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Inference of the Mean Outcome under an Optimal Treatment Regime in High Dimensions

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Data from medical studies are often characterized by some degree of heterogeneity. Patients may differ significantly in the way they respond to the treatment. Over the past few decades, there have been extensive developments of individualized optimal treatment regimes (OTRs) to account for patients' heterogeneity in response to treatment. However, less has been done to quantify the impact of implementing an OTR, especially when there exists a large number of prognostic variables. This paper is concerned with statistical inference of the mean outcome under an OTR (the optimal value function) under the ultrahigh dimensional setting where the number of prognostic covariates is allowed to grow exponentially fast with respect to the sample size. We propose to estimate the heterogeneous treatment effects by some regularized regression methods and develop a valid inference procedure for the optimal value function based on the (augmented) inverse propensity-score weighted estimator ((A)IPWE). The proposed estimator for the optimal value function is $\sqrt{n}$-consistent as long as the penalized regression estimator satisfies certain convergence rates and geometrical constraints. It also preserves the double robustness property of AIPWE under the ultrahigh dimensional setting. Moreover, it is semiparametric efficient when the propensity score function and the conditional mean functions are correctly specified.

Keywords: optimal treatment regime, ultrahigh dimensions, statistical inference, optimal value function, double robustness, semiparametric efficiency.

1. Introduction

Data from medical studies are often characterized by some degree of heterogeneity. Patients may differ significantly in the way they respond to the treatment. In contrast to the classical "one size fits all" approach, precision medicine focuses on finding the most effective treatment decision that assigns the best treatment to individual patient based on his/her covariates. Such a decision rule is formally referred to as an individualized optimal treatment regime (OTR). In the literature, there have been extensive developments of OTRs to account for patient's heterogeneity in response to treatment. Some popular approaches include Q-learning [20, 3], A-learning [14, 11], outcome weighted learning [24] and value-search methods [22]. To handle high dimensional covariates, [13] proposed to construct the OTR by estimating the conditional mean of the response with ℓ_1 penalty function. [8] introduced a penalized least square regression framework for identifying important variables that are involved in an OTR. [15] developed a penalized A-learning method for deriving the OTR.

Prior to adopting any OTR in clinical practice, it is crucial to know the impact of implementing such an OTR in the population. This requires to evaluate the mean outcome (value function) under the OTR. Despite the popularity of estimating the OTR in the existing literature, less attention has been devoted to statistical inference of the value function. [22] developed an inference procedure for the optimal value function in restricted classes where the treatment regimes are indexed by a finite-dimensional vector. [10] proposed a two-step procedure to construct a calibrated OTR and developed an inference procedure to evaluate its value. [2] considered inference for the value of an estimated OTR using the m-out-of-n bootstrap. [9] proposed to construct the CI for the optimal value function based on their proposed one-step online estimator. However, all these methods assume the number of the covariates is fixed and it remains unknown whether these procedures are valid in high dimensions. [16] extended [8]'s proposal to a high dimensional setup. They proposed to construct the OTR by penalized least square estimation and evaluate the optimal value function using the empirical average of the advantage function [11]. However, the asymptotic normality of their estimated optimal value function relies crucially on the oracle property (i.e, model selection consistency and asymptotic normality) of the penalized least square estimator. Besides, their procedure requires correct specification of the propensity score model and is not double robust.

This paper is concerned with statistical inference of the optimal value function under the ultrahigh dimensional setting, assuming a well-specified linear model on the conditional treatment effects. The number of pretreatment variables is allowed to grow exponentially fast with respect to the sample size. Given linearity and sparsity assumptions, the conditional treatment effects can be consistently estimated by some regularized regression procedures. We propose to estimate the optimal value function using the (augmented) inverse propensity-score weighted estimator [(A)IPWE, 22]. The technical challenges in the derivation of the theoretical properties are summarized as follows. First, the asymptotic properties of IPWE is not standard due to the curse of dimensionality. When the dimension of the covariates is fixed, [22] outlined an argument showing that their estimated optimal value function based on IPWE is asymptotically the same as the same estimator had we known the OTR in advance. However, such results no longer hold in ultrahigh dimensions. Second, the computation of AIPWE requires the specification of propensity score and conditional mean functions. In observation studies, these functions are unknown and need to be estimated. The derivation of the asymptotic properties of AIPWE is thus complicated by the estimated propensity score and conditional mean functions. Moreover, it is well-known that AIPWE is doubly robust in fixed p case. However, it remains unknown whether the double robustness property preserves in ultrahigh dimensions.

One of the key findings of this paper is that, the asymptotic normality of (A)IPWE doesn't require the oracle property of the regularized estimator for the conditional treatment effects, nor does it require the estimator to be root-n consistent. Moreover, bias correction procedures [see for example, 18, 12] are not needed for the regularized estimator. We prove the proposed estimator for the optimal value function is asymptotically normal, as long as the regularized estimator satisfies certain convergence rates and geometrical constraints. We further propose a consistent variance estimator for the estimated value function. To deal with data from observational studies, we propose to estimate the propensity score and conditional mean functions via penalized regression with folded-concave penalty functions. Another technical contribution of this paper is to study the oracle properties of these penalized estimators in a random design setting, allowing the model to be misspecified. With the established oracle properties, we show the proposed estimated value function preserves the double robustness property even in ultrahigh dimensions.

The rest of the paper is organized as follows. In Section 2, we present the definitions of the OTR and the optimal value function. In Section 3, we introduce the proposed inference procedure for the optimal value function and study the asymptotic properties of the estimated optimal value function. We further propose a doubly robust inference procedure in Section 4. In Section 5, we study the oracle properties of the penalized estimators in propensity score and conditional mean models. Proofs of Theorem 3.1 and Theorem 4.1 are presented in Section 6. Finally, we conclude our paper by a discussion section. The proof of Theorem 5.1 is given in the supplementary material.

2. Optimal treatment regime

For simplicity, we only consider a single-stage study. Let $\boldsymbol{X}_0 \in \mathbb{R}^p$ be patient's baseline covariates, $A_0 \in \{0,1\}$ denote the binary treatment a patient receives, and Y_0 denote a patient's clinical outcome (the larger the better by convention). A treatment regime $d(\cdot)$ is a deterministic function that maps $\boldsymbol{X}_0$ to $\{0,1\}$. Let $Y_0^*(0)$ and $Y_0^*(1)$ be a patient's potential outcomes, representing the response he/she would get if treated by treatment 0 and 1, respectively. In addition, define the potential outcome

$$Y_0^*(d) = Y_0^*(0)\{1 - d(\boldsymbol{X}_0)\} + Y_0^*(1) d(\boldsymbol{X}_0),$$

representing the response a patient would have if treated according to a treatment regime d. Let $V(d) = \mathrm{E}\{Y_0^*(d)\}$. The optimal treatment regime (OTR) d^{opt} is defined as the maximizer of the expected potential outcome $V(d)$ among the set of all possible treatment regimes, i.e,

$$d^{opt} \equiv \arg\max_d V(d).$$

Assume the following two assumptions hold.

(A1.) SUTVA: $Y_0 = (1 - A_0)Y_0^*(0) + A_0 Y_0^*(1)$.
(A2.) No unmeasured confounders: $Y_0^*(0), Y_0^*(1) \perp\!\!\!\perp A_0 | \boldsymbol{X}_0$.
Then we can show

$$d^{opt}(\boldsymbol{x}) = I\{\tau(\boldsymbol{x}) > 0\}, \qquad \forall \boldsymbol{x} \in \mathbb{R}^p, \tag{2.1}$$

where $\tau(\cdot)$ is the contrast function,

$$\tau(\boldsymbol{x}) \equiv \mathrm{E}\{Y_0|A_0=1,\boldsymbol{X}_0=\boldsymbol{x}\} - \mathrm{E}\{Y_0|A_0=0,\boldsymbol{X}_0=\boldsymbol{x}\}.$$

We assume $\tau(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{\beta}_0$ for some $\boldsymbol{\beta}_0 \in \mathbb{R}^p$. Such a linearity assumption is strong, but in high dimensions it is needed for inference to be possible. By (2.1), it is immediate to see that finding the OTR is equivalent to estimating the high dimensional parameter $\boldsymbol{\beta}_0$. The focus of this paper is to construct a two-sided CI for the optimal value function $V_0 \equiv V(d^{opt})$.

3. Estimation and inference for the optimal value

Let $\pi(\cdot)$ be the propensity score function, i.e, $\pi(\boldsymbol{x}) = \Pr(A_0=1|\boldsymbol{X}_0=\boldsymbol{x})$. We assume $\pi(\cdot)$ is known throughout this section. Such an assumption is satisfied in a randomized study where $\pi(\cdot)$ is usually a known constant by design. In Section 4, we allow the propensity score to be estimated from data as in observational studies. Let $(\boldsymbol{X}_i, A_i, Y_i)_{i=1,\dots,n}$ be i.i.d copies of $(\boldsymbol{X}_0, A_, Y_0)$. We assume the dimension p is much larger than n and satisfies $\log p = O(n^{a_0})$ for some $0 < a_0 < 1$. Let $\pi_i = \pi(\boldsymbol{X}_i)$. For any treatment regime d, [22] proposed an inverse propensity-score weighted estimator (IPWE),

$$\widehat{V}(d) = \frac{1}{n}\sum_{i=1}^n \frac{A_i d(\boldsymbol{X}_i) + (1-A_i)\{1-d(\boldsymbol{X}_i)\}}{A_i\pi_i + (1-A_i)(1-\pi_i)} Y_i,$$

to estimate $V(d)$. If a treatment regime d takes the form $d(\boldsymbol{x}) = I(\boldsymbol{x}^T\boldsymbol{\beta} > 0)$ for some $\boldsymbol{\beta} \in \mathbb{R}^p$, we use a shorthand and write $V(d)$ as $V(\boldsymbol{\beta})$. We propose to estimate V_0 by $\widehat{V}(\widehat{\boldsymbol{\beta}})$, where $\widehat{\boldsymbol{\beta}}$ is some consistent estimator for $\boldsymbol{\beta}_0$ and satisfies some geometrical constraints.

For any vector $\boldsymbol{a} \in \mathbb{R}^q$, let $\|\boldsymbol{a}\|_0$ be the number of nonzero elements in $\boldsymbol{a}$. Define $\|\boldsymbol{a}\|_1 = \sum_{j=1}^q |a_j|$, $\|\boldsymbol{a}\|_2 = \sqrt{\sum_{j=1}^q a_j^2}$ and $\|\boldsymbol{a}\|_\infty = \max_{j=1}^q |a_j|$. Below, we establish the asymptotic normality of $\sqrt{n}\{\widehat{V}(\widehat{\boldsymbol{\beta}}) - V_0\}$. We need the following conditions.

(A3.) Assume $\Pr(\|\widehat{\boldsymbol{\beta}}\|_0 \le \kappa_n) \to 1$ for some $1 \le \kappa_n = o(n)$.
(A4.) Assume $\max(n^{1/4}, \kappa_n \log p)\|\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\|_1 = o_p(1)$.
(A5.) Assume $\|\boldsymbol{X}_0\|_\infty \le \omega_0$ for some constant $\omega_0 > 0$.
(A6.) Assume there exist some positive constants c_1, c_2 such that $0 < c_1 \le \pi(\boldsymbol{x}) \le c_2 < 1$, $\forall \boldsymbol{x} \in \mathbb{R}^p$.
(A7.) Assume $\sup_{\boldsymbol{x}} \mathrm{E}(Y_0^2|\boldsymbol{X}_0 = \boldsymbol{x}) \le c_0$ for some constant $c_0 > 0$. Further assume $\|Y_0\|_{\psi_1} \le \bar{\omega}$ for some constant $\bar{\omega} > 0$ where

$$\|Y_0\|_{\psi_1} \equiv \inf\left\{C > 0 : \mathrm{E}\exp\left(\frac{|Y_0|}{C}\right) \le 2\right\}.$$

(A8.) Assume there exist some positive constants $\bar{c}$ and δ_0 such that

$$\Pr(|\boldsymbol{X}_0^T\boldsymbol{\beta}_0| \le \varepsilon) \le \bar{c}\varepsilon,$$

for any $0 \leq \varepsilon \leq \delta_0$.

Condition (A3) requires the estimator $\widehat{\boldsymbol{\beta}}$ to be sparse. Note that we don't require $\boldsymbol{\beta}_0$ to be sparse. The parameter $\boldsymbol{\beta}_0$ is allowed to contain many small but nonzero coefficients. Condition (A4) assumes $\widehat{\boldsymbol{\beta}}$ satisfies certain convergence rates. It automatically holds if $\widehat{\boldsymbol{\beta}}$ is estimated by the penalized least square method [16] or the penalized A-learning method [15], under reasonable assumptions on the covariates. Condition (A8) holds when the random variable $\boldsymbol{X}_0^T \boldsymbol{\beta}_0$ has a bounded probability density function. This together with Condition (A4) and Condition (A5) guarantees that

$$\sqrt{n}\{V(\widehat{\boldsymbol{\beta}}) - V_0\} = o_p(1). \tag{3.1}$$

Besides, under the given conditions, we can show

$$\sqrt{n}\{\widehat{V}(\widehat{\boldsymbol{\beta}}) - \widehat{V}(\boldsymbol{\beta}_0) - V(\widehat{\boldsymbol{\beta}}) + V(\boldsymbol{\beta}_0)\} = o_p(1). \tag{3.2}$$

When the dimension of the covariates is fixed, (3.2) follows by some standard arguments in the empirical process theory [19]. However, such results no longer hold in the ultrahigh dimensions. A sufficient condition is given (A3) which requires $\widehat{\boldsymbol{\beta}}$ to be sparse. The number of the nonzero elements measures an estimator's "complexity", which is closely related to the notion of the VapnikChervonenkis (VC) index in the empirical process theory.

Combining (3.2) together with (3.1) yields $\sqrt{n}\{\widehat{V}(\widehat{\boldsymbol{\beta}}) - \widehat{V}(\boldsymbol{\beta}_0)\} = o_p(1)$. We summarize our results in the following theorem.

