F_K/F_π notes

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Notes on F_K/F_π extrapolation

I. STRATEGY

We want to perform several extrapolations of our F_K/F_π data and perform a weighted model average of these extrapolations. The six extrapolations to be performed are:

1. NLO MA + NNLO c.t.: ratio of F_K and F_π fit functions
    (a) Use full on-shell mixed meson mass as they appear
    (b) Use average mixed-meson mass splitting, where average is taken over all flavors and quark masses
2. NLO MA + NNLO c.t.: Taylor expanded ratio of F_K/F_π extrapolation functions
    (a) Use full on-shell mixed meson mass as they appear
    (b) Use average mixed-meson mass splitting, where average is taken over all flavors and quark masses
3. NLO χPT + discretization + NNLO c.t.: ratio of F_K and F_π fit functions
4. NLO χPT + discretization + NNLO c.t.: Taylor expanded ratio of F_K/F_π extrapolation functions

In all cases, the predicted NLO finite volume corrections should be used. Also - we can change these 6 fits into 18 by treating the F^2 terms that appear in the NLO expressions and definitions of the ϵ_ϕ^2 parameters as

$$F^2 \to \begin{cases} F_\pi^2 \\ F_\pi F_K \\ F_K^2 \end{cases} \tag{1.1}$$

A. Plots to make

We should make plots to show the following

1. Mixed meson mass splitting
2. size of finite volume corrections
3. light quark mass dependence

To make the last plot - what we do is, take all data, extrapolated to infinite volume, and then apply a correction after the fit that shifts the kaon mass to its physical value, leaving only light quark mass dependence. Then, for each lattice spacing, plot the curve vs ϵ_π and finally, plot the continuum extrapolated curve vs. ϵ_π as well.

B. Non analytic NLO loop integral functions

At NLO, the following functions arise in the extrapolation formulae (the ln terms arise from the infinite volume integrals and the sums over Bessel functions are the finite volume corrections):

$$\mathcal{I}(m) = \frac{m^2}{(4\pi)^2} \ln\left(\frac{m^2}{\mu^2}\right) + \frac{m^2}{4\pi^2} \sum_{|\mathbf{n}|\neq 0} \frac{c_n}{mL|\mathbf{n}|} K_1(mL|\mathbf{n}|) \tag{1.2}$$

$$\begin{aligned} d\mathcal{I}(m) &= \frac{1+\ln\left(\frac{m^2}{\mu^2}\right)}{(4\pi)^2} + \sum_{|\mathbf{n}|\neq 0} \frac{c_n}{(4\pi)^2}\left[\frac{2K_1(mL|\mathbf{n}|)}{mL|\mathbf{n}|} - K_0(mL|\mathbf{n}|) - K_2(mL|\mathbf{n}|)\right] \\ &= \frac{1}{(4\pi)^2} + \frac{\mathcal{I}(m)}{m^2} + \sum_{|\mathbf{n}|\neq 0} \frac{c_n}{(4\pi)^2}\left[\frac{2K_1(mL|\mathbf{n}|)}{mL|\mathbf{n}|} - K_0(mL|\mathbf{n}|) - K_2(mL|\mathbf{n}|)\right] \end{aligned} \tag{1.3}$$

$$\mathcal{K}(m, M) = \frac{1}{M^2 - m^2}\left[\mathcal{I}(M) - \mathcal{I}(m)\right] \tag{1.4}$$

$$\begin{aligned} \mathcal{K}_{21}(m, M) &= \int_R \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-m^2)^2} \frac{1}{k^2-M^2} \\ &= \frac{\partial}{\partial m^2} \int_R \frac{d^dk}{(2\pi)^d} \frac{1}{k^2-m^2} \frac{1}{k^2-M^2} \\ &= \frac{\partial}{\partial m^2} \mathcal{K}(m, M) \\ &= \frac{1}{(M^2-m^2)^2}\left[\mathcal{I}(M) - \mathcal{I}(m)\right] - \frac{1}{M^2-m^2} d\mathcal{I}(m) \end{aligned} \tag{1.5}$$

