Linear algebra assignment

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PartI

Let's consider the following set of 5 vectors with real components :

$$\{\begin{bmatrix}5\\0\end{bmatrix},\begin{bmatrix}0\\-1\end{bmatrix},\begin{bmatrix}11\\3\end{bmatrix},\begin{bmatrix}3\\2\end{bmatrix},\begin{bmatrix}5\\2\end{bmatrix}\}$$

Q1

Which subset of this set of vectors is a basis for the vector space of dimension 2 with real components? Why?

Solution: Since it is the set of vectors, it can be denoted in matrix format:

$$S = \begin{bmatrix}5 & 0 & 11 & 3 & 5\\0 & -1 & 3 & 2 & 2\end{bmatrix} \tag{1}$$

Using $r_j, j \in \mathbb{N}$ represent row, we reduce the row of this matrix: apply $r_1/5$ and $r_2 \cdot -1$, we get:

$$S = \begin{bmatrix}1 & 0 & 2.2 & 0.6 & 1\\0 & 1 & -3 & -2 & -2\end{bmatrix} \tag{2}$$

The pivot columns present the basis vector of the set of vectors. Specifically, the basis vectors are:

$$\{\begin{bmatrix}5\\0\end{bmatrix},\begin{bmatrix}0\\-1\end{bmatrix}\}$$

Q2

We define the 2x2 matrix, $A = \begin{bmatrix}3 & 5\\2 & 2\end{bmatrix}$, If we multiply A and a vector u, we get a vector Au which can be seen as a linear combination of 2 vectors : except for the canonical basis, what obvious set of 2 vectors can it be?

Solution: Obviously, Au can be considered as the transformation of the basis $\{\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}5\\2\end{bmatrix}\}$

Q3

Does A^{-1} exist? Why?

Solution: Since A is a squared matrix and the determinate of A is $\det(A) = 3*2-5*2 = -4 \neq 0$, so the inverse of A exists.

Q4

Use 2 different ways to find the vector x such as $Ax = \begin{bmatrix}2\\-4\end{bmatrix}$

Solution: Suppose $x = \begin{bmatrix}a\\b\end{bmatrix}$

S1. The product of the matrix and vector:

$$\begin{aligned}
Ax = \begin{bmatrix}3 & 5\\2 & 2\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}3a+5b\\2a+2b\end{bmatrix} = \begin{bmatrix}2\\-4\end{bmatrix} \quad (1)\\
\Rightarrow \begin{cases}3a+5b=2\\2a+2b=-4\end{cases} \quad (2)\\
\Rightarrow \begin{cases}a=-6\\b=4\end{cases} \quad (3)\\
\Rightarrow x = \begin{bmatrix}-6\\4\end{bmatrix} \quad (4)
\end{aligned}$$

S2. Use A^{-1}:

$$\begin{aligned}
Ax &= \begin{bmatrix}2\\-4\end{bmatrix} \quad (1)\\
Ax \cdot A^{-1} &= \begin{bmatrix}2\\-4\end{bmatrix} \cdot A^{-1} \quad (2)\\
x &= \begin{bmatrix}-0.5 & 1.25\\0.5 & -0.75\end{bmatrix} \cdot \begin{bmatrix}2\\-4\end{bmatrix} = \begin{bmatrix}-6\\4\end{bmatrix} \quad (3)
\end{aligned}$$

Q5

What are the eigenvalues and eigenvectors of A?

Solution: Suppose λ, u presents the eigenvalues and eigenvectors, we get: $Au = \lambda u$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 5 \\ 2 & 2-\lambda \end{bmatrix} = 0 \Rightarrow \lambda^2 - 5\lambda - 4 = 0 \Rightarrow \lambda = \frac{5-\sqrt{41}}{2}, \frac{5+\sqrt{41}}{2} \tag{1}$$

$$\text{for simplify the computation of eigenvectors, we propose: } \lambda = \begin{cases} -0.7, \\ 5.7 \end{cases} \tag{2}$$

$$\text{Let } u = \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

$$\text{For } \lambda = -0.7, (A - \lambda I)u = \begin{bmatrix} 3.7 & 5 \\ 2 & 2.7 \end{bmatrix} u = 0 \Rightarrow \begin{cases} 3.7x + 5y = 0 \\ 2x + 2.7y = 0 \end{cases} \Rightarrow \begin{cases} x = -1.35y \\ y = y \end{cases} \tag{4}$$

$$\text{Let } y = 1, \text{we get: } u = \begin{bmatrix} -1.35 \\ 1 \end{bmatrix} \tag{5}$$

$$\text{For } \lambda = 5.7, \text{after the same process, we get: } u = \begin{bmatrix} 1.85 \\ 1 \end{bmatrix} \tag{6}$$