Theorem 3.1. *Assume (A1)-(A8) hold and $\kappa_n \log p \log n = o(\sqrt{n})$. Then we have*

$$\sqrt{n}\{\widehat{V}(\widehat{\boldsymbol{\beta}}) - \widehat{V}(\boldsymbol{\beta}_0)\} = o_p(1).$$

Recall that $V_0 = V(\boldsymbol{\beta}_0)$. The term $\sqrt{n}\{\widehat{V}(\boldsymbol{\beta}_0) - V_0\}$ corresponds to a sum of i.i.d mean zero random variables. Assume

$$\liminf \mathrm{Var}\left(\frac{A_0 I(\boldsymbol{X}_0^T \boldsymbol{\beta}_0 > 0) + (1 - A_0) I(\boldsymbol{X}_0^T \boldsymbol{\beta}_0 \leq 0)}{A_0 \pi(\boldsymbol{X}_0) + (1 - A_0)\{1 - \pi(\boldsymbol{X}_0)\}} Y_0 \right) > 0.$$

By the central limit theorem, we have $\sqrt{n}\{\widehat{V}(\boldsymbol{\beta}_0) - V_0\} \stackrel{d}{\to} N(0, \sigma_0^2)$ for some $\sigma_0^2 > 0$. This together with Theorem 3.1 gives

$$\sqrt{n}\{\widehat{V}(\widehat{\boldsymbol{\beta}}) - V_0\} \stackrel{d}{\to} N(0, \sigma_0^2).$$

The standard deviation σ_0 can be consistently estimated by

$$\widehat{\sigma} \equiv \left\{ \frac{1}{n-1} \sum_{i=1}^{n} \left(\frac{A_i I(\boldsymbol{X}_i^T \widehat{\boldsymbol{\beta}} > 0) + (1 - A_i) I(\boldsymbol{X}_i^T \widehat{\boldsymbol{\beta}} \leq 0)}{A_i \pi_i + (1 - A_i)(1 - \pi_i)} - \widehat{V}(\widehat{\boldsymbol{\beta}}) \right)^2 \right\}^{1/2},$$

under the given conditions. Therefore, a two-sided $1 - \alpha$ CI for V_0 is given by

$$[\widehat{V}(\widehat{\boldsymbol{\beta}}) - z_{\alpha/2}\widehat{\sigma}, \widehat{V}(\widehat{\boldsymbol{\beta}}) + z_{\alpha/2}\widehat{\sigma}],$$

where $z_{\alpha/2}$ denotes the upper $\alpha/2$-th quantile of a standard normal distribution.

4. Doubly robust inference procedure

In an observational studies, the propensity score function is usually unknown. In practice, it can be estimated using parametric models. Here, we assume a logistic regression model for $\pi(\boldsymbol{x})$. For any $\boldsymbol{\alpha} \in \mathbb{R}^p$, let $\pi^*(\boldsymbol{x}, \boldsymbol{\alpha}) = \texttt{plogis}(\boldsymbol{x}^T\boldsymbol{\alpha})$ where $\texttt{plogis}(z) = \exp(z)/\{1+\exp(z)\}$. Define $\boldsymbol{\alpha}^*$ to be the p-dimensional parameter that minimizes the following Kullback-Leibler divergence,

$$\boldsymbol{\alpha}^* \equiv \arg\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \mathcal{D}\left(\pi^*(\cdot,\boldsymbol{\alpha})\|\pi(\cdot)\right) = \arg\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \mathrm{E}\left(\pi(\boldsymbol{X}_0)\log\frac{\pi^*(\boldsymbol{X}_0,\boldsymbol{\alpha})}{\pi(\boldsymbol{X}_0)} + \{1-\pi(\boldsymbol{X}_0)\}\frac{1-\pi^*(\boldsymbol{X}_0,\boldsymbol{\alpha})}{1-\pi(\boldsymbol{X}_0)}\right).$$

Take the derivative of $\mathcal{D}\left(\pi^*(\cdot,\boldsymbol{\alpha})\|\pi(\cdot)\right)$ with respect to $\boldsymbol{\alpha}$, it is immediate to see that $\boldsymbol{\alpha}^*$ is the solution to the following score equation:

$$\mathrm{E}\boldsymbol{X}_0\{\pi(\boldsymbol{X}_0) - \pi^*(\boldsymbol{X}_0,\boldsymbol{\alpha}^*)\} = \mathrm{E}\boldsymbol{X}_0\{A_0 - \pi^*(\boldsymbol{X}_0,\boldsymbol{\alpha}^*)\} = 0. \tag{4.1}$$

Let $h_0(\boldsymbol{x}) = \mathrm{E}(Y_0|A_0 = 0, \boldsymbol{X}_0 = \boldsymbol{x})$ and $h_1(\boldsymbol{x}) = \mathrm{E}(Y_0|A_0 = 1, \boldsymbol{X}_0 = \boldsymbol{x})$. We further posit the following models for these conditional mean functions,

$$h_0^*(\boldsymbol{x},\boldsymbol{\theta}) = \boldsymbol{x}^T\boldsymbol{\theta} \quad \text{and} \quad h_1^*(\boldsymbol{x},\boldsymbol{\eta}) = \boldsymbol{x}^T\boldsymbol{\eta}.$$

Define

$$\boldsymbol{\theta}^* \equiv \arg\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \mathrm{E}(1-A_0)\{h_0(\boldsymbol{X}_0) - h_0^*(\boldsymbol{X}_0,\boldsymbol{\theta})\}^2 \quad \text{and} \quad \boldsymbol{\eta}^* \equiv \arg\min_{\boldsymbol{\eta}\in\mathbb{R}^p} \mathrm{E}A_0\{h_1(\boldsymbol{X}_0) - h_1^*(\boldsymbol{X}_0,\boldsymbol{\eta})\}^2.$$

We have

$$\mathrm{E}\boldsymbol{X}_0(1-A_0)\{Y_0 - h_0^*(\boldsymbol{X}_0,\boldsymbol{\theta}^*)\} = 0 \quad \text{and} \quad \mathrm{E}\boldsymbol{X}_0A_0\{Y_0 - h_1^*(\boldsymbol{X}_0,\boldsymbol{\eta}^*)\} = 0. \tag{4.2}$$

The models π^*, h_0^* and h_1^* can be misspecified. However, we will require either the propensity score function or the conditional mean functions to be correctly specified. When these models are correct, we have $\pi(\cdot) = \pi^*(\cdot,\boldsymbol{\alpha}^*)$, $h_0(\cdot) = h_0^*(\cdot,\boldsymbol{\theta}^*)$ and $h_1(\cdot) = h_1^*(\cdot,\boldsymbol{\eta}^*)$. Otherwise, the parameters $\boldsymbol{\alpha}^*$, $\boldsymbol{\theta}^*$ and $\boldsymbol{\eta}^*$ correspond to some least false parameters in the misspecified models [21].

We assume $\boldsymbol{\alpha}^*$, $\boldsymbol{\theta}^*$ and $\boldsymbol{\eta}^*$ are sparse. Let

$$\mathcal{M}_\alpha = \{1\le j\le p : \alpha_j^* \neq 0\},\quad \mathcal{M}_\theta = \{1\le j\le p : \theta_j^* \neq 0\},\quad \mathcal{M}_\eta = \{1\le j\le p : \eta_j^* \neq 0\},$$

be the support of these parameters. Define $s_\alpha = \|\boldsymbol{\alpha}^*\|_0$, $s_\theta = \|\boldsymbol{\theta}^*\|_0$ and $s_{\boldsymbol{\eta}} = \|\boldsymbol{\eta}^*\|_0$.

To deal with high-dimensionality, we propose to estimate $\boldsymbol{\alpha}^*$, $\boldsymbol{\theta}^*$ and $\boldsymbol{\eta}^*$ via nonconcave penalized regression [6]. More specifically, define

$$\widehat{\boldsymbol{\alpha}} = \arg\min_{\boldsymbol{\alpha}\in\mathbb{R}^p} \left(\frac{1}{n}\sum_{i=1}^n [\log\{1+\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha})\} - A_i\boldsymbol{X}_i^T\boldsymbol{\alpha}] + \sum_{j=1}^p p_{\lambda_1}(|\alpha_j|) \right), \tag{4.3}$$

$$\widehat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}\in\mathbb{R}^p} \left(\frac{1}{2n}\sum_{i=1}^n (1-A_i)(Y_i - \boldsymbol{X}^T\boldsymbol{\theta})^2 + \sum_{j=1}^p p_{\lambda_2}(|\theta_j|) \right), \tag{4.4}$$

$$\widehat{\boldsymbol{\eta}} = \arg\min_{\boldsymbol{\eta}\in\mathbb{R}^p} \left(\frac{1}{2n}\sum_{i=1}^n A_i(Y_i - \boldsymbol{X}^T\boldsymbol{\eta})^2 + \sum_{j=1}^p p_{\lambda_3}(|\eta_j|) \right), \tag{4.5}$$

for some folded-concave penalty function $p_\lambda(\cdot)$ such as SCAD [5] and MCP [23].

Let $\hat{\pi}_i = \pi^*(\boldsymbol{X}_i, \widehat{\alpha}^*)$. For any $\boldsymbol{\beta} \in \mathbb{R}^p$, consider the following doubly robust estimator for $V(\boldsymbol{\beta}_0)$,

$$\widehat{V}^{dr}(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^n \left\{ \frac{A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta} > 0) + (1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta} \le 0)}{A_i\hat{\pi}_i + (1-A_i)(1-\hat{\pi}_i)} Y_i \right.$$
$$- \left. \left(\frac{A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta} > 0) + (1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta} \le 0)}{A_i\hat{\pi}_i + (1-A_i)(1-\hat{\pi}_i)} - 1 \right) \{\boldsymbol{X}_i^T\widehat{\boldsymbol{\eta}} I(\boldsymbol{X}_i^T\boldsymbol{\beta} > 0) + \boldsymbol{X}_i^T\widehat{\boldsymbol{\theta}} I(\boldsymbol{X}_i^T\boldsymbol{\beta} \le 0)\} \right\}.$$

We propose to estimate V_0 by $\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})$ for some regularized estimator $\widehat{\boldsymbol{\beta}}$ that satisfies Condition (A3) and (A4). For any matrix $\boldsymbol{\Psi} \in \mathbb{R}^{p\times p}$ and any $\mathcal{J}_1, \mathcal{J}_2 \subseteq \{1,\dots,p\}$, let $\boldsymbol{\Psi}_{\mathcal{J}_1,\mathcal{J}_2}$ denote the sub-matrix of $\boldsymbol{\Psi}$ formed by rows in $\mathcal{J}_1$ and columns in $\mathcal{J}_2$. For any p-dimensional vector $\boldsymbol{a}$, let $\boldsymbol{a}_{\mathcal{J}_1}$ denote the sub-vector of $\boldsymbol{a}$ formed by elements in $\mathcal{J}_1$. Let $\boldsymbol{\Omega} = \mathrm{E}\pi^*(\boldsymbol{X}_0, \boldsymbol{\alpha}^*)\{1-\pi^*(\boldsymbol{X}_0, \boldsymbol{\alpha}^*)\}\boldsymbol{X}_0\boldsymbol{X}_0^T$, $\boldsymbol{\Phi} = \mathrm{E}(1-A_0)\boldsymbol{X}_0\boldsymbol{X}_0^T$, $\boldsymbol{\Xi} = A_0\boldsymbol{X}_0\boldsymbol{X}_0^T$ and $\boldsymbol{\Sigma} = \mathrm{E}\boldsymbol{X}_0\boldsymbol{X}_0^T$. Let $\mathcal{J}_1^c$ be the complement of $\mathcal{J}_1$. To investigate the theoretical properties of $\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})$, we need the following "oracle properties" of $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\eta}}$.

(A9) Assume the following events hold with probability tending to 1,

$$\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha^c} = 0, \quad \widehat{\boldsymbol{\theta}}_{\mathcal{M}_\theta^c} = 0, \quad \widehat{\boldsymbol{\eta}}_{\mathcal{M}_\eta^c} = 0.$$

Besides, assume

$$\sqrt{n}(\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha} - \boldsymbol{\alpha}^*_{\mathcal{M}_\alpha}) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \boldsymbol{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{X}_{i,\mathcal{M}_\alpha}\{A_i - \pi^*(\boldsymbol{X}_i,\boldsymbol{\alpha}^*)\} + o_p(1),$$

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_{\mathcal{M}_\theta} - \boldsymbol{\theta}^*_{\mathcal{M}_\theta}) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \boldsymbol{\Phi}^{-1}_{\mathcal{M}_\theta,\mathcal{M}_\theta}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(1-A_i)(Y_i - X_i^T\boldsymbol{\theta}^*) + o_p(1),$$

$$\sqrt{n}(\widehat{\boldsymbol{\eta}}_{\mathcal{M}_\eta} - \boldsymbol{\eta}_{\mathcal{M}_\eta}) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \boldsymbol{\Xi}^{-1}_{\mathcal{M}_\eta,\mathcal{M}_\eta}\boldsymbol{X}_{i,\mathcal{M}_\eta}A_i(Y_i - \boldsymbol{X}_i^T\boldsymbol{\eta}^*) + o_p(1).$$

(A10) Assume there exists some constant $\omega^* > 0$ such that $|\boldsymbol{X}_0^T\boldsymbol{\alpha}^*| \le \omega^*$, $|\boldsymbol{X}_0^T\boldsymbol{\theta}^*| \le \omega^*$, $|\boldsymbol{X}_0^T\boldsymbol{\eta}^*| \le \omega^*$.

(A11) Assume there exist some constants $c_3, c_4 > 0$ such that $\lambda_{\min}(\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}) \geq c_3$, $\lambda_{\min}(\boldsymbol{\Phi}_{\mathcal{M}_\theta,\mathcal{M}_\theta}) \geq c_3$, $\lambda_{\min}(\boldsymbol{\Xi}_{\mathcal{M}_\eta,\mathcal{M}_\eta}) \geq c_3$, $\lambda_{\max}(\boldsymbol{\Sigma}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}) \leq c_4$, $\lambda_{\max}(\boldsymbol{\Sigma}_{\mathcal{M}_\theta,\mathcal{M}_\theta}) \leq c_4$, $\lambda_{\max}(\boldsymbol{\Sigma}_{\mathcal{M}_\eta,\mathcal{M}_\eta}) \leq c_4$.

