline intersections or set $\dot{\vec{x}} = \vec{0}$.
2. Figure out stability of f.p. – eigenvalues of Jacobian (linearization) or intuit stability f...
3. Determine qualitative flow everywhere

typically using nullclines, can intuit from linearization.

# Midterm Monday

covers through local stability in 2D (not nullclines)

Ex - Pendulum




$$ml^2\ddot{\theta} = -b\dot{\theta} - mgl\sin\theta$$

$$\ddot{\theta} = -\frac{b}{ml^2}\dot{\theta} - \boxed{\frac{g}{l}}\sin\theta$$

$\tau = \sqrt{l/g}$

$T = t/\tau = t/\sqrt{l/g}$

$\Rightarrow \ddot{\theta} = -a\dot{\theta} - \sin\theta$

$x_1 = \theta, \ x_2 = \dot{\theta} \Rightarrow \dot{x}_1 = x_2$

$\dot{x}_2 = -ax_2 - \sin x_1$

fixed points: $\vec{x}^* = \begin{bmatrix} 0, \pm\pi, \pm 2\pi, \ldots \\ 0 \end{bmatrix}$

Intuition says $\vec{x}_1^* = \begin{bmatrix} 0, \pm 2\pi, \pm 4\pi, \ldots \\ 0 \end{bmatrix}$ is a stable spiral/node

$\vec{x}_2^* = \begin{bmatrix} \pm\pi, \pm 3\pi, \ldots \\ 0 \end{bmatrix}$ is a saddle

Stability:

$$\overline{\overline{J}} = \begin{bmatrix} 0 & 1 \\ -\cos x & -a \end{bmatrix}$$

for $\vec{x}_1^* = \begin{bmatrix} 0, \pm 2\pi, \pm 4\pi, \ldots \\ 0 \end{bmatrix}$, $\overline{\overline{J}}_{|\vec{x}_1^*} = \begin{bmatrix} 0 & 1 \\ -1 & -a \end{bmatrix}$

$\vec{x}_2^* = \begin{bmatrix} \pm\pi, \pm 3\pi, \ldots \\ 0 \end{bmatrix}$, $\overline{\overline{J}}_{|\vec{x}_2^*} = \begin{bmatrix} 0 & 1 \\ 1 & -a \end{bmatrix}$

$\vec{x}_1^* \Rightarrow \lambda(\lambda + a) + 1 = 0 \Rightarrow \lambda^2 + a\lambda + 1 = 0 \Rightarrow \lambda = -\frac{a}{2} \pm \sqrt{(\frac{a}{2})^2 - 1}$

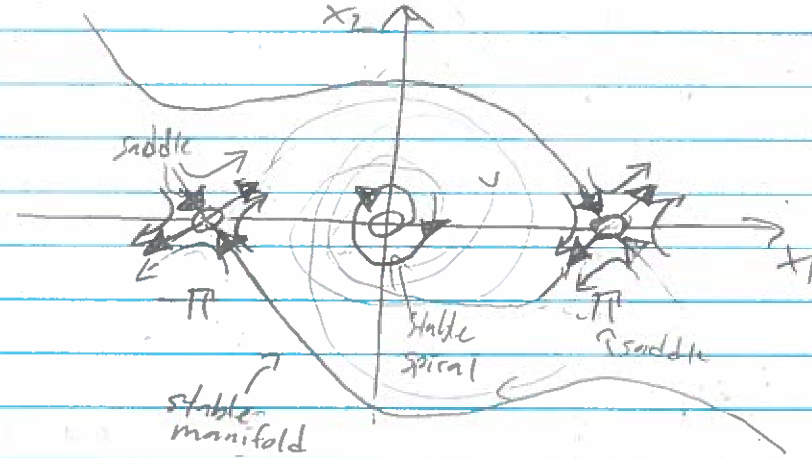
clearly, $\sqrt{(\frac{a}{2})^2 - 1} < \frac{a}{2}$ always, so $\lambda_{1,2} < 0$ always.

for $a < 2$, $\lambda_{1,2} \in \mathbb{C} \Rightarrow$ stable spiral. for $a > 2$, $\lambda_{1,2} \in \mathbb{R} \Rightarrow$ stable node.

$\vec{x}_2^* \Rightarrow \lambda^2 + a\lambda - 1 = 0 \Rightarrow \lambda_{1,2} = -\frac{a}{2} \pm \sqrt{(\frac{a}{2})^2 + 1}$

then we always have one positive $\lambda$, one negative $\lambda$ & always real $\Rightarrow$ always saddle.

Now suppose we want to sketch 2D phase portrait when $a = \varepsilon \ll 1$




for $\vec{x}_2^*$, $\lambda_1 \sim 1$, $\lambda_2 \sim -1$, so

$$\bar{\bar{J}}_{\vec{x}_2^*} - \lambda_1 \bar{\bar{I}} \sim \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \vec{\xi}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{\bar{J}}_{\vec{x}_2^*} + \lambda_2 \bar{\bar{I}} \sim \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow \vec{\xi}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

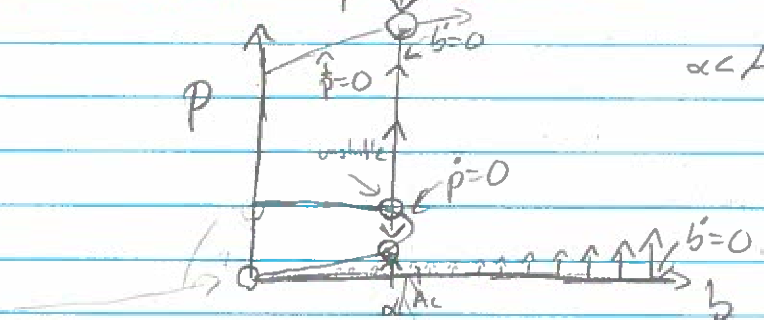
test spiral direction: $\bar{\bar{J}}_{\vec{x}_1^*} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

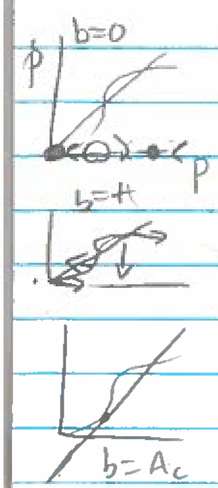
New tool to help us sketch 2D phase portraits $\Rightarrow$ stable manifold of saddles.

Ex: $\dot{p} = -p + 6\frac{p^2}{1+p^2} + b$ Sketch full 2-D phase portrait

$\dot{b} = -bp + \alpha p$ (*)

nullclines of (*) are $p = 0$, $b = \alpha$ (on these lines, $\dot{b} = 0$)

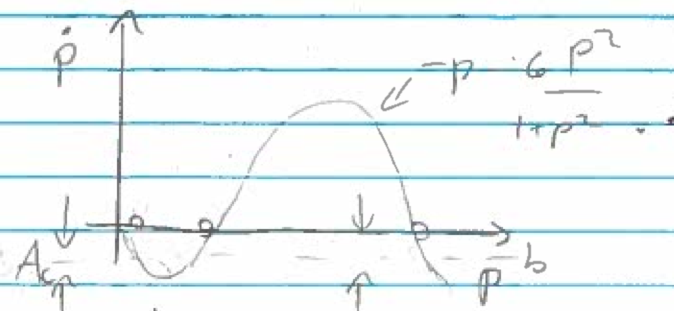


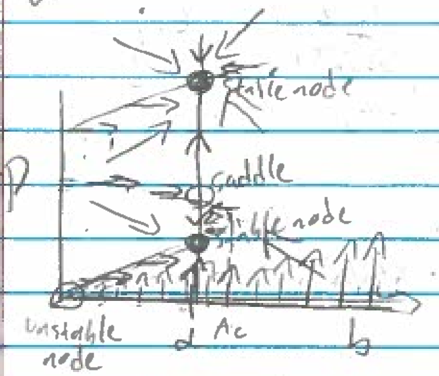

3 fixed points can only have real eigenvalues

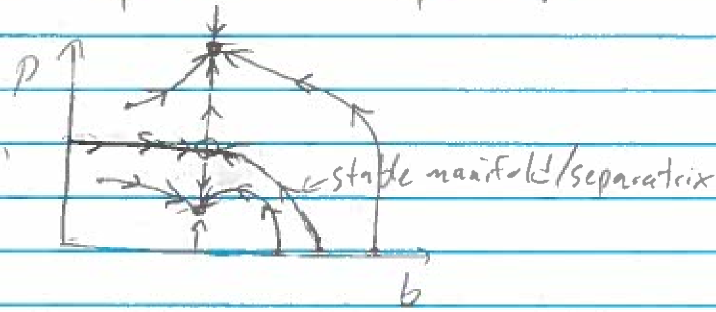
Recall:

$$\dot{p} = -p + 6\frac{p^2}{1+p^2} + b$$
$$\dot{b} = -bp + \alpha p$$




$\alpha < A_c$




Summary: 2 tools – local stability analysis / linear stability
– global analysis – nullclines & trajectories

Which tool you use depends on the problem.

