Problem 1. *Write a multigrid V-cycle code to solve the Poisson equation in two dimensions on the unit square with Dirichlet boundary conditions. Use full weighting for restriction, bilinear interpolation for prolongation, and red-black Gauss-Seidel for smoothing. Use this code to solve*

$$\Delta u = -\exp\left(-(x-0.25)^2 - (y-0.6)^2\right)$$

on the unit square $(0,1)\times(0,1)$ with homogeneous Dirichlet boundary conditionss for different grid spacings. How many steps of pre and postsmoothing did you use? What tolerance did you use? How many cycles did it take to converge? Compare the amount of work needed to reach convergence with your solvers from Homework 3 taking into account how much work is involved in a V-cycle.

For the given problem, I ran my multigrid V-cycle code with the stopping criterion max-norm of the residual relative to the max-norm of the RHS function, with a tolerance of 10^{-10}. Using my analysis from 2, I found that (1,1), (2,1), and (2,2) were top contenders for optimal pre-smoothing, post-smoothing step numbers, and the fastest results were obtained for 2 pre-smoothing steps and 1 post-smoothing step. The results show that iteration count for the multigrid algorithm is grid-independent, and are tabulated as follows.
For $(\nu_1, \nu_2) = (1, 1)$,

grid spacing h	iteration count	run time (seconds)
2^{-5}	12	0.499733
2^{-6}	12	1.848
2^{-7}	12	7.19456
2^{-8}	12	28.9125
2^{-9}	12	126.997

For $(\nu_1, \nu_2) = (2, 1)$,

grid spacing h	iteration count	run time (seconds)
2^{-5}	10	0.440417
2^{-6}	10	1.7189
2^{-7}	10	7.17908
2^{-8}	10	29.8294
2^{-9}	10	124.565

For $(\nu_1, \nu_2) = (2, 2)$,

grid spacing h	iteration count	run time (seconds)
2^{-5}	9	0.443553
2^{-6}	9	1.88315
2^{-7}	9	7.48722
2^{-8}	9	29.7502
2^{-9}	9	127.246

The choice of $(\nu_1, \nu_2) = (2, 1)$ performs slightly better than $(1, 1)$ and $(2, 2)$ on most grid sizes. Compared to the solvers from homework 3, i.e., the Jacobi, GS-lex, and SOR iterative methods, this program takes significantly fewer total iterations, and considerably less work overall. since each iteration consists of 8-21 GS-RB iterations, and 4-7 restrictions, interpolations, and residual calculations (comparable work to 1 GS-RB iteration each). Thus each iteration of my V-cycle is approximately equivalent to 20-42 GS iterations in work, so the overall work of my V-cycle multigrid code is comparable to about 200-400 iterations of GS. This is comparable work to the SOR method from homework 3, and much better than the Jacobi-GS methods. While the iteration counts are grid-independent for the multigrid method, the overall work is still comparable to SOR.

Problem 2. *Numerically estimate the average convergence factor,*

$$E_k = \left(\frac{\|e^{(k)}\|_\infty}{\|e^{(k)}\|_\infty} \right)^{1/k},$$

for different numbers of presmoothing steps, ν_1, and postsmoothing steps, ν_2, for $\nu = \nu_1 + \nu_2 \leq 4$. Be sure to use a small value of k because convergence may be reached very quickly. What test problem did you use? Do your results depend on the grid spacing? Report the results in a table, and discuss which choices of ν_1 and ν_2 give the most efficient solver.

I used my multigrid program from problem 1 to solve the problem

$$\Delta u = -2\sin(\pi x)\sin(\pi y)$$

on the unit square $(0, 1) \times (0, 1)$ with homogeneous Dirichlet boundary conditions, which has the known solution

$$u(x, y) = \sin(\pi x)\sin(\pi y).$$

I performed an analysis of all the different pairings (ν_1, ν_2) for grid spacings $h = 2^{-5}, 2^{-6}, 2^{-7}$ with stopping criterion relative iterate differences with tolerance 10^{-6}, and these all achieved relatively similar results. The results do not depend on the grid spacing, as the data will testify. I report the average convergence factors for 1-5 iterations (e.g., E_3 is the average convergence factor among 3 iterations, while E_5 is the average convergence factor among 5 iterations), and 5 was chosen as the largest k to consider since my lowest reported iteration count for a multigrid solve was 8. The following table is for $h = 2^{-5}$, tolerance 10^{-10}.