Condition (A9) assumes $\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha}$, $\widehat{\boldsymbol{\theta}}_{\mathcal{M}_\theta}$ and $\widehat{\boldsymbol{\eta}}_{\mathcal{M}_\eta}$ are asymptotically linear. Under certain regularity conditions, it implies that the convergence rates of these estimators are $O_p(\sqrt{s_\alpha/n})$, $O_p(\sqrt{s_\theta/n})$ and $O_p(\sqrt{s_\eta/n})$, respectively. Besides, they are asymptotically normally distributed. Moreover, Condition (A9) assumes these estimators are selection consistent. In Section 5, we will show Condition (A9) holds for these regularized estimators.

Let $\pi_0^* = \pi^*(\boldsymbol{X}_0, \boldsymbol{\alpha}^*)$ and $d_0 = I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 > 0)$. Define

$$
\begin{aligned}
v_1 &= \operatorname{Var}\left\{ \left(\frac{d_0}{\pi_0^*} - \boldsymbol{\xi}_{\mathcal{M}_\eta}^T \boldsymbol{\Xi}_{\mathcal{M}_\eta,\mathcal{M}_\eta}^{-1} \boldsymbol{X}_{0,\mathcal{M}_\eta} \right) A_0\{Y_0 - h_1(\boldsymbol{X}_0)\} \right. \\
&+ \left. \left(\frac{1-d_0}{1-\pi_0^*} - \boldsymbol{\phi}_{\mathcal{M}_\theta}^T \boldsymbol{\Phi}_{\mathcal{M}_\theta,\mathcal{M}_\theta}^{-1} \boldsymbol{X}_{0,\mathcal{M}_\theta} \right) (1-A_0)\{Y_0 - h_0(\boldsymbol{X}_0)\} \right\}, \\
v_2 &= \operatorname{Var}\left\{ \left(\frac{A_0}{\pi_0^*} - 1 \right) d_0\{h_1(\boldsymbol{X}_0) - \boldsymbol{X}_0^T\boldsymbol{\eta}^*\} + \left(\frac{1-A_0}{1-\pi_0^*} - 1 \right) (1-d_0)\{h_0(\boldsymbol{X}_0) - \boldsymbol{X}_0^T\boldsymbol{\theta}^*\} \right\} \\
&- \boldsymbol{\omega}_{\mathcal{M}_\alpha}^T \boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1} \boldsymbol{\omega}_{\mathcal{M}_\alpha}, \\
v_3 &= \operatorname{Var}\{h_0(\boldsymbol{X}_0)d_0 + h_1(\boldsymbol{X}_0)(1-d_0)\},
\end{aligned}
$$

where

$$
\begin{aligned}
&\boldsymbol{\omega} = \mathrm{E}[\{(1-A_0)(1-d_0)\exp(\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)(Y_0 - \boldsymbol{X}_0^T\boldsymbol{\theta}^*) - A_0 d_0 \exp(-\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)(Y_0 - \boldsymbol{X}_0^T\boldsymbol{\eta}^*)\}\boldsymbol{X}_0], \\
&\boldsymbol{\phi} = \mathrm{E}\left(\frac{1-A_0}{1-\pi_0^*} - 1 \right)(1-d_0)\boldsymbol{X}_0, \qquad \boldsymbol{\xi} = \mathrm{E}\left(\frac{A_0}{\pi_0^*} - 1 \right) d_0 \boldsymbol{X}_0.
\end{aligned}
$$

Theorem 4.1. *Assume Condition (A1)-(A11) hold,* $\liminf(v_1+v_2+v_3) > 0$, $\kappa_n \log p \log n = o(\sqrt{n})$ *and* $s_\alpha, s_\theta, s_\eta = o\{\sqrt{n/\log(n)}\}$*. Then, when either the propensity score model or the conditional mean models are correctly specified, we have*

$$
\frac{\sqrt{n}\{\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}}) - V_0\}}{\sqrt{v_1+v_2+v_3}} \xrightarrow{d} N(0,1).
$$

Theorem 4.1 proves the doubly robustness property of $\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})$. The asymptotic variance of $\sqrt{n}\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})$ consists of three parts. When the propensity score model is correct, we have $\boldsymbol{\phi} = \boldsymbol{\xi} = 0$ and hence $v_1 = v_1^*$ where

$$
v_1^* \equiv \operatorname{Var}\left(\frac{A_0 d_0}{\pi_0^*}\{Y_0 - h_1(\boldsymbol{X}_0)\} + \frac{(1-A_0)(1-d_0)}{1-\pi_0^*}\{Y_0 - h_0(\boldsymbol{X}_0)\} \right).
$$

The asymptotic variance of $\sqrt{n}\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})$ is then equal to $v_1^*+v_2+v_3$. When the conditional mean models are correct, we have $h_1(\boldsymbol{X}_0) = \boldsymbol{X}_0^T\boldsymbol{\eta}^*$, $h_0(\boldsymbol{X}_0) = \boldsymbol{X}_0^T\boldsymbol{\theta}^*$ and $\boldsymbol{\omega} = 0$. This implies $v_2 = 0$. As a result, the asymptotic variance of $\sqrt{n}\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})$ reduces to $v_1 + v_3$.

Corollary 4.1. *Assume Condition (A1)-(A11) hold, $\kappa_n \log p \log n = o(\sqrt{n})$ and $s_\alpha, s_\theta, s_\eta = o\{\sqrt{n/\log(n)}\}$. Assume the propensity score model is correctly specified, $E\{h_0(\boldsymbol{X}_0) - \boldsymbol{X}_0^T\boldsymbol{\theta}^*\}^2 \to 0$, $E\{h_1(\boldsymbol{X}_0) - \boldsymbol{X}_0^T\boldsymbol{\eta}^*\}^2 \to 0$ and $\liminf(v_1^* + v_3) > 0$. Then, we have*

$$\frac{\sqrt{n}\{\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}}) - V_0\}}{\sqrt{v_1^* + v_3}} \xrightarrow{d} N(0,1).$$

As we commented before, when the propensity score model is correct, the asymptotic variance of $\sqrt{n}\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})$ reduces to $v_1^* + v_2 + v_3$. As shown in Corollary 4.1, the variance component v_2 vanishes when the proposed conditional mean models are close to the true models in the sense that $\mathrm{E}\{h_0(\boldsymbol{X}_0) - \boldsymbol{X}_0^T\boldsymbol{\theta}^*\}^2 \to 0$, $\mathrm{E}\{h_1(\boldsymbol{X}_0) - \boldsymbol{X}_0^T\boldsymbol{\eta}^*\}^2 \to 0$. When this condition is satisfied, the resulting estimator $\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})$ achieves its smallest variance and is semiparametric efficient. In general, we gain efficiency by posing good working models for $h_0(\cdot)$ and $h_1(\cdot)$.

Assume the propensity score model is correct and $\boldsymbol{\alpha}^*$ is known ahead of time. Consider the following estimator for V_0,

$$\widetilde{V}(\widehat{\boldsymbol{\beta}}) = \frac{1}{n}\sum_{i=1}^n \left\{ \frac{A_i I(\boldsymbol{X}_i^T\widehat{\boldsymbol{\beta}} > 0) + (1-A_i) I(\boldsymbol{X}_i^T\widehat{\boldsymbol{\beta}} \le 0)}{A_i \pi_i^* + (1-A_i)(1-\pi_i^*)} Y_i \right. \tag{4.6}$$
$$- \left. \left(\frac{A_i I(\boldsymbol{X}_i^T\widehat{\boldsymbol{\beta}} > 0) + (1-A_i) I(\boldsymbol{X}_i^T\widehat{\boldsymbol{\beta}} \le 0)}{A_i \pi_i^* + (1-A_i)(1-\pi_i^*)} - 1 \right) \{\boldsymbol{X}_i^T\widehat{\boldsymbol{\eta}} I(\boldsymbol{X}_i^T\widehat{\boldsymbol{\beta}} > 0) + \boldsymbol{X}_i^T\widehat{\boldsymbol{\theta}} I(\boldsymbol{X}_i^T\widehat{\boldsymbol{\beta}} \le 0)\} \right\},$$

where $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\eta}}$ are defined in (4.4) and (4.5), respectively. Let

$$v_2^* = \mathrm{Var}\left\{ \left(\frac{A_0}{\pi_0^*} - 1\right) d_0\{h_1(\boldsymbol{X}_0) - \boldsymbol{X}_0^T\boldsymbol{\eta}^*\} + \left(\frac{1-A_0}{1-\pi_0^*} - 1\right)(1-d_0)\{h_0(\boldsymbol{X}_0) - \boldsymbol{X}_0^T\boldsymbol{\theta}^*\} \right\}.$$

We have the following results.

Corollary 4.2. *Assume Condition (A1)-(A11) hold, $\kappa_n \log p \log n = o(\sqrt{n})$ and $s_\alpha, s_\theta, s_\eta = o\{\sqrt{n/\log(n)}\}$. Assume the propensity score model is correctly specified, and $\liminf(v_1^* + v_2^* + v_3) > 0$. Then, we have*

$$\frac{\sqrt{n}\{\widetilde{V}(\widehat{\boldsymbol{\beta}}) - V_0\}}{\sqrt{v_1^* + v_2^* + v_3}} \xrightarrow{d} N(0,1),$$

where $\widetilde{V}(\widehat{\boldsymbol{\beta}})$ is defined in (4.6).

Corollary 4.2 suggests that the asymptotic variance of $\sqrt{n}\widetilde{V}(\widehat{\boldsymbol{\beta}})$ is equal to $v_1^* + v_2^* + v_3$. In contrast, when the propensity score model is correct, the asymptotic variance of $\sqrt{n}\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})$ is equal to $v_1^* + v_2 + v_3$. By definition, we have $v_2 \le v_2^*$. Corollary 4.2 thus implies that the asymptotic variance of the estimated optimal value function will be smaller than that of the same estimator had we known the propensity score in advance.

This is in line with the classical semiparametric theory [17] which states that the AIPWE is more efficient when we estimate the propensity score instead of using their true values.

Let $\hat{d}_i = I(\boldsymbol{X}_i^T \widehat{\boldsymbol{\beta}} > 0)$, for $i = 1, \dots, n$. Define

$$\begin{aligned}
\hat{\zeta}_i &= \left(\frac{A_i \hat{d}_i}{\hat{\pi}_i} + \frac{(1-A_i)(1-\hat{d}_i)}{1-\hat{\pi}_i} \right) Y_i - \left(\frac{A_i \hat{d}_i}{\hat{\pi}_i} + \frac{(1-A_i)(1-\hat{d}_i)}{1-\hat{\pi}_i} - 1 \right) \{ \boldsymbol{X}_i^T \widehat{\boldsymbol{\eta}} \hat{d}_i + \boldsymbol{X}_i^T \widehat{\boldsymbol{\theta}} (1-\hat{d}_i) \} \\
&+ \widehat{\boldsymbol{\omega}}_{\mathcal{M}_\alpha}^T \left(\frac{1}{n} \sum_{j=1}^n \boldsymbol{X}_{j,\mathcal{M}_\alpha} \hat{\pi}_j (1-\hat{\pi}_j) \boldsymbol{X}_{j,\mathcal{M}_\alpha}^T \right)^{-1} \boldsymbol{X}_{i,\mathcal{M}_\alpha} (A_i - \hat{\pi}_i) \\
&- \widehat{\boldsymbol{\xi}}_{\mathcal{M}_\eta}^T \left(\frac{1}{n} \sum_{j=1}^n A_j \boldsymbol{X}_{j,\mathcal{M}_\eta} \boldsymbol{X}_{j,\mathcal{M}_\eta}^T \right)^{-1} A_i \boldsymbol{X}_{i,\mathcal{M}_\eta} (Y_i - \boldsymbol{X}_i^T \widehat{\boldsymbol{\eta}}) \\
&- \widehat{\boldsymbol{\phi}}_{\mathcal{M}_\theta}^T \left(\frac{1}{n} \sum_{j=1}^n (1-\boldsymbol{A}_j) \boldsymbol{X}_{j,\mathcal{M}_\theta} \boldsymbol{X}_{j,\mathcal{M}_\theta}^T \right)^{-1} (1-A_i) \boldsymbol{X}_{i,\mathcal{M}_\theta} (Y_i - \boldsymbol{X}_i^T \widehat{\boldsymbol{\theta}}),
\end{aligned}$$

where

$$\widehat{\boldsymbol{\omega}} = \frac{1}{n} \sum_{i=1}^n \boldsymbol{X}_i \{ (1-A_i)(1-\hat{d}_i) \exp(\boldsymbol{X}_i^T \widehat{\boldsymbol{\alpha}})(Y_i - \boldsymbol{X}_i^T \widehat{\boldsymbol{\theta}}) - A_i \hat{d}_i \exp(-\boldsymbol{X}_i^T \widehat{\boldsymbol{\alpha}})(Y_i - \boldsymbol{X}_i^T \widehat{\boldsymbol{\eta}}) \},$$

$$\widehat{\boldsymbol{\phi}} = \frac{1}{n} \sum_{i=1}^n \left(\frac{1-A_i}{1-\hat{\pi}_i} - 1 \right) (1-\hat{d}_i) \boldsymbol{X}_i, \qquad \widehat{\boldsymbol{\xi}} = \frac{1}{n} \sum_{i=1}^n \left(\frac{A_i}{\hat{\pi}_i} - 1 \right) \hat{d}_i \boldsymbol{X}_i.$$

The asymptotic variance of $\sqrt{n} \widetilde{V}(\widehat{\boldsymbol{\beta}})$ can be consistently estimated by

$$(\widehat{\sigma}^{dr})^2 = \frac{1}{n-1} \sum_{i=1}^n \hat{\zeta}_i^2 - \frac{1}{n(n-1)} \left(\sum_{i=1}^n \hat{\zeta}_i \right)^2.$$

The corresponding two-sided CI for V_0 is given by

$$[\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}}) - z_{\alpha/2} \widehat{\sigma}^{dr}, \widehat{V}^{dr}(\widehat{\boldsymbol{\beta}}) + z_{\alpha/2} \widehat{\sigma}^{dr}].$$

5. Oracle properties of folded-concave type estimators

The asymptotic normality of $\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})$ relies on Condition (A9) which assumes the oracle properties of $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\eta}}$ defined in (4.3), (4.4) and (4.5). In this section, we show these oracle properties hold when $p_\lambda(\cdot)$ belongs to the class of folded-concave penalty functions. Let $\rho(t, \lambda) = p_\lambda(t)/\lambda$. Define

$$d_{n,\alpha} = \min_{j \in \mathcal{M}_\alpha} \frac{|\alpha_j^*|}{2}, \qquad d_{n,\theta} = \min_{j \in \mathcal{M}_\theta} \frac{|\theta_j^*|}{2}, \qquad d_{n,\eta} = \min_{j \in \mathcal{M}_\eta} \frac{|\eta_j^*|}{2}.$$

For any function $\rho : \mathbb{R} \to \mathbb{R}$ and any vector $\boldsymbol{a} \in \mathbb{R}^q$ such that $a_j \neq 0, \forall j = 1, \ldots, q$, define the local concavity function

$$\kappa(\rho, \boldsymbol{a}) = \lim_{\varepsilon \to 0} \sup_{t_1 < t_2 \in (|a_j| - \varepsilon, |a_j| + \varepsilon)} -\frac{\rho'(t_2) - \rho'(t_1)}{t_2 - t_1}.$$

We need the following conditions.

