Optimization assignment

Zihang Wang - zihang.wang@epita.fr

22/12/2023

PartI

In a factory, 2 products P1 and P2 are manufactured using 3 kinds of raw material M1, M2, M3, which are available in limited quantities : 18 units of M1, 8 units of M2, 14 units of M3.

The following constraints need to be satisfied:

- to build 1 unit of P1, 1 unit of M1, 1 unit of M2, 2 units of M3 are required
- to build 1 unit of P2, 3 units of M1, 1 unit of M2, 1 units of M3 are required

Selling 1 unit of P1 makes an average benefit of 1 €and selling 1 unit of P2 makes an average benefit of 3€.

The factory director wants to make the maximum possible benefit. Find the modelization of this problem and solve it.

Solution: According to the given information, we get:

$$\textit{Constraints:} \quad \begin{cases} x_1 + 3x_2 \leq 18 \\ x_1 + x_2 \leq 8 \\ 2x_1 + x_2 \leq 14 \end{cases}$$

$$\textit{Maximize:} \quad z = x_1 + 3x_2$$

$$\left[\begin{array}{ccccc|c|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & b \\ \hline 1 & 3 & 1 & 0 & 0 & 0 & 18 \\ 1 & 1 & 0 & 1 & 0 & 0 & 8 \\ 2 & 1 & 0 & 0 & 1 & 0 & 14 \\ \hline -1 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

The pivot value is $M_{12} = 3$, to make other values in the pivot column be $0 \Rightarrow$

$$3R_2 - R_1 \to R_2$$
$$3R_3 - R_1 \to R_3$$
$$R_4 + R_1 \to R_4 \Rightarrow$$

$$\left[\begin{array}{ccccc|c|c} x_1 & x_2 & s_1 & s_2 & s_3 & z & b \\ \hline 1/3 & 1 & 1/3 & 0 & 0 & 0 & 6 \\ 2 & 0 & -1 & 3 & 0 & 0 & 6 \\ 5 & 0 & -1 & 0 & 3 & 0 & 24 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 & 18 \end{array}\right]$$

The bases are$\{x_2, s_2\}$, the solutions are $\Rightarrow$

$$\begin{cases} x_1 = 0 \\ x_2 = 6 \\ s_2 = 2 \\ z = 18 \end{cases}$$

Listing 1: Check results in Python

```
from pulp import *

model = pulp.LpProblem('linear_programming', LpMaximize)

# get solver
solver = getSolver('PULP_CBC_CMD')

# declare decision variables
x1 = LpVariable('x1', lowBound = 0, cat = 'continuous')
x2 = LpVariable('x2', lowBound = 0, cat = 'continuous')

# declare objective
model += x1 + 3*x2

# declare constraints
model += x1 + 3*x2 <= 18
model += x1+x2 <= 8
model += 2*x1+x2 <= 14

# solve
results = model.solve(solver=solver)

# print results
if LpStatus[results] == 'Optimal': print('The solution is optimal.')
print(f'Objective value: z* = {value(model.objective)}')
print(f'Solution: x1* = {value(x1)}, x2* = {value(x2)}')
```

PartII

Solve the following problem: Max. w = -4x -10y -5z

$$\begin{cases} 5x + 20y + 15z \geq 4 \\ -2x + 2y \geq 10 \\ 5x - 15y + 10z \geq -8 \quad \text{with } x, y, z \geq 0 \end{cases}$$

Solution:
Applying Big-M method since the format is not the standard one, but we need to make

sure the right side of each equation is positive. We get:

$$\begin{cases} 5x+20y+15z-s_1+a_1=4 \\ -2x+2y-s_2+a_2=10 \\ -5x+15y-10z+s_3=8 \quad \text{with } x,y,z\geq 0 \end{cases}$$

to find the maximum of w in $w=-4x-10y-5z-Ma_1-Ma_2$. Create the simplex tableau:

$$\left[\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & a_1 & a_2 & b \\ \hline 5 & 20 & 15 & -1 & 0 & 0 & 1 & 0 & 4 \\ -2 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 10 \\ -5 & 15 & -10 & 0 & 0 & 1 & 0 & 0 & 8 \\ \hline -4 & -10 & -5 & 0 & 0 & 0 & -M & -M & 0 \end{array}\right]$$

