On the decidability of a fragment of preferential LTL

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Abstract

Linear Temporal Logic (LTL) has found extensive applications in Computer Science and Artificial Intelligence, notably as a formal framework for representing and verifying computer systems that vary over time. Non-monotonic reasoning, on the other hand, allows us to formalize and reason with exceptions and the dynamics of information. The goal of this paper is therefore to enrich temporal formalisms with non-monotonic reasoning features. We do so by investigating a preferential semantics for defeasible LTL along the lines of that extensively studied by Kraus et al. in the propositional case and recently extended to modal and description logics. The main contribution of the paper is a decidability result for a meaningful fragment of preferential LTL that can serve as the basis for further exploration of defeasibility in temporal formalisms.

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1 Introduction

Specification and verification of dynamic computer systems is an important task, given the increasing number of new computer technologies being developed. Recent examples include blockchain technology and various existing tools for home automation of the different production chains provided by Industry 4.0. Therefore, it is fundamental to ensure that systems based on them have the desired behavior but, above all, satisfy safety standards. This becomes even more critical with the increasing deployment of artificial intelligence techniques as well as the need to explain their behaviors.

Several approaches for qualitative analysis of computer systems have been developed. Among the most fruitful are the different families of temporal logic. The success of these is due mainly to their simplified syntax compared to that of first-order logic, their intuitive syntax, semantics and their good computational properties. One of the members of this family is Linear Temporal Logic [15, 19], known as LTL, is wildly used in formal verification and specification of computer programs.

Despite the success and wide use of linear temporal logic, it remains limited for modeling and reasoning about the real aspects of computer systems or those that depend on them. In fact, computer systems are not either 100% secure or 100% defective, and the properties we wish to check may have innocuous and tolerable exceptions, or conversely, exceptions that must be carefully addressed in order to guarantee the overall reliability of the system. Similarly, the expected behavior of a system may be correct not for all possible execution, but rather for its most “normal” or expected executions.

It turns out that LTL, because it is a logical formalism of the so-called classical type, whose underlying reasoning is that of mathematics and not that of common sense, does not allow at all to formalize the different nuances of the exceptions and even less to treat them. First of all, at the level of the object language (that of the logical symbols), it has operators behaving monotonically, and at the level of reasoning, posses a notion of logical consequence which is monotonic too, and consequently, it is not adapted to the evolution of defeasible facts.

Non-monotonic reasoning (NMR), on the other hand, allows to formalize and reason with exceptions, it has been widely studied by the AI community for over 40 years now. Such is the case of Kraus et al. [12] , known as the KLM approach.

However, the major contributions in this area are limited to the propositional framework. It is only recently that some approaches to non-monotonic reasoning, such as belief revision, default

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rules and preferential approaches, have been studied for more expressive logics than propositional logic, including modal [3, 5] and description logics [4]. The objective of our study is to establish a bridge between temporal formalisms for the specification and verification of computer systems and approaches to non-monotonic reasoning, in particular the preferential one, which satisfactorily solves the limitations raised above.

In this paper, we define a logical framework for reasoning about defeasible properties of program executions, we investigate the integration of preferential semantics in the case of LTL, hereby introducing preferential linear temporal logic $LTL^{\sim}$. The remainder of the present paper is structured as follows: In Section 3 we set up the notation and appropriate semantics of our language. In Sections 4, 5 and 6, we investigate the satisfiability problem of this formalism. The appendix contains proofs of results in this paper. The remaining proofs can be viewed anonymously in `https://github.com/calleann/Preferential_LTL`.

2 Preliminaries: LTL and the KLM approach to NMR

Let $\mathcal{P}$ be a finite set of *propositional atoms*. The set of operators in the *Linear Temporal Logic* can be split into two parts: the set of *Boolean connectives* $(\neg, \wedge)$, and that of *temporal operators* $(\Box, \Diamond, \bigcirc, \mathcal{U})$, where $\Box$ reads as *always*, $\Diamond$ as *eventually*, $\bigcirc$ as *next* and $\mathcal{U}$ as *until*. The set of well-formed sentences expressed in LTL is denoted by $\mathcal{L}$. Sentences of $\mathcal{L}$ are built up according to the following grammar: $\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box\alpha \mid \Diamond\alpha \mid \bigcirc\alpha \mid \alpha\mathcal{U}\alpha$.

Let the set of natural numbers $\mathbb{N}$ denote time points. A *temporal interpretation* I is a mapping function $V : \mathbb{N} \longrightarrow 2^{\mathcal{P}}$ which associates each time point $t \in \mathbb{N}$ with a set of propositional atoms $V(t)$ corresponding to the set of propositions that are true in t. (Propositions not belonging to $V(t)$ are assumed to be false at the given time point.) The truth conditions of LTL sentences are defined as follows, where I is a temporal interpretation and t a time point in I:

- $I, t \models p$ if $p \in V(t)$; $I, t \models \neg\alpha$ if $I, t \not\models \alpha$;
- $I, t \models \alpha \wedge \alpha'$ if $I, t \models \alpha$ and $I, t \models \alpha'$; $I, t \models \alpha \vee \alpha'$ if $I, t \models \alpha$ or $I, t \models \alpha'$;
- $I, t \models \Box\alpha$ if $I, t' \models \alpha$ for all $t' \in \mathbb{N}$ s.t. $t' \geq t$; $I, t \models \Diamond\alpha$ if $I, t' \models \alpha$ for some $t' \in \mathbb{N}$ s.t. $t' \geq t$;
- $I, t \models \bigcirc\alpha$ if $I, t+1 \models \alpha$;
- $I, t \models \alpha\mathcal{U}\alpha'$ if $I, t' \models \alpha'$ for some $t' \geq t$ and for all $t \leq t'' < t'$ we have $I, t'' \models \alpha$.

We say $\alpha \in \mathcal{L}$ is *satisfiable* if there are I and $t \in \mathbb{N}$ such that $I, t \models \alpha$.

We now give a brief outline to Kraus et al.'s [12] approach to non-monotonic reasoning. A propositional *defeasible consequence relation* $\mid\!\sim$ [12] is defined as a binary relation on sentences of an underlying propositional logic. The semantics of preferential consequence relation is in terms of *preferential models*: A preferential model on a set of atomic propositions $\mathcal{P}$ is a tuple $\mathscr{P} \stackrel{\text{def}}{=} (S, l, \prec)$ where S is a set of elements called states, $l : S \longrightarrow 2^{\mathcal{P}}$ is a mapping which assigns to each state s a single world $m \in 2^{\mathcal{P}}$ and $\prec$ is a *strict partial* order on S satisfying smoothness condition. Intuitively, the states that are lower down in the ordering are more plausible, normal or in a general case preferred, than those that are higher up. A statement of the form $\alpha \mid\!\sim \beta$ holds in a preferential model iff he minimal α-states are also β-states.

3 Preferential LTL

In this paper, we introduce a new formalism for reasoning about time that is able to distinguish between normal and exceptional points of time. We do so by investigating a defeasible extension of LTL with a preferential semantics. The following example introduces a case scenario we shall be using in the remainder of this section, with the purpose of giving a motivation for this formalism and better illustrating the definitions in what follows.

▶ **Example 1.** We have a computer program in which the values of its variables change with time. In particular, the agent wants to check two parameters, say x and y. These two variables take one and only one value between 1 and 3 on each iteration of the program. We represent the set of atomic propositions by $\mathcal{P} = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ where x_i (resp. y_i) for all $i \in \{1, 2, 3\}$ is true iff the variable x (resp. y) has the value i in a current iteration. Figure 1 depicts a temporal interpretation corresponding to a possible behaviour of such a program:

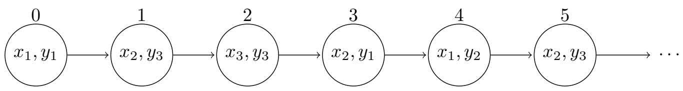

Figure 1 LTL interpretation V (for $t > 5$, $V(t) = V(5) = \{x_2, y_3\}$)

Under normal circumstances, the program assigns the value 3 to y whenever $x = 2$. We can express this fact using classical LTL as follows: $\Box(x_2 \to y_3)$, with $x_2 \to y_3$ is defined by $\neg x_2 \vee y_3$. Nevertheless, the agent notices that there is one exceptional iteration (Iteration 3) where the program assigns the value 1 to y when $x = 2$.

Some might consider that the current program is defective at some points of time. In LTL, the statement $\Box(x_2 \to y_3) \wedge \Diamond(x_2 \wedge y_1)$ will always be false, since y cannot have two different values in an iteration where $x = 2$. Nonetheless we want to propose a logical framework that is exception tolerant for reasoning about a system's behaviour. In order to express this general tendency $(x_2 \to y_3)$ while taking into account that there might be some exceptional iterations which do not crash the program. We base our semantic constructions on the preferential approach [16, 12].

3.1 Introducing defeasible temporal operators

Britz & Varzinczak [5] introduced new modal operators called defeasible modalities. In their setting, defeasible operators, unlike their classical counterparts, are able to single out normal worlds from those that are less normal or exceptional in the reasoner's mind. Here we extend the vocabulary of classical LTL with the *defeasible temporal operators* $\tilde{\Box}$ and $\tilde{\Diamond}$. Sentences of the resulting logic $LTL^{\sim}$ are built up according to the following grammar:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box\alpha \mid \Diamond\alpha \mid \bigcirc\alpha \mid \alpha\mathcal{U}\alpha \mid \tilde{\Box}\alpha \mid \tilde{\Diamond}\alpha$$

The intuition behind these new operators is the following: $\tilde{\Box}$ reads as *defeasible always* and $\tilde{\Diamond}$ reads as *defeasible eventuality*.

▶ **Example 2.** Going back to our example 1, we can describe the normal behaviour of the program using the statement $\tilde{\Box}(x_2 \to y_3) \wedge \Diamond(x_2 \wedge y_1)$. In all normal future time points, the program assigns the value 3 to y when $x = 2$. Although unlikely, there are some exceptional time points in the future where $x = 2$ and $y = 1$. But those are 'ignored' by the defeasible always operator.

The set of all well-formed $LTL^{\sim}$ sentences is denoted by $\mathcal{L}^{\sim}$. It is worth to mention that any well-formed sentence $\alpha \in \mathcal{L}$ is a sentence of $\mathcal{L}^{\sim}$. We denote a subset of our language that contains only Boolean connectives, the two defeasible operators $\tilde{\Box}$, $\tilde{\Diamond}$ and their classical counterparts by $\mathcal{L}^{\star}$. Next we shall discuss how to interpret statements that have this defeasible aspect and how to determine the truth values of each well-formed sentence in $\mathcal{L}^{\sim}$.

3.2 Preferential semantics

First of all, in order to interpret the sentences of $\mathcal{L}^{\sim}$ we consider, as stated on the preliminaries, $(\mathbb{N}, <)$ to be a temporal structure. Hence, a temporal interpretation that associates each time point t with a truth assignment of all propositional atoms.

The preferential component of the interpretation of our language is directly inspired by the preferential semantics proposed by Shoham [17] and used in the KLM approach [12]. The preference relation $\prec$ is a strict partial order on our points of time. Following Kraus et al. [12], $t \prec t'$ means that t is more preferred than t'. The reasoner has now the tools to express the preference between points of time by comparing them w.r.t. each other, with time points lower down the order being more preferred than those higher up.

▶ **Definition 3** **(Minimality w.r.t. $\prec$).** *Let $\prec$ be a strict partial order on a set $\mathbb{N}$ and $N \subseteq \mathbb{N}$. The set of the minimal elements of N w.r.t. $\prec$, denoted by $min_{\prec}(N)$, is defined by $min_{\prec}(N) \stackrel{\text{def}}{=} \{t \in N \mid$ there is no $t' \in N$ such that $t' \prec t\}$.*

▶ **Definition 4** **(Well-founded set).** *Let $\prec$ be a strict partial order on a set $\mathbb{N}$. We say $\mathbb{N}$ is* well-founded w.r.t. $\prec$ *iff $min_{\prec}(N) \neq \emptyset$ for every $\emptyset \neq N \subseteq \mathbb{N}$.*

▶ **Definition 5** **(Preferential temporal interpretation).** *An $LTL^{\sim}$ interpretation on a set of propositional atoms $\mathcal{P}$, also called preferential temporal interpretation on $\mathcal{P}$, is a pair $I \stackrel{\text{def}}{=} (V, \prec)$ where V is a temporal interpretation on $\mathcal{P}$, and $\prec \subseteq \mathbb{N} \times \mathbb{N}$ is a strict partial order on $\mathbb{N}$ such that $\mathbb{N}$ is well-founded w.r.t. $\prec$. We denote the set of preferential temporal interpretations by $\mathfrak{I}$.*

In what follows, given a preference relation $\prec$ and a time point $t \in \mathbb{N}$, the set of *preferred time points relative to* t is the set $min_{\prec}([t, +\infty[)$ which is denoted in short by $min_{\prec}(t)$. It is also worth to point out that given a preferential interpretation $I = (V, \prec)$ and $\mathbb{N}$, the set $min_{\prec}(t)$ is always a non-empty subset of $[t, +\infty[$ at any time point $t \in \mathbb{N}$.

Preferential temporal interpretations provide us with an intuitive way of interpreting sentences of $\mathcal{L}^{\sim}$. Let $\alpha \in \mathcal{L}^{\sim}$, let $I = (V, \prec)$ be a preferential interpretation, and let t be a time point in I in $\mathbb{N}$. Satisfaction of α at t in I, denoted $I, t \models \alpha$, is defined as follows:

- $I, t \models \Box\!\!\!\sim\alpha$ if $I, t' \models \alpha$ for *all* $t' \in min_{\prec}(t)$;
- $I, t \models \Diamond\!\!\!\sim\alpha$ if $I, t' \models \alpha$ for *some* $t' \in min_{\prec}(t)$.

The truth values of Boolean connectives and classical modalities are defined as in LTL. The intuition behind a sentence like $\Box\!\!\!\sim\alpha$ is that α holds in *all* preferred time points that come after t. $\Diamond\!\!\!\sim\alpha$ intuitively means that α holds on at least one preferred time point relative in the future of t.

We say $\alpha \in \mathcal{L}^{\sim}$ is *preferentially satisfiable* if there is a preferential temporal interpretation I and a time point t in $\mathbb{N}$ such that $I, t \models \alpha$. We can show that $\alpha \in \mathcal{L}^{\sim}$ is *preferentially satisfiable* iff there is a preferential temporal interpretation I s.t. $I, 0 \models \alpha$. A sentence $\alpha \in \mathcal{L}^{\sim}$ is *valid* (denoted by $\models \alpha$) iff for all temporal interpretation I and time points t in $\mathbb{N}$, we have $I, t \models \alpha$.

▶ **Example 6.** Going back to Example 1, we can see that the time points 5 and 1 are more "normal" than iteration 3. By adding preferential preference $\prec := \{(5, 3), (1, 3)\}$, we denote the preferential temporal interpretation by $I = (V, \prec)$. We have that $I, 0 \not\models \Box(x_2 \rightarrow y_3) \wedge \Diamond(x_2 \wedge y_1)$ and $I, 0 \models \Box\!\!\!\sim(x_2 \rightarrow y_3) \wedge \Diamond(x_2 \wedge y_1)$.

We can see that the addition of $\prec$ relation preserves the truth values of all classical temporal sentences. Moreover, for every $\alpha \in \mathcal{L}$, we have that α is satisfiable in LTL if and only if α is preferentially satisfiable in $LTL^{\sim}$.

We discuss some properties of these defeasible modalities next. In what follows, let α, β be well-formed sentences in $\mathcal{L}^{\sim}$. We have duality between our defeasible operators: $\models \Box\!\!\!\sim\alpha \leftrightarrow \neg \Diamond\!\!\!\sim\neg\alpha$. We also have $\models \Box\alpha \rightarrow \Box\!\!\!\sim\alpha$ and $\models \Diamond\!\!\!\sim\alpha \rightarrow \Diamond\alpha$. Intuitively, This property states that if a statement holds in all of future time points of any given point of time t, it holds on all our *future preferred* time points. As intended, this property establishes the defeasible always as "weaker" than the classical always. It can commonly be accepted since the set of all preferred future states are in the future. This is why we named $\widetilde{\Box}$ *defeasible always*. On the other hand, we see that $\widetilde{\Diamond}$ is "stronger" than classical eventually, the statement within $\widetilde{\Diamond}$ holds at a preferable future.

The axiom of distributivity (K) can be stated in terms of our defeasible operators. We can also verify the validity of these two statements $\models \widetilde{\Box}(\alpha \wedge \beta) \leftrightarrow (\widetilde{\Box}\alpha \wedge \widetilde{\Box}\beta)$ and $\models (\widetilde{\Box}\alpha \vee \widetilde{\Box}\beta) \rightarrow \widetilde{\Box}(\alpha \vee \beta)$, the converse of the second statement is not always true.

The reflexivity axiom (T) for the classical operators does not hold in the case of defeasible modalities. We can easily find an interpretation $I = (V, \prec)$ where $I, t \not\models \widetilde{\Box}\alpha \rightarrow \alpha$. Indeed, since we can have $t \notin min_{\prec}(t)$ for a temporal point t, we can have $I, t \models \widetilde{\Box}\alpha$ and $I, t \models \neg\alpha$.