Linear Stability

E-vals

| Re | Im | |
|---|---|---|
| +/+ | =0 | unstable node |
| +/− | =0 | saddle |
| −/− | =0 | stable node |
| +/+ | ≠0 | unstable spiral |
| −/− | ≠0 | stable spiral |
| 0 | ≠0 | center (linearization can fail). |

When is a center really a center?

Non-linear centers – look at pendulum

# When is a center really a center?

## Non-linear centers

Pendulum ex:


$\Rightarrow \ddot{\theta} = -a\dot{\theta} - \sin\theta$, let $a=0$ (frictionless hinge)

This is now a conservative system!

Lin. stab:

$$x_1 = \theta,\ x_2 = \dot{\theta} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin(x_1) \end{cases}$$

$$\vec{x}_1^* = \begin{bmatrix} 0, \pm 2\pi, \pm 4\pi, \dots \\ 0 \end{bmatrix},\quad \vec{x}_2^* = \begin{bmatrix} \pm\pi, \pm 3\pi, \dots \\ 0 \end{bmatrix}$$

$$\overset{=}{J} = \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & 0 \end{bmatrix},\quad \overset{=}{J}\big|_{\vec{x}_1^*} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda^2 + 1 = 0$$

$\lambda = \pm i$ center.

$$\overset{=}{J}\big|_{\vec{x}_2^*} = \begin{bmatrix} 0 & 1 \\ +1 & 0 \end{bmatrix} \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \text{ saddle}$$

Linearization predicts a center, and physical intuition tells us it's true b/c energy is conserv.

$$E = \underbrace{mgh(x_1)}_{\text{Potential}} + \underbrace{\tfrac{1}{2}m\left[v(x_1, x_2)\right]^2}_{\text{Kinetic}}$$

$$= mg(l - l\cos x_1) + \tfrac{1}{2}m\left[l x_2\right]^2$$

$$\dot{E} = mgl\sin x_1 \cdot \dot{x}_1 + ml^2 x_2 \dot{x}_2$$

$$= \cancel{mgl}(\sin x_1)x_2 + \cancel{ml^2} x_2(-\sin(x_1))$$

$$\Rightarrow \dot{E} = x_2 \sin x_1 - x_2 \sin x_1 = 0$$

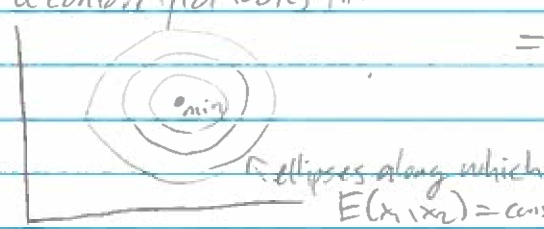
forgot to nondimensionalize the energy eqn.

B/c $\dot{E} = 0$ along trajectories, so energy is conserved.

In general, suppose we find a conserved quantity $E(x_1, x_2)$.

Now let's imagine an (isolated) minimum of E.

Then a contour plot looks like




$\Rightarrow$ minimum is a real center.

$\Rightarrow$ max also a real center.

(saddles are saddles)

10/30

$ml^2\ddot{\theta} = -mgl\sin\theta$

$E = mgl(1-\cos\theta) + \frac{1}{2}ml^2\dot{\theta}^2$

$\dot{E} = mgl\sin\theta\dot{\theta} + ml^2\dot{\theta}\ddot{\theta} = mgl\sin\theta\dot{\theta} - mgl\sin\theta\dot{\theta} = 0$

$\dot{x} = x_2$

$\dot{x}_2 = -\sin(x_1)$ $\longrightarrow$ $E = \frac{1}{2}x_2^2 + (1-\cos x_1)$

near min. $x_1 = x_2 = 0$

series of circles $\longrightarrow$ $E \sim \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 + O(x^4)$

w/ radius $\sqrt{2E}$

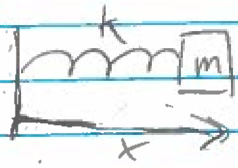
So this is a center.

Q's

1) When does a system have a conserved quantity & why?

2) Given a particular type of conserved quantity, show that there are centers & saddles.

3) Words of warning! (Booo! For Halloween!)




$m\ddot{x} = -kx$

$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$

$\dot{E} = m\dot{x}\ddot{x} + kx\dot{x} = -kx\dot{x} + kx\dot{x} = 0$

If $\ddot{x} = -\frac{dV}{dx}$, then I have a conserved quantity.

$E = \frac{1}{2}\dot{x}^2 + V(x)$

$\Rightarrow \dot{E} = \dot{x}\ddot{x} + \frac{dV}{dx}\dot{x} = \dot{x}\left(-c\frac{dV}{dx}\right) + \frac{dV}{dx}\dot{x}$

If $c = 1$, then $\dot{E} = 0$

So, given that $\ddot{x} = -\frac{dV}{dx}$, I want to show that the minima of E are centers & saddles of E are saddles.

(1) Linearize about fixed points to show we can only get centers & saddles

$$x_1 = x \qquad \Rightarrow \qquad \dot{x}_1 = x_2 \qquad \qquad \vec{x}^* = \begin{bmatrix} \text{crit. pts. of } V \\ 0 \end{bmatrix}$$

$$x_2 = \dot{x} \qquad \qquad \dot{x}_2 = -\frac{dV(x_1)}{dx_1}$$

$$\overline{\overline{J}} = \begin{bmatrix} 0 & 1 \\ -\frac{d^2V}{dx_1^2} & 0 \end{bmatrix} \Rightarrow \overline{\overline{J}}\Big|_{\vec{x}^*} \Rightarrow \lambda^2 + \frac{d^2V}{dx_1^2}\Big|_{x_{1c}} = 0$$

$$\Rightarrow \lambda = \pm\sqrt{-\frac{d^2V}{dx_1^2}\Big|_{x_{1c}}}$$

So if $\frac{d^2V}{dx_1^2}\Big|_{x_{1c}} > 0$, linearization says $\vec{x}^*$ is a center

if $\frac{d^2V}{dx_1^2}\Big|_{x_{1c}} < 0$, $\vec{x}^*$ is a saddle

$\frac{d^2V}{dx_1^2}\Big|_{x_{1c}} > 0$ at a minimum of $V$

and $\frac{d^2V}{dx_1^2}\Big|_{x_{1c}} < 0$ at a max. of $V$.

at $\vec{x}^*$, $E(x_1, x_2)$ has a crit. pt. $\left(E = \frac{1}{2}x_2^2 + V(x_1)\right)$

$\underbrace{\frac{\partial E}{\partial x_1} = 0}_{\frac{dV}{dx_1}=0}$, $\underbrace{\frac{\partial E}{\partial x_2} = 0}_{x_2=0}$ are clearly satisfied.

$$H = \begin{bmatrix} \frac{\partial^2 E}{\partial x_1^2} & \frac{\partial^2 E}{\partial x_1 \partial x_2} \\ \frac{\partial^2 E}{\partial x_2 \partial x_1} & \frac{\partial^2 E}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} \frac{d^2V}{dx_1^2} & 0 \\ 0 & 1 \end{bmatrix}$$

When $\frac{d^2V}{dx_1^2} < 0$, lin. gives saddle <u>and</u> E has a saddle.

When $\frac{d^2V}{dx_1^2} > 0$, lin. gives center <u>and</u> E has a <u>minimum</u>.

---

In Mechanics, systems that can be written as $\ddot{x} = -\frac{dV}{dx}$ are called conservative Hamiltonian systems. Their fixed pts are either centers or saddles.

<u>2 Warnings!</u>

(1) ~~Energy is "never" really conserved~~ It's often useful to think about energy".

(2) Systems can conserve energy <u>and</u> have asymptotically stable fixed pts.

## 11/2 – Conserved Quantities

Recall: When is a center really a center?

→ If there's a conserved quantity, then fixed pts are saddles are saddles of that conserved quantity and centers are min/max of that conserved quantity.