Eliminate M in the artificial columns $\Rightarrow$

$$R_3-(-M\cdot R_1)\to R_3$$

$$R_3-(-M\cdot R_2)\to R_3$$

$$\left[\begin{array}{cccccccc|c} x & y & z & s_1 & s_2 & s_3 & a_1 & a_2 & b \\ \hline 5 & 20 & 15 & -1 & 0 & 0 & 1 & 0 & 4 \\ -2 & 2 & 0 & 0 & -1 & 0 & 0 & 1 & 10 \\ -5 & 15 & -10 & 0 & 0 & 1 & 0 & 0 & 8 \\ \hline -3M+4 & -22M+10 & -15M+5 & M & M & 0 & 0 & 0 & -14M \end{array}\right]$$

The pivot is $M_{12}=20\Rightarrow$

$$R_1/20\to R_1$$
$$R_2-2R_1\to R_2$$
$$R_3-15R_1\to R_3$$
$$R_4-(-22M+10)R_1\to R_4\Rightarrow$$

$$\left[\begin{array}{cccccccc|c}
x & y & z & s_1 & s_2 & s_3 & a_1 & a_2 & b \\
\hline
1/4 & 1 & 3/4 & -1/20 & 0 & 0 & 1/20 & 0 & 1/5 \\
-5/2 & 0 & -3/2 & 1/10 & -1 & 0 & -1/10 & 1 & 48/5 \\
-35/4 & 0 & -85/4 & 3/4 & 0 & 1 & -3/4 & 0 & 5 \\
\hline
5/2M + 3/2 & 0 & 3/2M - 5/2 & -1/10M + 1/2 & M & 0 & 11/10M - 1/2 & 0 & -48/5M - 2
\end{array}\right]$$

The new pivot is $M_{43} = 3/4 \Rightarrow$

$$R_3/3/4 \to R_3$$
$$R_1 + 1/20R_3 \to R_1$$
$$R_2 - 1/10R_3 \to R_2$$
$$R_4 - (-\frac{1}{10}M + \frac{1}{2})R_3 \to R_4 \Rightarrow$$

$$\left[\begin{array}{cccccccc|c}
x & y & z & s_1 & s_2 & s_3 & a_1 & a_2 & b \\
\hline
-1/3 & 1 & -2/3 & 0 & 0 & 1/15 & 0 & 0 & 8/15 \\
-4/3 & 0 & 4/3 & 0 & -1 & -2/15 & 0 & 1 & 134/15 \\
-35/3 & 0 & -85/3 & 1 & 0 & 4/3 & -1 & 0 & 20/3 \\
\hline
4/3M + 22/3 & 0 & -4/3M + 35/3 & 0 & M & 2/15M - 2/3 & M & 0 & -134/15M - 16/3
\end{array}\right]$$

The new pivot is $M_{33} = 4/3 \Rightarrow$

$$R_2/4/3 \to R_2$$
$$R_1 + 2/3R_2 \to R_1$$
$$R_3 + 85/3R_2 \to R_3$$
$$R_4 - (-\frac{4}{3}M + \frac{35}{3})R_2 \to R_4 \Rightarrow$$

$$\left[\begin{array}{cccccccc|c}
x & y & z & s_1 & s_2 & s_3 & a_1 & a_2 & b \\
\hline
-1 & 1 & 0 & 0 & -1/2 & 0 & 0 & 1/2 & 5 \\
-1 & 0 & 1 & 0 & -3/4 & -1/10 & 0 & 3/4 & 67/10 \\
-40 & 0 & 0 & 1 & -85/4 & -3/2 & -1 & 85/4 & 393/2 \\
\hline
19 & 0 & 0 & 0 & 35/4 & 1/2 & M & M - 35/4 & -167/2
\end{array}\right]$$

The bases are $\{x_2, x_3, s_1\}$, the solutions are $\Rightarrow$

$$\begin{cases} x = 0 \\ y = 5 \\ z = 6.7 \\ s_1 = 393/2 \\ w = -167/2 \end{cases}$$

Listing 2: Check results in Python

```python
from pulp import *

# Create a Linear Programming model
model = pulp.LpProblem('linear_programming', LpMaximize)

# Get the solver
solver = getSolver('PULP_CBC_CMD')

# Declare decision variables
x1 = LpVariable('x1', lowBound=0, cat='Continuous')
x2 = LpVariable('x2', lowBound=0, cat='Continuous')
x3 = LpVariable('x3', lowBound=0, cat='Continuous')

# Declare the objective function
model += -4 * x1 + (-5 * x2) + (-10 * x3)

# Declare constraints
model += 5 * x1 + 20 * x2 + 15 * x3 >= 4
model += -2 * x1 + 2 * x2 + 0 * x3 >= 10
model += 5 * x1 + (-15 * x2) + 10 * x3 >= -8

# Solve the model
results = model.solve(solver=solver)

# Print results
if LpStatus[results] == 'Optimal':
    print('The solution is optimal.')
print(f'Objective value: z* = {value(model.objective)}')
print(f'Solution: x1* = {value(x1)}, x2* = {value(x2)}, x3* = {value(x3)}')
```