One thing worth pointing out is the set of future preferred time points changes dynamically as we move forward in time. Given three time points $t_1 \leq t_2 \leq t_3$, $t_3 \notin min_{\prec}(t_1)$ whilst $t_3 \in min_{\prec}(t_2)$ could be true in some cases. Hence, if $I, t \models \widetilde{\Box}\widetilde{\Box}\alpha$ does not imply that for all $t' \in min_{\prec}(t)$, $I, t' \models \widetilde{\Box}\alpha$. Therefore, the transitivity axiom (4) does not hold also in our defeasible modalities. On the other hand, given those three time points, $t_3 \notin min_{\prec}(t_1)$ implies that $t_3 \notin min_{\prec}(t_2)$.

And since we do not have a version of the axioms (T) and (4) for our defeasible operators, we do not have the collapsing property on the case $\widetilde{\Box}$, $\widetilde{\Diamond}$. Redundant sentences in the case modal sentences such as $\Box\Box \ldots \Box\alpha$ can be reduced to $\Box\alpha$. It is not the case for our preferential operators $\widetilde{\Box}$ and $\widetilde{\Diamond}$.

3.3 State-dependent preferential interpretations

We define a class of well-behaved $LTL^{\sim}$ interpretations that are useful in the remainder of the paper.

▶ **Definition 7** **(State-dependent preferential interpretations).** *Let $I = (V, \prec) \in \mathfrak{I}$. I is state-dependent preferential interpretation iff for every $i, j, i', j' \in \mathbb{N}$, if $V(i') = V(i)$ and $V(j') = V(j)$, then $(i, j) \in \prec$ iff $(i', j') \in \prec$.*

In what follows, $\mathfrak{I}^{sd}$ denotes the set of all state-dependent interpretations. The intuition behind setting up this restriction is to have a more compact form of expressing preference over time points. In a way, time points with similar valuations are considered to be identical with regards to $\prec$, they express the same preferences towards other time points. Moreover, we have some interesting properties that do not in the general case. In particular, we have the following property :

▶ **Proposition 8.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i, i', j, j' \in \mathbb{N}$ s.t. $i \leq i'$, $i' \leq j'$ and $j \in min_{\prec}(i)$. If $V(j) = V(j')$, then $j' \in min_{\prec}(i')$.*

This property is specific to the class of state-dependent interpretations. However, the following proposition is true for every $I \in \mathfrak{I}$.

▶ **Proposition 9.** *Let $I = (V, \prec) \in \mathfrak{I}$ and let $i, j \in \mathbb{N}$ s.t. $j \in min_{\prec}(i)$. For all $i \leq i' \leq j$, we have $j \in min_{\prec}(i')$.*

4 A useful representation of preferential structures

One of the objectives of this paper is to establish some computational properties about the satisfiability problem. In order to do this, we introduce into the sequel different structures inspired by the approach followed by Sistla and Clarke in [18]. They observe that in every LTL interpretation, there is a time point t after which every t-successor's valuation occurs infinitely many times. This is an obvious consequence of having an infinite set of time points and a finite number of possible valuations. That is the case also for $LTL^{\sim}$ interpretations.

▶ **Lemma 10.** *Let $I = (V, \prec) \in \mathfrak{I}$. There exists a $t \in \mathbb{N}$ s.t. for all $l \in [t, +\infty[$, there is a $k > l$ where $V(l) = V(k)$.*

For an interpretation $I \in \mathfrak{I}$, we denote the first time point where the condition set in Lemma 10 is satisfied by $\mathtt{t}_I$. We can split each temporal structure into two intervals: an initial and a final part.

▶ **Definition 11.** *Let $I = (V, \prec) \in \mathfrak{I}$. We define: $init(I) \stackrel{\text{def}}{=} [0, \mathtt{t}_I[$; $final(I) \stackrel{\text{def}}{=} [\mathtt{t}_I, +\infty[$; $range(I) \stackrel{\text{def}}{=} \{V(i) \mid i \in final(I)\}$; $val(I) \stackrel{\text{def}}{=} \{V(i) \mid i \in \mathbb{N}\}$; $size(I) \stackrel{\text{def}}{=} length(init(I)) + card(range(I))$, where $length(\cdot)$ denotes the length of a sequence and $card(\cdot)$ set cardinality.*

In the size of I we count the number of time points in the initial part and the number of valuations contained in the final part. In what follows, we discuss some properties concerning these notions and state dependent interpretations.

▶ **Proposition 12.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i \leq j \leq i' \leq j'$ be time points in $final(I)$ s.t. $V(j) = V(j')$. Then we have $j \in min_{\prec}(i)$ iff $j' \in min_{\prec}(i')$.*

▶ **Lemma 13.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $i \leq i'$ be time points of $final(I)$ where $V(i) = V(i')$. Then for every $\alpha \in \mathcal{L}^\star$, we have $I, i \models \alpha$ iff $I, i' \models \alpha$.*

What we have in Lemma 13 is that given an interpretation $I \in \mathfrak{I}^{sd}$, points of time in $final(I)$ that have the same valuations satisfy exactly the same sentences.

▶ **Definition 14** (**Faithful Interpretations**)**.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two interpretations over the same set of atoms $\mathcal{P}$. We say that I, I' are faithful interpretations if $val(I) = val(I')$ and, for all $i, j, i', j' \in \mathbb{N}$ s.t. $V'(i') = V(i)$ and $V'(j') = V(j)$, we have $(i, j) \in \prec$ iff $(i', j') \in \prec'$.*

Throughout this paper, we write $init(I) \doteq init(I')$ as shorthand for the condition that states: $length(init(I)) = length(init(I'))$ and for each $i \in init(I)$ we have $V(i) = V'(i)$.

▶ **Lemma 15.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $V'(0) = V(0)$ (in case $init(I)$ is empty), $init(I) \doteq init(I')$, and $range(I) = range(I')$. Then for all $\alpha \in \mathcal{L}^\star$, we have that $I, 0 \models \alpha$ iff $I', 0 \models \alpha$.*

Lemma 15 implies that the ordering of time points in $final(\cdot)$ does not matter, and what matters is the $range(\cdot)$ of valuations contained within it. It is worth to mention that Lemma 13 and 15 hold only in the case interpretations in $\mathfrak{I}^{sd}$ and they are not always true in the general case.

Sistla & Clarke [18] introduced the notion of acceptable sequences. The general purpose behind it is the ability to build, from an initial interpretation, other interpretations. We adapt this notion for preferential temporal structures. We then introduce the notion of pseudo-interpretations that will come in handy in showing decidability of the satisfiability problem in $\mathcal{L}^\star$ in the upcoming section.

In the sequel, the term temporal sequence or sequence in short, will denote a sequence of ordered integer numbers. A sequence allows to represent a set of time points. Sometimes, we will consider integer intervals as sequences. Moreover, given two sequences N_1, N_2, the union of N_1 and N_2, denoted by $N_1 \cup N_2$, is the sequence containing only elements of N_1 and N_2. An acceptable sequence is a temporal sequence that is built relatively to a preferential temporal interpretation I as follows:

▶ **Definition 16** (**Acceptable sequence w.r.t. I**)**.** *Let $I = (V, \prec) \in \mathfrak{I}$ and N be a sequence of temporal time points. N is an acceptable sequence w.r.t. I iff for all $i \in N \cap final(I)$ and for all $j \in final(I)$ s.t. $V(i) = V(j)$, we have $j \in N$.*

The particularity we are looking for is that any picked time point in $init(\cdot)$ (resp. $final(\cdot)$) will remain in the initial (resp. final) part of the new interpretation. It is worth pointing out that an acceptable sequence w.r.t. a preferential temporal interpretation can be either finite or infinite.

Moreover, $\mathbb{N}$ is an acceptable sequence w.r.t. any interpretation $I \in \mathfrak{I}$. The purpose behind the notion of acceptable sequence is to construct new interpretations starting from an $LTL^{\sim}$ interpretation.

Given N an acceptable sequence w.r.t. I, if N has a time point t in $\mathit{final}(I)$, then all time points t' that have the same valuation as t must be in N. Thus, we have an infinite sequence of time points. As such, we can define an initial part and a final part, in a similar way as $LTL^{\sim}$ interpretations. We let $\mathit{init}(I, N)$ be the largest subsequence of N that is a subsequence of $\mathit{init}(I)$. Note that if N does not contain any time point of $\mathit{final}(I)$, then N is finite.

We now define the notions $\mathit{init}(\cdot)$, $\mathit{final}(\cdot)$, $\mathit{range}(\cdot)$, and $\mathit{size}(\cdot)$ for acceptable sequences.

▶ **Definition 17.** *Let $I = (V, \prec) \in \mathfrak{I}$, and let N be an acceptable sequence w.r.t. I. We define: $\mathit{init}(I, N) \stackrel{\text{def}}{=} N \cap \mathit{init}(I)$; $\mathit{final}(I, N) \stackrel{\text{def}}{=} N \setminus \mathit{init}(I, N)$; $\mathit{range}(I, N) \stackrel{\text{def}}{=} \{V(t) \mid t \in \mathit{final}(I, N)\}$; $\mathit{val}(I, N) \stackrel{\text{def}}{=} \{V(t) \mid t \in N\}$; $\mathit{size}(I, N) \stackrel{\text{def}}{=} \mathit{length}(\mathit{init}(I, N)) + \mathit{card}(\mathit{range}(I, N))$.*

It is worth mentioning that, thanks to Definition 16, given an acceptable sequence w.r.t. I, we have $\mathit{size}(I, N) \leq \mathit{size}(I)$.

▶ **Definition 18** (**Pseudo-interpretation over N**)**.** *Let $I = (V, \prec) \in \mathfrak{I}$ and N be an acceptable sequence w.r.t. I. The pseudo-interpretation over N is the tuple $I^N \stackrel{\text{def}}{=} (N, V^N, \prec^N)$ where:*

- *$V^N : N \longrightarrow 2^{\mathcal{P}}$ is a valuation function over N, where for all $i \in N$, we have $V^N(i) = V(i)$,*
- *$\prec^N \subseteq N \times N$, where for all $(i, j) \in N^2$, we have $(i, j) \in \prec^N$ iff $(i, j) \in \prec$.*

The truth values of $\mathcal{L}^{\star}$ sentences in pseudo-interpretations are defined in a similar fashion as for preferential temporal interpretations. With $\models_{\mathscr{P}}$ we denote the truth values of sentences in a pseudo-interpretation. We highlight truth values for classical and defeasible modalities.

- $I^N, t \models_{\mathscr{P}} \Box\alpha$ if $I^N, t' \models_{\mathscr{P}} \alpha$ for all $t' \in N$ s.t. $t' \geq t$;
- $I^N, t \models_{\mathscr{P}} \Diamond\alpha$ if $I^N, t' \models_{\mathscr{P}} \alpha$ for some $t' \in N$ s.t. $t' \geq t$;
- $I^N, t \models_{\mathscr{P}} \boxminus\alpha$ if for all $t' \in N$ s.t. $t' \in \mathit{min}_{\prec^N}(t)$, we have $I^N, t \models_{\mathscr{P}} \alpha$;
- $I^N, t \models_{\mathscr{P}} \Diamondminus\alpha$ if $I^N, t' \models_{\mathscr{P}} \alpha$ for some $t' \in N$ s.t. $t' \in \mathit{min}_{\prec^N}(t)$.

▶ **Proposition 19.** *Let $I = (V, \prec) \in \mathfrak{I}$, N_1, N_2 be two acceptable sequences w.r.t. I. Then $N_1 \cup N_2$ is an* acceptable sequence *w.r.t. I s.t. $\mathit{size}(I, N_1 \cup N_2) \leq \mathit{size}(I, N_1) + \mathit{size}(I, N_2)$.*

▶ **Proposition 20.** *Let $I = (V, \prec) \in \mathfrak{I}$ and N be an acceptable sequence w.r.t. I. If for all distinct $t, t' \in N$, we have $V(t') = V(t)$* only *when both $t, t' \in \mathit{final}(I, N)$, then $\mathit{size}(I, N) \leq 2^{|\mathcal{P}|}$.*

5 Bounded-model property

The main contribution of this paper is to establish certain computational properties regarding the satisfiability problem in $\mathcal{L}^{\star}$. The algorithmic problem is as follows: Given an input sentence $\alpha \in \mathcal{L}^{\star}$, decide whether α is preferentially satisfiable. In this section, we show that this problem is decidable.

The proof is based on the one given by Sistla and Clarke to show the complexity of propositional linear temporal logic [18]. Let $\mathcal{L}^{\star}$ be the fragment of $\mathcal{L}^{\sim}$ that contains only Boolean connectives and temporal operators ($\Box$, $\boxminus$, $\Diamond$, $\Diamondminus$). Let $\alpha \in \mathcal{L}^{\star}$, with $|\alpha|$ we denote the number of symbols within α. The main result of the present paper is summarized in the following theorem, of which the proof will be given in the remainder of the section.

▶ **Theorem 21** (**Bounded-model property**)**.** *If $\alpha \in \mathcal{L}^{\star}$ is $\mathfrak{I}^{sd}$-satisfiable, then we can find an interpretation $I \in \mathfrak{I}^{sd}$ such that $I, 0 \models \alpha$ and $\mathit{size}(I) \leq |\alpha| \times 2^{|\mathcal{P}|}$.*

Hence, given a satisfiable sentence $\alpha \in \mathcal{L}^{\star}$, there is an interpretation satisfying α of which the size is bounded. Since α is $\mathfrak{I}^{sd}$-satisfiable, we know $I, 0 \models \alpha$. From I we can construct an interpretation I' also satisfying α, i.e., $I', 0 \models \alpha$, which is bounded on its size by $|\alpha| \times 2^{|\mathcal{P}|}$.

The goal of this section is to show how to build said bounded interpretation. Let $\alpha \in \mathcal{L}^\star$ and let $I \in \mathfrak{I}^{sd}$ be s.t. $I, 0 \models \alpha$. The first step is to characterize an acceptable sequence N w.r.t. I such that N is bounded first of all, and "keeps" the satisfiability of the sub-sentences α_1 contained in α i.e., if $I, t \models \alpha_1$, then $I^N, t \models_{\mathscr{P}} \alpha_1$ (see Definition 18). We do so by building inductively a bounded pseudo-interpretation step by step by selecting what to take from the initial interpretation I for each sub-sentence α_1 contained in α to be satisfied. In what follows, we introduce the notion of $Anchors(\cdot)$ as a strategy for picking out the desired time points from I. Definitions 23–25 tell us how to pick said time points.

▶ **Definition 22** (**Acceptable sequence transformation**)**.** *Let $I = (V, \prec) \in \mathfrak{I}$ and let N be a sequence of time points. Let N' be the sequence of all time points t' in $\mathit{final}(I)$ for which there is $t \in N \cap \mathit{final}(I)$ with $V(t') = V(t)$. With $AS(I, N) \stackrel{\text{def}}{=} N \cup N'$ we denote the* acceptable sequence transformation *of N w.r.t. I.*

The sequence $AS(I, N)$ is the acceptable sequence transformation of N w.r.t. I. In the previous definition, N' is the sequence of all time points t' having the same valuation as some time point $t \in N$ that is in $\mathit{final}(I)$. It is also worth to point out that N' can be empty in the case of there being no time point $t \in N$ that is in $\mathit{final}(I)$. N is then a finite acceptable sequence w.r.t. I where $AS(I, N) = N$. This notation is mainly used to ensure that we are using the acceptable version of any sequence.

▶ **Definition 23** (**Chosen occurrence w.r.t.** α)**.** *Let $I = (V, \prec) \in \mathfrak{I}$, $\alpha \in \mathcal{L}^{\sim}$ and N be an acceptable sequence w.r.t. I s.t. there exists a time point t in N with $I, t \models \alpha$. The* chosen occurrence *satisfying α in N, denoted by $\mathfrak{t}_\alpha^{I,N}$, is defined as follows:*

$$\mathfrak{t}_\alpha^{I,N} \stackrel{\text{def}}{=} \begin{cases} \min_<\{t \in \mathit{final}(I, N) \mid I, t \models \alpha\}, & \textit{if } \{t \in \mathit{final}(I, N) \mid I, t \models \alpha\} \neq \emptyset \\ \max_<\{t \in \mathit{init}(I, N) \mid I, t \models \alpha\}, & \textit{otherwise} \end{cases}$$

Notice that $<$ above denotes the natural ordering of the underlying temporal structure

The strategy to pick out a time point satisfying a given sentence α in N is as follows. If said sentence is in the final part, we pick the first time point that satisfies it, since we have the guarantee to find infinitely many time points having the same valuations as $\mathfrak{t}_\alpha^{I,N}$ that also satisfy α (see Lemma 13). If not, we pick the last occurrence in the initial part that satisfies α. Thanks to Definition 23, we can limit the number of time points taken that satisfy the same sentence.