(C1.) Assume $\rho(t, \lambda)$ is increasing and concave in $t \in [0, \infty)$, and has a continuous derivative $\rho'(t, \lambda)$ with $\rho'(0+, \lambda) > 0$. In addition, assume $\rho'(t, \lambda)$ is increasing in $\lambda \in [0, \infty)$ and $\rho'(0+, \lambda)$ is independent of λ.
(C2.) Assume $d_{n,\alpha} \gg \lambda_1 \gg \max(\sqrt{s_\alpha / n}, \sqrt{\log p / n})$, $p'_{\lambda_1}(d_{n,\alpha}) = o(s_\alpha^{-1/2} n^{-1/2})$, and $\lambda_1 \kappa_\alpha = o(1)$ where

$$\kappa_\alpha = \sup_{\|\boldsymbol{\alpha}_{\mathcal{M}_\alpha} - \boldsymbol{\alpha}^*_{\mathcal{M}_\alpha}\|_\infty \leq d_{n,\alpha}} \kappa(\rho(\cdot, \lambda_1), \boldsymbol{\alpha}_{\mathcal{M}_\alpha}).$$

(C3.) Assume $d_{n,\theta} \gg \lambda_2 \gg \max(\sqrt{s_\theta / n}, \sqrt{\log p / n})$, $p'_{\lambda_2}(d_{n,\theta}) = o(s_\theta^{-1/2} n^{-1/2})$, and $\lambda_2 \kappa_\theta = o(1)$ where

$$\kappa_\theta = \sup_{\|\boldsymbol{\theta}_{\mathcal{M}_\theta} - \boldsymbol{\theta}^*_{\mathcal{M}_\theta}\|_\infty \leq d_{n,\theta}} \kappa(\rho(\cdot, \lambda_2), \boldsymbol{\alpha}_{\mathcal{M}_\theta}).$$

(C4.) Assume $d_{n,\eta} \gg \lambda_3 \gg \max(\sqrt{s_\eta / n}, \sqrt{\log p / n})$, $p'_{\lambda_3}(d_{n,\eta}) = o(s_\eta^{-1/2} n^{-1/2})$, and $\lambda_3 \kappa_\eta = o(1)$ where

$$\kappa_\eta = \sup_{\|\boldsymbol{\eta}_{\mathcal{M}_\eta} - \boldsymbol{\eta}^*_{\mathcal{M}_\eta}\|_\infty \leq d_{n,\eta}} \kappa(\rho(\cdot, \lambda_3), \boldsymbol{\eta}_{\mathcal{M}_\eta}).$$

Popular penalty functions, such as SCAD and MCP, satisfy Condition (C1). Assume $d_{n,\alpha} \gg \lambda_1 \gg \max(\sqrt{s_\alpha / n}$, $d_{n,\theta} \gg \lambda_2 \gg \max(\sqrt{s_\theta / n}, \sqrt{\log p / n})$, $d_{n,\eta} \gg \lambda_3 \gg \max(\sqrt{s_\eta / n}, \sqrt{\log p / n})$, then Condition (C2),(C3) and (C4) are automatically satisfied for the SCAD penalty function, since we have $p'_{\lambda_1}(d_{n,\alpha}) = p'_{\lambda_2}(d_{n,\theta}) = p'_{\lambda_3}(d_{n,\eta}) = 0$ and $\kappa_\alpha = \kappa_\theta = \kappa_\eta = 0$ for sufficiently large n.

Theorem 5.1. *Assume Condition (A5), (A7), (A11), (C1), (C2), (C3) and (C4) hold. Assume $s_\alpha = o(n^{1/3})$ and $s_\theta, s_\eta = o\{\sqrt{n / \log(n)}\}$. Then there exist some local minimizers $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\theta}}$ and $\widehat{\boldsymbol{\eta}}$ that satisfy Condition (A9).*

6. Technical proofs

This section consists of proofs of Theorem 3.1 and Theorem 4.1.

Proof of Theorem 3.1. It suffices to prove (3.1) and (3.2). We first prove (3.1). By the linearity assumption, we have

$$\begin{aligned}
& \mathrm{E}\{\widehat{V}(\boldsymbol{\beta})-\widehat{V}(\boldsymbol{\beta}_0)\} \\
= \; & \mathrm{E}\left(\frac{A_1}{\pi_1}Y_1\{I(\boldsymbol{X}_1^T\boldsymbol{\beta}>0)-I(\boldsymbol{X}_1^T\boldsymbol{\beta}_0>0)\}\right)+\mathrm{E}\left(\frac{1-A_1}{1-\pi_1}Y_1\{I(\boldsymbol{X}_1^T\boldsymbol{\beta}\le 0)-I(\boldsymbol{X}_1^T\boldsymbol{\beta}_0\le 0)\}\right) \\
= \; & \mathrm{E}\left(\frac{A_1}{\pi_1}Y_1-\frac{1-A_1}{1-\pi_1}Y_1\right)\{I(\boldsymbol{X}_1^T\boldsymbol{\beta}>0)-I(\boldsymbol{X}_1^T\boldsymbol{\beta}_0>0)\} \\
= \; & \mathrm{E}\left(\frac{A_1}{\pi_1}\mathrm{E}(Y_1|A_1=1,\boldsymbol{X}_1)-\frac{1-A_1}{1-\pi_1}\mathrm{E}(Y_1|A_1=0,\boldsymbol{X}_1)\right)\{I(\boldsymbol{X}_1^T\boldsymbol{\beta}>0)-I(\boldsymbol{X}_1^T\boldsymbol{\beta}_0>0)\} \\
= \; & \mathrm{E}\{\mathrm{E}(Y_1|A_1=1,\boldsymbol{X}_1)-\mathrm{E}(Y_1|A_1=0,\boldsymbol{X}_1)\}\{I(\boldsymbol{X}_1^T\boldsymbol{\beta}>0)-I(\boldsymbol{X}_1^T\boldsymbol{\beta}_0>0)\} \\
= \; & \mathrm{E}(\boldsymbol{X}_1^T\boldsymbol{\beta}_0)\{I(\boldsymbol{X}_1^T\boldsymbol{\beta}>0)-I(\boldsymbol{X}_1^T\boldsymbol{\beta}_0>0)\}=V(\boldsymbol{\beta})-V(\boldsymbol{\beta}_0). \qquad (6.1)
\end{aligned}$$

Note that for any $a,b\in\mathbb{R}$, $I(a>0)\neq I(b>0)$ only when $|a|\le|a-b|$. As a result, we have for any $\boldsymbol{\beta}$ such that $\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le\delta_0/\omega_0$,

$$\begin{aligned}
& V(\boldsymbol{\beta})-V(\boldsymbol{\beta}_0)=\mathrm{E}(\boldsymbol{X}_1^T\boldsymbol{\beta}_0)\{I(\boldsymbol{X}_1^T\boldsymbol{\beta}>0)-I(\boldsymbol{X}_1^T\boldsymbol{\beta}_0>0)\} \qquad (6.2) \\
\le \; & \mathrm{E}|\boldsymbol{X}_1^T\boldsymbol{\beta}_0|I(|\boldsymbol{X}_1^T\boldsymbol{\beta}_0|\le|\boldsymbol{X}_1^T(\boldsymbol{\beta}-\boldsymbol{\beta}_0)|)\le\mathrm{E}|\boldsymbol{X}_1^T\boldsymbol{\beta}_0|I(|\boldsymbol{X}_1^T\boldsymbol{\beta}_0|\le\omega_0\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1) \\
\le \; & \omega_0\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\Pr(|\boldsymbol{X}_1^T\boldsymbol{\beta}_0|\le\omega_0\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1)\le\bar{c}\omega_0^2\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1^2,
\end{aligned}$$

where the second inequality is due to Condition (A5) and the last inequality is due to Condition (A8). By Condition (A4), we obtain $V(\widehat{\boldsymbol{\beta}})-V(\boldsymbol{\beta}_0)=o_p(n^{-1/2})$. This proves (3.1). It remains to prove (3.2). Let

$$f_{\boldsymbol{\beta}}(\boldsymbol{x},a,y)=\left(\frac{a}{\pi(\boldsymbol{x})}-\frac{1-a}{1-\pi(\boldsymbol{x})}\right)\{I(\boldsymbol{x}^T\boldsymbol{\beta}>0)-I(\boldsymbol{x}^T\boldsymbol{\beta}_0>0)\}y.$$

Consider the following empirical process indexed by $\boldsymbol{\beta}$:

$$\mathbb{G}_nf_{\boldsymbol{\beta}}\equiv\frac{1}{\sqrt{n}}\sum_{i=1}^n\{f_{\boldsymbol{\beta}}(\boldsymbol{X}_i,A_i,Y_i)-\mathrm{E}f_{\boldsymbol{\beta}}(\boldsymbol{X}_i,A_i,Y_i)\}.$$

Note that we have $\mathbb{G}_nf_{\boldsymbol{\beta}}=\sqrt{n}\{\widehat{V}(\boldsymbol{\beta})-V(\boldsymbol{\beta})\}$. For a given subset $\mathcal{M}\subseteq\{1,\dots,p\}$ and any $\varepsilon>0$, define $\mathcal{F}_{\mathcal{M},\varepsilon}=\{f_{\boldsymbol{\beta}}:\boldsymbol{\beta}\in\mathbb{R}^p,\boldsymbol{\beta}_{\mathcal{M}^c}=0,\max(n^{1/4},\kappa_n\log p)\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le\varepsilon\}$. We first provide an upper bound for $\mathrm{E}\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_nf|$.

Let $F(\boldsymbol{x},a,y)=|y|\{1/c_1+1/(1-c_2)\}$. By Condition (A6), we have for any $\boldsymbol{\beta}\in\mathbb{R}^p$,

$$\begin{aligned}
|f_{\boldsymbol{\beta}}(\boldsymbol{x},a,y)| \; & \le \; \left|\frac{a}{\pi(\boldsymbol{x})}y\{I(\boldsymbol{x}^T\boldsymbol{\beta}>0)-I(\boldsymbol{x}^T\boldsymbol{\beta}_0>0)\}\right|+\left|\frac{1-a}{1-\pi(\boldsymbol{x})}y\{I(\boldsymbol{x}^T\boldsymbol{\beta}>0)-I(\boldsymbol{x}^T\boldsymbol{\beta}_0>0)\}\right| \\
& \le \; \frac{1}{c_1}|y|+\frac{1}{1-c_2}|y|=F(x,a,y).
\end{aligned}$$

By the definition of Orlicz norm and Taylor's theorem, we have for any random variable $\mathbb{Z}$,

$$1+\mathrm{E}\frac{\mathbb{Z}^2}{2\|\mathbb{Z}\|_{\psi_1}^2}\le\mathrm{E}\exp\left(\frac{|\mathbb{Z}|}{\|\mathbb{Z}\|_{\psi_1}}\right)\le 2. \qquad (6.3)$$

Hence, we have $\mathrm{E}\mathbb{Z}^2 \le 2\|\mathbb{Z}\|_{\psi_1}^2$. Let $c_* = 1/c_1 + 1/(1-c_2)$. By Condition (A7), we have

$$\mathrm{E}F(\boldsymbol{X}_0, A_0, Y_0)^2 \le 2c_*^2\bar{\omega}^2. \tag{6.4}$$