PartII

Q1

What are the eigenvalues and eigenvectors of the matrix

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution:

$$(M - \lambda I)u = 0 \Rightarrow \det(M - \lambda I) = 0 \Rightarrow \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0 \Rightarrow \begin{cases} \lambda = 1 \\ \lambda = 2 \\ \lambda = 3 \end{cases} \tag{1}$$

$$\text{For } \lambda = 1 : (M - I)u = 0 \Rightarrow u = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \text{ let } x = 1, \text{we get: } u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

$$\text{For } \lambda = 2 : u = \begin{bmatrix} x \\ x \\ 0 \end{bmatrix}, \text{ let } x = 1, \text{we get: } u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \tag{3}$$

$$\text{For } \lambda = 3 : u = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}, \text{ let } z = 1, \text{we get: } u = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

Q2

Is the matrix M diagonalizable and why? If this is the case, what does a diagonal matrix similar to it looks like?

Solution:
Since M is a squared matrix, we got the eigenvalues of M and they are unique, so we can say M is diagonalizable.
The similar one is its corresponding diagonal matrix D (Descending order of Eigenvalues)

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q3

Calculate M^5

Solution:

$$
\begin{aligned}
\text{Diagonalize } M: \quad M &= CDC^{-1} && (1) \\
C &= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (2) \\
C^{-1} &= \frac{1}{\det(C)} \cdot C^T = -1 \cdot \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} && (3) \\
M^5 &= CDC^5C^{-1} && (4) \\
&= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 243 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} && (5) \\
&= \begin{bmatrix} 0 & 32 & 1 \\ 0 & 32 & 0 \\ 243 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} && (6) \\
&= \begin{bmatrix} 1 & 31 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{bmatrix} && (7)
\end{aligned}
$$

Q4

Based on c), what is M^4?

Solution:

$$
\begin{aligned}
M^4 &= CDC^4C^{-1} && (1) \\
&= \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 81 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} && (2) \\
&= \begin{bmatrix} 1 & 15 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix} && (3)
\end{aligned}
$$

PartIII

We define the 2×2 matrix $A = \begin{bmatrix} -5 & 3 \\ 6 & 2 \end{bmatrix}$

Q1

If we multiply A and a vector u, we get a vector Au which can be seen as a linear combination of 2 vectors : except for the canonical basis, what obvious set of 2 vectors can it be?

Solution:
Obiviously, the set of 2 basis vectors are $\{\begin{bmatrix}-5\\6\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}\}$

Q2

What are the eigenvalues and eigenvectors of A?

Solution:

$$(A-\lambda I)u = 0 \tag{1}$$

$$\det\left(\begin{bmatrix}-5-\lambda & 3\\ 6 & 2-\lambda\end{bmatrix}\right) = 0 \Rightarrow \begin{cases}\lambda & = -7\\ \lambda & = 4\end{cases} \tag{2}$$

$$\text{For } \lambda = -7, (A-(-7)I)u = 0 \Rightarrow x = -1.5y, \text{ let } y = 1, \text{we get:} \tag{3}$$

$$u = \begin{bmatrix}-1.5\\1\end{bmatrix} \tag{4}$$

$$\text{For } \lambda = 4, (A-4I)u = 0 \Rightarrow y = 3x, \text{ let } x = 1, \text{we get:} \tag{5}$$

$$u = \begin{bmatrix}1\\3\end{bmatrix} \tag{6}$$

Q3

Prove that A is diagonalizable.

Solution:
Obviously, A is a squared matrix and the eigenvalues of A are unique. So, we can say A is diagonalizable

Q4

Deduce from b) and c) that there exists a matrix B such as $B^3 = A$

Solution:

$$B^3 = A \Rightarrow \tag{1}$$

$$B = A^{\frac{1}{3}} = CD^{\frac{1}{3}}C^{-1} \tag{2}$$

$$= \begin{bmatrix} 1 & -1.5 \\ 3 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 4 & 0 \\ 0 & -7 \end{bmatrix} \right)^{\frac{1}{3}} \cdot \begin{bmatrix} \frac{2}{11} & \frac{3}{11} \\ \frac{-6}{11} & \frac{2}{11} \end{bmatrix} \tag{3}$$

$$\text{Obviously, there exists a solution} \tag{4}$$

$$\text{In conclusion, } B \text{ exists} \tag{5}$$