PartIII

Q1

Try to solve the problem using the simplex method : what happens? How do you explain this? $\max z = 2x + y$

$$\begin{cases} x - 2y \leq 2 \\ -2x + y \leq 2 \\ x, y \geq 0 \end{cases}$$

Solution:

$$\left[\begin{array}{cccc|c} x & y & s_1 & s_2 & b \\ \hline 1 & -2 & 1 & 0 & 2 \\ -2 & 1 & 0 & 1 & 2 \\ \hline -2 & -1 & 0 & 0 & 0 \end{array}\right]$$

The pivot is $M_{12} = -2 \Rightarrow$

$R_2/2 \to R_2$

$R_1 - R_2 \to R_1$

$R_3 - 2R_2 \to R_3 \Rightarrow$

$$\left[\begin{array}{cccc|c} x & y & s_1 & s_2 & b \\ \hline 0 & -1.5 & 1 & 0.5 & 1 \\ -1 & 0.5 & 0 & 0.5 & 1 \\ \hline 0 & -2 & 0 & 0 & 0 \end{array}\right]$$

The pivot is $M_{21} = -1.5 \Rightarrow$

$R_1/-1.5 \to R_1$

$2R_2 - R_1 \to R_2$

$R_3 - 2R_1 \to R_3 \Rightarrow$

$$\left[\begin{array}{cccc|c} x & y & s_1 & s_2 & b \\ \hline 0 & 1 & -2/3 & 1/3 & 2/3 \\ -2 & 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 2 & 1 & 2 \end{array}\right]$$

The basic values are 0, so there is a degeneracy. We can test the constraints firstly by $2R_1 + R_2 \Rightarrow y \leq -4/3$. It is contradictory with R_3

Q2

You're solving a linear problem by using the simplex method and at some point you get the tableau:

$$\left[\begin{array}{ccccccc|c} a & b & c & d & e & f & z & sol \\ \hline 0 & 0 & 0 & 1 & 1/2 & 0 & 0 & 1/2 \\ 0 & 2 & -4 & 0 & -3 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & -2 & 0 & 0 & 0 \\ \hline 0 & -2 & 1 & 0 & 4 & 0 & 1 & 4 \end{array}\right]$$

Explain what happens. Could this situation have been avoided?

Solution: The pivot column is b, but all values of sol/b is negative or 0 or NULL, that means the solution of this LPP is unbounded. Since the given information is not in detail, we can suppose this LPP can be transformed to its dual LPP. Once we find out the minimum optimal solution of the dual LPP, we find out the maximum of it.

PartIIII

Solve the following problem: $\max z = 12x + 20y$

$$\begin{cases} 6x + 10y \geq 60 \\ 8x + 25y \geq 200 \\ 2x + 8y \leq 80 \\ x, y \geq 0 \end{cases}$$

Solution:
After Big-M, we get:

$$\begin{cases} 6x + 10y - s_1 + a_1 = 60 \\ 8x + 25y - s_2 + a_2 = 200 \\ 2x + 8y + s_3 = 80 \quad \text{with } x, y \geq 0 \end{cases}$$

to find the maximum of w in $z = 12x + 20y - Ma_1 - Ma_2$. Create the simplex tableau:

$$\left[\begin{array}{ccccccc|c} x & y & s_1 & s_2 & s_3 & a_1 & a_2 & b \\ \hline 6 & 10 & -1 & 0 & 0 & 1 & 0 & 60 \\ 8 & 25 & 0 & -1 & 0 & 0 & 1 & 200 \\ 2 & 8 & 0 & 0 & 1 & 0 & 0 & 80 \\ \hline 12 & 20 & 0 & 0 & 0 & -M & -M & 0 \end{array}\right]$$