Let $M_0 = \max_{i=1,2,\dots,n} |F(\boldsymbol{X}_i, A_i, Y_i)|$. It follows from Lemma 2.2.2 in [19] that

$$\|M_0\|_{\psi_1} \le K_0 \max_{i=1,\dots,n} \|F(\boldsymbol{X}_i, A_i, Y_i)\|_{\psi_1} \log(1+n) \le K_0 c_* \bar{\omega} \log(1+n), \tag{6.5}$$

for some constant $K_0 > 0$. By (6.3), we have

$$\mathrm{E}M_0^2 \le c_*^2 K_0^2 c_*^2 \bar{\omega}^2 \log^2(1+n). \tag{6.6}$$

Moreover, following the arguments in (6.2), by Condition (A5), (A6) and (A8), we have

$$\begin{aligned} & \sup_{\max(n^{1/4},\kappa_n \log p)\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1 \le \varepsilon} \mathrm{E}|f_{\boldsymbol{\beta}}(\boldsymbol{X}_0, A_0, Y_0)|^2 \qquad (6.7) \\ \le\ & c_*^2 \sup_{\max(n^{1/4},\kappa_n \log p)\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1 \le \varepsilon} \mathrm{E}Y_0^2 |I(\boldsymbol{X}_0^T\boldsymbol{\beta} > 0) - I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 > 0)| \\ \le\ & c_*^2 \sup_{\max(n^{1/4},\kappa_n \log p)\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1 \le \varepsilon} \mathrm{E}[\{\mathrm{E}(Y_0^2|\boldsymbol{X}_0)\}|I(\boldsymbol{X}_0^T\boldsymbol{\beta} > 0) - I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 > 0)|] \\ \le\ & c_0 c_*^2 \sup_{\max(n^{1/4},\kappa_n \log p)\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1 \le \varepsilon} \mathrm{E}|I(\boldsymbol{X}_0^T\boldsymbol{\beta} > 0) - I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 > 0)| \\ \le\ & \underbrace{\bar{c} c_0 c_*^2 \omega_0 \varepsilon \min(n^{-1/4}, \kappa_n^{-1} \log^{-1} p)}_{\sigma_\varepsilon^2}, \end{aligned}$$

for any $\varepsilon > 0$ such that $\varepsilon \le n^{1/4}\delta_0$. For any $\mathcal{M} \subseteq \{1,\dots,p\}$ such that $|\mathcal{M}| = \kappa_n$ and any sufficiently small $\varepsilon > 0$, the VC index of the class of functions in $\mathcal{F}_{\mathcal{M},\varepsilon}$ is upper bounded by $\kappa_n + 2$. It follows from Corollary 5.1 in [4] that there exist some constant t_0 and $\bar{c}_0 > 0$,

$$\sup_{\substack{\mathcal{M}\subseteq\{1,\dots,p\} \\ |\mathcal{M}|=\kappa_n}} \mathrm{E} \sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}} |\mathbb{G}_n f| \le \bar{c}_0 \left\{ (\kappa_n+2)\sigma_\varepsilon^2 \log\left(\frac{t_0\sqrt{\mathrm{E}F^2}}{\sigma_\varepsilon}\right) \right\}^{1/2} + \bar{c}_0 \frac{(\kappa_n+2)\sqrt{\mathrm{E}M_0^2}}{\sqrt{n}} \log\left(\frac{t_0\sqrt{\mathrm{E}F^2}}{\sigma_\varepsilon}\right),$$

where σ_ε is defined in (6.7). By assumption, we have $\kappa_n = o(n)$ and $\log p = o(n)$. Hence,

$$\sigma_\varepsilon^2 \ge \varepsilon \bar{c} c_0 c_*^2 \omega_0 n^{-2}. \tag{6.8}$$

By (6.4), (6.6) and (6.8), we have

$$\begin{aligned} \sup_{\substack{\mathcal{M}\subseteq\{1,\dots,p\} \\ |\mathcal{M}|=\kappa_n}} \mathrm{E} \sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}} |\mathbb{G}_n f| \ \le\ & C_0 \varepsilon^{1/2} \kappa_n^{-1/2} \log^{-1/2} p \sqrt{\kappa_n+2} \{\log(C_0) + \log(1/\varepsilon) + \log(n)\}^{1/2} \\ +\ & C_0 \frac{(\kappa_n+2)\log(1+n)}{\sqrt{n}} \{\log(C_0) + \log(1/\varepsilon) + \log(n)\}, \qquad (6.9) \end{aligned}$$

for some constant $C_0>0$ and any sufficiently small $\varepsilon>0$.

Besides, it follows from Theorem 4 in [1] that for any $t>0$,

$$\sup_{\substack{\mathcal{M}\subseteq\{1,\dots,p\}\\|\mathcal{M}|=\kappa_n}}\Pr\left(\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_n f|-\frac{3}{2}\mathrm{E}\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_n f|\geq t\right)\leq\exp\left(-\frac{t^2}{3\sigma_\varepsilon^2}\right)+3\exp\left(-\frac{\sqrt{n}t}{C\|M_0\|_{\psi_1}}\right),$$

for some constant $C>0$. Set $t_*=3\max(\sigma_\varepsilon\sqrt{\kappa_n\log p}, C\|M_0\|_{\psi_1}n^{-1/2}\kappa_n\log p)$, we have

$$\sup_{\substack{\mathcal{M}\subseteq\{1,\dots,p\}\\|\mathcal{M}|=\kappa_n}}\Pr\left(\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_n f|-\frac{3}{2}\mathrm{E}\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_n f|\geq t_*\right)\leq 4\exp\left(-3\kappa_n\log p\right).$$

The number of subsets $\mathcal{M}$ satisfying $|\mathcal{M}|\leq\kappa_n$ is upper bounded by p^{κ_n}. Hence, by Bonferroni's inequality, we have

$$\begin{aligned}&\Pr\left\{\sup_{\substack{\mathcal{M}\subseteq\{1,\dots,p\}\\|\mathcal{M}|=\kappa_n}}\left(\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_n|-\frac{3}{2}\mathrm{E}\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_n f|\right)\geq t_*\right\}\\ \leq\ & p^{\kappa_n}\sup_{\substack{\mathcal{M}\subseteq\{1,\dots,p\}\\|\mathcal{M}|=\kappa_n}}\Pr\left(\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_n f|-\frac{3}{2}\mathrm{E}\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_n f|\geq t_*\right)\leq 4\exp(-2\kappa_n\log p)\to 0.\end{aligned}$$

This together with (6.9) gives that

$$\begin{aligned}\sup_{\substack{\mathcal{M}\subseteq\{1,\dots,p\}\\|\mathcal{M}|=\kappa_n}}\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_n f|\ &\leq\ t_*+\frac{3}{2}C_0\varepsilon^{1/2}\kappa_n^{-1/2}\log^{-1/2}p\sqrt{\kappa_n+2}\{\log(C_0)+\log(1/\varepsilon)+\log(n)\}^{1/2}\\ &+\ \frac{3}{2}C_0\frac{(\kappa_n+2)\log(1+n)}{\sqrt{n}}\{\log(C_0)+\log(1/\varepsilon)+\log(n)\},\end{aligned}\tag{6.10}$$

with probability tending to 1. By condition, we have $\kappa_n\log p\log n=o(n^{1/2})$ and $p\gg n$. This further implies that $\kappa_n\log^2 n=o(n^{1/2})$. By (6.5), (6.7) and the definition of t_*, there exists some constant $\bar{C}_0>0$ such that

$$\sup_{\substack{\mathcal{M}\subseteq\{1,\dots,p\}\\|\mathcal{M}|=\kappa_n}}\sup_{f\in\mathcal{F}_{\mathcal{M},\varepsilon}}|\mathbb{G}_n f|\leq\bar{C}_0\varepsilon^{1/2},$$

for any $\varepsilon>0$, with probability tending to 1. By Condition (A3) and (A4), we obtain

$$\Pr\left(\sqrt{n}\{\widehat{V}(\widehat{\boldsymbol\beta})-\widehat{V}(\boldsymbol\beta_0)-V(\widehat{\boldsymbol\beta})+V(\boldsymbol\beta_0)\}\leq\bar{C}_0\varepsilon^{1/2}\right)\to 1.$$

Let $\varepsilon\to 0$, we have

$$\sqrt{n}\{\widehat{V}(\widehat{\boldsymbol\beta})-\widehat{V}(\boldsymbol\beta_0)-V(\widehat{\boldsymbol\beta})+V(\boldsymbol\beta_0)\}=o_p(1).$$

This completes the proof. □

Proof of Theorem 4.1. By Condition (A11), the minimum eigenvalue of $\mathbf{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}$ is lower bounded by $c_3 > 0$. Hence, the maximum eigenvalue of $\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}$ is upper bounded by c_3^{-1}. Therefore, we have

$$
\begin{aligned}
& \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha}\\ \|\boldsymbol{a}\|_2=1}} \mathrm{Var}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \boldsymbol{a}^T\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{X}_{i,\mathcal{M}_\alpha}\{A_i-\pi^*(\boldsymbol{X}_i,\boldsymbol{\alpha}^*)\}\right)\\
= & \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha}\\ \|\boldsymbol{a}\|_2=1}} \mathrm{Var}\left(\boldsymbol{a}^T\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{X}_{0,\mathcal{M}_\alpha}\{A_0-\pi^*(\boldsymbol{X}_0,\boldsymbol{\alpha}^*)\}\right)\\
\le & \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha}\\ \|\boldsymbol{a}\|_2=1}} \mathrm{E}\left(\boldsymbol{a}^T\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{X}_{0,\mathcal{M}_\alpha}\{A_0-\pi^*(\boldsymbol{X}_0,\boldsymbol{\alpha}^*)\}\right)^2\\
= & \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha}\\ \|\boldsymbol{a}\|_2=1}} \mathrm{E}\left(\boldsymbol{a}^T\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{X}_{0,\mathcal{M}_\alpha}\{A_0-\pi^*(\boldsymbol{X}_0,\boldsymbol{\alpha}^*)\}^2\boldsymbol{X}^T_{0,\mathcal{M}_\alpha}\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{a}\right)\\
\le & \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha}\\ \|\boldsymbol{a}\|_2=1}} \mathrm{E}\left(\boldsymbol{a}^T\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{X}_{0,\mathcal{M}_\alpha}\boldsymbol{X}^T_{0,\mathcal{M}_\alpha}\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{a}\right) = \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha}\\ \|\boldsymbol{a}\|_2=1}} \boldsymbol{a}^T\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\mathbf{\Sigma}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{a}\\
\le & \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha}\\ \|\boldsymbol{a}\|_2=1}} \lambda_{\max}(\mathbf{\Sigma}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha})\|\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{a}\|_2^2 \le \lambda_{\max}(\mathbf{\Sigma}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha})\lambda^2_{\max}(\mathbf{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}) \le c_3^{-2}c_4.
\end{aligned}
$$

By Condition (A9), this further yields

$$\|\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha}-\boldsymbol{\alpha}^*_{\mathcal{M}_\alpha}\|_2 = O_p(\sqrt{s_\alpha/n}). \tag{6.11}$$

Similarly, we can show

$$\|\widehat{\boldsymbol{\theta}}_{\mathcal{M}_\theta}-\boldsymbol{\theta}^*_{\mathcal{M}_\theta}\|_2 = O_p(\sqrt{s_\theta/n}) \quad \text{and} \quad \|\widehat{\boldsymbol{\eta}}_{\mathcal{M}_\eta}-\boldsymbol{\eta}^*_{\mathcal{M}_\theta}\|_2 = O_p(\sqrt{s_\eta/n}). \tag{6.12}$$

By Condition (A10), we have

$$\max_{i=1}^n \|\boldsymbol{X}_i^T\boldsymbol{\alpha}^*\| \le \omega^*. \tag{6.13}$$

Besides, by Condition (A9), we have with probability tending to 1 that

$$\max_{i=1}^n \|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)\| = \max_{i=1}^n \|\boldsymbol{X}^T_{i,\mathcal{M}_\alpha}(\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha}-\boldsymbol{\alpha}^*_{\mathcal{M}_\alpha})\|. \tag{6.14}$$

By Condition (A5) and (6.11), we have

$$\max_{i=1}^n \|\boldsymbol{X}^T_{i,\mathcal{M}_\alpha}(\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha}-\boldsymbol{\alpha}^*_{\mathcal{M}_\alpha})\| \le \max_{i=1}^n \|\boldsymbol{X}_i\|_\infty\|\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha}-\boldsymbol{\alpha}^*_{\mathcal{M}_\alpha}\|_1 \le \omega_0\sqrt{s_\alpha}\|\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha}-\boldsymbol{\alpha}^*_{\mathcal{M}_\alpha}\|_2 = O_p(s_\alpha/\sqrt{n}).$$

This together with (6.14) gives

$$\max_{i=1}^n \|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)\| = O_p(s_\alpha/\sqrt{n}).$$

Under the condition $s_\alpha = o(\sqrt{n})$, we obtain $\max_{i=1}^n \|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*)\| = o_p(1)$. By (6.13), we have with probability tending to 1,

$$\max_{i=1}^n \|\boldsymbol{X}_i^T \widehat{\boldsymbol{\alpha}}\| \le 2\omega^*. \tag{6.15}$$

It follows from Taylor's theorem that

$$\frac{1}{\hat{\pi}_i} = \frac{1}{\pi^*(\boldsymbol{X}_i, \boldsymbol{\alpha}^*)} - \exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*) + \frac{1}{2}\exp(-\boldsymbol{X}_i^T\widetilde{\boldsymbol{\alpha}}_i)\|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*)\|_2^2, \tag{6.16}$$

for some $\widetilde{\boldsymbol{\alpha}}_i$ lying on the line segment jointing $\widehat{\boldsymbol{\alpha}}$ and $\boldsymbol{\alpha}^*$. By (6.13) and (6.15), we have

$$\max_{i=1}^n \|\boldsymbol{X}_i^T \widetilde{\boldsymbol{\alpha}}_i\| \le \max(\max_{i=1}^n \|\boldsymbol{X}_i^T \boldsymbol{\alpha}^*\|, \max_{i=1}^n \|\boldsymbol{X}_i^T \widehat{\boldsymbol{\alpha}}\|) \le 2\omega^*, \tag{6.17}$$

with probability tending to 1. Combining this together with (6.16) yields

$$\left| \frac{1}{\hat{\pi}_i} - \frac{1}{\pi^*(\boldsymbol{X}_i, \boldsymbol{\alpha}^*)} + \exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*) \right| \le \frac{\exp(2\omega^*)}{2}\|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*)\|_2^2,$$

with probability tending to 1. Therefore, under the event defined in (6.17), we have

$$\begin{aligned} &\sup_{\boldsymbol{\beta}\in\mathbb{R}^p} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta} > 0) Y_i \left(\frac{1}{\hat{\pi}_i} - \frac{1}{\pi_i^*} \right) + \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta} > 0)}{\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)} Y_i \boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*) \right| \\ \le &\sup_{\boldsymbol{\beta}\in\mathbb{R}^p} \frac{\exp(2\omega^*)}{2\sqrt{n}} \sum_{i=1}^n A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta} > 0)|Y_i|\|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*)\|_2^2 \le \frac{\exp(2\omega^*)}{2\sqrt{n}} \sum_{i=1}^n |Y_i|\|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*)\|_2^2, \end{aligned} \tag{6.18}$$

where π_i^* is a shorthand for $\pi^*(\boldsymbol{X}_i, \boldsymbol{\alpha}^*)$. In the following, we focus on proving

$$\sup_{\boldsymbol{\beta}\in\mathbb{R}^p} \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n \left\{ A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta} > 0) Y_i \left(\frac{1}{\hat{\pi}_i} - \frac{1}{\pi_i^*} \right) + \frac{A_i I(\boldsymbol{X}_i^T\widehat{\boldsymbol{\beta}} > 0)}{\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)} Y_i \boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*) \right\} \right| = o_p(1). \tag{6.19}$$