$$\begin{aligned} \mathcal{K}(m_1, m_2, m_3) &= \frac{1}{m_1^2-m_2^2}\frac{1}{m_1^2-m_3^2}\mathcal{I}(m_1) + \frac{1}{m_2^2-m_1^2}\frac{1}{m_2^2-m_3^2}\mathcal{I}(m_2) \\ &+ \frac{1}{m_3^2-m_1^2}\frac{1}{m_3^2-m_2^2}\mathcal{I}(m_3) \end{aligned} \tag{1.6}$$

C. NLO chiral extrapolation formulae

In these expressions, the η meson mass can be approximated with the $SU(3)$ Gell-Mann–Okubo formula

$$m_\eta^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2\,. \tag{1.7}$$

The X mass appearing in the MA formulae is

$$m_X^2 = m_\eta^2 + a^2\Delta_{\mathrm{I}} \tag{1.8}$$

where $a^2\Delta_{\mathrm{I}}$ is the taste-identity mass splitting which MILC has determined [1].

Given the quark mass tuning we have done, at LO in MA EFT [2], all partial quenching parameters are given by the taste-identity splitting

$$\Delta_{rs}^2 = \Delta_{ju}^2 = a^2\Delta_{\mathrm{I}}\,. \tag{1.9}$$

Also, all the mixed meson masses are given by

$$m^2_{val,sea} = \frac{1}{2}m^2_{val,val} + \frac{1}{2}m^2_{sea,sea,5} + a^2\Delta_{\rm Mix} \simeq m^2_{val,val} + a^2\Delta_{\rm Mix}\,. \tag{1.10}$$

We can plot the splitting over flavors and pion masses, and see how well this LO relation holds. We will also fit using the full on-shell masses as they appear in the MA formula as well as use an average mixed-meson mass splitting, $a^2\Delta_{\rm mix}$, averaged over the flavors and ensembles (possibly replace the average with extrapolation).

1. *NLO MA + NNLO c.t. ratio*

For these fits, we will do

$$\frac{F_K}{F_\pi} = \frac{F_K^{\rm nlo}}{F_\pi^{\rm nlo}} + \delta^{\rm NNLO}_{c.t.} \tag{1.11}$$

where

$$\frac{F_\pi^{\rm nlo}}{F_0} = 1 - \frac{\mathcal{I}(m_{ju})}{F^2} - \frac{\mathcal{I}(m_{ru})}{2F^2} + 4\epsilon_\pi^2(4\pi)^2(L_4+L_5) + 8\epsilon_K^2(4\pi)^2L_4 \tag{1.12}$$

$$\begin{aligned}
\frac{F_K^{\rm nlo}}{F_0} = 1 &- \frac{\mathcal{I}(m_{ju})}{2F^2} + \frac{\mathcal{I}(m_\pi)}{8F^2} - \frac{\mathcal{I}(m_{ru})}{4F^2} - \frac{\mathcal{I}(m_{sj})}{2F^2} - \frac{\mathcal{I}(m_{rs})}{4F^2} + \frac{\mathcal{I}(m_{ss})}{4F^2} - \frac{3\mathcal{I}(m_X)}{8F^2}\\
&+ 4\epsilon_\pi^2 L_4 + 4\epsilon_K^2(L_5+2L_4)\\
&+ \Delta_{ju}^2\left[-\frac{d\mathcal{I}(m_\pi)}{8F^2} + \frac{\mathcal{K}(m_\pi,m_X)}{4F^2}\right] - \Delta_{ju}^4\frac{\mathcal{K}_{21}(m_\pi,m_X)}{24F^2}\\
&+ \Delta_{ju}^2\Delta_{rs}^2\left[\frac{\mathcal{K}(m_\pi,m_{ss},m_X)}{6F^2} + \frac{\mathcal{K}_{21}(m_{ss},m_X)}{12F^2}\right]\\
&+ \Delta_{rs}^2\left[\frac{\mathcal{K}(m_{ss},m_X)}{4F^2} - \frac{\mathcal{K}_{21}(m_{ss},m_X)m_K^2}{6F^2} + \frac{\mathcal{K}_{21}(m_{ss},m_X)m_\pi^2}{6F^2}\right]
\end{aligned} \tag{1.13}$$

