Reinforcement Learning II
Dynamic Programming and Model-free predictions

Antoine SYLVAIN

EPITA

2022

Contents

Policies

- Specifies the behaviour of the agent
- A **deterministic policy** associates an action to each state
- It can be represented by a table
-

s_0	a_1
s_1	a_0
s_2	a_1

- A **stochastic policy** associates various actions with a probability for each state
- The sum of probabilities of the actions for a given state equals 1

Policies

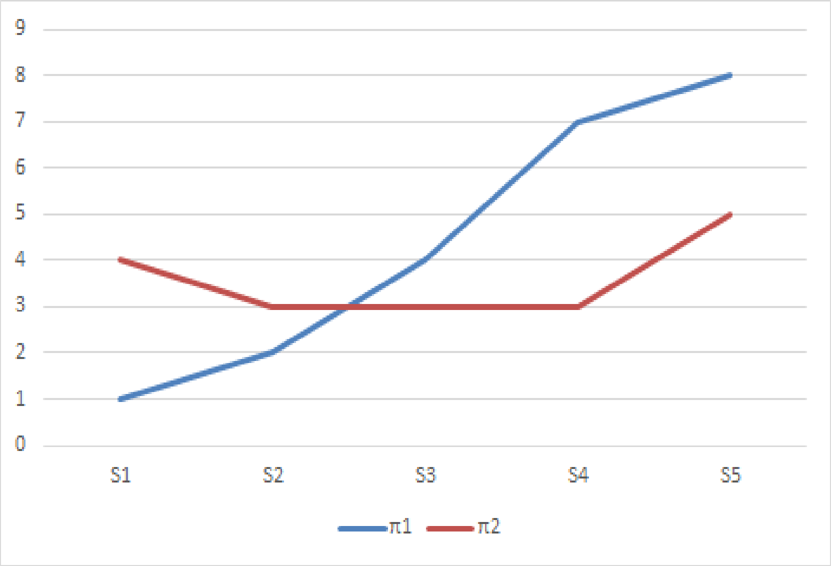
- Deterministic policy:
  $\pi(s) = a$
- Stochastic policy:
  $\pi(a|s) = \mathbb{P}[\mathcal{A}_t = a|\mathcal{S}_t = s]$

Policies

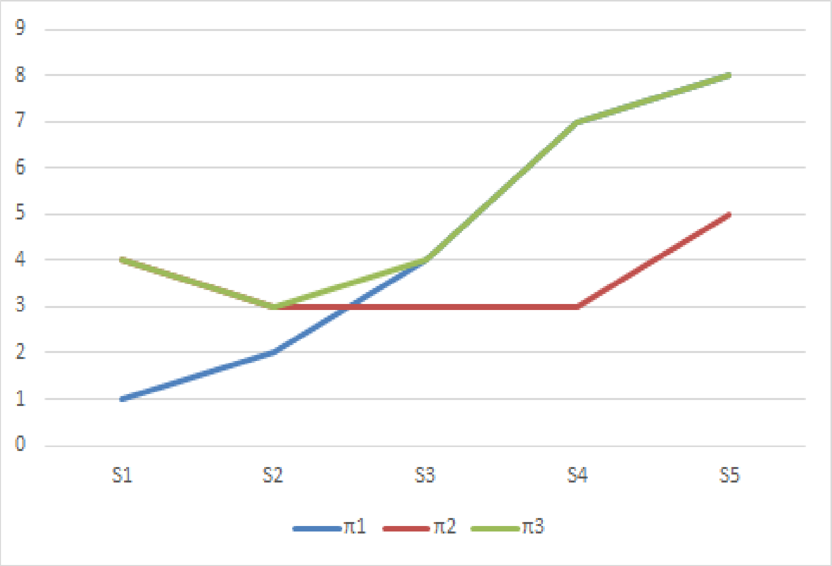
The best policy is the one that achieves the highest value

Is there always an optimal policy ? (i.e. that returns a value greater or equal than the other policies for each state)

