Problem 1. *Let L be the linear operator $Lu = u_{xx}$, $u_x(0) = u_x(1) = 0$.*

(a) Find the eigenfunctions and corresponding eigenvalues of L.

$$u_{xx} - \lambda u = 0, u_x(0) = u_x(1) = 0$$

Assume $u = e^{kx}$, then the characteristic equation is $k^2 - \lambda = 0$, so $u = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$. Then $u_x = A\sqrt{\lambda}e^{\sqrt{\lambda}x} - B\sqrt{\lambda}e^{-\sqrt{\lambda}x}$, so $u_x(0) = A\sqrt{\lambda} - B\sqrt{\lambda} = 0$ and $u_x(1) = A\sqrt{\lambda}e^{\sqrt{\lambda}} - B\sqrt{\lambda}e^{-\sqrt{\lambda}} = 0$. The first boundary condition implies that $A = B$, then the second boundary condition is

$$A\sqrt{\lambda}\left(e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}}\right) = 0.$$

This is only possible if $\lambda = 0$ or if $\lambda < 0$, in which case

$$2A\sqrt{\lambda}\cos\left(\sqrt{\lambda}\right) = 0.$$

Thus for $\lambda < 0$, we must have that $\lambda = -k^2\pi^2$, $k > 0$.

Thus we have our eigenvalues $\lambda_k = -k^2\pi^2$ and eigenfunctions $u_k(x) = \cos(k\pi x)$, $k = 0, 1, \ldots$.

(b) Show that the eigenfunctions are orthogonal in the $L^2[0, 1]$ inner product.

Consider u_n and u_m eigenfunctions for $n \neq m$. Then

$$\begin{aligned}
\langle u_n, u_m \rangle &= \int_0^1 \cos(n\pi x)\cos(m\pi x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \cos((n+m)\pi x) + \cos((n-m)\pi x)\mathrm{d}x \\
&= \frac{1}{2}\left[-\frac{1}{(n+m)\pi}\sin((n+m)\pi x) - \frac{1}{(n-m)\pi}\sin((n-m)\pi x)\right]_0^1 \\
&= 0.
\end{aligned}$$

Where the last equality holds because $n+m$ and $n-m$ are both integers, and sin of any integer multiple of π is 0.

For $n = m$, then

$$\begin{aligned}
\langle u_n, u_m \rangle &= \int_0^1 \cos^2(n\pi x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 1 + \cos(2n\pi x)\mathrm{d}x \\
&= \frac{1}{2}\left[x - \frac{1}{2n\pi}\sin(2n\pi x)\right]_0^1 \\
&= \frac{1}{2}.
\end{aligned}$$

(c) It can be shown that the eigenfunctions $\phi_j(x)$, form a complete set in $L^2[0,1]$. Express the solution to

$$u_{xx} = f, u_x(0) = u_x(1) = 0,$$

as a series solution of the eigenfunctions.

Since the eigenfunctions form a complete set, we can express f as series

$$f(x) = \sum_{j=0}^{\infty} f_j \phi_j(x).$$

where

$$f_j = \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}.$$

Then assuming the solution has the form

$$u(x) = \sum_{j=0}^{\infty} a_j \phi_j(x),$$

we have that

$$u_{xx} = f$$
$$\sum_{j=0}^{\infty} \lambda_j a_j \phi_j(x) = \sum_{j=0}^{\infty} f_j \phi_j(x).$$

If we take the inner product of the above equation with any ϕ_j, by orthogonality we will obtain the single terms

$$\lambda_j a_j \langle \phi_j, \phi_j \rangle = f_j \langle \phi_j, \phi_j \rangle.$$

Thus

$$a_j = \frac{f_j}{\lambda_j} = \frac{\langle f, \phi_j \rangle}{-j^2\pi^2 \langle \phi_j, \phi_j \rangle},$$

so that

$$u(x) = \sum_{j=0}^{\infty} \frac{\langle f, \phi_j \rangle}{-j^2\pi^2 \langle \phi_j, \phi_j \rangle} \phi_j(x).$$

(d) Note that the BVP does not have a solution for all f. Express the condition for existence of a solution in terms of the eigenfunctions of L.

