# PROPER CLASSES OF MAXIMAL $\theta$ -INDEPENDENT FAMILIES FROM LARGE CARDINALS

### CALLIOPE RYAN-SMITH

ABSTRACT. While maximal independent families can be constructed in ZFC via Zorn's lemma, the presence of a maximal  $\sigma$ -independent family already gives an inner model with a measurable cardinal, and Kunen has shown that from a measurable cardinal one can construct a forcing extension in which there is a maximal  $\sigma$ -independent family. We extend this technique to construct proper classes of maximal  $\theta$ -independent families for various uncountable  $\theta$ . In the first instance, a single  $\theta^+$ -strongly compact cardinal has a set-generic extension with a proper class of maximal  $\theta$ -independent families. In the second, we take a class-generic extension of a model with a proper class of measurable cardinals to obtain a proper class of  $\theta$  for which there is a maximal  $\theta$ -independent family.

## 1. Introduction

In 1983 Kunen published [11], a paper exhibiting the equiconsistency of a single measurable cardinal and a single maximal  $\sigma$ -independent family. For an infinite cardinal  $\theta$  and an infinite set X,  $\mathcal{A} \subseteq \mathscr{P}(X)$  is  $\theta$ -independent if  $|\mathcal{A}| \geq \theta$  and for all partial functions  $p \colon \mathcal{A} \to 2$  with  $|p| < \theta$ ,

$$\bigcap\{A\mid p(A)=1\}\cap\bigcap\{X\setminus A\mid p(A)=0\}\neq\emptyset,$$

where we say  $\sigma$ -independent to mean  $\aleph_1$ -independent. By Zorn's Lemma one obtains maximal  $\omega$ -independent families from ZFC alone, but the existence of even a single  $\sigma$ -independent family entails an inner model with a measurable cardinal, a fascinating increase in consistency strength.

The proof that converts a measurable cardinal into a maximal  $\sigma$ -independent family can be extended to larger cardinal properties, something that was known at the time: In [11] Kunen comments that a single strongly compact cardinal  $\kappa$  would beget, in a forcing extension, maximal  $\sigma$ -independent families  $\mathcal{A} \subseteq \mathscr{P}(\lambda)$  for all  $\lambda$  such that  $\mathrm{cf}(\lambda) \geq \kappa$ . We shall prove this result whilst reducing the consistency strength requirement to  $\kappa$  merely being  $\aleph_1$ -strongly compact, and generalise the setting to  $\theta$ -independence.

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**Theorem A** (Section 4.1). Let  $\kappa$  be a strong limit and  $\theta^+$ -strongly compact for some regular uncountable  $\theta < \kappa$ , and let G be V-generic for  $Add(\theta, \kappa)$ . In V[G], for all  $\lambda \geq \kappa$  with  $cf(\lambda) \geq \kappa$ , there is a maximal  $\theta$ -independent family  $A \subseteq \mathcal{P}(\lambda)$ .

We also extend the technique to the case that there is a proper class of measurable cardinals, iterating the process to induce a model in which, for all ground-model measurable cardinals  $\kappa$ , there is a maximal  $\kappa$ -independent family  $\mathcal{A} \subseteq \mathscr{P}(\kappa)$  in the forcing extension. An analysis of the iteration also shows that the Mitchell rank of cardinals is very nearly preserved.

**Theorem B** (Section 4.2). Let V be a model of  $\mathsf{ZFC} + \mathsf{GCH}$ . There is a classlength forcing iteration  $\mathbb P$  preserving  $\mathsf{ZFC} + \mathsf{GCH}$  such that, if  $G \subseteq \mathbb P$  is V-generic, then whenever  $\kappa$  is a measurable cardinal in V there is a maximal  $\kappa$ -independent family  $\mathcal A \subseteq \mathscr P(\kappa)$  in V[G]. Furthermore, whenever  $\kappa$  is a measurable cardinal in V,  $o(\kappa)^V = 1 + o(\kappa)^{V[G]}$ , and whenever  $\kappa$  is non-measurable in V it remains non-measurable in V[G].

1.1. Structure of the paper. In Section 2 we go over preliminaries, partially to set our notational conventions. An existing familiarity with forcing and ultrapower embeddings will be helpful for understanding the content of the proofs. In Section 3 we present two powerful tools, our proverbial hammers, for constructing models with maximal  $\theta$ -independent families. In Section 4 we apply our hammers to various nails: Kunen's theorem (Theorem 4.1) and our extensions (Theorem A and Theorem B). We end in Section 5 with some open questions.

## 2. Preliminaries

We work in ZFC. For cardinals  $\kappa \leq \lambda$ , let  $\mathscr{P}_{\kappa}(\lambda) = \{X \subseteq \lambda \mid |X| < \kappa\}$ . If V is a model of ZFC then by  $(\kappa^{\lambda})^{V}$ ,  $\mathscr{P}(X)^{V}$ , etc. we mean respectively the set of functions  $\lambda \to \kappa$  in V, the set of subsets of X in V, etc. id is the identity function on an appropriate domain.

2.1. **Independent families.** For regular  $\theta$  and  $|X| \geq \theta$ , a  $\theta$ -independent family on X is  $A \subseteq \mathcal{P}(X)$  such that  $|A| \geq \theta$  and, for all partial functions  $p \colon A \to 2$  with  $|p| < \theta$ ,

$$\mathcal{A}^p := \bigcap \{A \mid p(A) = 1\} \cap \bigcap \{X \setminus A \mid p(A) = 0\} \neq \emptyset.$$

 $\mathcal{A}$  is a maximal  $\theta$ -independent family if, for all  $\theta$ -independent  $\mathcal{A}' \supseteq \mathcal{A}$  on  $X, \mathcal{A}' = \mathcal{A}$ .

2.2. **Ideals.**  $\mathcal{I} \subseteq \mathscr{P}(X)$  is an *ideal* (on X) if  $\mathcal{I} \cap \{\emptyset, X\} = \{\emptyset\}$  and  $\mathcal{I}$  is closed under subsets of elements and finite unions of elements. We say that  $\mathcal{I}$  is:  $\lambda$ -complete if  $\mathcal{I}$  is closed under unions of fewer than  $\lambda$  many elements;  $\lambda$ -saturated if, for all  $\{A_{\alpha} \mid \alpha < \lambda\} \subseteq \mathscr{P}(X) \setminus \mathcal{I}$ , there are  $\alpha < \beta < \lambda$  such that  $A_{\alpha} \cap A_{\beta} \notin \mathcal{I}$ ; non-trivial if  $[X]^1 \subseteq \mathcal{I}$ ; and prime if for all  $A \subseteq X$ ,  $A \in \mathcal{I}$  or  $X \setminus A \in \mathcal{I}$ .  $\mathcal{F} \subseteq \mathscr{P}(X)$  is a filter (on X) if  $\mathcal{F}^* := \{X \setminus A \mid A \in \mathcal{F}\}$  is an ideal on X.

 $\mathcal{F}\subseteq\mathscr{P}(X)$  is a filter (on X) if  $\mathcal{F}^*:=\{X\setminus A\mid A\in\mathcal{F}\}$  is an ideal on X. We say that  $\mathcal{F}$  is  $\lambda$ -complete if  $\mathcal{F}^*$  is  $\lambda$ -complete (as an ideal), and an ultrafilter if  $\mathcal{F}^*$  is prime. An ultrafilter  $\mathcal{U}$  on a set  $S\subseteq\mathscr{P}(X)$  is fine if for all  $x\in X$ ,  $\{A\in S\mid x\in A\}\in\mathcal{U}$ . We shall say that an ultrafilter  $\mathcal{U}$  on a cardinal  $\kappa$  is a measure if it is  $\kappa$ -complete, and that a measure  $\mathcal{U}$  on  $\kappa$  is normal if for all  $A\in\mathcal{U}$  and  $f\in\prod A$  there is  $B\in\mathcal{U}$  such that  $f\upharpoonright B$  is constant.

Given two models  $V \subseteq W$  of ZFC and an ideal  $\mathcal{I} \in V$  on X, we denote by  $\langle \mathcal{I} \rangle^W$  the ideal generated by  $\mathcal{I}$  in the extension:

$$\langle \mathcal{I} \rangle^W = \big\{ A \in \mathscr{P}(X)^W \bigm| (\exists Y \in \mathcal{I}) \, A \subseteq Y \big\}.$$

2.3. Large cardinals. We briefly revise measurable cardinals and the construction of ultrapower embeddings in order to fix notation, though our presentation is standard. Given a  $\sigma$ -complete ultrafilter  $\mathcal{U}$  on an infinite set X and functions  $f,g\colon X\to V$  we say that  $f=_{\mathcal{U}}g$  if  $\{x\in X\mid f(x)=g(x)\}\in \mathcal{U}$ , and denote by  $[f]_{\mathcal{U}}$  the  $=_{\mathcal{U}}$ -equivalence class of f. We then endow these classes with the relation  $\in_{\mathcal{U}}$  given by  $[f]_{\mathcal{U}}\in_{\mathcal{U}}[g]_{\mathcal{U}}$  if  $\{x\in X\mid f(x)\in g(x)\}\in \mathcal{U}$ . Finally, we identify this ultrapower construction

$$Ult(V, \mathcal{U}) = (\{[f]_{\mathcal{U}} \mid f \colon X \to V\}, \in_{\mathcal{U}})$$

and the Mostowski collapse M of this structure, going as far as to say  $a = [f]_{\mathcal{U}}$  to mean that a is the element of M associated with  $[f]_{\mathcal{U}}$  under the collapse. We then refer to the elementary embedding  $j_{\mathcal{U}} \colon V \to M$  given by  $j_{\mathcal{U}}(a) = [c_a]_{\mathcal{U}}$ , where  $c_a$  is the constant function  $X \to \{a\}$ , as the associated ultrapower embedding of  $\mathcal{U}$ . The critical point of an elementary embedding j, denoted  $\mathrm{crit}(j)$  is  $\min\{\alpha \mid \alpha < j(\alpha)\}$ . If  $\mathcal{U} \in V$  then  $\mathrm{crit}(j_{\mathcal{U}})$  is measurable in V (and hence a regular strong limit). We say that a transitive inner model M is  $\lambda$ -closed to mean that  $M^{<\lambda} \subseteq M$ . We may say that M is  $\lambda$ -closed in V to emphasise that we specifically mean  $M^{<\lambda} \cap V \subseteq M \subseteq V$ .

