

# Tails, Fears, and Risk Premia

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## ABSTRACT

We show that the compensation for rare events accounts for a large fraction of the average equity and variance risk premia. Exploiting the special structure of the jump tails and the pricing thereof, we identify and estimate a new Investor Fears index. The index reveals large time-varying compensation for fears of disasters. Our empirical investigations involve new extreme value theory approximations and high-frequency intraday data for estimating the expected jump tails under the statistical probability measure, and short maturity out-of-the-money options and new model-free implied variation measures for estimating the corresponding risk-neutral expectations.

*So what are policymakers to do? First and foremost, reduce uncertainty. Do so by removing tail risks, and the perception of tail risks.*

—Olivier Blanchard, chief economist, IMF, *The Economist*, January 31, 2009.

LACK OF INVESTOR CONFIDENCE is frequently singled out as one of the main culprits behind the massive losses in market values in the advent of the Fall 2008 financial crises. At the time, the portrayal in the financial news media of different doomsday scenarios in which the stock market would have declined even further were also quite commonplace. Motivated by these observations, we provide a new theoretical framework for better understanding the way in

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which the market prices and perceives jump tail risks. Our estimates rely on the use of actual high-frequency intraday data and short maturity out-of-the-money options. Our empirical results based on data for the S&P 500 index spanning the period from 1990 to mid-2007 show that the market generally incorporates the possible occurrence of rare disasters in the way it prices risky payoffs, and the fear of such events accounts for a surprisingly large fraction of the historically observed equity and variance risk premia. Extending our results through the end of 2008, we document an even larger role of investor fears during the recent financial market crises. As such, our findings implicitly support the conjecture quoted above in that removing the perception of jump tail risk was indeed crucial in restoring asset values at the time.

More precisely, by exploiting the special structure of the jump tails and the pricing thereof, we identify and estimate a new Investors Fears index.<sup>1</sup> Our identification of investors' fears is based on the distinctly different economic roles played by the compensation for the pathwise variation in asset prices versus the compensation for the possible occurrence of "large" rare jump events. These two separate sources of variation have traditionally been treated as latent constituents of the market risk premia. Our findings suggest that compensation for the former, which is naturally associated with temporal variation in the investment opportunity set, is rather modest, while the compensation attributable to the latter, and the fear of rare events, is both time varying and often quite large.<sup>2</sup>

Our empirical estimates rely on a series of new statistical procedures for backing out the objective expectations of jump tail events and the market's pricing thereof. In particular, building on new extreme value theory (EVT) developed in Bollerslev and Todorov (2011), we use the relatively frequent "medium" sized jumps in high-frequency intraday returns for reliably estimating the expected interdaily tail events under the statistical probability measure, thereby explicitly avoiding peso type problems in inferring the actual jump tails.

Similarly, our estimates for the market risk-neutral expectations are based on actual short maturity out-of-the-money options and the model-free variation measures originally used by Carr and Wu (2003) and further developed here to recover the complete risk-neutral tail jump density. Intuitively, while the continuous price variation and the possibility of jumps both affect short maturity options, their relative importance varies across different strikes, which allows us to separate the valuation of the two risks in a model-free manner.

<sup>1</sup> Our Investor Fears index is conceptually different from standard measures of tail risks that only depend on the actual probabilities for the occurrences of tail events but not the pricing thereof; see, for example, the discussion in Artzner et al. (1999). Further, most traditional tail risk measures apply to any portfolio, while the Investor Fears index pertains explicitly to the pricing of jump tail risk in the aggregate market portfolio.

<sup>2</sup> The relatively minor role played by temporal variation in the investment opportunity set for explaining the risk premia suggested by our results is also consistent with the evidence in Chacko and Viceira (2005), who find only modest gains from standard intertemporal hedging in the context of dynamic consumption and portfolio choice.