▶ **Definition 24** (**Selected time points**)**.** *Let $I = (V, \prec) \in \mathfrak{I}$, N be an acceptable sequence w.r.t. I and $\alpha \in \mathcal{L}^{\sim}$ s.t. there is t in N s.t. $I, t \models \alpha$. With $ST(I, N, \alpha) \stackrel{\text{def}}{=} AS(I, (\mathfrak{t}_\alpha^{I,N}))$ we denote the* selected time points *of N and α w.r.t. I. (Note that $(\mathfrak{t}_\alpha^{I,N})$ is a sequence of only one element.)*

Given a sentence $\alpha \in \mathcal{L}^{\sim}$ and an acceptable sequence N w.r.t. I s.t. there is at least one time point t where $I, t \models \alpha$, the sequence $ST(I, N, \alpha)$ is the acceptable sequence transformation of the sequence $(\mathfrak{t}_\alpha^{I,N})$. If $\mathfrak{t}_\alpha^{I,N} \in \mathit{init}(I)$, the sequence $ST(I, N, \alpha)$ is the sequence $(\mathfrak{t}_\alpha^{I,N})$. Otherwise, the sequence $ST(I, N, \alpha)$ is the sequence of all time points t in $\mathit{final}(I)$ that have the same valuation as $\mathfrak{t}_\alpha^{I,N}$. In both cases, we can see that $\mathit{size}(I, ST(I, N, \alpha)) = 1$.

Given an interpretation $I = (V, \prec)$ and N an acceptable sequence w.r.t I, the *representative sentence* of a valuation v is formally defined as $\alpha_v \stackrel{\text{def}}{=} \bigwedge\{p \mid p \in v\} \wedge \bigwedge\{\neg p \mid p \notin v\}$.

▶ **Definition 25** (**Distinctive reduction**)**.** *Let $I = (V, \prec) \in \mathfrak{I}$ and let N be an acceptable sequence w.r.t. I. With $DR(I, N) \stackrel{\text{def}}{=} \bigcup_{v \in val(I,N)} ST(I, N, \alpha_v)$ we denote the distinctive reduction of N.*

Given an acceptable sequence N w.r.t. I, $DR(I, N)$ is the sequence containing the chosen occurrence $\mathfrak{t}_{\alpha_v}^{I,N}$ that satisfies the representative α_v in N for each $v \in val(I, N)$. In other words, we pick the selected time points for each possible valuation in $val(I, N)$. There are two interesting results with regard to $DR(I,N)$. The first one is that $DR(I,N)$ is an acceptable sequence w.r.t. I. This can easily be proven since $ST(I,N,\alpha_v)$ is also an acceptable sequence w.r.t. I, and the union of all $ST(I,N,\alpha_v)$ is an acceptable sequence w.r.t. I (see Proposition 19). The second result is that $size(I,DR(I,N)) \le 2^{|\mathcal{P}|}$. Indeed, thanks to Proposition 19, we can see that $size(I,DR(I,N)) \le \sum_{v\in val(I,N)} size(ST(I,N,\alpha_v))$. Moreover, we have $size(I,ST(I,N,\alpha_v)) = 1$ for each $v \in val(I,N)$. On the other hand, there are at most $2^{|\mathcal{P}|}$ possible valuations in $val(I,N)$. Thus, we can assert that $\sum_{v\in val(I,N)} size(I,ST(I,N,\alpha_v)) \le 2^{|\mathcal{P}|}$, and then we have $size(I,DR(I,N)) \le 2^{|\mathcal{P}|}$.

▶ **Definition 26** **(Anchors).** *Let a sentence $\alpha \in \mathcal{L}^\star$ starting with a temporal operator, let $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I,t \models \alpha$. The sequence $Anchors(I,T,\alpha)$ is defined as: Let $\alpha_1 \in \mathcal{L}^\star$.*

$$
\begin{aligned}
Anchors(I,T,\Diamond\alpha_1) &\stackrel{\text{def}}{=} ST(I,\mathbb{N},\alpha_1);\\
Anchors(I,T,\Box\alpha_1) &\stackrel{\text{def}}{=} \emptyset;\\
Anchors(I,T,\Diamond\!\!\!\!\sim\alpha_1) &\stackrel{\text{def}}{=} \textstyle\bigcup_{t\in T} ST(I,AS(I,min_\prec(t)),\alpha_1);\\
Anchors(I,T,\Box\!\!\!\!\sim\alpha_1) &\stackrel{\text{def}}{=} DR(I,\textstyle\bigcup_{t\in T} AS(I,min_\prec(t))).
\end{aligned}
$$

Given an acceptable sequence T w.r.t. $I \in \mathfrak{I}^{sd}$ where all of its time points satisfy α, where α is a sentence starting with a temporal operator, $Anchors(I,T,\alpha)$ is an acceptable sequence w.r.t. I. This is due thanks to the notion of selected time points and distinctive reduction (see Definition 24 and 25). $Anchors(I,T,\alpha)$ contains the selected time points satisfying the sub-sentence α_1 of α (except for $\Box\alpha_1$). Our goal is to have the selected time points that satisfy α_1 for each $t \in T$.

It is worth to point out that the choice of $Anchors(I,T,\Box\alpha_1) = \emptyset$ is due to the fact α_1 is satisfied starting from the first time $t_0 \in T$ i.e., for all $t \ge t_0$, we have $I,t \models \alpha$. So no matter what time point t we pick after t_0, we have $I,t \models \alpha_1$. On the other hand, by the nature of the semantics of $\Box\!\!\!\!\sim\alpha_1$, all $t \in \bigcup_{t_i\in T} AS(I,min_\prec(t_i))$ satisfy α_1. The acceptable sequence $Anchors(I,T,\Box\!\!\!\!\sim\alpha_1)$ contains only the selected time points for each distinct valuation in $\bigcup_{t_i\in T} AS(I,min_\prec(t_i))$.

The following are some properties of $Anchors(\cdot)$ that are worth mentioning:

▶ **Lemma 27.** *Let $\alpha_1 \in \mathcal{L}^\star$ be a sentence starting with a temporal operator, $I = (V,\prec) \in \mathfrak{I}^{sd}$ and let T be a non-empty acceptable sequence w.r.t. I where for all $t \in T$ we have $I,t \models \Diamond\!\!\!\!\sim\alpha_1$. Then for all $t,t' \in Anchors(I,T,\Diamond\!\!\!\!\sim\alpha_1)$ s.t. $V(t) = V(t')$ and $t \neq t'$, we have $t,t' \in final(I, Anchors(I,T,\Diamond\!\!\!\!\sim\alpha_1))$.*

▶ **Proposition 28.** *Let $\alpha \in \mathcal{L}^\star$ be a sentence starting with a temporal operator, $I = (V,\prec) \in \mathfrak{I}^{sd}$. Let T be a non-empty acceptable sequence w.r.t. I where for all $t \in T$ we have $I,t \models \alpha$. Then, we have $size(I, Anchors(I,T,\alpha)) \le 2^{|\mathcal{P}|}$.*

▶ **Proposition 29.** *Let $\alpha_1 \in \mathcal{L}^\star$, $I = (V,\prec) \in \mathfrak{I}^{sd}$, let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I,t \models \Box\!\!\!\!\sim\alpha_1$, with $\alpha_1 \in \mathcal{L}^\star$. For all acceptable sequences N w.r.t. I s.t. $Anchors(I,T,\Box\!\!\!\!\sim\alpha_1) \subseteq N$ and for all $t_i \in N \cap T$, we have the following: Let $I^N = (V^N, \prec^N)$ be the pseudo-interpretation over N and $t' \in N$, if $t' \notin min_\prec(t_i)$, then $t' \notin min_{\prec^N}(t_i)$.*

Proposition 29 helps us mitigate the dynamic nature of $min_\prec(t_i)$. The selected time points help us circumvent adding time points that were not originally "preferred" w.r.t. t_i in I, and becoming preferred in the reduced structure I^N that we want to build. The strategy of building $Anchors(\cdot)$ is explained by the fact that we want to preserve the truth values of defeasible sub-sentences of α in the bounded interpretation.

With $Anchors(\cdot)$ defined, we introduce the notion of $Keep(\cdot)$. $Keep(\cdot)$ will help us compute recursively starting from the initial satisfiable sentence α down to its literals, the selected time points to pick in order to build our pseudo-interpretation.

▶ **Definition 30** (**Keep**). *Let $\alpha \in \mathcal{L}^\star$ be in NNF, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be an acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. The sequence $\mathit{Keep}(I, T, \alpha)$ is defined as $\emptyset$, if $T = \emptyset$; otherwise it is recursively defined as follows:*

- *$\mathit{Keep}(I, T, \ell) \stackrel{\text{def}}{=} \emptyset$, where ℓ is a literal;*
- *$\mathit{Keep}(I, T, \alpha_1 \wedge \alpha_2) \stackrel{\text{def}}{=} \mathit{Keep}(I, T, \alpha_1) \cup \mathit{Keep}(I, T, \alpha_2)$;*
- *$\mathit{Keep}(I, T, \alpha_1 \vee \alpha_2) \stackrel{\text{def}}{=} \mathit{Keep}(I, T_1, \alpha_1) \cup \mathit{Keep}(I, T_2, \alpha_2)$, where $T_1 \subseteq T$ (resp. $T_2 \subseteq T$) is the sequence of all $t_1 \in T$ (resp. $t_2 \in T$) s.t. $I, t_1 \models \alpha_1$ (resp. $I, t_2 \models \alpha_2$);*
- *$\mathit{Keep}(I, T, \Diamond\alpha_1) \stackrel{\text{def}}{=} \mathit{Anchors}(I, T, \Diamond\alpha_1) \cup \mathit{Keep}(I, \mathit{Anchors}(I, T, \Diamond\alpha_1), \alpha_1)$;*
- *$\mathit{Keep}(I, T, \Box\alpha_1) \stackrel{\text{def}}{=} \mathit{Keep}(I, T, \alpha_1)$;*
- *$\mathit{Keep}(I, T, \diamondminus\alpha_1) \stackrel{\text{def}}{=} \mathit{Anchors}(I, T, \diamondminus\alpha_1) \cup \mathit{Keep}(I, \mathit{Anchors}(I, T, \diamondminus\alpha_1), \alpha_1)$;*
- *$\mathit{Keep}(I, T, \boxminus\alpha_1) \stackrel{\text{def}}{=} \mathit{Anchors}(I, T, \boxminus\alpha_1) \cup \mathit{Keep}(I, T', \alpha_1)$, where $T' = \bigcup_{t_i \in T} AS(I, \mathit{min}_{\prec}(t_i))$.*

With $\mu(\alpha)$ we denote the number of classical and non-monotonic modalities in α.

▶ **Proposition 31.** *Let $\alpha \in \mathcal{L}^\star$ be in NNF, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. Then, we have $\mathit{size}(I, \mathit{Keep}(I, T, \alpha)) \leq \mu(\alpha) \times 2^{|\mathcal{P}|}$.*

Given an acceptable sequence N w.r.t. I, we need to make sure when a time point $t \in N$ in our acceptable sequence s.t. $I, t \models \alpha$, then $I^N, t \models_{\mathscr{P}} \alpha$. The function $\mathit{Keep}(I, T, \alpha)$ returns the acceptable sequence of time s.t. if $\mathit{Keep}(I, T, \alpha) \subseteq N$ and $t \in T$, then said condition is met. We prove this in Lemma 32.

▶ **Lemma 32.** *Let $\alpha \in \mathcal{L}^\star$ be in NNF, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. For all acceptable sequences N w.r.t. I, if $\mathit{Keep}(I, T, \alpha) \subseteq N$, then for every $t \in N \cap T$, we have $I^N, t \models_{\mathscr{P}} \alpha$.*

Since we build our pseudo-interpretation I^N by adding selected time points for each sub-sentence α_1 of α, we need to make sure that said sub-sentence remains satisfied in I^N. Lemma 32 ensures that.

▶ **Definition 33** (**Pseudo-interpretation transformation**). *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let N be an infinite acceptable sequence w.r.t. I. The pseudo-interpretation $I^N = (V^N, \prec^N)$ can be transformed into a preferential interpretation $I' = (V', \prec') \in \mathfrak{I}^{sd}$ as follows:*

- *for all $i \geq 0$, we have $V'(i) = V^N(t_i)$;*
- *for all $i, j \geq 0$, $t_i, t_j \in N$, we have $(t_i, t_j) \in \prec^N$ iff $(i, j) \in \prec'$.*

We can now prove our bounded-model theorem.

Proof of Theorem 21. We assume that $\alpha \in \mathcal{L}^\star$ is $\mathfrak{I}^{sd}$-satisfiable. The first thing we notice is that $|\alpha| \geq \mu(\alpha) + 1$. Let α' be the NNF of the sentence α. As a consequence of the duality rules of $\mathcal{L}^\star$, we can deduce that $\mu(\alpha') = \mu(\alpha)$. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ s.t. $I, 0 \models \alpha'$. Let $T_0 = AS(I, (0))$ be an acceptable sequence w.r.t. I. We can see that $\mathit{size}(I, T_0) = 1$. Since for all $t \in T_0$ we have $I, t \models \alpha'$ (see Lemma 13), we can compute recursively $U = \mathit{Keep}(I, T_0, \alpha')$. Thanks to Proposition 31, we conclude that U is an acceptable sequence w.r.t. I s.t. $\mathit{size}(I, U) \leq \mu(\alpha') \times 2^{|\mathcal{P}|}$. Let $N = T_0 \cup U$ be the union of T_0 and U and let $I^N = (N, V^N, \prec^N)$ be its pseudo-interpretation over N. Thanks to Proposition 19, we have $\mathit{size}(I, N) \leq 1 + \mu(\alpha') \times 2^{|\mathcal{P}|}$. Thanks to Lemma 32, since $0 \in N \cap T_0$ and $\mathit{Keep}(I, T_0, \alpha') \subseteq N$, we have $I^N, 0 \models_{\mathscr{P}} \alpha'$. In case N is finite, we replicate the last time point t_n infinitely many times. Notice that $\mathit{size}(I, N)$ does not change if we replicate the last element. We can transform the pseudo interpretation I^N to $I' \in \mathfrak{I}^{sd}$ by changing the labels of N into a sequence of natural numbers minding the order of time points in N (see Definition 33). We can see that $\mathit{size}(I') = \mathit{size}(I, N)$ and $I', 0 \models \alpha$. Consequently, we have $\mathit{size}(I') \leq 1 + \mu(\alpha') \times 2^{|\mathcal{P}|}$. Hence, from a given interpretation I s.t. $I, 0 \models \alpha$ we can build an interpretation I' s.t. $I', 0 \models \alpha$ and $\mathit{size}(I') \leq 1 + \mu(\alpha') \times 2^{|\mathcal{P}|}$. Without loss of generality, we conclude that $\mathit{size}(I') \leq |\alpha| \times 2^{|\mathcal{P}|}$. ◀

6 The satisfiability problem in $\mathcal{L}^\star$

We now provide an algorithm allowing to decide whether a sentence $\alpha \in \mathcal{L}^\star$ is $\mathfrak{I}^{sd}$-satisfiable or not. For this purpose, first we focus on particular interpretations of the class $\mathfrak{I}^{sd}$, namely the ultimately periodic interpretations (UPI in short), and a finite representation of these interpretations, called ultimately periodic pseudo-interpretation (UPPI in short). As we will see in the second part of this section, to decide the $\mathfrak{I}^{sd}$-satisfiability of a sentence $\alpha \in \mathcal{L}^\star$, the proposed algorithm guesses a bounded UPPI in a first step. Then, it checks the satisfiability of α by the UPI of the guessed UPPI.

▶ **Definition 34 (UPI).** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $\pi = card(range(I))$. We say I is an* ultimately periodic interpretation *if:*
- *for every $t, t' \in [\mathsf{t}_I, \mathsf{t}_I + \pi[$ s.t. $t \neq t'$ (see Definition 10), we have $V(t) \neq V(t')$,*
- *for every $t \in [\mathsf{t}_I, +\infty[$, we have $V(t) = V(\mathsf{t}_I + (t - \mathsf{t}_I) \bmod \pi)$.*

A UPI I is a state dependent interpretation s.t. each time point's valuation in $\mathit{final}(I)$ is replicated periodically. Given a UPI, $\pi = \mathit{card}(\mathit{range}(I))$ denotes the length of the period and the interval $[\mathsf{t}_I, \mathsf{t}_I + \pi[$ is the first period which is replicated periodically throughout the final part. It is worth pointing out that for every $t \in \mathit{final}(I)$, we have $V(t) \in \{V(t') \mid t' \in [\mathsf{t}_I, \mathsf{t}_I + \pi[\}$, which is one of the consequences of the definition above. Thanks to Lemma 15, we can prove the following proposition.

▶ **Proposition 35.** *Let $\mathcal{P}$ be a set of atomic propositions, $I = (V. \prec) \in \mathfrak{I}^{sd}$, $i = length(init(I))$ and $\pi = card(range(I))$. There exists an ultimately periodic interpretation $I' = (V', \prec') \in \mathfrak{I}^{sd}$ s.t. I, I' are faithful interpretations over $\mathcal{P}$ (Definition 14), $init(I') \doteq init(I)$, $range(I') = range(I)$ and $V'(0) = V(0)$. Moreover, for all $\alpha \in \mathcal{L}^\star$, we have $I, 0 \models \alpha$ iff $I', 0 \models \alpha$.*

It is worth to point out that the size of an interpretation and that of its UPI counterparts are equal. It can easily be seen that these interpretations have the same initial part and the same range of valuations in the final part. I' from the aforementioned proposition is obtained from I by keeping the same initial part, and placing each distinct valuation of $\mathit{range}(I)$ in the interval $[\mathsf{t}_I, \mathsf{t}_I + \pi[$ and replicating this interval infinitely many times. Moreover, the preference relation $\prec'$ arranges valuations in the same way as $\prec$. We can see that I and I' are faithful and that $\mathit{init}(I') \doteq \mathit{init}(I)$, $\mathit{range}(I') = \mathit{range}(I)$ and $V'(0) = V(0)$. Therefore, I and I' satisfy the same sentences.