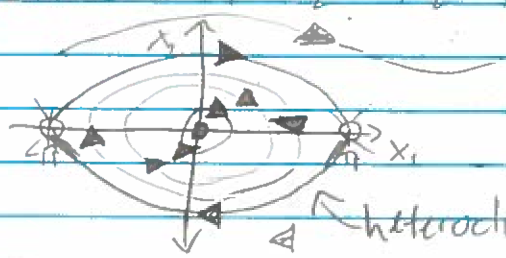
→ If there's a conserved quantity, say E, then E = const. is a trajectory.

→ Even if quantity is not conserved, the conservative system can provide insight.

Ex: pendulum on spring $m l^2 \ddot{\theta} = -mgl\sin\theta$ ] Dim'l

$E = \frac{1}{2} m l^2 \dot{\theta}^2 + mg(l - l\cos\theta)$

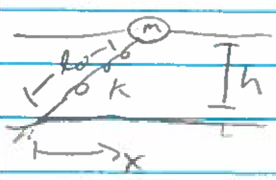
nondim'l $\Rightarrow \dot{x}_1 = x_2,\ \dot{x}_2 = -\sin(x_1) \Rightarrow E = \frac{1}{2}x_2^2 + 1 - \cos x_1$




conservative pendulum

heteroclinic orbit (start at 1 f.p., end at another f.p.).

Ex:




$m\ddot{x} = -k\left(x - \frac{x l_0}{\sqrt{x^2+h^2}}\right) = f(x)$

$\Rightarrow \dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{k}{m}\left(x_1 - \frac{x_1 l_0}{\sqrt{x_1^2+h^2}}\right) = f(x_1)$

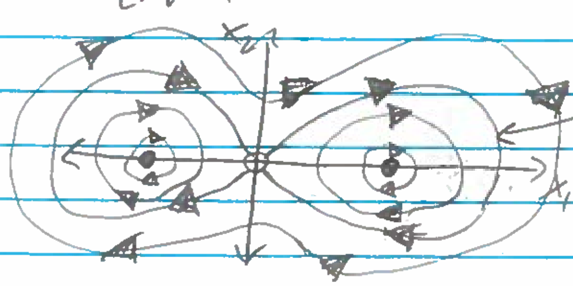
$\Rightarrow V(x_1) = -\int f(x_1)\,dx$

so $E = \frac{1}{2}x_2^2 + V(x_1)$

$\frac{dE}{dt} = x_2\dot{x}_2 + \frac{dV}{dx_1}\dot{x}_1 = x_2 f(x) - f(x_1)x_2 = 0$ E is conserved!

$V(x_1) = \frac{1}{2}\frac{k}{m}x_1^2 - \frac{k}{m}l_0\sqrt{x_1^2+h^2}$

$\Rightarrow E = \frac{1}{2}x_2^2 + V(x_1)$ is conserved.




homoclinic orbit (start at f.p., end at f.p.) same f.p.

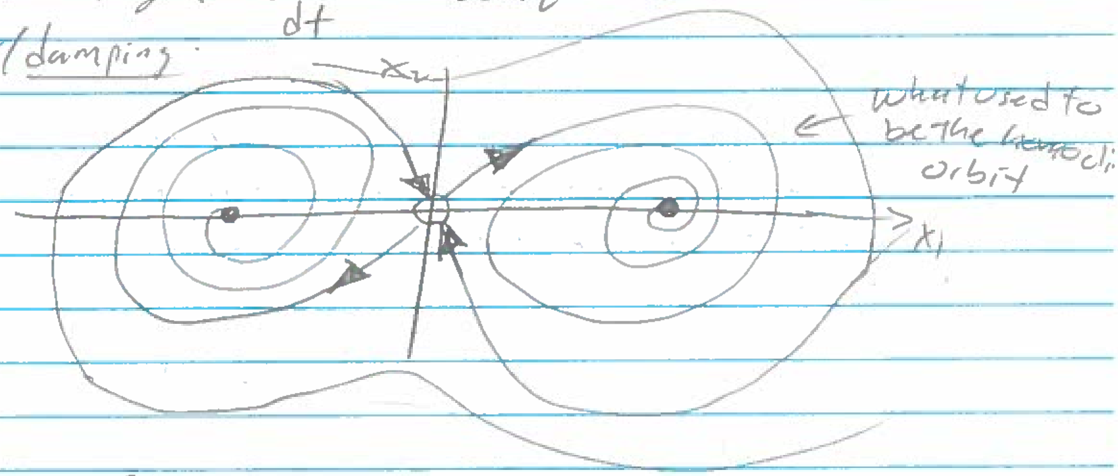
What if we add damping?

$$m\ddot{x} = -k\left(x - \frac{x}{\sqrt{x^2+h^2}}\right) - a\dot{x} \Rightarrow E = \frac{1}{2}x_2^2 + V(x_1)$$

w/o damping $(a=0)$, $\frac{dE}{dt} = 0$

w/ damping, $\frac{dE}{dt} = -a x_2^2 \le 0$

w/ damping:



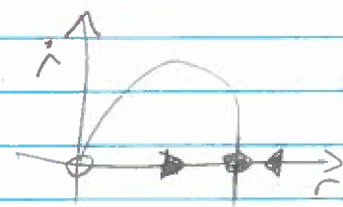

§7: Ex: $\dot{\theta} = 1$

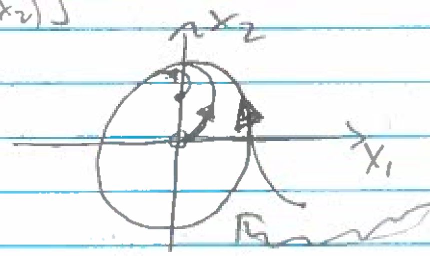
$\dot{r} = r(1-r)$

Supp. we have an ODE

$\dot{x}_1 = f(x_1, x_2)$ } polar coords

$\dot{x}_2 = g(x_1, x_2)$




Notice
trajectories don't head into a fixed point but rather a fixed orbit

Limit cycle

limit cycles $\neq$ centers

↳ isolated closed trajectory — ↳ set of closed trajectories

↳ different IC's give same oscillations — ↳ diff IC's give diff. oscillations

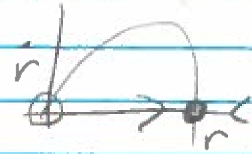
↳ local analysis won't work

11/4

Recall – limit cycles

$\dot{\theta} = 1$

$\dot{r} = r(1-r)$



Isolated closed orbit

Ex: Van der Pol oscillator

$$\ddot{x} = -x - \mu(x^2-1)\dot{x} \qquad \frac{dE}{dt} = -b(\dot{x})^2$$

compare w/ $\ddot{x} = -x - b\dot{x}$ damped spring → if $b>0$, system loses energy

so $b_{eff} = \mu(x^2-1)$ is some sort of damping

near $x=0$, $b_{eff} \sim -\mu$, system gains energy

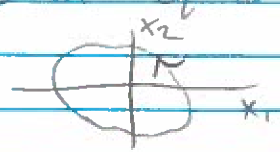
⇒ expect $x^* = 0$ to be an unstable spiral

Conversely, if $x$ is large, $b_{eff} \sim \mu x^2$ ← positive ⇒ system loses energy

⇒ Might expect a limit cycle.

Ruling out limit cycles

Given an equation/system, when can we say a limit cycle doesn't exist?



If a closed orbit exists, then if we have a fn. $E(x_1(t), x_2(t))$, then $E(x_1(t+T), x_2(t+T)) = E(x_1(t), x_2(t))$

$T$ = period of limit cycle

So if we can find a fn. $E(x_1, x_2)$ that is monotonic in $x_1$ & $x_2$, then the above cannot be true & there are no closed orbits.

This is called a Lyapunov fn.

Ex: $\ddot{x} = -x - b\dot{x}$ has Lyapunov fn. $E$, $\frac{dE}{dt} = -b(\dot{x})^2$ ⇒ $E$ monotonic

Q: How do you know if a Lyapunov fn. exists?

A: Usually it's hard to tell. In general, you'd have to solve something really hard – like a conserved quantity but harder.

If $E(x_1, x_2)$ is conserved,

then $\frac{\partial E}{\partial x_1}\dot{x}_1 + \frac{\partial E}{\partial x_2}\dot{x}_2 = 0$ is a hard PDE to solve.

(Solving for Lyapunov is the same eqn but now $\neq 0$.)

However, there are special cases where Lyapunov fns. are obvious

$\dot{\vec{x}} = -\nabla V$ ← if your system looks like this,

then $\frac{dV}{dt} = \nabla V \cdot \dot{\vec{x}} = -(\dot{\vec{x}})^2$

↖ always decreasing except at fixed points

$\Rightarrow V$ is a Lyapunov fn.