Comparing two policies



Combining two policies



Contents

What is Dynamic Programming

- A collection of algorithms that can be used to compute optimal policies
- They require a perfect model of the environment as a MDP
- Optimal substructures
- Divide the problem into sub-problems
- Combine the solutions to the sub-problems

Policy Evaluation and Control

- **Policy Evaluation**: Evaluation of the value function of a given policy
- **Policy Control**: Finding the best policy (i.e. maximize value function)
- Control is the overall objective
- But, to find the best policy, you first need to be able to evaluate how good it is

Iterative Policy Evaluation

- Starts with arbitrary values for each non-terminal state
- At each iteration, update the values of these states with the Bellman equation and the updated values
- Evaluations converge to the value function

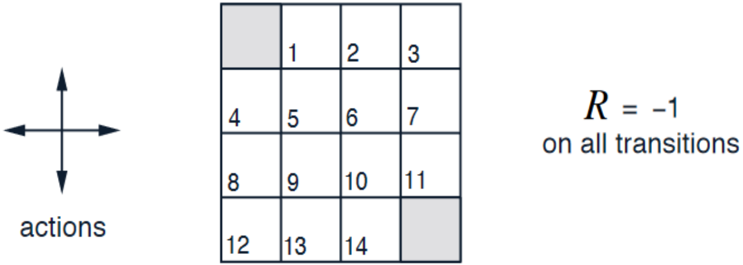
Iterative Policy Evaluation

Input: π (the policy to be evaluated)
Initialize an array $V(s) = 0$ for all $s \in \mathcal{S}^+$
Repeat:
    $\Delta \leftarrow 0$
    For each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)[r + \gamma V(s')]$
        $\Delta \leftarrow max(\Delta, |v - V(s)|)$
until $\Delta < \theta$ (small positive number)
Output: $V \approx v_\pi$

Example

- Small grid world
- Four possible actions (up, down, left, right)
- Reward is -1 until a terminal state is reached
- Uniform random policy (0.25 for each action)
- Undiscounted ($\gamma = 1$)

Example



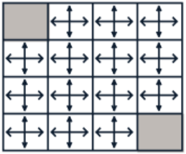
Example

v_k for the Random Policy

Greedy Policy w.r.t. v_k

$k = 0$

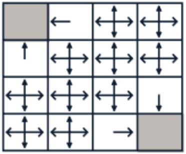
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



random policy

$k = 1$

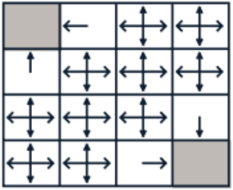
0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0



Example

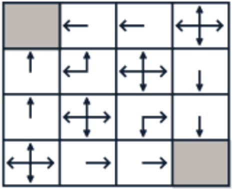
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0



$k = 2$

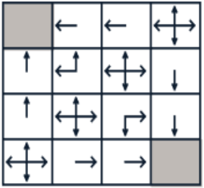
0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0



Example

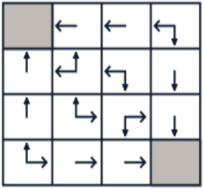
$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0



$k = 3$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0



Example

$k = 3$

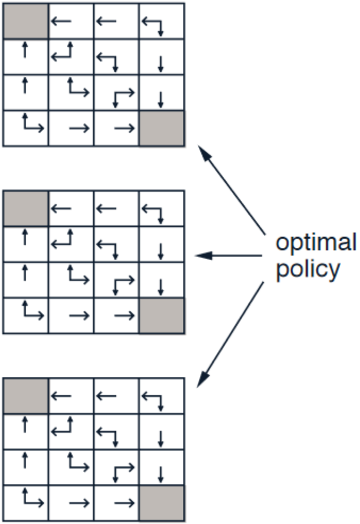
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

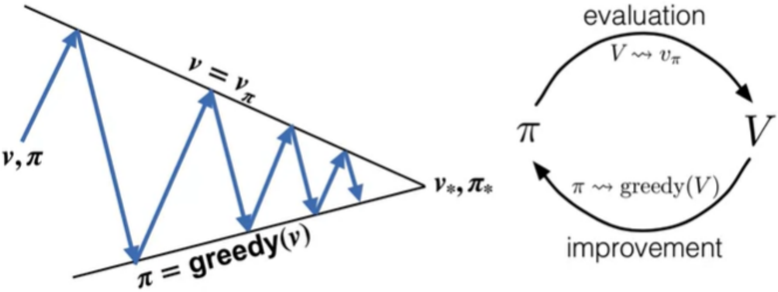


How to improve a policy ?

