Applied Math w/ Hunter

Fréchet vs. Gateaux Derivatives

Def: A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if $\exists$ a linear map $A: \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$f(x+h) = f(x) + Ah + o(h) \quad \text{as } h \to 0$$

where "little oh" $o(h)$ stands for an error term $r(h)$ s.t. $r: \mathbb{R}^n \to \mathbb{R}^m$ and $\lim_{h \to 0} \frac{\|r(h)\|}{\|h\|} = 0$

"big oh" $O(h)$ stands for $r(h)$ s.t. $\exists\, c, \delta > 0$ s.t. $\|r(h)\| \le c$ for $\|h\| < \delta$.

When f is diff. at x, write $A = f'(x)$ for $df(x)$, $Df(x)$, $f': \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. f' is the Fréchet derivative of f

Prop: If $f: \mathbb{R}^n \to \mathbb{R}$ is Fréchet differentiable at $\vec{x} \in \mathbb{R}^n$, then $f(\vec{x} + \vec{h}) \to f(\vec{x})$ as $\vec{h} \to 0$ so f is cont's at $\vec{x}$.

Def: If $f: \mathbb{R}^n \to \mathbb{R}^m$ and $\vec{h} \in \mathbb{R}^n$, then the directional derivative of f at $\vec{x} \in \mathbb{R}^n$ in the direction $\vec{h}$ is

$$df(\vec{x}; \vec{h}) = \frac{d}{d\varepsilon} f(\vec{x} + \varepsilon \vec{h}) \Big|_{\varepsilon = 0}$$

Ex: If $h = e_i$ is the unit vector in the i^{th} direction, $df(\vec{x}; e_i) = \frac{\partial f}{\partial x_i}(\vec{x})$

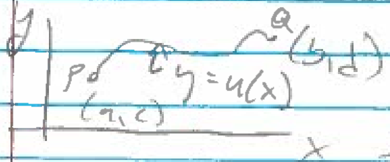
Rmk: If f is Fréchet differentiable, then $f(x + \varepsilon h) = f(x) + \varepsilon A h + o(\varepsilon)$

$$\Rightarrow \frac{d}{d\varepsilon} f(x + \varepsilon h)\Big|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon h) - f(x)}{\varepsilon} = Ah$$

Prop: Fréchet diff $\Rightarrow$ all directional derivs exist & are linear in direction h, called Gateaux differentiable

Gateaux diff $\not\Rightarrow$ Fréchet diff.

Calculus of Variations



Want to find the shortest curve btwn two points

$$\underbrace{J(u)}_{\text{functional of } u} = \int_a^b \underbrace{\sqrt{1+u'(x)^2}}_{\text{arclength function of curve } u}\, dx$$

Want to find a function/curve u to minimize $J(u)$.

solution: straight line

Fermat's principle: Particle moves in x,y plane w/ velocity $c(x,y)>0$

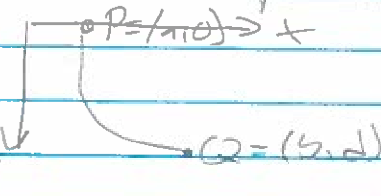
Maximize time to go from P to Q.

$$J(u) = \int_a^b \frac{\sqrt{1+u'(x)^2}}{c(x,u(x))}\,dx, \quad u(a)=c, \quad u(b)=d.$$

Brachistochrone: Curve of fastest descent btwn two points under gravity

$\frac{1}{2}mv^2 = mgy$ (no friction)

$\Rightarrow v = \sqrt{2gy}$



$$J(u) = \int_a^b \sqrt{\frac{1+(u'(x))^2}{2g\,u(x)}}\,dx, \quad u(a)=0, \quad u(b)=d.$$

Two Basic approaches:

1) Direct approach - look for solutions as limit of minimizing sequences

2) Indirect approach - look for critical points of functional

↳ leads to an Euler-Lagrange eqn & reduces to solving an ODE.

In a finite dimensional problem, $f:\mathbb{R}^n \to \mathbb{R}$, we find $\hat{x}\in\mathbb{R}^n$ s.t. $f(\hat{x}) = \inf_{\vec{x}\in\mathbb{R}^n} f(\vec{x})$. $\hat{x}$ is a local minimum if $\exists\ \delta>0$ s.t.

$$f(\hat{x}) \le f(\vec{x}) \quad \forall \vec{x}\in\mathbb{R}^n \text{ s.t. } \|\vec{x}-\hat{x}\|<\delta.$$

If $f(\vec{x})$ has a local min. at $\hat{x}$ and f is diff., then $\nabla f(\hat{x})=0$.

This is a necessary, not sufficient condition for a local min.

Similarly, we will look for u s.t. $\frac{d}{d\varepsilon} J(u;\varepsilon)\big|_{\varepsilon=0} = 0$

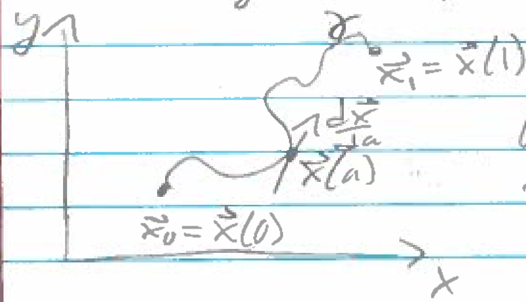
for where J has a zero Fréchet derivative at u.

Applied Math w/ Biello
1/8/16

Calculus of Variations

Point: Optimization problems give rise to DE
- Boundary value problems (ODEs or PDEs)

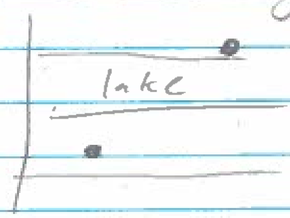
$\mathbb{R}^2$:



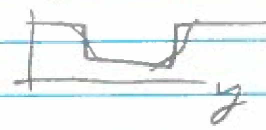
w/ $\vec{x} = x\hat{\imath} + y\hat{\jmath}$

What is the curve γ that goes from $\vec{x}_0$ to $\vec{x}_1$ in the (least amount of time) given some constraints Σ?

Imagine running then swimming in a lake



$c(\vec{x})$ speed depends on location $\vec{x}$



smooth w/ hyperbolic tangents.

let "a" be the parameter along a curve

$\vec{x} = x(a)\hat{\imath} + y(a)\hat{\jmath}$

$\frac{d\vec{x}}{da} = x_a\hat{\imath} + y_a\hat{\jmath}$ $\quad |\vec{x}_a| = (x_a^2 + y_a^2)^{1/2}$

the unit tangent is $\dfrac{\frac{d\vec{x}}{da}}{\left|\frac{d\vec{x}}{da}\right|} = \hat{t} = \dfrac{\vec{x}_a}{|\vec{x}_a|}$

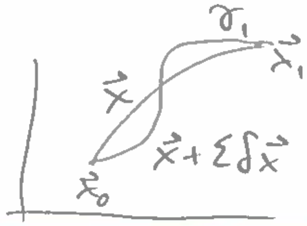
arc length $ds = |\vec{x}_a| da$ (units length)

time $dt = \dfrac{|\vec{x}_a| da}{c(\vec{x})}$

$$T[\vec{x}] = \int_0^1 \frac{|\vec{x}_a|}{c(\vec{x})} da = \int_0^1 \frac{(x_a^2 + y_a^2)^{1/2}}{c(\vec{x})} da$$

$$\Rightarrow T[\vec{x}] = \int_0^1 f(\vec{x}, \vec{x}_a) da$$

call this $T_0 = T[\vec{x}]$



recall $(1+\text{small})^{\alpha} \simeq 1+\alpha\,\text{small}$

$$T_\varepsilon = T[\vec{x}+\varepsilon\delta\vec{x}] = \int_0^1 f(\vec{x}+\varepsilon\delta\vec{x},\ \vec{x}_a+\varepsilon\delta\vec{x}_a)\,da$$

Then $T_\varepsilon - T_0 \simeq \varepsilon\,\delta T[\vec{x}, \delta\vec{x}; \varepsilon]$

So $\delta T[x, \delta x; 0]$ = Directional derivative of T in the direction $\delta\vec{x}$.
$= \mathcal{L}[\delta\vec{x}]$.

Ex: Let $c(\vec{x}) = c_0$

$$T_\varepsilon - T_0 = \frac{1}{c_0}\int_0^1 |\vec{x}_a+\varepsilon\delta\vec{x}_a| - |\vec{x}_a|\,da$$
$$= \frac{1}{c_0}\int_0^1 \left(|\vec{x}_a|^2 + 2(\vec{x}_a\cdot\delta\vec{x}_a)\varepsilon + |\delta\vec{x}|^2\varepsilon^2\right)^{1/2} - |\vec{x}_a|^2\,da$$

Note: $(u^2+v^2\varepsilon)^{1/2} = u\left(1+\frac{v^2\varepsilon}{u^2}\right)^{1/2} \simeq u + \frac{v^2\varepsilon}{2u} + \mathcal{O}(\varepsilon^2)$

$$\Rightarrow T_\varepsilon - T_0 = \frac{1}{c_0}\int_0^1 \varepsilon\left[\frac{\vec{x}_a\cdot\delta\vec{x}_a}{|\vec{x}_a|} + \frac{|\delta x|^2}{2}\varepsilon\right] + \mathcal{O}(\varepsilon^2)\,da$$
$$= \frac{\varepsilon}{c_0}\int_0^1 \frac{\vec{x}_a\cdot\delta\vec{x}_a}{|\vec{x}_a|}\,da + \mathcal{O}(\varepsilon^2)$$

So $\mathcal{L}[\delta\vec{x}] = \frac{1}{c_0}\int_0^1 \frac{\vec{x}_a}{|\vec{x}_a|}\cdot\delta\vec{x}_a\,da$

To extremize the time taken, one must require $\mathcal{L}[\delta\vec{x}] = 0 \ \forall\delta\vec{x}$
and with $\delta\vec{x}(0) = \delta\vec{x}(1) = 0$.

Then $0 = \frac{\vec{x}_a\cdot\delta\vec{x}}{|\vec{x}_a|}\Big|_0^1 - \int_0^1\left(\frac{\vec{x}_a}{|\vec{x}_a|}\right)_a\cdot\delta\vec{x}\,da$

By the b.C.'s, $\delta\vec{x}(0) = \delta\vec{x}(1) = 0$ (Fund. Thm. of Variational Calculus)

So $0 = \int_0^1\left(\frac{\vec{x}_a}{|\vec{x}_a|}\right)_a\cdot\delta\vec{x}\,da \quad \forall\,\delta\vec{x}\in C^2(\mathbb{R})$

by FTVC: $\left(\frac{\vec{x}_a}{|\vec{x}_a|}\right)_a = 0 \Rightarrow \hat{t}_a = 0$

unit tangent should always point in same direction (ie. straight line)

Principle of least time for arbitrary speed fns.

$$T_0 = T[\vec{x}] = \int_0^1 \frac{|\vec{x}_a|}{c(\vec{x})}\, da$$

$$f(\vec{x}, \vec{x}_a) = \frac{|\vec{x}_a|}{c(\vec{x})}$$

$$T_\varepsilon = T[\vec{x} + \varepsilon\delta\vec{x}] = \int_0^1 f(\vec{x} + \varepsilon\delta\vec{x}, \vec{x}_a + \varepsilon\delta\vec{x}_a)\, da$$

$$T_\varepsilon - T_0 = \varepsilon \int_0^1 \frac{\partial f}{\partial \varepsilon}\Big|_{\varepsilon=0} da + O(\varepsilon^2)$$

$$= \varepsilon \int_0^1 \left(\frac{\partial f}{\partial \vec{x}} \cdot \delta\vec{x} + \frac{\partial f}{\partial \vec{x}_a} \cdot \delta\vec{x}_a\right) da + O(\varepsilon^2)$$

$$\mathcal{J}[\delta\vec{x}] = \int_0^1 \left(\frac{\partial f}{\partial \vec{x}} \cdot \delta\vec{x} + \frac{\partial f}{\partial \vec{x}_a} \cdot \delta\vec{x}_a\right) da$$

$$= \frac{\partial f}{\partial \vec{x}_a} \cdot \delta\vec{x}\,\Big|_0^1 - \int_0^1 \left[\frac{\partial f}{\partial \vec{x}} - \frac{\partial}{\partial a}\left(\frac{\partial f}{\partial \vec{x}_a}\right)\right] \cdot \delta\vec{x}\, da$$

1st term 0 by B.C., 2nd term 0 by extremization $\forall \delta\vec{x}$, so by F.T.V.C.

$$\frac{\partial f}{\partial \vec{x}} - \frac{\partial}{\partial a}\left(\frac{\partial f}{\partial \vec{x}_a}\right) = 0$$

This is the Euler-Lagrange eqn.

1. HW problem find Euler-Lagrange eqn. for princ. of least time
2. use $c(x)$ w/ walking then swimming lake (smooth it)
3. create a neat $c(x)$ & get a Euler-Lagrange eqn.

Saito lecture 1: Variational Problems I

Newton's Eqns. of Motion:

Notation: $\mathbb{r} = \mathbb{r}(t) = (x_1, \dots, x_n)^T \in \mathbb{R}^n = (x_1(t), \dots, x_n(t))^T \in \mathbb{R}^n$
$\mapsto$ a point or particle in $\mathbb{R}^n$ (position vector)
$\mathbb{e}_i = (0, 0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{R}^n$ – i^{th} canonical basis vector of $\mathbb{R}^n$
$\dot{\mathbb{r}} = \frac{d\mathbb{r}}{dt} = (\dot{x}_1, \dots, \dot{x}_n)^T$ – velocity of particle (vector velocity)

Particle of mass m w/ pos. $\mathbb{r}(t) \in \mathbb{R}^3$ moving under external force $\mathbb{F} \in \mathbb{R}^3$
satisfies Newton's Eqn. of Motion: $m\ddot{\mathbb{r}} = \mathbb{F}$
Can solve for $\mathbb{r}(t)$ given $\mathbb{r}(0)$ and $\dot{\mathbb{r}}(0)$.
Consider special force $\mathbb{F} = -\nabla V$, $V = V(x, y, z)$ (V a potential fn)
$$= \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z}\right)^T$$
Suppose particle moves under such a potential field V w/ no other ext. forces
Supp. don't know Newton's eqn. what can we say?

Def The kinetic energy of a particle is $T := \frac{1}{2} m |\dot{\mathbb{r}}|^2$ ($|\cdot|$ Euclid. norm)
The total energy of particle is $E := T + V$.
If $E \equiv$ const. indep. of time t, the field is called conservative.

Claim: If particle satisfies Newton's eqn. $m\ddot{\mathbb{r}} = -\nabla V$, then V is conservative.
Pf: $T = \frac{1}{2} m |\dot{\mathbb{r}}|^2$, $-\nabla V = m\ddot{\mathbb{r}}$
$\Rightarrow E = T + V \Rightarrow \nabla E = \nabla T + \nabla V = 2\left(\frac{1}{2} m |\dot{\mathbb{r}}|\right)\ddot{\mathbb{r}} - m\ddot{\mathbb{r}} = 0$ //

Def: The action of a particle along a cont's path γ starting at $\mathbb{r}(t_1)$ and arriving at $\mathbb{r}(t_2)$, $t_1 \le t_2$ is defined as
$$\underbrace{A}_{\text{path dependent \#}} := \int_{t_1}^{t_2} (T - V)\, dt = \int_{t_1}^{t_2} L\, dt$$
L ← Lagrange fn (Lagrangian)
$A = A(\gamma)$

Euler & Lagrange thought an actual path $= \underset{\gamma \in \Gamma}{\arg\min}\, A(\gamma)$
w/ Γ: a set of admissible curves
$$\Gamma = \left\{ \begin{array}{l} \gamma(t) : (x(t), y(t), z(t))^T,\ t_1 \le t \le t_2 \\ \gamma(t_i) = \mathbb{r}(t_i)\ (i = 1, 2),\ x, y, z \in C^1[t_1, t_2] \end{array} \right\}$$

Rmk: Under a conservative field, $E = T + V = C$ (constant)

so $L = T - V = 2T - C$

$\Rightarrow \underset{\gamma\in\Gamma}{\operatorname{argmin}} A(\gamma) = \underset{\gamma\in\Gamma}{\operatorname{argmin}} \int_{t_1}^{t_2} T\,dt$: the minimization of the time integral of kinetic energy T.

Principle of Least Action

a particle w/ mass m & pot. energy $V(\mathbf{r})$ takes the path γ^* in $[t_1, t_2]$

s.t. $\gamma^* = \underset{\gamma\in\Gamma}{\operatorname{argmin}} A(\gamma) = \underset{\gamma\in\Gamma}{\operatorname{argmin}} \int_{t_1}^{t_2} (T - V)\,dt$

Check:

$\mathbf{r}(t) = (x(t), y(t), z(t))^T$

$\Rightarrow A = \int_{t_1}^{t_2} \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z)\,dt$

let $\mathbf{r}(t_i) = \underline{P}_i = (x_i, y_i, z_i)^T$, $i = 1, 2$

Derive necessary cond. for path γ^* to minimize $A(\gamma)$.

Let $\gamma^* = \gamma^*(t) = (x^*(t), y^*(t), z^*(t))^T$, $t \in [t_1, t_2]$, be the minimizer & assume $\gamma^* \in \Gamma$. γ^* is said to be an extremal.

Consider a deviating path from γ^*, say $\gamma^*_\varepsilon = (x^*(t) + \varepsilon\xi(t), y^*(t), z^*(t))$

so $\xi(t_i) = 0$, $i = 1, 2$. γ^*_ε is a variation of γ^*

Then

$A(\gamma^*_\varepsilon) = A(\varepsilon) = \int_{t_1}^{t_2} \frac{m}{2}\left((\dot{x} + \varepsilon\dot{\xi})^2 + \dot{y}^2 + \dot{z}^2\right) - V(x^* + \varepsilon\xi, y^*, z^*)\,dt$

Since $\gamma^* = \gamma^*_0$ is the extremal, minimizer, must have $\frac{dA}{d\varepsilon}\Big|_{\varepsilon=0} = 0$

$\delta A := \frac{dA}{d\varepsilon}\Big|_{\varepsilon=0}$ is the first variation of A.

$\delta A = \frac{d}{d\varepsilon}\int_{t_1}^{t_2} \frac{m}{2}(\dot{x}^{*2} + 2\varepsilon\dot{x}^*\dot{\xi} + \varepsilon^2\dot{\xi}^2 + \dot{y}^{*2} + \dot{z}^{*2}) - V(x^* + \varepsilon\xi, y^*, z^*)\,dt$

$= \int_{t_1}^{t_2} m\dot{x}^*\dot{\xi} - \xi\frac{\partial V}{\partial x}(x^*, y^*, z^*)\,dt$ $\quad u = m\dot{x}^* \quad dv = \dot{\xi}$, $\quad du = m\ddot{x}^* \quad v = \xi$

$= \underbrace{m\dot{x}^*\xi\Big|_{t_1}^{t_2}}_{=0 \text{ by } \xi(t_i) = 0} - \int_{t_1}^{t_2}\left(m\ddot{x}^*\xi + \xi\frac{\partial V}{\partial x}\right)dt = 0$

$\Rightarrow \int_{t_1}^{t_2} \xi\left(m\ddot{x}^* + \frac{\partial V}{\partial x}\right)dt = 0$ must hold $\forall \xi \in \Gamma$

So by FTVC, must have $m\ddot{x}^* + \frac{\partial V}{\partial x} = 0$

hence $m\ddot{x}^* = -\frac{\partial V}{\partial x}(x^*, y^*, z^*)$ $(F=ma)$

Similarly, by varying γ^* w/ the variables y^*, z^*, recover Newton's eqns!

$$m\ddot{\mathbf{r}}(t) = -\nabla V(\mathbf{r}).$$

Rmk: $\delta A = 0$ is necessary condition. Sufficient cond. for minimizer involves the second variation, won't consider in this course!

Rmk: For Newton's eqn. only necessity was $\delta A = \frac{dA}{d\varepsilon}\big|_{\varepsilon=0} = 0$ to derive! Didn't need an absolute minimizer, just a stationary path.

Def: If a path $\gamma \in \Gamma$ satisfies $\delta A = 0$, γ is a stationary path & the action takes on a stationary value.

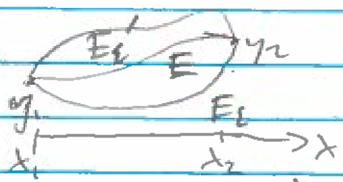
Sait's Notes – Lecture 2 – The Euler-Lagrange Eqn.

Consider a more general problem than the particle system:

~~Let $\Gamma := \{y \in C^1[x_1, x_2] : y(x_i) = y_i,\ i=1,2\}$~~

Find $y \in \Gamma$ s.t. (*) $I = \int_{x_1}^{x_2} f(x, y, y')\,dx \to$ minimized!
w/ $f \in C^2$ in each variable.