<u>Recall</u>: <u>Limit Cycles</u>

<u>When can we rule out a limit cycle?</u>

If $\frac{dE}{dt} < 0$ along closed trajectories, Then a limit cycle cannot exist.

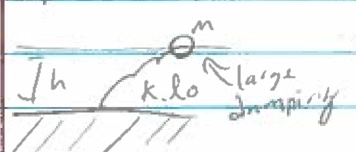
E is a Lyapunov fn.

How do you find E? "Divine inspiration is usually required, but sometimes one can work backwards".

✗ Special class of dynamical systems that have easy to find Lyapunov fns.

Gradient systems, $\dot{\vec{x}} = -\nabla V$, V is the Lyapunov fn.

Intuition is an overdamped mechanical system.




$\dot{x} = f(x)$

<u>Overdamped systems</u>:

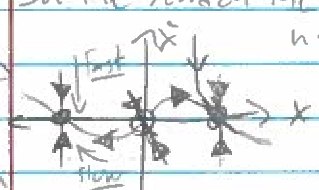
$\ddot{\vec{x}} = -a\dot{\vec{x}} + f(\vec{x}) \underset{\text{approx}}{\rightsquigarrow} \dot{\vec{x}} \sim \frac{1}{a} f(\vec{x})$

need $\dot{\vec{x}}(0)$ and $\vec{x}(0)$ to solve, but in approx. just need $\vec{x}(0)$

(assuming $\ddot{\vec{x}}$ is negligible, slower timescale?)

In the bead on the wire system full 2D phase plane looks like

lin. stab. would tell me ① there is a big imbalance ② fast eigendirection is vertical




$h < l_0$, damping relatively large (stable f.p.'s are nodes)

as damping increases, fast eigendirection becomes vertical & $\lambda \to \infty$, slow eigendirection becomes horizontal & $\lambda \to 0^-$, so system is collapsing onto the 1D phase line.

on super short (fast) timescale, $\ddot{\vec{x}}$ is huge! But on the long time (slow) timescale,

$\dot{\vec{x}} = -\nabla V$

(1) Show that $V$ decreases along closed trajectories.

$\frac{dV}{dt} = \nabla V \cdot \dot{\vec{x}} = -(\dot{x})^2 < 0$ except at fixed points.

Thm: 2D $\underbrace{\dot{x} = f(x,y),\ \dot{y} = g(x,y)}_{\text{smooth}}$ is a gradient system iff $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$

Pf:

$\Rightarrow$: $\dot{\vec{x}} = -\nabla V$ then $-\frac{\partial V}{\partial x} = f(x,y)$. If $\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}$ (which it should be)

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad -\frac{\partial V}{\partial y} = g(x,y)$

then $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

$\Leftarrow$: $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} = -\frac{\partial V}{\partial x \partial y}$. Then $\int \frac{\partial f}{\partial y} dy \overset{\text{some smoothness assumed}}{=} f(x,y) = -\frac{\partial V}{\partial x} + C(x)$

and $\int \frac{\partial g}{\partial x} dx = g(x,y) = -\frac{\partial V}{\partial y} + C(y)$

Let $E = V - \int C(x)dx - \int C(y)dy$

Then $-\nabla E = \begin{bmatrix} -\frac{\partial V}{\partial x} + C(x) \\ -\frac{\partial V}{\partial y} + C'(y) \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \dot{\vec{x}}$ $\square$

Summary: If you want to find a Lyapunov fn, check if it's a gradient system by checking if $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ (given that kind of system).

Ex: $\dot{y} = -x - ay = g(x,y)$ (mass on spring eqns.)

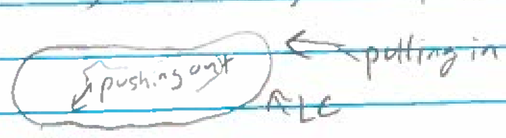
$\dot{x} = y \quad = f(x,y)$

$\frac{\partial f}{\partial y} = 1$, $\frac{\partial g}{\partial x} = -1$ $\Rightarrow$ not a gradient system.

Generally, a good first guess for a Lyapunov fn. is $E \sim x^2 + y^2$

$\frac{dE}{dt} = 2x\dot{x} + 2y\dot{y} = 2xy - 2xy - 2ay^2 = -2ay^2 < 0$

☆ In order to have a limit cycle, you need something pushing you out towards the cycle when your amplitude is too small & vice versa.



11/4

## Recall: Limit Cycles (closed, isolated orbits)

– do not exist if $\frac{dE(\vec{x})}{dt} < 0$ along closed trajectories ($E(\vec{x})$ is a Lyapunov fn.)

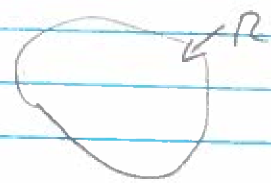
– do not exist if $\exists$ a fn. $g(\vec{x})$ s.t. $\nabla \cdot g\dot{\vec{x}}$ has one sign [Dulac's crit

$\Rightarrow$ $$\iint_A \nabla \cdot g\dot{\vec{x}}\, dA = \oint_{\text{closed orbit}} g\dot{\vec{x}} \cdot \hat{n}\, d\ell = 0 \quad \text{since } \dot{\vec{x}} \perp \hat{n}$$

But $\nabla \cdot g\dot{\vec{x}}$ has one sign, so $\iint_A \nabla \cdot g\dot{\vec{x}}\, dA \neq 0$

This is a contradiction, so there are no closed orbits

## Proving existence of a limit cycle (2D)




if we can construct a "trapping region" $R$ s.t. trajectories stay in $R$ then either $\exists$ an f.p in $R$ or a closed orbi

## Poincaré – Bendixson Thm

Ex: Oscillating Chemical Reaction – chemical clock ← looks like an isolated closed orbit & not a center

Expect a limit cycle.

2 color-system:
$$\dot{x} = a - x - \frac{4xy}{1+x^2} \qquad x, y \geq 0$$
$$\dot{y} = bx\left(1 - y/(1+x^2)\right) \qquad a, b > 0$$

Does a limit cycle exist? If so, for what combos of $a, b$?

Turns out, Poincaré-Bendixson is what we should use.

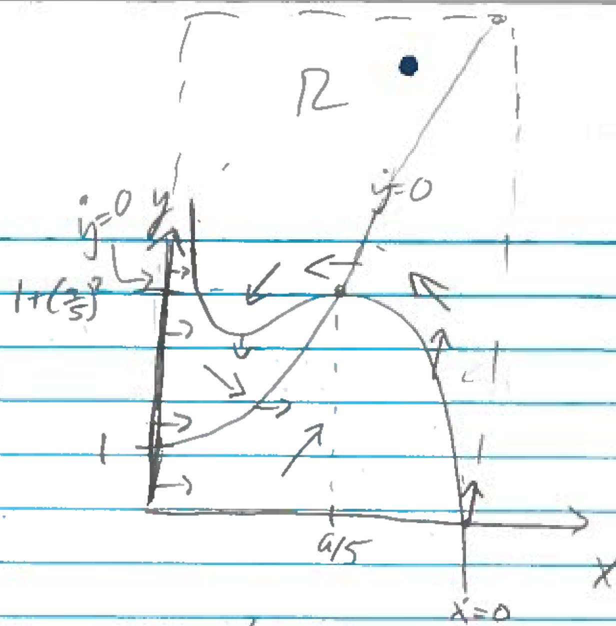
So how do we construct our trapping region?

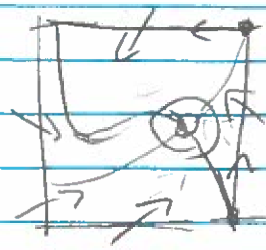
Sketch the nullclines and determine qualitative flow and find $R$ that has flow inwards

$\dot{y} = 0$ along $x = 0$ and $y = 1 + x^2$ nullclines

$\dot{x} = 0$ along $a - x = \frac{4xy}{1+x^2}$
$$\frac{-x^3 + ax^2 - x + a}{4x} = y$$

$y = 1 + x^2 \Rightarrow a - x = \frac{4xy}{y} \Rightarrow x = a/5,\ y = 1 + (a/5)^2$



flow is into this box except at 2 points where the flow is tangent.
trajectories stay in box.
Cut out hole around f.p., check if flow is outwards of hole.

We have a trapping region so long as the f.p. is unstable (node or spiral)


✱ Cannot have a saddle inside a limit cycle ← index theory.