- Given a policy π
  - Evaluate the policy π
    $v_\pi(s) = \mathbb{E}_\pi[\mathcal{R}_{t+1} + \gamma\mathcal{R}_{t+2} + \gamma^2\mathcal{R}_{t+3} + ...|\mathcal{S}_t = s]$
  - Improve the policy by acting greedily with respect to v_π
    $\pi' = greedy(v_\pi)$
- In the Small Gridworld example, the improved policy was optimal, so $\pi' = \pi^*$
- But in most cases, we will need more iterations of evaluation/improvement
- Nonetheless, this process of policy evaluation always converge to π^*

Policy Iteration

Policy Iteration



Policy iteration for estimating $\pi \approx \pi^*$

Initialization: $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s) \forall s \in \mathcal{S}$

Policy Evaluation

Repeat:

  $\Delta \leftarrow 0$

  For each $s \in \mathcal{S}$:

    $v \leftarrow V(s)$

    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)[r + \gamma V(s')]$

    $\Delta \leftarrow max(\Delta, |v - V(s)|)$

until $\Delta < \theta$ (small positive number)

Policy Improvement

policy-stable $\leftarrow$ True

For each $s \in \mathcal{S}$:

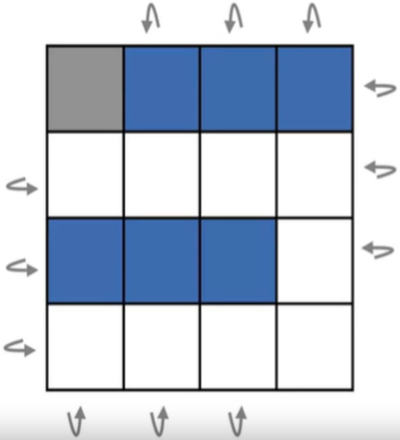
  old-action $\leftarrow \pi(s)$

  $\pi(s) \leftarrow argmax_a \sum_{s',r} p(s', r|s, a)[r + \gamma V(s')]$

  If old-action $\neq \pi(s)$, then policy-stable $\leftarrow$ False

If policy-stable, return $V \approx v^*$ and $\pi \approx \pi^*$, else **Policy Evaluation**

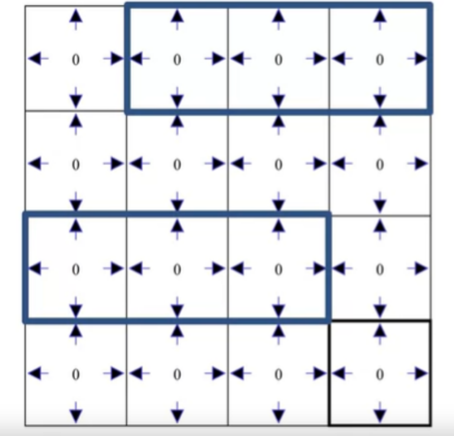
Example



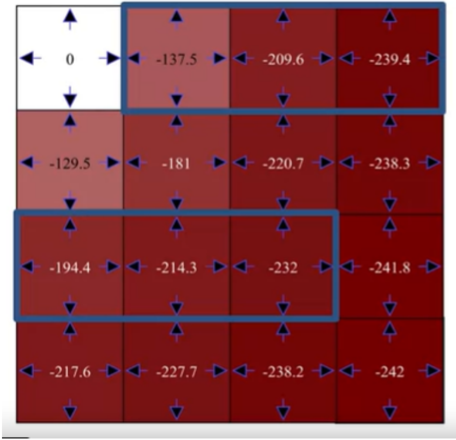
$$R = \begin{cases} -10 & \textbf{Blue states} \\ -1 & \textbf{Other states} \end{cases}$$

