Propositional logic

Discrete mathematics

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Bibliography

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Definition A *proposition* (pl. zdanie matematyczne) is a declarative sentence that is either true or false but never both.

Examples

1. $1 + 2 = 4$
2. $3 < 4$
3. $\mathbb{N} \subseteq \mathbb{Z}$
4. There exists integer x that $x^2 = -4$
5. Warszawa is the capitol of Poland
6. Today sun is shining.
7. Mary has a little lamb.
8. Any noise annoys an oyster, but a noisy noise annoys an oyster more.

Definition *proposition* is a declarative sentence that is either true or false bu never both.

Wrong examples

- $x + 2 = 4$
- $x^2 + y^2 = z^2$
- What time is it?
- Have fun.
- How much wood would a woodchuck chuck?

Value function of proposition.

$$w(p) = \begin{cases} 1 & \text{for true } p \\ 0 & \text{for false } p \end{cases}$$

Compound propositions (pl. zdania złożone)

Examples

- $2+2=4$ and $2\cdot 2=4$,
- If 123456789 is divisible by 1357 then $\frac{123456789}{1357}\in\mathbb{N}$,
- It is not the case that $\emptyset\subseteq\emptyset$,
- $1234567\in\mathbb{N}$ or $1234567\notin\mathbb{N}$.

Logical operators (connectives) (pl. spójniki).

- $\vee$ - disjunction (pl. alternatywa),
- $\wedge$ - conjunction (pl. koniunkcja),
- $\sim$ - negation (pl. negacja),
- $\oplus$ - exclusive or (pl. alternatywa wykluczająca),
- $\rightarrow$ - conditional statement, implication (pl. implikacja)
- $\leftrightarrow$ - biconditional statement, bi-implication, equivalence (pl. równoważność)

Truth table

$w(p)$	$w(q)$	$w(p \vee q)$	$w(p \wedge q)$	$w(p \rightarrow q)$	$w(p \leftrightarrow q)$	$w(p \oplus q)$	$\sim p$
0	0	0	0	1	1	0	1
0	1	1	0	1	0	1	1
1	0	1	0	0	0	1	0
1	1	1	1	1	1	0	0

Polyadic operators (pl. operatory wieloargumentowe)

$$(w(NAND(p_1, p_2, ..., p_n)) = 1) \iff \text{at least one } p_i \text{ is false}$$

For $p \to q$

- p is hypothesis, antecedent, premise (pl. poprzednik, założenie, warunek wystarczający)

- q conclusion, consequence (pl. następnik, wniosek, warunek konieczny)

Definition *Compound proposition* (again) (pl. schemat logiczny) is an expression formed from propositional variables using logical operators.

Definition A compound proposition $S(p_1, p_2, ..., p_n)$ that is always true, no matter what are the valuse of propositional variables $p_1, p_2, ..., p_n$, is called a *tautology* (pl. tautologia). Then we write:

$$\models S(p_1, p_2, ..., p_n)$$

By analogy compound proposition that is always false is called a *contradiction* (pl. sprzeczny). We write:

$$\models \sim S(p_1, p_2, ..., p_n)$$

The remaining compound propositions are called *contingency*.

Examples of tautologies

- $\models p \leftrightarrow p$

- $\models p \vee \sim p$

- $\models \sim (p \wedge \sim p)$

- $\models (p \wedge \sim p) \rightarrow q$

Truth Table for $p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$

$w(p)$	$w(q)$	$w(r)$	$w(q \wedge r)$	$w(p \vee (q \wedge r))$	$w(p \vee q)$	$w(p \vee r)$	$w((p \vee q) \wedge (p \vee r))$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Equivalence

$$(\models S \leftrightarrow R) \text{ iff } (S \Leftrightarrow R)$$

Implication

$$(\models S \rightarrow R) \text{ iff } (S \Rightarrow R)$$

Chosen logical equivalences

1. Commutative laws (pl. prawa przemienności)

   - $\models (p \vee q) \leftrightarrow (q \vee p)$
   - $\models (p \wedge q) \leftrightarrow (q \wedge p)$
   - $\models (p \oplus q) \leftrightarrow (q \oplus p)$
   - $\models (p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$

2. Associative laws (pl. prawa łączności)

- $\models ((p \vee q) \vee r) \leftrightarrow (p \vee (q \vee r))$
- $\models ((p \wedge q) \wedge r) \leftrightarrow (p \wedge (q \wedge r))$
- $\models ((p \oplus q) \oplus r) \leftrightarrow (p \oplus (q \oplus r))$

3. Distributive laws (pl. rozdzielności)

- $\models (p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$
- $\models (p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$

4. Transitivity laws (pl. prawa przechodniości)

- $\models ((p \leftrightarrow q) \wedge (q \leftrightarrow r)) \rightarrow (p \leftrightarrow r)$
- $\models ((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

5. De Morgan's laws (pl. prawa De Morgana)

- $\models \sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$
- $\models \sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$

6. Duble negation law (pl. prawo podwójnego przeczenia)

   - $\models (\sim\sim p) \leftrightarrow p$

7. Law of excluded middle (pl. prawo wyłączonego środka)

   - $\models (p \vee \sim p)$

8. Contraposition law (pl. pawo kontrapozycji)

   - $\models (p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

9. Identity laws (pl. prawa identyczności)

   - $\models (p \wedge T) \leftrightarrow (p)$
   - $\models (p \vee F) \leftrightarrow (p)$

10. Domination laws (pl. prawa dominacji)

   - $\models (p \vee T) \leftrightarrow (T)$
   - $\models (p \wedge F) \leftrightarrow (F)$

11. Idempotent laws (pl. prawa idempotentności)

- $\models (p \wedge p) \leftrightarrow (p)$
- $\models (p \vee p) \leftrightarrow (p)$

12. Negation laws (pl. prawa negacji)

- $\models (p \wedge \sim p) \leftrightarrow (F)$
- $\models (p \vee \sim p) \leftrightarrow (T)$

How to define all operators using $\sim$ and $\wedge$?

CNF

Given compound formula $S(p_1, p_2, ..., p_n)$. There is a set of literals $l_{ij} = p_i \vee l_{ij} = \sim p_i$

$S(p_1, p_2, ..., p_n) \Leftrightarrow (l_{11} \vee l_{21} \vee ... \vee l_{n1}) \wedge (l_{12} \vee l_{22} \vee ... \vee l_{n2})) \wedge ... \wedge (l_{1m} \vee l_{2m} \vee ... \vee l_{nm})$

DNF

How to define all operators using $\sim$ and $\rightarrow$?