2. *NLO MA + NNLO c.t. Taylor expanded ratio*

$$\begin{aligned}
\frac{F_K}{F_\pi} = 1 &+ \frac{\mathcal{I}(m_{ju})}{2F^2} + \frac{\mathcal{I}(m_\pi)}{8F^2} + \frac{\mathcal{I}(m_{ru})}{4F^2} - \frac{\mathcal{I}(m_{sj})}{2F^2} + \frac{\mathcal{I}(m_{ss})}{4F^2} - \frac{\mathcal{I}(m_{rs})}{4F^2} - \frac{3\mathcal{I}(m_X)}{8F^2}\\
&+ \Delta_{ju}^2\left[-\frac{d\mathcal{I}(m_\pi)}{8F^2} + \frac{\mathcal{K}(m_\pi,m_X)}{4F^2}\right] - \Delta_{ju}^4\frac{\mathcal{K}_{21}(m_\pi,m_X)}{24F^2}\\
&+ \Delta_{ju}^2\Delta_{rs}^2\left[\frac{\mathcal{K}(m_\pi,m_{ss},m_X)}{6F^2} + \frac{\mathcal{K}_{21}(m_{ss},m_X)}{12F^2}\right]\\
&+ \Delta_{rs}^2\left[\frac{\mathcal{K}(m_{ss},m_X)}{4F^2} - \frac{\mathcal{K}_{21}(m_{ss},m_X)m_K^2}{6F^2} + \frac{\mathcal{K}_{21}(m_{ss},m_X)m_\pi^2}{6F^2}\right]\\
&+ 4(4\pi)^2L_5(\mu)\frac{m_K^2-m_\pi^2}{(4\pi F)^2}
\end{aligned} \tag{1.14}$$

3. NLO χPT + NNLO c.t. ratio

For these fits, we will do

$$\frac{F_K}{F_\pi} = \frac{F_K^{\rm nlo}}{F_\pi^{\rm nlo}} + \delta_{c.t.}^{\rm NNLO} \tag{1.15}$$

where

$$\frac{F_\pi^{\rm nlo}}{F_0} = 1 - \frac{\mathcal{I}(m_\pi)}{F^2} - \frac{1}{2}\frac{\mathcal{I}(m_K)}{F^2} + 4\epsilon_\pi^2(4\pi)^2(L_4+L_5) + 8\epsilon_K^2(4\pi)^2 L_4 \tag{1.16}$$

and

$$\frac{F_K^{\rm nlo}}{F_0} = 1 - \frac{3}{8}\frac{\mathcal{I}(m_\pi)}{F^2} - \frac{3}{4}\frac{\mathcal{I}(m_K)}{F^2} - \frac{3\mathcal{I}(m_\eta)}{8F^2} + 4\epsilon_\pi^2 L_4 + 4\epsilon_K^2(L_5+2L_4) \tag{1.17}$$

4. NLO χPT + NNLO c.t. Taylor expansion

$$\frac{F_K}{F_\pi} = 1 + \frac{5}{8}\frac{\mathcal{I}(m_\pi)}{F^2} - \frac{1}{4}\frac{\mathcal{I}(m_K)}{F^2} - \frac{3}{8}\frac{\mathcal{I}(m_\eta)}{F^2} + 4(\epsilon_K^2-\epsilon_\pi^2)(4\pi)^2 L_5(\Lambda_\chi) + \delta_{c.t.}^{\rm NNLO} \tag{1.18}$$