▶ **Definition 36 (UPPI).** *A* model structure *is a tuple $M = (i, \pi, V_M, \prec_M)$ where: i, π are two integers such that $i \geq 0$ and $\pi > 0$ (where i is intended to be the starting point of the period, π is the length of the period); $V_M : [0, i + \pi[\longrightarrow 2^{\mathcal{P}}$, and $\prec_M \subseteq 2^{\mathcal{P}} \times 2^{\mathcal{P}}$ is a strict partial order. Moreover, (I) for all $t \in [i, i + \pi[$, we have $V_M(t) \neq V_M(i - 1)$; and (II) for all distinct $t, t' \in [i, i + \pi[$, we have $V_M(t) \neq V_M(t')$.*

The reason behind setting properties (I) and (II) is that we can build a UPPI from a UPI, and back.

Given a UPPI $M = (i, \pi, V_M, \prec_M)$, we define the *size of* M by $\mathit{size}(M) \stackrel{\text{def}}{=} i + \pi$. From a UPPI we define a UPI in the following way:

▶ **Definition 37.** *Given a UPPI $M = (i, \pi, V_M, \prec_M)$, let $\mathsf{I}(M) \stackrel{\text{def}}{=} (V, \prec)$, where for every $t \geq 0$, $V(t) \stackrel{\text{def}}{=} V_M(t)$, if $t < i$, and $V(t) \stackrel{\text{def}}{=} V_M(i + (t - i) \bmod \pi)$, otherwise, and $\prec \stackrel{\text{def}}{=} \{(t, t') \mid (V(t), V(t')) \in \prec_M\}$.*

Given a UPPI $M = (i, \pi, V_M, \prec_M)$, the interval $[0, i[$ of a UPPI corresponds to the initial temporal part of the underlying interpretation $\mathsf{I}(M)$ and $[i, i + \pi[$ represents a temporal period that is infinitely replicated in order to determine the final temporal part of the interpretation.

It is worth to point out that given a UPPI M, $\mathsf{I}(M) = (V, \prec)$ is a UPI. Moreover, we have $\mathit{size}(\mathsf{I}(M)) = \mathit{size}(M)$.

Now we extend the notion of preferred time points w.r.t a time point for a UPPI :

▶ **Definition 38** (**UPPI's preferred time points**). *Let $M = (i, \pi, V_M, \prec_M)$ be a UPPI and a time point $t \in [0, i+\pi[$. The set of preferred time points of t w.r.t. M, denoted by $min_{\prec_M}(t)$, is defined as follows: $min_{\prec_M}(t) \stackrel{\text{def}}{=} \{t' \in [min_{<}\{t, i\}, i+\pi[\mid$ there is no $t'' \in [min_{<}\{t, i\}, i+\pi[$ with $(V_M(t''), V_M(t')) \in \prec_M\}$.*

▶ **Proposition 39.** *Let $M = (i, \pi, V_M, \prec_M)$ be a UPPI, $\mathsf{I}(M) = (V, \prec)$ and $t, t', t_M, t'_M \in \mathbb{N}$ s.t.:*

$$t_M = \begin{cases} t, \text{ if } t < i; \\ i + (t-i) \bmod \pi, \text{ otherwise.} \end{cases} \qquad t'_M = \begin{cases} t', \text{ if } t' < i; \\ i + (t'-i) \bmod \pi, \text{ otherwise.} \end{cases}$$

We have the following: $t' \in min_{\prec}(t)$ iff $t'_M \in min_{\prec_M}(t_M)$.

Now that UPPI is defined, we can move to the task of checking the satisfiability of a sentence α. We define for a UPPI $M = (i, \pi, V_M, \prec_M)$ and a sentence $\alpha \in \mathcal{L}^\star$ a labelling function $lab^M_\alpha(\cdot)$ which associates a set of sub-sentences of α to each $t \in [0, i+\pi[$.

▶ **Definition 40** (**Labelling function**). *Let $M = (i, \pi, V_M, \prec_M)$ be a UPPI, $\alpha \in \mathcal{L}^\star$. The set of sub-sentences of α for $t \in [0, i+\pi[$, denoted by $lab^M_\alpha(t)$, is defined as follows:*

- $p \in lab^M_\alpha(t)$ *iff* $p \in V_M(t)$; $\neg\alpha_1 \in lab^M_\alpha(t)$ *iff* $\alpha_1 \notin lab^M_\alpha(t)$;
- $\alpha_1 \wedge \alpha_2 \in lab^M_\alpha(t)$ *iff* $\alpha_1, \alpha_2 \in lab^M_\alpha(t)$; $\alpha_1 \vee \alpha_2 \in lab^M_\alpha(t)$ *iff* $\alpha_1 \in lab^M_\alpha(t)$ *or* $\alpha_2 \in lab^M_\alpha(t)$;
- $\Diamond\alpha_1 \in lab^M_\alpha(t)$ *iff* $\alpha_1 \in lab^M_\alpha(t')$ *for some* $t' \in [min_{<}\{t, i\}, i+\pi[$;
- $\Box\alpha_1 \in lab^M_\alpha(t)$ *iff* $\alpha_1 \in lab^M_\alpha(t')$ *for all* $t' \in [min_{<}\{t, i\}, i+\pi[$;
- $\mathrel{\Diamond\!\!\!\!\sim}\alpha_1 \in lab^M_\alpha(t)$ *iff* $\alpha_1 \in lab^M_\alpha(t')$ *for some* $t' \in min_{\prec_M}(t)$;
- $\mathrel{\Box\!\!\!\!\sim}\alpha_1 \in lab^M_\alpha(t)$ *iff* $\alpha_1 \in lab^M_\alpha(t')$ *for all* $t' \in min_{\prec_M}(t)$.

▶ **Lemma 41.** *Let a UPPI $M = (i, \pi, V_M, \prec_M)$, $\alpha \in \mathcal{L}^\star$ and $t \in \mathbb{N}$, $\mathsf{I}(M), 0 \models \alpha$ iff $\alpha \in lab^M_\alpha(0)$.*

We accept a UPPI M as a model for $\alpha \in \mathcal{L}^\star$ iff $\alpha \in lab^M_\alpha(0)$. Otherwise, M is rejected.

▶ **Proposition 42.** *Let $\alpha \in \mathcal{L}^\star$. We have that α is $\mathfrak{I}^{sd}$-satisfiable iff there exists a UPPI M such that $\mathsf{I}(M), 0 \models \alpha$ and $size(\mathsf{I}(M)) \leq |\alpha| \times 2^{|\mathcal{P}|}$.*

Hence, to decide the satisfiability of a sentence $\alpha \in \mathcal{L}^\star$, we can first guess a UPPI M bounded by $|\alpha| \times 2^{|\mathcal{P}|}$. Next, using the labelling function of M, we check the satisfiability of α by the UPI $\mathsf{I}(M)$.

▶ **Theorem 43.** *$\mathfrak{I}^{sd}$-satisfiability problem for $\mathcal{L}^\star$ sentences is* decidable.

7 Concluding remarks

The contributions of this paper are as follows: we introduced the formalism of $LTL^{\sim}$ with its expressive syntax and intuitive semantics. We defined also the class of state-dependent interpretations $\mathfrak{I}^{sd}$ and the fragment $\mathcal{L}^\star$. We then showed that $\mathfrak{I}^{sd}$-satisfiability in $\mathcal{L}^\star$ is a decidable problem.

It is worth pointing out that it is hard to define a tableaux method for our logic similar to Wolper's [19]. The main reason is that we do not have defeasible versions of the axioms (T) and (4), and therefore nested defeasible modalities cannot be reduced as in the classical case. Furthermore, at present we have $\not\models \mathrel{\Box\!\!\!\!\sim}\alpha \leftrightarrow \alpha \wedge \bigcirc \mathrel{\Box\!\!\!\!\sim}\alpha$ and $\not\models \mathrel{\Diamond\!\!\!\!\sim}\alpha \leftrightarrow \alpha \vee \bigcirc \mathrel{\Diamond\!\!\!\!\sim}\alpha$. That is why we decided to tackle the satisfiability problem of our logic before establishing a semantic tableaux for $LTL^{\sim}$.

Among the immediate next steps is the introduction of defeasible counterparts to $\bigcirc$ and $\mathcal{U}$. We shall also investigate the addition of $\mathrel{|\!\!\sim}$-statementes to our logic.

8 The fragment $\mathcal{L}_1$

In this section, we have another fragment whose satisfiability problem is decidable. In the following fragment, we chose to omit $\boxminus$ operator , and only allow propositional Boolean sentences within a $\Box$ sentences. Though we reduce the expressivity of the language on this fragment, we have a polynomial upper bound w.r.t. to the size of the input formula α.

The vocabulary of the fragment $\mathcal{L}_1$ consists of a finite set of atomic propositions $\mathscr{P}$. The set of operators consists of $(\wedge, \vee, \Diamond, \Box, \Diamond\!\!\!\sim)$. Sentence in $\mathcal{L}_1$ are in negation normal form, which means that negation is only applied to atomic propositions. Furthermore, only Boolean connectors are allowed within the scope of $\Box$ sentences. Temporal operators, classical or non-monotonic, are not permitted in the range of $\Box$ sentences.

In what follows, we describe well formed sentences of $\mathcal{L}_1$. In order to do that, we define first the set of Boolean sentences $\mathcal{L}_{bool}$. Let $p \in \mathscr{P}$, sentences $\alpha_{bool} \in \mathcal{L}_{bool}$ are defined recursively as such:

$$\alpha_{bool} ::= \top \mid \bot \mid p \mid \neg p \mid \alpha_{bool} \wedge \alpha_{bool} \mid \alpha_{bool} \vee \alpha_{bool}$$

Where $\top$ is an abbreviation of $p \vee \neg p$, and $\bot$ is an abbreviation of $p \wedge \neg p$. Next, let $\alpha_{bool} \in \mathcal{L}_{bool}$, sentences in $\mathcal{L}_1$ are recursively defined as such:

$$\alpha ::= \alpha_{bool} \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Diamond\alpha \mid \Box\alpha_{bool} \mid \bigcirc\alpha \mid \Diamond\!\!\!\sim\alpha$$

Given a $\mathfrak{I}$-satisfiable sentence in $\alpha \in \mathcal{L}_1$, there exists an interpretation $I \in \mathfrak{I}$ s.t. $I, 0 \models \alpha$. From I, we can construct an interpretation I' also satisfying α, i.e., $I', 0 \models \alpha$, which is bounded on its size. We define a structure, named pseudo-interpretation, which is a restriction of the original interpretation over a sequence of time points.

The goal is to show that we can find a finite sequence of time points on which the truth values of sentences in the generated pseudo-interpretation are kept the same as the original interpretation. We define firsthand, subsequences and pseudo-interpretations.

▶ **Definition 44** (Sub-sequence)**.** *Let N, N' be two ordered sequences of natural numbers, N' is a subsequence of (written as $N' \subseteq N$) N iff for all $i \in N'$, we have $i \in N$.*

▶ **Definition 45** (Pseudo-interpretation over N)**.** *Let $I = (V, \prec) \in \mathfrak{I}$ and N be a sequence of $\mathbb{N}$. The pseudo-interpretation over N is the pair $I^N \stackrel{\text{def}}{=} (V^N, \prec^N)$ where:*

- *$V^N : N \longrightarrow 2^{\mathcal{P}}$ is a valuation function over N, where for all $i \in N$, we have $V^N(i) = V(i)$,*
- *$\prec^N \subseteq N \times N$, where for all $(i, j) \in N^2$, we have $(i, j) \in \prec^N$ iff $(i, j) \in \prec$*

The size of a pseudo interpretation is the number of time points in N. Namely, the size of I^N is $size(I, N) \stackrel{\text{def}}{=} length(N)$.

The truth values of $\mathcal{L}^\star$ sentences in pseudo-interpretations are defined in a similar fashion as for preferential temporal interpretations. With $\models$ we denote the truth values of sentences in a pseudo-interpretation. Let $t \in N$:

- $I^N, t \models p$ if $p \in V^N(t)$;
- $I^N, t \models \neg p$ if $p \notin V^N(t)$;
- $I^N, t \models \Diamond\alpha$ if $I^N, t' \models \alpha$ for some $t' \in N$ s.t. $t' \geq t$;
- $I^N, t \models \Box\alpha$ if $I^N, t' \models \alpha$ for all $t' \in N$ s.t. $t' \geq t$;
- $I^N, t \models \bigcirc\alpha$ if we have $t + 1 \in N$ and $I^N, t + 1 \models \alpha$;
- $I^N, t \models \Diamond\!\!\!\sim\alpha$ if $I^N, t' \models \alpha$ for some $t' \in N$ s.t. $t' \in min_{\prec^N}(t)$.
-

▶ **Proposition 46.** *Let $\alpha_{bool} \in \mathcal{L}_{bool}$, let $I = (V, \prec) \in \mathfrak{I}$ and N be a sequence containing t s.t. $I^N, t \models \alpha_{bool}$, then for all $N' \subseteq N$ containing t, we have $I^{N'}, t \models \alpha_{bool}$.*

Proof. Let $\alpha_{bool} \in \mathcal{L}_{bool}$, let $I = (V, \prec) \in \mathfrak{I}$ and N be a sequence containing t s.t. $I^N, t \models \alpha_{bool}$, let N' be a subsequence of N that contains t, we use structural induction based on α_{bool}.

- $\alpha_{bool} := p$. Since $I^N, t \models p$, we know that $p \in V^N(t)$ and therefore $p \in V(t)$. On the other hand, since we have $t \in N'$ and $p \in V(t)$, then we have $p \in V^{N'}(t)$. Therefore, we have $I^{N'}, t \models p$.
- $\alpha_{bool} := \neg p$. Since $I^N, t \models \neg p$, we know that $p \notin V^N(t)$ and therefore $p \notin V(t)$. On the other hand, since we have $t \in N'$ and $p \notin V(t)$, then we have $p \in V^{N'}(t)$. Therefore, we have $I^{N'}, t \models \neg p$.
- $\alpha_{bool} := \alpha_1 \wedge \alpha_2$. We have $I^N, t \models \alpha_1 \wedge \alpha_2$, that means $I^N, t \models \alpha_1$ and $I^N, t \models \alpha_2$. Since N' is a subsequence of N containing t, and using the induction hypothesis on α_1 and α_2, we have $I^{N'}, t \models \alpha_1$ and $I^{N'}, t \models \alpha_2$. Therefore, we have $I^{N'}, t \models \alpha_1 \wedge \alpha_2$.
- $\alpha_{bool} := \alpha_1 \vee \alpha_2$. We have $I^N, t \models \alpha_1 \vee \alpha_2$, that means either $I^N, t \models \alpha_1$ or $I^N, t \models \alpha_2$. We suppose that $I^N, t \models \alpha_1$. Since N' is a subsequence of N containing t, and using the induction hypothesis on α_1, we have $I^{N'}, t \models \alpha_1$. Therefore, we have $I^{N'}, t \models \alpha_1 \vee \alpha_2$. Same reasoning applies when $I^N, t \models \alpha_2$.

◀

▶ **Lemma 47.** *Let $\alpha \in \mathcal{L}_1$, $I = (V, \prec) \in \mathfrak{I}$ and $N \subseteq \mathbb{N}$ s.t. $I^N, t \models \alpha$; there exists a* ***finite*** *sequence M containing t such that:*

I *$M \subseteq N$;*
II *$size(I, M) \leq |\alpha|$;*
III *for all sequences Q where $M \subseteq Q \subseteq N$, we have $I^Q, t \models \alpha$.*

Proof. Let $\alpha \in \mathcal{L}_1$, $I = (V, \prec) \in \mathfrak{I}$ and $N \subseteq \mathbb{N}$ s.t. $I^N, t \models \alpha$; we use structural induction on the length of α.

- $\alpha := p$. Let $M := (t)$ be sequence containing only t, then M is a finite sequence such that:

  I Since $t \in N$, then $M \subseteq N$;
  II $size(I, M) = 1 \leq |p|$;
  III Since $I^N, t \models p$, then we have $p \in V(t)$. Let Q b a sequence s.t. $M \subseteq Q \subseteq N$, we have $t \in Q$. Therefore, we have $p \in V^Q(t)$ and $I^Q, t \models p$.

- $\alpha := \neg p$. Let $M := (t)$ be sequence containing only t, then M is a finite sequence such that:

  I Since $t \in N$, then $M \subseteq N$;
  II $size(I, M) = 1 \leq |\neg p|$;
  III Since $I^N, t \models \neg p$, then we have $p \notin V(t)$. Let Q b a sequence s.t. $M \subseteq Q \subseteq N$, we have $t \in Q$. Therefore, we have $p \notin V^Q(t)$ and $I^Q, t \models \neg p$.

- $\alpha := \alpha_1 \wedge \alpha_2$. Since $I^N, t \models \alpha_1 \wedge \alpha_2$, we then have $I^N, t \models \alpha_1$ and $I^N, t \models \alpha_2$. Using the induction hypothesis on α_1, there exists a finite sequence M_1 containing t such that:

  I $M_1 \subseteq N$;
  II $size(I, M_1) \leq |\alpha_1|$;
  III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t \models \alpha_1$.