$$\gamma = 1$$

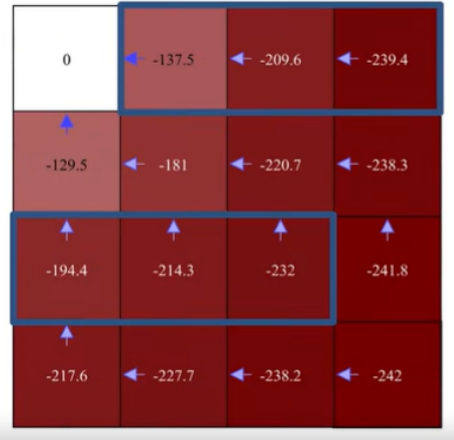
Example



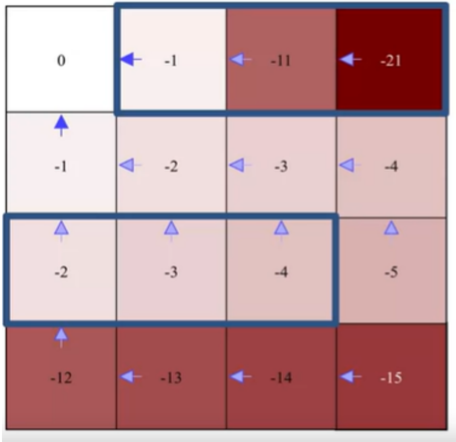
Example


0
-137.5
-209.6
-239.4
-129.5
-181
-220.7
-238.3
-194.4
-214.3
-232
-241.8
-217.6
-227.7
-238.2
-242

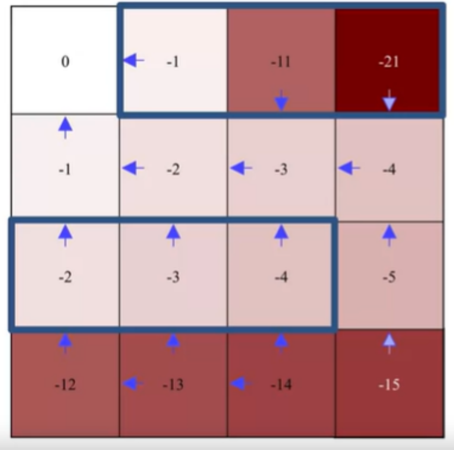
Example


0
-137.5
-209.6
-239.4
-129.5
-181
-220.7
-238.3
-194.4
-214.3
-232
-241.8
-217.6
-227.7
-238.2
-242