As before, assume $E := y = y^*(x)$ is the extremal (minimizer) & consider a one-parameter variation of E, ie, $E_\varepsilon : y = y^*(x,\varepsilon) = y^*_\varepsilon(x)$
w/ $\frac{\partial y^*}{\partial x}, \frac{\partial y^*}{\partial \varepsilon}, \frac{\partial^2 y^*}{\partial x \partial \varepsilon}$ all constant fns. in x and
$y^*(x,0) = y^*(x)$, $y^*(x_i, \varepsilon) = y_i$ $i=1,2$ (given constants)



Let $\frac{\partial y^*}{\partial \varepsilon}(x,\varepsilon) = \xi(x,\varepsilon)$, then $\xi(x_i,\varepsilon) = 0$ $i=1,2$.
put into eqn (*) to get:

Then $I(\varepsilon) = \int_{x_1}^{x_2} f(x, y^*(x,\varepsilon), y^{*\prime}(x,\varepsilon))\,dx$

~~For (*) to be minimized,~~ $\delta I = \frac{dI}{d\varepsilon}(0) = 0$ must be satisfied

$$\frac{dI}{d\varepsilon} = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}(x, y^*_\varepsilon, y^{*\prime}_\varepsilon)\cdot\frac{\partial y^*_\varepsilon}{\partial \varepsilon} + \frac{\partial f}{\partial y'}(x, y^*_\varepsilon, y^{*\prime}_\varepsilon)\cdot\frac{\partial y^{*\prime}_\varepsilon}{\partial \varepsilon}\,dx$$

$$= \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}(x, y^*_\varepsilon, y^{*\prime}_\varepsilon)\cdot\xi + \frac{\partial f}{\partial y'}(x, y^*_\varepsilon, y^{*\prime}_\varepsilon)\,\xi'\right)dx$$

$$= \int_{x_1}^{x_2} \frac{\partial f}{\partial y}(x, y^*_\varepsilon, y^{*\prime}_\varepsilon)\cdot\xi\,dx + \underbrace{\xi\frac{\partial f}{\partial y'}(x, y^*_\varepsilon, y^{*\prime}_\varepsilon)\Big|_{x_1}^{x_2}}_{=0} - \int_{x_1}^{x_2} \xi\,\frac{d}{dx}\frac{\partial f}{\partial y'}(x, y^*_\varepsilon, y^{*\prime}_\varepsilon)\,dx$$

$$= \int_{x_1}^{x_2} \xi \left(\frac{\partial f}{\partial y}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) - \frac{d}{dx}\frac{\partial f}{\partial y'}(x, y_\varepsilon^*, y_\varepsilon^{*\prime}) \right) dx$$

w/ $\varepsilon = 0$, $\int_{x_1}^{x_2} \xi \left(\frac{\partial f}{\partial y}(x, y^*, y^{*\prime}) - \frac{d}{dx}\frac{\partial f}{\partial y'}(x, y^*, y^{*\prime}) \right) dx = 0$

Since ξ is arbitrary (w/ right endpoints), then by FLCV must have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \qquad \text{(The Euler-Lagrange Eqn.)}$$

1/11/16

Hunter is back!

Euler-Lagrange Eqn: Consider the functional J:

$$J(u) = \int_a^b F(x, u(x), u'(x))\,dx$$

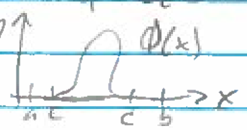
where $F(x,y,z)$ fn. of 3 variables $F: \mathbb{R}^3 \to \mathbb{R}$

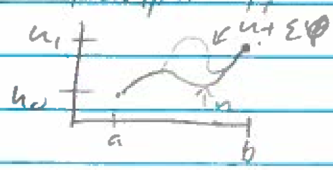
$F_u = \dfrac{\partial F}{\partial u}(x, u, u')$ etc. (u not a fn. here, just a variable)

$$X = \{u \in C^\infty[a,b] : u(a) = u_0,\ u(b) = u_1\}$$

Suppose $u \in X$ minimizes $J(u)$ over X, i.e. $J(u) = m$, $J(v) \geq m\ \forall v \in X$.

Suppose $\varphi \in C_c^\infty(a,b)$ smooth fn. w/ compact support ($\varphi = 0$ outside some $[c,d] \subset (a,b)$)





Then $u + \varepsilon\varphi$ is some small deviation from the minimizer u.

Then $j(\varepsilon) = J(u + \varepsilon\varphi)$ has minimum as fn. of ε at $\varepsilon = 0$.

It follows that $dJ(u;\varphi) = \dfrac{d}{d\varepsilon} J(u+\varepsilon\varphi)\Big|_{\varepsilon=0} = 0$

Gateaux derivative at point u in direction φ

$$dJ(u;\varphi) = \frac{d}{d\varepsilon}\int_a^b F(x, u+\varepsilon\varphi, u'+\varepsilon\varphi')\,dx\Big|_{\varepsilon=0}$$

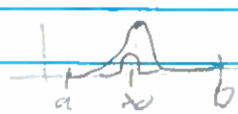
$$\Rightarrow 0 = \int_a^b F_u(x,u,u')\varphi + F_{u'}(x,u,u')\varphi'\,dx$$

$$\Rightarrow 0 = \int_a^b \left\{F_u(x,u,u') - \frac{d}{dx}F_{u'}(x,u,u')\right\}\varphi\,dx + \cancel{F_{u'}(x,u,u')\varphi\Big|_a^b}^{\,0}$$

This holds $\forall \varphi \in C_c^\infty(a,b)$. Note: boundary terms are zero since $\varphi = 0$ at a, b.

Fundamental Lemma of Calculus of Variations: If $f: [a,b] \to \mathbb{R}$ is a cont's fn. and $\int_a^b f(x)\varphi(x)\,dx = 0\ \forall \varphi \in C_c^\infty[a,b]$, then $f(x) = 0$.

Pf: Supp $f(x_0) \neq 0$:



f is cont's, so f is $\neq 0$ & same sign near x_0.

Can pick a φ s.t. $\int_a^b \varphi(x)\,dx = 1$ and $\varphi(x) \geq 0$ then $\int_a^b f(x)\varphi(x) \geq \varepsilon\int_a^b \varphi(x)\,dx = \varepsilon$.

Rmk: Do require $f(x)$ is cont's! Consider $f(x) = \begin{cases} 1 & x=0 \\ 0 & x \neq 0\end{cases}$ $\Rightarrow$ has zero integral! $\square$

Conclude that $\boxed{-\dfrac{d}{dx}F_{u'} + F_u = 0}$ is a necessary condition for u to minimize $J(u)$.

Euler-Lagrange eqn. $\rightarrow$

$F = F(x, u(x), u'(x))$, so Euler-Lagrange Eqn. becomes

$-\frac{d}{dx} F_{u'} + F_u = 0$

$\hookrightarrow -F_{u'u'} u'' - F_{uu'} u' - F_{u'x} + F_u = 0$

Ex: $J(u) = \int_a^b \left\{ \frac{1}{2} p(x) (u')^2 + \frac{1}{2} q(x) u^2 \right\} dx$

$F(x, y, z) = \frac{1}{2} p(x) z^2 + \frac{1}{2} q(x) y^2$

Euler-Lagrange eqn: $-\frac{d}{dx}\left(p(x) \frac{du}{dx}\right) + q(x) u = 0$

$\rightarrow -p(x) \frac{d^2u}{dx^2} - p'(x) \frac{du}{dx} + q(x) u = 0$

Quadratic functional $\Rightarrow$ linear Euler-Lagrange eqn.

This eqn. is self-adjoint & is a Sturm-Liouville ODE

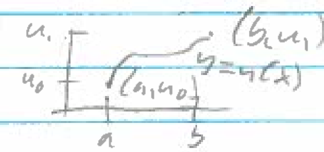
$\hookrightarrow$ will have real eigenvalues & eigenfns.

Obs: Suppose $F = F(x, u')$ independent of u:

$J(u) = \int_a^b F(x, u') dx$, $F_u = 0$

Euler-Lagrange eqn: $-\frac{d}{dx} F_{u'} = 0$

$\Rightarrow F_{u'}(x, u') = \text{constant}$.



Ex: Shortest curve between 2 points

$J(u) = \int_a^b \sqrt{1 + u'^2}\, dx$

it follows that $F_{u'}(x, u') = \text{const.}$

$\Rightarrow \frac{u'}{\sqrt{1 + u'^2}} = c_1 \Rightarrow u' = c_2 \Rightarrow y = u(x)$ is a straight line

Obs. Suppose $F(u, u')$ independent of x

$J(u) = \int_a^b F(u, u') dx \rightarrow -\frac{d}{dx} F_{u'} + F_u = 0$

$\Rightarrow -u' \frac{d}{dx} F_{u'} + u' F_u = 0$

$\Rightarrow -\frac{d}{dx}[u' F_{u'}] + \underbrace{u'' F_{u'} + u' F_u}_{\frac{d}{dx} F(u, u')} = 0$ $\rightarrow \frac{d}{dx}(F - u' F_{u'}) = 0$ so $-u' F_{u'} + F = \text{constant}$

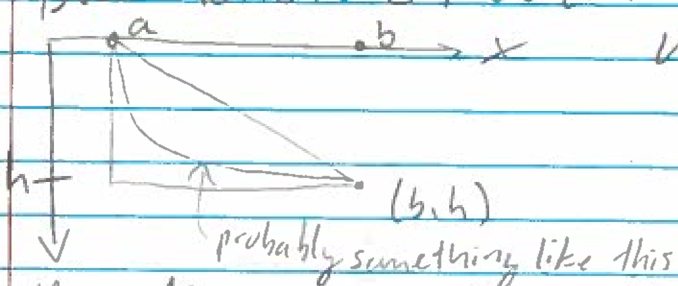
Rewording, when $F(u,u')$ ind. of x, euler-lagrange eqn. becomes
$\frac{d}{dx}(-u'F_{u'}+F)=0$, so $-u'F_{u'}+F=$ constant.

Rmk: This is an example of Noether's thm:

Invariance of Lagrangian (under transformation) $\longleftrightarrow$ Conservation law/ first integration

1/13/16

Brachistichrone Problem



Want a curve from $(a,0)$ to (b,h) that minimizes the time (rolling a mass down it)

Minimize time for mass to drop under gravity from $(a,0)$ to (b,h)

Minimize

$$J(u)=\int_a^b \sqrt{\frac{1+u'^2}{u}}\,dx$$

$$=\int_a^b F(u,u')\,dx \quad w/\ F(u,u')=\sqrt{\frac{1+u'^2}{u}}$$

Euler-lagrange: $\begin{cases} -\frac{d}{dx}F_{u'}+F_u=0 \\ u(a)=0,\ u(b)=h \end{cases}$

F is independent of x, so 1^{st} integral $\Rightarrow \int \frac{d}{dx}(F-u'F_{u'})\,dx=0$

$\Rightarrow -u'F_{u'}+F=1/c_1$

$$-u'\frac{u'}{\sqrt{u(1+u'^2)}}+\frac{\sqrt{1+u'^2}}{\sqrt{u}}=\frac{1}{c_1}$$

algebra $\hookrightarrow$ $u'=\sqrt{\frac{c_1^2-u}{u}}$ is a separable 1^{st} order ODE

$\Rightarrow \int\sqrt{\frac{u}{c_1^2-u}}\,du=\int dx=x+c_2$

$\Rightarrow u=\frac{1}{2}c_1^2(1-\cos t)=c_1^2\sin^2(\frac{t}{2}) \Rightarrow \sqrt{\frac{u}{c_1^2-u}}=\tan t/2$

$c_1^2-u=c_1^2(1-\frac{1}{2}+\frac{1}{2}\cos t)=c_1^2\cos^2(t/2)$

w/ substitution $u = c_1^2 \sin^2(t/2)$

$c_1^2 - u = c_1^2 \cos^2 t/2$

Then, $\int \sqrt{\frac{u}{c_1^2 - u}}\, du = \int \tan(t/2) \cdot \frac{1}{2} c_1^2 \sin(t)\, dt$

$= \int \tan(\frac{t}{2}) \cdot \frac{1}{2} c_1^2\, 2 \sin(\frac{t}{2}) \cos(\frac{t}{2})\, dt$

$= c_1^2 \int \sin^2(t/2)\, dt$

$= \frac{1}{2} c_1^2 \int 1 - \cos t\, dt$

$x = \frac{1}{2} c_1^2 (t - \sin t) + c_2$

$\Rightarrow u = \frac{1}{2} c_1^2 (1 - \cos t)$

$x = \frac{1}{2} c_1^2 (t - \sin t) + c_2$

$r \quad \frac{1}{2} c_1^2$

$u = r(1 - \cos t)$

$x = r(t - \sin t) + c$

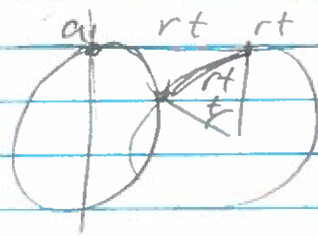
parametric description of our minimizing curve

a cycloid

$u(a) = 0 \Rightarrow t = 0 \Rightarrow x = a \Rightarrow c = a$

$u(b) = h \Rightarrow h = r(1 - \cos t)$

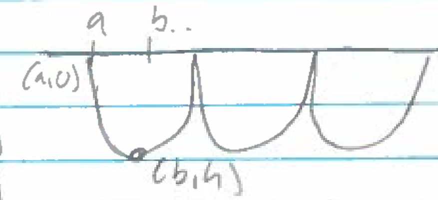
$b = r(t - \sin t)$



$\Rightarrow$

$x = r(t - \sin t)$

$y = r(1 - \cos t)$



Bernoulli already solved this in 1600s (John or Jacob)

(or Jingleheimerschmidt)

Natural Boundary Conditions

Suppose we don't impose any boundary conditions on u, and try to minimize (or maximize) $J(u) = \int_a^b F(x, u, u')\, dx$ over all (smooth) fns $u(x)$.

If u is a minimizer,

$$\frac{d}{d\varepsilon} J(u + \varepsilon \varphi) \Big|_{\varepsilon = 0} = 0 \quad \forall \varphi \in C^\infty([a,b])$$

If u is a minimizer (& we have no imposed B.C)

$$\frac{d}{d\varepsilon} J(u+\varepsilon\varphi)\big|_{\varepsilon=0}=0 \quad \forall \varphi \in C^\infty([a,b])$$

OH 2:30-3:45 W

Then $\Rightarrow \int_a^b [F_u\varphi + F_{u'}\varphi']dx = 0$

$$\Rightarrow \int_a^b \left\{F_u\varphi + \frac{d}{dx}(F_{u'}\varphi) - \frac{d}{dx}(F_{u'})\varphi\right\}dx = 0$$

$$\Rightarrow \int_a^b \left(-\frac{d}{dx}F_{u'} + F_u\right)\varphi\, dx + [F_{u'}\varphi]_a^b = 0$$

1) consider $\varphi \in C_c^\infty([a,b])$, then

$$\int_a^b \left(-\frac{d}{dx}F_{u'}+F_u\right)\varphi\, dx = 0 \Rightarrow \underline{-\frac{d}{dx}F_{u'}+F_u = 0}$$

2) Follows that $[F_{u'}\varphi]_a^b \quad \forall \varphi \in C^\infty([a,b])$

$$\Rightarrow F_{u'}|_a = 0, \quad F_{u'}|_b = 0$$

Thus if u is a smooth minimizer of $J(u)$ then

$$\begin{cases} -\frac{d}{dx}F_{u'} + F_u = 0 & a<x<b \\ F_{u'} = 0 & \text{at } x=a,b \end{cases} \quad \text{(natural b.c.'s)}$$

On the other hand, if we impose B.C.s $u(a)=u_0$, $u(b)=u_1$ on our admissible fns, then

$$\begin{cases} -d/dx\, F_{u'} + F_u = 0 \\ u(a)=u_0, \quad u(b)=u_1 \end{cases}$$

Ex: $J(u) = \int \{\frac{1}{2}p(x)(u')^2 + \frac{1}{2}q(x)u^2 - f(x)u\}dx$

Euler-Lagrange eqn:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + qu = f(x)$$

Impose B.C's: $u(a)=u_0$, $u(b)=u_1$ $\to$ Dirichlet B.C.'s

Natural B.C's: $u'(a)=0$, $u'(b)=0$ $\Rightarrow$ Neumann B.C.'s (provided $p(x)\neq 0$)

nonsingular

$p = F_{u'u'} \Rightarrow -\frac{d}{dx}F_{u'} + F_u = -F_{u'u'}\frac{d^2u}{dx^2} + \dots$ $\quad F_{u'u'} \neq 0 \to$ problem

$F_{u'u'} > 0 \Rightarrow F$ is convex fn. of u'

Rmk: Convexity is a crucial property for minimization problems.

HW 1 up Midterm 2/19

1/16 **Functional Derivatives**

$f: \mathbb{R}^n \to \mathbb{R}$ smooth

$$df(x;h) = \frac{d}{d\varepsilon} f(x+\varepsilon h)\Big|_{\varepsilon=0}$$

$\langle \cdot,\cdot \rangle$ inner product on $\mathbb{R}^n$ ($\langle x,y\rangle = x\cdot y = \sum_{i=1} x_i y_i$)

there's a vector $\nabla f(x)$ s.t.

$$df(x;h) = \langle \nabla f(x), h\rangle = \nabla f(x)\cdot h$$

Level curves

Consider functionals $J: X \to \mathbb{R}$ X = space of functions

e.g. $X = \{u: [a,b] \to \mathbb{R}$, u smooth + BC$\}$

Define L^2-inner product on X

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$$

$\frac{d}{d\varepsilon} J(u+\varepsilon h)\big|_{\varepsilon=0}$ = directional derivative of J at u in direction h

Suppose there is a fn. $\frac{\delta J}{\delta u}$ s.t.

$$\frac{d}{d\varepsilon} J(u+\varepsilon h)\Big|_{\varepsilon=0} = \left\langle \frac{\delta J}{\delta u(x)}, h\right\rangle = \int_a^b \frac{\delta J}{\delta u(x)} h(x)\,dx$$

※ gradient depends not only on the functional J but also the inner product

Ex: $J(u) = \int_a^b F(x,u,u')\,dx$

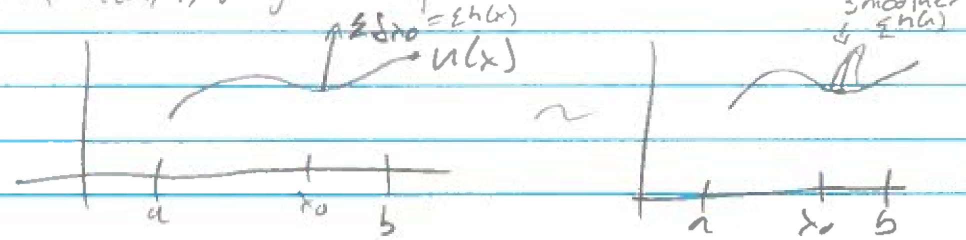
Euler-Lagrange $\Rightarrow$ $dJ(u;h) = \int_a^b \left\{-\frac{d}{dx}F_{u'} + F_u\right\} h(x)\,dx$

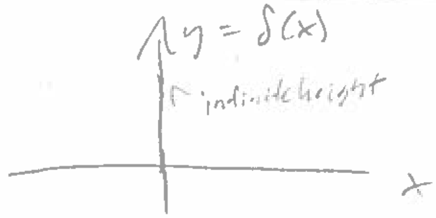
Then $\frac{\delta J}{\delta u} = -\frac{d}{dx}F_{u'} + F_u$

Euler-Lagrange: $\frac{\delta J}{\delta u} = 0$ (functional derivative $0 \to$ minimum!)

$\frac{\delta J}{\delta u}$ is the functional/variational derivative of $J(u)$.

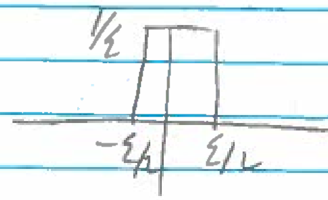
What if $h(x)$ is only at one point?





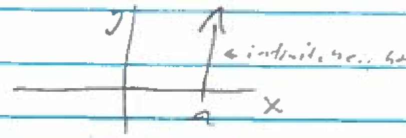
Delta function

formal def. $\begin{cases} \delta(x) = 0, & x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)\,dx = 1 \end{cases}$



$f_\varepsilon(x) = \begin{cases} 1/\varepsilon & |x| < \varepsilon/2 \\ 0 & |x| \geq \varepsilon/2 \end{cases}$

Then $f_\varepsilon \to \delta$ as $\varepsilon \to 0$



$\delta_a(x) = \delta(x-a)$: δ fn. supported at a

"density of unit point source located at a"

Formally, suppose f is a continuous fn: $\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)\int_{-\infty}^{\infty}\delta(x-a)\,dx$
$= f(a)\cdot 1$

(we use a convolution integral w/ δ fn to pick out a specific value of f)

Formally, "$\frac{d}{d\varepsilon} J(u+\varepsilon\delta_{x_0})\big|_{\varepsilon=0}$" $= \int_a^b \frac{\delta J}{\delta u(x)}\cdot\delta(x-x_0)\,dx$

$= \frac{\delta J}{\delta u(x_0)}$ given $a < x_0 < b$

$\frac{\delta J}{\delta u(x_0)}$ tells how $J(u)$ changes when u is perturbed only at the point x_0.

continuous analog of the gradient of fns.

Then if smooth fn. u minimizes $J(u)$, then (*) $\frac{\delta J(u)}{\delta u} = 0$ (Euler-Lagrange)

Any soln. of (*) is called a stationary point or extremal of $J(u)$ (even if $J(u)$ doesn't have extreme values - e.g. saddles)

Constrained Variational problems:

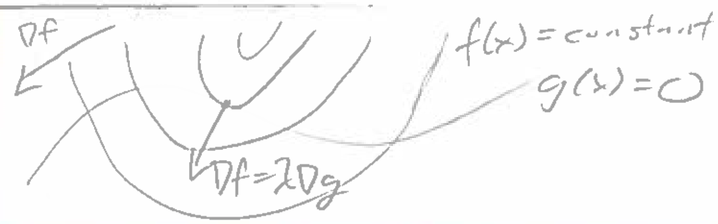
$f, g : \mathbb{R}^n \to \mathbb{R}$ (smooth)

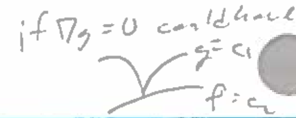
Find min/max of $f(x)$ on $g(x) = 0$

$\min f(x)$ subject to constraint $g(x) = 0$

Thm: (Lagrange multiplier) If $f(x)$ attains a min. at $x \in \mathbb{R}^n$ on $g(x) = 0$ and $\nabla g(x) \neq 0$, then $\exists$ constant λ s.t. $\nabla f(x) = \lambda \nabla g(x)$

or $\nabla(f - \lambda g) = 0$





Consider corresponding problem in Calc. of variations

~~$\min J(u) = 0, \; K(u) = 0$~~

$J(u) = \int_a^b F(x, u, u')\,dx, \quad K(u) = \int_a^b G(x, u, u')\,dx$

$u(a) = u_0, \quad u(b) = u_1$

Claim: If $u(x)$ is a smooth minimizer of J and $\frac{\delta K}{\delta u} \neq 0$, then $\exists\, \lambda$ s.t.

$$\frac{\delta J}{\delta u} = \lambda \frac{\delta K}{\delta u}$$

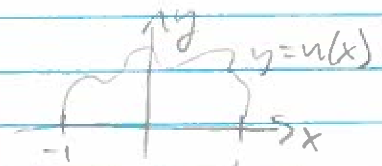
For previous functionals:

$$\begin{cases} -\frac{d}{dx} F_{u'} + F_u = \lambda \left[-\frac{d}{dx} G_{u'} + G_u \right] \\ u(a) = u_0, \quad u(b) = u_1 \end{cases}$$

1/20 Isoperimetric Problems

Find extreme value of $J(u) = \int_a^b F(x, u, u')\,dx$
subject to constraint $K(u) = \int_a^b G(x, u, u')\,dx = \text{constant}$
on e.g. $X = \{ u : [a,b] \to \mathbb{R}$ smooth w/ $u(a) = u_0, \; u(b) = u_1 \}$.

