Operations Research 2: Optimization

Zihang Wang - zihang.wang@epita.fr

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PartI

Q1

What are the critical points (and their nature) of the function h defined on $\mathbb{R}^2$, by $h(x,y) = (x-1)^2 + 3y^2$

Solution: since $h(x)$ is continuous on $\mathbb{R}^2$, to find the critical points, we have:

$$
\begin{aligned}
&h_x(x,y) = 2x - 2 = 0 \\
&h_y(x,y) = 6y = 0 \\
&\Rightarrow x = 1, y = 0 \\
&\textit{Second deravative test:} \\
&h_{xx}(x,y) = 2 \\
&h_{yy}(x,y) = 6 \\
&h_{xy}(x,y) = 0 \\
&\textit{for the critical point } (1,0): \\
&H = h_{xx}(1,0) f_{yy}(1,0) - f_{xy}^2(1,0) = 2 \times 6 - 0^2 = 12
\end{aligned}
$$

In conclusion, Since $H > 0$, $h_{xx}(x,y) > 0$, $h(x,y)$ has a local minimum at $(1,0)$

Q2

We define a function f by $f(x,y) = \dfrac{x^2+y^2}{x+y}$,What is the domain D of f? Can f reach a max or min somewhere in D?

Solution: Obviously, the domain is $D = \{(x,y) \in \mathbb{R}^2 : x \neq -y\}$

Suppose there exsists critical points:

$$\begin{aligned}
&f_x(x,y) = \frac{2x(x+y)-(x^2+y^2)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2} = 0 \\
&f_y(x,y) = \frac{2y(x+y)-(x^2+y^2)}{(x+y)^2} = \frac{-x^2+2xy+y^2}{(x+y)^2} = 0 \\
&\Rightarrow x,y = 0
\end{aligned}$$

Since the point $(0,0) \notin D$, it is no extreme for this function.

Q3

We define a function g on $\mathbb{R}^2$ by $g(x) = (x^2+y^2)e^{-x}$,What are the local extrema (max/min) of g on $\mathbb{R}^2$

Solution: since $g(x)$ is continuous on $\mathbb{R}^2$, to find the critical points, we have:

$$\begin{aligned}
&g_x(x,y) = 2xe^{-x} - (x^2+y^2)e^{-x} = 0 \\
&g_y(x,y) = 2e^{-x}y = 0 \\
&\Rightarrow x,y = 0 \vee x = 2, y = 0 \\
&\textit{Second derivative test:} \\
&g_{xx}(x,y) = -2xe^{-x} + 2e^{-x} - (2xe^{-x} - x^2e^{-x} - y^2e^{-x}) \\
&= (x^2 - 4x + y^2 + 2)e^{-x} \\
&g_{yy}(x,y) = 2e^{-x} \\
&g_{xy}(x,y) = -2e^{-x}y \\
&\textit{for the critical point } (0,0): \\
&H = g_{xx}(0,0)f_{yy}(0,0) - f_{xy}^2(0,0) = 2 \times 2 - 0^2 = 4 \\
&\textit{for the critical point } (2,0): \\
&H = g_{xx}(2,0)f_{yy}(2,0) - f_{xy}^2(2,0) = -2e^{-2} \times 2e^{-2} - 0^2 = -4e^{-4}
\end{aligned}$$

In conclusion, for the critical point $(0,0), H > 0, g_{xx} > 0$, it is the local minimum; for the critical point $(2,0), H < 0, g_{xx} < 0$, it is the saddle point.

PartII

We are in a 3D space and we assume the function, $f(x,y) = \frac{5}{3}x^3 - xy^2 + y^3 - y^2 + 9$, gives the altitude at the point (x,y).

Q1

We assume we are at the point $(3, 1)$. In what direction should we go to increase altitude the most? What is the slope in this direction?

Solution:

Gradient is the direction of steepest ascent, calculate:

$$
\begin{aligned}
&f_x(3,1) = 5x^2 - y^2 = 5 \times 3^2 - 1^2 = 44 \\
&f_y(3,1) = -2xy + 3y^2 - 2y = -2 \times 3 \times 1 + 3 \times 1^2 - 2 \times 1 = -5 \\
&\nabla f(3,1) = \begin{pmatrix} 44 \\ -5 \end{pmatrix} \\
&k = \frac{f_y(3,1)}{f_x(3,1)} = -\frac{5}{44}
\end{aligned}
$$

Q2

If we are at the point $(2, -2)$ and we go in the direction given by the vector $\begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix}$,what is the slope and what interpretation can be made of this value?

Solution:

From the previous calculation, we get:

$$
\begin{aligned}
&\nabla f(2,-2) = \begin{pmatrix} 16 \\ 24 \end{pmatrix} \\
&k = \nabla f(2,-2) \times \frac{\vec{u}}{|\vec{u}|} = \begin{pmatrix} 16 \\ 24 \end{pmatrix} \cdot \begin{pmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{pmatrix} = -\frac{8}{5}
\end{aligned}
$$

In conclusion, while we are in the 3D space, the slope $-\frac{8}{5}$ is the change of the height by moving along with the direction $< \frac{4}{5}, -\frac{3}{5} >$ at the point $(2, -2)$

Q3

We are now at the point $(3, 3)$. In which direction should we go to follow a contour line of f?

Solution:

From the previous calculation, we get:

$$\nabla f(3,3) = \begin{pmatrix} 36 \\ 3 \end{pmatrix}$$

Since the gradient of each point is orthogonal to the contour line, so

$$\nabla f(3,3) \cdot \vec{r} = 0$$

$$\Rightarrow \vec{r} = span\{< -3, 36 >, < 3, -36 >\}$$

PartIII

Solve the following optimization problem

$$Max\ f(x,y) = -5x^2 – 5y^2 + 2xy + 3x + 3y + 1000$$
$$s.t. x + y = 20$$

Solution:
Applying Lagrangian Method, we have:

$$f(x,y) = -5x^2 – 5y^2 + 2xy + 3x + 3y + 1000$$
$$h(x,y) = x + y - 20 = 0$$
$$\mathbf{l}(x,y,\lambda) = -5x^2 – 5y^2 + 2xy + 3x + 3y + 1000 + \lambda(x + y - 20)$$
$$\nabla \mathbf{l}(x,y,\lambda) = \begin{pmatrix} -10x + 2y + 3 + \lambda \\ -10y + 2x + 3 + \lambda \\ x + y - 20 \end{pmatrix} = 0$$
$$\Rightarrow \begin{cases} y = x \\ x = y = 10 \end{cases}$$
$$\Rightarrow f(10,10) = 260$$

To determine whether it is a maximum or minimum, select a point in the D :

$$f(9,11) = 248$$

In conclusion, we found the critical point is $(10,10)$ by setting the gradient to 0 (Gradient is defined everywhere in the domain), and $f(10,10) > f(9,11)$, it is the maximum