Check if f.p. is unstable node/spiral.

$$J = \begin{pmatrix} -1 - \frac{4y}{1+x^2} + \frac{8x^2y}{(1+x^2)^2} & -\frac{4x}{1+x^2} \\ b - \frac{by}{1+x^2} + \frac{2bx^2y}{(1+x^2)^2} & -\frac{bx}{1+x^2} \end{pmatrix}, \quad x^* = a/5, \quad y^* = 1+(x^*)^2$$

$$J|_{x^*} = \begin{pmatrix} -5 + \frac{8(x^*)^2}{1+(x^*)^2} & -\frac{4x^*}{1+x^{*2}} \\ \frac{2b(x^*)^2}{1+x^{*2}} & \frac{-bx^*}{1+x^{*2}} \end{pmatrix} \qquad \text{let } \alpha = \frac{x^*}{1+x^{*2}} \geq 0$$

$$J|_{x^*} = \begin{pmatrix} -5+8\alpha x^* & -4\alpha \\ 2b\alpha x^* & -b\alpha \end{pmatrix} \Rightarrow (5-8\alpha x^*+\lambda)(b\alpha+\lambda) + 8bx^*\alpha^2 = 0$$

$$\lambda^2 + \underbrace{(b\alpha+5-8\alpha)}_{\tau}\lambda + b\alpha(5-8\alpha) + 8bx^*\alpha^2 = 0$$

if this is positive, we get stable f.p (node/spiral)
if negative, get unstable node/spiral

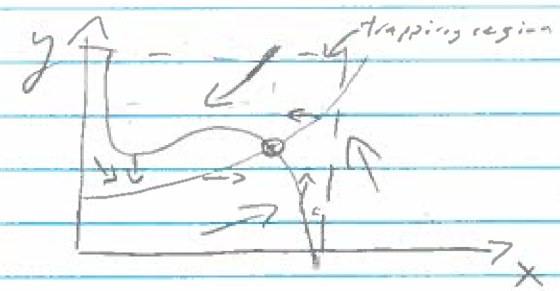
Recall: Poincaré–Bendixson (2D)

A "trapping region" that contains trajectories for all time, but contains no fixed points, must contain a closed orbit.

Ex: Oscillating chain rxn

$$\dot{x} = a - x - \frac{4xy}{1+x^2}$$

$$\dot{y} = bx\left(1 - \frac{y}{1+x^2}\right)$$




Find a, b s.t. the fixed point is an unstable node/spiral => limit cycle exists.

In this example, as we vary one parameter, the limit cycle approaches
↳ Hopf bifurcation

## Bifurcations, revisited, in $2^+$D (Ch 8)

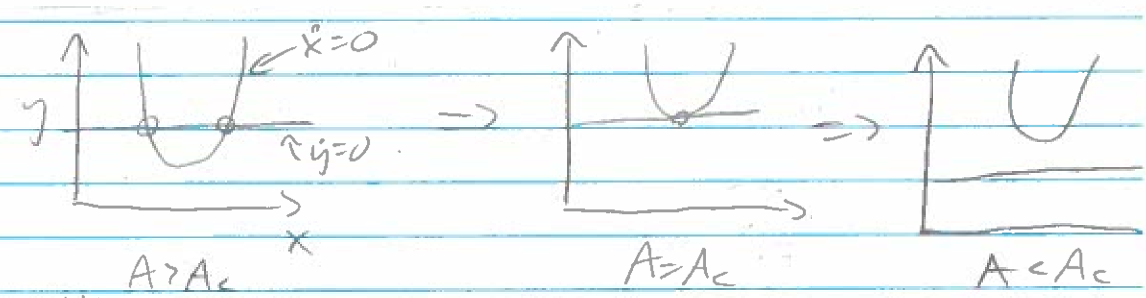
Recall from 1D

① saddle node

② pitchfork < subcrit / supercrit

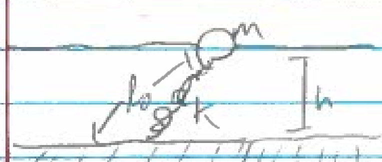
transcritical

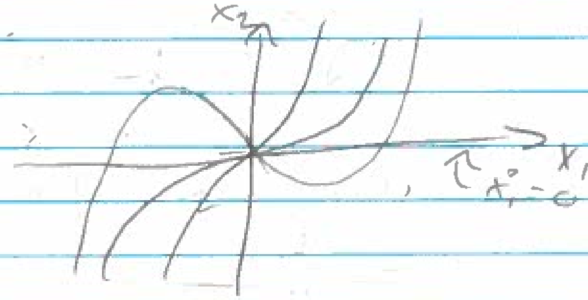
①




saddle node at $A_c$.

②




$$\ddot{x} = -\dot{x} + f(x, h)$$

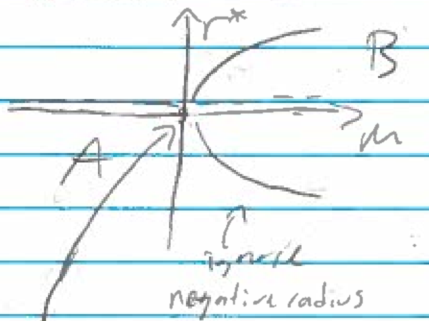
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 + f(x, h)$$

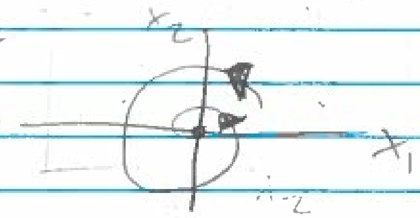
All 1-D bifurcations happen in $2^+$D, & basically look the

All 1D bifurcations happen in 2+D & basically look the same because all the "action" happens along one dimension.
↳ local picture the same w/ correct change of coordinates.

Ex: $\dot{r} = \mu r - r^3$    $w, b > 0$ fixed    ☆ Hopf is like a
$\dot{\theta} = \omega + b r^2$    vary $\mu$.    spinning pitchfork




A:

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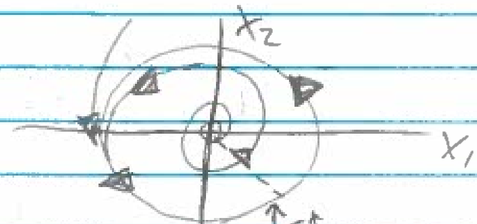
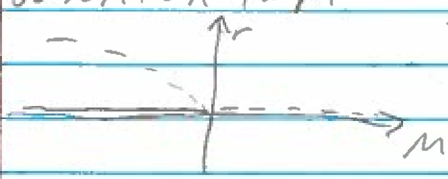
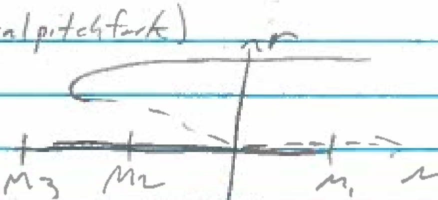



supercritical pitchfork bif
⇒ supercrit. Hopf bif.
Spiral goes from stable to unstable
⇒ Poincaré–Bendixson yields limit cycle now.

Subcritical Hopf. (spinning a subcritical pitchfork)

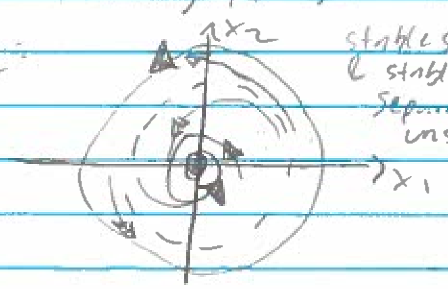



typically rare, more common to have hysteresis



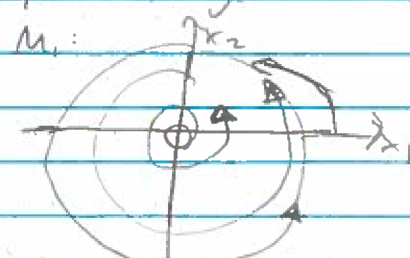

To avoid blowing up, higher order terms would impact stability

$\mu_2$:



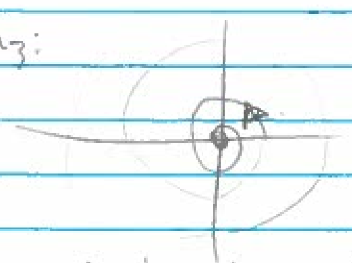

stable spiral & stable limit cycle separated by unstable limit cycle

$\mu_1$:




unstable spiral going into stable limit cycle.