Ex:



Find curve $y = u(x)$ from $(-1, 0)$ to $(1, 0)$ that encloses the max. area w/ fixed length

Maximize $J(u) = \int_{-1}^{1} u\,dx$

Constraint $K(u) = \int_{-1}^{1} \sqrt{1 + u'^2}\,dx = L$

u is stationary point of $J(u)$ on $K(u) = L$ if $\exists\, \lambda \in \mathbb{R}$ s.t.

$$\frac{\delta J}{\delta u} = \lambda \frac{\delta K}{\delta u} \quad \left(\text{provided } \frac{\delta K}{\delta u} \neq 0\right)$$

$$\underbrace{\frac{\delta J}{\delta u} = 1}_{\text{since } F = u}, \quad \frac{\delta K}{\delta u} = -\frac{d}{dx}\left[\frac{u'}{\sqrt{1 + u'^2}}\right]$$

Ex cont'd

$$\Rightarrow 1 = -\lambda \frac{d}{dx}\left[\frac{u'}{\sqrt{1+u'^2}}\right]$$ ← Could also integrate ~~at this step twice to get eqn of circle arc.~~

$$-\frac{1}{\lambda} = \frac{u''}{\sqrt{1+u'^2}} - \frac{u'^2 u''}{(1+u'^2)^{3/2}}$$

$$\underbrace{-\frac{1}{\lambda}}_{\text{constant}} = \underbrace{\frac{u''}{(1+u'^2)^{3/2}}}_{\text{curvature of } y=u(x)}$$

Follows that $y=u(x)$ must be the arc of a circle.

Constraints will give us the exact arc.

Note that $2 < L \leq \pi$

when $L < 2$, can't connect points so no curves to pick from

when $L > \pi$, max. area still a circle ◯ but no longer a graph!

Any graph attempts will not be maximal / stationary

What have we proved?

Euler-Lagrange eqn is only a necessary condition for extremals

We have proved that if a smooth curve $y=u(x)$ of fixed length encloses a maximal area then it must be the arc of a circle.

Derivation of Lagrange Multiplier Condition:

Suppose $u(x)$ gives an extreme value of $J(u) = \int_a^b F(x,u,u')dx$

subject to the constraint $K(u) = 0 = \int_a^b G(x,u,u')dx$, $\frac{\delta K}{\delta u} \neq 0$. (at extremal)

Since $\frac{\delta K}{\delta u} \neq 0$, $\exists \psi \in C_c^\infty(a,b)$ s.t. $dK(u;\psi) = \int_a^b \frac{\delta K}{\delta u(x)}\psi(x)dx \neq 0$.

Consider varying $u(x)$ to $u(x) + \varepsilon\varphi(x) + \eta\psi(x)$, w/ $\varphi \in C_c^\infty(a,b)$.

$f(\varepsilon,\eta) = J(u+\varepsilon\varphi+\eta\psi)$

$g(\varepsilon,\eta) = K(u+\varepsilon\varphi+\eta\psi)$

Know that $f(\varepsilon,\eta)$ has an extreme value at $\varepsilon=\eta=0$ subject to constraint $g(\varepsilon,\eta)=0$.

$\Rightarrow \nabla f = (f_\varepsilon, f_\eta)|_{(\varepsilon,\eta)=0}$. $f_\varepsilon(0,0) = \frac{d}{d\varepsilon}f(\varepsilon,0)|_{\varepsilon=0} = \frac{d}{d\varepsilon}J(u+\varepsilon\varphi)|_{\varepsilon=0} = \int_a^b \frac{\delta J}{\delta u}\varphi dx$

$f_\eta(0,0) = \frac{d}{d\eta}f(0,\eta)|_{\eta=0} = \frac{d}{d\eta}J(u+\eta\psi)|_{\eta=0} = \int_a^b \frac{\delta J}{\delta u}\psi(x)dx$

Similarly, $g_\varepsilon(0,0) = \int_a^b \frac{\delta K}{\delta u}\varphi dx$, $g_\eta(0,0) = \int_a^b \frac{\delta K}{\delta u}\psi dx$

$$f_\varepsilon(0,0) = \int_a^b \frac{\delta J}{\delta u}\varphi\,dx \qquad g_\varepsilon(0,0) = \int_a^b \frac{\delta K}{\delta u}\varphi\,dx$$

$$f_\eta(0,0) = \int_a^b \frac{\delta J}{\delta u}\psi\,dx \qquad g_\eta(0,0) = \int_a^b \frac{\delta K}{\delta u}\psi\,dx$$

By Lagrange multiplier thm for $f(\varepsilon,\eta)$, $g(\varepsilon,\eta)$:

$\exists\, \lambda \in \mathbb{R}$ s.t. $\nabla f = \lambda \nabla g$ at $(\varepsilon,\eta)=(0,0)$

or $\begin{cases} f_\varepsilon = \lambda g_\varepsilon \\ f_\eta = \lambda g_\eta \end{cases}$ since $g_\eta \neq 0$ (so $\nabla g \neq 0$)
(since $\frac{\delta K}{\delta u} \neq 0$)

Hence $\displaystyle\int_a^b \frac{\delta J}{\delta u}\varphi\,dx = \lambda \int_a^b \frac{\delta K}{\delta u}\varphi\,dx \quad \forall \varphi.$

where $\lambda = \dfrac{f_\eta(0,0)}{g_\eta(0,0)}$ (constant; independent of φ)

ie. $\displaystyle\int_a^b \left[\frac{\delta J}{\delta u} - \lambda \frac{\delta K}{\delta u}\right]\varphi\,dx \quad \forall \varphi$

so by FLVC, $\dfrac{\delta J}{\delta u} = \lambda \dfrac{\delta K}{\delta u}$ $\square$

Basic Generalization: Systems

$$J(u_1,u_2,\ldots,u_n) = \int_a^b F(x,u_1,\ldots,u_n,u_1',\ldots,u_n')\,dx$$

Euler-Lagrange eq:

$$-\frac{d}{dx}F_{u_i'} + F_{u_i} = 0 \qquad 1 \le i \le n$$

$$\Rightarrow -\frac{d}{dx}F_{\vec{u}'} + F_{\vec{u}} = 0 \qquad \vec{u} = (u_1,\ldots,u_n$$

1/22

Review of vector calculus

Gradient: $\phi(x) = \phi(x_1, x_2, \ldots, x_n)$ is a scalar field $\phi : \mathbb{R}^n \to \mathbb{R}$

Then gradient $\nabla \phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \ldots, \frac{\partial \phi}{\partial x_n} \right)$

Divergence: $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(\vec{x}) = (F_1(\vec{x}), F_2(\vec{x}), \ldots, F_n(\vec{x}))$

divergence $\nabla \cdot F = \text{div} F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i}$

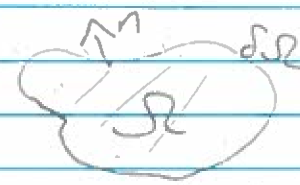
summation convention: sum over repeated indices: $\nabla \cdot F = \frac{\partial F_i}{\partial x_i}$

$F \cdot G = F_i G_i$

(assuming $x = (x_1, x_2, \ldots, x_n)$ are Cartesian coordinates)

Laplacian: $\Delta \phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x_i \partial x_i}$ w/ ϕ a scalar field

Divergence Thm: $\Omega \subset \mathbb{R}^n$ bounded region w/ smooth boundary $\partial \Omega$



$$\int_\Omega \nabla \cdot F \, dx = \int_{\partial \Omega} F \cdot n \, ds$$

n = unit outward normal

$\int_{\partial \Omega} \ldots ds$ = integral wrt. surface area on $\partial \Omega$

Vector identities

ϕ scalar field, F vector field $F = (F_1, \ldots, F_n)$

$$\nabla \cdot (\phi F) = \frac{\partial}{\partial x_i}(\phi F_i) = \frac{\partial \phi}{\partial x_i} F_i + \phi \frac{\partial F_i}{\partial x_i} = (\nabla \phi) \cdot F + \phi (\nabla \cdot F)$$

then $\int_\Omega \nabla \cdot (\phi F) \, dx = \int_{\partial \Omega} \phi F \cdot n \, ds$

Ex: $F = f e_i$, f scalar then $\nabla \cdot (\phi F) = \nabla \cdot (\phi f e_i) = \frac{\partial (\phi f)}{\partial x_i}$

$$= \frac{\partial \phi}{\partial x_i} f + \phi \frac{\partial f}{\partial x_i}$$

$$\int_\Omega \nabla \cdot (\phi f e_i) = \int_{\partial \Omega} \phi f n_i \, ds \implies \int_\Omega \phi \frac{\partial f}{\partial x_i} dx = \int_{\partial \Omega} \phi f n_i \, ds - \int_\Omega f \frac{\partial \phi}{\partial x_i} dx$$

This yields multidimensional integration by parts

$$\int_{\Omega} \phi \frac{\partial f}{\partial x_i} dx = \underbrace{\int_{\partial\Omega} \phi f n_i \, ds}_{\text{boundary term}} - \underbrace{\int_{\Omega} f \frac{\partial \phi}{\partial x_i} dx}_{\text{simpler integral}}$$

Green's Identities:

Take $F = \nabla \psi \quad \psi : \mathbb{R}^n \to \mathbb{R}$

$$\nabla \cdot (\phi \nabla \psi) = \nabla \phi \nabla \psi + \phi \Delta \psi$$

(1) By divergence thm, Green's first identity $\displaystyle\int_{\Omega} \nabla \phi \cdot \nabla \psi \, dx = \int_{\partial\Omega} \phi \nabla \psi \cdot n \, ds - \int_{\Omega} \phi \Delta \psi \, dx$

Also write $\nabla \psi \cdot n = \frac{\partial \psi}{\partial n}$ normal derivative of ψ on $\partial\Omega$

$$\nabla \cdot (\psi \nabla \phi) = \nabla \phi \nabla \psi + \psi \Delta \phi$$

$$\to \nabla \cdot (\phi \nabla \psi - \psi \nabla \phi) = \phi \Delta \psi - \psi \Delta \phi$$

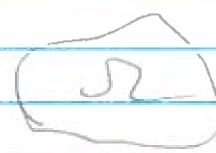
(2) $$\int_{\Omega} \phi \Delta \psi - \psi \Delta \phi \, dx = \int_{\partial\Omega} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) ds$$

Ex: In 1-D, get $\displaystyle\int_a^b \phi' \psi' dx = [\phi \psi']_a^b - \int_a^b \phi \psi'' dx$

and $\displaystyle\int_a^b \phi \psi'' - \psi \phi'' dx = [\phi \psi' - \psi \phi']_a^b$

Green's identities are generalizations of these to multidimensions.

Laplace's Equation: $u : \Omega \to \mathbb{R} \quad \Omega \subset \mathbb{R}^n$



$J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx \qquad \& \ u = g$ on $\partial\Omega$

Suppose u smooth function that minimizes $J(u)$ over all functions whose boundary values are $u = g$ on $\partial\Omega$.

Suppose $\phi \in C_c^{\infty}(\Omega)$ (ϕ smooth, compact support on Ω)

Then $dJ(u; \phi) = \frac{d}{d\varepsilon} J(u + \varepsilon \phi)|_{\varepsilon = 0} = 0.$

$$dJ(u;\varphi) = \frac{d}{d\varepsilon} \frac{1}{2}\int_\Omega |\nabla(u+\varepsilon\varphi)|^2 dx \Big|_{\varepsilon=0}$$

$$|\nabla(u+\varepsilon\varphi)|^2 = |\nabla u + \varepsilon\nabla\varphi|^2 = (\nabla u \cdot \varepsilon\nabla\varphi)\cdot(\nabla u + \varepsilon\nabla\varphi)$$
$$= \nabla u\nabla u + 2\varepsilon\nabla u\nabla\varphi + \varepsilon^2\nabla\varphi\nabla\varphi$$

Then $dJ(u,\varphi) = \frac{d}{d\varepsilon}\left\{\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \varepsilon\int_\Omega \nabla u\nabla\varphi\, dx + \frac{1}{2}\varepsilon^2\int_\Omega |\nabla\varphi|^2 dx\right\}\Big|_{\varepsilon=0}$

$$= \int_\Omega \nabla u \cdot \nabla\varphi\, dx$$

Necessary condition for u to minimize $J(u)$ is

$$\int_\Omega \nabla u \cdot \nabla\varphi\, dx = 0 \quad \forall\, \varphi \in C_c^\infty(\Omega) \quad \varphi = 0 \text{ on } \delta\Omega$$

$$\Rightarrow \cancel{\int_{\delta\Omega} \varphi \nabla u \cdot n\, ds} - \int_\Omega \Delta u \cdot \varphi\, dx = 0$$

or

$$\Rightarrow \boxed{-\Delta u = 0 \quad \text{in } \Omega} \quad \Leftarrow \text{Euler Lagrange eqn. of } J(u)$$

Showed that $dJ(u;\varphi) = -\int_\Omega \Delta u \cdot \varphi\, dx = \int_\Omega \frac{\delta J}{\delta u}\varphi\, dx$

$$\Rightarrow \frac{\delta J}{\delta u} = -\Delta u$$

Ex: In 1-D, $J(u) = \frac{1}{2}\int_a^b (u')^2 dx$, E-L: $-u'' = 0$

Minimizer u satisfies Dirichlet problem for Laplacian:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \delta\Omega \end{cases}$$

HW due Fri

1/25

(more)
General functionals of several variables:

~~$J(u) = \int_\Omega F(x, u, \nabla u)\,dx$~~ $dx = dx_1 \cdots dx_n$

$$= \int_\Omega F(x, u, u_{x_1}, \dots u_{x_n})\,dx$$

$$\frac{d}{d\varepsilon} J(u+\varepsilon h)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_\Omega F(x, u+\varepsilon h, u_{x_1}+\varepsilon h_{x_1}, \dots, u_{x_n}+\varepsilon h_{x_n})\,dx\Big|_\varepsilon$$

$h \in C_c^\infty(\Omega)$ & zero near boundary of Ω

$$= \int_\Omega \Big\{F_u(x,u,\nabla u)h + \sum_{i=1}^n F_{u_{x_i}}(x,u,\nabla u)h_{x_i}\Big\}\,dx$$

$$= \int_\Omega \Big\{F_u(x,u,\nabla u) - \sum_{i=1}^n \frac{\partial}{\partial x_i} F_{u_{x_i}}(x,u,\nabla u)\Big\} h\,dx$$

Hence $\dfrac{\delta J}{\delta u} = -\sum_{i=1}^n \dfrac{\partial}{\partial x_i} F_{u_{x_i}}(x,u,\nabla u) + F_u(x,u,\nabla u)$

Euler-Lagrange: $\delta J/\delta u = 0$.

Direct methods in Calc. of Variations:

Ex. (Bolza): $J(u) = \int_0^1 \{[(u')^2-1]^2 + u^4\}\,dx$

w/ $X = \{u \in C^1([0,1]) : u(0) = 0 = u(1)\}$ or $X = W^{1,4}(0,$

$(\int_0^1 u^4 dx, \int (u')^4 d$

Clearly, $J(u) \geq 0$ $\forall u \in X$, so J is bounded from below.

$J(u) \to \infty$ as $\|u\|_X \to \infty$

$J(u)$ is a quartic fn. of u (smooth)

For $m = \inf_{u\in X} J(u)$, does there exist a u w/ $J(u) = m$?

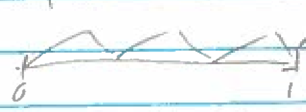
Euler-Lagrange not useful since $F_{u'u'} = 0$ at some points.

If we set $u=0$, $u'=0$, then $J(0) = \int_0^1 1 = 1$

So $1 \geq m \geq 0$.

Want to get $((u')^2-1)^2$ small as possible & u^4 small as pos

Need $(u')^2$ close to 1 and u^4 close to 0.

Use "zigzag soln"  slopes $1,-1$

smooth out to be C^1 or C^∞

Then can get close to 0, so $\inf J(u) = 0$, but we know that $J(u) = 0 \Rightarrow u = 0$ and $(u')^2 = 1$ on $[0,1]$, which is impossible!

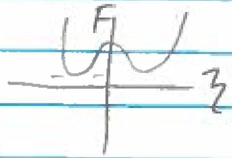
So $J(u)$ doesn't attain a minimum value.

Let u_n be the soln w/ n zig zags

The problem is that $J(u_n) \to 0$ but u_n do not converge to anything in $(X, \|\cdot\|_{C^1})$ ($\nexists$ fn. w/ $u=0$ & $u'=1$!!)

Arises because $J(u) = \int_0^1 F(u,u')\,dx$

$F(u,\xi) = (\xi^2-1)^2 + u^4$



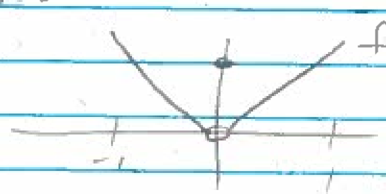
F not convex fn. of ξ.

(F convex in $\xi \Rightarrow F(\lambda\xi + (1-\lambda)\psi) \le \lambda F(\xi) + (1-\lambda)F(\psi)$, $0 \le \lambda \le 1$).

Weierstrass Thm: Consider $f: X \to \mathbb{R}$ where $X \subset \mathbb{R}^n$ is compact (If (x_n) is a seq. in X, then $\exists$ convergent subseq (x_{n_k}) s.t. $x_{n_k} \to x \in X$) & f cont's.
Then f attains a min. on X.

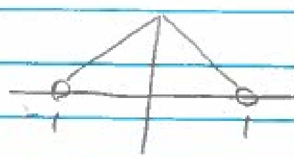
Sketch of pf: Let $m = \inf_{x \in X} f(x)$ m finite. Choose $x_n \in X$ s.t. $f(x_n) \to m$.
Then (x_n) has a conv. subseq. $x_{n_k} \to x \in X$ and then $f(x) = \lim_{k\to\infty} f(x_{n_k})$ since f cont's, hence $f(x) = m$, so f attains its minimum at x.

Need f to be cont's



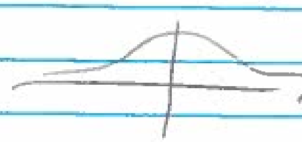
← doesn't attain min!

Need X closed



← Miss mins!

Need X bounded



↑ never reach a min. value

1/27

Basic PDE's

$$\Delta u = 0 \qquad \text{Laplace's eqn.}$$

2d: $\Delta u = u_{xx} + u_{yy}$

3d: $\Delta u = u_{xx} + u_{yy} + u_{zz}$ $\qquad \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ n-d

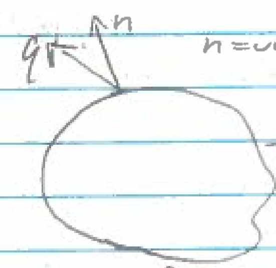
Nice linear eqn. Here's more basic linear homogeneous PDEs:

$$u_t = \Delta u \qquad \text{diffusion (heat) eqn.}$$

$$u_{tt} = \Delta u \qquad \text{wave eqn.}$$

$$i u_t = -\Delta u + V(x) u \qquad \text{Schrödinger eqn.}$$

Derivation of heat eqn.



n = unit normal vector to $\delta\Omega$

$\Omega \subset \mathbb{R}^n$

$e(x,t)$ = density of heat energy (per unit volume)

$q(x,t)$ = heat flux vector

$q \cdot n$ = rate at which heat energy flows across $\delta\Omega$ per unit surface area

$f(x,t)$ = rate of generation of heat energy by internal sources (per unit vol.)

Assume energy is conserved.

$\frac{d}{dt}$(energy in Ω) = $-$rate energy flows out of Ω through $\delta\Omega$ + rate energy is generated in Ω by internal sources

$$\frac{d}{dt}\int_\Omega e(x,t)\,dx = -\int_{\delta\Omega} q(x,t)\cdot n(t)\,dS(x) + \int_\Omega f(x,t)\,dx$$

This is the integral form of conservation of energy (should hold for any Ω)

Write in one integral $= 0$ to get the differential form $\Rightarrow$

$$\int_\Omega e_t\,dx = -\int_\Omega \nabla\cdot q\,dx + \int_\Omega f(x,t)\,dx$$

$$\Rightarrow \int_\Omega (e_t + \nabla\cdot q - f)\,dx = 0 \quad \forall \Omega$$

If integrand is continuous then must be 0, so

$$e_t + \nabla\cdot q = f \qquad \text{(Differential form of conservation of energy)}$$

To get a closed eqn, we need to add to add constitutive relations that specify e, q, f. Assume f is a given heat source, given fn.

For heat flow: introduce temperature $u(x,t)$.