- 2.3.1. The Mitchell order. Given normal measures  $\mathcal{U}$  and  $\mathcal{V}$  on  $\kappa$ , say that  $\mathcal{V} \triangleleft \mathcal{U}$  if  $\mathcal{V} \in \mathrm{Ult}(\mathcal{U}, \mathcal{V})$ . The relation  $\triangleleft$  here is called the Mitchell order. This order was introduced and proved to be well-founded by Mitchell in [13], so we may therefore endow such measures with their Mitchell rank  $o(\mathcal{U})$ , the order type of  $\{\mathcal{V} \mid \mathcal{V} \triangleleft \mathcal{U}\}$ . Similarly we define the Mitchell rank of a cardinal  $\kappa$  to be the height of the tree induced by  $\triangleleft$ , denoted  $o(\kappa)$ . In particular, our convention is that  $o(\kappa) > 0$  if and only if  $\kappa$  is measurable. Also,  $o(\mathcal{U}) = \alpha$  if and only if  $\{\lambda < \kappa \mid o(\lambda) = \alpha\} \in \mathcal{U}$ .
- 2.3.2.  $\theta$ -strongly compact cardinals. We also require a large cardinal property that was introduced in [1].

**Definition 2.1.** For  $\theta \le \kappa$  we say that  $\kappa$  is  $\theta$ -strongly compact if every  $\kappa$ -complete filter on an arbitrary set X can be extended to a  $\theta$ -complete ultrafilter on X.

Note that sometimes in the literature a " $\theta$ -strongly compact" cardinal refers to a cardinal  $\kappa \leq \theta$  such that there is a  $\kappa$ -complete fine ultrafilter on  $\mathscr{P}_{\kappa}(\theta)$ . We shall not make use of this other definition.

**Theorem 2.2** ([1, Theorem 4.7]). The following are equivalent:

- (1)  $\kappa$  is  $\theta$ -strongly compact.
- (2) For all  $\alpha \geq \kappa$  there is an elementary embedding  $j \colon V \to M$ , where M is a transitive inner model of ZFC, such that  $\mathrm{crit}(j) \geq \theta$  and, for some  $D \in M$ ,  $j``\alpha \subseteq D$  and  $M \vDash |D| < j(\kappa)$ .
- (3) For all  $\alpha \geq \kappa$  there is a fine  $\theta$ -complete ultrafilter on  $\mathscr{P}_{\kappa}(\alpha)$ .

**Fact.** If  $\mathcal{U}$  is a fine  $\theta$ -complete ultrafilter on  $\mathscr{P}_{\kappa}(\alpha)$  then  $j_{\mathcal{U}}$  satisfies Item (2) with  $D = [\mathrm{id}]_{\mathcal{U}}$ .

Corollary 2.3, remarked upon in [1], is immediate but important.

<sup>&</sup>lt;sup>1</sup>This is easiest to prove by noting  $j_{\mathcal{U}}(\langle o(\alpha) \mid \alpha < \kappa \rangle)(\kappa) = o(\mathcal{U})$  (from [14]).

Corollary 2.3. Let  $\kappa$  be  $\theta$ -strongly compact.

- (i) There is a measurable cardinal  $\mu$  such that  $\theta \leq \mu \leq \kappa$ . If  $\mu$  is the least measurable cardinal greater than or equal to  $\theta$  then  $\kappa$  is  $\mu$ -strongly compact.
- (ii) For all  $\lambda \geq \kappa$ ,  $\lambda$  is  $\theta$ -strongly compact. In particular, there is a  $\theta$ -strongly compact cardinal  $\lambda$  such that  $2^{<\lambda} = \lambda$ .

*Proof. Item* (i). Firstly, by Item (2) in Theorem 2.2 with  $\alpha = \kappa^+$ , we obtain  $j \colon V \to M$  with  $\theta \le \operatorname{crit}(j)$ . Since there is an injection  $D \to j(\kappa)$  in M with  $j "\kappa^+ \subseteq D$ , we must have  $\kappa^+ \le j(\kappa)$ , so  $\operatorname{crit}(j) \le \kappa$ . Hence  $\operatorname{crit}(j)$  is measurable with  $\theta \le \operatorname{crit}(j) \le \kappa$ .

Let  $\mu$  be the least measurable cardinal with  $\mu \geq \theta$ . Then for each  $\alpha \geq \kappa$  there is  $j \colon V \to M$  such that  $\operatorname{crit}(j) \geq \theta$  and, for some  $D \in M$ ,  $j``\alpha \subseteq D$  and  $M \vDash |D| < j(\kappa)$ . However, in this case we must also have  $\operatorname{crit}(j) \geq \mu$ . Hence  $\kappa$  is  $\mu$ -strongly compact.

Item (ii). For all  $\lambda \geq \kappa$ , any  $\lambda$ -complete filter  $\mathcal{F}$  on any set X is also  $\kappa$ -complete. Hence, by the definition of  $\kappa$  being  $\theta$ -strongly compact, the filter  $\mathcal{F}$  extends to a  $\theta$ -complete ultrafilter  $\mathcal{U}$  on X. Setting  $\lambda = \beth_{\omega}(\kappa) = \sup\{\kappa, 2^{\kappa}, 2^{2^{\kappa}}, \ldots\} > \kappa$  we have  $2^{<\lambda} = \lambda$  and  $\lambda$  is  $\theta$ -strongly compact.

2.4. **Forcing.** Our presentation of forcing is standard. For a more in-depth introduction to forcing there are a wealth of excellent texts, such as [8, Ch. 14]. By a notion of forcing we mean a partial order  $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$  with a maximum  $\mathbb{1}_{\mathbb{P}}$ , though subscripts will be omitted when clear from context. We force downwards, so if  $q \leq p$  then we say that q is an extension or stronger condition than p.  $\mathbb{P}$  has the  $\kappa$ -chain condition, or  $\kappa$ -c.c., if every antichain in  $\mathbb{P}$  has cardinality less than  $\kappa$ . We say that  $\mathbb{P}$  is  $\kappa$ -closed if for every  $\gamma < \kappa$  and descending chain  $\{p_{\alpha} \mid \alpha < \gamma\} \subseteq \mathbb{P}$  there is  $q \in \mathbb{P}$  such that  $q \leq p_{\alpha}$  for all  $\alpha < \gamma$ . By a V-generic filter for  $\mathbb{P}$  we mean a filter  $G \subseteq \mathbb{P}$  such that for all dense  $D \subseteq \mathbb{P}$  with  $D \in V$ ,  $G \cap D \neq \emptyset$ .

We say that  $p, q \in \mathbb{P}$  are *compatible*, denoted  $p \parallel_{\mathbb{P}} q$ , if there is  $r \in \mathbb{P}$  with  $r \leq p, q$ . Otherwise, we say that p and q are *incompatible*, denoted  $p \perp_{\mathbb{P}} q$ . Again, subscripts are omitted when clear from context.

Given a set of  $\mathbb{P}$ -names X we define the bullet name of X, denoted  $X^{\bullet}$ , to be  $\{\langle \mathbb{1}_{\mathbb{P}}, \dot{x} \rangle \mid \dot{x} \in X\}$ . Then  $X^{\bullet}$  is always interpreted as the set  $\{\dot{x}^G \mid \dot{x} \in X\}$  in the extension V[G]. This notation extends to tuples, functions, etc. with ground model domains. For  $X \in V$  we define the check name of X, denoted  $\check{X}$ , to be  $\{\check{x} \mid x \in X\}^{\bullet}$ . That is,  $\check{X}^G = X \in V[G]$  for all V-generic  $G \subseteq \mathbb{P}$ . We may alternatively use the bullet notation to define a canonical name of a definable object. For example,  $\mathscr{P}(\check{X})^{\bullet}$  is the canonical name for the power set of X in the forcing extension.

