Problem 1. *Write a multigrid V-cycle code to solve the Poisson equation in two dimensions on the unit square with Dirichlet boundary conditions. Use full weighting for restriction, bilinear interpolation for prolongation, and red-black Gauss-Seidel for smoothing. Use this code to solve*

$$\Delta u = -\exp\left(-(x-0.25)^2 - (y-0.6)^2\right)$$

on the unit square $(0,1)\times(0,1)$ with homogeneous Dirichlet boundary conditionss for different grid spacings. How many steps of pre and postsmoothing did you use? What tolerance did you use? How many cycles did it take to converge? Compare the amount of work needed to reach convergence with your solvers from Homework 3 taking into account how much work is involved in a V-cycle.

For the given problem, I ran my multigrid V-cycle code with stopping criterion relative tolerance of 10^{-7}. The results are tabulated as follows.

grid spacing h	iteration count	run time (seconds)
2^{-5}	16	0.703004
2^{-6}	16	2.82203
2^{-7}	15	10.7262
2^{-8}	14	40.7734

Testing my code on the thing from p2, I got the following data:

grid spacing h	iteration count	run time (seconds)	max. relative errors
0.03125	15	0.661239	0.130899
0.015625	15	2.63253	0.0809393
0.0078125	15	10.6599	0.0456318
0.00390625	14	40.5742	0.0248325

The errors are not looking too good.

Problem 2. *Numerically estimate the average convergence factor,*

$$E_k = \left(\frac{\|e^{(k)}\|_\infty}{\|e^{(k)}\|_\infty}\right)^{1/k},$$

for different numbers of presmoothing steps, ν_1, and postsmoothing steps, ν_2, for $\nu = \nu_1 + \nu_2 \leq 4$. Be sure to use a small value of k because convergence may be reached very quickly. What test problem did you use? Do your results depend on the grid spacing? Report the results in a table, and discuss which choices of ν_1 and ν_2 give the most efficient solver.

I used my multigrid program from problem 1 to solve the problem

$$\Delta u = -2\sin(\pi x)\sin(\pi y)$$

on the unit square $(0,1)\times(0,1)$ with homogeneous Dirichlet boundary conditions, which has the known solution $u(x,y) = \sin(\pi x)\sin(\pi y)$. I performed an analysis of all the different pairings (ν_1, ν_2) for grid spacings $h = 2^{-5}, 2^{-6}, 2^{-7}$ with stopping criterion relative iterate differences with tolerance 10^{-6}, and these all achieved relatively similar results. The results do not depend

(ν_1, ν_2)	E_1	$\frac{1}{2}\sum_{i=1}^{2} E_i$	$\frac{1}{3}\sum_{i=1}^{3} E_i$	$\frac{1}{4}\sum_{i=1}^{4} E_i$	$\frac{1}{5}\sum_{i=1}^{5} E_i$	iterations
(0, 0)	0.396875	0.396875	0.454525	0.50511	0.552891	14
(0, 1)	0.305393	0.305393	0.288137	0.318529	0.371915	23
(1, 0)	0.240477	0.240477	0.244411	0.316576	0.384936	20
(1, 1)	0.0798504	0.0798504	0.181037	0.282042	0.360163	13
(0, 2)	0.179352	0.179352	0.185095	0.262303	0.332694	16
(2, 0)	0.0987041	0.0987041	0.212508	0.311878	0.388076	13
(1, 2)	0.0556845	0.0556845	0.166925	0.267998	0.345461	12
(2, 1)	0.00870737	0.00870737	0.164355	0.276806	0.358282	10
(3, 0)	0.0844652	0.0844652	0.218674	0.3192	0.393989	10
(0, 3)	0.121375	0.121375	0.166698	0.251102	0.322984	13
(2, 2)	0.0115739	0.0115739	0.158641	0.267094	0.346734	10
(1, 3)	0.043239	0.043239	0.158128	0.258661	0.33563	11
(3, 1)	0.0376952	0.0376952	0.186759	0.293044	0.37053	9
(4, 0)	0.117969	0.117969	0.240874	0.334553	0.405141	8
(0, 4)	0.0914259	0.0914259	0.15609	0.24315	0.315229	12

on the grid spacing, as the data will testify. I report the average convergence factors for 1-5 iterations (e.g., E_3 is the average convergence factor among 3 iterations, while E_5 is the average convergence factor among 5 iterations), and 5 was chosen as the largest k to consider since my lowest reported iteration count for a multigrid solve was 8. The following table is for $h = 2^{-5}$, tolerance 10^{-6}.