$e = cu$, c = heat capacity

$q = -k\nabla u$, k = thermal conductivity (Fourier's Law)

Hence our conservation of energy differential eqn. becomes

$$cu_t - k\Delta u = f$$

$$\Rightarrow u_t - D\Delta u = g \qquad g(x,t) = \frac{1}{c}f(x,t)$$

If $g=0$ (no sources of heat), then $\qquad D = \frac{k}{c}$ ← diffusion coefficient

$$\underline{u_t = D\Delta u} \quad \leftarrow \text{heat eqn.}$$

lets say Θ = temp. unit, l = length unit, T = time unit

$[u] = \Theta \quad [x] = l, \quad [t] = T$

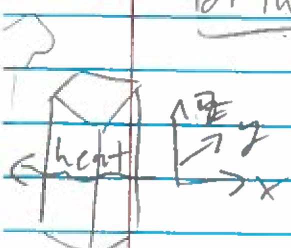
$[u_t] = [D][\Delta u]$

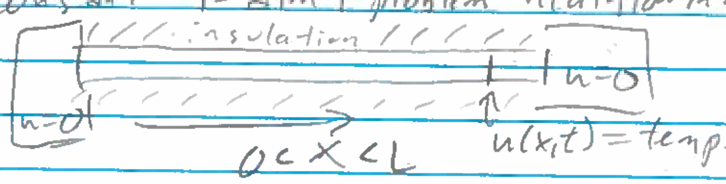
$\frac{\Theta}{T} = [D]\frac{\Theta}{L^2} \Rightarrow [D] = \frac{L^2}{T}$

Expect temperature differences to diffuse a distance L after time T where $L \sim \sqrt{DT}$ (like $\sqrt{T}$)

1/29 An Initial-Boundary Value problem (IBVP) for the heat eqn.

Consider 1-dim'l problem: heat flow in a rod





-wall-inside
1-D problem

$$\begin{cases} u_t = D u_{xx} & 0<x<L,\ t>0 \quad \text{(PDE)} \\ u(0,t) = 0 = u(L,t) & \text{(BC)} \\ u(x,0) = f(x) & \text{(IC)} \end{cases}$$

Parameters: Length L, Diffusivity D, Temperature Θ (from $f(x)$)

Units: Length L, Time t, temperature Θ

Nondim'lize: $T = \frac{L^2}{D}$ ← also time it takes to diffuse the length of the rod

$x = \frac{\bar{x}}{L}$, $t = \frac{\bar{t}}{T}$, $u(x,t) = \Theta\,\bar{u}(\bar{x},\bar{t})$

$\partial_x = \frac{1}{L}\partial_{\bar{x}}$, $\partial_t = \frac{1}{T}\partial_{\bar{t}}$

$\Rightarrow$ PDE: $\frac{\Theta}{T}\bar{u}_{\bar{t}} = \frac{D\Theta}{L^2}\bar{u}_{\bar{x}\bar{x}}$

$\Rightarrow \bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}}$

BC: $\bar{u}(0,\bar{t}) = 0 = \bar{u}(1,\bar{t})$

IC: $\Theta\bar{u}(\bar{x},0) = f(L\bar{x})$

$\rightarrow f(L\bar{x})/\Theta = \bar{f}(\bar{x})$

$\cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2}$, $\sinh\theta = \dfrac{e^{\theta}-e^{-\theta}}{2}$

$\cosh(i\theta) = \cos\theta$, $\sinh(i\theta) = i\sin\theta$

Drop "—"'s from nondim'lized IBVP:

IBVP $\left\{ \begin{array}{lll} \text{PDE:} & u_t = u_{xx} & 0<x<1,\ 0<t \\ \text{BC:} & u(0,t)=0 & t>0 \\ & u(1,t)=0 & \\ \text{IC:} & u(x,0)=f(x) & 0<x<1 \end{array} \right.$

Separation of Variables

Look for separated solutions of $\begin{cases} u_t = u_{xx} \\ u(0,t)=u(1,t)=0 \end{cases}$

Say $u(x,t) = F(x)G(t)$ ← may not exist in general

then $FG' = F''G$

$\Rightarrow \underbrace{\dfrac{G'}{G}}_{\text{fn. of } t} = \underbrace{\dfrac{F''}{F}}_{\text{fn. of } x} \Rightarrow$ must equal a constant $\overbrace{-\lambda}^{\text{separation constant}}$

$\Rightarrow \dfrac{F''}{F} = -\lambda$ | $\dfrac{G'}{G} = -\lambda$

$-F'' = \lambda F$ | $G' = -\lambda G$

Need $F(x)$ to satisfy | $\Rightarrow G(t) = G_0 e^{-\lambda t}$

the eigenvalue problem:

$\begin{cases} -F'' = \lambda F & 0<x<1 \\ F(0)=0,\ F(1)=0 \end{cases}$ basic example of a Sturm-Liouville EVP

guess $F = e^{rx}$

$\Rightarrow -r^2 = \lambda$ characteri

(1) $\lambda<0$: $\lambda = -k^2 \Rightarrow r = \pm k$

(2) $\lambda = 0$: $r=0 \Rightarrow 1, x$

(3) $\lambda>0$: $\lambda = k^2 \Rightarrow r = \pm ik \Rightarrow \cos kx, \sin kx$

(1) $\lambda = -k^2 < 0$, have $F(x) = \hat{c}_1 e^{kx} + \hat{c}_2 e^{-kx}$

$= c_1\cosh(kx) + c_2\sinh(kx)$

$F(0)=0 \Rightarrow c_1 = 0$ $F(1)=0 \Rightarrow c_2\sinh k = 0 \Rightarrow c_2 = 0$

So general soln is $F(x)=0$.

2) $\lambda = 0$: $F(x) = c_1 + c_2 x$

$F(0)=0 \Rightarrow c_1 = 0$ $F(1)=0 \Rightarrow c_2 = 0$

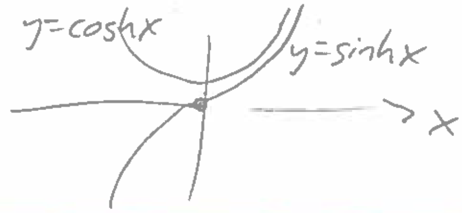
So gen. soln is $F(x)=0$

So no nonzero solns for $\lambda \le 0$!

$(\cosh x)' = \sinh x$

$(\sinh x)' = \cosh x$

$\cosh^2 x - \sinh^2 x = 1$



$\cosh(0) = 1$

$\sinh(0) = 0$

Note: $-F'' = \lambda F \qquad F(0) = F(1) = 0$

$-F'F'' = \lambda F F'$

2/1 Assume $\lambda > 0$, i.e. $\lambda = k^2$ then $-F'' = k^2 F$

Assume $F(x) = e^{rx} \Rightarrow -r^2 = k^2$ characteristic

$\Rightarrow r = \pm ik \Rightarrow F(x) = c_1 \cos kx + c_2 \sin kx$

$F(0) = 0 \Leftrightarrow c_1 = 0$

$F(1) = 0 \Leftrightarrow c_2 \sin k = 0$

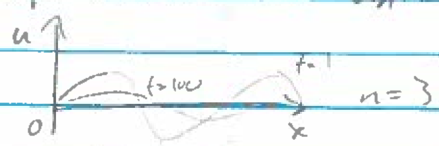
$\Leftrightarrow \sin k = 0$ if $c_2 \neq 0$

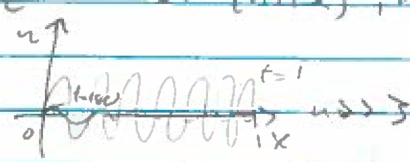
take $k = n\pi$, $n \in \mathbb{N}$ ($-n \Rightarrow k = -n\pi$ but $\lambda = k^2$ so can ignore)

then Eigvalues $\lambda = \lambda_n$: $\lambda_n = n^2\pi^2$, $n = 1, 2, \ldots$

Eigenfunctions $F(x) = F_n(x)$ $\quad F_n(x) = c_n \sin(n\pi x)$

Then separated solns are $u_n(x,t) = e^{-n^2\pi^2 t} \sin(n\pi x)$, $n = 1, 2, 3, \ldots$





By superposition (IBVP is linear + homogeneous)

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t} \sin(n\pi x)}$$

To satisfy IC, need to choose c_n s.t. $\boxed{f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)}$

Note that $\langle \sin(n\pi x), \sin(m\pi x) \rangle = \int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 0 & n \neq m \\ 1/2 & n = m \end{cases}$

So then $\langle f(x), \sin n\pi x \rangle = \sum_{m=1}^{\infty} c_m \langle \sin(m\pi x), \sin n\pi x \rangle = \frac{1}{2} c_n$

$$\Rightarrow \boxed{c_n = 2\langle f, \sin n\pi x \rangle = 2\int_0^1 f(x)\sin(n\pi x)\,dx}$$

Smoothness of $f(x)$ $\longleftrightarrow$ decay of Fourier coefficients

$u(x,t) \in C^\infty$ for $t > 0$ $\longleftrightarrow$ exponential decay of Fourier coefs as $n \to \infty$

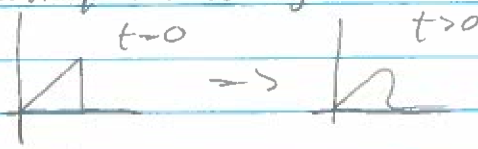
$$\left\{ \begin{array}{l} f \in L^2(0,1) \\ \int_0^1 f(x)^2\,dx < \infty \end{array} \right. \Rightarrow \begin{array}{l} u(\cdot, t) \in C^\infty(0,1) \\ \forall t > 0 \end{array}$$

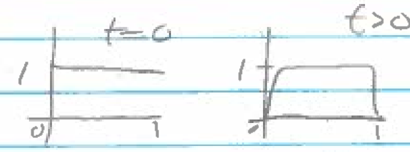
As $t\to\infty$ $u(x,t)\to 0$ (since holding ends at 0, heat equilibriates / flows out ends of rod)

For large t, $u(x,t) \sim c_1 e^{-\pi^2 t}\sin(\pi x)$

(given $c_1 \neq 0$, o/w go to first nonzero term)

Heat eqn is smoothing!





Ex. Supp. $f(x)=1 \quad 0<x<1$

then $c_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx = 2\int_0^1 \sin n\pi x\,dx = 2\left[\frac{-\cos n\pi x}{n\pi}\right]_0^1$

$$= \frac{2}{n\pi}\left[-\cos(n\pi)+1\right] = \frac{2}{n\pi}\left[1-(-1)^n\right] = \begin{cases} 4/n\pi & n \text{ odd} \\ 0 & n \text{ even}\end{cases}$$

so $u(x,t) = \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n} e^{-n^2\pi^2 t}\sin(n\pi x)$

$$= \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1} e^{-(2k-1)^2\pi^2 t}\sin((2k-1)\pi x)$$

<u>Eigenvalue Problems:</u>

$A:\mathbb{R}^n \to \mathbb{R}^n$ linear map $A(cx+y) = cAx + Ay$ $\quad x,y\in\mathbb{R}^n$, $c\in\mathbb{R}$

analogous concept in $\mathbb{C}^n$:

$Ax = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn}\end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_n\end{pmatrix}$ $\quad a_{ij} = A_{ij}$

$\langle x,y\rangle = x\cdot y = \sum_{i=1}^n x_i y_i$ inner prod. on $\mathbb{R}^n$

adjoint matrix = transpose conjugate

Define A^T by

$$\langle x, Ay\rangle = \langle A^T x, y\rangle \Rightarrow (A^T)_{ij} = A_{ji}$$

Hermitian = self-adjoint ~ symm.

A symmetric if $A^T = A \Rightarrow \langle x, Ay\rangle = \langle Ax, y\rangle$

Eigenvalues of A are $\lambda\in\mathbb{C}$ s.t.

$$Ax = \lambda x \quad x\neq 0$$

corresponding eigenspace of eigenvectors x w/ eigenvalue λ

Thm: If A is symmetric (or Hermitian) then all eigenvalues are real & eigenvectors w/ diff. eigenvalues are orthogonal. Furthermore, there is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

2/3

Complex inner product spaces

Def: X vector space over $\mathbb{C}$. An inner product on X is a fn $\langle \cdot,\cdot \rangle : X \times X \to \mathbb{C}$

s.t. a) $\langle x, y+z \rangle = \langle x,y \rangle + \langle x,z \rangle \quad \forall x,y,z \in X$

b) $\langle x, cy \rangle = c\langle x,y \rangle \quad \forall x,y \in X \ \forall c \in \mathbb{C}$ | sesqui. linear form

c) $\langle x,y \rangle = \overline{\langle y,x \rangle}$

d) $\langle x,x \rangle \geq 0 \ \forall x \in X$ & $\langle x,x \rangle = 0 \Rightarrow x = 0$

$\|x\| = \sqrt{\langle x,x \rangle}$ is a norm on $x \in X$ (has Δ ineq: $\|x+y\| \leq \|x\| + \|y\|$)

Ex: $X = \mathbb{C}^n$, $x = (x_1, \dots, x_n)$, $x_k \in \mathbb{C}$

$\langle x,y \rangle = \sum_{k=1}^{n} \overline{x_k} y_k$ and $\|x\| = \sqrt{\sum_{k=1}^{n} |x_k|^2}$

Ex: $X = L^2(a,b)$ $u : (a,b) \to \mathbb{C}$ s.t. $\int_a^b |u(x)|^2 dx < \infty$

$\langle u,v \rangle = \int_a^b \overline{u(x)} v(x)\, dx$, $\|u\| = \sqrt{\int_a^b |u(x)|^2 dx}$

Def: A linear operator on X is a linear map $A : X \to X$

a) $A(x+y) = Ax + Ay$ b) $A(cx) = cAx$

$\|A\| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Ax\|}{\|x\|}$ is the operator norm

Ex: $X = \mathbb{C}^n$, $y = Ax$: $y_j = \sum_{k=1}^{n} a_{jk} x_k \Rightarrow (a_{jk})$ is matrix of A w.r.t. standard basis of $\mathbb{C}^n$

Ex: $A = -\frac{d^2}{dx^2}$, let $X = C^\infty([a,b])$, so $A : X \to X$ is a linear differential operator.

Alternative: $H^k([a,b])$ = space of fns $u : (a,b) \to \mathbb{C}$ s.t. $u, u', \dots, u^{(k)} \in L^2([a,b])$ weak derivatives

Then $-\frac{d^2}{dx^2} : H^2([a,b]) \longrightarrow L^2([a,b])$

Basic problem: $A = -\frac{d^2}{dx^2}$ is an unbounded linear operator!

$\hookrightarrow u_n = \sin(n\pi x) \in L^2(0,1) \Rightarrow Au_n = n^2\pi^2 \sin(n\pi x)$

$\Rightarrow \|Au_n\| = n^2\pi^2 \|u_n\| \Rightarrow \|A\| = \sup \frac{\|Au_n\|}{\|u_n\|} = \infty$

Self-adjoint operators

Def: Linear op. $A: X \to X$ is self-adjoint (Hermitian - $\mathbb{C}$, symmetric - ...
if $\langle x, Ay \rangle = \langle Ax, y \rangle$

Def: In general def. adjoint A^* of A by $\langle A^* x, y \rangle = \langle x, Ay \rangle$
(in $\mathbb{C}$, $A^* = (\overline{a_{ji}})$ for A (a_{ij}), and in $\mathbb{R}$ $A^* = (a_{ji})$).

Ex: $A: \mathbb{C}^n \to \mathbb{C}^n$ $\quad \langle x, Ay \rangle = \overline{x}_j a_{jk} y_k$ (summation convention)
$= \overline{\overline{a_{jk}} x_j} y_k = \langle A^* x, y \rangle$
hence matrix of A^* is the conjugate transpose of the matrix A.

Ex: $A: X \to X$ $\quad A = -\frac{d^2}{dx^2}$ $\quad X = C^\infty([a,b])$
$\langle u, Av \rangle - \langle Au, v \rangle = \int_a^b (-\overline{u} v'' + \overline{u}'' v) dx$
$= \int_a^b \frac{d}{dx} [-\overline{u} v' + \overline{u}' v] dx = [-\overline{u} v' + \overline{u}' v]_a^b$
No reason for last be zero $\forall u, v$, so A is not self-adjoint on X.

↳ But we say $\frac{d^2}{dx^2}$ is formally self-adjoint when we include some boundary conditions to make A self-adjoint.

1) Dirichlet conditions: $u(a)=0, u(b)=0, v(a)=0, v(b)=0$
$\Rightarrow \langle u, Av \rangle - \langle Au, v \rangle = [-\overline{u} v' + \overline{u}' v]_a^b = 0 \Rightarrow \langle u, Av \rangle = \langle Au, v \rangle$
A self-adjoint

2) Neumann conditions: $u'(a) = u'(b) = 0$, $v'(a) = v'(b) = 0$
$\Rightarrow \langle u, Av \rangle = \langle Au, v \rangle$ $\quad$ A self-adjoint

3) Mixed conditions $\alpha u(a) + \beta u'(a) = 0$ etc. $\alpha, \beta \neq 0$
$\Rightarrow A$ self-adjoint

4) Periodic boundary conditions: $u(a) = u(b)$, $u'(a) = u'(b)$; $v(a) = v(b)$, $v'(a) = v'(b)$ } not separ...
$\Rightarrow A$ self-adjoint.

Initial conditions: $u(a) = u'(a) = v(a) = v'(a) = 0$ but no cond. at b
does NOT yield A self-adjoint. $\langle u, Av \rangle - \langle Au, v \rangle = -\overline{u} v' + \overline{u}' v |_{x=b}$

If $u(a)=0$, $u'(a)=0$, need v to satisfy adjoint B.C's: $v(b)=0$, $v'(b)=0$
Adjoint of initial conditions are final conditions

2/5 **Eigenvalues & Eigenvectors**

$$A: D(A) \subset X \to X \qquad X \text{ a Hilbert space}$$

Ex: $A = -\frac{d^2}{dx^2}$ $\quad X = L^2(a,b)$

$D(A) = \{u \in H^2(a,b) : u(a)=0,\ u(b)=0\} = H_0^1(a,b) \cap H^2(a,b)$
where $\quad H_0^k(a,b) = \{u \in H^k(a,b) : u, u', \dots, u^{(k-1)} = 0 \text{ at } x=a,b\}$

$u \in H^2(a,b)$ if $u, u', u'' \in L^2(a,b)$

Suppose $A = A^*$ self adjoint. Eigenvalue of A is $\lambda \in \mathbb{C}$ s.t.
$\quad Au = \lambda u$ for some eigenvector $u \in D(A)$, $u \neq 0$.

Thm: If A is selfadjoint, then every eigenvalue $\lambda \in \mathbb{R}$ of A is real & eigenvectors with different eigenvalues are orthogonal

Pf: Supp. $Au = \lambda u$ $\quad u \neq 0$. Then $\langle u, Au \rangle = \langle u, \lambda u \rangle = \lambda \langle u, u \rangle = \lambda \|u\|^2$
Also $\langle u, Au \rangle = \langle Au, u \rangle = \langle \lambda u, u \rangle = \bar{\lambda} \|u\|^2$
Hence $\lambda = \bar{\lambda}$ so $\lambda \in \mathbb{R}$.
Now consider $Au = \lambda u$, $Av = \mu v$, $\lambda \neq \mu$
Then $\langle u, Av \rangle = \langle u, \mu v \rangle = \mu \langle u, v \rangle$
$\quad \langle u, Av \rangle = \langle Au, v \rangle = \langle \lambda u, v \rangle = \lambda \langle u, v \rangle$
$\Rightarrow \lambda \langle u, v \rangle = \mu \langle u, v \rangle$ but $\lambda \neq \mu$
hence $\langle u, v \rangle = 0$. $\quad \square$

Ex: $A = -\frac{d^2}{dx^2}$, $u(0)=u(1)=0$. $\lambda_n = n^2\pi^2$, $u_n(x) = \sin n\pi x$

Then $\langle u_n, u_m \rangle = \int_0^1 \sin n\pi x \sin m\pi x \, dx = \begin{cases} 1/2 & m=n \\ 0 & m \neq n \end{cases}$

when $u'(0)=u'(1)=0$ are the B.C.s, $\lambda_n = n^2\pi^2$, $u_n(x) = \cos n\pi x$
same reason for orthogonality

Sturm-Louiville Operators

Q: ~~What is the most general formally self-adjoint second-order real differential linear operator?~~

$$A = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$$, a, b, c real valued fns.

$$Au = au'' + bu' + cu$$

Then $\langle u, v\rangle = \int_\alpha^\beta u(x)v(x)\,dx$ (real fns, real inner prod)

So $\langle u, Av\rangle = \int_\alpha^\beta u\{av'' + bv' + cv\}dx$

$$= \int_\alpha^\beta \{-(au)'v' - (bu)'v + cuv\}\,dx + [auv' + buv]_\alpha^\beta$$

$$= \int_\alpha^\beta \{(au)''v - (bu)'v + cuv\}\,dx + [auv' - (au)'v + buv]_\alpha^\beta$$

If boundary terms vanish, then

$\langle u, Av\rangle = \langle A^* u, v\rangle$

w/ $A^* = \frac{d^2}{dx^2}a(x) - \frac{d}{dx}b(x) + c(x)$

eg: $A^*u = (au)'' - (bu)' + cu$

$= au'' + 2a'u' + ua'' - bu' - b'u + cu$

$= au'' + (2a' - b)u' + (a'' - b' + c)u$

When is $A = A^*$? Need $b = 2a' - b \Rightarrow \boxed{b = a'}$

then $a'' - b' + c = a'' - a'' + c = c$ ✓ and obv. $a = a$.

Standard notation: $a(x) = -p(x)$, $c(x) = q(x)$, $b = -p'$

$$A = -p(x)\frac{d^2}{dx^2} - p'(x)\frac{d}{dx} + q(x) = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q$$

$$Au = -pu'' - p'u' + qu$$

$$= -\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu$$ ← general Sturm-Louiville operator

$$(ABC)^* = C^* B^* A^*$$

Note that $(ABA)^* = A^* B^* A^*$

so if $A = \frac{d}{dx}$, $B = p$

$A^* = -d/dx$, $B^* = p$ $\Rightarrow \left(\frac{d}{dx} p \frac{d}{dx}\right)^* = \frac{d}{dx} p \frac{d}{dx}$

So sturm-Liouville op. is self-adjoint!

$$Au = -\frac{d}{dx}\left(p \frac{du}{dx}\right) + q u$$

If u satisfies BC $Bu = 0$ (eg. $u(\alpha) = 0$, $u(\beta) = 0$)

Then adjoint BC's on v $B^* v = 0$

are defined so that boundary terms $= 0$ $\forall u, v$ s.t. $Bu = 0$, $B^* v = 0$

Ex: Separated B.Cs $\begin{cases} c_1 u'(\alpha) + c_2 u(\alpha) = 0 & c_1^2 + c_2^2 \neq 0 \\ c_3 u'(\beta) + c_4 u(\beta) = 0 & c_3^2 + c_4^2 \neq 0 \end{cases}$

These are self-adjoint B.Cs for the Sturm Liouville op.