Given non-empty sets A and B we denote by  $\mathrm{Add}(B,A)$  the notion of forcing given by partial functions  $p\colon A\times B\to 2$  such that  $|p|<|B|.^2$  Note that if  $\kappa$  is a cardinal and A is non-empty then  $\mathrm{Add}(\kappa,A)$  is  $\mathrm{cf}(\kappa)$ -closed and has the  $(\kappa^{<\kappa})^+$ -c.c.

**Definition 2.4.** Given a forcing iteration  $\langle \mathbb{P}_{\alpha}, \dot{\mathbb{Q}}_{\alpha} \mid \alpha < \delta \rangle$  of limit length  $\delta$ , the *inverse limit* of the system is the notion of forcing  $\mathbb{P}$  with conditions given by functions p with domain  $\delta$  such that, for all  $\gamma < \delta$ ,  $p \upharpoonright \gamma \in \mathbb{P}_{\gamma}$ . The ordering is given by  $q \leq p$  if for all  $\gamma < \delta$ ,  $q \upharpoonright \gamma \leq_{\gamma} p \upharpoonright \gamma$ . The *direct limit* of the system is the

<sup>&</sup>lt;sup>2</sup>When  $|A| \ge |B|$  we denote by  $\operatorname{Fn}(A, 2, B)$  the notion of forcing with conditions that are partial functions  $p: A \to 2$  such that |p| < |B|. In this case  $\operatorname{Fn}(A, 2, B)$  is isomorphic to  $\operatorname{Add}(B, A)$ . While  $\operatorname{Fn}(A, 2, B)$  is not used in this paper, it is used in [11], to which we sometimes refer.

notion of forcing  $\mathbb{P}$  with conditions given by functions p with domain  $\gamma$  for some  $\gamma < \delta$  such that  $p \in \mathbb{P}_{\gamma}$ . The ordering is given by  $q \leq p$  if  $\gamma = \text{dom}(p) \leq \text{dom}(q)$  and  $q \upharpoonright \gamma \leq_{\gamma} p$ . By an *Easton support iteration* we mean that for all limit ordinals  $\alpha$ ,  $\mathbb{P}_{\alpha}$  is taken as an inverse limit if  $\alpha$  is singular, and a direct limit otherwise.

**Definition 2.5.** Given two notions of forcing  $\mathbb{P}$  and  $\mathbb{Q}$ , a function  $\psi \colon \mathbb{P} \to \mathbb{Q}$  is a dense embedding if:

- (i) For all  $p,p'\in\mathbb{P},\,p\leq_{\mathbb{P}}p'$  if and only if  $\psi(p)\leq_{\mathbb{Q}}\psi(p');$  and
- (ii) for all  $q \in \mathbb{Q}$  there is  $p \in \mathbb{P}$  such that  $\psi(p) \leq_{\mathbb{Q}} q$ .

Note that in this case we also have  $p \perp_{\mathbb{P}} p'$  if and only if  $\psi(p) \perp_{\mathbb{Q}} \psi(p')$ .

2.5. Closure points and elementary embeddings. In Section 4.2 we shall wish to consider which elementary embeddings are found in a forcing extension. In our instance we are making use of an Easton-support iteration that, as is often the case for Easton-support iterations, admits no elementary embeddings that do not extend a ground-model embedding. To show this, we will make use of *closure points*, as explored in [7].

**Definition 2.6.** A notion of forcing has a *closure point* at  $\delta$  if it can be factored as  $\mathbb{P}_0 * \dot{\mathbb{P}}_1$ , where  $\mathbb{P}_0$  is atomless,  $|\mathbb{P}_0| \leq \delta$ , and  $\mathbb{1} \Vdash_{\mathbb{P}_0}$  " $\dot{\mathbb{P}}_1$  is  $\leq \delta$  strategically closed".<sup>3</sup>

The following result is a combination of Lemma 13 and Theorem 10 in [7]. While Lemma 13 and Theorem 10 in [7] are more powerful than what we present here, the Proposition 2.7 is all that we will need.

**Proposition 2.7** (Hamkins). If  $\mathbb{P}$  has a closure point at  $\delta$ ,  $G \subseteq \mathbb{P}$  is V-generic, and  $U \in V[G]$  is a normal measure on  $\kappa > \delta$ , then  $U \cap V \in V$  is a normal measure on  $\kappa$ .

# 3. Hammers

We have two main tools at our disposal that work together to produce models in which there is a  $\theta$ -independent family on a set X. Let us begin with Lemma 3.1, a method for showing that maximal  $\theta$ -independent families exist contingent on the presence of a particular ideal  $\mathcal{I}$ . The proof and result are essentially due to Kunen in the form of [11, Lemma 2.1], but we have softened the requirements.

**Lemma 3.1.** Let  $\theta$  be a regular uncountable cardinal, X a set with  $|X| \geq \theta$ , and  $\mathcal{I}$  a  $\theta$ -complete ideal over X such that  $\mathrm{Add}(\theta, 2^X)$  densely embeds into  $\mathscr{P}(X)/\mathcal{I}$ . Then there is a maximal  $\theta$ -independent family  $\mathcal{A} \subseteq \mathscr{P}(X)$ .

*Proof.* Let  $\mathbb{P} = \operatorname{Add}(\theta, 2^X)$  and  $\psi \colon \mathbb{P} \to \mathscr{P}(X)/\mathcal{I}$  be a dense embedding. For all  $f \in 2^X$  and  $\delta < \theta$  choose  $A_{f,\delta} \subseteq X$  such that  $[A_{f,\delta}] = \psi(\{\langle f, \delta, 1 \rangle\})$  and define  $\varphi \colon \mathbb{P} \to \mathscr{P}(X)$  by

$$(*) \hspace{1cm} \varphi(p) = \bigcap \{A_{f,\delta} \mid p(f,\delta) = 1\} \cap \bigcap \{X \setminus A_{f,\delta} \mid p(f,\delta) = 0\}.$$

Claim 3.1.1. For all  $p \in \mathbb{P}$ ,  $\psi(p) = [\varphi(p)]$ .

<sup>&</sup>lt;sup>3</sup>We shall not define strategic closure here. A  $\delta^+$ -closed notion of forcing is ≤ $\delta$  strategically closed, and we will only ever use this case. A definition can be found in [3, Ch. 12, Definition 5.15].

Proof of Claim. We shall first show that  $\psi(\{\langle f, \delta, 0 \rangle\}) = [X \setminus A_{f,\delta}]$ . Let  $B_{f,\delta}$  be chosen so that  $\psi(\{\langle f, \delta, 0 \rangle\}) = [B_{f,\delta}]$ . Since  $\{\langle f, \delta, 0 \rangle\} \perp \{\langle f, \delta, 1 \rangle\}$ ,  $A_{f,\delta} \cap B_{f,\delta} \in \mathcal{I}$ . For all  $A \notin \mathcal{I}$  there is  $p \in \mathbb{P}$  such that  $\psi(p) \leq [A]$ , and either  $p \parallel \{\langle f, \delta, 0 \rangle\}$  or  $p \parallel \{\langle f, \delta, 1 \rangle\}$ . Hence, setting  $[C] = \psi(p)$ , we have that  $C \cap A_{f,\delta} \notin \mathcal{I}$  or  $C \cap B_{f,\delta} \notin \mathcal{I}$ . In particular, letting  $D = X \setminus (A_{f,\delta} \cup B_{f,\delta})$ , we have  $D \cap A_{f,\delta} = D \cap B_{f,\delta} = \emptyset \in \mathcal{I}$ , so  $D \in \mathcal{I}$ . That is,  $[A_{f,\delta} \cup B_{f,\delta}] = [X]$  and so  $[B_{f,\delta}] = [X \setminus A_{f,\delta}]$  as required.

Therefore, for all  $p \in \mathbb{P}$ ,  $\psi(p) \leq [A_{f,\delta}]$  whenever  $p(f,\delta) = 1$  and  $\psi(p) \leq [X \setminus A_{f,\delta}]$  whenever  $p(f,\delta) = 0$ . Given that  $|p| < \theta$  and  $\mathcal{I}$  is  $\theta$ -complete, this means that  $\psi(p) \leq [\varphi(p)]$ . Setting  $\psi(p) = [A]$ , if  $[A] < [\varphi(p)]$  then  $\varphi(p) \setminus A \notin \mathcal{I}$  and so there is  $q \in \mathbb{P}$  such that  $\psi(q) \leq [\varphi(p) \setminus A]$ . In particular,  $\psi(q) \leq [A_{f,\delta}] = \psi(\{\langle f, \delta, 1 \rangle\})$  whenever  $p(f,\delta) = 1$  and  $\psi(q) \leq [X \setminus A_{f,\delta}] = \psi(\{\langle f, \delta, 0 \rangle\})$  whenever  $p(f,\delta) = 0$ . That is,  $q \leq p$  and thus  $\psi(q) \leq [A]$ . However, this cannot be the case since  $[\varphi(p) \setminus A] \perp [A]$ .