The general idea behind the paper is related to an earlier literature that seeks to explain the observed differences in the time series behavior of options prices and the prices of the underlying asset through the pricing of jump risk; see, for example, Andersen, Benzoni, and Lund (2002), Bates (1996, 2000), Broadie, Chernov, and Johannes (2009a), Eraker (2004), and Pan (2002). The paper is also related to the work of Aït-Sahalia, Wang, and Yared (2001), who compare nonparametric estimates of the risk-neutral state price densities from time series of returns to the densities estimated directly from options prices. All of these earlier studies rely on specific, typically affine, parametric stochastic volatility jump diffusion models, or an assumption of no unspanned stochastic volatility. Our approach is distinctly different in that we rely on flexible nonparametric procedures that are able to accommodate rather complex dynamic tail dependencies as well as time-varying stochastic volatility. Our focus on the aggregate market risk premia and the notion of investor fears implicit in the wedge between the estimated objective and risk-neutral jump tails, along with the richer data sources used in the estimation, also set the paper apart from this earlier literature.

Our empirical results suggest that on average close to 5% of the equity premium (in absolute terms) may be attributed to the compensation for rare disaster events. Related option-based estimates for the part of the equity premium due to compensation for jump risk have previously been reported in the literature by Broadie, Chernov, and Johannes (2009a), Eraker (2004), Pan (2002), and Santa-Clara and Yan (2010), among others. However, as noted above, these earlier studies are based on fairly tightly parameterized jump-diffusion models and restrictive assumptions about the shapes and dependencies in the tails.<sup>3</sup> Counter to these assumptions, our new nonparametric approach reveals a series of intriguing dynamic dependencies in the tail probabilities, including time-varying jump intensities and a tendency for fatter tails in periods of high overall volatility. At the same time, our results also suggest that the risk premium for tail events cannot solely be explained by the level of the volatility.

In parallel to the equity premium, defined as the difference between the statistical and risk-neutral expectation of the aggregate market returns, the variance risk premium is naturally defined as the difference between the statistical and risk-neutral expectation of the corresponding forward variation.<sup>4</sup> Taking our analysis one step further, we show that on average more than half of the historically observed variance risk premium is directly attributable to disaster risk, or differences in the jump tails of the risk-neutral and objective

<sup>3</sup> Anderson, Hansen, and Sargent (2003) and Broadie, Chernov, and Johannes (2009) both forcefully argue from very different perspectives that misspecified parametric models can result in highly misleading estimates for the risk premia.

<sup>4</sup> Several recent studies seek to explain the existence of a variance risk premium, and how the premium correlates with the underlying returns, within the context of various equilibrium-based pricing models; see, for example, Bakshi and Kapadia (2003), Bakshi and Madan (2006), Bollerslev, Tauchen, and Zhou (2009), Carr and Wu (2009), Drechsler and Yaron (2011), Eraker (2008), and Todorov (2010).

distributions. Moreover, while the left and right jump tails in the statistical distribution are close to symmetric, the contribution to the overall risk-neutral variation coming from the left tail associated with dramatic market declines is several times larger than that attributable to the right tail. Therefore, even though the Chicago Board Options Exchange (CBoE) VIX volatility index formally contains compensation for different risks, that is, time-varying volatilities and jump intensities as well as fears for jump tail events, our separation shows that a nontrivial portion of the index may be attributed to the latter component and notions of investor fear.<sup>5</sup>

At a more general level, our results hold the promise of better understanding the economics behind the historically large average equity and variance risk premia observed in the data. Most previous equilibrium-based explanations put forth in the literature effectively treat the two risk premia in isolation, relying on different mechanisms for mapping the dynamics of the underlying macroeconomic fundamentals into asset prices. Instead, our analysis shows that high-frequency aggregate market and derivative prices together contain sufficient information to jointly identify both premia in an essentially model-free manner, and that the fear of rare events plays an important role in determining both. As such, any satisfactory equilibrium-based asset pricing model must be able to generate large and time-varying compensations for the possible occurrence of rare disasters.<sup>6</sup>

The plan for the rest of the paper is as follows. We begin in the next section with a discussion of the basic setup and assumptions, along with the relevant formal definitions of the equity and variance risk premia. Section II outlines the procedures that we use for separating jumps and continuous price variation under the risk-neutral distribution. Section III details the methods that we use for the comparable separation under the statistical measure. Section IV discusses the data and our initial empirical results. Our main empirical findings related to the tail risks and the risk premia are given in Section V. Section VI reports on various robustness checks related to the dependencies in the tail measures and the accuracy of the asymptotic approximations underlying our results. Section VII concludes. Most of the technical proofs are deferred to the Appendix, as are some of the details concerning our handling of the options and intraday data, as well as the econometric modeling procedures.