(i.e. the boundary terms $= 0$ w/ u & v satisfying the above BCs)

Corresponding SL eigenvalue problem:

$$-\frac{d}{dx}\left(p(x) \frac{d}{dx}\right) + q(x) u = \lambda u \qquad a < x < b$$

$$\begin{cases} c_1 u'(a) + c_2 u(a) = 0 \\ c_3 u'(b) + c_4 u(b) = 0 \end{cases}$$

know $\lambda \in \mathbb{R}$ & eigenvectors orthogonal if things behave nicely

Regular Sturm-Liouville problem:

1) $[a,b]$ finite interval
2) $p, p', q \in C([a,b])$
3) $p > 0$ on $[a,b]$

Thm: The (normalized) eigenfunctions of a regular Sturm-Liouville problem form an orthonormal basis of $L^2(a,b)$. Each eigenvalue is simple, i.e. the dimension of each eigenvalue's eigenspace is 1.

For $f \in L^2(a,b)$, $f(x) = \sum_{n=1}^{\infty} c_n u_n(x)$ w/ $Au_n = \lambda_n u_n$, $n = 1, 2, 3, \ldots$

$\|u_n\| = 1$, $c_n = \langle u_n, f \rangle = \int_a^b u_n(x) f(x) dx$

$\|f - \sum_{n=1}^{N} c_n u_n\|_{L^2} \to 0$ as $N \to \infty$

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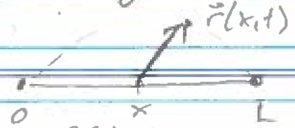
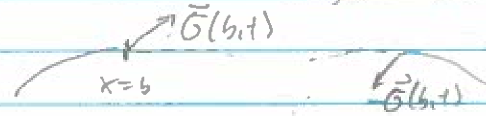
Vibrating Strings

reference configuration



elastic string (stretch, no resistance to bending)

$0 \le x \le L$ $\quad \rho_0(x) =$ mass density per unit length of string in the reference config.



$\vec G$ = force of one part of string on remaining part

Newton's 2^{nd} law on segment w/ $a \le x \le b$

rate of change of momentum = force $\qquad$ external force $\vec f(x,t)$ per unit length

$$\int_a^b \rho_0(x)\,\vec r_{tt}(x,t)\,dx = \vec G(b,t) - \vec G(a,t) + \int_a^b \vec f(x,t)\,dx$$

$$\int_a^b \{\rho_0 \vec r_{tt} - \vec G_x - \vec f\}\,dx = 0 \quad \text{w/ } \vec G(b,t) - \vec G(a,t) = \int_a^b \vec G_x\,dx$$

Since this holds $\forall$ intervals $[a,b]$, we have (assuming integrand conts)

$$\boxed{\rho_0 \vec r_{tt} = \vec G_x + \vec f} \qquad \text{Balance of momentum}$$

Add a constitutive relation for $\vec G$ as a function of $\vec r$

For elastic string (force tangent to stretch) $\quad \vec G = T(|\vec r_x|)\dfrac{\vec r_x}{|\vec r_x|}$

hence

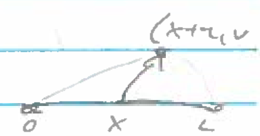
$$\rho_0 \vec r_{tt} = \left(T(|\vec r_x|)\frac{\vec r_x}{|\vec r_x|}\right)_x + \vec f$$

Linearize for small displacements & restrict to planar motion of the string

Reference config.

$\vec r(x,t) = (x,0) \leftarrow$ x-y coordinates

Deformed config. $\quad \vec r(x,t) = (x+u(x,t),\ v(x,t))$



where (u,v) is displacement from reference config.

Assume u, v & their derivatives are small

$$\vec r_x = (1+u_x,\ v_x) \qquad |\vec r_x|^2 = (1+u_x)^2 + v_x^2 \approx (1+u_x)^2$$

$$\Rightarrow |\vec r_x| \approx 1+u_x \quad \rightarrow \quad \frac{\vec r_x}{|\vec r_x|} \approx \left(1, \frac{v_x}{1+u_x}\right) \approx (1, v_x)$$

$$T(|\vec r_x|) \approx T(1+u_x) = T(1) + T'(1)u_x + O(u_x^2) \approx T_0 + T_0' u_x$$

$$\Rightarrow \rho_0(u_{tt}, v_{tt}) = \{(T_0 + T_0' u_x)(1, v_x)\}_x + (f, g)$$

So up to linear terms, $\begin{cases} \rho_0 u_{tt} = T_0' u_{xx} + f & \leftarrow \text{longitudinal vibrations} \\ \rho_0 v_{tt} = T_0 v_{xx} + g & \leftarrow \text{transverse vibrations} \end{cases}$

In either case, (neglecting external force) we get a wave eqn.

$$u_{tt} = c_0^2 u_{xx} \qquad \Big(\underbrace{c_0^2 = \frac{T_0'}{\rho_0}}_{\text{longitudinal}} \text{ or } \underbrace{c_0^2 = T_0/\rho_0}_{\text{transverse}} \Big)$$

$[c_0] = L/T$ (velocity)

Check: $[T_0] = \text{Force} = \frac{ML}{T^2}$ $\Rightarrow \left[\frac{T_0}{\rho_0}\right] = \frac{L^2}{T^2}$

$[\rho_0] = \text{density} = M/L$

Transverse Vibrations of a plucked string:

PDE $u_{tt} = c_0^2 u_{xx}$ $\quad 0 < x < L \quad t > 0$

BC $u(0,t) = u(L,t) = 0$

IC $u(x,0) = f(x)$, $u_t(x,0) = g(x)$

Solve by separation of variables:

$u(x,t) = F(x)G(t) \Rightarrow FG'' = c_0^2 F'' G$

$\Rightarrow \frac{F''}{F} = \frac{G''}{G c_0^2} = -\lambda$

SLEVP: $\begin{cases} -F'' = \lambda F \\ F(0) = F(L) = 0 \end{cases} \Rightarrow F_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, \dots$

$\lambda_n = n^2\pi^2/L^2 = k_n^2$ w/ $k_n = n\pi/L$

$-G'' = \lambda c_0^2 G = k^2 c_0^2 G$

$\Rightarrow G(t) = a\cos(k c_0 t) + b\sin(k c_0 t)$

Sep. solutions:

$$u_n(x,t) = \begin{cases} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c_0}{L} t\right) \\ \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c_0}{L} t\right) \end{cases}$$

wavenumber $\frac{n\pi}{L}$ $\qquad$ frequency $\frac{n\pi c_0}{L}$

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Adjoints

$A =$ differential operator acting in $L^2(a,b)$

$$A = \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k}$$

boundary conditions $Bu = 0$ (at $x = a, b$)

$$\langle u, Av\rangle - \langle A^* u, v\rangle = [\partial(u,v)]_a^b$$

$$A^* = \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} a_k(x) \quad \text{formal adjoint of } A$$

A is formally self-adjoint if $A = A^*$

Ex: $A = -\frac{d}{dx} p(x) \frac{d}{dx} + q(x) = -p(x)\frac{d^2}{dx^2} - p'(x)\frac{d}{dx} + q(x)$

$A^* = A$ formally self adjoint

$$\langle u, Av\rangle - \langle Au, v\rangle = \int_a^b u\{-(pv')' + qv\} - v\{-(pu')' + qu\}\, dx$$
$$= -\int_a^b u(pv')' - v(pu')'\, dx = -\int_a^b \{puv' - pvu'\}'\, dx$$
$$= -[p(uv' - vu')]_a^b = [\partial(u,v)]_a^b$$

Include BC's: $A : D(A) \subset L^2(a,b) \to L^2(a,b)$

$D(A) = \{v \in L^2(a,b) : Av \in L^2(a,b),\ Bv = 0\}$

Define adjoint BC's $B^* u = 0 \iff [\partial(u,v)]_a^b = 0$
$\forall v$ s.t. $Bv = 0$

then $\langle u, Av\rangle = \langle A^* u, v\rangle \quad \forall u \in D(A^*),\ v \in D(A)$

Ex: $A = -\frac{d}{dx} p \frac{d}{dx} + q$, $Bv = \begin{pmatrix} v(a) \\ v(b) \end{pmatrix}$ Dirichlet BC's $Bv = 0$

$\partial(u,v) = [puv']_a^b = p(b)u(b)v'(b) - p(a)u(a)v'(a)$

conclude that $B^* u = \begin{pmatrix} u(a) \\ u(b) \end{pmatrix}$ (assuming $p(a), p(b) \neq 0$, regular Sturm-Liouville op)

Then A is self-adjoint.

In general, A is self-adjoint if $A = A^*$ (formally self-adjoint) and $D(A) = D(A^*)$ (i.e., $B = B^*$)

Ex: Sturm Liouville - A $\quad Bv = \begin{pmatrix} v(a) \\ v'(a) \\ v(b) \\ v'(b) \end{pmatrix}$, $\quad B^* u = 0$ (i.e. $B^* = 0$)

Then even though A is formally self-adjoint, A is not self-adjoint.

Other BC's.

self-adjoint: (separated BC's) $\begin{cases} \alpha_0 u(a) + \alpha_1 u'(a) = 0 & \alpha_0^2 + \alpha_1^2 \neq 0 \\ \beta_0 u(b) + \beta_1 u'(b) = 0 & \beta_0^2 + \beta_1^2 \neq 0 \end{cases}$

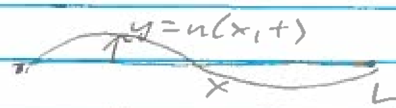
(periodic) $\begin{cases} u(a) = u(b) \\ u'(a) = u'(b) \end{cases}$

Not self-adjoint $\quad$ initial conditions $\quad\quad$ final conditions

$$\begin{cases} u(a) = 0 \\ u'(a) = 0 \end{cases} \to Bu = \begin{pmatrix} u(a) \\ u'(a) \end{pmatrix} \quad \to B^* u = \begin{pmatrix} u(b) \\ u'(b) \end{pmatrix}$$

Self-adjoint operators have real eigenvalues & eigenvectors/functions. For the regular Sturm-Liouville operators, self-adjointedness ensures a complete set of orthogonal eigenfunctions.

Wave Equation IBVP



$c_0^2 = \frac{T}{\rho_0}$

$$\begin{cases} u_{tt} = c_0^2 u_{xx} \\ u(0,t) = 0, \; u(L,t) = 0 \\ u(x,0) = f(x), \; u_t(x,0) = g(x) \end{cases}$$

Separated solutions $\quad$ eigenfunctions

$$u(x,t) = \begin{cases} \cos\left(\frac{\pi n c_0 t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\ \sin\left(\frac{\pi n c_0 t}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \end{cases}$$

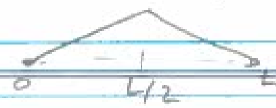
General soln of PDEs + BCs is

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi c_0 t}{L}\right) + b_n \sin\left(\frac{n\pi c_0 t}{L}\right) \right\} \sin \frac{n\pi x}{L}$$

To satisfy ICs: $f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \to a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

$$g(x) = \sum_{n=1}^{\infty} \left(\frac{n\pi c_0}{L}\right) b_n \sin\left(\frac{n\pi x}{L}\right) \to b_n = \frac{2}{n\pi c_0} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

plucked string



Ex: $f(x) = \begin{cases} 2x/L & 0 \le x \le L/2 \\ \frac{2(L-x)}{L} & L/2 < x \le L \end{cases}$

$g(x) = 0$

$\Rightarrow b_n = 0, \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$

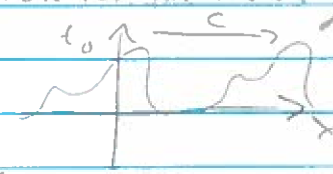
$\Rightarrow u(x,t) = \frac{8}{\pi^2}\Big\{ \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi c_0 t}{L}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi c_0 t}{L}\right)$
$+ \frac{1}{5^2}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{\pi c_0 t}{L}\right) - \dots \Big\}$

No smoothing in time, all modes keep oscillating in time, no damping

D'Alembert's Solution

$$\begin{cases} u_{tt} = c_0^2 u_{xx} & -\infty < x < \infty,\ t > 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Look for the traveling wave solutions: $u(x,t) = F(x-ct)$



$u_{tt} = (-c)^2 F''$

$u_{xx} = F''$

$\Rightarrow u_{tt} = c_0^2 u_{xx}$ if $c^2 = c_0^2 \Rightarrow c = \pm$

Superposing traveling waves

$u(x,t) = F(x - c_0 t) + G(x + c_0 t)$

To satisfy IC:

$F(x) + G(x) = f(x)$

$-c_0 F'(x) + c_0 G'(x) = g(x)$

$$\begin{cases} -F(x) + G(x) = \frac{1}{c_0}\int_{x_0}^{x} g(\xi)\,d\xi \\ F(x) + G(x) = f(x) \end{cases}$$

$\Rightarrow F(x) = \frac{1}{2}f(x) - \frac{1}{2c_0}\int_{x_0}^{x} g(\xi)\,d\xi$

$G(x) = \frac{1}{2}f(x) + \frac{1}{2c_0}\int_{x_0}^{x} g(\xi)\,d\xi$

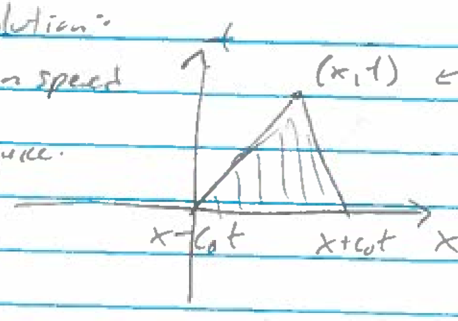
$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[f(x - c_0 t) + f(x + c_0 t)\right] + \frac{1}{2c_0}\int_{x-c_0 t}^{x+c_0 t} g(\xi)\,d\xi}$$

D'Alembert's solution

Three features of this solution:

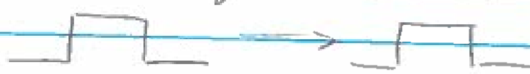
1) finite propagation speed

Similarly, domain of dependence.



← Solution $u(x,t)$ depends only on data in this triangle

2) No smoothing



3) Reversible in time

Schrödinger eqn: $i\hbar \psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi$

describes a (nonrelativistic) quantum particle moving in a potential $V(x)$.

Classical analog: $m\frac{d^2x}{dt^2} = -\nabla V(x)$

$\psi(x,t)$ complex-valued $\rightarrow$ $\int_{\mathbb{R}^n} |\psi|^2 dx = 1$

$\Rightarrow$ $\int_\Omega |\psi|^2 dx$ = probability of observing particle in Ω at time t.

$\hbar$ = Planck's constant

Dimensions: $[V] = \frac{ML^2}{T^2}$ (energy)

$[\hbar] = \frac{ML^2}{T}$ = energy·time = momentum·space = action

Look for separated solutions:

$\psi(x,t) = u(x)e^{-iEt/\hbar}$

$-\frac{\hbar^2}{2m}\Delta u + V(x)u = Eu(x)$

$Hu = Eu$ w/ $H = \underbrace{-\frac{\hbar^2}{2m}\Delta + V(x)}_{\text{Hamiltonian operator}}$

Quantum harmonic oscillator:

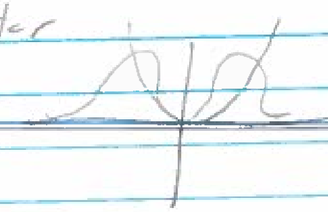
$-\infty < x < \infty$ $V(x) = \frac{1}{2}kx^2$

In classical $m\frac{d^2x}{dt^2} = -kx$

$x = A\cos\omega t + B\sin\omega t$, $\omega^2 = k/m$

Quantum harmonic oscillator

$-\infty < x < \infty \qquad V(x) = \frac{1}{2}kx^2$



$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + \frac{1}{2}kx^2u = Eu$$

$u \in L^2(\mathbb{R})$. This is a Sturm-Liouville operator w/ variable coefficients but it's singular since over $\mathbb{R}$ not a finite interval.

Nondimensionalized problem: $T = 2\sqrt{\frac{m}{k}}$ since $[m] = M$, $[k] = \frac{M}{T}$

$L^2 = \hbar/\sqrt{mk}$ since $[\hbar] = ML^2/T^2$

$$\Rightarrow \boxed{\begin{array}{c} -u'' + x^2u = \lambda u \\ u \in L^2(\mathbb{R}) \end{array}} \qquad \omega = \_\_ \cdot E$$

← solve this by "Operator methods"

$$Hu = \lambda u, \quad H = -\frac{d^2}{dx^2} + x^2$$

$a = \left(-i\frac{d}{dx} - ix\right)\frac{1}{\sqrt{2}}$ annihilation operator

$a^* = \frac{1}{\sqrt{2}}\left(-i\frac{d}{dx} + ix\right)$ creation operator.

] ladder operators

Note $[A, B] = AB - BA$

$$\left[-i\frac{d}{dx}, x\right] = -i\frac{d}{dx}x + ix\frac{d}{dx}$$

$$\frac{d}{dx}(xu) = \frac{d}{dx}x \cdot u + x\frac{du}{dx}$$

$$= -i\left(x\frac{d}{dx} + 1\right) + ix\frac{d}{dx}$$

$$= -i$$

$$2[a, a^*] = aa^* - a^*a = +i^2\left[\frac{d}{dx} + x\right]\left[\frac{d}{dx} - x\right] - i^2\left[\frac{d}{dx} - x\right]\left[\frac{d}{dx}\right.$$

$$= -\frac{d^2}{dx^2} - x\frac{d}{dx} + \frac{d}{dx}x + x^2 + \frac{d^2}{dx^2} - x\frac{d}{dx} + \frac{d}{dx}x - x^2$$

$$= 2\left[\frac{d}{dx}x - x\frac{d}{dx}\right] = 2\left[\frac{d}{dx}, x\right] = 2$$

$$\left[\frac{d}{dx}, x\right] = \frac{d}{dx}x - x\frac{d}{dx} = x\frac{d}{dx} + 1 - x\frac{d}{dx} = 1$$

$$\Rightarrow a = \frac{-i}{\sqrt{2}}\left[\frac{d}{dx} + x\right]$$

$$a^* = \frac{-i}{\sqrt{2}}\left[\frac{d}{dx} - x\right]$$

$$[a, a^*] = 1$$

$$H = -\frac{d^2}{dx^2} + x^2 = 2a^*a + 1$$

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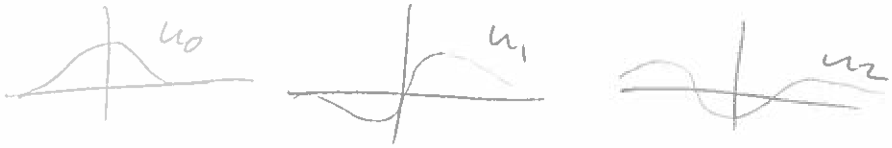
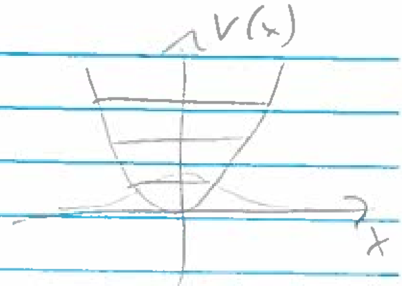




Then $[H, a^*] = (2a^*a+1)a^* - a^*(2a^*a+1)$

$= 2a^*aa^* + \cancel{a^*} - 2(a^*)^2 a - \cancel{a^*}$

$= 2a^*(a^*a + [a,a^*]) - 2(a^*)^2 a$

$= 2a^*[a,a^*] = 2a^*$

$$\begin{cases} -u'' + x^2 u = \lambda u & -\infty < x < \infty \\ u \in L^2(\mathbb{R}) & , \; Hu = \lambda u \end{cases}$$

1) $au = 0 \Rightarrow u' + xu = 0$ (only obvious choice to get an eigenfunction for H)

$e^{\frac{1}{2}x^2}u' + xe^{\frac{1}{2}x^2}u = 0$

$(e^{\frac{1}{2}x^2}u)' = 0$

$u = ce^{-x^2/2} \Rightarrow u_0(x) = e^{-x^2/2}$

$Hu_0 = (2a^*a+1)u_0 = 2a^*(au_0) + u_0 = u_0.$

$\Rightarrow Hu_0 = 1 \cdot u_0$, u_0 is an eigenfn. w/ eigenvalue $\lambda = 1$.

2) $u_1 = a^* u_0$

$a^*a u_1 = a^*aa^*u_0 = a^*([a,a^*] + a^*\underbrace{a)u_0}_{0}$

$= a^*u_0 = u_1$

$\Rightarrow Hu_1 = (2a^*a+1)u_1 = 2u_1 + u_1 = 3u_1$

u_1 is an eigenfn. w/ eigenvalue $\lambda = 3$.

Then $Hu_n = \lambda_n u_n$ $\qquad a^*a\, u_n = \frac{1}{2}(\lambda_n - 1)u_n$

$u_{n+1} = a^* u_n$

$Hu_{n+1} = 2a^*a\,u_{n+1} + u_{n+1} = 2a^*aa^*u_n + u_{n+1}$

$= 2a^*([a,a^*] + a^*a)u_n + u_{n+1}$

$= 2a^*(1 + \frac{1}{2}(\lambda_n - 1))u_n + u_{n+1}$

$= (2 + \lambda_n - 1 + 1)u_{n+1} = (2 + \lambda_n)u_{n+1}$

Hence $\lambda_n = 2n+1$

& $u_n(x) = (a^*)^n u_0 \propto \left(-\frac{d}{dx} + x\right)^n e^{-\frac{1}{2}x^2}$

$u_n(x) = \frac{1}{\sqrt{2^n n!}} \frac{1}{\pi^{1/4}} \underbrace{H_n(x)}_{\text{Hermite polys.}} e^{-x^2/2}$ $\qquad$ (note $\int_{-\infty}^{\infty} |u_n|^2 dx = 1$)

$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})$

Hermite polys. are eigenfunctions of Fourier transform.

