CS5800 Algorithms	*Out: 18 September 2021*
Problem Set 2	
Ravi Sundaram	*Due: 28 September 2021*

Problem 1 (Recurrences) **12+8**

a. (a) $T(n) = 3T(n/2) + n$ (This is the recursion for Karatsuba's Algorithm)

(b) $T(n) = 3T(n/2) + n^2$

(c) $T(n) = 2T(n/4) + T(n/2) + n$ (Hint: Think of the number of nodes at each level. You can determine it by solving a recurrence).

(d) $T(n) = 5T(n/4) + n$

b. Use the **recursion tree method** to solve the following recurrence:

$$T(n) = 3T(n/3) + n; \quad T(1) = 1$$

Solution:

a. (a) $T(n) = \Theta(n^{\log_2 3})$
Proof: $a = 3, b = 2 \Rightarrow n^{\log_b a} = n^{\log_2 3}; f(n) = n$.
Case 1 of Master Theorem: $f(n) = O(n^{\log_2 3 - \epsilon})$ where $\epsilon = \log_2 3 - 1$, then $T(n) = \Theta(n^{\log_2 3})$.

(b) $T(n) = \Theta(n^2)$
Proof: $a = 3, b = 2 \Rightarrow n^{\log_b a} = n^{\log_2 3}; f(n) = n^2$.
Case 3 of Master Theorem: $f(n) = \Omega(n^{\log_2 3+\epsilon})$ where $\epsilon = 2 - \log_2 3$ and $3f(n/2) \leq \frac{3}{4}f(n)$ where $c = \frac{3}{4}$, then $T(n) = \Theta(n^2)$.

(c) $T(n) = \Theta(n \log n)$

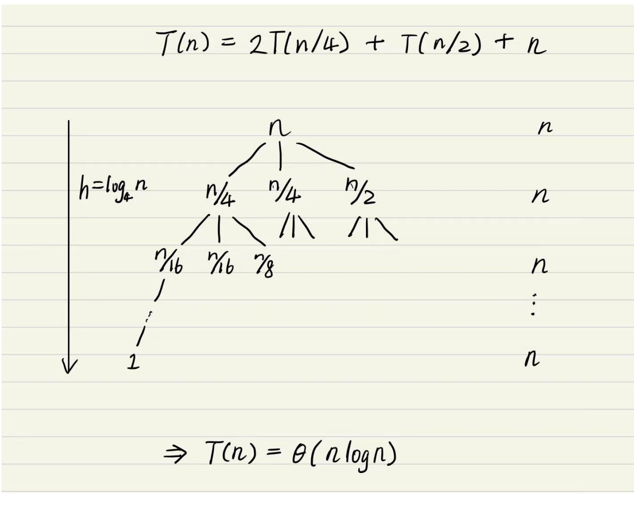

Figure 1: The recursion tree of Problem 1a(c).

(d) $T(n) = \Theta(n^{\log_4 5})$
Proof: $a = 5, b = 4 \Rightarrow n^{\log_b a} = n^{\log_4 5}; f(n) = n$.
Case 1 of Master Theorem: $f(n) = O(n^{\log_4 5 - \epsilon})$ where $\epsilon = \log_4 5 - 1$, then $T(n) = \Theta(n^{\log_4 5})$.

b. $T(n) = \Theta(n \log n)$
Proof:

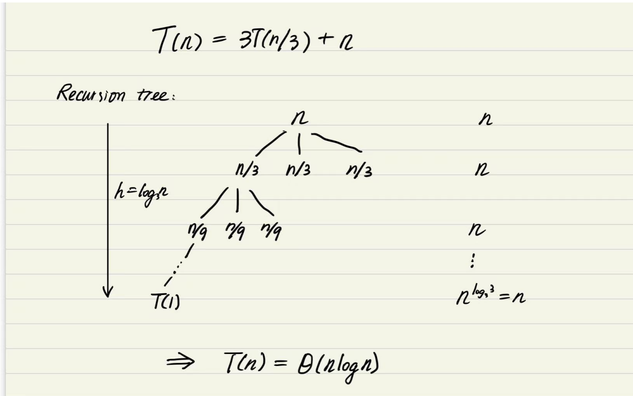

Figure 2: The recursion tree of Problem 1b.

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Problem 2 (Rotated and Sorted) 10+10

Suppose you are given a array of n distinct numbers which was sorted and then rotated k indices, for some unknown integer k between 1 and $n-1$. That is, you are given an array $A[1..n]$ such that some prefix $A[1..k]$ is sorted in increasing order, the corresponding suffix $A[k+1..n]$ is also sorted in increasing order, and $A[n] < A[1]$. For example, you might be given the following 16-element array (where $k = 10$): $[9, 13, 16, 20, 21, 24, 25, 26, 27, 30, ||, 1, 2, 3, 5, 6, 8]$

a) Describe and analyze an algorithm to compute the value of k in $\Theta(\log n)$ time.

b) Describe and analyze an algorithm to determine if the given array contains a given number x that runs in $\Theta(\log n)$ time assuming you already know the value of k.

