Mathematical induction

Discrete mathematics

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Bibliography

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Axiom of mathematical induction

$$\models \Big(P(1) \land \forall_{k\in\mathbb{N}}\big(P(k) \to P(k+1)\big)\Big) \to \forall_{k\in\mathbb{N}}\big(P(k)\big)$$

Axiom of mathematical induction, version 1.

$$\models \Big(P(1) \wedge \forall_{k\in\mathbb{N}}\big(P(k) \to P(k+1)\big)\Big) \to \forall_{k\in\mathbb{N}}\big(P(k)\big)$$

In other words: To prove that $P(k)$ is true for arbitrary positive integer k it is enough to verify that $P(1)$ is true and that conditional statement $(P(k) \to P(k+1))$ holds for all positive integers.

Procedure

1. Define predicate $P(k)$

2. Prove $P(1)$

3. Prove $P(k) \to P(k+1)$, for $k \geq 1$.

Example

$\forall_{n \in \mathbb{N}} \left(1 + 2 + ... + k = \frac{k(k+1)}{2} \right)$

Stronger induction, version 2.

$$\models \Big(P(s) \wedge \forall_{k\in\mathbb{N},k\geq s}\big(P(k) \to P(k+1)\big)\Big) \to \forall_{k\in\mathbb{N},k\geq s}\big(P(k)\big)$$

In other words: To prove that $P(k)$ is true for all positive integer k greater or equal s, it is enough to verify that $P(s)$ is true and that conditional statement $(P(k) \to P(k+1))$ holds for all positive integers greater or equal s.

Example
$2^k \geq k^2$ for all $k \in \mathbb{N}$, $k > 3$.

Strong induction, version 3.

$$\models \left(P(s) \wedge \forall_{k\in\mathbb{N},k\geq s}\Big(\Big(\forall_{s\leq r\leq k} P(r)\Big) \rightarrow P(k+1)\Big)\right) \rightarrow \forall_{k\in\mathbb{N},k\geq s}\Big(P(k)\Big)$$

In other words: To prove that $P(k)$ is true for all positive integer k greater or equal s, it is enough to verify that $P(s)$ is true and that conditional statement $\Big(\forall_{s\leq r\leq k} P(r)\Big) \rightarrow P(k+1)$ holds for all positive integers k greater or equal s.

Example

The archipelago of islands is operated by the airline in such a way that for each pair of islands there is a connection but only to the other side. Prove that it is possible to plan a trip on the islands in such a way that each one is visited exactly once.

Induction for other (then positive integers) infinite sets.
If we are supposed to prove that $P(k)$ holds for $k \in A$. We need a surjective mapping (pl. funkcja “na”) $f : \mathbb{N} \to A$ and prove quantifier $Q(n) \Leftrightarrow P(f(n))$.

Example
Prove $\left(1 + 3 + 5 + ... + k = \frac{(k+1)^2}{4}\right)$ for odd numbers.

Polyadic predicates

$$\models \Big(P(1,1) \wedge \forall_{k,r\in\mathbb{N}}\big(P(k,r) \to P(k+1,r) \wedge P(k,r+1)\big)\Big) \to \forall_{k,r\in\mathbb{N}}\big(P(k,r)\big)$$

Example

Prove that $3|k^3+n^3+2k-n$ for $n,k \in \mathbb{N}$.

Errors in inductive reasoning

One more example For $n \in \mathbb{N}$ and arbitrary $c_1, c_2, ...c_n \in \mathbb{R}_+$

$$\sqrt[n]{c_1 c_2 ... c_n} \leq \frac{c_1 + c_2 + ... + c_n}{n}$$

Prove version 2 using version 1.