Rmk: $\int_{-\infty}^{\infty} u_n u_m \, dx = \delta_{n,m}$

or $\int_{-\infty}^{\infty} H_n(x) H_m(x) e^{-x^2} dx = 0 \quad n \neq m$

H_n orthogonal wrt. weight e^{-x^2}

This is a singular Sturm-Louiville problem that behaves regularly in that it has inf. # eigenvalues & complete set of eigenfns. that form an orthonormal basis of $L^2(\mathbb{R})$

<u>Free Quantum Particle</u>

$V(x) = 0$

$H = -\frac{d^2}{dx^2} \qquad Hu = \lambda u \Rightarrow \begin{cases} -u'' = \lambda u, & -\infty < x < \infty \\ u \in L^2(\mathbb{R}) \end{cases}$

Look for eigenfns. $u \in L^2(\mathbb{R})$

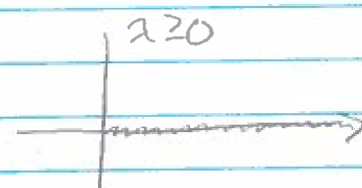
$\lambda = k^2 > 0, \quad -u'' = k^2 u$

$u(x) = Ae^{+ikx} + Be^{-ikx} \qquad u \notin L^2(\mathbb{R})$

No eigenfunctions!

Have continuous spectrum

$\sigma(-d^2/dx^2) = [0, \infty)$



Instead of eigenfunction expansion, get integral transform

$u(x) = \int_{-\infty}^{\infty} \hat{u}(k) e^{ikx} dk.$

<u>ODE'S</u>

$A = p_0 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_2$

1) Initial value problems (IVPs)

$\begin{cases} p_0 u'' + p_1 u' + p_2 u = f(x) \\ u(x_0) = u_0, \quad u'(x_0) = u_0' \end{cases}$ always a unique solution

2) Boundary value problems (BVPs)

$\begin{cases} p_0 u'' + p_1 u' + p_2 u = f(x) \\ Bu = 0 \quad \text{e.g. } u(a) = 0, u(b) = 0 \end{cases}$ not always a unique solution or even a solution

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3) Eigenvalue problems (EVPs)

$$\begin{cases} p_0 u'' + p_1 u' + p_2 u = \lambda u \\ Bu = 0 \qquad \text{on } a < x < b \end{cases}$$

Initial Value Problems:

Assume $p_0(x) \neq 0$ in $|x - x_0| \leq a$ (no dropping of order in ODE - singularity)

Take $p_0 = 1$ (o/w get singular points)

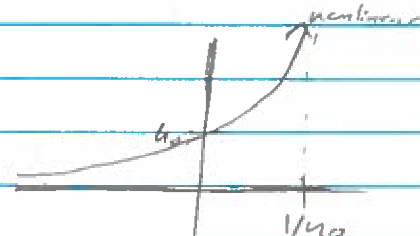
Thm: If p_1, p_2, f are cont's real-valued fns in $|x - x_0| \leq a$, then $\exists!\, u(x) \in C^2$ in $|x - x_0| \leq a$ solution of IVP: $u'' + p_1 u' + p_2 u = f$

$\forall u_0, u_0' \in \mathbb{R}$ $\qquad u(x_0) = u_0, \quad u'(x_0) = u_0'$

Note: Consider nonlinear IVP: $\begin{cases} u_t = u^2 \\ u(0) = u_0 \end{cases}$

$$\int \frac{du}{u^2} = \int dt \Rightarrow -\frac{1}{u} = t - \frac{1}{u_0}$$

$$\Rightarrow u(t) = \frac{1}{1/u_0 - t}$$



solution "blows up" in finite time

Compare w/ $\begin{cases} u_t = u \\ u(0) = u_0 \end{cases} \Rightarrow u(t) = u_0 e^t$

blows up in infinite time.

Sketch of proof of thm: $\begin{cases} u' = v \\ v' = -p_1 v - p_2 u + f \end{cases}$

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -p_2 & -p_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix} = F(u, v)$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} + \int_{x_0}^{x} F(u, v)\, dx \qquad |x - x_0| \leq a$$

Apply contraction mapping (or Picard iteration)

Then $\begin{cases} Au = f \\ u(x_0) = u_0 \\ u'(x_0) = u_0' \end{cases}$ so by superposition

$$u(x) = u_p(x) + u_0 u_1(x) + u_0' u_2(x)$$

w/ $\begin{cases} Au_p = f \\ u_p(x_0) = 0 \\ u_p'(x_0) = 0 \end{cases}$ and $\begin{cases} Au_1 = 0 \\ u_1(x_0) = 1 \\ u_1'(x_0) = 0 \end{cases}$ and $\begin{cases} Au_2 = 0 \\ u_2(x_0) = 0 \\ u_2'(x_0) = 1 \end{cases}$

$Au = Au_p + u_0 Au_1 + u_0' Au_2 = f + 0 + 0$

$u(x_0) = 0 + u_0 + 0 \qquad u'(x_0) = 0 + 0 + u_0'$

Wronskian

$Au_1 = 0 \quad Au_2 = 0$

Def: $W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_2 u_1'$

$\{u_1, u_2\}$ fundamental pair of solutions if $W \neq 0$.
↳ can sum to match any IC of IVP.

We have that $W' + p_1 W = 0 \Rightarrow W(x) = W(x_0) \exp\left[-\int_{x_0}^{x} p_1(t)\,dt\right]$

so Wronskian is either nonzero $\forall\, |x - x_0| \leq a$ or not.

If $Au = -(pu')' + qu$ is Sturm-Liouville op. $(p \neq 0)$

$Au = -pu'' - p'u' + qu$

$\Rightarrow p_1 = p'/p$, $W = W_0 \exp\left[-\int_{x_0}^{x} p'/p \, dx\right]$

$W(x) = W_0 / p(x)$

$\Rightarrow p(x) W(x) = W_0$.

Boundary Value Problems:

$Au = f \qquad a \leq x \leq b$

$Bu = 0 \quad$ at $x = a, b$ e.g. $u(a) = 0$, $u(b) = 0$

In contrast to IVP, we may have no soln. or we may have multiple solns

Ex: $-u'' = f(x) \quad 0 < x < 1$

$u'(0) = 0$, $u'(1) = 0$

$\Rightarrow -\int_0^1 u'' \, dx = \int_0^1 f(x)$

$-[u'(1) - u'(0)] = \int_0^1 f(x)$

$\Rightarrow \int_0^1 f \, dx = 0$.

If $\int_0^1 f\,dx = 0$ and u is a soln., then $u(x) + C$ is also a solution $\forall C \in \mathbb{R}$, hence solutions nonunique.

Note that $A = -\frac{d^2}{dx^2}$ + Neumann BC'S is singular!

$\varphi(x) = 1$, $\varphi'(0) = \varphi'(1) = 0$ $\qquad (\lambda = 0$ eigenvalue$)$

$A\varphi = 0$

Solvability condition $\langle \varphi, f \rangle = \int_0^1 f(x)\,dx = 0$

If a solution u exists, then $u + C\varphi$ is also a soln.

Ex: $-u'' - \pi^2 u = f(x) \quad 0 < x < 1$
$u(0) = 0, \ u(1) = 0$

Solvable only if $\langle \sin \pi x, f(x) \rangle = 0$

And in that case, $u = u_p(x) + C \sin \pi x$.

(If BVP is nonsingular, will have unique solns.)?

Green's Functions:

Linear problem: $Au = f$ (+ BC's)

Introduce δ-function: $\delta(x) = 0, \ x \neq 0$ and $\int_{-\varepsilon}^{\varepsilon} \delta(x) = 1 \ \forall \varepsilon > 0$.

$\int \delta(x - \xi) f(\xi) \, d\xi = f(x)$

convolution w/ δ-fn is the identity operator

Define Green's fn. $G(x, \xi)$ as solution of

$$A\left(\frac{d}{dx}\right) G(x, \xi) = \delta(x - \xi) \qquad \begin{matrix} a < x < b \\ a < \xi < b \end{matrix}$$

$G(x, \xi)$ = response of solution to a point source located at ξ.

$$u(x) = \int_a^b G(x, \xi) f(\xi) \, d\xi$$

$$A\left(\frac{d}{dx}\right) u = \int_a^b A\left(\frac{d}{dx}\right) G(x, \xi) f(\xi) \, d\xi = \int_a^b \delta(x - \xi) f(\xi) \, d\xi = f(x).$$

Ex: $\begin{cases} -u'' = f(x) & 0 < x < 1 \\ u(0) = 0, \ u(1) = 0 \end{cases}$

$Au = f$, $\quad A = -\frac{d^2}{dx^2}$, $\quad D(A) = \{ u \in H^2(0,1) : u(0) = u(1) = 0 \}$

Define Green's function $G(x, \xi)$ $\quad 0 \le x, \xi \le 1$

as solution of $\begin{cases} -G_{xx} = \delta(x - \xi) & 0 < x < 1 \\ G(0, \xi) = 0, \ G(1, \xi) = 0 \end{cases}$

Satisfies:

(a) $G_{xx} = 0 \quad \begin{cases} 0 < x < \xi \\ \xi < x < 1 \end{cases}$ + BC's

(b) $G(\xi^+, \xi) = G(\xi^-, \xi)$ ← limits from left & right at ξ agree, i.e. G cont's at ξ

(c) $[-G_x]_{x=\xi} = -[G_x(\xi^+, \xi) - G_x(\xi^-, \xi)] = 1$ (G_x is step-fn. w/ step at ξ)

$G_{xx} = 0 \implies G(x, \xi) = A(\xi) x + B(\xi)$

Ex. w/ Green's fns.

a) $G_{xx} = 0 \Rightarrow G(x,\xi) = A(\xi)x + B(\xi)$

(i) $0 \le x < \xi$: $G(0,\xi) = 0 \Rightarrow B(\xi) = 0$

$\Rightarrow G(x,\xi) = A(\xi).x$

(ii) $\xi < x \le 1$: $G(1,\xi) = 0 \Rightarrow A(\xi) = -B(\xi)$

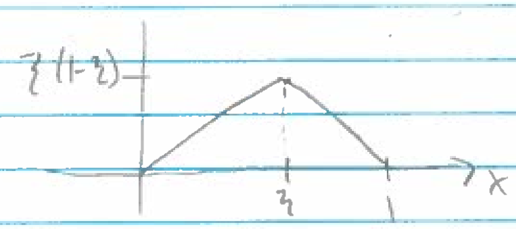
$\Rightarrow G(x,\xi) = B(\xi)(1-x)$

b) $\xi A(\xi) = (1-\xi) B(\xi)$

c) $-[-B - A] = 1 \Rightarrow A(\xi) + B(\xi) = 1$] solve this 2×2 system

$\Rightarrow A(\xi) = 1-\xi, \quad B(\xi) = \xi.$

So $G(x,\xi) = \begin{cases} (1-\xi)x & , 0 \le x \le \xi \\ \xi(1-x) & , \xi < x < 1 \end{cases}$



Physical Interpretation:

Ex: Steady heat flow. $\begin{cases} u_t = u_{xx} + f(x) \\ u(0,t) = u(1,t) = 0 \end{cases}$

Steady soln. $u = u(x) \quad (u_t = 0)$

$$\begin{cases} -u'' = f(x) \\ u(0) = u(1) = 0 \end{cases}$$

$f(x) = \delta(x-\xi)$

$G(x,\xi)$ = temperature at x due to point source at ξ.

Ex: Force on a String $\begin{cases} u_{tt} = u_{xx} + f(x) \\ u(0,t) = u(1,t) = 0 \end{cases}$

Steady soln same idea

$G(x,\xi)$ = displacement of string due to point force at ξ.

(literally image of a plucked string)

Green's identity for self-adjoint $A = -\frac{d^2}{dx^2}$

$$\int_0^1 (uAv - vAu)\,dx = 0$$

Let $v(x) = G(x,\xi)$, $Au = f$

$$\hookrightarrow \int_0^1 \{u(x)A(\tfrac{d}{dx})G(x,\xi) - G(x,\xi)A(\tfrac{d}{dx})u(x)\}\,dx = 0$$

$$\Rightarrow \int_0^1 \{u(x)\delta(x-\xi) - G(x,\xi)f(x)\}\,dx = 0$$

$$\Rightarrow \quad u(\xi) = \int_0^1 G(x,\xi)f(x)\,dx$$

$$\Rightarrow \quad u(x) = \int_0^1 G(\xi,x)f(\xi)\,d\xi$$

Note $G(x,\xi) = G(\xi,x)$ is symmetric (self-adjoint)

since $G(x,\xi) = (1-x_>)x_<$

w/ $x_> = \max\{x,\xi\}$, $x_< = \min\{x,\xi\}$

$$\Rightarrow \boxed{u(x) = \int_0^1 G(x,\xi)f(\xi)\,d\xi .}$$

$G(x,\xi) = G(\xi,x)$ reciprocity condition

response at x due to point source at ξ = response at ξ due to point source at x.

Verify this solution holds:

$$u(x) = \int_0^x (1-x)\xi f(\xi)\,d\xi + \int_x^1 x(1-\xi)f(\xi)\,d\xi$$

$$= (1-x)\int_0^x \xi f(\xi)\,d\xi + x\int_x^1 (1-\xi)f(\xi)\,d\xi$$

$u(0) = u(1) = 0$ satisfies BCs ✓

$$u'(x) = -\int_0^x \xi f(\xi)\,d\xi + (1-x)xf(x) + \int_x^1 (1-\xi)f(\xi)\,d\xi - x(1-x)f(x)$$

$$= \int_x^1 f(\xi)\,d\xi - \int_0^1 \xi f(\xi)\,d\xi$$

$u''(x) = -f(x) \Rightarrow -u'' = f$ ✓

What have we done?

$A : D(A) \subset L^2([0,1]) \to L^2([0,1])$ $\qquad Gf(x) = \int_0^1 G(x,\xi)f(\xi)\,d\xi$

$G : L^2([0,1]) \to D(A) \subseteq L^2([0,1])$ ← integral operator whose kernel is $G(x,\xi)$

$GA : D(A) \to D(A)$, $AG : L^2([0,1]) \to L^2([0,1])$ are identity maps

Inverse of differential operator A is a compact integral operator G.

G : $Gf(x) = \int_0^1 G(x,\xi) f(\xi)\, d\xi$

G maps bounded sets in L^2 to bounded sets in H^2 (which are precompact in L^2)

EVP $Au = \lambda u$, $u \in D(A)$, $\lambda \neq 0$

$GAu = \lambda Gu$

$\frac{1}{\lambda} u = Gu \Rightarrow \mu u = Gu$, $\mu = 1/\lambda$

Spectral theory of compact operators

$\Rightarrow$ ON basis of eigenfunctions of G

$\{\varphi_n : n \in \mathbb{N}\}$ w/ eigenvalues $\{\mu_n \in \mathbb{R} : n \in \mathbb{N}\}$ s.t. $\mu_n \xrightarrow{n\to\infty} 0$

(hence $\lambda_n \to \infty$ as $n \to \infty$)

2/26 A general Sturm-Liouville Problem:

$$-(pu')' + qu = f \qquad a < x < b$$

Separated BC's: $\cos\alpha\, u'(a) + \sin\alpha\, u(a) = 0$

$\cos\beta\, u'(b) + \sin\beta\, u(b) = 0$

Regular SL eqn:

$p, p', q, f : [a,b] \to \mathbb{R}$ cont's

$p(x) > 0 \quad a \le x \le b$ to prevent degeneration of order

Define $G(x,\xi)$ $(G : [a,b] \times [a,b] \to \mathbb{R})$

Solution of:

$$\begin{cases} -(pG_x)_x + qG = \delta(x-\xi) & a < x < b \\ + \text{BC's at } x=a,\ x=b & (\text{same as above but with } G, G_x) \end{cases}$$

G satisfies:

1) $-(pG_x)_x + qG = 0 \quad a < x < \xi \ \& \ \xi < x < b$

2) Satisfy BC at $x=a$, $x=b$

3) $G(x,\xi)$ cont's at $x=\xi$

4) $-[pG_x]_{x=\xi^-}^{x=\xi^+} = 1$

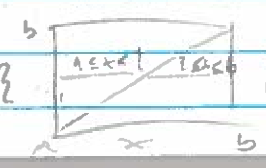
1) Let $y_1(x)$ be soln of IVP

$$\begin{cases} -(py_1')' + qy_1 = 0 \\ y_1(a) = \cos\alpha \\ y_1'(a) = -\sin\alpha \end{cases}$$

Let $y_2(x)$ be soln. of Final VP

$$\begin{cases} -(py_2')' + qy_2 = 0 \\ y_2(b) = \cos\beta \\ y_2'(b) = -\sin\beta \end{cases}$$

$$\Rightarrow G(x,\xi) = \begin{cases} A(\xi) y_1(x) & a \le x < \xi \\ B(\xi) y_2(x) & \xi < x \le b \end{cases}$$



$$G(x,\xi) = \begin{cases} A(\xi)\, y_1(x), & a \le x < \xi \\ B(\xi)\, y_2(x), & \xi < x \le b \end{cases}$$ satisfies (2)

3) $G(x,\xi^+) = G(x,\xi^-)$

$B(\xi)\,y_2(\xi) = A(\xi)\,y_1(\xi) \implies A(\xi) = y_2(\xi)\cdot C(\xi)$
$B(\xi) = y_1(\xi)\cdot C(\xi)$

$$\implies G(x,\xi) = \begin{cases} C(\xi)\, y_2(\xi)\, y_1(x), & a \le x < \xi \\ C(\xi)\, y_1(\xi)\, y_2(x), & \xi < x \le b \end{cases}$$

4) $-[pG_x]_{\xi^-}^{\xi^+} = -\left[p(\xi)C(\xi)y_1(\xi)y_2'(\xi) - p(\xi)C(\xi)y_2(\xi)y_1'(\xi)\right]$

$= -C(\xi)\,p(\xi)\,[\underbrace{y_1(\xi)y_2'(\xi) - y_2(\xi)y_1'(\xi)}_{W(\xi) = \text{Wronskian of } y_1, y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}] = 1$

Want to choose $C(\xi)$ s.t. $p(\xi)\,W(\xi)\,C(\xi) = -1$

$$\implies C(\xi) = \frac{-1}{p(\xi)W(\xi)}$$

If $W(\xi) = 0$, then y_1 & y_2 are linearly dependent and y_1 is a singular soln. of the original homogeneous BVP.
Assuming $W(\xi) \neq 0$, which is the case iff $\nexists$ no nonzero soln of the homogeneous BVP, $C(\xi)$ is well-defined.

For SL equation, $pW = $ constant, so $C = \frac{-1}{pW}$ is constant.
hence
$$G(x,\xi) = \begin{cases} C\, y_1(x)\, y_2(\xi), & a \le x < \xi \\ C\, y_1(\xi)\, y_2(x), & \xi < x \le b \end{cases}$$

$x_< = \min(x,\xi)$
$x_> = \max(x,\xi)$

$$= C\, y_1(x_<)\, y_2(x_>)$$

Then solution $u(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi$ (integral representation)

Note that $G(x,\xi) = G(\xi,x)$ is symmetric.
Green's operator is self-adjoint (in $L^2([a,b])$)

Consider $k: [a,b] \times [a,b] \to \mathbb{C}$ sq.-cont's w/ $\int_a^b \int_a^b |k(x,\xi)|^2\, dx\, d\xi < \infty$ Hilbert-Schmidt ?

$Kf(x) = \int_a^b k(x,\xi) f(\xi)\, d\xi \quad : L^2([a,b])$?

$\langle f, Kg\rangle = \int_a^b \overline{f(x)}\, Kg(x)\, dx = \int_a^b \int_a^b \overline{f(x)}\, k(x,\xi)\, g(\xi)\, d\xi\, dx$

$= \int_a^b \left(\int_a^b \overline{\overline{k(x,\xi)} f(x)\, dx}\right) g(\xi)\, d\xi = \langle K^* f, g\rangle$

w/ $(K^* f)(x) = \int_a^b \overline{k(\xi,x)}\, f(\xi)\, d\xi$. Adjoint of K is another integral op. w/ conjugate transpose kernel.

In particular, $K = K^*$ is self-adjoint if $k(x,\xi) = \overline{k(\xi,x)}$

~~For k real-valued, need $k(x,\xi) = k(\xi,x)$~~

So Green's operator for SL problem is self-adjoint

so $A = A^* \Rightarrow (A^{-1})^* = (A^*)^{-1} = A^{-1}$.

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Green's fns:

Consider regular S-L BVP:

$$\begin{cases} -(pu')' + qu = f(x) & a < x < b \\ \cos\alpha\, u(a) + \sin\alpha\, u'(a) = 0 \\ \cos\beta\, u(b) + \sin\beta\, u'(b) = 0 \end{cases}$$

If there is no non-zero soln of the homogeneous problem ($f=0$) then have a unique solution

$$u(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi \quad w/ \quad G(x,\xi) = \begin{cases} \frac{1}{C} u_1(x) u_2(\xi) & x \le \xi \\ \frac{1}{C} u_1(\xi) u_2(x) & x \ge \xi \end{cases}$$

Nonseparated BC's:

Consider e.g. periodic BC's: $u(a) = u(b)$, $u'(a) = u'(b)$.