Let  $\mathcal{A} = \{A_{f,\delta} \mid f \in 2^X, \delta < \theta\}$ . Then for all  $p \in \mathbb{P}$ ,  $[\varphi(p)] = \psi(p) \neq [\emptyset]$ , so  $\varphi(p) \notin \mathcal{I}$  and thus  $\mathcal{A}$  is  $\theta$ -independent. Furthermore, for all  $A \notin \mathcal{I}$  there is  $p \in \mathbb{P}$  such that  $\psi(p) = [\varphi(p)] \leq [A]$ , and thus for all  $A \subseteq X$  there is  $p \in \mathbb{P}$  such that  $[\varphi(p)] \leq [A]$  or  $[\varphi(p)] \leq [X \setminus A]$ . However, this is not quite true maximality, as we would require that for all  $A \subseteq X$  there is  $p \in \mathbb{P}$  such that  $\varphi(p) \subseteq A$  or  $\varphi(p) \subseteq X \setminus A$ . To achieve this we alter  $\mathcal{A}$  slightly.

Enumerate  $\mathcal{I}$  as  $\{C_f \mid f \in 2^X\}$  (with repeat entries if necessary) and define  $A'_{f,\delta} = A_{f,\delta} \setminus C_f$ . Since the representatives  $A_{f,\delta} \in \psi(\{\langle f,\delta,1\rangle\})$  were chosen arbitrarily, Claim 3.1.1 still holds for  $\mathcal{A}' = \{A'_{f,\delta} \mid f \in 2^X, \delta < \theta\}$ , where we define  $\varphi'$  analogously to  $\varphi$  in Equation (\*). Hence, if  $A \notin \mathcal{I}$  then there is  $p \in \mathbb{P}$  such that  $[\varphi'(p)] \leq [A]$ , so  $\varphi'(p) \setminus A = C_f \in \mathcal{I}$ . Since  $|p| < \theta$  there is  $\delta < \theta$  such that  $\langle f, \delta \rangle \notin \text{dom}(p)$ , and hence  $\varphi'(p \cup \{\langle f, \delta, 1 \rangle\}) \subseteq \varphi'(p) \setminus C_f \subseteq A$  as required.  $\square$ 

**Remark.** The statement of Lemma 3.1 is, on the surface, a strengthening of Kunen's result, as we have removed two requirements (both  $2^{<\theta} = \theta$  and that  $\mathcal{I}$  is  $\theta^+$ -saturated) and weakened further requirements (we only need  $\mathcal{I}$  to be  $\theta$ -complete, rather than |X|-complete, and only demand that there is a dense embedding of  $\mathrm{Add}(\theta,2^X)$  into  $\mathscr{P}(X)/\mathcal{I}$ , rather than an isomorphism). This weakening is partially illusory. The proof of Lemma 3.1 in [11] makes little use of some of these extraneous assumptions, and some of these requirements that we have altered are consequences: By Theorem 4.2 it will be the case that  $2^{<\theta} = \theta$  and, since  $\mathrm{Add}(\theta,2^X)$  densely embeds into  $\mathscr{P}(X)/\mathcal{I}$ , we recover that  $\mathcal{I}$  is  $\theta^+$ -saturated by the chain condition.

Our second tool is an old technique present in [11] (among many other places) for obtaining ideals  $\mathcal{I}$  on X such that  $\mathscr{P}(X)/\mathcal{I}$  is a complete Boolean algebra isomorphic to a desired notion of forcing. This method is closely tied to the idea of lifting elementary embeddings: If  $j \colon V \to M$  is an elementary embedding and G is V-generic, then one can lift the elementary embedding to  $\hat{\jmath} \colon V[G] \to M[j(G)]$ , where j(G) is an appropriate M-generic filter. However, if j was definable in V and  $j(G) \notin V[G]$ , then we may be unable to lift the embedding definably in V[G]. Theorem 3.2 can be understood intuitively as the idea that if  $j = j_{\mathcal{U}}$  is an ultrapower embedding and  $G \subseteq \mathbb{P}$  is V-generic, then  $\mathcal{I} = \langle \mathcal{U}^* \rangle^{V[G]}$  will be a prime ideal only if j lifts to V[G], and if it does not then  $\mathscr{P}(X)/\mathcal{I}$  is the extra amount of forcing required to lift the embedding:  $\mathbb{P} * \mathscr{P}(X)/\mathcal{I} \cong j(\mathbb{P})$ .

This technique has been the subject of much refinement, culminating in Foreman's *Duality Theorem*, from [2], which bring precipitous ideals and a more refined definition of  $\mathcal{I}$  into the fold. We do not need quite the level of complexity that the Duality Theorem affords, and so we shall present a specialised version that is localised to prime ideals, the scenario that we have described. For the case that  $\dot{\mathbb{R}}$  is forced to be trivial, one can follow the technique of [11] in which the special case of  $\mathbb{P} = \mathrm{Add}(\omega_1, \kappa)$  was applied<sup>4</sup> to see a proof.

**Theorem 3.2.** Let  $\mathcal{U}$  be a  $\sigma$ -complete ultrafilter on a set X with ultrapower embedding  $j = j_{\mathcal{U}} \colon V \to M$ . Let  $\mathbb{P}$  be a notion of forcing such that  $j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$  by an isomorphism  $\pi$  satisfying  $\pi(j(p)) = \langle p, \mathbb{1}, \mathbb{1} \rangle$ . Suppose that, for all V-generic  $G * H \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ , there is M[G \* H]-generic  $F \subseteq \dot{\mathbb{R}}^{G*H}$  with  $F \in V[G * H]$ . Then there is a  $\mathbb{P}$ -name for an ideal  $\dot{\mathcal{I}}$  on X such that  $\mathbb{1} \Vdash_{\mathbb{P}} \mathscr{P}(\check{X})^{\bullet}/\dot{\mathcal{I}} \cong B(\dot{\mathbb{Q}})$ .

In the special case that  $\mathbb{R}$  is forced to be the trivial forcing,  $\mathcal{I}$  is in fact  $\langle \mathcal{U}^* \rangle^{V[G]}$ , the ideal generated by  $\mathcal{U}^*$  in the extension. Hence, if  $\mathbb{P}$  is  $\theta$ -distributive and  $\mathcal{U}$  is  $\theta$ -complete then  $\mathcal{I}$  will be  $\theta$ -complete as well.

## 4. Nails

Let us now apply our tools to produce examples of maximal  $\theta$ -independent families, beginning with Theorem 4.1. Our presentation of this result is a slight extension of Kunen's original method to allow for general regular uncountable  $\theta$ .

**Theorem 4.1** ([11, Theorem 2]). Let  $\kappa$  be a measurable cardinal,  $\theta < \kappa$  be uncountable and regular, and G be V-generic for  $Add(\theta, \kappa)$ . Then there is a maximal  $\theta$ -independent family  $\mathcal{A} \subseteq \mathcal{P}(2^{\theta})$  in V[G].

*Proof.* Let  $\kappa$  have measure  $\mathcal{U}$  and ultrapower embedding  $j=j_{\mathcal{U}}\colon V\to M$ , so  $\mathrm{crit}(j)=\kappa$ . Then in V[G] we have  $\theta^{<\theta}=\theta$ ,  $2^{\theta}=\kappa$ , and

$$\begin{split} j(\mathrm{Add}(\theta,\kappa)) &= \mathrm{Add}(\theta,j(\kappa)) \\ &\cong \mathrm{Add}(\theta,j``\kappa) \times \mathrm{Add}(\theta,j(\kappa) \setminus j``\kappa) \\ &\cong \mathrm{Add}(\theta,\kappa) \times \mathrm{Add}(\theta,j(\kappa) \setminus \kappa) \times \{1\!\!1\}. \end{split}$$

Since each  $p \in \operatorname{Add}(\theta, \kappa)$  has cardinality less than  $\theta$  we have that  $j(p) = j^*p = p$ . Furthermore, since M is  $\kappa^+$ -closed,  $\operatorname{Add}(\theta, j(\kappa))^M = \operatorname{Add}(\theta, j(\kappa))^V$ . By Theorem 3.2, setting  $\mathcal{I} = \langle \mathcal{U}^* \rangle^{V[G]}$ , we have  $\mathscr{P}(\kappa)/\mathcal{I} \cong B(\operatorname{Add}(\theta, j(\kappa) \setminus \kappa))$ . Note here that since  $\operatorname{Add}(\theta, \kappa)$  is  $\theta$ -closed,  $\operatorname{Add}(\theta, X)^V = \operatorname{Add}(\theta, X)^{V[G]}$  for all  $X \in V$  and so the isomorphism class of  $\operatorname{Add}(\theta, X)$  (in either V or V[G]) depends only on the cardinality of X. Furthermore, since  $\operatorname{Add}(\theta, \kappa)$  is  $\theta$ -closed,  $\mathcal{I}$  is  $\theta$ -complete.

Finally, since  $\kappa$  is measurable,  $2^{\kappa} < j(\kappa) < (2^{\kappa})^+$  and so  $Add(\theta, j(\kappa) \setminus \kappa)$  is isomorphic to  $Add(\theta, 2^{\kappa})$ . Therefore, by Lemma 3.1, there is a maximal  $\theta$ -independent family  $\mathcal{A} \subseteq \mathscr{P}(\kappa) = \mathscr{P}(2^{\theta})$  in V[G].

Hence, if ZFC plus the existence of a measurable cardinal is consistent, then so is ZFC plus the existence of a maximal  $\sigma$ -independent family on  $2^{\omega_1}$ . Furthering this, Kunen recovers the consistency of a measurable cardinal from the consistency of a maximal  $\theta$ -independent family.