The pivot is $M_{21} = 10 \Rightarrow$

$$R_1/10 \to R_1$$
$$R_2 - 25R_2 \to R_2$$
$$R_3 - 8R_1 \to R_3$$
$$R_4 - 10R_1 \to R_4 \Rightarrow$$

$$\left[\begin{array}{cccccccc|c} x & y & s_1 & s_2 & s_3 & a_1 & a_2 & & b \\ \hline 3/5 & 1 & -1/10 & 0 & 0 & 1/10 & 0 & & 6 \\ -7 & 0 & 5/2 & -1 & 0 & -5/2 & 1 & & 50 \\ -14/5 & 0 & 4/5 & 0 & 1 & -4/5 & 0 & & 32 \\ \hline 7M & 0 & -5/2M-2 & M & 0 & 7/2M+2 & 0 & & -50M+120 \end{array}\right]$$

The pivot is $M_{32} = 5/2 \Rightarrow$

$$R_2/5/2 \to R_2$$
$$R_1 + 1/10R_2 \to R_1$$
$$R_3 + 8/10R_2 \to R_3$$
$$R_4 - (-5/2M - 2)R_2 \to R_4 \Rightarrow$$

$$\left[\begin{array}{ccccccc|c} x & y & s_1 & s_2 & s_3 & a_1 & a_2 & b \\ \hline 8/25 & 1 & 0 & -1/25 & 0 & 0 & 1/25 & 8 \\ -14/5 & 0 & 1 & -2/5 & 0 & -1 & 2/5 & 20 \\ -14/25 & 0 & 0 & 8/25 & 1 & 0 & -8/25 & 16 \\ \hline -8/25 & 0 & 0 & -4/5 & 0 & M & M+4/5 & 160 \end{array}\right]$$

The pivot is $M_{11} = 8/25 \Rightarrow$

$$R_1/8/25 \to R_1$$
$$R_2 + 14/5R_1 \to R_2$$
$$R_3 + 14/25R_1 \to R_3$$
$$R_4 - (-8/25R_1) \to R_4 \Rightarrow$$

$$\left[\begin{array}{ccccccc|c} x & y & s_1 & s_2 & s_3 & a_1 & a_2 & b \\ \hline 1 & 25/8 & 0 & -1/8 & 0 & 0 & 1/8 & 25 \\ 0 & 35/4 & 1 & -3/4 & 0 & -1 & 3/4 & 90 \\ 0 & 7/4 & 0 & 1/4 & 1 & 0 & -1/4 & 30 \\ \hline 0 & 35/2 & 0 & -3/2 & 0 & M & M+3/2 & 300 \end{array}\right]$$

The pivot is $M_{33} = 1/4 \Rightarrow$

$$R_3/1/4 \to R_3$$
$$R_1 + 1/8R_3 \to R_1$$
$$R_2 + 3/4R_3 \to R_2$$
$$R_4 - (-3/2R_3) \to R_4 \Rightarrow$$

$$\left[\begin{array}{ccccccc|c}
x & y & s_1 & s_2 & s_3 & a_1 & a_2 & b \\
\hline
1 & 4 & 0 & 0 & 1/2 & 0 & 0 & 40 \\
0 & 14 & 1 & 0 & 3 & -1 & 0 & 180 \\
0 & 7 & 0 & 1 & 4 & 0 & -1 & 120 \\
\hline
0 & 35/2 & 0 & -3/2 & 0 & M & M+3/2 & 300
\end{array}\right]$$

The bases are $\{x_1, s_1, s_2\}$, the solutions are $\Rightarrow$

$$\begin{cases} x = 40 \\ y = 0 \\ s_1 = 180 \\ s_2 = 120 \\ z = 480 \end{cases}$$

Listing 3: Check results in Python

```
from pulp import *

# Create a Linear Programming model
model = pulp.LpProblem('linear_programming', LpMaximize)

# Get the solver
solver = getSolver('PULP_CBC_CMD')

# Declare decision variables
x1 = LpVariable('x1', lowBound=0, cat='Continuous')
x2 = LpVariable('x2', lowBound=0, cat='Continuous')

# Declare the objective function
model += 12 * x1 + 20 * x2

# Declare constraints
model += 6 * x1 + 10 * x2 >= 60
model += 8 * x1 + 25 * x2 >= 200
model += 2 * x1 + 8 * x2 <= 80

# Solve the model
results = model.solve(solver=solver)

# Print results
if LpStatus[results] == 'Optimal':
    print('The solution is optimal.')
print(f'Objective value: z* = {value(model.objective)}')
print(f'Solution: x1* = {value(x1)}, x2* = {value(x2)}')
```

Memo

8 hours work, 30 A4 pages for calculations...
Life is short, you need Python