<sup>5</sup> As such, this does lend some credence to the common use of the term “investor’s fear gauge” as a moniker for the VIX volatility index, albeit a rather imperfect proxy at that; see, for example, Whaley (2009). Instead, the new Investor Fears index developed here “cleanly” isolates the fear component.

<sup>6</sup> The idea that rare disasters, or tail events, may help explain the equity premium and other empirical puzzles in asset pricing finance dates back at least to Rietz (1988). While the original work by Rietz (1988) is based on a peso type explanation and probabilities of severe events that exceed those materialized in sample, Barro (2006) argues that, when calibrated to international data, actual disaster events do appear frequent enough to meaningfully impact the size of the equity premium. Further building on these ideas, Gabaix (2010) and Wachter (2010) have shown that explicitly incorporating time-varying risks of rare disasters in otherwise standard equilibrium-based asset pricing models may help explain the apparent excess volatility of aggregate market returns.

## I. Risk Premia and Jumps

The continuous-time no-arbitrage framework that underlies our empirical investigations is very general and essentially model-free. It includes all parametric models previously analyzed and estimated in the literature as special cases. We begin with a discussion of the basic setup and assumptions.

### A. Setup and Assumptions

To lay out the notation, let  $F_t$  denote the futures price for the aggregate market portfolio expiring at some undetermined future date, and let the corresponding logarithmic price be denoted by  $f_t \equiv \log(F_t)$ . The absence of arbitrage implies that the futures price should be a semimartingale.<sup>7</sup> Restricting this just slightly by assuming an Itô semimartingale along with jumps of finite variation, the dynamics of the futures price may be expressed as<sup>8</sup>

$$\frac{dF_t}{F_t} = \alpha_t dt + \sigma_t dW_t + \int_{\mathbb{R}} (e^x - 1) \tilde{\mu}(dt, dx), \quad (1)$$

where  $\alpha_t$  and  $\sigma_t$  denote locally bounded but otherwise unspecified drift and instantaneous volatility processes, respectively, and  $W_t$  is a standard Brownian motion. The last term accounts for any jumps, or discontinuities, in the price process through the so-called compensated jump measure  $\tilde{\mu}(dt, dx) \equiv \mu(dt, dx) - v_t^{\mathbb{P}}(dx) dt$ , where  $\mu(dt, dx)$  is a simple counting measure for the jumps and  $v_t^{\mathbb{P}}(dx) dt$  denotes the compensator (stochastic intensity) of the jumps for  $v_t^{\mathbb{P}}(dx)$  being predictable and such that  $\int_{\mathbb{R}} (x^2 \wedge 1) v_t^{\mathbb{P}}(dx)$  is locally integrable.<sup>9</sup> This is a very general specification, and, aside from the implicit weak integrability conditions required for the existence of some of the expressions discussed below, we do not make any further assumptions about the properties of the jumps or the form of the stochastic volatility process.<sup>10</sup>

To better understand the impact and pricing of the two separate components, it is instructive to consider the total variation associated with the market price. Specifically, let  $QV_{[t, T]}$  denote the quadratic variation of the log-price process over the  $[t, T]$  time interval,

$$QV_{[t, T]} \equiv \int_t^T \sigma_s^2 ds + \int_t^T \int_{\mathbb{R}} x^2 \mu(ds, dx). \quad (2)$$

<sup>7</sup> For the connection between the semimartingale assumption for the price and the notion of no arbitrage, see, for example, Delbaen and Schachermayer (1994).

<sup>8</sup> The assumption of finite variation for the jumps is inconsequential and for expositional purposes only. The “big” jumps are always of finite activity, that is, only a finite number over any finite time interval.

<sup>9</sup> Intuitively, this renders the “demeaned” sum of the jumps  $\int_{\mathbb{R}} (e^x - 1) \tilde{\mu}(dt, dx)$  a (local) martingale.

<sup>10</sup> The specification in (1) does exclude semimartingale processes whose characteristics are not absolutely continuous with respect to Lebesgue measure, for example, certain time-changed Brownian motions where the time change is a discontinuous process.