In order for $Lu = f$ to have a solution, f must be in the range of L. Since

$$L^2[0,1] = \operatorname{ran} L \bigoplus \ker L^*,$$

f must be orthogonal to $\ker L^*$. Since L is self-adjoint, f must be orthogonal to $\ker L$. Note that $\ker L$ is spanned by $\phi_0(x) = 1$, since $\phi_0(x)$ has eigenvalue 0. Thus

$$\langle f, \phi_0 \rangle = \langle f, 1 \rangle = \int_0^1 f(x) \mathrm{d}x = 0.$$

Problem 2. *Define the functional $F : X \to \mathbb{R}$ by*

$$F(u) = \int_0^1 \frac{1}{2}(u_x)^2 + fu\mathrm{d}x,$$

where X is the space of real valued functions on $[0,1]$ that have at least one continuous derivative and are zero at $x = 0, 1$. The Frechet derivative of F at a point u is defined to be the linear operator $F'(u)$ for which

$$F(u + v) = F(u) + F'(u)v + R(v),$$

where

$$\lim_{\|v\|\to 0} \frac{\|R(v)\|}{\|v\|} = 0.$$

One way to compute the derivative is

$$F'(u)v = \lim_{\varepsilon\to 0} \frac{F(u + \varepsilon v) - F(u)}{\varepsilon}.$$

Note that this looks just like a directional derivative.

(a) Compute the Frechet derivative of F.

$$\begin{aligned}
F'(u)v &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^1 \frac{1}{2}(u_x + \varepsilon v_x)^2 + f(u + \varepsilon v) - \frac{1}{2}(u_x)^2 - fu\mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^1 \varepsilon u_x v_x + \frac{1}{2}\varepsilon^2 v_x^2 + \varepsilon f v\mathrm{d}x \\
&= \int_0^1 u_x v_x + fv\mathrm{d}x \\
&= \cancelto{0}{[u_x v]_0^1} + \int_0^1 fv - u_{xx}v \ \mathrm{d}x \quad \text{by } v \in X \\
&= \int_0^1 (f - u_{xx})v \ \mathrm{d}x.
\end{aligned}$$

(b) $u \in X$ is a critical point of F if $F'(u)v = 0$ for all $v \in X$. Show that if u is a solution to the Poisson equation

$$u_{xx} = f, u(0) = u(1) = 0,$$

then it is a critical point of F.

If u solves

$$u_{xx} = f, u(0) = u(1) = 0,$$

then for all $v \in X$,

$$F'(u)v = \int_0^1 (f - u_{xx})v \ \mathrm{d}x = \int_0^1 0v \ \mathrm{d}x = 0.$$

Hence u is a critical point of F.

(c) Let X_h be a finite dimensional subspace of X, and let $\{\phi_i(x)\}$ be a basis for X_h. This means that all $u_h \in X_h$ can be expressed as $u_h(x) = \sum_i u_i\phi_i(x)$ for some constants u_i. Thus we can identify the elements of X_h with vectors $\vec{u}$ that have components u_i. Let $G(\vec{u}) = F(u_h)$. Show that the gradient of G (whos components are $(\boldsymbol{\nabla} G)_j = \frac{\partial G}{\partial u_j}$) is of the form $\boldsymbol{\nabla} G(\vec{u}) = A\vec{u} + \vec{b}$, and write expressions for the elements of the matrix A and the vector $\vec{b}$.

$$\begin{aligned}
F(u_h) &= \int_0^1 \frac{1}{2}(u_{h,x}^2 + fu)\ \mathrm{d}x \\
G(\vec{u}) &= \int_0^1 \frac{1}{2}\left(\sum_i u_i\phi_i'\right)^2 + f(\vec{u}\cdot\vec{\phi})\ \mathrm{d}x \\
\implies \frac{\partial G}{\partial u_j}(\vec{u}) &= \int_0^1 \phi_j'(\vec{u}\cdot\vec{\phi}') + f\phi_j\ \mathrm{d}x \\
&= \Big[(\vec{u}\cdot\vec{\phi}')\phi_j\Big]_0^1 + \int_0^1 (f - \frac{\mathrm{d}}{\mathrm{d}x}(\vec{u}\cdot\vec{\phi}'))\phi_j\ \mathrm{d}x \\
&= \int_0^1 \Big[f - \vec{u}\cdot\vec{\phi}''\Big]\phi_j\ \mathrm{d}x,
\end{aligned}$$

where the boundary terms vanish since $\phi_j \in X$.