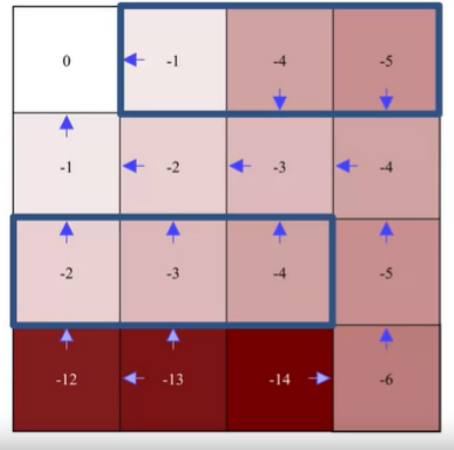
Example



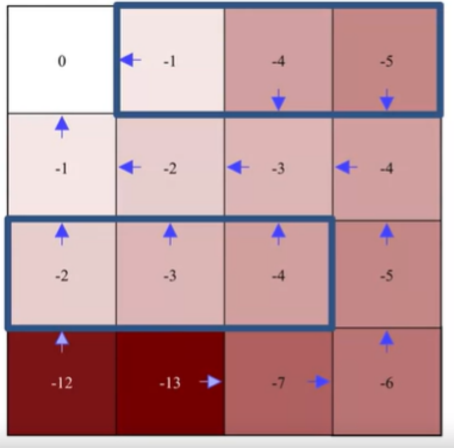
Example



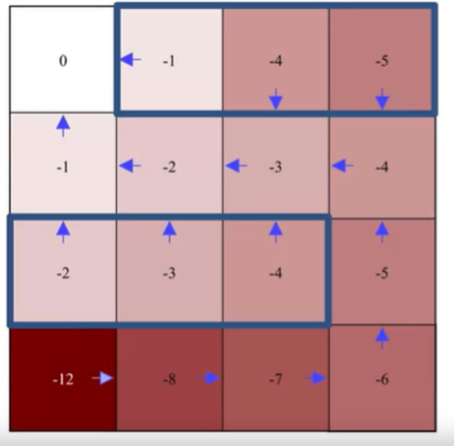
Example



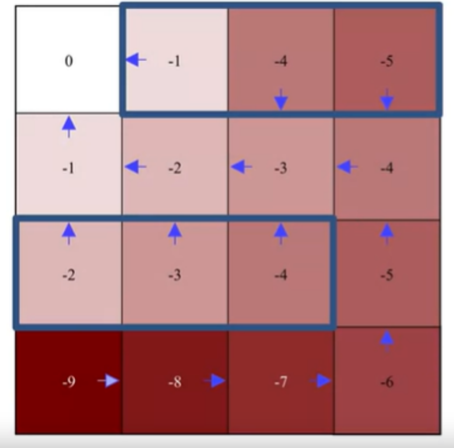
Example



Example



Example



Contents

Model-free algorithms

- Dynamic Programming enables us to find optimal policies in a Markov Decision Process
- But it has a costly pre-requisite : we need a complete knowledge of the MDP
- What if we have a limited knowledge of the MDP ?
- Can we make predictions based on our experience ?

Monte-Carlo Learning

- Methods that learn directly from experience
- Model-free: we have no previous knowledge of the MDP transitions and reward
- Learns from complete episodes: no bootstraping, all episodes must terminate
- Idea: value = mean return

Monte-Carlo Policy Evaluation

- Goal: learn v_π from episodes of experience under policy π : $\mathcal{S}_1, \mathcal{A}_1, \mathcal{R}_2, .., \mathcal{S}_k \sim \pi$
- Return is the total discounted reward : $\mathcal{G}_t = \mathcal{R}_{t+1} + \gamma \mathcal{R}_{t+2} + ... + \gamma^{\tau-1} \mathcal{R}_\tau$
- Value function is the expected return : $v_\pi(s) = \mathbb{E}_\pi[\mathcal{G}_t | \mathcal{S}_t = s]$
- Instead of expecting return, Monte-Carlo uses **empirical mean**

First-visit Monte-Carlo Method

- Estimates the value $v_\pi(s)$ of state s under policy π
- Each occurrence of state s in an episode is called a visit to s
- The first occurrence of s is thus the first visit to s
- The first-visit MC-method estimates $v_\pi(s)$ as the average of the returns following the first visit to s:
  $V(s) = S(s)/N(s)$
  with
  $S(s) \leftarrow S(s) + \mathcal{G}_t$, the incremented total return
  $N(s)$, the visits counter

First-visit Monte-Carlo Prediction

Input: π, a policy to be evaluated
Initialization:

$V(s) \in \mathbb{R} \forall s \in S$, arbitrarily
$Returns(s) \leftarrow$ an empty list, for all $s \in S$

Loop (for each episode):

Generate an episode following π: $S_0, A_0, R_1, ..., R_\tau$
$\mathcal{G} \leftarrow 0$
Loop for each step of episode $t = T-1, T-2, .., 0$:
$\mathcal{G} \leftarrow \gamma\mathcal{G} + R_{t+1}$
Unless S_t appears in $S_0, S_1, ..., S_{t-1}$:
Append $\mathcal{G}$ to $Returns(s)$
$V(S_t) \leftarrow avg(Returns(S_t))$