D. NNLO counter terms

The NNLO and NNNLO counter term expressions are

$$\delta_{c.t.}^{\rm NNLO} = \epsilon_a^2(\epsilon_K^2-\epsilon_\pi^2)A_{2,2} + (\epsilon_K^2-\epsilon_\pi^2)^2 M_{4,0,0} + \epsilon_K^2(\epsilon_K^2-\epsilon_\pi^2)M_{2,2,0} + \epsilon_\pi^2(\epsilon_K^2-\epsilon_\pi^2)M_{2,0,2} \tag{1.19}$$

$$\delta_{c.t.}^{\rm NNNLO} = \epsilon_a^4(\epsilon_K^2-\epsilon_\pi^2)A_{4,2} + \epsilon_a^2(\epsilon_K^2-\epsilon_\pi^2)^2 A_{2,4} + \epsilon_a^2\epsilon_K^2(\epsilon_K^2-\epsilon_\pi^2)A_{2,2,2,0} + \epsilon_a^2\epsilon_\pi^2(\epsilon_K^2-\epsilon_\pi^2)A_{2,2,0,2} \tag{1.20}$$

$\lvert\mathbf{n}\rvert$	1	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{4}$	$\sqrt{5}$	$\sqrt{6}$	$\sqrt{7}$	$\sqrt{8}$	$\sqrt{9}$	$\sqrt{10}$
c_n	6	12	8	6	24	24	0	12	30	24

TABLE I. Finite volume weight factors for the first few finite volume modes.

II. OLD

In this expression We can estimate the FV corrections by tracking only the terms with π and ju meson loops. Introducing shorthand notation, we find