Solution:

1. Use binary search over the array to find k, as the pseudo code indicates. The time complexity is $T(n) = T(n/2) + O(1) = O(\log n)$.

```
start = 0, end = len(A) - 1
while start < end:
  mid = (start + end) / 2
  if A[mid] > A[end]:
      start = mid + 1
  else:
      end = mid
return start
```

2. Use binary search twice over the subarrays $A[1..k]$ and $A[k+1..n]$. If the number x is found in either subarray, return that the array contains x. Otherwise, return that the array does not contain x. The time complexity of this approach is $T(n) = T(k) + T(n-k) = O(\log k) + O(\log(n-k)) = O(\log n)$.

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Problem 3 (Hex) 20

Hex is a two-player game played on a diamond-shaped board made up of hexagons. In the game, one player plays white pieces, while the other plays black, with play alternating between players and placement only allowed on unoccupied hexagons. Alternate sides of the board are designated white and black as shown above, and the goal of the game is to complete a chain of pieces between one player's two sides.

Give an algorithm that takes as input the black pieces and white pieces on a board of length n on each side, and determines whether that configuration is a winning configuration for either player. Your algorithm must employ the Union-Find data structure. See accompanying figure for an illustration.

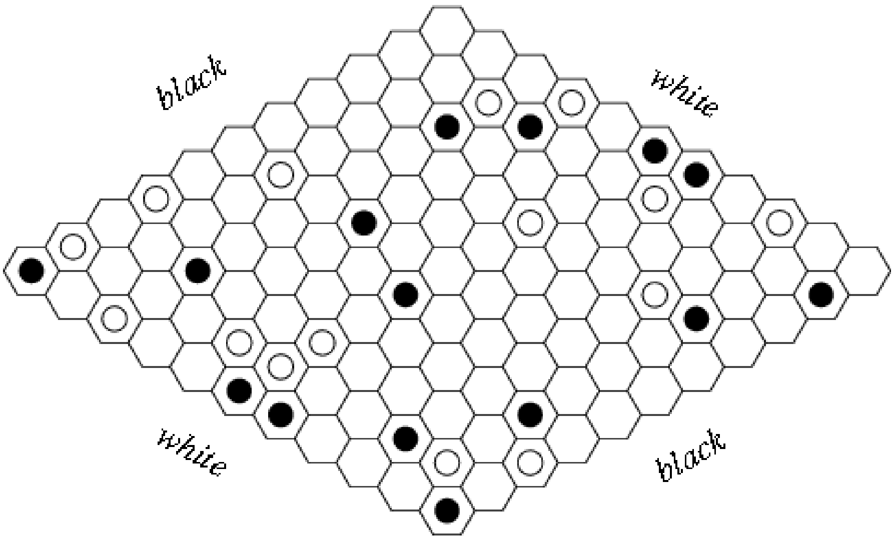

Figure 3: The game is usually played on a boards of size 11 on a side, for a total of 121 hexagons. Reference: http://mathworld.wolfram.com/GameofHex.html

Problem 4 (Thresholded Inversions and Doctor Manhattan) 20+20

a. You'll be given an array of integers a_i and a threshold value t as input, describe and analyze an algorithm to determine the number of threshold inversions in the array. An inversion between indices $i < j$ is a threshold inversion if $a_i > t * a_j$, where t is the threshold value given as input.

   Hint: Understanding counting inversions (normal inversions, without a threshold) will be useful to figure out how to do threshold inversions. We will be covering that in the recitations.

b. **Description**: Assist Doctor Manhattan in his attempt to preemptively save the world.

   Problem Statement: Doctor Manhattan does not see time as we know it. Due to his perception of time, he sees his past, present and future simultaneously. He has recorded tachyon levels in the present and near future. A drastic change in tachyon levels at two points in time is a critical event that should be brought to his attention since it may be cause for immense concern. It is your job to help Doctor Manhattan find these critical events. Given an array of tachyon levels in distinct points of time, compute the number of critical events.

You are given an array a which denotes tachyon levels for different points of time and a threshold factor t. For two tachyon levels, it is considered a critical event if for some $i < j$ we have $a[i] > t * a[j]$. You have to count the total number of such critical events. **Input Format**: First Line contains a single number which represent the threshold level. i.e. t

Second Line contains the size of the array i.e. n

Third line contains n elements separated by space i.e. $a[1]\ a[2] \ldots\ a[n]$

Constraints: $0 < n < 10^5$

$0 < a[i] < 10^9$

$1 < t < 500$

Output Format: Just a single number i.e. total count of critical events

Print the answer modulo: $10^9 + 7$.

Solution:

HankerRank username: **zhang_lei1**. ■