Similarly, using the induction hypothesis on α_2, there exists a finite sequence M_2 such that:

- I $M_2 \subseteq N$;
- II $\mathit{size}(I, M_2) \leq |\alpha_2|$;
- III for all sequences Q where $M_2 \subseteq Q \subseteq N$, we have $I^Q, t \models \alpha_2$.

Let $M := M_1 \cup M_2$. Since M_1 and M_2 contain t, then M is a finite sequence that contains t. We also have:

- I Since $M_1 \subseteq N$ and $M_2 \subseteq N$, then we have $M_1 \cup M_2 \subseteq N$;
- II $\mathit{size}(I, M) = \mathit{size}(M_1 \cup M_2) \leq \mathit{size}(I, M_1) + \mathit{size}(I, M_2) \leq |\alpha_1| + |\alpha_2| \leq |\alpha_1 \wedge \alpha_2|$;
- III Let $M \subseteq Q \subseteq N$. Since $M_1 \subseteq Q \subseteq N$, then we have $I^Q, t \models \alpha_1$. Similarly, Since $M_2 \subseteq Q \subseteq N$, then we have $I^Q, t \models \alpha_2$. Therefore, we have $I^Q, t \models \alpha_1 \wedge \alpha_2$.

- $\alpha := \alpha_1 \vee \alpha_2$. We have either $I^N, t \models \alpha_1$ or $I^N, t \models \alpha_2$. Using the induction hypothesis on α_1, there exists a finite sequence M_1 containing t such that:

- I $M_1 \subseteq N$;
- II $\mathit{size}(I, M_1) \leq |\alpha_1|$;
- III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t \models \alpha_1$.

Let $M := M_1$. Since M_1 contains t, then M is a finite sequence that contains t. We also have:

- I Since $M = M_1 \subseteq N$;
- II $\mathit{size}(I, M) = \mathit{size}(M_1) \leq |\alpha_1| \leq |\alpha_1 \vee \alpha_2|$;
- III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t \models \alpha_1$. Therefore, $I, t \models \alpha_1 \vee \alpha_2$.

The reasoning is the same when $I^N, t \models \alpha_2$.

- $\alpha := \bigcirc\alpha_1$. Since $I^N, t \models \bigcirc\alpha_1$, then $t+1 \in N$ and $I^N, t+1 \models \alpha_1$. Using the induction hypothesis on α_1, there exists a finite sequence sequence containing $t+1$ such that:

- I $M_1 \subseteq N$;
- II $\mathit{size}(I, M_1) \leq |\alpha_1|$;
- III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t+1 \models \alpha_1$.

Let $M := (t) \cup M_1$; then M is a finite sequence containing t such that:

- I Since $M_1 \subseteq N$ and $t \in N$, then we have $M \subseteq N$;
- II $\mathit{size}(I, M) = 1 + \mathit{size}(I, M_1) \leq |\bigcirc\alpha_1|$;
- III Let Q be a sequence such that $M \subseteq Q \subseteq N$, we have $t, t+1 \in M$. Since $M_1 \subseteq Q \subseteq N$, then $I^Q, t+1 \models \alpha_1$. Therefore, we have $I^Q, t \models \bigcirc\alpha_1$.

- $\alpha := \Diamond\alpha_1$. Since $I^N, t \models \Diamond\alpha_1$, then $t' \in N$ and $I^N, t' \models \alpha_1$. Using the induction hypothesis on α_1, there exists a finite sequence sequence containing t' such that:

- I $M_1 \subseteq N$;
- II $\mathit{size}(I, M_1) \leq |\alpha_1|$;
- III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t' \models \alpha_1$.

Let $M := (t) \cup M_1$; then M is a finite sequence containing t such that:

I Since $M_1 \subseteq N$ and $t \in N$, then we have $M \subseteq N$;
II $size(I, M) = 1 + size(I, M_1) \leq |\Diamond\alpha_1|$;
III Let Q be a sequence such that $M \subseteq Q \subseteq N$, we have $t, t' \in M$. Since $M_1 \subseteq Q \subseteq N$ and $t' \in M_1$, then $I^Q, t' \models \alpha_1$. Therefore, we have $I^Q, t \models \Diamond\alpha_1$.

- $\alpha := \Diamond\!\!\!\!\prec\alpha_1$. Since $I^N, t \models \Diamond\!\!\!\!\prec\alpha_1$, there exists $t' \in N$ s.t. $t' \in min_{\prec^N}(t)$. Using the induction hypothesis on α_1, there exists a finite sequence M_1 containing t' such that:

I $M_1 \subseteq N$;
II $size(I, M_1) \leq |\alpha_1|$;
III for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t' \models \alpha_1$.

Let $M := (t) \cup M_1$; then M is a finite sequence containing t such that:

I Since $M_1 \subseteq N$ and $t \in N$, then we have $M \subseteq N$;
II $size(I, M) = 1 + size(I, M_1) \leq |\Diamond\!\!\!\!\prec\alpha_1|$;
III Let Q be a sequence such that $M \subseteq Q \subseteq N$, we have $t, t' \in M$. Since $M_1 \subseteq Q \subseteq N$ and $t' \in M_1$, then (i) $I^Q, t' \models \alpha_1$.
We suppose that $t' \notin min_{\prec^Q}(t)$, there exists $t'' \in Q$ s.t. $(t'', t') \in \prec^Q$. Following this supposition, we have $(t'', t') \in \prec$. Since $t', t'' \in N$, we have $(t'', t') \in \prec^N$, thus $t' \notin min_{\prec^N}(t)$. This supposition conflicts with our assumption that $t' \in min_{\prec^N}(t)$. Therefore we have (ii) $t' \in min_{\prec^Q}(t)$. From (i) and (ii), we conclude that $I^Q, t \models \Diamond\!\!\!\!\prec\alpha_1$.

- $\alpha := \Box\alpha_{bool}$. Since $I^N, t \models \Box\alpha_{bool}$, we have $I^N, t \models \alpha_{bool}$ for all $t' \in N$ s.t. $t' \geq t$. Consider that $M = (t)$, we have the following:

I $M \subseteq N$;
II $size(I, M) = 1 \leq |\Box\alpha_{bool}|$;
III Let $M \subseteq Q \subseteq N$, we need to prove that $I^Q, t \models \Box\alpha_{bool}$. Let assume that $I^Q, t \not\models \Box\alpha_{bool}$, it means that there exists $t' \in Q$ s.t. $t' \geq t$ and $I^Q, t' \not\models \alpha_{bool}$.
On the hand, since $t' \in Q$, and $Q \subseteq N$, we have $t' \in N$. We know that $I^N, t \models \Box\alpha_{bool}$, and $t' \geq t$, therefore $I^N, t' \models \alpha_{bool}$. Thanks to Proposition 46, since $\alpha_{bool} \in \mathcal{L}_{bool}$, $t' \in Q \subseteq N$ and $I^N, t' \models \alpha_{bool}$, then we have $I^Q, t' \models \alpha_{bool}$, which raises a contradiction with our assumption. Thus, there is no $t' \in Q$ s.t. $t' \geq t$ and $I^Q, t' \not\models \alpha_{bool}$. We conclude that $I^Q, t' \models \Box\alpha_{bool}$.

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▶ **Corollary 48.** *Let $\alpha \in \mathcal{L}_1$ and $I = (V, \prec) \in \mathfrak{I}$ s.t. $I, t \models \alpha$, then there exists a finite sequence M containing t s.t. $I^M, t \models \alpha$ and $size(I, M) \leq |\alpha|$.*

▶ **Definition 49** (Pseudo-interpretation transformation). *Let $I = (V, \prec) \in \mathfrak{I}$, let $N = (t_0, t_1, t_2, \ldots, t_{n-1})$ be a finite sequence. The pseudo-interpretation $I^N = (V^N, \prec^N)$ can be transformed into a preferential interpretation $I' \overset{\text{def}}{=} (V', \prec') \in \mathfrak{I}$ as follows:*

$$V' : \begin{cases} V'(i) := V^N(t_i), \text{ if } 0 \leq i < n; \\ V'(i) := V^N(n-1), \text{ otherwise.} \end{cases}$$

And for all $0 \leq i, j < n$ s.t. $(t_i, t_j) \in \prec^N$, we have $(i, j) \in \prec'$.

▶ **Theorem 50** (Bounded Model property). *Let $\alpha \in \mathcal{L}_1$ be $\mathfrak{I}$-satisfiable, there exists $I = (V, \prec) \in \mathfrak{I}$ s.t. $size(I) \leq |\alpha|$ and $I, 0 \models \alpha$.*

Let $I^N := (V^N, \prec^N)$ be a pseudo-interpretation and let $I = (V', \prec')$ be its transformed interpretation. We can see that $size(I') \leq size(I, M)$. The size of the initial part of I' is the sequence N and the final part has one distinct valuation which is the valuation of the last element of the sequence N.

We can also see that truth value of sub-sentences is unchanged with the transformation of the pseudo-interpretation. In other words, $I^N, t_i \models \alpha$ entails $I', i \models \alpha$.

Proof. Let α be a $\mathfrak{I}$-satisfiable sentence and let $I = (V, \prec) \in \mathfrak{I}$ where $I, 0 \models \alpha$ be an interpretation that satisfies α. Thanks to Lemma 47, since $\mathbb{N}$ is a sequence and $0 \in \mathbb{N}$ s.t. $I, 0 \models \alpha$, then there is a sequence $M \subseteq \mathbb{N}$ containing 0 where $size(I, M) \leq |\alpha|$ and $I^M, 0 \models \alpha$. We can transform it then to $I' = (V', \prec)$ by changing the labels of M into a sequence of natural numbers and looping the valuation of the last element of M. We can see that $I', 0 \models \alpha$ and $size(I) \leq |\alpha|$. ◀

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A Proofs of results in Section 3 and Section 4

▶ **Proposition 8.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i, i', j, j' \in \mathbb{N}$ s.t. $i \leq i'$, $i' \leq j'$ and $j \in min_{\prec}(i)$. If $V(j) = V(j')$, then $j' \in min_{\prec}(i')$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let i, j, i', j' be four time points s.t. $i \leq i'$, $i' \leq j'$ and $j \in min_{\prec}(i)$. We assume that $V(j) = V(j')$ and we suppose that $j' \notin min_{\prec}(i')$. Following our supposition, $j' \notin min_{\prec}(i')$ means that there exists $k \in [i', +\infty[$ where $(k, j') \in \prec$. From Definition 7, if $(k, j') \in \prec$ and $V(j) = V(j')$, then $(k, j) \in \prec$. Since $(k, j) \in \prec$, we have $j \notin min_{\prec}(i)$. This conflicts with our assumption of $j \in min_{\prec}(i)$. We conclude that if $V(j) = V(j')$ then $j' \in min_{\prec}(i')$. ◀

▶ **Proposition 9.** *Let $I = (V, \prec) \in \mathfrak{I}$ and let $i, j \in \mathbb{N}$ s.t. $j \in min_{\prec}(i)$. For all $i \leq i' \leq j$, we have $j \in min_{\prec}(i')$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}$ and let $i, i', j \in \mathbb{N}$ s.t. $j \in min_{\prec}(i)$ and $i \leq i' \leq j$. Since $j \in min_{\prec}(i)$, there is no $j' \in [i, +\infty[$ s.t. $(j', j) \in \prec$. Moreover, we have $i \leq i'$, we conclude that there is no $j' \in [i', +\infty[$ s.t. $(j', j) \in \prec$. Therefore, we have $j \in min_{\prec}(i')$. ◀

▶ **Proposition 12.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i \leq j \leq i' \leq j'$ be time points in final(I) s.t. $V(j) = V(j')$. Then we have $j \in min_{\prec}(i)$ iff $j' \in min_{\prec}(i')$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$. We have four time points $i \leq j \leq i' \leq j' \in final(I)$, this proof is divided in two parts:

- For the only-if part, we suppose that $j \in min_{\prec}(i)$ and we prove that $j' \in min_{\prec}(i')$. We have $i \leq i'$, $i' \leq j'$, $V(j) = V(j')$ and $j \in min_{\prec}(i)$. Thanks to Proposition 8, $j' \in min_{\prec}(i')$.
- For the if part, we suppose that $j' \in min_{\prec}(i')$ and we prove that $j \in min_{\prec}(i)$. We use a proof by contradiction. We assume that $j' \in min_{\prec}(i')$ and we suppose that $j \notin min_{\prec}(i)$. This implies that there exists $k \in [i, +\infty[$ such that $(k, j) \in \prec$.
  - Case 1: $k \in [i', +\infty[$. From Definition 7, since $V(j) = V(j')$ and $(k, j) \in \prec$, then $(k, j') \in \prec$ thus $j' \notin min_{\prec}(i')$. This conflicts with our assumption that $j' \in min_{\prec}(i')$.
  - Case 2: $k \in [i, i'[$. From Lemma 10, since $k \in final(I)$, then there exists $k' \in [i', +\infty[$ such that $V(k') = V(k)$. From Definition 7, since we have $V(j') = V(j)$, $V(k') = V(k)$ and $(k, j) \in \prec$, then $(k', j') \in \prec$, thus $j' \notin min_{\prec}(i')$. This conflicts with our assumption that $j' \in min_{\prec}(i')$.

◀

▶ **Lemma 13.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $i \leq i'$ be time points of $\mathit{final}(I)$ where $V(i) = V(i')$. Then for every $\alpha \in \mathcal{L}^\star$, we have $I, i \models \alpha$ iff $I, i' \models \alpha$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $i \leq i'$ in $\mathit{final}(I)$ such that $V(i) = V(i')$. We prove that $I, i \models \alpha$ iff $I, i' \models \alpha$ using structural induction on α.

- Base: α is an atomic proposition p. For the only-if part, we know that $I, i \models p$ iff $p \in V(i)$. Since $V(i) = V(i')$, we have $p \in V(i')$, thus $I, i' \models p$. Same reasoning applies for the if part.
- $\alpha = \neg\alpha_1$. For the only-if part, we assume that $I, i \models \neg\alpha_1$ and suppose that $I, i' \not\models \neg\alpha_1$. $I, i' \not\models \neg\alpha_1$ implies $I, i' \models \alpha_1$. Since the Lemma holds on α_1 and $I, i' \models \alpha_1$, we conclude that $I, i \models \alpha_1$, conflicting with our assumption. We follow the same reasoning for the if part.
- $\alpha = \alpha_1 \wedge \alpha_2$. For the only-if part, $I, i \models \alpha_1 \wedge \alpha_2$ means that $I, i \models \alpha_1$ and $I, i \models \alpha_2$. Since the Lemma holds on both α_1 and α_2, we have $I, i' \models \alpha_1$ and $I, i' \models \alpha_2$. Thus $I, i' \models \alpha_1 \wedge \alpha_2$. The same reasoning applies for the if part.
- $\alpha = \Diamond\alpha_1$. For the only-if part, we assume that $I, i \models \Diamond\alpha_1$. Following our assumption, it means that there exists $j \in [i, +\infty[$ s.t. $I, j \models \alpha_1$. Thanks to Lemma 10. Since $j \in \mathit{final}(I)$, there exists $j' \in [i', +\infty[$ where $V(j') = V(j)$. Thanks to the induction hypothesis, if $V(j) = V(j')$ and $I, j \models \alpha_1$ then $I, j' \models \alpha_1$, we conclude that $I, i' \models \Diamond\alpha_1$.
  For the if part, we assume that $I, i' \models \Diamond\alpha_1$. $I, i' \models \Diamond\alpha_1$ means that there exists $j' \in [i', +\infty[$ s.t. $I, j' \models \alpha_1$. We know that $[i', +\infty[\subseteq [i, +\infty[$, we conclude that $I, i \models \Diamond\alpha_1$.
- $\alpha = \Diamond\!\!\!\!\sim\alpha_1$. For the only-if part, we assume that $I, i \models \Diamond\!\!\!\!\sim\alpha_1$. Following our assumption, $I, i \models \Diamond\!\!\!\!\sim\alpha_1$ means that there exists $j \in [i, +\infty[$ s.t. $j \in min_\prec(i)$ and $I, j \models \alpha_1$. Thanks to Lemma 10. Since $j \in \mathit{final}(I)$, there exists $j' \in [i', +\infty[$ such that $V(j') = V(j)$. Thanks to the induction hypothesis, if $V(j) = V(j')$ and $I, j \models \alpha_1$ then (I) $I, j' \models \alpha_1$. Thanks to Proposition 8, $V(j) = V(j')$, $i \leq i'$, $i' \leq j'$ and $j \in min_\prec(i)$ means that (II) $j' \in min_\prec(i')$. From (I) and (II), we conclude that $I, i' \models \Diamond\!\!\!\!\sim\alpha_1$.
  For the if part, we assume that $I, i' \models \Diamond\!\!\!\!\sim\alpha_1$. $I, i' \models \Diamond\!\!\!\!\sim\alpha_1$ means that there is a $j' \in [i', +\infty[$ such that $j' \in min_\prec(i')$ and (I) $I, j' \models \alpha_1$. We need to prove that $j' \in min_\prec(i)$. We suppose that $j' \notin min_\prec(i)$. It means that there exists $k \in [i, +\infty[$ such that $(k, j') \in \prec$. From Lemma 10, since $k \in \mathit{final}(I)$, that means there is $k' \in [i', +\infty[$ such that $V(k) = V(k')$. Following the condition set in Definition 7, since $(k, j') \in \prec$ and $V(k') = V(k)$, then $(k', j') \in \prec$ and thus $j' \notin min_\prec(i')$, conflicting with our assumption of $j' \in min_\prec(i')$, thus (II) $j' \in min_\prec(i)$. From (I) and (II), we conclude that $I, i \models \Diamond\!\!\!\!\sim\alpha$.