The first term on the right-hand side represents the variation attributable to the continuous-time stochastic volatility process  $\sigma_s$ , that is, the variation due to “small” price moves. The second term measures the variation due to jumps, that is, large price moves. These risks are fundamentally different and present different challenges from a risk management perspective. While diffusive risks can be hedged (managed) by continuously rebalancing a portfolio exposed to those risks, the locally unpredictable nature of jumps means that such a strategy will not work. Consequently, these two types of risks may also demand different compensation, as manifest in the form of different contributions to the aggregate equity and variance risk premia.

### B. Equity and Variance Risk Premia

The equity risk premium represents the compensation directly associated with the uncertainty about the future price level. The variance risk premium refers to the compensation for the risk associated with temporal changes in the variation of the price level.<sup>11</sup>

In particular, let  $\mathbb{Q}$  denote the risk-neutral distribution associated with the general dynamics in equation (1). The equity and variance risk premia are then formally defined by<sup>12</sup>

$$ERP_t \equiv \frac{1}{T-t} \left( \mathbb{E}_t^{\mathbb{P}} \left( \frac{F_T - F_t}{F_t} \right) - \mathbb{E}_t^{\mathbb{Q}} \left( \frac{F_T - F_t}{F_t} \right) \right), \quad (3)$$

and

$$VRP_t \equiv \frac{1}{T-t} (\mathbb{E}_t^{\mathbb{P}}(QV_{[t,T]}) - \mathbb{E}_t^{\mathbb{Q}}(QV_{[t,T]})), \quad (4)$$

which correspond to the expected excess return on the market and the expected payoff on a (long) variance swap on the market portfolio, respectively. Empirically, of course, the sample equity risk premium is generally positive and historically large, while the sample variance risk premium as defined above is on average negative and equally large and puzzling.<sup>13</sup>

<sup>11</sup> Investor desire to hedge against intertemporal shifts in riskiness has led to the recent advent of many new financial instruments with their payoffs directly tied to various notions of realized price variation. Especially prominent among these are variance swap, or forward, contracts on future realized variances. Exotic so-called gap options explicitly designed to hedge against large price moves over short daily time intervals have also recently been introduced on the OTC market.

<sup>12</sup> From here on we will suppress the dependence on the horizon  $T$  for notational convenience. Also, the discrete-time equity premium defined here should not be confused with the instantaneous premium, or drift, defined in equation (5) below. The latter serves a particular convenient role in explicitly separating the different types of risks, as manifest in equation (12).

<sup>13</sup> The equity premium puzzle and the failure of standard consumption-based asset pricing models to explain the sample  $ERP_t$  is arguably one of the most studied issues in academic finance over the past two decades; see, for example, the discussion in Campbell and Cochrane (1999) and Bansal and Yaron (2004) and the many references therein. Representative agent equilibrium models involving time-separable utility also rule out priced volatility risk and corresponding hedging demands, and in turn imply that  $VRP_t = 0$ . Recent rational equilibrium-based pricing

To disentangle the distributional features that account for these well-documented differences in the  $\mathbb{P}$  and  $\mathbb{Q}$  expectations, let  $v_t^{\mathbb{Q}}(dx)dt$  denote the compensator, or intensity, for the jumps under  $\mathbb{Q}$  (recall our absolute continuity assumption for the  $\mathbb{P}$  jump compensator and the fact that the probability measures  $\mathbb{P}$  and  $\mathbb{Q}$  are equivalent). Also, let  $\lambda_t$  denote the drift that turns  $W_t$  into a Brownian motion under  $\mathbb{Q}$ , that is,  $W_t^{\mathbb{Q}} = W_t + \int_0^t \lambda_s ds$ .<sup>14</sup> The drift term for the futures price in the  $\mathbb{P}$  distribution in equation (1) must therefore satisfy

$$\alpha_t = \lambda_t \sigma_t + \int_{\mathbb{R}} (e^x - 1) v_t^{\mathbb{P}}(dx) - \int_{\mathbb{R}} (e^x - 1) v_t^{\mathbb{Q}}(dx). \quad (5)$$

This general expression directly manifests the compensation that is required by investors to accept the different types of equity price-related risks.