<sup>&</sup>lt;sup>4</sup>Rather, the special case  $\mathbb{P} = \operatorname{Fn}(\kappa, 2, \omega_1)$ , but these are isomorphic.

**Theorem 4.2** ([11, Theorem 1]). Let  $\theta$  be an uncountable regular cardinal such that there is a maximal  $\theta$ -independent family  $\mathcal{A} \subseteq \mathscr{P}(\lambda)$ . Then  $2^{<\theta} = \theta$  and, for some  $\kappa$  such that  $\sup\{(2^{\alpha})^+ \mid \alpha < \theta\} \leq \kappa \leq \min\{\lambda, 2^{\theta}\}$ , there is a non-trivial  $\theta^+$ -saturated  $\kappa$ -complete ideal over  $\kappa$ .

The full proof may be found in [11], with the roles of  $\kappa$  and  $\lambda$  swapped, but we shall sketch it here.

Sketch proof. We say that a maximal  $\theta$ -independent family  $\mathcal{A} \subseteq \mathscr{P}(\lambda)$  is globally maximal if, setting P to be the set of partial functions  $p: \mathcal{A} \to 2$  with  $|p| < \theta$ ,

$$(\forall p \in P)(\forall X \subseteq \mathcal{A}^p)(\exists q \supseteq p)(\mathcal{A}^q \subseteq X \vee \mathcal{A}^q \cap X = \emptyset).$$

**Fact.** There is  $p \in P$  such that  $A/p = \{A \cap A^p \mid A \in A \setminus \text{dom}(p)\}$  is globally maximal  $\theta$ -independent.

Hence, replacing  $\lambda$  by some  $\lambda' < \lambda$  if necessary, we may assume that  $\mathcal{A}$  is globally maximal  $\theta$ -independent on  $\lambda$ . Let

$$\mathcal{I}_{\mathcal{A}} = \{ X \subseteq \lambda \mid (\forall p \in P) \mathcal{A}^p \not\subseteq X \}.$$

Then  $\mathcal{I}_{\mathcal{A}}$  is  $\theta^+$ -saturated and  $(2^{\alpha})^+$ -complete for all  $\alpha < \theta$ . Setting  $\kappa$  be least such that  $\mathcal{I}_{\mathcal{A}}$  is not  $\kappa^+$ -complete, we can refine  $\mathcal{I}_{\mathcal{A}}$  to a  $\kappa$ -complete  $\theta^+$ -saturated ideal on  $\kappa$ . We immediately have  $\sup\{(2^{\alpha})^+ \mid \alpha < \theta\} \leq \kappa \leq \lambda$ . On the other hand, if  $\mathcal{A}_0 \in [\mathcal{A}]^{\theta}$  then  $P_0 = \{\mathcal{A}^p \mid p \text{ a total function } \mathcal{A}_0 \to 2\} \subseteq \mathcal{I}_{\mathcal{A}}$ , but  $\bigcup P_0 = \lambda \notin \mathcal{I}_{\mathcal{A}}$  and so  $\kappa \leq 2^{\theta}$ .

**Remark.** The maximal  $\theta$ -independent families constructed by Lemma 3.1 are globally maximal  $\theta$ -independent. Furthermore, when we later construct maximal  $\kappa$ -independent families on  $\kappa$ , the bounds directly give us that there is a  $\kappa$ -complete and  $\kappa^+$ -saturated ideal on  $\kappa$ .

Corollary 4.3. If there is a maximal  $\theta$ -independent family for some uncountable regular  $\theta$  then there is an inner model containing a measurable cardinal.

*Proof.* By [10, Theorem 11.13], if  $\kappa$  carries a  $\kappa$ -complete,  $\kappa^+$ -saturated ideal then there is an inner model in which  $\kappa$  is measurable. Since  $\kappa \geq \theta$ ,  $\theta^+$ -saturated implies  $\kappa^+$ -saturated.

In fact, [10, Theorem 11.13] can be extended to any finite collection of saturated ideals on increasing cardinals, as noted in [15].

**Lemma 4.4** ([15]). If  $\kappa_0 < \cdots < \kappa_{n-1}$  are uncountable regular cardinals such that for all i < n there is a normal  $\kappa_i$ -complete,  $\kappa_i^+$ -saturated ideal  $\mathcal{I}_i \subseteq \mathscr{P}(\kappa_i)$ , then in  $L[\mathcal{I}_0, \ldots, \mathcal{I}_{n-1}]$ ,  $\kappa_i$  is measurable for all i.

It follows from Theorem 4.2 and Lemma 4.4 that if there is a maximal  $\theta$ -independent family on  $\lambda$  and a maximal  $\theta$ -independent family on  $\lambda'$  such that  $\min\{\lambda, 2^{\theta}\} < \sup\{(2^{\alpha})^{+} \mid \alpha < \theta'\}$ , then there is an inner model with two measurable cardinals, and indeed this pattern holds for all finite collections of such families. Therefore, a corollary of Theorem A is that the consistency of an  $\aleph_1$ -strongly compact cardinal implies the consistency of any finite number of measurable cardinals (though this is already known).

Kunen briefly sketches how to obtain a maximal  $\kappa$ -independent family on inaccessible  $\kappa$ , starting with a model in which  $\kappa$  is measurable. This requires a slightly

more delicate use of Theorem 3.2 to obtain the result. We have also included additional content regarding lifting normal measures, which will be useful when proving Theorem B.

**Proposition 4.5.** Let  $\kappa$  be measurable with normal measure  $\mathcal{U}$ ,  $2^{\kappa} = \kappa^+$ , and  $A \in \mathcal{U}$  be a set of regular cardinals. Let G be V-generic for the Easton-support iteration  $\mathbb{P} = \bigstar_{\alpha \in A} \operatorname{Add}(\alpha, \alpha^+)$ . Then in V[G] there is a maximal  $\kappa$ -independent family  $A \subseteq \mathscr{P}(\kappa)$ . Furthermore, if  $V \in V$  is a normal measure on  $\kappa$  such that  $A \notin \mathcal{V}$  then there is a normal measure  $\hat{\mathcal{V}} \supseteq \mathcal{V}$  on  $\kappa$  in V[G].

*Proof.* Let  $j = j_{\mathcal{U}} \colon V \to M$ , and  $H \subseteq \operatorname{Add}(\kappa, \kappa^+)$  be V[G]-generic. Note that  $j(\kappa) < (2^{\kappa})^+ = \kappa^{++}$ . Furthermore, M is  $\kappa^+$ -closed, and this is preserved by the forcing  $(\mathbb{P} * \operatorname{Add}(\kappa, \kappa^+) \in M)$ , so M[G \* H] is also  $\kappa^+$ -closed.

Let  $\dot{\mathbb{R}} = j(\mathbb{P})/(\mathbb{P} * \mathrm{Add}(\kappa, \kappa^+)^{\bullet})$ , noting that due to the Easton support we truly have  $j(\mathbb{P}) \cong \mathbb{P} * \mathrm{Add}(\kappa, \kappa^+) * \dot{\mathbb{R}}$  as required in Theorem 3.2. Let

$$\mathbb{R} = \dot{\mathbb{R}}^{G*H} = \underset{\alpha \in j(A) \setminus \kappa^+}{*} \operatorname{Add}(\alpha, (\alpha^+)^M)^M.$$

 $|\mathbb{P}|^V = \kappa$ , so  $|j(\mathbb{P})|^M = j(\kappa)$ , and hence  $|\mathbb{R}|^{V[G]} = \kappa^+$ . Furthermore, each iterand of  $\mathbb{R}$  is  $\alpha$ -closed according to M for some  $\alpha \geq \kappa^+$ . Since M[G\*H] is  $\kappa^+$ -closed, this means that each iterand of  $\mathbb{R}$  is  $\kappa^+$ -closed (in V[G\*H]) and, since it is an iteration of length  $j(\kappa) \geq \kappa^+$ ,  $\mathbb{R}$  itself is  $\kappa^+$ -closed. However,  $\mathbb{P}$  has only  $\kappa$ -many maximal antichains: Each iterand is of cardinality less than  $\kappa$  and so has fewer than  $\kappa$ -many antichains. Hence  $M[G*H] \models$  " $\mathbb{R}$  has only  $j(\kappa)$ -many maximal antichains". Since  $j(\kappa) < \kappa^{++}$  we can build an M[G\*H]-generic filter  $F \subseteq \mathbb{R}$  in V[G\*H]. Hence, by Theorem 3.2, in V[G] there is an ideal  $\mathcal{I}$  on  $\kappa$  such that  $\mathscr{P}(\kappa)/\mathcal{I} \cong B(\mathrm{Add}(\kappa,\kappa^+))^{V[G]}$ . This ideal can be expressed as

$$\mathcal{I} := \Big\{ \dot{A}^G \subseteq \kappa \ \Big| \ \mathbbm{1} \Vdash_{\mathbb{P}*\mathrm{Add}(\kappa,\kappa^+)*\hat{\mathbb{R}}/\dot{F}} \check{\kappa} \not\in j(\dot{A}) \Big\},$$

where  $\dot{F}$  is a  $\mathbb{P}*\mathrm{Add}(\kappa,\kappa^+)$ -name for an M[G\*H]-generic ideal  $F\subseteq\mathbb{R}$ . In this case, if  $\{\dot{A}_\alpha\mid\alpha<\gamma\}\subseteq\mathcal{I}$  for some  $\gamma<\kappa$  then, since  $\mathrm{crit}(j)=\kappa,\ j(\bigcup\dot{A}_\alpha)=\bigcup j(\dot{A}_\alpha)$ , and so  $\bigcup\dot{A}_\alpha^G\in\mathcal{I}$ . Hence,  $\mathcal{I}$  is  $\kappa$ -complete as required, and so by Lemma 3.1 there is a maximal  $\kappa$ -independent family on  $\kappa$  in V[G].