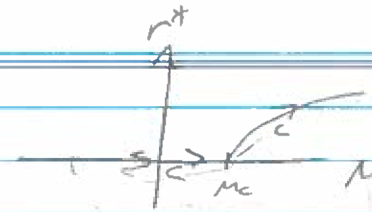
$\mu_3$:



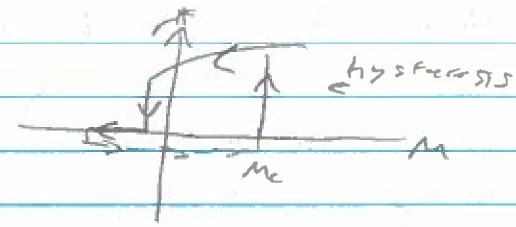
no limit cycles, just stable spiral

Q: Supp. we vary $\mu$ & observe a limit cycle behavior above a critical value, $\mu_c$. ⇒ Expect Hopf.
Supercritical: amplitude $\sim (\mu - \mu_c)^{1/2}$
No hysteresis

Supercritical Hopf bif:
amplitude $\sim (\mu - \mu_c)^{1/2}$
no hysteresis




Subcritical Hopf
· Amplitude ~ constant
doesn't go to 0 cont'sly
· hysteresis




11/18

Lyapunov Fns

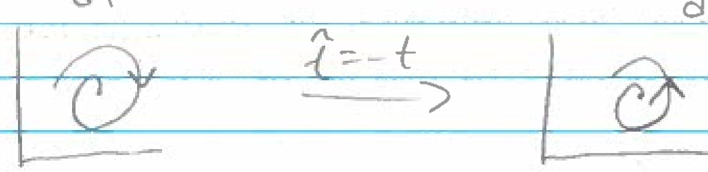
Book: $V(\vec{x})$ s.t. $\frac{dV}{dt} < 0$ for $\vec{x} \neq \vec{x}^*$ & $V(x)$ has a global min. at
then $\vec{x}^*$ is globally asymptotically stable

We can relax this def. by only considering some region $R$
Also can relax $\frac{dV}{dt} \leq 0$ a little bit, as long as not on orbits/f.p.s.

Suppose we can find $V(\vec{x})$ s.t. $\frac{dV}{dt} \leq 0$

Poincaré-Bendixson & Time Reversal

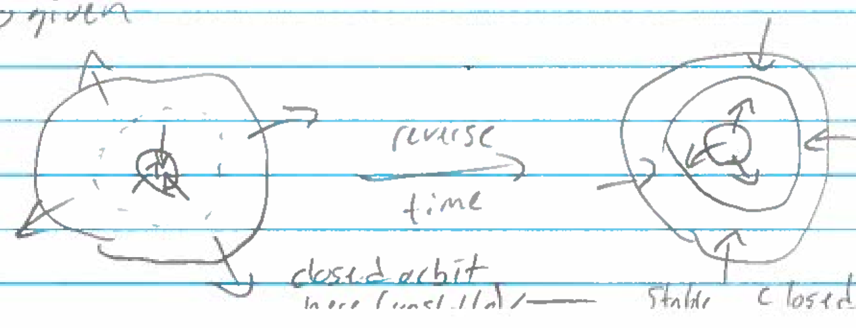
Supp. $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x})$. Let $\hat{t} = -t \Rightarrow \frac{dx}{d\hat{t}} = -f(\vec{x})$




stable f.p./l.c. → unstable
unstable f.p./l.c. → stable

saddles → stay saddles
but manifolds swap stabilities

so given



## Recall: Bifurcations in $2^+$D (ch. 8)

- 1-D bifurcations (saddle node, transcrit, pitchfork) also occur in $2^+$D along 1-D curves
- New stuff: Hopf (need $2^+$D b/c involves limit cycle)

Hopf is like a radial pitchfork in polar coordinates

↳ Supercritical & subcritical Hopf bifurcations exist

↳ Amplitude $\sim\sqrt{\mu}$, no hysteresis ↳ cons. amplitude, hysteresis, bistability [typically in physical systems]

In both, a spiral changes stability

Local bifurcations – a fixed pt. is changing stability or appearing/disappearing

ex: Hopf, saddle, tc, pf

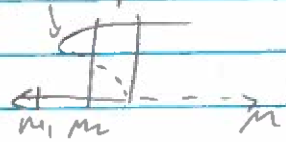
Global bifurcations

Ex: subcrit hopf (not global)

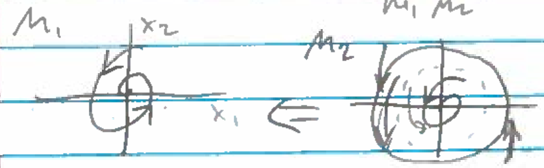
$\dot{r}$ = pitchfork $\Rightarrow$

$\dot{\theta}$ = const.

saddle node pt ← here's some global bif.









$\mu_2 \to \mu_1$ limit cycles collide & annihilate

## Saddle Node in Cycles Bifurcation (SNIC)

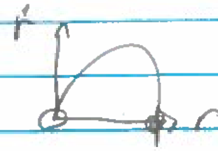
→ 2 limit cycles collide & annihilate

This is not a local bifurcation, no one point at which this bif. occurs.

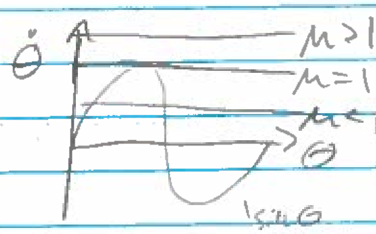
$\Rightarrow$ we cannot use linearization about a point.

## Infinite Period Bifurcation

$$\dot{r} = r(1-r)$$
$$\dot{\theta} = \mu - \sin\theta$$

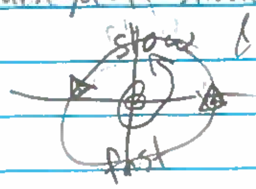








when $\mu>1$, stable limit cycle & unstable spiral



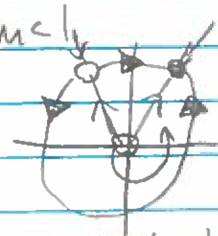


when $\mu=1$

we go infinitely slow in the slow region

$\Rightarrow$ infinite period

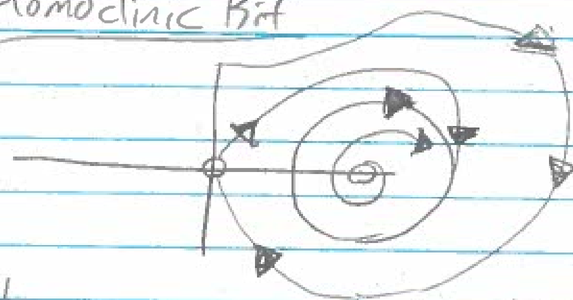
$\mu<1$



← stable node

heteroclinic ... unstable node

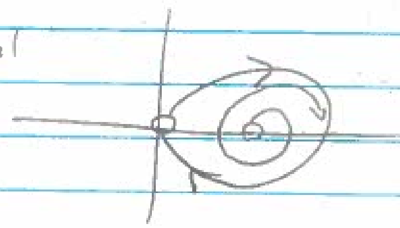
## Homoclinic Bif



saddle
stable limit cycle
unstable spiral

increase $\mu$, limit cycle grows until

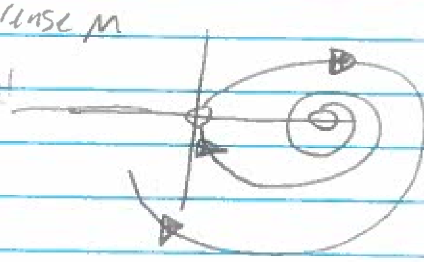
US spiral
saddle



limit cycle collides w/ saddle, forms homoclinic orbit

increase $\mu$

US spiral
saddle



this has no separatrix
no orbits

## 11/20 – Discrete Dynamical Systems ((Ch 10)

$\vec{x}_{n+1} = \vec{f}(\vec{x}_n)$ ← discrete dynamical systems

→ Smoothly dynamical systems are a special case of this

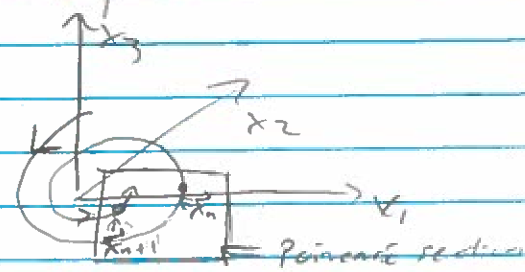
$$\frac{d\vec{x}}{dt} = g(\vec{x}) \Rightarrow \lim_{\Delta t \to 0} \frac{\vec{x}(t+\Delta t) - \vec{x}(t)}{\Delta t} = g(\vec{x})$$

$$\Rightarrow \lim_{\Delta t \to 0} \vec{x}(t+\Delta t) = \lim_{\Delta t \to 0} g(\vec{x})\Delta t + \vec{x}(t)$$

Smooth: $\vec{x}_{n+1} = \vec{x}(t+\Delta t)$, $\vec{x}_n = \vec{x}(t)$, $f(\vec{x}) = g(\vec{x})\Delta t + \vec{x}$

→ Arises from discrete processes

→ Poincaré Sections
((Ch 8.7)




Suppose we have $\vec{x}_{n+1} = \vec{f}(\vec{x}_n)$ Let's look at fixed pts & stability

If at a fixed point, $\vec{x}_{n+1} = \vec{x}_n = \vec{x}^*$

$\Rightarrow \vec{x}^* = \vec{f}(\vec{x}^*)$. This eqn. allows us to find fixed pts.