Using this result, the equity risk premium is naturally decomposed as

$$ERP_t = ERP_t^c + ERP_t^d, \quad (6)$$

where

$$ERP_t^c \equiv \frac{1}{T-t} \mathbb{E}_t^{\mathbb{P}} \left( \int_t^T \frac{F_s}{F_t} \lambda_s \sigma_s ds \right), \quad (7)$$

and

$$ERP_t^d \equiv \frac{1}{T-t} \mathbb{E}_t^{\mathbb{P}} \left( \int_t^T \int_{\mathbb{R}} \frac{F_s}{F_t} (e^x - 1) v_s^{\mathbb{P}}(dx) ds - \int_t^T \int_{\mathbb{R}} \frac{F_s}{F_t} (e^x - 1) v_s^{\mathbb{Q}}(dx) ds \right), \quad (8)$$

with the two terms representing the unique contribution to the premium coming from diffusive and jump risk, respectively.

Similarly, the variance risk premium may be expressed as

$$VRP_t = VRP_t^c + VRP_t^d, \quad (9)$$

where

$$VRP_t^c = \frac{1}{T-t} \left( \mathbb{E}_t^{\mathbb{P}} \left( \int_t^T \sigma_s^2 ds \right) - \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T \sigma_s^2 ds \right) \right), \quad (10)$$

and

$$VRP_t^d = \frac{1}{T-t} \left( \mathbb{E}_t^{\mathbb{P}} \left( \int_t^T \int_{\mathbb{R}} x^2 v_s^{\mathbb{P}}(dx) ds \right) - \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T \int_{\mathbb{R}} x^2 v_s^{\mathbb{Q}}(dx) ds \right) \right), \quad (11)$$

models designed to help explain the positive variance risk premium by incorporating jumps and/or time-varying volatility-of-volatility in the consumption growth process of the representative agent include Bollerslev, Tauchen, and Zhou (2009), Drechsler and Yaron (2011), and Eraker (2008).

<sup>14</sup> Standard asset pricing theory implies that all futures (and forward) contracts must be martingales under the risk-neutral  $\mathbb{Q}$  measure; see, for example, Duffie (2001). This result is subject to a boundedness condition on the interest rate process, but such assumption can be further relaxed; see Pozdnyakov and Steele (2004).

representing the diffusive and jump risk components, respectively. As equations (10) and (11) make clear, the presence of jumps may result in a nonzero premium even with i.i.d. returns.<sup>15</sup>

The jump risk directly links the equity and variance risk premia. By contrast, the price of diffusive price risk does not enter the variance risk premium. Previous attempts in the literature to separately identify and estimate the risk premia due to jumps have been based on explicit and somewhat restrictive parametric models that result in different and sometimes conflicting numerical findings; see, for example, the discussion in chapter 15 of Singleton (2006). Meanwhile, as equations (6) and (9) make clear, the compensation for jumps generally affects the equity and variance risk premia in distinctly different, and statistically readily identifiable, ways.

In particular, using actual high-frequency data and short maturity options, it is possible to nonparametrically estimate<sup>16</sup>

$$\begin{aligned} ERP_t(k) = & \frac{1}{T-t} \left( \mathbb{E}_t^{\mathbb{P}} \left( \int_t^T \int_{|x|>k} (e^x - 1) v_s^{\mathbb{P}}(dx) ds \right) \right. \\ & \left. - \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T \int_{|x|>k} (e^x - 1) v_s^{\mathbb{Q}}(dx) ds \right) \right), \end{aligned} \quad (12)$$

and

$$VRP_t(k) = \frac{1}{T-t} \left( \mathbb{E}_t^{\mathbb{P}} \left( \int_t^T \int_{|x|>k} x^2 v_s^{\mathbb{P}}(dx) ds \right) - \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T \int_{|x|>k} x^2 v_s^{\mathbb{Q}}(dx) ds \right) \right), \quad (13)$$

where  $k > 0$  refers to the threshold that separates large from small jumps.<sup>17</sup> Comparing these estimates with the corresponding estimates for  $ERP_t$  with  $VRP_t$ , we may therefore gauge the importance of fears and rare events in explaining the equity and variance risk premia from directly observable financial data. We begin our discussion of the relevant empirical procedures with the way in which we quantify the risk-neutral  $\mathbb{Q}$  measures.