Suppose $\{u_1, u_2\}$ fundamental pair of solutions of homogeneous ODE

$$-(pu')' + qu = 0 \qquad \text{(without BC's)}$$

Then $G(x,\xi) = \begin{cases} A_1(\xi) u_1(x) + A_2(\xi) u_2(x) & a \le x \le \xi \\ B_1(\xi) u_1(x) + B_2(\xi) u_2(x) & \xi \le x \le b \end{cases}$

Need

1) $A_1(\xi)u_1(a) + A_2(\xi)u_2(a) = B_1(\xi)u_1(b) + B_2(\xi)u_2(b)$ — BC's

2) $A_1(\xi)u_1'(a) + A_2(\xi)u_2'(a) = B_1(\xi)u_1'(b) + B_2(\xi)u_2'(b)$

3) $A_1(\xi)u_1(\xi) + A_2(\xi)u_2(\xi) = B_1(\xi)u_1(\xi) + B_2(\xi)u_2(\xi)$ continu[ity]

4) $-[pG_x]_{x=\xi^-}^{x=\xi^+} = 1$:

$$A_1(\xi)u_1'(\xi) + A_2(\xi)u_2'(\xi) - B_1(\xi)u_1'(\xi) - B_2(\xi)u_2'(\xi) = -1/p(\xi)$$

If no solution to the homogeneous problem, solve these equations to get $G(x,\xi) = G(\xi,x)$.

To see symmetry of $G(x,\xi)$ formally:

Green's id: $\int_a^b \{G(x,\xi) A(\tfrac{d}{dx}) G(x,\eta) - A(\tfrac{d}{dx}) G(x,\xi) \cdot G(x,\eta)\}\, dx = 0$ since A self-[adjoint]

$= \int_a^b G(x,\xi)\delta(x-\eta) - \delta(x-\xi) G(x,\eta)\, dx = G(\eta,\xi) - G(\xi,\eta).$

Why do we look at $-[pG_x]_{x=\xi^-}^{x=\xi^+} = 1$?
Instead of say $-[pG_x + \xi]$ or something?
$-(pG_x)_x + qG = \delta(x-\xi)$
$\varepsilon > 0$: $\int_{\xi-\varepsilon}^{\xi+\varepsilon} \{-(pG_x)_x + qG\} dx = 1$

$\Rightarrow -[pG_x]_{\xi-\varepsilon}^{\xi+\varepsilon} + \int_{\xi-\varepsilon}^{\xi+\varepsilon} qG\, dx = 1$

Let $\varepsilon \to 0^+$
$-[pG_x]_{\xi^-}^{\xi^+} + 0 = 1$

Generalized (modified) Green's Function:

$Au = f$
$u \in D(A) = \{u \in \mathcal{H}^2(a,b) : Bu = 0\}$

Suppose $A\varphi = 0$ for some $\varphi \neq 0$
1) Solvability condition: $\langle \varphi, Au \rangle = \langle \varphi, f \rangle$
$\Rightarrow \langle A\varphi, u \rangle = \langle \varphi, f \rangle$
$\Rightarrow \langle \varphi, f \rangle = 0$
only have a solution if $f \perp \varphi$.
2) If there's a solution v, not unique b/c
$u = v + c\varphi$ is also a soln. for any c since
$Au = Av + cA\varphi = f$

Define a "pseudoinverse" of A by solving:
$$\begin{cases} Au = f - \langle \varphi, f \rangle \varphi & \text{assuming } \|\varphi\|^2 = \langle \varphi, \varphi \rangle = 1 \\ u \perp \varphi \end{cases}$$

$\Rightarrow u = Gf$, G is the pseudoinverse of A (or Moore-Penrose inverse)
$Gf(x) = \int_a^b G(x,\xi) f(\xi) d\xi$
generalized Green's function

1) Project RHS f onto range of A
2) Choose soln orthogonal to $\ker A$ (unique by projection thm.)

steady temperature distribution in a completely isolated rod

steady soln is any constant temp.

Ex: $\begin{cases} -u'' = f(x) & 0 < x < 1 \\ u'(0) = u'(1) = 0 \end{cases}$

Homogeneous problem $-u'' = 0$

$\Rightarrow u = c_1 + c_2 x$ ($1, x$ fundamental soln. pair)

$u'(0) = 0 \Rightarrow c_2 = 0$ $u'(1) = 0$ ok

soln of homogeneous problem $\varphi(x) = 1$

$\int_0^1 \varphi(x)^2 dx = 1$ so its normalized func.

solvability condition $\langle \varphi, f \rangle = \int_0^1 1 \cdot f(x) dx = 0$

suppose $\int_0^1 f dx$ is not zero

Projected eqn: $\begin{cases} -u'' = f(x) - \int_0^1 f(x) dx \cdot 1 \\ u'(0) = u'(1) = 0 \end{cases}$

Generalized green's fn satisfies:

$\begin{cases} -G_{xx} = \delta(x-\xi) - 1 \\ G_x(0,\xi) = G_x(1,\xi) = 0 \end{cases}$ (so that solvability cond'n satisfied)

Need:

1) $\begin{cases} G_{xx} = 1 & 0 \le x < \xi \\ G_x(0,\xi) = 0 \end{cases}$, $\begin{cases} G_{xx} = 1 & \xi < x \le 1 \\ G_x(1,\xi) = 0 \end{cases}$

$G(x,\xi) = \frac{1}{2}x^2 + c_1 + c_2 x$, $G_x(0,\xi) = 0 \Rightarrow c_2 = 0$

$G(x,\xi) = \frac{1}{2}x^2 + A(\xi)$, $0 \le x < \xi$

similarly, $G_x(1,\xi) = 0 \Rightarrow c_2 = -1$

so $G(x,\xi) = \frac{1}{2}x^2 - x + B(\xi)$ $\xi < x \le 1$

2) Continuity: $\frac{1}{2}\xi^2 + A(\xi) = \frac{1}{2}\xi^2 - \xi + B(\xi)$

so $A(\xi) = B(\xi) - \xi$

3) Jump cond'n $-[G_x]_{\xi^-}^{\xi^+} = 1$

$-(\xi - 1 - \xi) = 1$ ✓

$$G(x,\xi) = \begin{cases} \frac{1}{2}x^2 + A(\xi), & 0 \le x < \xi \\ \frac{1}{2}x^2 - x + \xi + A(\xi), & \xi < x \le 1 \end{cases}$$

4) Last condition $G(\cdot,\xi) \perp 1 \Rightarrow A$ and G symmetric.

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Generalized Green's functions

Specify unique soln. of projection eqn. by $\langle u, \varphi \rangle = 0$ $(u \perp \varphi)$
in this case $\int_0^1 u(x)\,dx = 0$

4) Require $G(x,\xi)$ to satisfy $G(\cdot,\xi) \perp \varphi \Rightarrow \langle G, \varphi \rangle = 0$

$$\Rightarrow \int_0^1 G(x,\xi)\,dx = 0$$

$$\int_0^\xi \tfrac{1}{2}x^2 + A(\xi)\,dx + \int_\xi^1 -x+\xi\,dx = \int_0^1 G(x,\xi)\,dx$$

$$\frac{1}{6} + A(\xi) + \left[-\frac{x^2}{2} + \xi x\right]_\xi^1 = 0$$

$$\tfrac{1}{6} + A(\xi) + \left(-\tfrac{1}{2} + \xi\right) - \left(-\tfrac{\xi^2}{2} + \xi^2\right) = 0$$

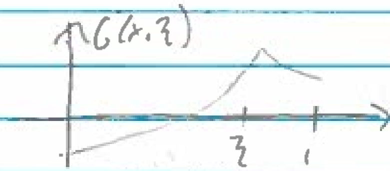
$$-\tfrac{1}{3} - \tfrac{1}{2}\xi^2 + \xi + A(\xi) = 0$$

$$\Rightarrow A(\xi) = \tfrac{1}{3} + \tfrac{1}{2}\xi^2 + \xi$$

$$\Rightarrow G(x,\xi) = \begin{cases} \tfrac{1}{2}(x^2+\xi^2) - \xi + 1/3 & 0 \le x < \xi \\ \tfrac{1}{2}(x^2+\xi^2) - x + 1/3 & \xi < x \le 1 \end{cases}$$

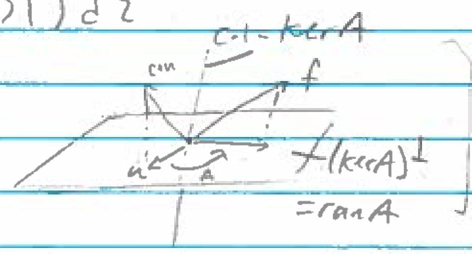
$$= \tfrac{1}{2}(x_>^2 + x_<^2) - x_> + 1/3$$

$G_{xx} = \delta(x-\xi) - 1$



$$u(x) = \int_0^1 G(x,\xi)\left(f(\xi) - \langle f, 1\rangle 1\right)d\xi$$

$A = -\frac{d^2}{dx^2}$ + Neumann BC's



General BVP

$$-(pu')' + qu = f(x)$$

$$\cos\alpha\, u(a) + \sin\alpha\, u'(a) = 0$$

$$\cos\beta\, u(b) + \sin\beta\, u'(b) = 0$$

In nonsingular BVP, each boundary condition gives a solution to the homogeneous IVP/FVP but will not solve the other, so we have a fundamental pair of solutions and a two-dimensional kernel of $A = -\frac{d}{dx}\left(p\frac{d}{dx}\right) + q$

General BVP

$-(pu')' + qu = f(x) \qquad a < x < b$

$\cos\alpha\, u(a) + \sin\alpha\, u'(a) = 0$
$\cos\beta\, u(b) + \sin\beta\, u'(b) = 0$) solves both IVP & FVP!

Suppose solution $u = \varphi(x)$ of homogeneous BVP, $(f=0)$, i.e., have a one-dimensional nullspace: $\|\varphi\|^2 = \int_a^b \varphi^2 dx = 1$

Modified/Projected Problem:

$-(pu')' + qu = f(x) - \langle \varphi, f \rangle \varphi(x) \quad + BCs$

Green's function $G(x,\xi)$ solves

$$\begin{cases} -(pG_x)_x + qG = \delta(x-\xi) - \varphi(\xi)\varphi(x) \\ \cos\alpha\, G(a,\xi) + \sin\beta\, G_x(a,\xi) = 0 \\ \cos\beta\, G(b,\xi) + \sin\beta\, G_x(b,\xi) = 0 \end{cases}$$

solution not unique, so specify

$\langle \varphi, G \rangle = \int_a^b G(x,\xi)\varphi(x)\,dx = 0$

Let $\eta(x)$ be a solution of nonhomogeneous ODE

$-(p\eta')' + q\eta = \varphi(x)$

Let $\{u_1, u_2\}$ be a fundamental pair of solutions of homogeneous ODE

$-(pu_i')' + qu_i = 0$

Then

$$G(x,\xi) = \begin{cases} A_1(\xi)u_1(x) + A_2(\xi)u_2(x) - \varphi(\xi)\eta(x) & 0 \le x < \xi \\ B_1(\xi)u_1(x) + B_2(\xi)u_2(x) - \varphi(\xi)\eta(x) & \xi < x \le 1 \end{cases}$$

(2 eqns) Impose BC's at $x=a$, $x=b$

(1 eqn) Impose jump conditions at $x=\xi$ (continuity or jump - linearly dep.

(1 eqn) Impose orthogonality condition for uniqueness

Then solution of modified problem

$u(x) = \int_a^b G(x,\xi) f(\xi)\, d\xi$

Spectral Theory & Green's Functions

$Au_n = \lambda_n u_n \quad n=1,2,\ldots \quad \|u_n\|_{L^2}=1$

$-(pu')'+qu = \lambda u + f(x) \quad + \text{BCs}$

Green's function $G(x,\xi;\lambda) = \sum_{n=1}^{\infty}\left[\frac{u_n(x)u_n(\xi)}{\lambda_n-\lambda}\right]$

in a bilinear expansion

G has poles at $\lambda=\lambda_n$

3/4: $Au = -(pu')'+qu = \lambda u + f(x) \quad a<x<b$

$B_\alpha u = \cos\alpha\, u(a) + \sin\alpha\, u'(a) = 0$

$B_\beta u = \cos\beta\, u(b) + \sin\beta\, u'(b) = 0$

Regular SL problem p, p', q, f cont's and $p(x)>0 \quad a\le x\le b$

$Au = \lambda u + f, \quad u\in D(A) = \{u\in H^2(a,b) : B_\alpha u=0,\ B_\beta u=0\}$

Complete set of eigenfunctions u_n w/ simple eigenvalues $\lambda_n\in\mathbb{C}$:

$\begin{cases} Au_n = \lambda_n u_n \\ u_n\in D(A)\end{cases}$

Then $u(x) = \sum_{n=1}^{\infty} c_n u_n(x)$ w/ $c_n = \langle u_n, u\rangle$ w/ $\|u_n\|=1$

$f(x) = \sum_{n=1}^{\infty} f_n u_n(x)$ w/ $f_n = \langle u_n, f\rangle$

$Au = \sum_{n=1}^{\infty}\lambda_n c_n u_n(x)$

$\sum_{n=1}^{\infty}\lambda_n c_n u_n(x) = \sum_{n=1}^{\infty}\lambda c_n u_n(x) + \sum_{n=1}^{\infty} f_n u_n(x)$

$\Rightarrow (\lambda-\lambda_n)c_n = f_n \Rightarrow c_n = \frac{f_n}{\lambda_n-\lambda}$

$u(x) = \sum_{n=1}^{\infty}\frac{f_n}{\lambda_n-\lambda}u_n(x) = \sum_{n=1}^{\infty}\frac{1}{\lambda_n-\lambda}\left[\int_a^b \overline{u_n(\xi)}f(\xi)\,d\xi\right]u_n(x)$

$= \int_a^b\left(\sum_{n=1}^{\infty}\frac{\overline{u_n(\xi)}u_n(x)}{\lambda_n-\lambda}\right)f(\xi)\,d\xi = \int_a^b G(x,\xi)f(\xi)\,d\xi$

where $G(x,\xi) = \sum_{n=1}^{\infty}\frac{\overline{u_n(\xi)}u_n(x)}{\lambda_n-\lambda}$

Alternatively, $-(pG_x)_x + qG = \lambda G + \delta(x-\xi)$

$\delta(x-\xi) = \sum_{n=1}^{\infty}\langle u_n, \delta(x-\xi)\rangle u_n(x) = \sum_{n=1}^{\infty}\overline{u_n(\xi)}u_n(x) \Rightarrow$ get same expression.

Note: $\delta(x-\xi) = \sum_{n=1}^{\infty} u_n(x)\overline{u_n(\xi)}$ doesn't converge in L^2,
think of as distribution, not a function.

But $G(x,\xi) = \sum_{n=1}^{\infty} \frac{u_n(x)\overline{u_n(\xi)}}{\lambda_n - \lambda}$ does converge in L^2, so don't worry!

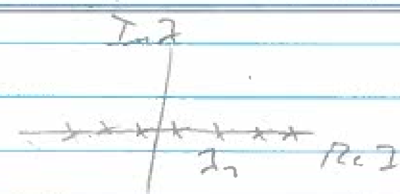
$Au = \lambda u + f$ solvable for every $f \in L^2$ provided $\lambda \neq \lambda_n$ f.a.s

Spectrum of A

$\sigma(A) = \{\lambda_n \in \mathbb{R} : n \in \mathbb{N}\}$

Resolvent set of A

$\rho(A) = \mathbb{C} \setminus \sigma(A)$



So for $\lambda \in \rho(A)$, $A - \lambda I$ is invertible

recover eigenfunctions from known Green's fn. & λ's.

$G(x,\xi;\lambda)$ has a simple pole at $\lambda = \lambda_n$ with residue $-u_n(x)\overline{u_n(\xi)}$

$\Rightarrow \frac{1}{2\pi i}\oint_{C_n} G(x,\xi;\lambda)\,d\lambda = -u_n(x)\overline{u_n(\xi)}$

$(\lambda I - A)^{-1} = R_A(\lambda)$ resolvent operator of A def. when $\lambda \in \rho(A)$

$(A - \lambda I)^{-1} f(x) = \int_a^b G(x,\xi;\lambda) f(\xi)\,d\xi$

At $\lambda = \lambda_n$, define generalized Green's function:

$$G_M(x,\xi;\lambda_n) = \sum_{m \neq n} \frac{u_m(x)\overline{u_m(\xi)}}{\lambda_m - \lambda_n}$$

need $\overline{u_n(\xi)} = \langle f, u_n \rangle = 0$

So can use to solve

$$Au = \lambda_n u + f - (f, u_n)u_n$$

One Last SL example

$$\begin{cases} -u'' = \lambda u + f(x) & -\infty < x < \infty \\ u \in L^2(\mathbb{R}) \end{cases}$$

$\leftarrow$ not regular! (singular SL problem)

Consider Green's fn. $G(x,\xi;\lambda)$:

$-G_{xx} = \lambda G + \delta(x-\xi) \quad -\infty < x < \infty$

On inf. interval, $G(x,\xi;\lambda) = G(x-\xi;\lambda)$, so

$G(x;\lambda)$ soln. of $\begin{cases} -G_{xx} = \lambda G + \delta(x) & -\infty < x < \infty \\ G(x;\lambda) \in L^2(\mathbb{R}) \end{cases}$

HW due next Monday!
last day of class

First, suppose $\lambda = -k^2 < 0$ $\quad$ wLOG $(k>0)$

$-G_{xx} = -k^2 G + \delta(x)$

1) $-G_{xx} = -k^2 G \qquad -\infty < x < 0$

2) $-G_{xx} = -k^2 G \qquad 0 < x < \infty$

3) $G(0^+,\lambda) = G(0^-,\lambda)$ $\quad$ cont's at $x=0$

4) $-[G_x]_{0^-}^{0^+} = 1$ $\quad$ jump condition

1/2) $-G_{xx} + k^2 G = 0 \Rightarrow G(x) = Be^{kx} + Ae^{-kx}$

$$G(x) = \begin{cases} Ae^{-kx}, & 0<x<\infty \\ Be^{kx}, & -\infty<x<0 \end{cases} \quad \text{satisfy } G \in L^2(\mathbb{R})$$

3) $Ae^0 = Be^0 \Rightarrow A = B$

4) $-[G_x]_{0^-}^{0^+} = G_x(0^-) - G_x(0^+) = kA - (-kA) = 1$

$A = 1/2k$

$$G(x,\lambda) = \frac{1}{2k} e^{-k|x|}, \quad \lambda = -k^2$$

$$u(x) = \int_{-\infty}^{\infty} G(x-\xi,\lambda) f(\xi)\, d\xi = \frac{1}{2k}\int_{-\infty}^{\infty} e^{-k|x-\xi|} f(\xi)\, d\xi$$

An example w/ a Continuous Spectrum

$$\begin{cases} -u'' = \lambda u + f(x) & -\infty < x < \infty \\ u \in L^2(\mathbb{R}) & \lambda \in \mathbb{C} \end{cases}$$

Green's function $G(x;\lambda)$

$-G_{xx} = \lambda G + \delta(x)$

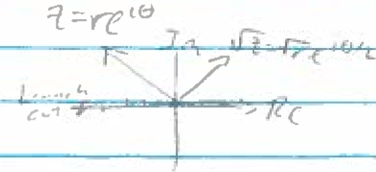
Solves $-G_{xx} = \lambda G$ on $-\infty < x < 0$ & $0 < x < \infty$

$G(x) = e^{rx} \qquad -r^2 = \lambda, \quad r = \pm\sqrt{-\lambda}$

Choose standard branch of $\sqrt{z}$: $\mathrm{Re}\sqrt{z} \geq 0$

$z = re^{i\theta} \quad -\pi < \theta < \pi \qquad \sqrt{z} = \sqrt{r}\, e^{i\theta/2}$

here we look at $\sqrt{-\lambda}$, so



~~branch~~ branch cut

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$G(x) = -\frac{1}{2\sqrt{-\lambda}} e^{-\sqrt{-\lambda}|x|}$

$\Rightarrow u(x) = \int_{-\infty}^{\infty} G(x-\xi;\lambda) f(\xi)\, d\xi = (G * f)(x)$

provided $\lambda \notin [0,\infty)$ (since thats where we put the branch cut)

$\Rightarrow$ Spectrum of $\underbrace{-\frac{d^2}{dx^2}}_{=A}$ on $L^2(\mathbb{R})$ is $[0,\infty) = \sigma(A)$

Green's fn. $G(x;\lambda)$ for A has a branch cut on cont's spectrum $\sigma(A)$

Corresponding expansion is Fourier transform

$u(x) = \int_{-\infty}^{\infty} \hat{u}(k) e^{ikx}\, dk$

Laplace's Equation:

$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$

$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = \nabla \cdot (\nabla u)$

Laplace's eqn: $\Delta u = 0$ homogeneous

Poisson's eq: $-\Delta u = f(x)$ nonhomogeneous

Dirichlet + Neumann Problem:

$\Omega \in \mathbb{R}^n$ bounded domain

$$\begin{cases} -\Delta u = f(x) & x \in \Omega \\ u = g(x) & x \in \delta\Omega \leftarrow \text{Dirichlet} \end{cases}$$

or $\frac{\partial u}{\partial n} = g(x)$ $x \in \delta\Omega$ $\leftarrow$ Neumann

$n \leftarrow$ unit normal vector

Dirichlet Green's function $G(x,\xi)$, $x \in \Omega$, $\xi \in \Omega$ soln of

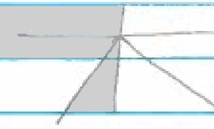
$$\begin{cases} -\Delta G = \delta(x-\xi) & x \in \Omega \\ G(x,\xi) = 0 & x \in \delta\Omega \end{cases} \quad \left(\begin{array}{l} \delta(x-\xi) = 0, \quad x \neq \xi \\ \int_{|x-\xi|<\epsilon} \delta(x-\xi)\, dx = 1 \end{array} \right)$$

Free-space Green's function:

$-\Delta G = \delta(x)$, $x \in \mathbb{R}^n$

In 1-D: $-G_{xx} = \delta(x)$

$G(x) = -\frac{1}{2}|x| + (c_1 + c_2 x)$



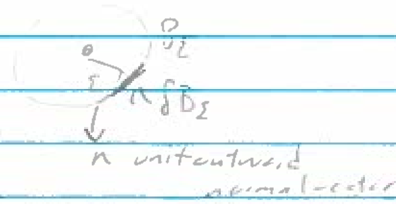
$-\Delta G = \delta(x) \qquad x \in \mathbb{R}^n$

soln. will be unique up to a multiple of homogeneous eqn.