◀

The result of Lemma 15 can be found in Section D.

▶ **Proposition 19.** *Let $I = (V, \prec) \in \mathfrak{I}$, N_1, N_2 be two acceptable sequences w.r.t. I. Then $N_1 \cup N_2$ is an* acceptable sequence *w.r.t. I s.t. $\mathit{size}(I, N_1 \cup N_2) \leq \mathit{size}(I, N_1) + \mathit{size}(I, N_2)$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}$, N_1, N_2 be two acceptable sequences w.r.t. I and let $I^{N_1} = (V^{N_1}, \prec^{N_1})$, $I^{N_2} = (V^{N_1}, \prec^{N_2})$ be two pseudo interpretations over N_1 and N_2 respectively. We assume that $N = N_1 \cup N_2$.

We suppose that N is not an acceptable sequence w.r.t. I. It means that there exist two time points $t, t' \in \mathit{final}(I)$ s.t. $V(t) = V(t')$ where $t \in N$ and $t' \notin N$. Since $t \in N$, t is either an element of N_1 or N_2. We consider that $t \in N_1$. By Definition 16, since $t \in N_1$ and N_1 is an acceptable sequence w.r.t. I, all time points of $\mathit{final}(I)$ that have the same valuation as t are in N_1. Since $t' \in \mathit{final}(I)$ and $V(t') = V(t)$, then $t' \in N_1$, and therefore $t' \in N$. This conflicts with the supposition of $t' \notin N$. Same reasoning applies if we take $t \in N_2$. We conclude that for all $t \in N$ s.t. $t \in \mathit{final}(I)$, all $t' \in \mathit{final}(I)$ s.t. $V(t') = V(t)$ are also in N. Thus, N is an acceptable sequence w.r.t. I.

In order to prove that $\mathit{size}(I, N) \leq \mathit{size}(I, N_1) + \mathit{size}(I, N_2)$, we need to prove that $\mathit{init}(I, N) \subseteq \mathit{init}(I, N_1) \cup \mathit{init}(I, N_2)$ and $\mathit{range}(I, N) \subseteq \mathit{range}(I, N_1) \cup \mathit{range}(I, N_2)$. Let $t \in N$ be a time point s.t. $t \in init(I, N)$. By the definition of $init(I, N)$, we know that $t \in init(I)$. Since N is a sequence containing only elements of N_1 or N_2, the time point t is either in N_1 or N_2. By definition of $init(I, N_1)$, if $t \in N_1$ and $t \in init(I)$, then $t \in init(I, N_1)$. The same goes in the case of $t \in N_2$. We conclude that if $t \in init(I, N)$, then $t \in init(I, N_1) \cup init(I, N_2)$.

Following the same line of thought, we can prove that $final(I, N) \subseteq final(I, N_1) \cup final(I, N_2)$ and consequently we can prove that $range(I, N) \subseteq range(I, N_1) \cup range(I, N_2)$.

Since $init(I^N) \subseteq init(I^{N_1}) \cup init(I^{N_2})$, then $length(init(I^N)) \leq length(init(I^{N_1})) + length(init(I^{N_2}))$. Similarly, if $range(I^N) \subseteq range(I^{N_1}) \cup range(I^{N_2})$, then $card(range(I^N)) \leq card(range(I^{N_1})) + card(range(I^{N_2}))$. Therefore, we conclude that $size(I^N) \leq size(I^{N_1}) + size(I^{N_2})$. ◀

▶ **Proposition 20.** *Let $I = (V, \prec) \in \mathfrak{I}$ and N be an acceptable sequence w.r.t. I. If for all distinct $t, t' \in N$, we have $V(t') = V(t)$* only *when both $t, t' \in final(I, N)$, then $size(I, N) \leq 2^{|\mathcal{P}|}$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}$ and N be an acceptable sequence w.r.t. I. We assume that for all $t, t' \in N$ s.t. we have $V(t') = V(t)$ only when both $t, t' \in final(N)$. Two cases are possible:

- $init(I, N)$ is empty. Since $card(range(I, N)) \leq 2^{|\mathcal{P}|}$, we conclude that $size(I, N) \leq 2^{|\mathcal{P}|}$.
- $init(I, N)$ is not empty. Going back to our assumption, we can see that for all $t \in init(I, N)$ and $t' \in N$ s.t. $t' \neq t$ we have $V(t') \neq V(t)$. If $init(I, N)$ has n time points having distinct valuations, then $range(final(I, N))$ has at most $2^{|\mathcal{P}|} - n$ valuations. Therefore, we have $size(I, N) \leq 2^{|\mathcal{P}|}$.

◀

B Proofs of results in Section 5

▶ **Lemma 27.** *Let $\alpha_1 \in \mathcal{L}^\star$ be a sentence starting with a temporal operator, $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let T be a non-empty acceptable sequence w.r.t. I where for all $t \in T$ we have $I, t \models \diamondminus\alpha_1$. Then for all $t, t' \in Anchors(I, T, \diamondminus\alpha_1)$ s.t. $V(t) = V(t')$ and $t \neq t'$, we have $t, t' \in final(I, Anchors(I, T, \diamondminus\alpha_1))$.*

Proof. Let $\alpha_1 \in \mathcal{L}^\star$, let T be a non-empty acceptable sequence w.r.t. $I \in \mathfrak{I}^{sd}$ where for all $t \in T$ we have $I, t \models \diamondminus\alpha_1$. Just as a reminder, we have $Anchors(I, T, \diamondminus\alpha_1) = \bigcup_{t_i \in T} ST(I, AS(I, min_\prec(t_i)), \alpha_1)$. Thus, there exists $t_i \in T$ such that $t \in ST(I, AS(I, min_\prec(t_i)), \alpha_1)$. Suppose that there exist $t, t' \in Anchors(I, T, \diamondminus\alpha_1)$ with $t \neq t'$ such that t is in $init(I, Anchors(I, T, \diamondminus\alpha_1))$ and $V(t) = V(t')$. Notice that $t \in init(I)$, since $t \in init(I, Anchors(I, T, \diamondminus\alpha_1))$. Without loss of generality, we assume that $t < t'$. From Definition 24, we have $t \in AS(I, (\mathrm{t}_{\alpha_1}^{I, AS(I, min_\prec(t_i))}))$. Thanks to Definition 22 and Definition 23, the fact that $t' \in init(I)$, we can see that : (1) there is no $t'' \in final(I, AS(I, min_\prec(t_i)))$ s.t. $I, t'' \models \alpha_1$ and (2) $t = \mathrm{t}_{\alpha_1}^{I, AS(I, min_\prec(t_i))} = max_<\{t'' \in init(I, AS(I, min_\prec(t_i))) \mid I, t'' \models \alpha_1\}$. On the other hand, thanks to Proposition 8, since $t' < t''$ and $t' \in min_\prec(t_i)$, we have $t'' \in min_\prec(t_i)$. Hence $t'' \in AS(I, min_\prec(t_i))$. Since $t'' \in Anchors(I, T, \diamondminus\alpha_1)$, we also have $I, t'' \models \alpha_1$. From this and the property (1) we can assert that t' does not belong to $final(I, AS(I, min_\prec(t_i)))$. It follows that $t' \in init(I, AS(I, min_\prec(t_i)))$. From the property (2) we can assert that $t \geq t'$, which leads to a contradiction since $t < t'$. Therefore, for all $t, t' \in Anchors(I, T, \diamondminus\alpha_1)$ s.t. $V(t) = V(t')$, we must have $t, t' \in final(Anchors(I, T, \diamondminus\alpha_1))$. ◀

▶ **Proposition 28.** *Let $\alpha \in \mathcal{L}^\star$ be a sentence starting with a temporal operator, $I = (V, \prec) \in \mathfrak{I}^{sd}$. Let T be a non-empty acceptable sequence w.r.t. I where for all $t \in T$ we have $I, t \models \alpha$. Then, we have $size(I, Anchors(I, T, \alpha)) \leq 2^{|\mathcal{P}|}$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. . We show that is the case for our temporal operators:

- Since $size(I, Anchors(I, T, \Box\alpha_1)) = size(I, \emptyset) = 0$, we conclude that $size(I, Anchors(I, T, \Box\alpha_1)) \leq 2^{|\mathcal{P}|}$.
- Since $size(I, Anchors(I, T, \Diamond\alpha_1)) = size(I, ST(I, \mathbb{N}, \alpha_1)) = 1$, we conclude that $size(I, Anchors(I, T, \Diamond\alpha_1)) \leq 2^{|\mathcal{P}|}$.
- T is an acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \diamondminus\alpha_1$. From Proposition 27, for all $t'_i, t'_j \in Anchors(I, T, \diamondminus\alpha_1)$ s.t. $V(t'_i) = V(t'_j)$ we have $t'_i, t'_j \in final(I, Anchors(I, T, \diamondminus\alpha_1))$. From Proposition 20, we can conclude that $size(Anchors(I, T, \diamondminus\alpha_1)) \leq 2^{|\mathcal{P}|}$.
- Going back to Definition 26, we have $Anchors(I, T, \boxminus\alpha_1) = DR(I, \bigcup_{t_i \in T} AS(I, min_{\prec}(t_i)))$. We denote the acceptable sequence $\bigcup_{t_i \in T} AS(I, min_{\prec}(t_i))$ by N. From Definition 25 we have $Anchors(I, T, \boxminus\alpha_1) = DR(I, N) = \bigcup_{v \in val(I,N)} ST(I, N, \alpha_v)$. Moreover, we know that $size(ST(I, N, \alpha_v)) = 1$ for all $v \in val(I, N)$. Consequently, thanks to Proposition 19, we have $size(\bigcup_{v \in val(I,N)} ST(I, N, \alpha_v)) \leq card(val(I, N))$. We can see that $card(val(I, N)) \leq 2^{|\mathcal{P}|}$, we can conclude that $size(Anchors(I, T, \boxminus\alpha_1)) = size(\bigcup_{v \in val(I,N)} ST(I, N, \alpha_v)) \leq 2^{|\mathcal{P}|}$.

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▶ **Proposition 29.** *Let $\alpha_1 \in \mathcal{L}^\star$, $I = (V, \prec) \in \mathfrak{I}^{sd}$, let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \boxminus\alpha_1$, with $\alpha_1 \in \mathcal{L}^\star$. For all acceptable sequences N w.r.t. I s.t. $Anchors(I, T, \boxminus\alpha_1) \subseteq N$ and for all $t_i \in N \cap T$, we have the following: Let $I^N = (V^N, \prec^N)$ be the pseudo-interpretation over N and $t' \in N$, if $t' \notin min_{\prec}(t_i)$, then $t' \notin min_{\prec^N}(t_i)$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \boxminus\alpha_1$, with $\alpha_1 \in \mathcal{L}^\star$. Let N be an acceptable sequence w.r.t. I s.t. $Anchors(I, T, \boxminus\alpha_1) \subseteq N$. Let $t_i \in N \cap T$. Let $t' \in N$ be a time point s.t. $t' \notin min_{\prec}(t_i)$, we discuss these two cases:

- $t' \notin [t_i, +\infty[$: Since $t' \notin [t_i, +\infty[$, then $t' \notin [t_i, +\infty[\cap N$. Therefore, we conclude that $t' \notin min_{\prec^N}(t_i)$.
- $t' \in [t_i, +\infty[$: Since $\prec$ satisfies the well-foundedness condition, $t' \notin min_{\prec}(t_i)$ implies that there exists a time point $t'' \in min_{\prec}(t_i)$ s.t. $(t'', t') \in \prec$. Let $\alpha_{t''}$ be the representative sentence of $V(t'')$. For the sake of readability, we shall denote the sequence $\bigcup_{t \in T} AS(I, min_{\prec}(t))$ with M. Notice that there exists $V \in val(I, M)$ such that $V = V(t'')$ since $t_i \in T$ and $t'' \in min_{\prec}(t_i)$. Thanks to Definition 25, since $DR(I, M) = \bigcup_{v \in val(I,M)} ST(I, M, \alpha_v)$ and $V(t'') \in val(I, M)$, we can find $t''' \in ST(I, M, \alpha_{t''})$ where $t''' \in DR(I, M) \subseteq N$, $V(t''') = V$ and $t''' \geq t''$. Since $(t'', t') \in \prec$, $I \in \mathfrak{I}^{sd}$ and $V(t''') = V(t'')$, we have $(t''', t') \in \prec$. Moreover, we have $t''', t' \in N$, and therefore $(t''', t') \in \prec^N$. Since $t''' \in [t_i, +\infty[\cap N$ and $(t''', t') \in \prec^N$, we conclude that $t' \notin min_{\prec^N}(t_i)$.

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▶ **Proposition 31.** *Let $\alpha \in \mathcal{L}^\star$ be in NNF, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. Then, we have $size(I, Keep(I, T, \alpha)) \leq \mu(\alpha) \times 2^{|\mathcal{P}|}$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$ which $\alpha \in \mathcal{L}^\star$.

We use structural induction on T and α in order to prove this property.

- Base $\alpha = p$ or $\alpha = \neg p$. $Keep(I, T, \alpha) = \emptyset$. Since $size(I, \emptyset) = 0 \leq \mu(\alpha) \times 2^{|\mathcal{P}|} = 0$, then the property holds on atomic propositions.
- $\alpha = \alpha_1 \wedge \alpha_2$. Since $I, t \models \alpha_1 \wedge \alpha_2$ for all $t \in T$, we can assert that $I, t \models \alpha_1$ and $I, t \models \alpha_2$. By applying the induction hypothesis on T, α_1 and α_2, we have $size(I, Keep(I, T, \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$ and $size(I, Keep(I, T, \alpha_2)) \leq \mu(\alpha_2) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 19, $size(Keep(I, T, \alpha_1 \wedge \alpha_2)) \leq (\mu(\alpha_1) + \mu(\alpha_2)) \times 2^{|\mathcal{P}|}$. We conclude that $size(I, Keep(I, T, \alpha_1 \wedge \alpha_2)) \leq (\mu(\alpha_1 \wedge \alpha_2)) \times 2^{|\mathcal{P}|}$.

- $\alpha = \alpha_1 \vee \alpha_2$. Since $I, t \models \alpha_1 \vee \alpha_2$ for all $t \in T$ then $I, t \models \alpha_1$ or $I, t \models \alpha_2$. Consider the sequence T_1 (resp. T_2) containing all $t_1 \in T$ (resp.$t_2 \in T$) s.t. $I, t_1 \models \alpha_1$ (resp. $I, t_2 \models \alpha_2$). Using induction hypothesis on T_1, T_2, α_1 and α_2, we have $\mathit{size}(I, \mathit{Keep}(I, T_1, \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$ and $\mathit{size}(I, \mathit{Keep}(I, T_2, \alpha_2)) \leq \mu(\alpha_2) \times 2^{|\mathcal{P}|}$. We conclude in the same way as the last case that $\mathit{size}(I, \mathit{Keep}(I, T, \alpha_1 \vee \alpha_2)) \leq (\mu(\alpha_1 \vee \alpha_2)) \times 2^{|\mathcal{P}|}$.
- $\alpha = \Diamond\alpha_1$. First of all, we proved in Proposition 28 that (I) $\mathit{size}(I, \mathit{Anchors}(I, T, \Diamond\alpha_1)) \leq 2^{|\mathcal{P}|}$. On the other hand, thanks to Definition 26 it is easy to see that $\mathit{size}(I, \mathit{Anchors}(I, T, \Diamond\alpha_1))$ is a non-empty acceptable sequence w.r.t. I s.t. for all $t' \in \mathit{Anchors}(I, T, \Diamond\alpha_1)$ we have $I, t' \models \alpha_1$. By the induction hypothesis on $\mathit{Anchors}(I, T, \Diamond\alpha_1)$ and α_1, we have (II) $\mathit{size}(I, \mathit{Keep}(I, \mathit{Anchors}(I, T, \Diamond\alpha_1), \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 19, from (I) and (II) we conclude that $\mathit{size}(I, \mathit{Keep}(I, T, \Diamond\alpha_1)) \leq (1 + \mu(\alpha_1)) \times 2^{|\mathcal{P}|} = \mu(\Diamond\alpha_1) \times 2^{|\mathcal{P}|}$.
- $\alpha = \Box\alpha_1$. As a result of semantics of the $\Box$ operator, we can see that for all $t \in T$ we have $I, t \models \alpha_1$. By the induction hypothesis on T and α_1, we have $\mathit{size}(I, \mathit{Keep}(I, T, \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. Since $\mathit{Keep}(I, T, \alpha_1) = \mathit{Keep}(I, T, \Box\alpha_1)$ then $\mathit{size}(I, \mathit{Keep}(I, T, \Box\alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. We conclude that $\mathit{size}(I, \mathit{Keep}(I, T, \Box\alpha_1)) \leq \mu(\Box\alpha_1) \times 2^{|\mathcal{P}|}$.
- $\alpha = \diamondminus\alpha_1$. First of all, we proved in Proposition 28 that (I) $\mathit{size}(I, \mathit{Anchors}(I, T, \diamondminus\alpha_1)) \leq 2^{|\mathcal{P}|}$. On the other hand, thanks to Definition 26 it is easy to see that $\mathit{Anchors}(I, T, \diamondminus\alpha_1)$ is a non-empty acceptable sequence w.r.t. I s.t. for all $t' \in \mathit{Anchors}(I, T, \diamondminus\alpha_1)$ we have $I, t' \models \alpha_1$. By the induction hypothesis on $\mathit{Anchors}(I, T, \diamondminus\alpha_1)$ and α_1, we have (II) $\mathit{size}(I, \mathit{Keep}(I, \mathit{Anchors}(I, T, \diamondminus\alpha_1), \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 19, from (I) and (II), we conclude that $\mathit{size}(I, \mathit{Keep}(I, T, \diamondminus\alpha_1)) \leq (1 + \mu(\alpha_1)) \times 2^{|\mathcal{P}|} = \mu(\diamondminus\alpha_1) \times 2^{|\mathcal{P}|}$.
- $\alpha = \boxminus\alpha_1$. First of all, we proved in Proposition 28 that (I) $\mathit{size}(I, \mathit{Anchors}(I, T, \boxminus\alpha_1)) \leq 2^{|\mathcal{P}|}$. On the other hand, from definition30, we have $T' = \bigcup_{t_i \in T} AS(I, min_{\prec}(t_i))$. It is easy to see that for all $t' \in T'$ we have $I, t' \models \alpha_1$ and that T' is a non-empty acceptable sequence w.r.t. I. By the induction hypothesis on T' and α_1, we have (II) $\mathit{size}(I, \mathit{Keep}(I, T', \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 19, form (I) and (II) we conclude that $\mathit{size}(I, \mathit{Keep}(I, T, \boxminus\alpha_1)) \leq (1 + \mu(\alpha_1)) \times 2^{|\mathcal{P}|} = \mu(\boxminus\alpha_1) \times 2^{|\mathcal{P}|}$.