In view of (6.18), we need to show

$$\sum_{i=1}^n |Y_i|\|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}^*)\|_2^2 = o_p(\sqrt{n}).$$

By Condition (A9), it suffices to show

$$\sum_{i=1}^n |Y_i|\|\boldsymbol{X}_{i,\mathcal{M}_\alpha}^T(\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha} - \boldsymbol{\alpha}^*_{\mathcal{M}_\alpha})\|_2^2 = o_p(\sqrt{n}). \tag{6.20}$$

It follows from Cauchy-Schwarz inequality that

$$\sum_{i=1}^n |Y_i|\|\boldsymbol{X}_{i,\mathcal{M}_\alpha}^T(\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha} - \boldsymbol{\alpha}^*_{\mathcal{M}_\alpha})\|_2^2 \le \|\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha} - \boldsymbol{\alpha}^*_{\mathcal{M}_\alpha}\|_2^2 \lambda_{\max}\left(\sum_{i=1}^n |Y_i| \boldsymbol{X}_{i,\mathcal{M}_\alpha}\boldsymbol{X}_{i,\mathcal{M}_\alpha}^T \right).$$

By (6.11) and the condition $s_\alpha = o(\sqrt{n})$, it suffices to show

$$\lambda_{\max}\left(\sum_{i=1}^n |Y_i| \boldsymbol{X}_{i,\mathcal{M}_\alpha} \boldsymbol{X}_{i,\mathcal{M}_\alpha}^T\right) = O_p(n). \tag{6.21}$$

Note that

$$\begin{aligned}
&\lambda_{\max}\left(\sum_{i=1}^n |Y_i| \boldsymbol{X}_{i,\mathcal{M}_\alpha} \boldsymbol{X}_{i,\mathcal{M}_\alpha}^T\right) = \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha} \\ \|\boldsymbol{a}\|_2=1}} \sum_{i=1}^n |Y_i| (\boldsymbol{X}_{i,\mathcal{M}_\alpha}^T \boldsymbol{a})^2 \\
\leq\ & n \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha} \\ \|\boldsymbol{a}\|_2=1}} \mathrm{E}|Y_0| (\boldsymbol{X}_{0,\mathcal{M}_\alpha}^T \boldsymbol{a})^2 + \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha} \\ \|\boldsymbol{a}\|_2=1}} \left| \boldsymbol{a}^T \left(\sum_{i=1}^n |Y_i| \boldsymbol{X}_{i,\mathcal{M}_\alpha} \boldsymbol{X}_{i,\mathcal{M}_\alpha}^T - n\mathrm{E}|Y_0| \boldsymbol{X}_{0,\mathcal{M}_\alpha} \boldsymbol{X}_{0,\mathcal{M}_\alpha}^T\right) \boldsymbol{a} \right| \\
\leq\ & n \underbrace{\lambda_{\max}\left(\mathrm{E}|Y_0| \boldsymbol{X}_{0,\mathcal{M}_\alpha} \boldsymbol{X}_{0,\mathcal{M}_\alpha}^T\right)}_{I_1} + \underbrace{\left\| \sum_{i=1}^n |Y_i| \boldsymbol{X}_{i,\mathcal{M}_\alpha} \boldsymbol{X}_{i,\mathcal{M}_\alpha}^T - n\mathrm{E}|Y_0| \boldsymbol{X}_{0,\mathcal{M}_\alpha} \boldsymbol{X}_{0,\mathcal{M}_\alpha}^T \right\|_2}_{I_2}.
\end{aligned}$$

By Condition (A7), (A11) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
& I_1 \leq \lambda_{\max}\left(\mathrm{E}(\mathrm{E}|Y_0||\boldsymbol{X}_0) \boldsymbol{X}_{0,\mathcal{M}_\alpha} \boldsymbol{X}_{0,\mathcal{M}_\alpha}^T\right) \qquad (6.22) \\
\leq\ & \lambda_{\max}\left(\mathrm{E}\sqrt{\mathrm{E}|Y_0|^2|\boldsymbol{X}_0} \boldsymbol{X}_{0,\mathcal{M}_\alpha} \boldsymbol{X}_{0,\mathcal{M}_\alpha}^T\right) \leq \sqrt{c}_0 \lambda_{\max}(\boldsymbol{\Sigma}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}) \leq \sqrt{c_0} c_4.
\end{aligned}$$

Besides, for any symmetric matrix $\boldsymbol{A}$, we have $\|\boldsymbol{A}\|_2 \leq \sqrt{\|\boldsymbol{A}\|_1 \|\boldsymbol{A}\|_\infty} = \|\boldsymbol{A}\|_\infty$. Therefore,

$$\begin{aligned}
I_2 &\leq \max_{j_1\in\mathcal{M}_\alpha} \sum_{j_2\in\mathcal{M}_\alpha} \left| \sum_{i=1}^n |Y_i| X_{i,j_1} X_{i,j_2} - n\mathrm{E}|Y_0| X_{0,j_1} X_{0,j_2} \right| \qquad (6.23) \\
&\leq s_\alpha \max_{j_1,j_2\in\mathcal{M}_\alpha} \left| \sum_{i=1}^n |Y_i| X_{i,j_1} X_{i,j_2} - n\mathrm{E}|Y_0| X_{0,j_1} X_{0,j_2} \right|.
\end{aligned}$$

By Condition (A5) and (A7), we have

$$\max_{j_1,j_2\in\mathcal{M}_\alpha} \||Y_0| X_{0,j_1} X_{0,j_2}\|_{\psi_1} \leq \omega_0^2 \|Y_0\|_{\psi_1} \leq \omega_0^2 \bar{\omega}.$$

It follows from Jensen's inequality that

$$\max_{j_1,j_2\in\mathcal{M}_\alpha} \|\mathrm{E}|Y_0| X_{0,j_1} X_{0,j_2}\|_{\psi_1} \leq \max_{j_1,j_2\in\mathcal{M}_\alpha} \||Y_0| X_{0,j_1} X_{0,j_2}\|_{\psi_1} \leq \omega_0^2 \bar{\omega}.$$

Hence, we have

$$\max_{j_1,j_2\in\mathcal{M}_\alpha} \||Y_0| X_{0,j_1} X_{0,j_2} - \mathrm{E}|Y_0| X_{0,j_1} X_{0,j_2}\|_{\psi_1} \leq 2\omega_0^2 \bar{\omega}.$$

Therefore, it follows from the Bernstein's inequality [see Theorem 3.1 in 7] that

$$\max_{j_1,j_2\in\mathcal{M}_\alpha}\Pr\left(\left|\sum_{i=1}^n |Y_i|X_{i,j_1}X_{i,j_2}-n\mathrm{E}|Y_0|X_{0,j_1}X_{0,j_2}\right|\ge t\right)\le 2\exp\left\{-C_1\min\left(\frac{t^2}{4n\omega_0^4\bar{\omega}^2},\frac{t}{2\omega_0^2\bar{\omega}}\right)\right\},$$

for some constant $C_1>0$ and any $t>0$. Let $t_0=3\omega_0^2\bar{\omega}\sqrt{n\log n}/\sqrt{C_1}$. Note that

$$\frac{t_0^2}{4\omega_0^4\bar{\omega}^2}=\frac{9\log n}{4C_1}\ll\frac{3\sqrt{n\log n}}{2\sqrt{C_1}}=\frac{t_0}{2n\omega_0^2\bar{\omega}}.$$

Hence, for sufficiently large n, we obtain

$$\begin{aligned}\max_{j_1,j_2\in\mathcal{M}_\alpha}\Pr\left(\left|\sum_{i=1}^n |Y_i|X_{i,j_1}X_{i,j_2}-n\mathrm{E}|Y_0|X_{0,j_1}X_{0,j_2}\right|\ge t_0\right)&\le 2\exp\left(-C_1\frac{9\log n}{4C_1}\right)\\&=2\exp(-9\log n/4).\end{aligned}$$

Since $s_\alpha=o(n)$, it follows from Bonferroni's inequality that

$$\begin{aligned}&\Pr\left(\max_{j_1,j_2\in\mathcal{M}_\alpha}\left|\sum_{i=1}^n |Y_i|X_{i,j_1}X_{i,j_2}-n\mathrm{E}|Y_0|X_{0,j_1}X_{0,j_2}\right|\ge t_0\right)\qquad(6.24)\\ \le\;& s_\alpha^2\max_{j_1,j_2\in\mathcal{M}_\alpha}\Pr\left(\left|\sum_{i=1}^n |Y_i|X_{i,j_1}X_{i,j_2}-n\mathrm{E}|Y_0|X_{0,j_1}X_{0,j_2}\right|\ge t_0\right)\\ \le\;& 2s_\alpha^2\exp(-9\log n/4)\le 2\exp(-9\log n/4+2\log n)=2\exp(-\log n/4)\to 0.\end{aligned}$$

By (6.23), this yields

$$\Pr\left(I_2\le 3\omega_0^2\bar{\omega}s_\alpha\sqrt{n\log n}/\sqrt{C_1}\right)\to 1. \qquad (6.25)$$

Combining this together with (6.22), we obtain that

$$\lambda_{\max}\left(\sum_{i=1}^n |Y_i|\boldsymbol{X}_{i,\mathcal{M}_\alpha}\boldsymbol{X}_{i,\mathcal{M}_\alpha}^T\right)=O(n)+O_p(s_\alpha\sqrt{n\log n})=O_p(n), \qquad (6.26)$$

where the last equality is due to the condition that $s_\alpha=o(\sqrt{n/\log n})$. Therefore, (6.19) is proven.

Similarly, we can show

$$\sup_{\boldsymbol{\beta}\in\mathbb{R}^p}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}\le 0)Y_i\left(\frac{1}{1-\hat{\pi}_i}-\frac{1}{1-\pi_i^*}-\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)\right)\right|=o_p(1),$$

$$\sup_{\boldsymbol{\beta}\in\mathbb{R}^p}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n A_iI(\boldsymbol{X}_i^T\boldsymbol{\beta}>0)\boldsymbol{X}_i^T\widehat{\boldsymbol{\eta}}\left(\frac{1}{\hat{\pi}_i}-\frac{1}{\pi_i^*}+\exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)\right)\right|=o_p(1),$$

$$\sup_{\boldsymbol{\beta}\in\mathbb{R}^p}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}\le 0)\boldsymbol{X}_i^T\widehat{\boldsymbol{\theta}}\left(\frac{1}{1-\hat{\pi}_i}-\frac{1}{1-\pi_i^*}-\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)\right)\right|=o_p(1).$$

Combining these together with (6.19) implies that $\sup_{\boldsymbol{\beta}\in\mathbb{R}^p}|\sqrt{n}\widehat{V}^{dr}(\boldsymbol{\beta})-\sqrt{n}\widehat{V}^{dr,*}(\boldsymbol{\beta})|=o_p(1)$ where

$$\widehat{V}^{dr,*}(\boldsymbol{\beta}) = \underbrace{\frac{1}{n}\sum_{i=1}^n\left(\frac{A_iI(\boldsymbol{X}_i^T\boldsymbol{\beta}>0)}{\pi_i^*}+\frac{(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}\le 0)}{1-\pi_i^*}\right)Y_i}_{I_3(\boldsymbol{\beta})}$$

$$+ \underbrace{\frac{1}{n}\sum_{i=1}^n\{(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}\le 0)\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)-A_iI(\boldsymbol{X}_i^T\boldsymbol{\beta}>0)\exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\}Y_i\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)}_{I_4(\boldsymbol{\beta})}$$

$$- \underbrace{\frac{1}{n}\sum_{i=1}^n\left(\frac{A_iI(\boldsymbol{X}_i^T\boldsymbol{\beta}>0)}{\pi_i^*}+\frac{(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}\le 0)}{1-\pi_i^*}-1\right)\{\boldsymbol{X}_i^T\boldsymbol{\eta}^*I(\boldsymbol{X}_i^T\boldsymbol{\beta}>0)+\boldsymbol{X}_i^T\boldsymbol{\theta}^*I(\boldsymbol{X}_i^T\boldsymbol{\beta}\le 0)\}}_{I_5(\boldsymbol{\beta})}$$

$$- \underbrace{\frac{1}{n}\sum_{i=1}^n\left(\frac{A_i}{\pi_i^*}-1\right)I(\boldsymbol{X}_i^T\boldsymbol{\beta}>0)\boldsymbol{X}_i^T(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta}^*)}_{I_6(\boldsymbol{\beta})} - \underbrace{\frac{1}{n}\sum_{i=1}^n\left(\frac{1-A_i}{1-\pi_i^*}-1\right)I(\boldsymbol{X}_i^T\boldsymbol{\beta}\le 0)\boldsymbol{X}_i^T(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*)}_{I_7(\boldsymbol{\beta})}$$

$$+ \underbrace{\frac{1}{n}\sum_{i=1}^n\left(A_iI(\boldsymbol{X}_i^T\boldsymbol{\beta}>0)\exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\boldsymbol{X}_i^T\widehat{\boldsymbol{\eta}}-(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}\le 0)\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\boldsymbol{X}_i^T\widehat{\boldsymbol{\theta}}\right)\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)}_{I_8(\boldsymbol{\beta})}.$$