(ν_1, ν_2)	E_1	E_2	E_3	E_4	iterations
(0, 1)	0.305393	0.301592	0.297944	0.291537	20
(1, 0)	0.293605	0.288987	0.291686	0.291592	24
(1, 1)	0.11991	0.116894	0.0975584	0.156273	12
(0, 2)	0.179352	0.176355	0.1692	0.118281	14
(2, 0)	0.193867	0.186207	0.180232	0.149008	16
(1, 2)	0.0805982	0.0760598	0.0647618	0.166059	10
(2, 1)	0.0809173	0.0763635	0.064411	0.166031	10
(3, 0)	0.140947	0.131789	0.117586	0.162013	13
(0, 3)	0.121375	0.117422	0.0979251	0.156212	12
(2, 2)	0.0606403	0.0543182	0.0831176	0.167628	9
(1, 3)	0.0606286	0.0543059	0.0831232	0.167629	9
(3, 1)	0.0607858	0.0544584	0.0830569	0.167623	9
(4, 0)	0.109314	0.100546	0.0728343	0.166327	12
(0, 4)	0.0914259	0.0863728	0.0404893	0.164779	10

For $h = 2^{-6}$, tolerance 10^{-10},

(ν_1, ν_2)	E_1	E_2	E_3	E_4	iterations
(0, 1)	0.305759	0.303893	0.302047	0.29956	20
(1, 0)	0.294261	0.295252	0.298011	0.300046	26
(1, 1)	0.12071	0.119945	0.115927	0.0577091	12
(0, 2)	0.180233	0.179461	0.177762	0.170408	14
(2, 0)	0.194353	0.191018	0.188791	0.182702	17
(1, 2)	0.0815659	0.0804702	0.0700874	0.111815	10
(2, 1)	0.0818544	0.0807546	0.070458	0.11171	10
(3, 0)	0.141378	0.137511	0.133371	0.111955	14
(0, 3)	0.122455	0.121482	0.117529	0.0682134	12
(2, 2)	0.0616633	0.0601744	0.0327965	0.11682	9
(1, 3)	0.0616527	0.0601636	0.0327591	0.116821	9
(3, 1)	0.061792	0.0603017	0.0332196	0.116802	9
(4, 0)	0.109726	0.106782	0.100253	0.113894	13
(0, 4)	0.0926053	0.0913771	0.0837286	0.106415	10

For $h = 2^{-7}$, tolerance 10^{-10}:

(ν_1, ν_2)	E_1	E_2	E_3	E_4	iterations
(0, 1)	0.305848	0.304463	0.303061	0.301479	21
(1, 0)	0.294425	0.297873	0.300711	0.302654	27
(1, 1)	0.120909	0.120695	0.119705	0.112979	12
(0, 2)	0.180453	0.180232	0.179786	0.178067	14
(2, 0)	0.194475	0.192975	0.192209	0.191775	18
(1, 2)	0.0818076	0.0815357	0.0792587	0.0481262	10
(2, 1)	0.0820882	0.0818156	0.0795544	0.0470683	10
(3, 0)	0.141487	0.139291	0.13751	0.133174	15
(0, 3)	0.122724	0.122481	0.121534	0.115184	12
(2, 2)	0.0619189	0.0615518	0.0572464	0.0771726	9
(1, 3)	0.0619085	0.0615412	0.057234	0.0771781	9
(3, 1)	0.0620431	0.0616766	0.0573914	0.0771083	9
(4, 0)	0.10983	0.10803	0.10601	0.095701	13
(0, 4)	0.0928997	0.0925947	0.0908535	0.0700194	10