For $h = 2^{-6}$, tolerance 10^{-6},

For $h = 2^{-7}$, tolerance 10^{-6},

Including more average convergence factors for $h = 2^{-7}$, tol 10^{-6}:

(ν_1, ν_2)	E_1	$\frac{1}{2}\sum_{i=1}^{2} E_i$	$\frac{1}{3}\sum_{i=1}^{3} E_i$	$\frac{1}{4}\sum_{i=1}^{4} E_i$	$\frac{1}{5}\sum_{i=1}^{5} E_i$	iterations
(0, 0)	0.398287	0.398287	0.457284	0.508628	0.552266	14
(0, 1)	0.305759	0.305759	0.297381	0.303017	0.34304	24
(1, 0)	0.24123	0.24123	0.221681	0.278559	0.338408	21
(1, 1)	0.080841	0.080841	0.152614	0.2388	0.310719	13
(0, 2)	0.180233	0.180233	0.162842	0.224646	0.288112	16
(2, 0)	0.099502	0.099502	0.187086	0.270098	0.339213	12
(1, 2)	0.0568028	0.0568028	0.140532	0.228899	0.301117	12
(2, 1)	0.00804287	0.00804287	0.136884	0.235023	0.310355	10
(3, 0)	0.082447	0.082447	0.18937	0.274838	0.343387	9
(0, 3)	0.122455	0.122455	0.142858	0.213729	0.279597	14
(2, 2)	0.0129262	0.0129262	0.13398	0.230136	0.304531	9
(1, 3)	0.0443836	0.0443836	0.133365	0.22225	0.294305	11
(3, 1)	0.0363218	0.0363218	0.158598	0.251192	0.322932	8
(4, 0)	0.115599	0.115599	0.209191	0.288809	0.353647	8
(0, 4)	0.0926053	0.0926053	0.131747	0.206181	0.27286	12

(ν_1, ν_2)	E_1	$\frac{1}{2}\sum_{i=1}^{2} E_i$	$\frac{1}{3}\sum_{i=1}^{3} E_i$	$\frac{1}{4}\sum_{i=1}^{4} E_i$	$\frac{1}{5}\sum_{i=1}^{5} E_i$	iterations
(0, 0)	0.39859	0.39859	0.45838	0.510181	0.553731	14
(0, 1)	0.305848	0.305848	0.301483	0.290371	0.316898	24
(1, 0)	0.241423	0.241423	0.22874	0.2646	0.311	20
(1, 1)	0.0810938	0.0810938	0.129929	0.201638	0.265662	13
(0, 2)	0.180453	0.180453	0.171162	0.209334	0.260305	16
(2, 0)	0.0997174	0.0997174	0.166077	0.234016	0.294026	12
(1, 2)	0.0570882	0.0570882	0.118255	0.192737	0.257747	11
(2, 1)	0.0082937	0.0082937	0.113843	0.19745	0.265399	9
(3, 0)	0.0819103	0.0819103	0.16469	0.234523	0.295157	10
(0, 3)	0.122724	0.122724	0.121915	0.178623	0.237253	14
(2, 2)	0.0132801	0.0132801	0.112315	0.194746	0.262212	9
(1, 3)	0.0446754	0.0446754	0.111415	0.186938	0.252265	10
(3, 1)	0.0359415	0.0359415	0.134041	0.212362	0.277185	8
(4, 0)	0.114961	0.114961	0.182013	0.246734	0.304525	9
(0, 4)	0.0928997	0.0928997	0.110056	0.171221	0.231142	12

v1, v2	ave of 3	average of 4	average of 5	average of 6	average of 7	iterations
(0, 0)	0.510181	0.553731	0.590725	0.622112	0.648662	14
(0, 1)	0.290371	0.316898	0.352501	0.388052	0.421029	24
(1, 0)	0.2646	0.311	0.356142	0.39694	0.432952	20
(1, 1)	0.201638	0.265662	0.32007	0.366267	0.405829	13
(0, 2)	0.209334	0.260305	0.308686	0.351767	0.389619	16
(2, 0)	0.234016	0.294026	0.345739	0.389893	0.427825	12
(1, 2)	0.192737	0.257747	0.31265	0.359168	0.398994	11
(2, 1)	0.19745	0.265399	0.321472	0.368514	0.408563	9
(3, 0)	0.234523	0.295157	0.346653	0.390554	0.428289	10
(0, 3)	0.178623	0.237253	0.28954	0.334963	0.374407	14
(2, 2)	0.194746	0.262212	0.318064	0.365009	0.405033	9
(1, 3)	0.186938	0.252265	0.307332	0.353987	0.393951	10
(3, 1)	0.212362	0.277185	0.331328	0.377056	0.416145	8
(4, 0)	0.246734	0.304525	0.354362	0.397149	0.434076	9
(0, 4)	0.171221	0.231142	0.28387	0.329496	0.369077	12