On the other hand, let  $\mathcal{V} \in V$  be a normal measure on  $\kappa$  such that  $A \notin \mathcal{V}$ , with ultrapower embedding  $i \colon V \to N$ . Then  $i(\mathbb{P}) \cong \mathbb{P} * \mathbb{R}$  (without the  $\mathrm{Add}(\kappa, \kappa^+)$  iterand), and so  $F \in V[G]$ . Hence we may lift i to  $i \colon V[G] \to M[G * F]$  in V[G] and obtain normal measure  $\hat{\mathcal{V}} = \{B \subseteq \kappa \mid \kappa \in \hat{i}(B)\}$  on  $\kappa$  extending  $\mathcal{V}$ .

4.1. A  $\theta^+$ -strongly compact cardinal. These techniques are ripe for transfer to other large cardinal properties. In the following we shall find that  $\kappa$  being  $\theta^+$ -strongly compact<sup>5</sup> for uncountable regular  $\theta$  is sufficient to produce the ultrapower embeddings  $j \colon V \to M$  that give rise to a proper class of  $\lambda$  such that there is a maximal  $\theta$ -independent families  $A \subseteq \mathcal{P}(\lambda)$ . The transfer is not entirely clean, as we additionally require that  $2^{<\kappa} = \kappa$ , but as noted in Corollary 2.3 this does not increase the consistency strength of the assumption.

**Theorem A.** Let  $\kappa$  be  $\theta^+$ -strongly compact for some uncountable regular  $\theta < \kappa$ , with  $2^{<\kappa} = \kappa$ , and let G be V-generic for  $Add(\theta, \kappa)$ . In V[G], for all  $\lambda \geq \kappa$  with  $cf(\lambda) \geq \kappa$ , there is a maximal  $\theta$ -independent family  $A \subseteq \mathscr{P}(\lambda)$ .

<sup>&</sup>lt;sup>5</sup>Note that, for fixed  $\theta$ , there is  $\mu > \theta$  such that  $\kappa$  is  $\mu$ -strongly compact if and only if  $\kappa$  is  $\theta^+$ -strongly compact.

*Proof.* Let  $\lambda$  be such that  $\operatorname{cf}(\lambda) \geq \kappa$ . We wish to use Lemma 3.1 to show that there is a maximal  $\theta$ -independent family  $\mathcal{A} \subseteq \mathscr{P}(X)$ , where  $X = \mathscr{P}_{\kappa}(\lambda)^V$  (noting that  $|X| = \lambda$ ). We therefore require a  $\theta$ -complete ideal  $\mathcal{I}$  over X such that  $B(\operatorname{Add}(\theta, 2^X))$  is isomorphic to  $\mathscr{P}(X)/\mathcal{I}$  in V[G], which we shall obtain through Theorem 3.2.

Let  $\mathcal{U} \in V$  be a fine  $\theta$ -complete ultrafilter on X and  $j = j_{\mathcal{U}} \colon V \to M$ . Since  $\kappa \geq \operatorname{crit}(j) > \theta$ ,

$$\begin{split} j(\mathrm{Add}(\theta,\kappa)) &= \mathrm{Add}(\theta,j(\kappa)) \\ &\cong \mathrm{Add}(\theta,j"\kappa) \times \mathrm{Add}(\theta,j(\kappa) \setminus j"\kappa) \\ &\cong \mathrm{Add}(\theta,\kappa) \times \mathrm{Add}(\theta,j(\kappa) \setminus \kappa) \times \{1\!\!1\}. \end{split}$$

Furthermore, each  $p \in \operatorname{Add}(\theta, \kappa)$  is of cardinality less than  $\theta$  and thus  $j(p) = j^*p$ , so the isomorphism extends  $j(p) \mapsto \langle p, \mathbb{1}, \mathbb{1} \rangle$  as required. Hence, setting  $\mathcal{I} = \langle \mathcal{U}^* \rangle^{V[G]}$ , we have  $B(\operatorname{Add}(\theta, j(\kappa) \setminus \kappa)^V) \cong \mathscr{P}(X)/\mathcal{I}$  in V[G] by Theorem 3.2. To finish we therefore need only show that

$$\operatorname{Add}(\theta, j(\kappa) \setminus \kappa)^V \cong \operatorname{Add}(\theta, 2^X)^{V[G]}.$$

Add $(\theta, \kappa)$  is  $\theta$ -closed so, for all  $Y \in V$ , Add $(\theta, Y)^V = \text{Add}(\theta, Y)^{V[G]}$  and so it is sufficient to prove that  $|j(\kappa) \setminus \kappa| = |(2^{\lambda})^{V[G]}|$ .

Add $(\theta, \kappa)$  is  $(\theta^{<\theta})^+$ -c.c. and  $\theta^{<\theta} \le \theta^{< \operatorname{crit}(j)} = \operatorname{crit}(j) \le \kappa$ , so Add $(\theta, \kappa)$  is  $\kappa^+$ -c.c.

 $\operatorname{Add}(\theta,\kappa)$  is  $(\theta^{<\theta})^+$ -c.c. and  $\theta^{<\theta} \leq \theta^{<\operatorname{crit}(j)} = \operatorname{crit}(j) \leq \kappa$ , so  $\operatorname{Add}(\theta,\kappa)$  is  $\kappa^+$ -c.c. By standard techniques,  $(2^{\lambda})^{V[G]} \leq (|\operatorname{Add}(\theta,\kappa)|^{\kappa \times \lambda})^V$ .  $|\operatorname{Add}(\theta,\kappa)| \leq \kappa^{\theta} \leq \lambda^{\lambda}$ , so we get that  $|(2^{\lambda})^{V[G]}| \leq |(2^{\lambda})^V|$ . Certainly  $(2^{\lambda})^V \subseteq (2^{\lambda})^{V[G]}$  so we conclude that  $|(2^{\lambda})^V| = |(2^{\lambda})^{V[G]}|$ . It is therefore sufficient to show that  $|2^{\lambda}| = |j(\kappa) \setminus \kappa|$  in V. To that end, we work in V for the remainder of the proof.

Since  $2^{\lambda} > \kappa$  it is sufficient to show that  $2^{\lambda} \leq j(\kappa) < (2^{\lambda})^+$ . Let  $D = [\mathrm{id}]_{\mathcal{U}}$  in M. By the fineness of  $\mathcal{U}$ ,  $j^{\mu} \leq D$  and  $M \models |D| < j(\kappa)$ . By elementarity,

$$M \vDash (\forall \gamma < j(\kappa))2^{\gamma} \le j(\kappa)$$
 and hence  $M \vDash |\mathscr{P}(D)^{M}| \le j(\kappa)$ .

 $2^{\lambda} \leq |\mathscr{P}(D)^{M}|$  as follows: Consider the function  $f \colon \mathscr{P}(\lambda) \to \mathscr{P}(D)^{M}$  given by  $f(A) = j(A) \cap D$ . Since  $j``\lambda \subseteq D$  we have that if f(A) = f(B) then j``A = j``B and so A = B. Hence f is an injection and  $2^{\lambda} \leq |j(\kappa)|$ .

On the other hand,  $j(\kappa) = \{[f]_{\mathcal{U}} \mid f \colon X \to \kappa\}$  and so  $j(\kappa) < (\kappa^{\lambda})^{+} = (2^{\lambda})^{+}$ . Thus  $2^{\lambda} \leq j(\kappa) < (2^{\lambda})^{+}$  as required.

4.2. A class of measurable cardinals. Assume GCH and suppose that  $\kappa < \lambda$  are the two smallest measurable cardinals. By [12], if G is V-generic for some  $\mathbb{P}$ , where  $|\mathbb{P}| < \lambda$ , then  $\lambda$  is still measurable in V[G]. Hence, as in Proposition 4.5, if we force with  $\bigstar_{\alpha \in A} \operatorname{Add}(\alpha, \alpha^+)$ , where  $A = \{\alpha < \kappa \mid \alpha \text{ is regular}\}$ , then there will be a maximal  $\kappa$ -independent family  $A \subseteq \mathscr{P}(\kappa)$  in the forcing extension. Furthermore, this forcing has cardinality  $\kappa$  and so  $\lambda$  will still be measurable, and GCH will still hold. If we were to repeat this, say letting  $\mathbb{P}' = \bigstar_{\alpha \in A'} \operatorname{Add}(\alpha, \alpha^+)$  in the forcing extension, where  $A' = \{\alpha < \lambda \mid \kappa < \alpha \wedge \alpha \text{ is regular}\}$ , then again Proposition 4.5 shows that in a new forcing extension by  $\mathbb{P}'$  there is a maximal  $\lambda$ -independent family  $A' \subseteq \mathscr{P}(\lambda)$ . However, since  $\mathbb{P}'$  is  $\kappa^+$ -closed, no new subsets of  $\kappa$  nor sequences of length  $\kappa$  in A have been added, so A is still maximal  $\kappa$ -independent in the second forcing extension. One may reasonably expect that we can continue iterating this

<sup>&</sup>lt;sup>6</sup>One could adapt the proof of [8, Lemma 15.1] to incorporate chain conditions, for example.