This can be represented as

$$\nabla G(\vec{u}) = A\vec{u} + \vec{b},$$

where

$$(A)_{ij} = -\int_0^1 \phi_i\phi_j''\ \mathrm{d}x$$

and

$$(\vec{b})_j = \int_0^1 f\phi_j\ \mathrm{d}x.$$

(d) Divide the unit interval into a set of $N+1$ equal length intervals $I_i = (x_i, x_{i+1})$ for $i = 0, \dots, N$. The enpoints of the intervals are $x_i = ih$, where $h = \frac{1}{N+1}$. Let X_h be the subspace of X such that the elements u_h of X_h are linear on each interval, continuous on $[0,1]$, and satisfy $u_h(0) = u_h(1) = 0$. X_h is an N dimensional space with basis elements

$$\phi_i(x) = \begin{cases} 1 - h^{-1}|x - x_i| & \text{if } |x - x_i| < h, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for $i = 1, \dots, N$. Compute the matrix A from the previous problem that appears in the gradient.

Note that

$$\phi_i' = \begin{cases} 1/h, & x_{i-1} < x < x_i \\ -1/h, & x_i < x < x_{i+1} \\ 0, & \text{o/w.} \end{cases}$$

So that

$$\phi_i'' = \delta(x - x_i).$$

Then

$$(A)_{ij} = -\int_0^1 \phi_i(x)\phi_j''(x)\ \mathrm{d}x = \int_0^1 \phi_i(x)\delta(x - x_j)\ \mathrm{d}x = -\phi_i(x_j).$$

Thus $(A)_{ij} = 0$ for $j \neq i$, and $(A)_{ii} = -1$, so $A = -I$.

Problem 3.

(a) Using a Taylor expansion, derive the finite difference formula to approximate the second derivative at x using function values at $x - \frac{h}{2}$, x, and $x + h$. How accurate is the finite difference approximation?

We begin by Taylor expanding $u(x - \dfrac{h}{2})$ and $u(x + h)$.

$$u(x - \frac{h}{2}) = u(x) - \frac{h}{2}u'(x) + \frac{h^2}{8}u''(x) - \frac{h^3}{48}u'''(x) + \dots\ .$$

$$u(x + h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \dots\ .$$

Then clearly,

$$u(x - \frac{h}{2}) + \frac{1}{2}u(x + h) = \frac{3}{2}u(x) + u''(x)\left(\frac{h^2}{8} + \frac{h^2}{4}\right) + u'''(x)\left(\frac{h^3}{12} + \frac{h^3}{48}\right) + \dots\ .$$

So

$$u(x - \frac{h}{2}) + \frac{1}{2}u(x + h) - \frac{3}{2}u(x) = \frac{3h^2}{8}u''(x) + \frac{3h^3}{48}u'''(x) + \dots\ .$$

Let

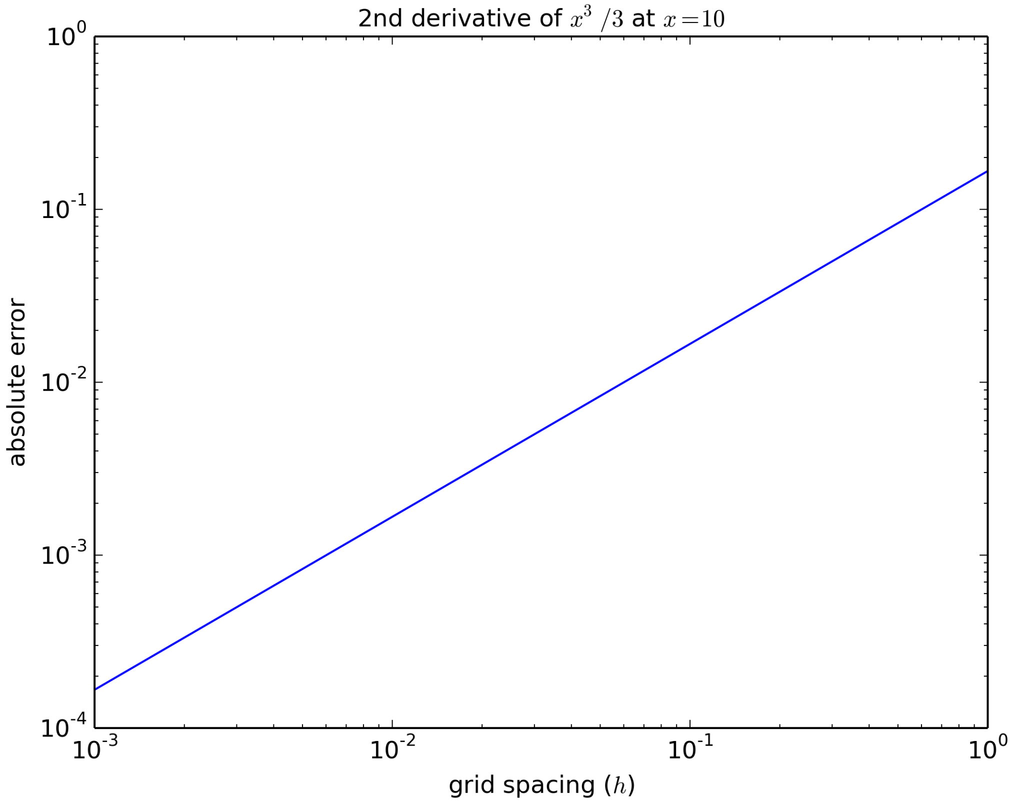
$$Du = \frac{4}{3h^2}\left[2u(x - \frac{h}{2}) - 3u(x) + u(x + h)\right].$$