<sup>15</sup> It is possible to further separate  $VRP_t^c$  into two components: diffusive and jump-type changes in the stochastic volatility. We purposely do not explore this, however, as our main focus is on the separation of the fear of disasters from the premia attached to changes in the investment opportunity set, and the type of changes underlying the evolution in  $\sigma_t$  has no effect on that separation. Further decomposing  $VRP_t^c$  would also necessitate stronger parametric assumptions than those employed here.

<sup>16</sup> The integrands in equations (8) and (12) obviously differ by  $F_s/F_t$ . Alternatively, we could have defined the equity risk premium in terms of logarithmic returns, in which case  $F_s/F_t$  would not appear in equation (8). In that case, however, the basic definition would instead involve convexity adjustment terms.

<sup>17</sup> The choice of  $k$  and other practical implementation issues are discussed further in the Appendix below.

## II. Risk-Neutral Measures

There is long history of using option prices to infer market expectations and corresponding option implied volatilities. Traditionally, these estimates have been based on specific assumptions about the underlying  $\mathbb{P}$  distribution and the pricing of systematic risk(s). However, as emphasized in the literature more recently, the risk-neutral expectation of the total quadratic variation of the price process in equation (2), normalized by the horizon  $T - t$ ,

$$QV_t^{\mathbb{Q}} \equiv \frac{1}{T-t} \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T \sigma_s^2 ds \right) + \frac{1}{T-t} \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T \int_{\mathbb{R}} x^2 \mu(ds, dx) \right), \quad (14)$$

may be estimated in a completely model-free manner by a portfolio of options with different strikes; see, for example, Bakshi, Kapadia, and Madan (2003), Britten-Jones and Neuberger (2000), and Carr and Wu (2009).<sup>18</sup> Meanwhile, as the expression in equation (14) makes clear, and in line with the discussion above, the expected quadratic variation under  $\mathbb{Q}$  is naturally decomposed into the expected variation associated with diffusive, or small price moves, and discontinuous, or large price moves. Building on these insights, we show below how option prices may be used in estimating these two separate parts of the risk-neutral distribution in an essentially model-free fashion.<sup>19</sup>

### A. Jumps and Tails

Our estimates for the jump risk perceived by investors are based on close-to-maturity deep out-of-the-money options. The use of deep out-of-the-money options to more effectively isolate jump risk has previously been applied in the estimation of specific parametric affine jump-diffusion stochastic volatility models by Bakshi, Cao, and Chen (1997), Bates (1996, 2000), Pan (2002), and Eraker (2004) among others, while Pan and Liu (2003) and Pan, Liu, and Longstaff (2003) consider similar fully parametric approaches in the context of hedging portfolio jump risks.<sup>20</sup> In contrast to these studies, our estimates for the jump risk premia are entirely nonparametric.

<sup>18</sup> This also mirrors the way in which the CBOE now calculates the VIX volatility index for the S&P 500. The approximation used by the CBOE formally differs from the expression in equation (14) by  $\frac{1}{T-t} \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T \int_{\mathbb{R}} 2(e^x - 1 - x - \frac{x^2}{2}) \mu(ds, dx) \right)$ ; for further discussion of the approximation errors involved in this calculation see the white paper on the CBOE website, as well as Carr and Wu (2009) and Jiang and Tian (2005, 2007).

<sup>19</sup> Previous attempts at nonparametrically estimating the risk-neutral distribution implied by options prices include Aït-Sahalia and Lo (1998, 2000), Bakshi, Kapadia, and Madan (2003), and Rosenberg and Engle (2002), among others. In particular, Aït-Sahalia, Wang, and Yared (2001) and seek to quantify the amount of additional jump risk needed to make the  $\mathbb{P}$  tails comparable to the options-implied  $\mathbb{Q}$  tails, by calibrating the jump frequency to match the estimated skewness (or kurtosis) of the state price density. To the best of our knowledge, however, none of these studies have sought to estimate the portion of the two risk premia that are directly attributable to the jump tails and the fears thereof.

<sup>20</sup> As previously noted, the recent exotic so-called gap options, or gap risk swaps, are explicitly designed to hedge against gap events, or jumps, in the form of a one-day large price move in the

In particular, building on the ideas in Carr and Wu (2003), who use the different rates of decay of the diffusive and jump risks for options with various levels of moneyness as a way to test for jumps, we rely on short-maturity options to actually identify the compensation for jump risk. Intuitively, short-maturity out-of-the-money options remain worthless unless a rare event, or a big jump, occurs before expiration. As such, this allows us to infer the relevant risk-neutral expectations through appropriately scaled option prices.