<sup>&</sup>lt;sup>7</sup>This method is similar to [9, Lemma 3.3.2], but could be older. We are grateful for Goldberg's help in [6] for this result.

procedure to produce a (potentially class-size) forcing extension V[G] such that, whenever  $\kappa$  is measurable in V, there is a maximal  $\kappa$ -independent family on  $\kappa$ .

The naïve approach to this argument has us construct the Easton-support iteration  $*_{\alpha \in A} \operatorname{Add}(\alpha, \alpha^+)$ , where A is the class of all regular non-measurable cardinals. We would then hope to use Proposition 4.5 to show that if  $G \subseteq \mathbb{P}$  is V-generic and  $\mathcal{U}$  is a normal measure on  $\kappa$  then we can construct a maximal  $\kappa$ -independent family on  $\kappa$  in V[G]. While this may work, one must be careful of the condition that  $A \in \mathcal{U}$  found in Proposition 4.5. If  $\mathcal{U}$  was such that  $\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in \mathcal{U}$ , then  $\kappa \notin j_{\mathcal{U}}(A \cap \kappa)$  and so  $j_{\mathcal{U}}(\mathbb{P}_{\kappa})$  is not isomorphic to  $\mathbb{P}_{\kappa} * \operatorname{Add}(\kappa, \kappa^+) * \dot{\mathbb{R}}$  as desired. Instead  $j_{\mathcal{U}} \colon V \to M$  may be lifted to  $\hat{\jmath} \colon V[G \upharpoonright \kappa] \to M[G \upharpoonright \kappa * F]$  in  $V[G \upharpoonright \kappa]$  (and then can be lifted to  $\tilde{\jmath} \colon V[G] \to M[j(G)]$  in V[G] by the closure of  $\mathbb{P}/\mathbb{P}_{\kappa}$ ). Therefore we must be sure to use  $\mathcal{U}$  with  $A \cap \kappa \in \mathcal{U}$  in our argument, which is to say  $o(\mathcal{U}) = 0$ . Fortunately, such such measures always exist by the well-foundedness of  $\triangleleft$ .

Continuing along our lifting argument, if  $\mathcal{U} \in V$  is a normal measure on  $\kappa$  and  $o(\mathcal{U})^V > 0$  then there is a normal measure  $\hat{\mathcal{U}} \supseteq \mathcal{U}$  in V[G]. This allows us to show that if  $o(\kappa)^V > \alpha$  then  $o(\kappa)^{V[G]} \ge \alpha$ . Though not all Mitchell ranks are preserved (we shall see that if  $o(\kappa)^V = 1$  then  $o(\kappa)^{V[G]} = 0$ ), the reduction shall be 'minimal': A closure point argument à la Proposition 2.7 gives us that if  $\mathcal{U} \in V[G]$  is a normal measure in the forcing extension then  $\mathcal{U} \cap V \in V$  is a normal measure in V. Hence if  $o(\kappa)^V > 0$  then  $o(\kappa)^V = 1 + o(\kappa)^{V[G]}$  exactly. That is,  $o(\kappa)^V = o(\kappa)^{V[G]} - 1$  if  $o(\kappa)^V$  is positive and finite, and otherwise  $o(\kappa)^V = o(\kappa)^{V[G]}$ . This operation warrants some ad-hoc notation. For  $\alpha \in \mathrm{Ord}$ , let

$$-\alpha := \begin{cases} 0 & \alpha = 0 \\ \alpha - 1 & 0 < \alpha < \omega \\ \alpha & \omega \le \alpha. \end{cases}$$

Our suggested interpretation of this operation is that, given some well-founded relation  $\langle X, \prec \rangle$ , we may produce a new relation  $\langle {}^-X, \prec \rangle$  by setting  ${}^-X$  to be those  $x \in X$  that are not minimal with respect to  $\prec$ . Then if  $\alpha$  is the height of  $\prec$  on X,  ${}^-\alpha$  is the height of  $\prec$  restricted to  ${}^-X$ .

The only other consideration is GCH. However, this is easy to force while preserving the Mitchell rank of all cardinals, such as with the Easton-support iteration  $*_{o(\kappa)>0} \operatorname{Add}(\kappa^+, 1)$ .

**Theorem B.** Let V be a model of ZFC+GCH. Then there is a class-length forcing iteration  $\mathbb{P}$  preserving ZFC+GCH such that, if  $G \subseteq \mathbb{P}$  is V-generic, then whenever  $\kappa$  is a measurable cardinal in V there is a maximal  $\kappa$ -independent family  $\mathcal{A} \subseteq \mathscr{P}(\kappa)$  in V[G]. Furthermore, whenever  $\kappa$  is a measurable cardinal in V,  $o(\kappa)^V = 1 + o(\kappa)^{V[G]}$ , and whenever  $\kappa$  is non-measurable in V it remains non-measurable in V[G].

*Proof.* Let us first define our iteration system  $\langle \mathbb{P}_{\alpha}, \dot{\mathbb{Q}}_{\alpha} \mid \alpha \in \text{Ord} \rangle$ . For all regular non-measurable cardinals  $\alpha$ , let  $\dot{\mathbb{Q}}_{\alpha}$  be a  $\mathbb{P}_{\alpha}$ -name for  $\text{Add}(\alpha, \alpha^{+})$  in the extension, and let  $\dot{\mathbb{Q}}_{\alpha} = \{1\}$  otherwise. We iterate this with Easton support: At limit stage  $\alpha$ , if  $\alpha$  is regular, let  $\mathbb{P}_{\alpha}$  be the direct limit of all  $\mathbb{P}_{\beta}$  for  $\beta < \alpha$ , otherwise let  $\mathbb{P}_{\alpha}$  be the inverse limit. Let  $\mathbb{P}$  be the direct limit of all  $\mathbb{P}_{\alpha}$ .

Note that for all regular  $\alpha$ ,  $\mathbb{P} = \mathbb{P}_{\alpha} * (\mathbb{P}/\mathbb{P}_{\alpha})$ , where  $\mathbb{P}_{\alpha}$  is  $\alpha^+$ -c.c. and  $\mathbb{P}/\mathbb{P}_{\alpha}$  is forced to be  $\alpha$ -closed. By a standard application of forcing techniques we have

that  $\mathbb{P}$  is tame and thus will preserve ZFC.<sup>8</sup> For measurable  $\kappa$ , we also have that  $\mathbb{P} = \mathbb{P}_{\kappa} * (\mathbb{P}/\mathbb{P}_{\kappa})$  with  $\mathbb{P}/\mathbb{P}_{\kappa}$  forced to be  $\kappa^+$ -complete. Therefore, if  $\mathbb{P}_{\kappa}$  adds a maximal  $\kappa$ -independent family  $\mathcal{A} \subseteq \mathscr{P}(\kappa)$  then, after forcing with  $\mathbb{P}/\mathbb{P}_{\kappa}$ ,  $\mathcal{A}$  will still be maximal  $\kappa$ -independent. It therefore remains to show that  $\mathbb{P}_{\kappa}$  does indeed add such a family, in the manner of Proposition 4.5. However, after taking care to pick a measure  $\mathcal{U} \in V$  with  $o(\mathcal{U})^V = 0$ , we may apply Proposition 4.5 without modification. In this case,  $A = {\lambda \in V \mid o(\lambda)^V > 0}$ , so  $A \cap \kappa \in \mathcal{U}$ .

The rest of the proof will be spent showing that, for all  $\kappa$ ,  $o(\kappa)^{V[G]} = {}^-o(\kappa)^V$ . Let us begin by noting that for all  $\kappa$ ,  $\mathbb{P}/\mathbb{P}_{\kappa^+}$  is  $\kappa$ -closed, and  $\mathbb{P}_{\kappa^+} = \mathbb{P}_{\omega} * (\mathbb{P}_{\kappa^+}/\mathbb{P}_{\omega})$ , where  $\mathbb{P}_{\omega} = \operatorname{Add}(\omega, \omega_1)$  and  $\mathbb{P}_{\kappa^+}/\mathbb{P}_{\omega}$  is  $\sigma$ -closed. That is,  $\mathbb{P}_{\kappa^+}$  has a closure point at  $\omega$ . Therefore, if  $\mathcal{U} \in V[G \upharpoonright \kappa^+]$  is a normal measure on  $\kappa$  then, by Proposition 2.7,  $\mathcal{U} \cap V \in V$  is a normal measure on  $\kappa$  in V. Since  $\mathbb{P}/\mathbb{P}_{\kappa^+}$  is  $\kappa^+$ -closed, any normal measure on  $\kappa$  in V[G] must have already been present in  $V[G \upharpoonright \kappa^+] = V[G \upharpoonright \kappa]$ . In particular, if  $o(\kappa)^V = 0$  then  $o(\kappa)^{V[G]} = 0$ . Having established this, the following claim will be helpful for our lifting arguments.