Need $\boxed{\Delta G = 0 \qquad x \neq 0}$

Let $B_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}$

$\partial B_\varepsilon = \{x \in \mathbb{R}^n : |x| = \varepsilon\}$



$$\int_{B_\varepsilon} \delta(x)\,dx = 1 \quad \forall \varepsilon > 0$$

$$-\int_{B_\varepsilon} \Delta G\,dx = \int_{B_\varepsilon} \delta(x)\,dx = 1 \quad \forall \varepsilon > 0$$

$$-\int_{B_\varepsilon} \nabla\cdot(\nabla G)\,dx = 1 \quad \forall \varepsilon > 0$$

jump condition analog

$$\boxed{-\int_{\partial B_\varepsilon} \frac{\partial G}{\partial n}\,ds = 1}\quad , \text{ where } \frac{\partial G}{\partial n} = \nabla G \cdot n$$

Look for spherically symmetric solution (rotationally invariant)

$$G(x) = \tilde{G}(|x|) \quad , \quad r = |x|$$

$$\Delta G(r) = \frac{1}{r^{n-1}} \frac{d}{dr}\left(r^{n-1} \frac{dG}{dr}\right) = 0 \qquad \forall r \neq 0$$

$$\Rightarrow r^{n-1} \frac{dG}{dr} = C_1 \Rightarrow \frac{dG}{dr} = \frac{C_1}{r^{n-1}}$$

$$\Rightarrow G(r) = \begin{cases} C_1 r + C_2 & n = 1 \\ C_1 \ln r + C_2 & n = 2 \\ \frac{1}{2-n} \cdot \frac{C_1}{r^{n-2}} + C_2 & n \geq 3 \end{cases}$$

WLOG, take $c_2 = 0$

Rmk: when $n=3$, $G(r) = \frac{1}{r} \Rightarrow \nabla G = -\frac{1}{r^2}\hat{x}$ $\left(\hat{x} = \frac{x}{r}\right)$ is Newton's inverse square law of gravity

To determine c_1:

$$\int_{\partial B_\varepsilon} \frac{\partial G}{\partial n}\,ds = \int_{\partial B_\varepsilon} \frac{\partial G}{\partial r}\,ds = \int_{\partial B_\varepsilon} \frac{c_1}{\varepsilon^{n-1}}\,ds = \frac{c_1}{\varepsilon^{n-1}} \cdot \text{Area}(\partial B_\varepsilon)$$

$= (c_1/\varepsilon^{n-1}) \cdot (w_n \cdot \varepsilon^{n-1}) = c_1 w_n = -1$, w/ w_n = Area of ball w/ rad 1

$\Rightarrow c_1 = -1/w_n$

$$\Rightarrow G(x) = \begin{cases} \frac{1}{2}|x| & n=1 \\ -\frac{1}{2\pi}\log r & n=2 \\ \frac{1}{n-2}\,\frac{1}{w_n}\,\frac{1}{r^{n-2}} & n=3 \end{cases}$$

$n \geq 1 \quad \int_0^1 \frac{1}{r^\alpha} r^{n-1}\,dr = \int_0^1 \frac{1}{r^{\alpha-n+1}}\,dr < \infty$ when $\alpha - n + 1 < 1$, $\alpha < n$

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Summarize: Green's Function for Laplacian

$G(x)$ = free space Green's function

$$-\Delta G = \delta(x) \qquad x \in \mathbb{R}^n$$

(1) $G = G(r) \qquad r = |x|$

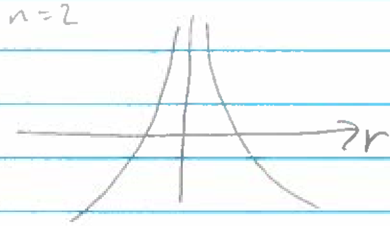
(2) $\Delta G = 0 \qquad x \neq 0$

(3) $-\int_{\partial B_\varepsilon(0)} \frac{\partial G}{\partial n}\,ds = 1$

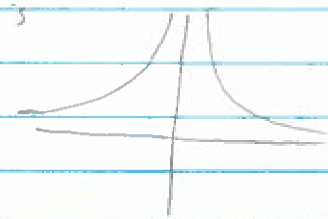
Then $-\Delta G(x-\xi) = \delta(x-\xi)$

$$G(x) = \begin{cases} -\frac{1}{2\pi}\log r & n=2 \\ \frac{1}{4\pi r} & n=3 \end{cases}$$

$n=2$



$n=3$



$-\Delta u = f(x) \qquad x \in \mathbb{R}^n$

$u(x) = \int_{\mathbb{R}^n} G(x-\xi) f(\xi)\,d\xi = (G * f)(x)$

In $n=3$:

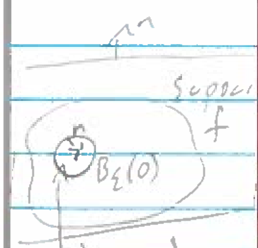
$u(x) = \int_{\mathbb{R}^n} \frac{f(\xi)}{4\pi|x-\xi|}\,d\xi$

differentiate by chopping out ε-ball!

formally, $-\Delta u(x) = \int -\Delta G(x-\xi) f(\xi)\,d\xi = \int \delta(x-\xi) f(\xi)\,d\xi = f(x)$

$u(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} G(x-\xi) f(\xi)\,d\xi$ by LDCT

$-\Delta \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} G(x-\xi) f(\xi)\,d\xi = -\Delta \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} G(\xi) f(x-\xi)\,d\xi$



$= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} G(\xi)\left[-\Delta_x f(x-\xi)\right] d\xi = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} G(\xi)\left[-\Delta_\xi f(x-\xi)\right] d\xi$

$= \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \underbrace{-\Delta_\xi G(\xi)}_{\substack{=0 \\ \text{by (2)}}} \cdot f(x-\xi)\,d\xi + \int_{\partial B_\varepsilon(0)} \left[G(\xi)\frac{\partial f}{\partial \rho}(x-\xi) - f(x-\xi)\frac{\partial G}{\partial \rho}\right] ds$

$\frac{d}{dn} = -\frac{d}{d\rho}$

$\rho = |\xi|$

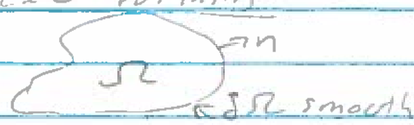
if $n=3$: $\sim \frac{1}{\varepsilon} \cdot M \cdot \varepsilon^2 \to 0$ as $\varepsilon \to 0$; $f(x)$, $\frac{1}{\varepsilon^2} \cdot \varepsilon^2$

hence as $\varepsilon \to 0^+$ $\longrightarrow 0 - \int_{\partial B_\varepsilon(0)} f(x) \frac{\partial G}{\partial \rho}(\xi)\, d\xi$

$$= - f(x) \underbrace{\int_{\partial B_\varepsilon(0)} \frac{\partial G}{\partial \rho}(\xi)\, d\xi}_{=-1 \text{ by } (3)} = f(x).$$

Green's Functions on Bounded Domains

$\Omega \subset \mathbb{R}^n$



Dirichlet problem

$$\begin{cases} -\Delta u = f(x) & x \in \Omega \\ u = g(x) & x \in \partial\Omega \end{cases}$$

Suppose $G(x,\xi)$ Green's function s.t.

$$\begin{cases} -\Delta G = \delta(x-\xi) & x \in \Omega \\ G(x,\xi) = 0 & x \in \partial\Omega \end{cases}$$

Green's identity:

$$\int_\Omega (u\Delta G - G\Delta u)\,dx = \int_{\partial\Omega} \left(u \frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS(x)$$

$$\int_\Omega -u(x)\delta(x-\xi) + G(x,\xi)f(x)\,dx = \int_{\partial\Omega} g(x) \frac{\partial G}{\partial n(x)}(x,\xi) - 0 \, dS(x)$$

$$-u(\xi) + \int_\Omega G(x,\xi) f(x)\,dx = \int_{\partial\Omega} g(x)\frac{\partial G}{\partial n(x)}(x,\xi)\, dS(x)$$

Switch $x \to \xi$, $\xi \to x$ & use fact that $G(x,\xi) = G(\xi,x)$

$$\boxed{u(x) = \int_\Omega G(x,\xi) f(\xi)\,d\xi - \int_{\partial\Omega} g(\xi) \frac{\partial G(x,\xi)}{\partial n(\xi)}\, dS(\xi)}$$

Again. $-\Delta G(x,\xi) = \delta(x-\xi)$

$$\int [G(x,\xi)\Delta G(x,\eta) - G(x,\eta)\Delta G(x,\xi)]\,dx = \int_{\partial\Omega} \overset{=0}{G(x,\xi)} \frac{\partial G}{\partial n(\eta)} - \overset{=0}{G(x,\eta)} \frac{\partial G}{\partial n(\eta)}\, dS = 0$$

$$\Rightarrow \int_\Omega G(x,\xi)[-\delta(x-\eta)] - G(x,\eta)[-\delta(x-\xi)]\,dx = 0$$

$$-G(\eta,\xi) + G(\xi,\eta) = 0 \Rightarrow \boxed{G(\xi,\eta) = G(\eta,\xi)}.$$

reciprocity principal.

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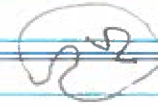
Dirichlet problem for the Laplacian

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$



Green's fn $G(x,\xi)$ satisfies

$$\begin{cases} -\Delta G = \delta(x-\xi) & x \in \Omega \\ G(x,\xi) = 0 & x \in \partial\Omega \end{cases}$$

By Green's thm:

$$\int_\Omega \left\{ G(x,\xi)\Delta u(x) - u(x)\Delta G(x,\xi) \right\} dx$$

$$= \int_{\partial\Omega} \left\{ G(x,\xi)\frac{\partial u}{\partial n(x)} - u(x)\frac{\partial G}{\partial n(x)}(x,\xi) \right\} dS$$

$$\Rightarrow \int_\Omega u(x)\delta(x-\xi)\,dx = -\int_{\partial\Omega} g(x)\frac{\partial G}{\partial n(x)}(x,\xi)\,dS$$

$$\Rightarrow u(\xi) = -\int_{\partial\Omega} g(x)\frac{\partial G}{\partial n(x)}(x,\xi)\,dS(x)$$

Switch $x \leftrightarrow \xi$ and use symmetry $G(x,\xi) = G(\xi,x)$

$$\Rightarrow u(x) = -\int_{\partial\Omega} g(\xi)\frac{\partial G}{\partial n(\xi)}(x,\xi)\,dS(\xi)$$

Structure of the Green's function:

Write

$$G(x,\xi) = G_F(x-\xi) + \phi(x,\xi)$$

w/ G_F = free space Green's fn $= \begin{cases} -\frac{1}{2\pi}\log|x| & , n=2 \\ \frac{1}{4\pi|x|} & , n=3 \end{cases}$

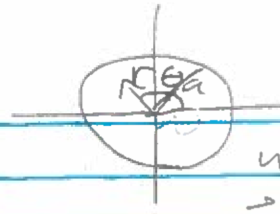
satisfies $-\Delta G_F = \delta(x)$

Then since $-\Delta G = -\Delta G_F(x-\xi) - \Delta\phi = \delta(x-\xi)$

we have $\begin{cases} \Delta\phi = 0 & \text{in } \Omega \\ \phi(x,\xi) = -G_F(x-\xi) & \text{on } \partial\Omega \end{cases}$

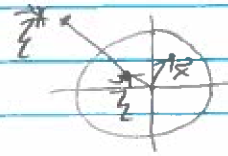
note: $\xi \in \Omega$ so $x-\xi \neq 0$ for $x \in \partial\Omega$

Dirichlet Problem on the disc



$$\begin{cases} \Delta u = 0 & r<a \\ u(a,\theta) = g(\theta) & r=a \end{cases}$$

$u = u(r,\theta)$



$\vec{x} = (x,y)$

$\vec{\xi} = (\xi,\eta)$

looking for Green's fn. satisfying

$$\begin{cases} -\Delta G = \delta(\vec{x}-\vec{\xi}) \\ G(\vec{x},\vec{\xi}) = 0 \text{ on } |\vec{x}|=a \end{cases}$$

Use method of images:

Image source at $\vec{\xi}^* = \dfrac{a^2}{|\vec{\xi}|^2}\cdot\vec{\xi}$ (want $|\vec{\xi}|\cdot|\vec{\xi}^*| = a^2$)

as $\vec{\xi} \to 0$, $|\vec{\xi}^*| \to \infty$

Assume $|\vec{x}|=a$ & consider

$$|\vec{x}-\vec{\xi}^*|^2 = (\vec{x}-\vec{\xi}^*)\cdot(\vec{x}-\vec{\xi}^*) = a^2 - 2\vec{x}\cdot\vec{\xi}^* + |\vec{\xi}^*|^2$$
$$= a^2 - 2\frac{a^2}{|\vec{\xi}|^2}\vec{x}\cdot\vec{\xi} + \frac{a^4}{|\vec{\xi}|^4}\vec{\xi}\cdot\vec{\xi}$$
$$= \frac{a^2}{|\vec{\xi}|^2}\left(\vec{\xi}\cdot\vec{\xi} - 2\vec{x}\cdot\vec{\xi} + \vec{x}\cdot\vec{x}\right) = \frac{a^2}{|\vec{\xi}|^2}|\vec{x}-\vec{\xi}|^2$$

so if $|\vec{x}|=a$, $|\vec{x}-\vec{\xi}| = \dfrac{|\vec{\xi}|}{a}|\vec{x}-\vec{\xi}^*|$

Consider $-\dfrac{1}{2\pi}\log|\vec{x}-\vec{\xi}| + \dfrac{1}{2\pi}\log|\vec{x}-\vec{\xi}^*|$ ← candidate for Green's fn.

$$= -\frac{1}{2\pi}\log\frac{|\vec{x}-\vec{\xi}|}{|\vec{x}-\vec{\xi}^*|} = -\frac{1}{2\pi}\log\left(\frac{|\vec{\xi}|}{a}\right) \text{ if } |\vec{x}|=a$$

← not zero on $\partial\Omega$

So $G(\vec{x},\vec{\xi}) = -\dfrac{1}{2\pi}\log\left(\dfrac{a}{|\vec{\xi}|}\cdot\dfrac{|\vec{x}-\vec{\xi}|}{|\vec{x}-\vec{\xi}^*|}\right)$ ← made zero on boundary by subtracting

Then $-\Delta G = \delta(\vec{x}-\vec{\xi})$ $x\in\Omega$

and $G(\vec{x},\vec{\xi}) = -\dfrac{1}{2\pi}\log\left(\dfrac{a}{|\vec{\xi}|}\cdot\dfrac{|\vec{\xi}|}{a}\right) = 0$ when $|\vec{x}|=a$, i.e. $x\in\partial\Omega$

Define $G(\vec{x},\vec{0}) = -\dfrac{1}{2\pi}\log(|\vec{x}|/a)$ $(\vec{\xi}=0)$

Then

$$u(\vec{x}) = -\frac{a}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\rho}\left[\log\left(\frac{a}{\rho}\,\frac{|\vec{x}-\vec{\xi}|}{|\vec{x}-\vec{\xi}^*|}\right)\right]_{\rho=a} g(\varphi)\,d\varphi$$

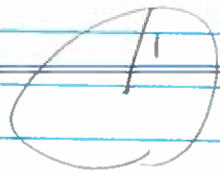
w/ (ρ,φ) polar coords for $\vec{\xi}$

$\vec{\xi} = \rho(\cos\varphi, \sin\varphi)$

$\vec{x} = r(\cos\theta, \sin\theta)$

Solution by separation of variables:

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$



$\Delta u = 0$, $u(r,\theta) = R(r)T(\theta)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)T + \frac{R}{r^2}\frac{d^2T}{d\theta^2} = 0$$

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{T}\frac{d^2T}{d\theta^2} = 0$$

$$\Rightarrow \frac{1}{T}\frac{d^2T}{d\theta^2} = -\lambda \quad , \quad \frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = +\lambda$$

need boundary conditions periodic in θ.

$$-\frac{d^2T}{d\theta^2} = \lambda T \quad , \quad T(0) = T(2\pi), \quad \frac{dT}{d\theta}(0) = \frac{dT}{d\theta}(2\pi)$$

$\Rightarrow T_n(\theta) = e^{in\theta}, e^{-in\theta}$, $\lambda_n = n^2$ $\quad n \in \mathbb{Z}$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - \lambda\frac{1}{r^2}R = 0$$

(*) $\quad r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = n^2 R$

[This is an Euler eq: $ar^2\frac{d^2u}{dr^2} + br\frac{du}{dr} + cu = 0$

try $u = r^m \Rightarrow ar^2 m(m-1)r^{m-2} + brmr^{m-1} + cr^m = 0$

$\Rightarrow am(m-1) + bm + c = 0$

$\Rightarrow u = Ar^{m_1} + Br^{m_2}$

(*) $\Rightarrow R = r^m \Rightarrow m^2 = n^2 \Rightarrow m = \pm n$

$n \neq 0 \Rightarrow R(r) = r^n, r^{-n}$ singular at the origin $r=0$

$n=0 \Rightarrow R(r) = 1, \log r$

Separated solns.

$$u(r,\theta) = \begin{cases} 1 & n=0 \\ r^n e^{in\theta}, r^n e^{-in\theta} & n \geq 1 \end{cases}$$

Superposing:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$

At $r=1$, $\sum_{n=-\infty}^{\infty} c_n e^{in\theta} = g(\theta)$ [boundary condition]

so $c_n = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, e^{-in\theta}\, d\theta$

thus $u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$ is the full soln via sep. of vars.

$\Rightarrow u(r,\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_0^{2\pi} g(\varphi)e^{-in\varphi}\,d\varphi\right)e^{in\theta}r^{|n|}$

(*) $= \frac{1}{2\pi}\int_0^{2\pi} g(\varphi)\left[\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-\varphi)}\right]d\varphi$

Note since $r<1$, $\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\varphi} = 1 + \sum_{n=1}^{\infty} r^n e^{in\varphi} + \sum_{n=1}^{\infty} r^n e^{-in\varphi}$

$= 1 + \frac{re^{i\varphi}}{1-re^{i\varphi}} + \frac{re^{-i\varphi}}{1-re^{-i\varphi}} = \frac{1-r^2}{1-2r\cos\varphi + r^2} = K_r(\varphi)$ $0\le r<1$

(*) ↘

$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} K_r(\theta-\varphi)\,g(\varphi)\,d\varphi$ for $r<1$

this is the Poisson integral formula & $K_r(\theta)$ is the Poisson kernel

same as what we found by Green's formula & method of images.

↳ will do again in complex analysis.

1) What happens when $r=0$? $K_0(\theta) = 1$

$u(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\varphi)\,d\varphi$ ← mean value of boundary values

Mean value property of harmonic functions

$\Delta u = 0$

value at center = avg. value in boundary

$\Rightarrow$ Maximum principle. If $u \neq$ constant, u has no interior max/min.

$u(r,\theta) = \frac{1}{2\pi} K_r * g$

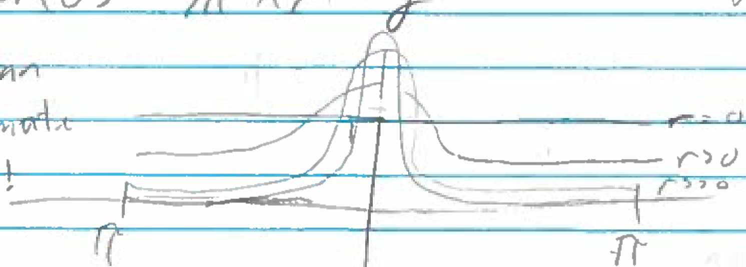
As $r\to 1^-$

$K_1(\theta) = 0$ for $\theta\neq 0$

& $\frac{1}{2\pi}\int_0^{2\pi} K_r(\theta)\,d\theta = 1$ for $r<1$

hence $K_r(\theta)\to\delta(\theta)$ as $r\to 1^-$

K_r is an approximate identity!



Applied w/ Guy

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Many problems we want to solve have a small parameter

$$F(u, \varepsilon) = 0$$

soln $\uparrow$ (u) parameter $\uparrow$ (ε) $\varepsilon \ll 1$

F may be an integral, algebraic eqn, ODE, PDE. Suppose that it's ~~"easy"~~ tractable to solve $F(u,0)=0$

This class is about how to do this

$$F(u,\varepsilon) = \underbrace{F(u,0)}_{\text{"easy"}} + \boxed{\text{corrections}(\varepsilon)}$$

Could approximate $F(u,\varepsilon)=0$ using
- numerical methods
  - ↳ fixed parameter, get soln
- asymptotic analysis
  - ↳ soln. structure
  - ↳ analytic approximation for how the soln depends on parameters

Satellite Problems

$$m\ddot{r} = -\frac{\gamma m M}{r^2}\hat{r}$$

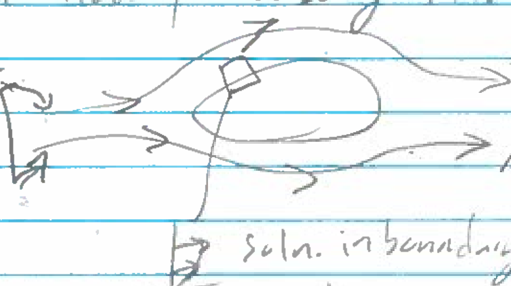
] Newton's 2nd law + Newton's gravitation law + we can solve it

+ air resistance
+ earth not a sphere
+ effect of the moon/others
+ relativistic effects

] corrections/perturbations

Fluid flow past a body w/ low viscosity

outer soln



Assume 2-D, inviscid, irrotational

Amounts to solving $\Delta \phi = 0$

soln. in boundary layer

put together using matching