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▶ **Lemma 32.** *Let $\alpha \in \mathcal{L}^\star$ be in NNF, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. For all acceptable sequences N w.r.t. I, if $\mathit{Keep}(I, T, \alpha) \subseteq N$, then for every $t \in N \cap T$, we have $I^N, t \models_{\mathscr{P}} \alpha$.*

Proof. Let $\alpha \in \mathcal{L}^\star$ be in NNF, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. We consider N to be an acceptable sequence w.r.t. I s.t. $\mathit{Keep}(I, T, \alpha) \subseteq N$ and $t \in N \cap T$. Let $I^N = (N, V^N, \prec^N)$ be the pseudo-interpretation over N.

We use structural induction on T and α in order to prove this property.

- $\alpha = p$ or $\alpha = \neg p$. Since $I, t \models p$ (resp. $\neg p$), it means that $p \in V(t)$ (resp. $p \notin V(t)$). We know that $V^N(t) = V(t)$. We conclude that $I^N, t \models_{\mathscr{P}} p$ (resp. $\neg p$).
- $\alpha = \alpha_1 \wedge \alpha_2$. Since $I, t \models \alpha_1 \wedge \alpha_2$ for all $t \in T$, we can assert that $I, t \models \alpha_1$ and $I, t \models \alpha_2$. By applying the induction hypothesis on T, α_1 and α_2, since $\mathit{Keep}(I, T, \alpha_1) \subseteq N$ and $\mathit{Keep}(I, T, \alpha_2) \subseteq N$, therefore we have $I^N, t \models_{\mathscr{P}} \alpha_1$ and $I^N, t \models_{\mathscr{P}} \alpha_2$. Thus, we have $I^N, t \models_{\mathscr{P}} \alpha_1 \wedge \alpha_2$.
- $\alpha = \alpha_1 \vee \alpha_2$. Suppose that $I, t \models \alpha_1$ (the case $I, t \models \alpha_2$ can be treated in a similar way) and consider the sequence T_1 containing all $t_1 \in T$ s.t. $I, t_1 \models \alpha_1$. Here, since $t \in T_1$, therefore T_1 is non-empty and $t \in T_1 \cap N$. We know that $\mathit{Keep}(I, T_1, \alpha_1) \cup \mathit{Keep}(I, T_2, \alpha_2) \subseteq N$. Consequently $\mathit{Keep}(I, T_1, \alpha_1) \subseteq N$. From the induction hypothesis, we have $I^N, t \models_{\mathscr{P}} \alpha_1$. Therefore, we have $I^N, t \models \alpha_1 \vee \alpha_2$.

- $\alpha = \Diamond\alpha_1$. We have $I, t \models \Diamond\alpha_1$ and we need to prove that $I^N, t \models_{\mathscr{P}} \Diamond\alpha_1$. $I, t \models \Diamond\alpha_1$ means that there exists $t' \in [t, \infty[$ such that $I, t' \models \alpha_1$, therefore $Anchors(I, T, \Diamond\alpha_1)$ is non-empty (see Definition 26). We know that $Anchors(I, T, \Diamond\alpha_1) \subseteq Keep(I, T, \Diamond\alpha_1) \subseteq N$, consequently $Anchors(I, T, \Diamond\alpha_1) \cap N$ is non-empty. Thanks to Definition 26 it is easy to see that for all $t_1 \in Anchors(I, T, \Diamond\alpha_1)$ we have $I, t_1 \models \alpha_1$. By the induction hypothesis on $Anchors(I, T, \Diamond\alpha_1)$ and α_1, since $Keep(I, Anchors(I, T, \Diamond\alpha_1), \alpha_1) \subseteq N$, $t' \in Anchors(I, T, \Diamond\alpha_1)$ (a non-empty acceptable sequence w.r.t I) and $I, t' \models \alpha_1$, thus $I^N, t' \models \alpha_1$. Therefore, we have $I^N, t \models_{\mathscr{P}} \Diamond\alpha_1$.
- $\alpha = \Box\alpha_1$. We have $I, t \models \Box\alpha_1$ and we need to prove that $I^N, t \models_{\mathscr{P}} \Box\alpha_1$. We know that for all $t' \geq t$ we have $I, t' \models \alpha_1$. We can assert that for all $t' \in N \cap T$ such that $t' \geq t$, we have $I^N, t' \models_{\mathscr{P}} \alpha_1$. By the induction hypothesis on T and α_1, $Keep(I, T, \alpha_1) = Keep(I, T, \Box\alpha_1)$. Consequently $Keep(I, T, \alpha_1) \subseteq N$ since for all $t' \in N \cap T$, we have $I^N, t' \models_{\mathscr{P}} \alpha_1$. We conclude that $I^N, t \models_{\mathscr{P}} \Box\alpha_1$.
- $\alpha = \diamondminus\alpha_1$. We have $I, t \models \diamondminus\alpha_1$ and we need to prove that $I^N, t \models_{\mathscr{P}} \diamondminus\alpha_1$. $I, t \models \diamondminus\alpha_1$ means that there exists $t' \in min_{\prec}(t)$ such that $I, t' \models \alpha_1$, therefore $Anchors(I, T, \diamondminus\alpha_1)$ is non-empty (see Definition 26). We know that $Anchors(I, T, \diamondminus\alpha_1) \subseteq Keep(I, T, \diamondminus\alpha_1) \subseteq N$, consequently $Anchors(I, T, \diamondminus\alpha_1) \cap N$ is non-empty. Thanks to Definition 26 it is easy to see that for all $t_1 \in Anchors(I, T, \diamondminus\alpha_1)$ we have $I, t_1 \models \alpha_1$. By the induction hypothesis on $Anchors(I, T, \diamondminus\alpha_1)$ and α_1, since $Keep(I, T_1, \alpha_1) \subseteq N$ with $T_1 = Anchors(I, T, \diamondminus\alpha_1)$, and T_1 is an acceptable sequence where $I, t' \models \alpha_1$ for all $t' \in T_1$, we conclude that $I^N, t' \models_{\mathscr{P}} \alpha_1$ (I). Thanks to the construction of the pseudo-interpretation I^N, since $t' \in min_{\prec^N}(t)$, therefore $t' \in min_{\prec}(t)$ (II). From (I) and (II), we conclude that $I^N, t \models_{\mathscr{P}} \diamondminus\alpha_1$.
- $\alpha = \boxminus\alpha_1$. We have $I, t \models \boxminus\alpha_1$ and we need to prove that $I^N, t \models_{\mathscr{P}} \boxminus\alpha_1$. $I, t \models \boxminus\alpha_1$ means that for all $t' \in min_{\prec}(t)$ we have $I, t' \models \alpha_1$, therefore for all $t' \in T' = \bigcup_{t_i \in T} AS(I, min_{\prec}(t_i))$ we have $I, t' \models \alpha_1$. In addition, thanks to the well-foundedness condition on $\prec$, T' is non-empty. We know that $Anchors(I, T, \boxminus\alpha_1) \subseteq Keep(I, T, \boxminus\alpha_1) \subseteq N$ and that $Anchors(I, T, \boxminus\alpha_1) = DR(I, T')$ consequently $T' \cap N$ is non-empty. We use proof by contradiction. Suppose that $I^N, t \not\models_{\mathscr{P}} \boxminus\alpha_1$, which means there exists $t' \in min_{\prec^N}(t_i)$ s.t. $I^N, t' \not\models_{\mathscr{P}} \alpha_1$. Thanks to Proposition 29, if $t' \in min_{\prec^N}(t_i)$, then $t' \in min_{\prec}(t_i)$. Just a reminder, we have $T' = \bigcup_{t_i \in T} AS(I, min_{\prec}(t_i))$ where for all $t'' \in T'$ we have $I, t'' \models \alpha_1$ (Note that T' is a non-empty acceptable sequence w.r.t. I). By the induction hypothesis on T' and α_1, since $Keep(I, T', \alpha_1) \subseteq N$, and $t' \in AS(I, min_{\prec}(t)) \subseteq T'$, therefore $I^N, t' \models_{\mathscr{P}} \alpha_1$. This conflicts with our supposition. We conclude that there is no $t' \in min_{\prec^N}(t)$ s.t. $I^N, t' \not\models_{\mathscr{P}} \alpha_1$, and therefore $I^N, t \models_{\mathscr{P}} \boxminus\alpha_1$.

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C Proof of results in Section 6

NB: The results marked $(*)$ are introduced here, while they are omitted in the main text.

▶ **Proposition 39.** *Let $M = (i, \pi, V_M, \prec_M)$ be a UPPI, $\mathsf{I}(M) = (V, \prec)$ and $t, t', t_M, t'_M \in \mathbb{N}$ s.t.:*

$$t_M = \begin{cases} t, \text{ if } t < i; \\ i + (t - i) \bmod \pi, \text{ otherwise.} \end{cases} \qquad t'_M = \begin{cases} t', \text{ if } t' < i; \\ i + (t' - i) \bmod \pi, \text{ otherwise.} \end{cases}$$

We have the following: $t' \in min_{\prec}(t)$ iff $t'_M \in min_{\prec_M}(t_M)$.

Proof. Let $M = (i, \pi, V_M, \prec_M)$ be a UPPI, $\mathsf{I}(M) = (V, \prec)$ and $t, t' \in \mathbb{N}$.

- For the only-if part, we assume that $t' \in min_{\prec}(t)$. Following our assumption, there is no $t'' \in [t, +\infty[$ s.t. $(t'', t') \in \prec$. We use a proof by contradiction. Suppose that $t'_M \notin min_{\prec_M}(t_M)$, which means there exists $t''_M \in [min_<\{t_M, i\}, i+\pi[$ with $(V_M(t''_M), V_M(t'_M)) \in \prec_M$. Going back to Definition 37, $V_M(t'_M) = V(t')$ and . Consequently, $(V(t''_M), V_M(t')) \in \prec_M$. Thanks to Definition 37, (I) $(t''_M, t') \in \prec$. There are two possible cases for t, . If $t \in [0, i[$ then $t_M = t$ and (II) $t''_M \in [t, i+\pi[$. From (I) and (II), there exists $t''_M > t$ such that $(t''_M, t') \in \prec$. This conflicts with our supposition. If $t \in [i, +\infty[$, then $t''_M \in [i, i+\pi[$ and t, t', t'' are in $\mathit{final}(I(M))$. Thanks to proposition 10, there exists $t'' > t$ such that $V(t'') = V(t_M)$. Since $I(M) \in \mathfrak{I}^{sd}$ and $(t''_M, t') \in \prec$ then $(t'', t) \in \prec$. Consequently, there exists $t'' > t$ such that $(t'', t) \in \prec$. This conflicts with our supposition.

- For the if part, we assume that $t'_M \in min_{\prec_M}(t_M)$. Following our assumption, there is no $t''_M \in [min_<\{t_M, i\}, i+\pi[$ with $(V_M(t''_M), V_M(t'_M)) \in \prec_M$. We use proof by contradiction. Suppose that $t' \notin min_\prec(t)$, which means there exists $t''' > t$ such that $(t''', t') \in \prec$. Let t'''_M be defined as follows:

$$t'''_M = \begin{cases} t''', \text{ if } t''' < i; \\ i + (t''' - i) \bmod \pi, \text{ otherwise.} \end{cases}$$

  Thanks to definition 37, $V(t''') = V_M(t'''_M)$, $V(t') = V_M(t'_M)$ and since $(t''', t') \in \prec$ then $(V(t'''), V(t')) \in \prec_M$. Consequently, (I) $(V(t'''_M), V(t'_M)) \in \prec_M$. . From (I) and (II), we have $t'_M \notin min_{\prec_M}(t_M)$. This conflicts with our supposition.

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▶ **Definition 51** (∗). *Given a UPI $I = (V, \prec)$, we define the UPPI $M(I) = (i, \pi, V_M, \prec_M)$ by:*

- $i = \mathit{length}(\mathit{init}(I))$, $\pi = \mathit{card}(\mathit{range}(I))$;
- $V_M(t) = V(t)$ *for all* $t \in [0, i+\pi[$;
- *for all* $t, t' \in [0, i+\pi[$, $(V(t), V(t')) \in \prec_M$ *iff* $(t, t') \in \prec$.

▶ **Proposition 42.** *Let $\alpha \in \mathcal{L}^\star$. We have that α is $\mathfrak{I}^{sd}$-satisfiable iff there exists a UPPI M such that $\mathsf{I}(M), 0 \models \alpha$ and $\mathit{size}(\mathsf{I}(M)) \leq |\alpha| \times 2^{|\mathcal{P}|}$.*

Proof. Let $\alpha \in \mathcal{L}^\star$.

- For the only if part, let α be $\mathfrak{I}^{sd}$-satisfiable. Thanks to Theorem 21 and Proposition 35, there exists a UPI $I = (V, \prec) \in \mathfrak{I}^{sd}$ s.t. $I, 0 \models \alpha$ and $\mathit{size}(I) \leq |\alpha| \times 2^{|\mathcal{P}|}$. We define the UPPI $M(I)$ from I. It can be checked that $\mathsf{I}(M(I)) = I$. Therefore, from $\mathfrak{I}^{sd}$-satisfiable sentence α, we can find a UPPI M such that $\mathsf{I}(M), 0 \models \alpha$ and $\mathit{size}(\mathsf{I}(M)) \leq |\alpha| \times 2^{|\mathcal{P}|}$.
- For the if part, let $M = (i, \pi, V_M, \prec_M)$ be a UPPI s.t. $\mathsf{I}(M), 0 \models \alpha$. Since $\mathsf{I}(M) \in \mathfrak{I}^{sd}$, therefore α is $\mathfrak{I}^{sd}$-satisfiable.

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Lemma 41 is a particular case of the following Lemma.

▶ **Lemma 52** (∗). *Let a UPPI $M = (i, \pi, V_M, \prec_M)$, $\alpha \in \mathcal{L}^\star$, $\alpha_1 \in SF(\alpha)$ and $t, t' \in \mathbb{N}$ such that:*

$$t' = \begin{cases} t, \textit{ if } t < i; \\ i + (t - i) \bmod \pi, \textit{ otherwise.} \end{cases}$$

We have $\mathsf{I}(M), t' \models \alpha$ iff $\alpha_1 \in lab^M_\alpha(t)$.

Proof. Let a UPPI $M = (i, \pi, V_M, \prec_M)$, $\alpha \in \mathcal{L}^\star$, $t \in \mathbb{N}$ and $\mathsf{I}(M) = (V, \prec)$. We use structural induction to prove the Lemma. Let t' be a time point s.t. $t' = t$ if $t \in [0, i[$, and $t' = i + (t - i) \bmod \pi$ if $t \in [i, +\infty[$.