Every-visit Monte-Carlo Method

Quite the same, but you evaluate the returns following every visit to s

Example of blackjack

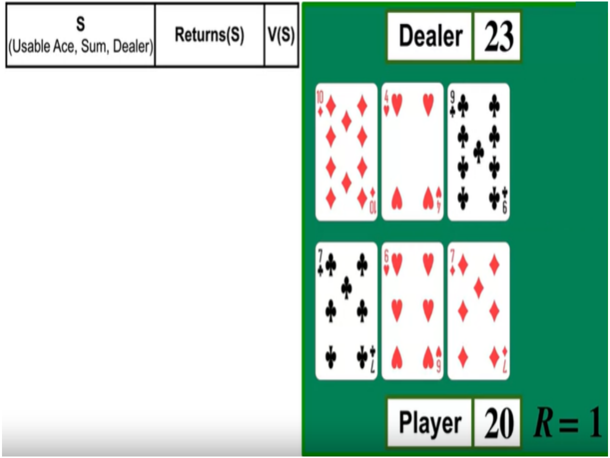
- 200 states:
  - Current sum (12-21)
  - Dealer's visible card (ace-10)
  - Usable ace ? (True or False)
- 2 actions:
  - **Stick**: Stop receiving cards and terminate
  - **Twist**: Take another card
- Stick reward:
  - $+1$ if sum of cards $>$ sum of dealer
  - 0 if sum of card $=$ sum of dealer
  - -1 if sum of card $<$ sum of dealer
- Twist reward:
  - -1 if sum of cards $>$ 21 (terminal)
  - 0 otherwise
- Transitions: automatically twist if sum of cards < 12
- Policy: Stick if sum $= 20$ or 21, else twist

Example of blackjack

Example of blackjack

Example of blackjack

Example of blackjack



Example of blackjack

S (Usable Ace, Sum, Dealer)	Returns(S)	V(S)
A	Returns(A) = [1]	1

A = (NoAce, 20, 10)

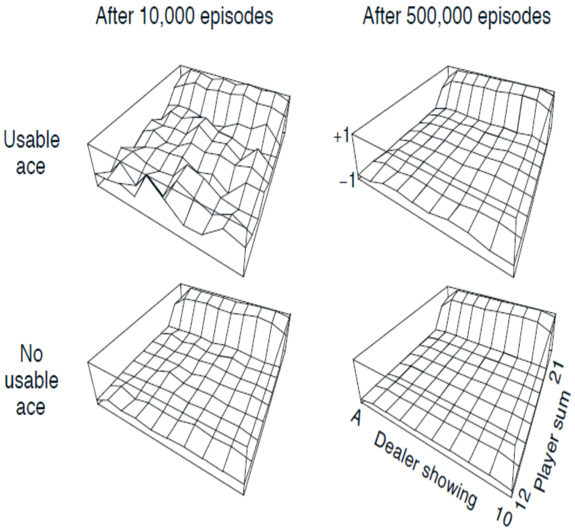
Example of blackjack

S (Usable Ace, Sum, Dealer)	Returns(S)	V(S)
A	Returns(A) = [1]	1
B	Returns(B) = [1]	1

A = (NoAce, 20, 10)

B = (NoAce, 13, 10)

Example of blackjack



Monte-Carlo Estimation of Action values

- If the model is not available, estimate (state-)action values rather than state values is particularly useful.
- With a model, like in dynamic-programming, state-value was enough. Without a model, it is not sufficient.
- So one of the primary goals of MC methods is to evaluate q^*
- The policy evaluation becomes an estimation of $q_\pi(s, a)$
- A state-action s, a is said to be visited in a an episode if state s is visited and action a is selected in it

Temporal-Difference Methods

- Learn directly from episodes of experience
- Model-free: we have no previous knowledge of the MDP transitions and reward
- Learn from incomplete episodes, by bootstraping
- Updates a guess towards a guess

Temporal Difference Methods

- Whereas MC methods need to wait until the end of the episode to update the value of $V(S_t)$
- Temporal-Difference methods makes the update at step $t+1$
- The simplest TD method is know as TD(0)
- $V(S_t) \leftarrow V(S_t) + \alpha[R_{t+1} + \gamma V(S_{t+1}) - V(S_t)]$
- α: stepsize parameter