In the following, we show $\sup_{\boldsymbol{\beta}\in\mathbb{R}^p}\sqrt{n}|I_8(\boldsymbol{\beta})-I_8^*(\boldsymbol{\beta})|=o_p(1)$ where

$$I_8^*(\boldsymbol{\beta})=\frac{1}{n}\sum_{i=1}^n\left(A_iI(\boldsymbol{X}_i^T\boldsymbol{\beta}>0)\exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\boldsymbol{X}_i^T\boldsymbol{\eta}^*-(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}\le 0)\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\boldsymbol{X}_i^T\boldsymbol{\theta}^*\right)\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*).$$

Notice that

$$\begin{aligned}
&\sqrt{n}|I_8(\boldsymbol{\beta})-I_8^*(\boldsymbol{\beta})|\\
\le\;& \frac{1}{\sqrt{n}}\sum_{i=1}^n\Big(\exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta}^*)||\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)|+\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*)||\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)|\Big)\\
\le\;& \frac{\exp(\omega^*)}{\sqrt{n}}\sum_{i=1}^n\Big(|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta}^*)||\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)|+|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*)||\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)|\Big)\\
\le\;& \frac{\exp(\omega^*)}{\sqrt{n}}\sum_{i=1}^n\left(|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)|^2+\frac{1}{2}|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*)|^2+\frac{1}{2}|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta}^*)|^2\right),
\end{aligned}$$

where the second inequality follows by (6.13), and the last inequality is due to Cauchy-Schwarz inequality. Similar to (6.20), we can show

$$\sum_{i=1}^n|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*)|^2=o_p(\sqrt{n}),\quad \sum_{i=1}^n|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}^*)|^2=o_p(\sqrt{n}),\quad \sum_{i=1}^n|\boldsymbol{X}_i^T(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta}^*)|^2=o_p(\sqrt{n}),$$

under the condition $s_\alpha, s_\theta, s_\eta = o\{\sqrt{n/\log(n)}\}$. This proves $\sqrt{n}|I_8(\boldsymbol{\beta}) - I_8^*(\boldsymbol{\beta})| = o_p(1)$. Hence, we've shown

$$\sup_{\boldsymbol{\beta}\in\mathbb{R}^p}\left|\sqrt{n}\widehat{V}^{dr}(\boldsymbol{\beta}) - \sqrt{n}\{I_3(\boldsymbol{\beta}) + I_4(\boldsymbol{\beta}) - I_5(\boldsymbol{\beta}) - I_6(\boldsymbol{\beta}) - I_7(\boldsymbol{\beta}) + I_8^*(\boldsymbol{\beta})\}\right| = o_p(1). \tag{6.27}$$

When the propensity score model is correctly specified, similar to (6.1), we can show $\mathrm{E}I_3(\boldsymbol{\beta}) = V(\boldsymbol{\beta})$ and $\mathrm{E}I_5(\boldsymbol{\beta}) = 0$. When the conditional mean models are correct, we have

$$\begin{aligned}
& \mathrm{E}\{I_3(\boldsymbol{\beta}) - I_5(\boldsymbol{\beta})\} = \mathrm{E}\left(\frac{A_0 I(\boldsymbol{X}_0^T\boldsymbol{\beta} > 0)}{\pi^*(\boldsymbol{X}_0, \boldsymbol{\alpha}^*)}(Y_0 - \boldsymbol{X}_0^T\boldsymbol{\eta}^*) + \frac{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta} \le 0)}{1-\pi^*(\boldsymbol{X}_0, \boldsymbol{\alpha}^*)}(Y_0 - \boldsymbol{X}_0^T\boldsymbol{\theta}^*)\right) \\
+ \; & \mathrm{E}\{\boldsymbol{X}_0^T\boldsymbol{\eta}^* I(\boldsymbol{X}_0^T\boldsymbol{\beta} > 0) + \boldsymbol{X}_0^T\boldsymbol{\theta}^* I(\boldsymbol{X}_0^T\boldsymbol{\beta} \le 0)\} \\
= \; & \mathrm{E}\left(\frac{A_0 I(\boldsymbol{X}_0^T\boldsymbol{\beta} > 0)}{\pi^*(\boldsymbol{X}_0, \boldsymbol{\alpha}^*)}\{Y_0^*(1) - \boldsymbol{X}_0^T\boldsymbol{\eta}^*\} + \frac{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta} \le 0)}{1-\pi^*(\boldsymbol{X}_0, \boldsymbol{\alpha}^*)}\{Y_0^*(0) - \boldsymbol{X}_0^T\boldsymbol{\theta}^*\}\right) \\
+ \; & \mathrm{E}\{\boldsymbol{X}_0^T\boldsymbol{\eta}^* I(\boldsymbol{X}_0^T\boldsymbol{\beta} > 0) + \boldsymbol{X}_0^T\boldsymbol{\theta}^* I(\boldsymbol{X}_0^T\boldsymbol{\beta} \le 0)\} \\
= \; & \mathrm{E}\left(\frac{A_0 I(\boldsymbol{X}_0^T\boldsymbol{\beta} > 0)}{\pi^*(\boldsymbol{X}_0, \boldsymbol{\alpha}^*)}[\mathrm{E}\{Y_0^*(1)|\boldsymbol{X}_0\} - \boldsymbol{X}_0^T\boldsymbol{\eta}^*] + \frac{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta} \le 0)}{1-\pi^*(\boldsymbol{X}_0, \boldsymbol{\alpha}^*)}[\mathrm{E}\{Y_0^*(0)|\boldsymbol{X}_0\} - \boldsymbol{X}_0^T\boldsymbol{\theta}^*]\right) \\
+ \; & \mathrm{E}\{\boldsymbol{X}_0^T\boldsymbol{\eta}^* I(\boldsymbol{X}_0^T\boldsymbol{\beta} > 0) + \boldsymbol{X}_0^T\boldsymbol{\theta}^* I(\boldsymbol{X}_0^T\boldsymbol{\beta} \le 0)\} = \mathrm{E}\{\boldsymbol{X}_0^T\boldsymbol{\eta}^* I(\boldsymbol{X}_0^T\boldsymbol{\beta} > 0) + \boldsymbol{X}_0^T\boldsymbol{\theta}^* I(\boldsymbol{X}_0^T\boldsymbol{\beta} \le 0)\} = V(\boldsymbol{\beta}),
\end{aligned}$$

where the second equality is due to Condition (A1) and the third equality is due to Condition (A2). Therefore, we have $\mathrm{E}\{I_3(\boldsymbol{\beta}) - I_5(\boldsymbol{\beta})\} = V(\boldsymbol{\beta})$. Following the arguments in the proof of Theorem 3.1, we can show

$$\sqrt{n}\{I_3(\widehat{\boldsymbol{\beta}}) - I_5(\widehat{\boldsymbol{\beta}}) - I_3(\boldsymbol{\beta}_0) + I_5(\boldsymbol{\beta}_0)\} = o_p(1). \tag{6.28}$$

In the following, we prove

$$\sqrt{n}|I_4(\widehat{\boldsymbol{\beta}}) - I_4(\boldsymbol{\beta}_0)| = o_p(1). \tag{6.29}$$

Let

$$I_{4,j}(\boldsymbol{\beta}) = \frac{1}{n}\sum_{i=1}^n \{(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta} \le 0)\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*) - A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta} > 0)\exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\}Y_i\boldsymbol{X}_{i,j}.$$

By Condition (A9), we have with probability tending to 1,

$$I_4(\widehat{\boldsymbol{\beta}}) - I_4(\boldsymbol{\beta}_0) = \sum_{j\in\mathcal{M}_\alpha}\{I_{4,j}(\widehat{\boldsymbol{\beta}}) - I_{4,j}(\boldsymbol{\beta}_0)\}(\widehat{\alpha}_j - \alpha_j^*). \tag{6.30}$$

Similar to (6.10), we can show

$$\max_{j\in\mathcal{M}_\alpha}\sup_{\substack{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1 \le \kappa_n^{-1}\log^{-1}p \\ \|\boldsymbol{\beta}\|_0 \le \kappa_n}} |I_{4,j}(\boldsymbol{\beta}) - I_{4,j}(\boldsymbol{\beta}_0) - \mathrm{E}I_{4,j}(\boldsymbol{\beta}) + \mathrm{E}I_{4,j}(\boldsymbol{\beta}_0)| = O_p(n^{-1/2}).$$

Combining this together with (6.11), we have

$$\sup_{\substack{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le\kappa_n^{-1}\log^{-1}p\\ \|\boldsymbol{\beta}\|_0\le\kappa_n}}\left|\sum_{j\in\mathcal{M}_\alpha}\{I_{4,j}(\boldsymbol{\beta})-I_{4,j}(\boldsymbol{\beta}_0)-\mathrm{E}I_{4,j}(\boldsymbol{\beta})+\mathrm{E}I_{4,j}(\boldsymbol{\beta}_0)\}(\widehat{\alpha}_j-\alpha_j^*)\right| \tag{6.31}$$
$$\le \max_{j\in\mathcal{M}_\alpha}\sup_{\substack{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le\kappa_n^{-1}\log^{-1}p\\ \|\boldsymbol{\beta}\|_0\le\kappa_n}}|I_{4,j}(\boldsymbol{\beta})-I_{4,j}(\boldsymbol{\beta}_0)-\mathrm{E}I_{4,j}(\boldsymbol{\beta})+\mathrm{E}I_{4,j}(\boldsymbol{\beta}_0)|\|\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*\|_1$$
$$\le \sqrt{s_\alpha}\max_{j\in\mathcal{M}_\alpha}\sup_{\substack{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le\kappa_n^{-1}\log^{-1}p\\ \|\boldsymbol{\beta}\|_0\le\kappa_n}}|I_{4,j}(\boldsymbol{\beta})-I_{4,j}(\boldsymbol{\beta}_0)-\mathrm{E}I_{4,j}(\boldsymbol{\beta})+\mathrm{E}I_{4,j}(\boldsymbol{\beta}_0)|\|\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}^*\|_2$$
$$= O_p(\sqrt{s_\alpha}n^{-1/2}\sqrt{s_\alpha/n})=o_p(n^{-1/2}),$$

where the last inequality is due to the condition that $s_\alpha=o(\sqrt{n})$.

Besides, using similar arguments in (6.7), we can show

$$\max_{j\in\mathcal{M}_\alpha}\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}}|\mathrm{E}I_{4,j}(\boldsymbol{\beta})-\mathrm{E}I_{4,j}(\boldsymbol{\beta}_0)| \tag{6.32}$$
$$\le \max_{j\in\mathcal{M}_\alpha}\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}}\mathrm{E}|Y_0||X_{0,j}||I(\boldsymbol{X}_0^T\boldsymbol{\beta}>0)-I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0>0)|\exp(|\boldsymbol{X}_0^T\boldsymbol{\alpha}^*|)\le C_2n^{-1/4},$$

for some constant $C_2>0$. Let

$$K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})=\mathrm{E}\{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta}\le 0)\exp(\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)-A_0I(\boldsymbol{X}_0^T\boldsymbol{\beta}>0)\exp(-\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)\}Y_0\boldsymbol{X}_{0,j}.$$

By (6.32) and the condition that $s_\alpha=o(\sqrt{n})$, we have

$$\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}}\|K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\|_2\le C_2\sqrt{s_\alpha}n^{-1/4}=o(1). \tag{6.33}$$

Therefore, by Condition (A9),

$$\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}}\left|\{K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\}^T\left(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0-\frac{1}{n}\sum_{i=1}^n\boldsymbol{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(A_i-\pi_i^*)\right)\right| \tag{6.34}$$
$$\le \sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}}\|K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\|_2\left\|\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0-\frac{1}{n}\sum_{i=1}^n\boldsymbol{\Omega}^{-1}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(A_i-\pi_i^*)\right\|_2=o_p(n^{-1/2}).$$

Moreover, by (6.33) and Condition (A11), we have

$$
\begin{aligned}
& \sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}} \mathrm{E}\left|\{K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\}^T\frac{1}{n}\sum_{i=1}^n \boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(A_i-\pi_i^*)\right|^2 \\
= & \frac{1}{n}\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}} \mathrm{E}[\{K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\}^T\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\boldsymbol{X}_{0,\mathcal{M}_\alpha}\{A_0-\pi^*(\boldsymbol{X}_0,\boldsymbol{\alpha}^*)\}]^2 \\
\le & \frac{1}{n}\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}} \mathrm{E}\{K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\}^T\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\boldsymbol{X}_{0,\mathcal{M}_\alpha}\boldsymbol{X}_{0,\mathcal{M}_\alpha}^T\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\{K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\} \\
= & \frac{1}{n}\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}} \{K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\}^T\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\boldsymbol{\Sigma}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\{K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\} \\
\le & \frac{1}{n}\lambda_{\max}(\boldsymbol{\Sigma}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha})\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}} \|\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\{K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\}\|_2^2 \\
\le & \frac{1}{n}\lambda_{\max}(\boldsymbol{\Sigma}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha})\lambda_{\max}^2(\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1})\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}} \|K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\|_2^2 \\
= & \frac{1}{n}\lambda_{\max}(\boldsymbol{\Sigma}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha})\lambda_{\min}^{-2}(\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha})\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}} \|K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\|_2^2 = o(\frac{1}{n}).
\end{aligned}
$$

It follows from Chebyshev's inequality that

$$
\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}} \left\|\{K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\}^T\frac{1}{n}\sum_{i=1}^n \boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(A_i-\pi_i^*)\right\|_2 = o_p(n^{-1/2}).
$$

Combining this together with (6.34) yields

$$
\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}} \left|\{K_{\mathcal{M}_\alpha}(\boldsymbol{\beta})-K_{\mathcal{M}_\alpha}(\boldsymbol{\beta}_0)\}^T(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0)\right| = o_p(n^{-1/2}),
$$

or equivalently,

$$
\sup_{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le n^{-1/4}} \left|\sum_{j\in\mathcal{M}_\alpha}\{\mathrm{E}I_{4,j}(\boldsymbol{\beta})-\mathrm{E}I_{4,j}(\boldsymbol{\beta}_0)\}(\widehat{\alpha}_j-\alpha_j^*)\right| = o_p(n^{-1/2}).
$$