Stability: Linearization: $\vec{x}_{n+1} = \vec{f}(\vec{x}^*) + \overrightarrow{J}_{|\vec{x}^*} \cdot (\vec{x}_n - \vec{x}^*)$

$$\vec{x}_{n+1} - \vec{x}^* = J_{|\vec{x}^*} (\vec{x}_n - \vec{x}^*)$$

$$\vec{\varepsilon}_{n+1} = J_{|x^*} \vec{\varepsilon}_n \quad \text{w/ } \vec{\varepsilon}_k = \vec{x}_k - \vec{x}^*$$

Stable if all $|\lambda_i| < 1$

explanation:

$\vec{x}_0$ ← starting point

$$\vec{x}_n = \left(J_{|\vec{x}^*}\right)^n \cdot \vec{x}_0$$

$$= (S \Lambda S^{-1})^n \cdot \vec{x}_0$$

$$= (S \Lambda^n S^{-1}) \cdot \vec{x}_0$$

$$= S \begin{bmatrix} \lambda_1^n & & \\ & \lambda_2^n & \\ & & \ddots \ \lambda^n \end{bmatrix} S^{-1} \vec{x}_0$$

population model $x \geq 0, r \geq 0$

Ex: $x_{n+1} = r x_n (1 - x_n)$

① fixed pts $\rightarrow$ $x^* = r x^* (1 - x^*) \Rightarrow x^* = 0$

$1 = r(1 - x^*) \Rightarrow x^* = \frac{r-1}{r}$

② stability: $f(x) = rx(1-x)$

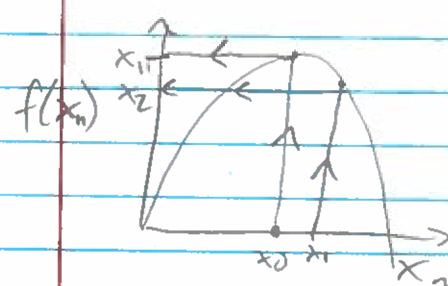
$f'(x) = r(1-x) - rx = r(1-2x)$

$f'(x^*=0) = r$, stable if $r < 1$ (since $r \geq 0$, don't need to write $|r|$)

$f'(\frac{r-1}{r}) = r(1 - \frac{2r-2}{r}) = -r+2$ stable if $1 < r < 3$

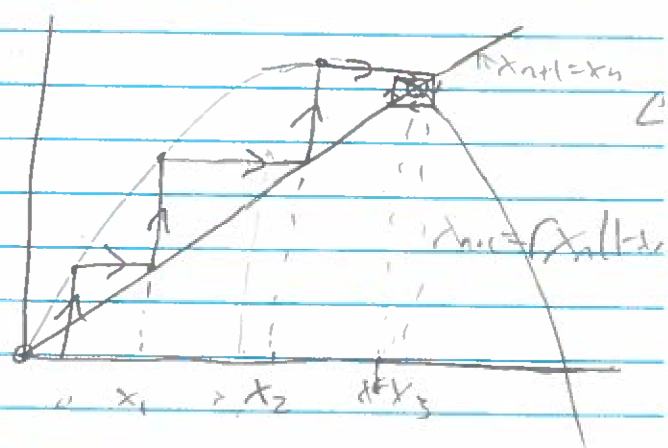
| $r$ | $x^* = 0$ | $x^* = \frac{r-1}{r}$ |
|---|---|---|
| $(0,1)$ | stable | unstable |
| $(1,3)$ | unstable | stable |
| $(3,\infty)$ | unstable | unstable |

$\rightarrow$ where do trajectories




long way

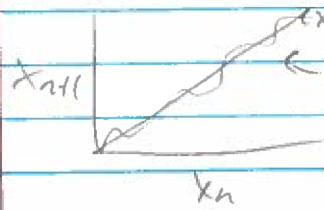
recall, to find f.p., we had to solve $f(x) = x$.



11/23

cont's dynamical system $\longrightarrow$ discrete

$\frac{dx}{dt} = g(x) \longrightarrow x_{n+1} = x_n + \Delta t\, g(x_n)$




← cont's systems constrained to near this line since $x_{n+1} = x_n + \underbrace{\Delta t g(x_n)}_{\text{something small}}$

Ex: $x_{n+1} = r x_n (1-x_n) = f(x_n)$

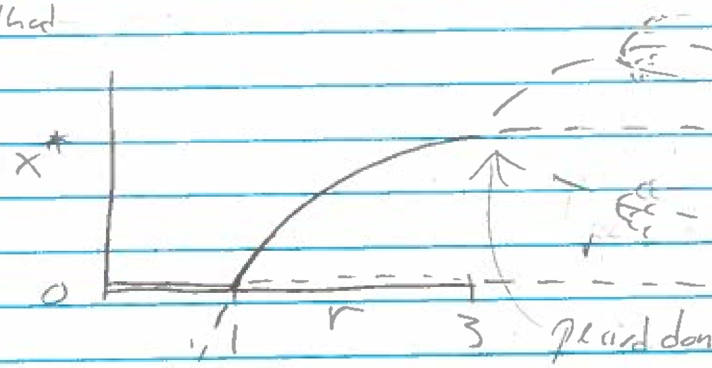
f.p.: $x^* = r x^*(1-x^*)$ stability - $\left|\frac{df}{dx_n}\right| < 1$

① $x^* = 0$, stable when $r < 1$

② $x^* = \frac{r-1}{r}$ stable when $1 < r < 3$

graphical method

Bif. diagram




$x_{n+1} = x_n$ ← period 1 motion

$x_{n+2} = x_n$ ← period 2

Since we found 2 before ∴ these are still solns, so at least 4 solns to this

$x_1^* = r x_1^*(1-x_1^*)$ (period 2)

$x_2^* = r(r x_2^*(1-x_2^*))(1 - r x_2^*(1-x_2^*))$ gives 4th order poly

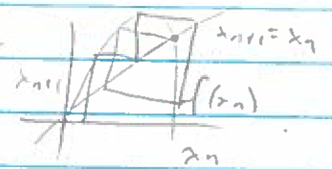
11/30

**Recall:** $x_{n+1} = f(x_n)$

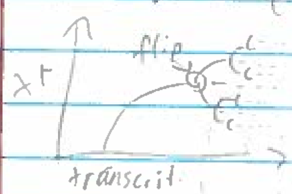
① fixed pts $x^* = f(x^*)$

② stability $|\text{eig}(\overrightarrow{\overrightarrow{J}}|_{x^*})| < 1$

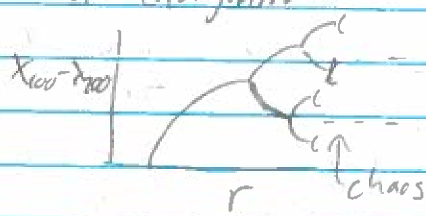
③ cobwebbing




④ bif. diagram (for $x_{n+1} = r x_n(1-x_n)$)




⑤ orbit diagram




Big picture: Maps – 1D Maps – Chaos

Aside: What is chaos? Can it occur in cont's systems?

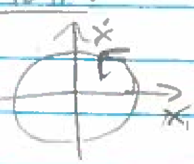
Yes it can occur in cont's systems!

**Ex:** Duffing's oscillator $\ddot{x} + \alpha\dot{x} + \beta x + \gamma x^3 = a\cos\omega t$

3D: $x_1 = x$, $x_2 = \dot{x}$, $x_3 = t$

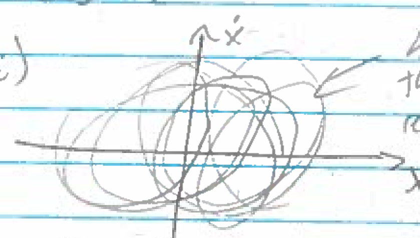
Solution?