Specifically, let  $C_t(K)$  and  $P_t(K)$  denote the price of a call and a put option, respectively, with a strike of  $K$  recorded at time  $t$  expiring at some future date,  $T$ . It follows that for  $T \downarrow t$  and  $K > F_{t-} \equiv \lim_{s \uparrow t} F_s$ , where  $\lim_{s \uparrow t}$  denotes the limit from the left,

$$e^{r(t,T)} C_t(K) \approx \int_t^T \mathbb{E}_t^{\mathbb{Q}} \left( \int_{\mathbb{R}} 1_{\{F_{s-} < K\}} (F_{s-} e^x - K)^+ v_s^{\mathbb{Q}}(dx) \right) ds, \quad (15)$$

while for  $K < F_{t-}$ ,

$$e^{r(t,T)} P_t(K) \approx \int_t^T \mathbb{E}_t^{\mathbb{Q}} \left( \int_{\mathbb{R}} 1_{\{F_{s-} > K\}} (K - F_{s-} e^x)^+ v_s^{\mathbb{Q}}(dx) \right) ds, \quad (16)$$

where  $x^+ \equiv \max\{0, x\}$ . Importantly, the expressions on the right-hand sides of (15) and (16) involve only the jump measure. Over short time intervals, changes in the price due to the continuous component are invariably small relative to the possible impact of large jumps, and the diffusive part may simply be ignored. Moreover, for  $T \downarrow t$  there will be at most one large jump. The right-hand sides of equations (15) and (16) represent the conditional expected values of the two different option payoffs under these approximations. The payoffs equal zero if no large jump occurred and otherwise are equal to the difference between the price after the jump and the strike price of the option. The proof of Proposition 1 below given in Subsection A of the Appendix contains a more formal analysis of the errors involved in the approximations and their orders of magnitude.

Going one step further, the expressions in equations (15) and (16) may be used in the construction of the following model-free risk-neutral jump tail measures:

$$RT_t^{\mathbb{Q}}(k) \equiv \frac{1}{T-t} \int_t^T \int_{\mathbb{R}} (e^x - e^k)^+ \mathbb{E}_t^{\mathbb{Q}}(v_s^{\mathbb{Q}}(dx)) ds \approx \frac{e^{r(t,T)} C_t(K)}{(T-t) F_{t-}}, \quad (17)$$

and

$$LT_t^{\mathbb{Q}}(k) \equiv \frac{1}{T-t} \int_t^T \int_{\mathbb{R}} (e^k - e^x)^+ \mathbb{E}_t^{\mathbb{Q}}(v_s^{\mathbb{Q}}(dx)) ds \approx \frac{e^{r(t,T)} P_t(K)}{(T-t) F_{t-}}, \quad (18)$$

where  $k \equiv \ln(\frac{K}{F_{t-}})$  denotes the log-moneyness. Unlike the variance replicating portfolio used in the construction of the VIX index and the approximation to

underlying; see Cont and Tankov (2009) and Tankov (2009) for further discussion on the pricing of these contracts using different parametric models as well as their hedging using standard exchanged-traded out-of-the-money options.

$QV_t^{\mathbb{Q}}$  in equation (14), which holds true for any value of  $T$ , the approximations for  $RT_t^{\mathbb{Q}}(k)$  and  $LT_t^{\mathbb{Q}}(k)$  in equations (17) and (18) depend on  $T \downarrow t$ .

In practice, of course, we invariably work with close-to-maturity options and a finite horizon  $T - t$ , so the effect of the diffusive volatility might be nontrivial. In Section VI, we report representative results from an extensive Monte Carlo simulation study designed to investigate the “finite sample” accuracy of the approximations across different models, option strikes  $K$ , and time-to-maturity  $T - t$ . Taken as a whole, these results suggest that the error involved in approximating  $LT_t^{\mathbb{Q}}$  through equation (18) is in fact quite trivial for options and empirical settings designed to mimic those of the actual data. The estimation error for  $RT_t^{\mathbb{Q}}$  in equation (17) tends to be somewhat larger, albeit still fairly small in an absolute sense. Intuitively, investors primarily fear large negative price moves, making negative jumps in the risk-neutral distribution more pronounced and easier to separate from the diffusive component.