TD(0) Algorithm

Input: π, the policy to be evaluated
Parameter: stepsize parameter $\alpha \in [0, 1]$
Initialization: $V(s) \forall s \in \mathcal{S}^+$, arbitrarily, except $V(\mathit{terminal}) = 0$

Loop for each episode:
    $A \leftarrow$ action given by π for S
    Select action A, observe R, S'
    $V(S) \leftarrow V(S) + \alpha[R + \gamma V(S') - V(S)]$
    $S \leftarrow S'$
until S terminal

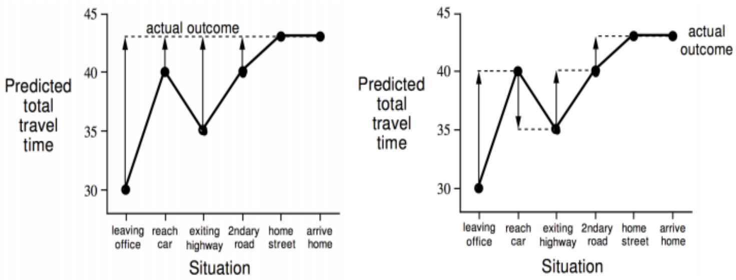
Example of driving home

State	Elapsed Time (minutes)	Predicted Time to Go	Predicted Total Time
leaving office	0	30	30
reach car, raining	5	35	40
exit highway	20	15	35
behind truck	30	10	40
home street	40	3	43
arrive home	43	0	43

Example of driving home: MC vs TD

Changes recommended by Monte Carlo methods (α=1)

Changes recommended by TD methods (α=1)



Comparison of MC and TD

- TD can learn before knowing the final outcome
- TD can learn online step by step
- MC have to wait until the end of episode to know the return
- TD can learn without the final outcome
- TD can learn from incomplete sequences
- MC can only learn from complete sequences
- TD works in continuing (non-terminating) environments
- MC only works form episodic (terminating) environments

Comparison of MC and TD

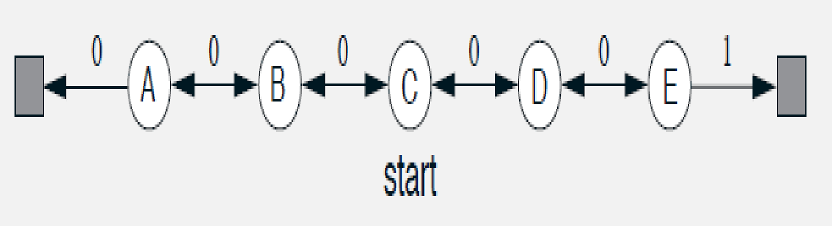
- MC return $G_t = R_{t+1} + \gamma R_{t+2} + ... + \gamma^{\tau-1} R_\tau$ is an unbiased estimate of $v_\pi(S_t)$
- TD target $R_{t+1} + \gamma V(S_{t+1})$ is an biased estimate of $v_\pi(S_t)$
- TD target has much lower variance than the return:
  - MC return depends on many random actions, transitions, rewards
  - TD target depends on one random action, transition, reward

Comparison of MC and TD

- MC has high variance, no bias
  - Good convergence properties
  - Not very sensitive to initial value
  - Very simple understanding and use
  - TD has low variance, but some bias
    - Usually more efficient than MC
    - TD(0) converge to $v_\pi(S_t)$
    - More sensitive to initial value

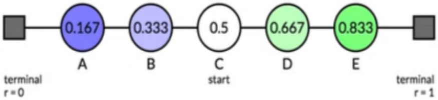
Random Walk Example

- All episodes start in the center state C
- Left and right states are terminal state
- Reward $+1$ if terminates on the right, else 0

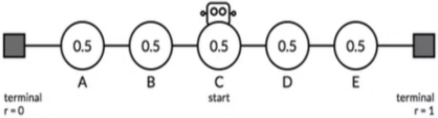


Random Walk Comparison

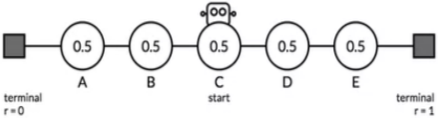
Target / Exact Values



Updates using TD Learning $V(S_t) \leftarrow V(S_t) + \alpha[\, R_{t+1} + \gamma V(S_{t+1}) \; - V(S_t)]$



