Sets

Discrete mathematics

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Bibliography

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Definition *Set* is an unordered collection of distinct objects, called elements or members of the set.

$a \in A$

- A contains a,
- a is an element (member) of A,

$a \notin A$

- A does not contains a,
- a is not an element (member) of A,

Examples

- $\{1, 2, 3, 4, 5, 6\} = \{1, 2, ..., 6\}$
- $\{a, b, c, d, e, f, g\}$
- $\{\{a, b\}, \{c, d\}, \{a, e, f, g\}\}$
- $\{``mathematics", ``is", ``beautiful"\}$
- Set of all students signed on the list.

- $\{1, 3, 2, 3, 6, 1\} = \{1, 3, 2, 6\}$

- $\{\}$, $\emptyset$

- $\mathbb{N} = \{1, 2, 3, ...\}$ - natural numbers

- $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ - integers

- $\{1, 3, 5, ...\} = \{x : x \in \mathbb{N} \land x \text{ mod } 2 = 1\} = \{x : x = 2k - 1, k \in \mathbb{N}\}$

- $\mathbb{R} = \{x : x = y + c_0 2^{-0} + c_1 2^{-1} + c_2 2^{-2} + ..., y \in \mathbb{Z}, c_i \in \{0, 1\}, i = 0, 1, 2, ...\}$ - real numbers

- $\mathbb{Q} = \{x : x = p/q, p \in \mathbb{Z}, q \in \mathbb{N}\}$ - rational numbers, p - numerator, q - denominator.

Definition The set A is a *subset* of B if and only if every element of A is also an element of B. Then A is included in B or A is a superset of B.

We write $A \subseteq B$.

Examples

- $\{3, 5\} \subseteq \{1, 2, 3, 4, 5, 6\}$
- $\{a, \{b, c, d\}, e, f, g\} \not\subseteq \{a, b, c, d, e, f, g\}$
- $\emptyset \subseteq \emptyset$
- For each set A, $\emptyset \subseteq A$.

Definition For two sets A and B $A = B$ (equals) $\iff A \subseteq B \wedge B \subseteq A$.

Definition The set A is a *proper subset* (pl. podzbiór właściwy) of B iff $A \subseteq B \wedge A \neq B$. We denote $A \subset B$.

Definition Set $A \cup B = \{x : x \in A \vee x \in B\}$ is called an *union* (pl. suma) of sets A and B.

Definition Set $A \cap B = \{x : x \in A \wedge x \in B\}$ is called an *intersection* (pl. iloczyn, przecięcie, część wspólna) of sets A and B.

Definition Set $A \backslash B = \{x : x \in A \wedge x \notin B\}$ is called a *difference* (pl. różnica) of sets A and B.

Definition Set $A \oplus B = \{x : (x \in A \vee x \in B) \wedge x \notin A \cap B\}$ is called a *symmetric difference* (pl. różnica symetryczna) of sets A and B.

Definition Given X that is an universal set. Set $A^c = X \backslash A$ is called a *complement* (pl. dopełnienie) of set A.

Examples

$\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}$

$\{1,2,3\} \cap \{3,4,5\} = \{3\}$

$\{1,2,3\} \backslash \{3,4,5\} = \{1,2\}$

$\{1,2,3\} \oplus \{3,4,5\} = \{1,2,4,5\}$

For universe $X = \mathbb{N}$, $\{1,2,3\}^c = \{4,5,6,...\}$

Set algebra laws

1. $A \cup B = B \cup A$ (commutative law) (pl. prawo przemienności),
2. $A \cap B = B \cap A$ (commutative law) (pl. prawo przemienności),
3. $(A \cup B) \cup C = A \cup (B \cup C)$ (associative law) (pl. prawo łączności),
4. $(A \cap B) \cap C = A \cap (B \cap C)$ (associative law) (pl. prawo łączności),
5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (distributive law) (pl. prawo rozdzielności),
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distributive law) (pl. prawo rozdzielności),
7. $A \cup A = A$ (idempotent law) (pl. prawo idempotentności),
8. $A \cap A = A$ (idempotent law) (pl. prawo idempotentności),
9. $A \cup \emptyset = A$ (identity law) (pl. prawo identycznosci),

10. $A \cup X = X$ (identity law) (pl. prawo identyczności),

11. $A \cap \emptyset = \emptyset$ (identity law) (pl. prawo identyczności),

12. $A \cap X = A$ (identity law) (pl. prawo identyczności),

13. $(A^c)^c = A$ (complementation law) (pl. prawo podwójnego dopełnienia),

14. $A \cup A^c = X$ (complement law),

15. $A \cap A^c = \emptyset$ (complement law),

16. $X^c = \emptyset$,

17. $\emptyset^c = X$,

18. $(A \cup B)^c = A^c \cap B^c$ (Demorgan's law) (pl. prawo De Morgana),

19. $(A \cap B)^c = A^c \cup B^c$ (Demorgan's law) (pl. prawo De Morgana),

20. $A \subseteq B \iff A \cap B = A$,

21. $A \backslash B = A \cap B^c$.

Definition The *ordered n-tuple* $(a_1, a_2, ..., a_n)$ is the ordered collection that has a_1 a the the fists element, a_2 at the second one, ... and a_n at the last element.

$$(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n) \iff a_1 = b_1, a_2 = b_2, ..., a_n = b_n$$

Definition An *ordered pair* is 2-tuple.

Definition Let A and B be sets. The *Cartesian product* (pl. iloczyn kartezjański) is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. We denote $A \times B$. In other words $A \times B = \{(a, b) : a \in A \wedge b \in B\}$.

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) : a_1 \in A_1, a_2 \in A_2, ..., a_n \in A_n$$

Example

$$\{1,2\} \times \{2,3\} \times \{1\} = \{(1,2,1),(1,3,1),(2,2,1),(2,3,1)\}$$

$$A^n = \{(a_1, a_2, ..., a_n) : a_i \in A, i = 1, 2, ..., n\}$$

Example

$\{0,1\}^3 = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$

Definition For a collection of sets A_1, A_2, ..., A_n we define *generalized union of sets*

$$A_1 \cup A_2 \cup ... \cup A_n = \bigcup_{i=1}^{n} A_i = \bigcup_{i \in T} A_i, T = \{1, 2, ..., n\}$$

and *generalized intersection of sets*

$$A_1 \cap A_2 \cap ... \cap A_n = \bigcap_{i=1}^{n} A_i = \bigcap_{i \in T} A_i, T = \{1, 2, ..., n\}$$

Example

Find

$$M = \bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}, j > i} \{i, i+1, ..., j\}$$

Definition The size of a finite set A is called its *cardinality* (pl. moc zbioru). We denote $|A|$.

Example $|\{0, 1, 3\}| = 3$

$|\emptyset| = 0$

What to do with infinite sets?

Definition For a givens set A, the power set (pl. zbiór potęgowy) of A is the set of all subsets of A. We write $P(A)$.

Example

For $A = \{1, 3\}$

$P(A) = \{\emptyset, \{1\}, \{3\}, \{1, 3\}\}$

$|P(A)| = 2^2$

Methods of proving

- Venn Diagrams
- From definition
- 01 technique
- Algebra laws

B. Russell's paradox

Let set (family) A be a set of all sets.

Let $B \subseteq A$ and $B = \{b : b \notin b\}$.