Predicates and quantifierers

Discrete mathematics

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Bibliography

- K. Piwakowski, Wykład, Algebra zbiorów
- K. H. Rosen, *Discrete Mathematics and Its Applications*
- K. A. Ross, C. R. Wright, *Discrete Mathematics*

Definition A *predicate* (pl. predykat, funkcja zdaniowa) is a propositional function, i.e. after replacing all the variables with values from its domain we get proposition.

Examples

- $P_1(x) \Leftrightarrow x > 4$
- $P_2(x, y) \Leftrightarrow x < y$
- $P_3(x) \Leftrightarrow x \in \mathbb{N} \leftrightarrow x + 1 \in \mathbb{N}$
- $P_4(x, y, z) \Leftrightarrow x^2 + y^2 = z^2$
- $P_5(x) \Leftrightarrow$ City x is the capitol of Poland
- $P_6(x, y) \Leftrightarrow$ City x is the capitol of country y.
- $P_7(A, B) \Leftrightarrow (A \cup B = \emptyset) \rightarrow (A = \emptyset)$.

What is the domain?

Take universe $X = \{x_1, x_2, ..., x_n\}$

Universal quantification (pl. kwantyfikator ogólny, uniwersalny) $\forall_x P(x)$ means $P(x_1) \wedge P(x_2) \wedge ... \wedge P(x_n)$

Existential quantification (pl. kwantyfikator szczególny, egzystencjonalny) $\exists_x P(x)$ means $P(x_1) \vee P(x_2) \vee ... \vee P(x_n)$

Examples

For universe X.

$$\forall_x P(x) \Leftrightarrow \{x : x \in X \wedge P(x)\} = X$$

$$\exists_x P(x) \Leftrightarrow \{x : x \in X \wedge P(x)\} \neq \emptyset$$

Quantifiers with restricted domain

$$\forall_{Q(x)} P(x) \Leftrightarrow \forall_x (Q(x) \rightarrow P(x))$$

$$\exists_{Q(x)} P(x) \Leftrightarrow \exists_x (Q(x) \wedge P(x))$$

Examples
$X = \mathbb{N}$

- $\forall_{x>1}(x - 1 > 0)$ - proposition,
- $\exists_{x>1}(x - 1 < 0.00000000000001)$ - proposition,
- $\forall_x(x > y)$ - predicate of variable y,
- $\exists_x \exists_y (x^2 + y^2 = z^2)$ - predicate of variable z
- $\forall_a(\exists_a(a \geq 3) \wedge a \geq 2 \wedge a \geq 1)$ - proposition
- $\forall_a(\exists_a(a \geq 3 \wedge a \geq 2) \wedge a \geq 1)$ - proposition

Predicate $P(x_1, ..., x_n)$ is alway true ($\models P(x_1, ..., x_n)$) means

$$\models \forall_{x_1}(\forall_{x_2}...\forall_{x_n}(P(x_1, x_2, ..., x_n)...))$$

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Chosen predicate algebra lows

Assume $X \leq \emptyset$, x, y are variables, $P()$ and Q() are propostional functions, Q if stands alone without variable is a proposition.

1. $\models \forall_x (P(x)) \rightarrow P(x)$

2. $\models P(x) \rightarrow \exists_x (P(x))$

   De Morgan's laws (pl. prawa De Morgana)

3. $\models \sim \exists_x P(x) \leftrightarrow \forall_x (\sim P(x))$

4. $\models \sim \forall_x P(x) \leftrightarrow \exists_x \sim P(x)$

   Factoring quantifiers out (pl. prawa wyłącznania kwantyfikatorów)

5. $\models \forall_x (P(x) \vee Q) \leftrightarrow (\forall_x P(x) \vee Q)$

6. $\models \exists_x (P(x) \vee Q) \leftrightarrow (\exists_x P(x) \vee Q)$

7. $\models \forall_x (P(x) \wedge Q) \leftrightarrow (\forall_x P(x) \wedge Q)$

8. $\models \exists_x (P(x) \wedge Q) \leftrightarrow (\exists_x P(x) \wedge Q)$

9. $\models \forall_x (P(x) \rightarrow Q) \leftrightarrow (\exists_x P(x) \rightarrow Q)$

10. $\models \exists_x (P(x) \rightarrow Q) \leftrightarrow (\forall_x P(x) \rightarrow Q)$

11. $\models \forall_x (Q \rightarrow P(x)) \leftrightarrow (Q \rightarrow \forall_x P(x))$

12. $\models \exists_x (Q \rightarrow P(x)) \leftrightarrow (Q \rightarrow \exists_x P(x))$

13. $\models \forall_x (P(x) \wedge Q(x)) \leftrightarrow (\forall_x P(x) \wedge \forall_x Q(x))$

14. $\models (\forall_x P(x) \vee \forall_x Q(x)) \rightarrow (\forall_x (P(x) \vee Q(x)))$

15. $\models \exists_x (P(x) \wedge Q(x)) \rightarrow (\exists_x P(x) \wedge \exists_x Q(x))$

16. $\models (\exists_x (P(x) \vee Q(x))) \leftrightarrow (\exists_x P(x) \vee \exists_x Q(x))$

17. $\models \forall_x (P(x) \rightarrow Q(x)) \rightarrow (\forall_x P(x) \rightarrow \forall_x Q(x))$

18. $\models (\forall_x (P(x) \rightarrow Q(x)) \rightarrow (\exists_x P(x) \rightarrow \exists_x Q(x))$

Variable replacement

19. $\models \forall_x P(x) \rightarrow \forall_y P(y)$

20. $\models \exists_x P(x) \rightarrow \exists_y P(y)$

Switching quantifiers

21. $\models \forall_x(\forall_y P(x,y)) \leftrightarrow \forall_y(\forall_x P(x,y))$

22. $\models \exists_x(\exists_y P(x,y)) \leftrightarrow \exists_y(\exists_x P(x,y))$

23. $\models \exists_x(\forall_y P(x,y)) \to \forall_y(\exists_x P(x,y))$

Order of operations, and brackets.