3) Predicates and quantifiers

[1/3] Consider universe $X = \{1, 2, 3, 4, 5\}$.
Define $P(x, y) \Leftrightarrow -1 \leq x - y \leq 1$.
Which propositions are true:

(a) $\exists_x(\exists_y P(x, y))$

(b) $\exists_x(\forall_y P(x, y))$

(c) $\forall_x(\exists_y P(x, y))$

For which elements of the universe the following predicates are true?

(e) $\exists_x(P(x, y))$

(f) $\forall_x(P(x, y))$

[2/3] Express the statements using predicates quantifiers and logic operators:

(a) "There is no maximal natural number". Assume $W(x, y)$ means $x > y$; $N(x)$ means $x \in \boldsymbol{N}$.
(b) "Barber shaves all those man and only those who do not shave themselves". Assume: $G(x, y)$ means, x shaves y, f is a barber (a person from the universe, a constant)
(c) "n is a multiple of 2 and 7"
(d) "n is not a square of integer"
(e) "Each natural number has a prime factor", assume $P(x)$ means x is prime number.
(f) "Each pair of distinct rational numbers there is a third number that is between them.

[3/3] Shift all the quantifiers to front without hanging the meaning of the statement. (Hint: rename variables if needed)

(1) $\forall_x(P(x)) \vee \forall_x(Q(x))$
(2) $\forall_x(P(x)) \rightarrow \forall_x(Q(x))$
(3) $\forall_x(P(x)) \rightarrow \forall_x(Q(x) \vee \forall_x(R(x)))$
(4) $\forall_{P(x)}[\ \exists_{Q(y)}(R(y)) \rightarrow \forall_{S(y)}(P(x, y))]$

[4/3] Which of these statements are tautologies of quantifiers calculus:

1. $\forall_x (P(x) \leftrightarrow \forall_x (Q(x))) \rightarrow \forall_x (P(x) \leftrightarrow Q(x))$
2. $\forall_x (P(x) \rightarrow Q) \leftrightarrow (\forall_x (P(x)) \rightarrow Q)$
3. $\forall_x (P(x) \rightarrow \forall_x (Q(x))) \rightarrow \forall_x (P(x) \rightarrow Q(x))$
4. $\forall_x (P(x) \vee Q(x)) \leftrightarrow (\forall_x (P(x)) \vee \forall_x(Q(x)))$
5. $\exists_x (P(x) \rightarrow Q(x)) \rightarrow (\exists_x (P(x)) \rightarrow \exists_x (Q(x)))$
6. $\forall_x(P(x) \rightarrow Q(x)) \leftrightarrow \forall_x (P(x) \rightarrow \forall_x(Q(x))$
7. $\forall_x (\forall_y P(x, y)) \rightarrow \forall_x P(x, x)$
8. $\exists_x (\exists_y P(x, y)) \rightarrow \exists_x P(x, x)$
9. $\forall_x (P(x) \rightarrow \forall_x (P(x)))$
10. $\forall_x (\exists_x (P(x)) \rightarrow P(x))$
11. $\exists_x(\forall_y(P(x,y)) \leftrightarrow \forall_y(\exists_x(P(x,y))$

[5/3] Take arbitrary predicate $P(n)$. Express the following statements using quantifiers:

- There is a finite number of numbers satisfying $P(n)$,
- $P(n)$ is satisfied for infinite number of numbers,
- $P(n)$ is satisfied for all but a finite number of numbers.

[6/3] Prove De Morgan's laws for restricted quantifiers:

$$\sim\forall_{P(x)}(Q(x)) \leftrightarrow \exists_{P(x)}(\sim Q(x)) \qquad \sim\exists_{P(x)}(Q(x)) \leftrightarrow \forall_{P(x)}(\sim Q(x))$$

Basing on general De Morgan's laws:

$$\sim\forall_x(Q(x)) \leftrightarrow \exists_x(\sim Q(x)) \qquad \sim\exists_x(Q(x)) \leftrightarrow \forall_x(\sim Q(x))$$

[7/3] Show that for arbitrary family of sets $\{A_n\}_{n\in N}$ and $\{B_n\}_{n\in N}$ The following inclusion holds $\bigcup_{n\in N}(A_n \setminus B_n) \subset \bigcup_{n\in N} A_n \setminus \bigcap_{n\in N} B_n$. Give a counter example for reverse inclusion. (Hint: express the problem in terms of quantifiers). Explicitly name each axiom used.