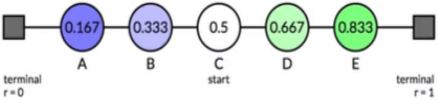
Updates using Monte Carlo



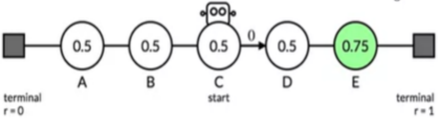
Random Walk Comparison

1st episode:
C,D,E,D,C,D,E,Right

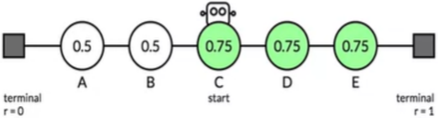
Target / Exact Values



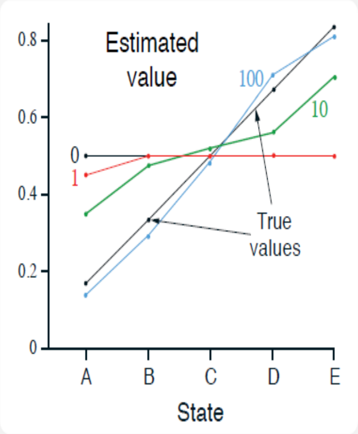
Updates using TD Learning $V(S_t) \leftarrow V(S_t) + \alpha[\underbrace{R_{t+1}}_{0} + \gamma \underbrace{V(S_{t+1})}_{0.5} - \underbrace{V(S_t)}_{0.5}]$

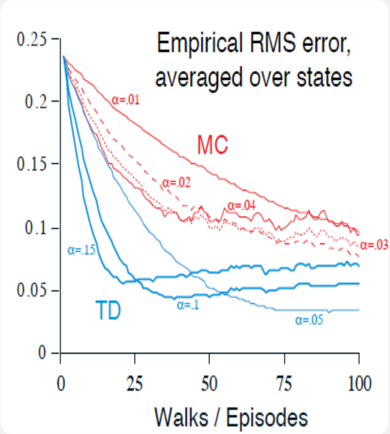


Updates using Monte Carlo



Random Walk Comparison





Batch MC and TD

- MC and TD converge
- But what about a batch experience of finite examples ?
- Repeatedly sample episodes
- Apply MC and TD(0) on these episodes

AB Example

Two states A and B, no discounting, 8 episodes of experience

B, 0

B, 1

B, 1

B, 1

B, 1

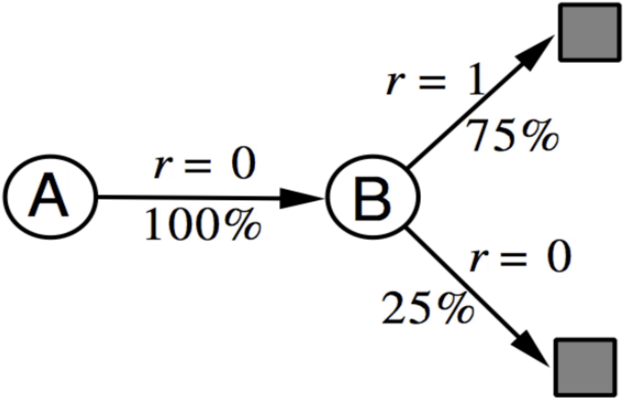
B, 1

B, 1

A, 0, B, 0

What are V(A) and V(B) ?

AB Example



Certainty Equivalence

- MC converges to solution with minimum mean-squared error
  - In the AB example, $V(A) = 0$
- TD(0) converges to solution of max likelihood Markov model
  - In the AB example, $V(A) = 0.75$

MC and TD

- TD exploits Markov property
- Usually more efficient in Markov environments
- MC does not exploit Markov property
- Usually more effective in non-Markov environments