This together with (6.31) gives

$$
\sup_{\substack{\|\boldsymbol{\beta}-\boldsymbol{\beta}_0\|_1\le \min(n^{-1/4},\kappa_n^{-1}\log^{-1}p)\\ \|\boldsymbol{\beta}\|_0\le\kappa_n}} \left|\sum_{j\in\mathcal{M}_\alpha}\{I_{4,j}(\boldsymbol{\beta})-I_{4,j}(\boldsymbol{\beta}_0)\}(\widehat{\alpha}_j-\alpha_j^*)\right| = o_p(n^{-1/2}).
$$

By Condition (A3) and (A4), we have

$$
\left|\sum_{j\in\mathcal{M}_\alpha}\{I_{4,j}(\widehat{\boldsymbol{\beta}})-I_{4,j}(\boldsymbol{\beta}_0)\}(\widehat{\alpha}_j-\alpha_j^*)\right| = o_p(n^{-1/2}).
$$

Combining this together with (6.30) yields (6.29). Similarly, we can show

$$\sqrt{n}|I_6(\widehat{\boldsymbol{\beta}}) - I_6(\boldsymbol{\beta}_0)| = o_p(1), \quad \sqrt{n}|I_7(\widehat{\boldsymbol{\beta}}) - I_7(\boldsymbol{\beta}_0)| = o_p(1), \quad \sqrt{n}|I_8^*(\widehat{\boldsymbol{\beta}}) - I_8^*(\boldsymbol{\beta}_0)| = o_p(1).$$

These together with (6.27), (6.28) and (6.29) yield

$$\sqrt{n}\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}}) = \sqrt{n}\{I_3(\boldsymbol{\beta}_0) + I_4(\boldsymbol{\beta}_0) - I_5(\boldsymbol{\beta}_0) - I_6(\boldsymbol{\beta}_0) - I_7(\boldsymbol{\beta}_0) + I_8^*(\boldsymbol{\beta}_0)\} + o_p(1). \tag{6.35}$$

By Condition (A9), we have with probability tending to 1 that,

$$I_4(\boldsymbol{\beta}_0) = \frac{1}{n}\sum_{i=1}^n \{(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}_0 \le 0)\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*) - A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta}_0 > 0)\exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\}Y_i\boldsymbol{X}_{i,\mathcal{M}_\alpha}^T(\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha} - \boldsymbol{\alpha}_{\mathcal{M}_\alpha}^*).$$

By Condition (A5), Condition (A7) and Condition (A10), we have

$$\begin{aligned}
&\mathrm{E}\left\|\frac{1}{n}\sum_{i=1}^n \{(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}_0 \le 0)\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*) - A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta}_0 > 0)\exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\}Y_i\boldsymbol{X}_{i,\mathcal{M}_\alpha}\right.\\
&\qquad \left. -\mathrm{E}\{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 \le 0)\exp(\boldsymbol{X}_0^T\boldsymbol{\alpha}^*) - A_0 I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 > 0)\exp(-\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)\}Y_0\boldsymbol{X}_{0,\mathcal{M}_\alpha}\right\|_2^2\\
&= \frac{1}{n^2}\mathrm{E}\sum_{i=1}^n \big\|\{(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}_0 \le 0)\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*) - A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta}_0 > 0)\exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\}Y_i\boldsymbol{X}_{i,\mathcal{M}_\alpha}\\
&\qquad -\mathrm{E}\{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 \le 0)\exp(\boldsymbol{X}_0^T\boldsymbol{\alpha}^*) - A_0 I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 > 0)\exp(-\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)\}Y_0\boldsymbol{X}_{0,\mathcal{M}_\alpha}\big\|_2^2\\
&\le \frac{1}{n}\mathrm{E}\,\big\|\{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 \le 0)\exp(\boldsymbol{X}_0^T\boldsymbol{\alpha}^*) - A_0 I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 > 0)\exp(-\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)\}Y_0\boldsymbol{X}_{0,\mathcal{M}_\alpha}\big\|_2^2\\
&\le \frac{1}{n}\mathrm{E}\left(\exp(2|\boldsymbol{X}_0^T\boldsymbol{\alpha}^*|)Y_0^2\|\boldsymbol{X}_{0,\mathcal{M}_\alpha}\|_2^2\right) \le \frac{1}{n}\mathrm{E}\exp(2|\boldsymbol{X}_0^T\boldsymbol{\alpha}^*|)\mathrm{E}(Y_0^2|\boldsymbol{X}_0)\|\boldsymbol{X}_{0,\mathcal{M}_\alpha}\|_2^2\\
&\le \frac{c_0\exp(2\omega^*)}{n}\mathrm{E}\|\boldsymbol{X}_{0,\mathcal{M}_\alpha}\|_2^2 \le \frac{s_\alpha c_0\omega_0^2\exp(2\omega^*)}{n}.
\end{aligned}$$

It follows from Chebyshev's inequality that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n \{(1-A_i)I(\boldsymbol{X}_i^T\boldsymbol{\beta}_0 \le 0)\exp(\boldsymbol{X}_i^T\boldsymbol{\alpha}^*) - A_i I(\boldsymbol{X}_i^T\boldsymbol{\beta}_0 > 0)\exp(-\boldsymbol{X}_i^T\boldsymbol{\alpha}^*)\}Y_i\boldsymbol{X}_{i,\mathcal{M}_\alpha}\\
= \;& \mathrm{E}\{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 \le 0)\exp(\boldsymbol{X}_0^T\boldsymbol{\alpha}^*) - A_0 I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 > 0)\exp(-\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)\}Y_0\boldsymbol{X}_{0,\mathcal{M}_\alpha} + O_p(s_\alpha^{1/2}n^{-1/2}).
\end{aligned}$$

By (6.11) and the condition that $s_\alpha = o(\sqrt{n})$, we have $\sqrt{n}I_4(\boldsymbol{\beta}_0) = \sqrt{n}I_4^*(\boldsymbol{\beta}_0) + o_p(1)$ where

$$I_4^*(\boldsymbol{\beta}_0) = (\widehat{\boldsymbol{\alpha}}_{\mathcal{M}_\alpha} - \boldsymbol{\alpha}_{\mathcal{M}_\alpha}^*)^T \underbrace{\mathrm{E}\{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 \le 0)\exp(\boldsymbol{X}_0^T\boldsymbol{\alpha}^*) - A_0 I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 > 0)\exp(-\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)\}Y_0\boldsymbol{X}_{0,\mathcal{M}_\alpha}}_{I_4^{**}}.$$

Note that

$$\|I_4^{**}\|_2 = \sup_{\substack{\boldsymbol{a}\in\mathbb{R}^{s_\alpha}\\ \|\boldsymbol{a}\|_2=1}} |\mathrm{E}\boldsymbol{a}^T\{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 \le 0)\exp(\boldsymbol{X}_0^T\boldsymbol{\alpha}^*) - A_0 I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0 > 0)\exp(-\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)\}Y_0\boldsymbol{X}_{0,\mathcal{M}_\alpha}|$$

By Condition (A7), Condition (A10) and Condition (A11), we have

$$
\begin{aligned}
&|\mathrm{E}\boldsymbol{a}^T\{(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0\le 0)\exp(\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)-A_0I(\boldsymbol{X}_0^T\boldsymbol{\beta}_0>0)\exp(-\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)\}Y_0\boldsymbol{X}_{0,\mathcal{M}_\alpha}| \\
&\qquad \le \exp(\omega^*)\mathrm{E}|Y_0||\boldsymbol{a}^T\boldsymbol{X}_{0,\mathcal{M}_\alpha}| \le \exp(\omega^*)\sqrt{\mathrm{E}|Y_0|^2}\sqrt{\mathrm{E}|\boldsymbol{a}^T\boldsymbol{X}_{0,\mathcal{M}_\alpha}|^2} \\
&\le \exp(\omega^*)\sqrt{\sup_{\boldsymbol{x}}\mathrm{E}(|Y_0|^2|\boldsymbol{X}_0=\boldsymbol{x})}\sqrt{\boldsymbol{a}^T\lambda_{\max}(\boldsymbol{\Sigma}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha})\boldsymbol{a}} \le \sqrt{c_0c_4}\exp(\omega^*)\|\boldsymbol{a}\|_2.
\end{aligned}
$$

Therefore,

$$
\|I_4^{**}\|_2 \le \sqrt{c_0c_4}\exp(\omega^*).
$$

By Condition (A9), $\sqrt{n}I_4^*(\boldsymbol{\beta}_0)$ is asymptotically equivalent to

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^n (I_4^{**})^T\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(A_i-\pi_i^*).
$$

To summarize, we have shown

$$
\sqrt{n}I_4(\boldsymbol{\beta}_0)=\frac{1}{\sqrt{n}}\sum_{i=1}^n (I_4^{**})^T\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(A_i-\pi_i^*)+o_p(1). \tag{6.36}
$$

Similarly, we can show

$$
\begin{aligned}
\sqrt{n}I_6(\boldsymbol{\beta}_0) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \boldsymbol{\xi}_{\mathcal{M}_\eta}^T\boldsymbol{\Xi}_{\mathcal{M}_\eta,\mathcal{M}_\eta}^{-1}\boldsymbol{X}_{i,\mathcal{M}_\eta}A_i(Y_i-\boldsymbol{X}_i^T\boldsymbol{\eta}^*)+o_p(1), \\
\sqrt{n}I_7(\boldsymbol{\beta}_0) &= \frac{1}{\sqrt{n}}\sum_{i=1}^n \boldsymbol{\phi}_{\mathcal{M}_\theta}^T\boldsymbol{\Phi}_{\mathcal{M}_\theta,\mathcal{M}_\theta}^{-1}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(1-A_i)(Y_i-X_i^T\boldsymbol{\theta}^*)+o_p(1),
\end{aligned}
$$

and

$$
\sqrt{n}I_8^*(\boldsymbol{\beta}_0)=\frac{1}{\sqrt{n}}\sum_{i=1}^n (I_8^{**})^T\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(A_i-\pi_i^*)+o_p(1),
$$

where

$$
I_8^{**}=\mathrm{E}\{A_0I(\boldsymbol{X}_0^T\boldsymbol{\beta}>0)\exp(-\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)\boldsymbol{X}_0^T\boldsymbol{\eta}^*-(1-A_0)I(\boldsymbol{X}_0^T\boldsymbol{\beta}\le 0)\exp(\boldsymbol{X}_0^T\boldsymbol{\alpha}^*)\boldsymbol{X}_0^T\boldsymbol{\theta}^*\}\boldsymbol{X}_{0,\mathcal{M}_\alpha}.
$$

Combining this together with (6.35) and (6.36) yields

$$
\begin{aligned}
&\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})=I_3(\boldsymbol{\beta}_0)-I_5(\boldsymbol{\beta}_0)-\frac{1}{n}\sum_{i=1}^n \boldsymbol{\phi}_{\mathcal{M}_\theta}^T\boldsymbol{\Phi}_{\mathcal{M}_\theta,\mathcal{M}_\theta}^{-1}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(1-A_i)(Y_i-X_i^T\boldsymbol{\theta}^*) \\
-\;&\frac{1}{n}\sum_{i=1}^n \boldsymbol{\xi}_{\mathcal{M}_\eta}^T\boldsymbol{\Xi}_{\mathcal{M}_\eta,\mathcal{M}_\eta}^{-1}\boldsymbol{X}_{i,\mathcal{M}_\eta}A_i(Y_i-\boldsymbol{X}_i^T\boldsymbol{\eta}^*)+\frac{1}{n}\sum_{i=1}^n \boldsymbol{\omega}_{\mathcal{M}_\alpha}^T\boldsymbol{\Omega}_{\mathcal{M}_\alpha,\mathcal{M}_\alpha}^{-1}\boldsymbol{X}_{i,\mathcal{M}_\alpha}(A_i-\pi_i^*)+o_p(n^{-1/2}).
\end{aligned}
$$

By (4.1) and (4.2), we have

$$\mathrm{E}\boldsymbol{X}_{0,\mathcal{M}_\theta}(1-A_0)(Y_0-\boldsymbol{X}_0^T\boldsymbol{\theta}^*)=0, \quad \mathrm{E}\boldsymbol{X}_{0,\mathcal{M}_\eta}A_0(Y_0-\boldsymbol{X}_0^T\boldsymbol{\eta}^*)=0, \quad \mathrm{E}\boldsymbol{X}_{0,\mathcal{M}_\alpha}(A_0-\pi_0^*)=0.$$

When either the propensity score or the conditional mean models are correct, we have

$$\mathrm{E}\{I_3(\boldsymbol{\beta}_0)-I_5(\boldsymbol{\beta}_0)\}=V_0.$$

Hence $\sqrt{n}\{\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})-V_0\}$ is asymptotically equivalent to the sum of n i.i.d mean zero random variables. The asymptotic normality of $\sqrt{n}\{\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})-V_0\}$ follows by the Lindeberg central limit theorem. With some calculations, we can show the asymptotic variance of $\sqrt{n}\{\widehat{V}^{dr}(\widehat{\boldsymbol{\beta}})-V_0\}$ is equal to $v_1+v_2+v_3$ where v_1, v_2, v_3 are defined in Theorem 4.1. The proof is hence completed. □

7. Discussion

In this paper, we propose a valid inference procedure for the optimal value function in ultrahigh dimensions. The proposed estimated optimal value function is double robust, in the sense that it is asymptotically normal as long as either the propensity score function, or the conditional mean functions are correctly specified. To derive its double robustness property, we establish the oracle properties of the regularized estimators in the propensity score and conditional mean models. Currently, we only consider a single stage study with two binary treatments. It is practically useful to extend the proposed procedure to multi-stages with multiple treatment options. These topics warrant further investigations.

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