Problem 1. *Write programs to solve the advection equation*

$$u_t + au_x = 0,$$

on $[0, 1]$ *with periodic boundary conditions using upwinding and Lax-Wendroff. For smooth solutions, we expect upwinding to be first-order accurate and Lax-Wendroff to be second-order accurate, but it is not clear what accuracy to expect for nonsmooth solutions.*

(a) Let $a = 1$ and solve the problem up to time $t = 1$. Perform a refinement study for both upwinding and Lax-Wendroff with $\Delta t = 0.9a\Delta x$ with a smooth initial condition. Compute the rate of convergence in the 1-norm, 2-norm, and max-norm. Note that the exact solution at time $t = 1$ is the initial condition, and so computing the error is easy.

(b) Repeat the previous problem with the discontinuous initial condition

$$u(x, 0) = \begin{cases} 1 & \text{if } |x - 1/2| < 1/4 \\ 0 & \text{otherwise} \end{cases}.$$

Problem 2. *For solving the heat equation we frequently use Crank-Nicolson, which is trapezoidal rule time integration with a second-order space discretization. The analogous scheme for the linear advection equation is*

$$u_{j+1}^{n+1} - u_j^n + \frac{\nu}{4}\left(u_{j+1}^n - u_{j-1}^n\right) + \frac{\nu}{4}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right) = 0,$$

where $\nu = a\Delta t/\Delta x$.

(a) Use von Neumann analysis to show that this scheme is unconditionally stable and that $\|u^n\|_2 = \|u^0\|_2$. This scheme is said to be nondissipative- i.e., there is no amplitude error. This seems reasonable because this is a property of the PDE.

(b) Solve the advection equation on the periodic domain $[0, 1]$ with the initial condition from problem 1(b). Show the solution and comment on your results.

(c) Compute the relative phase error as $\arg(g(\theta))/(-\nu\theta)$, where g is the amplification factor and $\theta = \xi\Delta x$, and plot it for $\theta \in [0, \pi]$. How does the relative phase error and lack of amplitude error relate to the numerical solutions you observed in part (b).