- $\alpha = p$. If $t \in [0, i[$, then we have $V_M(t') = V(t)$, thus we have $p \in V_M(t)$ iff $p \in V(t)$, and therefore $\mathsf{I}(M), t \models p$ iff $p \in lab^M_\alpha(t)$. If $t \in [i, +\infty[$, we have $V_M(t') = V(t)$. Following the same reasoning as the previous case, $\mathsf{I}(M), t \models p$ iff $p \in lab^M_\alpha(t')$.
- $\alpha = \neg\alpha_1$. By the induction hypothesis, we have $\mathsf{I}(M), t \models \alpha_1$ iff $\alpha_1 \in lab^M_\alpha(t')$, and therefore $\mathsf{I}(M), t \not\models \alpha_1$ iff $\alpha_1 \notin lab^M_\alpha(t')$. We conclude that $\mathsf{I}(M), t \models \neg\alpha_1$ iff $\neg\alpha_1 \in lab^M_\alpha(t')$.
- $\alpha = \alpha_1 \wedge \alpha_2$. By the induction hypothesis, we have $\mathsf{I}(M), t \models \alpha_1$ iff $\alpha_1 \in lab^M_\alpha(t')$ and $\mathsf{I}(M), t \models \alpha_2$ iff $\alpha_2 \in lab^M_\alpha(t')$, and therefore $\mathsf{I}(M), t \models \alpha_1 \wedge \alpha_2$ iff $\alpha_1 \wedge \alpha_2 \in lab^M_\alpha(t')$.
- $\alpha = \Diamond\alpha_1$.
  - For the only-if part, let $\mathsf{I}(M), t \models \Diamond\alpha_1$. There exists $t_1 \in [t, +\infty[$ s.t. $\mathsf{I}(M), t_1 \models \alpha_1$. For all $t_1 \in \mathbb{N}$, there is a t'_1 s.t. $t'_1 = t_1$ if $t_1 \in [0, i[$ and $t'_1 = i + (t_1 - i) \bmod \pi$ if $t_1 \in [i, +\infty[$. By the induction hypothesis, we have $\alpha_1 \in lab^M_\alpha(t'_1)$. If $t \in [0, i[$, there exists $t'_1 \geq t$ s.t. $\alpha_1 \in lab^M_\alpha(t'_1)$, and therefore $\Diamond\alpha_1 \in lab^M_\alpha(t)$. If $t \in [i, +\infty[$, there exists $t'_1 \in [i, i + \pi[$ s.t. $\alpha_1 \in lab^M_\alpha(t'_1)$, and therefore $\Diamond\alpha_1 \in lab^M_\alpha(t)$.
  - For the if part, let $\mathsf{I}(M), t \not\models \Diamond\alpha_1$. Following our assumption, $\mathsf{I}(M), t \models \neg\Diamond\alpha_1$ for all $t_1 \geq t$ we have $\mathsf{I}(M), t_1 \models \neg\alpha_1$. By the induction hypothesis, for all $t_1 \geq t$, we have $\neg\alpha_1 \in lab^M_\alpha(t'_1)$ where $t'_1 = t_1$ if $t_1 \in [0, i[$ and $t'_1 = i + (t_1 - i) \bmod \pi$ if $t_1 \in [i, +\infty[$. It is also worth to point out that for all , we have $\neg\alpha_1 \in lab^M_\alpha(t'_1)$. By Definition of $lab^M_\alpha(\cdot)$, we have $\neg\Diamond\alpha_1 \in lab^M_\alpha(t')$, and therefore $\Diamond\alpha_1 \notin lab^M_\alpha(t')$.
- $\alpha = \Diamond\!\!\!\prec\alpha_1$.
  - For the only-if part, let $\mathsf{I}(M), t \models \Diamond\!\!\!\prec\alpha_1$. There exists $t_1 \in min_\prec(t)$ s.t. $\mathsf{I}(M), t_1 \models \alpha_1$. For all $t_1 \in \mathbb{N}$, there is a t'_1 s.t. $t'_1 = t_1$ if $t_1 \in [0, i[$ and $t'_1 = i + (t_1 - i) \bmod \pi$ if $t_1 \in [i, +\infty[$. By the induction hypothesis, we have (I) $\alpha_1 \in lab^M_\alpha(t'_1)$. From Proposition 39, we can see that (II) $t_1 \in min_\prec(t)$ iff $t'_1 \in min_{\prec_M}(t')$. From (I) and (II), since there is $t'_1 \in min_{\prec_M}(t')$ where $\alpha_1 \in lab^M_\alpha(t'_1)$, we conclude that $\Diamond\!\!\!\prec\alpha_1 \in lab^M_\alpha(t')$.
  - For the if part, let $\mathsf{I}(M), t \not\models \Diamond\!\!\!\prec\alpha_1$. Following our assumption, $\mathsf{I}(M), t \models \neg\Diamond\!\!\!\prec\alpha_1$ for all $t_1 \in min_\prec(t)$ we have $\mathsf{I}(M), t_1 \models \neg\alpha_1$. By the induction hypothesis, for all $t_1 \in min_\prec(t)$, we have (I) $\neg\alpha_1 \in lab^M_\alpha(t'_1)$ where $t'_1 = t_1$ if $t_1 \in [0, i[$ and $t'_1 = i + (t_1 - i) \bmod \pi$ if $t_1 \in [i, +\infty[$. From Proposition 39, we can see that (II) $t_1 \in min_\prec(t)$ iff $t'_1 \in min_{\prec_M}(t')$. From (I) and (II), since there is no $t'_1 \in min_{\prec_M}(t')$ s.t. $\alpha_1 \in lab^M_\alpha(t'_1)$, we conclude that $\Diamond\!\!\!\prec\alpha_1 \notin lab^M_\alpha(t')$.

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D Proofs of results for Lemma 15

▶ **Proposition 53** (∗). *Let $I = (V, \prec) \in \mathfrak{I}$ and $i \in \mathit{final}(I)$. For all $j \in \mathit{final}(I)$, there exists $j' \geq j$ such that $V(j') = V(i)$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}$ and $i, j \in \mathit{final}(I)$. Let E be the set defined by $E = \{i' \in \mathit{final}(I) : V(i') = V(i)\}$. Since $i \in \mathit{final}(I)$, we have $E \neq \emptyset$. Suppose now that there does not exist $j' \geq j$ such that $V(j') = V(i)$. We have E is a non empty finite set of integers included in $[0, \ldots, j-1]$. Let k be the integer defined by $k = max\{k' \in E\}$. From the definitions of E and k, we have $k \in \mathit{final}(I)$ and there does not exist $k' > k$ such that $V(k') = V(k)$. This contradicts Lemma 10. We can conclude that there exists $j' \geq j$ such that $V(j') = V(i)$. ◀

▶ **Proposition 54** (∗). *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over the same set of atomic propositions $\mathcal{P}$ s.t. $range(I) = range(I')$. For all $i \in \mathit{final}(I)$ and $i' \in \mathit{final}(I')$ such that $V(i) = V'(i')$, we have :*

(1) for all $j \in [i, +\infty[$ there exists $j' \in [i', +\infty[$ such that $V'(j') = V(j)$.

(2) for all $j \in min_\prec(i)$ there exists $j' \in min_{\prec'}(i')$ such that $V(j) = V'(j')$.

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ s.t. $\mathit{range}(I) = \mathit{range}(I')$ and $i, i' \in \mathit{final}(I)$ such that $V(i) = V'(i')$.

(1) Let j belonging to $[i, +\infty[$. Since $i \in \mathit{final}(I)$, we have $j \in \mathit{final}(I)$. Moreover, from the equality $\mathit{range}(I) = \mathit{range}(I')$, we can assert that there exists $k \in \mathit{final}(I')$ such that $V'(k) = V(j)$. Hence, from Proposition 53, there exists $j' \geq i'$ such that $V'(j') = V'(k) = V(j)$.

(2) Let $j \in min_{\prec}(i)$. We have $j \in \mathit{final}(I)$. From Property (1), there exists $j' \geq i'$ such that $V'(j') = V(j)$. Suppose that $j' \notin min_{\prec'}(i')$. Since $j' \geq i'$, there exists $k' \geq i'$ such that $(k', j') \in \prec'$. From Property (1), there exists $k \geq i$ such that $V(k) = V'(k')$. Since $V(k) = V'(k')$, $V'(j') = V(j)$, $(k', j') \in \prec'$ and, I and I' are two faithful interpretations, we can assert that $(k, j) \in \prec$. Consequently, since $k \geq i$ and $(k, j) \prec$, we have $j \notin min_{\prec}(i)$. There is a contradiction. We can conclude that $j' \in min_{\prec'}(i')$.

◀

► **Proposition 55** $(*)$**.** *Let $\alpha \in \mathcal{L}^{\star}$, $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over the same set of atomic propositions $\mathcal{P}$ s.t. $\mathit{range}(I) = \mathit{range}(I')$. For $\alpha \in \mathcal{L}^{\star}$ $i \in \mathit{final}(I)$ and $i' \in \mathit{final}(I')$ s.t. $V(i) = V'(i')$, we have :*

$$I, i \models \alpha \text{ iff } I', i' \models \alpha.$$

Proof. Let $I = (V, \prec)$, $I' = (V', \prec')$ be two faithful interpretations belonging to $\mathfrak{I}^{sd}$. over the same set of atomic propositions $\mathcal{P}$ s.t. $\mathit{range}(I) = \mathit{range}(I')$. Let $\alpha \in \mathcal{L}^{\star}$, $i \in \mathit{final}(I)$ and $i' \in \mathit{final}(I')$ such that $V(i) = V'(i')$. Without loss of generality we suppose that α does not contain $\vee$, $\Box$ and $\boxminus$. This proposition can be proven by induction on the structure of the sentence α.

- Base case : $\alpha = p$ with $p \in \mathcal{P}$. Since $V(i) = V'(i')$, we have $p \in V(i)$ iff $p \in V'(i')$, thus $I, i \models p$ iff $I', i' \models p$.
- $\alpha = \diamondminus\alpha_1$. First we prove that $I, i \models \diamondminus\alpha_1$ implies $I', i' \models \diamondminus\alpha_1$. We assume that $I, i \models \diamondminus\alpha_1$. Hence, there exists $j \in [i, +\infty[$ s.t. $j \in min_{\prec}(i)$ and $I, j \models \alpha_1$. From Proposition 54 (2), there exists $j' \in min_{\prec'}(i')$ such that $V'(j') = V(j)$. By induction hypothesis, we have $I', j' \models \alpha_1$. We can conclude that $I', i' \models \diamondminus\alpha_1$. The if part can be proved with a similar reasoning.

◀

► **Corollary 56** $(*)$**.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over the same set of atomic propositions $\mathcal{P}$ s.t. $\mathit{range}(I) = \mathit{range}(I')$. For $i \in \mathit{final}(I)$ and $\alpha \in \mathcal{L}^{\star}$, we have : if $I, i \models \alpha$ then there exists $i' \in \mathit{final}(I')$ such that $I', i' \models \alpha$.*

► **Proposition 57** $(*)$**.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $\mathit{init}(I) \doteq \mathit{init}(I')$ and $\mathit{range}(I) = \mathit{range}(I')$. Then we have :*

$$\text{For all } t, t' \in \mathit{init}(I),\ t' \in min_{\prec}(t) \text{ iff } t' \in min_{\prec'}(t).$$

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $\mathit{init}(I) \doteq \mathit{init}(I')$ and $\mathit{range}(I) = \mathit{range}(I')$ and, $t, t' \in \mathit{init}(I)$ such that $t' \in min_{\prec}(t)$. Suppose that $t' \notin min_{\prec'}(t)$. Since $t' \geq t$, there exists $t'' \geq t$ such that $(t'', t') \in \prec'$. There are two possible cases.

- $t'' \in \mathit{init}(I')$. Since $\mathit{init}(I) \doteq \mathit{init}(I')$, we have $V'(t'') = V(t'')$. Moreover, since I and I' are two faithful interpretations and $V'(t') = V(t')$, we have $(t'', t') \in \prec$. Since $t'' \geq t$, it follows that $t' \notin min_{\prec}(t)$. There is a contradiction. We can conclude that $t' \in min_{\prec'}(t)$.
- $t'' \in \mathit{final}(I')$. Since $\mathit{range}(I) = \mathit{range}(I')$, there exists $t''' \in \mathit{final}(I)$ such that $V'(t'') = V(t''')$. Moreover, since I and I' are two faithful interpretations and $V'(t') = V(t')$, we have $(t''', t') \in \prec$. Since $t''' \geq t$, It follows that $t' \notin min_{\prec}(t)$. There is a contradiction. We can conclude that $t' \in min_{\prec'}(t)$.

Same reasoning can be applied to prove the if part. ◀

▶ **Proposition 58** ($*$). *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $init(I) \doteq init(I')$ and $range(I) = range(I')$. For all $t \in init(I)$ and $t' \in final(I)$ such that $t' \in min_{\prec}(t)$ we have $\{t'' \in final(I') : V'(t'') = V(t')\} \subseteq min_{\prec'}(t)$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $init(I) \doteq init(I')$ and $range(I) = range(I')$ and, $t \in init(I)$, $t' \in final(I)$, $t'' \in final(I')$ such that $t' \in min_{\prec}(t)$ and $V'(t'') = V(t')$. We will prove that $t'' \in min_{\prec'}(t)$.

Suppose that $t'' \notin min_{\prec'}(t)$. Since $t'' \geq t$, there exists $t''' \geq t$ such that $(t''', t'') \in \prec'$. There are two possible cases.

- $t''' \in init(I')$. Since $init(I) \doteq init(I')$, we have $V'(t''') = V(t''')$. Moreover, since I and I' are two faithful interpretations and $V'(t'') = V(t')$, we have $(t''', t') \in \prec$. Since $t''' \geq t$, it follows that $t' \notin min_{\prec}(t)$. There is a contradiction. We can conclude that $t'' \in min_{\prec'}(t)$.
- $t''' \in final(I')$. Since $range(I) = range(I')$, there exists $u \in final(I)$ such that $V'(t''') = V(u)$. Moreover, since I and I' are two faithful interpretations and $V'(t'') = V(t')$, we have $(u, t') \in \prec$. Since $u \geq t$, it follows that $t' \notin min_{\prec}(t)$. There is a contradiction. We can conclude that $t'' \in min_{\prec'}(t)$.

◀

▶ **Lemma 59** ($*$). *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $V'(0) = V(0)$, $init(I) \doteq init(I')$, and $range(I) = range(I')$. Then for all $\alpha \in \mathcal{L}^\star$, we have :*

$$\textit{For all } t \in init(I) \cup \{0\},\ I, t \models \alpha \textit{ iff } I', t \models \alpha.$$

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $V'(0) = V(0)$, $init(I) \doteq init(I')$, and $range(I) = range(I')$. Let $\alpha \in \mathcal{L}^\star$ and $t \in init(I) \cup \{0\}$. Without loss of generality we suppose that α does not contain $\vee$, $\Box$ and $\Box\!\!\!\!\prec$.

First, notice that in the case where $init(I)$ and $init(I')$ are empty intervals, we necessarily have $t = 0$. Moreover, since $t \in final(I)$ and $t \in final(I')$ and $V(0) = V'(0)$, from Proposition 55, we can assert that $I, t \models \alpha$ iff $I', t \models \alpha$. Consequently, the property to be proved is true. Now, we will suppose that $init(I)$ and $init(I')$ are non empty intervals. Hence, we have $t \in init(I)$ and $t \in init(I')$. We will prove that $I, t \models \alpha$ iff $I', t \models \alpha$ by structural induction on α.

- Base case : $\alpha = p$. Since $t \in init(I)$, we have $V(t) = V'(t)$. Hence, $p \in V(t)$ iff $p \in V'(t)$. Thus $I, t \models p$ iff $I', t \models p$.
- $\alpha = \neg\alpha_1$. By induction hypothesis, we have $I, t \models \alpha_1$ iff $I', t \models \alpha_1$. Hence, it is not the case that $I, t \models \alpha_1$ iff it is not the case that $I', t \models \alpha_1$. We can conclude that, $I, t \models \neg\alpha_1$ iff $I', t \models \neg\alpha_1$.
- $\alpha = \alpha_1 \wedge \alpha_2$. We have $I, t \models \alpha_1 \wedge \alpha_2$ iff $I, t \models \alpha_1$ and $I, t \models \alpha_2$. Using the induction hypothesis, it follows that $I, t \models \alpha_1$ and $I, t \models \alpha_2$ iff $I', t \models \alpha_1$ and $I', t \models \alpha_2$. We can conclude that $I, t \models \alpha_1 \wedge \alpha_2$ iff $I', t \models \alpha_1 \wedge \alpha_2$.
- $\alpha = \Diamond\alpha_1$. Suppose that $I, t \models \Diamond\alpha_1$. There exists a $t' \in [t, +\infty[$ s.t. $I, t' \models \alpha_1$. Two cases are possible w.r.t. t'.
  - $t' \in init(I)$. By induction hypothesis, we have $I', t' \models \alpha_1$. Hence, we can conclude that $I', t \models \Diamond\alpha_1$.
  - $t' \in final(I)$. Since $range(I) = range(I')$, there exists $t'' \in final(I')$ such that $V'(t'') = V(t')$. From Proposition 55, we have $I', t'' \models \alpha_1$. Since, $t'' > t$ we have $I', t \models \Diamond\alpha_1$.

  Same reasoning can be applied to prove the if part.
- $\alpha = \Diamond\!\!\!\!\prec\alpha_1$. Suppose that $I, t \models \Diamond\!\!\!\!\prec\alpha_1$. There exists $t' \in min_{\prec}(t)$ s.t. $I, t' \models \alpha_1$. Two cases are possible w.r.t. t'.

- $t' \in init(I)$. By induction hypothesis, we have $I', t' \models \alpha_1$. Moreover, from Proposition 57, we have $t' \in min_{\preceq'}(t)$. Hence, we can conclude that $I', t \models \Diamond\alpha_1$.
- $t' \in final(I)$. Since $range(I) = range(I')$, there exists $t'' \in final(I')$ such that $V'(t'') = V(t')$. From Proposition 55, we have $I', t'' \models \alpha_1$. From Proposition 58, we have $t'' \in min_{\preceq'}(t)$. Hence, we can conclude that $I', t \models \Diamond\alpha_1$.

Same reasoning can be applied to prove the if part.

◀

Lemma 15 is a direct result of result of Lemma 59.