$$\begin{aligned}
\delta\frac{F_K}{F_\pi} &= \frac{\delta\mathcal{I}(m_{ju})}{2F^2} + \frac{\delta\mathcal{I}(m_\pi)}{8F^2} + \Delta_{ju}^2\left[-\frac{\delta d\mathcal{I}(m_\pi)}{8F^2} + \frac{\delta\mathcal{K}(m_\pi, m_X)}{4F^2}\right] \\
&\quad - \Delta_{ju}^4\frac{\delta\mathcal{K}_{21}(m_\pi, m_X)}{24F^2} + \Delta_{ju}^2\Delta_{rs}^2\frac{\delta\mathcal{K}(m_\pi, m_{ss}, m_X)}{6F^2} \\
&= \sum_{|\mathbf{n}|\neq 0} c_n\Bigg\{2\frac{\epsilon_{ju}^2 K_1^{ju}}{x_{ju}^{Ln}} + \frac{1}{2}\frac{\epsilon_\pi^2 K_1^\pi}{x_\pi^{Ln}} - \frac{1}{8}\frac{\Delta_{ju}^2}{(4\pi F)^2}\left[\frac{2K_1^\pi}{x_\pi^{Ln}} - K_0^\pi - K_2^\pi\right] \\
&\quad + \frac{\Delta_{ju}^2}{4F^2}\frac{-1}{(m_X^2 - m_\pi^2)}\frac{m_\pi^2}{4\pi^2}\frac{K_1^\pi}{x_\pi^{Ln}} \\
&\quad - \frac{\Delta_{ju}^4}{24F^2}\left[\frac{1}{(m_X^2 - m_\pi^2)^2}\frac{-m_\pi^2}{4\pi^2}\frac{K_1^\pi}{x_\pi^{Ln}} - \frac{1}{m_X^2 - m_\pi^2}\frac{1}{(4\pi)^2}\left[\frac{2K_1^\pi}{x_\pi^{Ln}} - K_0^\pi - K_2^\pi\right]\right] \\
&\quad + \frac{\Delta_{ju}^2\Delta_{rs}^2}{6F^2}\frac{1}{(m_\pi^2 - m_X^2)(m_\pi^2 - m_{ss}^2)}\frac{m_\pi^2}{4\pi^2}\frac{K_1^\pi}{x_\pi^{Ln}}\Bigg\} \\
&= \sum_{|\mathbf{n}|\neq 0} c_n\Bigg\{2\epsilon_{ju}^2\frac{K_1^{ju}}{x_{ju}^{Ln}} + \frac{\epsilon_\pi^2}{2}\frac{K_1^\pi}{x_\pi^{Ln}} - \frac{\epsilon_{\Delta_{ju}}^2}{8}\left[\frac{2K_1^\pi}{x_\pi^{Ln}} - K_0^\pi - K_2^\pi\right] - \frac{\epsilon_{\Delta_{ju}}^2\epsilon_\pi^2}{\epsilon_X^2 - \epsilon_\pi^2}\frac{K_1^\pi}{x_\pi^{Ln}} \\
&\quad + \frac{\epsilon_{\Delta_{ju}}^4}{24}(4\pi)^4F^2\left[\frac{4F^2\epsilon_\pi^2/(4\pi F)^4}{(\epsilon_X^2 - \epsilon_\pi^2)^2}\frac{K_1^\pi}{x_\pi^{Ln}} + \frac{1/(4\pi F)^2}{\epsilon_X^2 - \epsilon_\pi^2}\frac{1}{(4\pi)^2}\left[\frac{2K_1^\pi}{x_\pi^{Ln}} - K_0^\pi - K_2^\pi\right]\right] \\
&\quad + \frac{\epsilon_{\Delta_{ju}}^2\epsilon_{\Delta_{rs}}^2 4\epsilon_\pi^2}{6(\epsilon_\pi^2 - \epsilon_X^2)(\epsilon_\pi^2 - \epsilon_{ss}^2)}\frac{K_1^\pi}{x_\pi^{Ln}}\Bigg\} \\
&= \sum_{|\mathbf{n}|\neq 0} c_n\Bigg\{2\epsilon_{ju}^2\frac{K_1^{ju}}{x_{ju}^{Ln}} + \frac{\epsilon_\pi^2}{2}\frac{K_1^\pi}{x_\pi^{Ln}} - \frac{\epsilon_{\Delta_{ju}}^2}{8}\left[\frac{2K_1^\pi}{x_\pi^{Ln}} - K_0^\pi - K_2^\pi\right] - \frac{\epsilon_{\Delta_{ju}}^2\epsilon_\pi^2}{\epsilon_X^2 - \epsilon_\pi^2}\frac{K_1^\pi}{x_\pi^{Ln}} \\
&\quad + \frac{\epsilon_{\Delta_{ju}}^4}{24}\left[\frac{4\epsilon_\pi^2}{(\epsilon_X^2 - \epsilon_\pi^2)^2}\frac{K_1^\pi}{x_\pi^{Ln}} + \frac{1}{\epsilon_X^2 - \epsilon_\pi^2}\left[\frac{2K_1^\pi}{x_\pi^{Ln}} - K_0^\pi - K_2^\pi\right]\right] \\
&\quad + \frac{2}{3}\frac{\epsilon_{\Delta_{ju}}^2\epsilon_{\Delta_{rs}}^2\epsilon_\pi^2}{(\epsilon_\pi^2 - \epsilon_X^2)(\epsilon_\pi^2 - \epsilon_{ss}^2)}\frac{K_1^\pi}{x_\pi^{Ln}}\Bigg\}
\end{aligned} \tag{2.1}$$

where $K_i^\phi = K_i(m_\phi L|\mathbf{n}|)$, $x_\phi^{LN} = m_\phi L|\mathbf{n}|$ and $\epsilon_Y^2 = m_Y^2/(4\pi F)^2$. The first 10 weights in the finite volume sum are listed in Table I.

[1] A. Bazavov *et al.* (MILC), "Lattice QCD ensembles with four flavors of highly improved staggered quarks," Phys. Rev. **D87**, 054505 (2013), arXiv:1212.4768 [hep-lat].

[2] Jiunn-Wei Chen, Donal O'Connell, and Andre Walker-Loud, "Two Meson Systems with Ginsparg-Wilson Valence Quarks," Phys. Rev. **D75**, 054501 (2007), arXiv:hep-lat/0611003 [hep-lat].