Then this is a first-order accurate approximation of $u''(x)$ with error

$$Du - u''(x) = \frac{8h}{48}u'''(x) + \dots = \mathcal{O}(h).$$

(b) Perform a refinement study to verify the accuracy of the difference formula you derived.

I applied this difference formula to the function $u(x) = \dfrac{x^3}{3}$ at $x = 10$ for various grid spacings h, and plotted the errors of our approximation on a log-log plot against the grid spacing.



Note that the slope appears to be 1, so this is indeed a first-order accurate approximation of $u''(x)$.

(c) Derive an expression for the quadratic polynomial that interpolates the data $\left(x-\frac{h}{2}, u\left(x-\frac{h}{2}\right)\right)$, $(x, u(x))$, and $(x+h, u(x+h))$. How is the finite difference formula you derived in problem 3a related to the interpolating polynomial?

We want to find a polynomial $u(x) = A + Bx + Cx^2$ interpolating these points. With $u_0 = u(x-\dfrac{h}{2})$, $u_1 = u(x)$, $u_2 = u(x+h)$, we solve the system of equations

$$\begin{aligned} u_0 &= A + B(x-\frac{h}{2}) + C(x-\frac{h}{2})^2 \\ u_1 &= A + Bx + Cx \\ u_2 &= A + B(x+h) + C(x+h)^2 \end{aligned}$$

using Maple to obtain:

$$A = \frac{3\,u_1\,h^2 + 4\,u_0\,xh - 3\,u_1\,xh - u_2\,xh + 4\,u_0\,x^2 - 6\,u_1\,x^2 + 2\,u_2\,x^2}{3h^2},$$

$$B = -\frac{4\,hu_0 - 3\,hu_1 - hu_2 + 8\,u_0\,x - 12\,u_1\,x + 4\,u_2\,x}{3h^2}$$

$$C = \frac{2}{3h^2}(2u_0 - 3u_1 + u_2).$$

Note that if we take the second derivative of this interpolating polynomial, we obtain

$$u''(x) = 2C = \frac{4}{3h^2}(2u_0 - 3u_1 + u_2) = Du.$$

So we have found a second way to construct our difference approximation operator.

Python Code used for 3b:

```
#hw_01_228A Refinement Study (problem 3 part b)
#Carter Johnson
#10/11/16

import numpy as np
from math import exp
import matplotlib.pyplot as plt

#my problem is x^3/3 -> second derivative is x
#evaluate at x=10
x=10
actual = 10

#2nd derivative approximate operator as function of grid spacing h
def D2_approx(h):
  return (2/(3*h**2))*(2*(x-h/2)**3/3 + (x+h)**3/3 - (x)**3)

#evaluate my 2nd derivative approximation for various h
H = np.linspace(1, .001, 100)
D2_approxes = [D2_approx(h) for h in H]

#find abs errors of these approximations
max_errors = [abs(approx - actual) for approx in D2_approxes]

#plot as log-log
plt.figure()
plt.loglog(H, max_errors, label="Absolute Error")
plt.title("2nd derivative of $x^3/3$ at $x=10$", fontsize=12)
plt.xlabel("grid spacing ($h$)")
plt.ylabel("absolute error")
plt.savefig("problem3_refinement_study.png", dpi=300)
plt.close()
```