Lissajous figure → (i)




(a small)

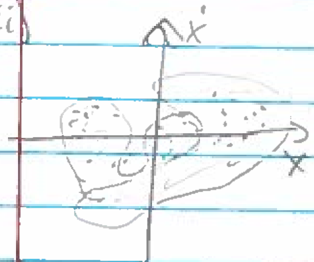
(ii)




how do we know the curve does not repeat itself.

a not small, right combo of parameters

(iii)




Big ideas

① no repetition

② There are regions with more dots
↳ still some underlying structure

③ sensitivity to initial conditions

Same as (ii) but t = ? ...

Ex: Walking — 1) system exhibits chaos, but not the point
2) another example of period-doubling to chaos
↳ logistic map is generic (sort of a normal form for chaos)

Context for the Walking model

Q: How much control is in locomotion?

Ex: Asimo vs. Boston dynamics

lots of control, looks unnatural → realistic-looking walking robots → energy cost similar to us.

~ one order of magnitude greater energy expenditure than us ↳ based on passive dynamics

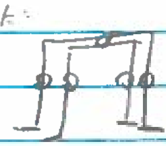
→ Stable underlying mechanical system

Idea: Maybe there's something we can analyze

Hopeful example: Tad McGeer ~ 1990


side

back:

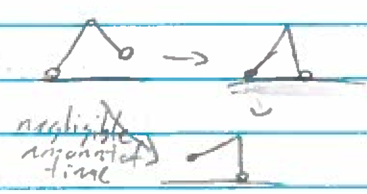


Q: 1) Is it really stable?
2) Is it stable like a bicycle or like a chair?
"dynamic stability" — need to move to be stable.

How do we address these questions? Build a simple model


$$\ddot{\theta}_1 = f_1(\dot{\theta}_1, \dot{\theta}_2, \theta_1, \theta_2)$$
$$\ddot{\theta}_2 = f_2(\dot{\theta}_1, \dot{\theta}_2, \theta_1, \theta_2)$$

Collisions → extremely large forces for very short times




↳ Impulse momentum → $\vec{F} \sim \delta(t)$

leads to
⇒ discontinuities in $\dot{\theta}_1, \dot{\theta}_2$

A fortuitous simplification: just after collision, $\theta_1 = -\theta_2 = \phi$, $\dot{\theta}_1 = -\dot{\theta}_2 = \dot{\phi}$ ] 2 variables just after collision

$$\vec{x}_i = \begin{bmatrix} \phi_i \\ \dot{\phi}_i \end{bmatrix} \xrightarrow{\text{simulate}} \vec{x}_{i+1} \Rightarrow \vec{x}_{i+1} = \vec{f}(\vec{x}_i)$$

$\vec{f}(\vec{x}_i)$

1) fixed pts. $\vec{x}^* = \vec{f}(\vec{x}^*)$

2) stability $\vec{f}\left(\vec{x}^* + \begin{bmatrix} 0 \\ \Delta \end{bmatrix}\right)$

$\vec{f}\left(\vec{x}^* + \begin{bmatrix} \Delta \\ 0 \end{bmatrix}\right)$

12/2

$\vec{x}_i = \begin{bmatrix} \theta \\ \dot\theta \end{bmatrix}$

$\vec{x}_{i+1} = \vec{f}(\vec{x}_i)$

(1) Simulate a double pendulum
(2) Simulate a collision

2-D map (1) are there fixed pts? (2) stability?
(1) root find $\vec{f}(\vec{x}^*) - \vec{x}^* = \vec{0}$
(2) linear stability - need Jacobian
Idea: numerically approx. Jacobian
$\vec{f}(\vec{x}^* + \varepsilon\hat{e}_1) \quad \hat{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} \longrightarrow \frac{df_1}{d\theta}, \frac{df_2}{d\theta}$
$\vec{f}(\vec{x}^* + \varepsilon\hat{e}_2), \quad \hat{e}_2 = \begin{bmatrix}0\\1\end{bmatrix} \longrightarrow \frac{df_1}{d\dot\theta}, \frac{df_2}{d\dot\theta}$

$$\vec{J}|_{x^*} \approx \begin{bmatrix} \vec{f}(\vec{x}^* + \varepsilon\hat{e}_1) - \vec{x}^* \\ \vec{f}(\vec{x}^* + \varepsilon\hat{e}_2) - \vec{x}^* \end{bmatrix} \cdot \frac{1}{\varepsilon}$$

2D map - period doubling to chaos is a characteristic of 1-D maps!
In the walker, even though we have a 2D map, we're really restricted to a 1-D manifold
☆ need a special rlnship btwn $\theta$ & $\dot\theta$ to get gaits
"cause walking is hard"

Who cares? What's interesting?
(1) There is a simple model for walking
(2) The simple model has fixed points (gaits)
(3) The fixed points (gaits) are stable!

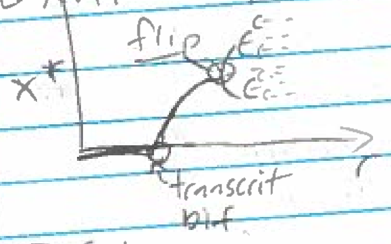
(1) Cockroaches w/ exploding backpacks
(2) Shift in robot design
(3) Energetics — prosthetic design

Final: Dec 9 8-10am here

**Maps**: Know how to:
- find fixed pts & determine stability

1-D maps → cobwebbing, bifurcation plots/orbit diagrams

Logistic map $x_{n+1} = r x_n (1 - x_n)$




Bif diagram


orbit diagram

Walker-Talk: Recall: conservative systems
↳ centers & saddles, no asymptotic stability

But ∃ systems that conserve energy & are asymp. stable (e.g. bicycle)
↳ constraints are non-integrable, so not a conservative system in a math sense

Given non linear ODE, what do? how many parameters
→ how many dimensions? non-dimensionalize!
→ fixed pts.? stability? approx. near equilibria
→ when/where are bifurcations? in 1 or 2D, sketch solns & phase portrait
→ If find center, is it really center? Is linearization wrong?
↳ Center → conserved quantity
↳ No center, Is there a Lyapunov fn? (no closed orbits)
↳ Is there a limit cycle? (In 2D try Poincaré Bendixson)
↳ 3+D, limit cycle → take Poincaré sections & make a map.

How would you make a model?
- Mathematical intuition → fixed pts of system, stability of f.p.
→ bifurcations, limit cycles

# Applied Math w/ Hunter

## Fréchet vs. Gateaux Derivatives

**Def:** A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if $\exists$ a linear map $A: \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$f(x+h) = f(x) + Ah + o(h) \quad \text{as } h \to 0$$

where "little oh" $o(h)$ stands for an error term $r(h)$ s.t. $r: \mathbb{R}^n \to \mathbb{R}^m$ and $\lim_{h\to 0} \frac{\|r(h)\|}{\|h\|} = 0$

"big oh" $O(h)$ stands for $r(h)$ s.t. $\exists\, c, \delta > 0$ s.t. $\|r(h)\| \le c$ for $\|h\| < \delta$.

When $f$ is diff. at $x$, write $A = f'(x)$ for $df(x), Df(x), \ldots$
$f': \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. $f'$ is the Fréchet derivative of $f$.

**Prop:** If $f: \mathbb{R}^n \to \mathbb{R}$ is Fréchet differentiable at $\vec{x} \in \mathbb{R}^n$, then $f(\vec{x}+\vec{h}) \to f(\vec{x})$ as $\vec{h} \to 0$ so $f$ is cont's at $\vec{x}$.

**Def:** If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{h} \in \mathbb{R}^n$, then the directional derivative of $f$ at $\vec{x} \in \mathbb{R}^n$ in the direction $\vec{h}$ is

$$df(\vec{x}; \vec{h}) = \frac{d}{d\varepsilon} f(\vec{x} + \varepsilon\vec{h})\Big|_{\varepsilon=0}$$

**Ex:** If $h = e_i$ is the unit vector in the $i^{th}$ direction, $df(\vec{x}; e_i) = \frac{\partial f}{\partial x_i}(\vec{x})$

**Rmk:** If $f$ is Fréchet differentiable, then $f(x+\varepsilon h) = f(x) + \varepsilon Ah + o(\varepsilon)$

$$\Rightarrow \frac{d}{d\varepsilon} f(x+\varepsilon h)\Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{f(x+\varepsilon h) - f(x)}{\varepsilon} = Ah$$

**Prop:** Fréchet diff $\Rightarrow$ all directional derivs. exist & are linear in direction $h$, called Gateaux differentiable.

Gateaux diff $\not\Rightarrow$ Fréchet diff.