Claim 4.5.1. If  $U \in V[G]$  is a normal measure on  $\kappa$ , then

$$\kappa \setminus A = \{\lambda < \kappa \mid o(\lambda)^V > 0\} \in \mathcal{U}.$$

*Proof of Claim.* By prior calculations let us work in  $V[G \upharpoonright \kappa]$  and let

$$j = j_{\mathcal{U}} \colon V[G \upharpoonright \kappa] \to N = M[j(G \upharpoonright \kappa)]$$

be the associated ultrapower embedding. Note that

$$j(\mathbb{P}_{\kappa}) = \underset{\alpha \in j(A \cap \kappa)}{*} \operatorname{Add}(\alpha, (\alpha^{+})^{N})^{N}.$$

By the  $\kappa^+$ -closure of N in  $V[G \upharpoonright \kappa]$ , if  $\kappa \in j(A)$  then  $\mathrm{Add}(\kappa, \kappa^+)^{V[G \upharpoonright \kappa]}$  is an iterand of  $j(\mathbb{P}_{\kappa})$  and we can extract from j(G) a  $V[G \upharpoonright \kappa]$ -generic filter for  $\mathrm{Add}(\kappa, \kappa^+)^{V[G \upharpoonright \kappa]}$ . However, j is definable in  $V[G \upharpoonright \kappa]$  and so certainly such an object cannot exist in  $V[G \upharpoonright \kappa]$ . Hence,  $\kappa \notin j(A)$  and so  $\kappa \setminus A \in \mathcal{U}$ .

The rest of the proof shall be spent showing the exact Mitchell ranks of cardinals in V[G]: For all  $\kappa$ ,  $o(\kappa)^{V[G]} = {}^-o(\kappa)^V$ . We shall do this by induction, so suppose that for all  $\lambda < \kappa$ ,  $o(\lambda)^{V[G]} = {}^-o(\lambda)^V$ . As we have shown that  $o(\kappa)^V = 0$  implies that  $o(\kappa)^{V[G]} = 0$ , let us assume that  $o(\kappa)^V > 0$ .

 $(o(\kappa)^{V[G]} \leq {}^-o(\kappa)^V)$ . Suppose that  $o(\kappa)^{V[G]} > {}^-o(\kappa)^V$ , witnessed by normal measure  $\mathcal{U} \in V[G]$  such that  $o(\mathcal{U})^{V[G]} = {}^-o(\kappa)^V$ . By Claim 4.5.1,  $\kappa \setminus A \in \mathcal{U} \cap V$ , and hence  $o(\mathcal{U} \cap V)^V > 0$  and  $o(\kappa)^V > 1$ . In particular, for any  $\alpha$ , if  ${}^-\alpha = {}^-o(\kappa)^V$  then  $\alpha = o(\kappa)^V$ . Therefore,

$$\{\lambda < \kappa \mid o(\lambda)^{V[G]} = {}^{-}o(\kappa)^{V}\} = \{\lambda < \kappa \mid {}^{-}o(\lambda)^{V} = {}^{-}o(\kappa)^{V}\}$$
$$= \{\lambda < \kappa \mid o(\lambda)^{V} = o(\kappa)^{V}\}$$
$$\in \mathcal{U} \cap V,$$

and so  $o(\mathcal{U} \cap V)^V = o(\kappa)^V$ , a contradiction.

 $(o(\kappa)^{V[G]} \geq o(\kappa)^{V})$ . By Proposition 4.5, if  $\mathcal{U} \in V$  is a normal measure on  $\kappa$  such that  $A \cap \kappa \notin \mathcal{U}$  (i.e.  $o(\mathcal{U})^{V} > 0$ ), there is  $\hat{\mathcal{U}} \supseteq \mathcal{U}$  a normal measure on  $\kappa$  in

 $<sup>^8</sup>$ [8, Ch. 15] provides a comprehensive overview of preservation of ZFC using class products. [4] has a deep treatment of class length forcing iterations.

<sup>&</sup>lt;sup>9</sup>Having found out that  $\kappa$  is measurable in V, we conclude that  $\dot{\mathbb{Q}}_{\kappa} = \{1\}^{\bullet}$ .

 $V[G \upharpoonright \kappa]$ . Furthermore, since  $\mathbb{P}/\mathbb{P}_{\kappa}$  is  $\kappa^+$ -closed,  $\hat{\mathcal{U}}$  is still a normal measure on  $\kappa$  in V[G]. Since  $o(\kappa)^{V[G]} \geq 0$  by definition, let us assume that  $o(\kappa)^V > 1$  and prove that  $o(\kappa)^{V[G]} \geq {}^-o(\kappa)^V$ . If  $\mathcal{U} \in V$  is such that  $o(\mathcal{U})^V > 0$  then

$$\{\lambda < \kappa \mid o(\lambda)^V = o(\mathcal{U})^V\} = \{\lambda < \kappa \mid o(\lambda)^{V[G]} = {}^-o(\mathcal{U})^V\} \in \hat{\mathcal{U}}.$$

Hence, for all 
$$\alpha < o(\kappa)^V$$
,  $-\alpha < o(\kappa)^{V[G]}$ , so  $-o(\kappa)^V \le o(\kappa)^{V[G]}$  as required.  $\square$ 

Note that this result on the Mitchell rank may not be reversible. Let  $\mathcal{U}, \mathcal{V} \in V$  be any two normal measures on some  $\kappa$  with  $o(\kappa)^V = 1$ , and  $A \in \mathcal{U} \setminus \mathcal{V}$  a set of regular cardinals. Then forcing with the Easton-support iteration  $*_{\alpha \in A} \operatorname{Add}(\alpha, \alpha^+)$  will produce a maximal  $\kappa$ -independent family  $A \subseteq \mathscr{P}(\kappa)$  thanks to  $\mathcal{U}$ , but  $\hat{\mathcal{V}}$  will witness that  $\kappa$  is measurable in the forcing extension. However, there need not be an inner model witnessing  $o(\kappa) > 1$ . On the other hand, if there is a normal measure  $\mathcal{U}$  on  $\kappa$  such that

$$A = \{\lambda < \kappa \mid (\exists A \subseteq \mathscr{P}(\lambda)) A \text{ is maximal } \lambda \text{-independent}\} \in \mathcal{U},$$

then it seems likely that  $o(\kappa) = 2$  in the model  $L[\langle \mathcal{I}_{\lambda} \mid \lambda \in A \rangle, \mathcal{I}_{\kappa}, \mathcal{U}]$ , where  $\mathcal{I}_{\lambda}$  is the  $\lambda^+$ -saturated,  $\lambda$ -complete ideal on  $\lambda$  given by Theorem 4.2 (see Question 5.6).

## 5. The future

Kunen's equiconsistency of measurable cardinals with maximal  $\sigma$ -independent families opens up an exciting correspondence between the consistency strength of large cardinals and corresponding collections of maximal  $\theta$ -independent families, either as a study in its own right or as an avenue to analyse other consistency strength relationships (such as determinacy or forcing axioms).

**Question 5.1.** Can we extend the methods of Theorem B and Proposition 4.5 to produce a proper class of maximal  $\mu$ -independent families when  $\kappa$  is  $\mu$ -strongly compact but not  $\mu^+$ -strongly compact? If  $\kappa$  is strongly compact then can we obtain a model in which there is a proper class of maximal  $\kappa$ -independent families?

**Question 5.2.** Can the technique of Theorem 4.1 be extended to general elementary embeddings, rather than ultrapower embeddings? If so, can we soften the requirements of Theorem A to only requiring, say, a  $\theta$ -strong cardinal?

**Question 5.3.** What is the consistency strength of a proper class of maximal  $\sigma$ -independent families? Of a proper class of  $\theta$  such that there is a maximal  $\theta$ -independent family? Of a proper class of  $\theta$  such that there is a maximal  $\theta$ -independent family on  $\theta$ ?

Our hope would be that one could investigate these questions without needing to develop an inner model theory for strongly compact cardinals, which would pose a non-trivial obstacle. Similarly, we would like to know more about  $\theta$ -strongly compact cardinals and where they fall in the large cardinal hierarchy.

**Question 5.4.** Is the consistency strength of a  $\theta$ -strongly compact cardinal strictly lower than that of a strongly compact cardinal?

**Question 5.5.** Is the least  $\theta$ -strongly compact cardinal a strong limit? In [5, Proposition 6.2] Gitik constructs (from  $\kappa < \lambda$  supercompact plus GCH) a cofinality-preserving forcing extensions such that  $2^{\kappa} = \lambda$  and  $\lambda$  is  $\kappa$ -strongly compact. Is  $\lambda$  (consistently) the least  $\kappa$ -strongly compact cardinal in this model?

Question 5.6. When can we recover Mitchel order from models with many  $\kappa$  such that there is a maximal  $\kappa$ -independent family on  $\kappa$ ? For example, is it sufficient to have  $\mathcal{A} \in \text{Ult}(V, \mathcal{U})$  to produce an inner model in which  $o(\kappa) \geq 2$ ?

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