Although the  $RT_t^{\mathbb{Q}}(k)$  and  $LT_t^{\mathbb{Q}}(k)$  measures are related to the tail jump density, they do not directly correspond to the risk-neutral expectations required for the calculation of the  $ERP_t(k)$  and  $VRP_t(k)$  jump premia. In order to get at these and the risk-neutral tail density, we rely on an additional Extreme Value Theory (EVT) type approximation.<sup>21</sup> This approximation allows us to easily extrapolate the tail behavior from the estimates of  $RT_t^{\mathbb{Q}}(k)$  and  $LT_t^{\mathbb{Q}}(k)$  for a range of different values of  $k$ . The only additional assumption required for these calculations concerns the format of the risk-neutral jump density.

Specifically, we assume that

$$\nu_t^{\mathbb{Q}}(dx) = (\varphi_t^+ 1_{\{x>0\}} + \varphi_t^- 1_{\{x<0\}}) \nu^{\mathbb{Q}}(x) dx, \quad (19)$$

where the unspecified  $\varphi_t^{\pm}$  stochastic processes allow for temporal variation in the jump arrivals and  $\nu^{\mathbb{Q}}(x)$  denotes any valid Lévy density. This is a very weak assumption on the general form of the tails, merely restricting the time variation to be the same across different jump sizes. This assumption is trivially satisfied by all of the previously estimated parametric jump-diffusion models referred to above, most of which restrict  $\varphi_t^{\pm} \equiv 1$  or  $\varphi_t^{\pm} \equiv \sigma_t^2$ .

The following proposition, the proof of which is deferred to Subsection A of the Appendix, formalizes the results underlying our estimation of  $\mathbb{E}_t^{\mathbb{Q}}(\int_t^T \int_{|x|>k} (e^x - 1) \nu_s^{\mathbb{Q}}(dx) ds)$  and  $\mathbb{E}_t^{\mathbb{Q}}(\int_t^T \int_{|x|>k} x^2 \nu_s^{\mathbb{Q}}(dx) ds)$  from  $RT_t^{\mathbb{Q}}(k)$  and  $LT_t^{\mathbb{Q}}(k)$ . Intuitively, by tracking the slope of the latter for increasingly deeper out-of-the-money options, it is possible to infer the tail behavior of  $\nu_t^{\mathbb{Q}}(dx)$ .<sup>22</sup>

**PROPOSITION 1:** *Assume that  $F_t$  and  $\nu_t^{\mathbb{Q}}(dx)$  satisfy equations (1) and (19), respectively. Denote  $\psi^+(x) = e^x - 1$  and  $\nu_{\psi}^{\mathbb{Q}+}(x) = \frac{\nu^{\mathbb{Q}}(\ln(x+1))}{x+1}$  for  $x > 0$ , and  $\psi^-(x) = e^{-x}$  and  $\nu_{\psi}^{\mathbb{Q}-}(x) = \frac{\nu^{\mathbb{Q}}(-\ln(x))}{x}$  for  $x < 0$ . Define  $\bar{\nu}_{\psi}^{\mathbb{Q}\pm}(x) = \int_x^{\infty} \nu_{\psi}^{\mathbb{Q}\pm}(u) du$ , and*

<sup>21</sup> For a general discussion of EVT, see Embrechts, Kluppelberg, and Mikosch (2001).

<sup>22</sup> Our actual estimation is slightly more general than the result of Proposition 1 in the sense that it does not restrict a scaling parameter; the details are in Appendix, Subsection A.

assume that  $\bar{v}_\psi^{\mathbb{Q}\pm}(x) = x^{-p_\mathbb{Q}^\pm} L^\pm(x)$  for  $p_\mathbb{Q}^\pm > 0$ , where  $L^\pm(x)$  satisfy  $L^\pm(tx)/L^\pm(x) = 1 + O(\tau^\pm(x))$  as  $x \uparrow \infty$  for each  $t > 0$ , and  $\tau^\pm(x) > 0$ ,  $\tau^\pm(x) \rightarrow 0$  as  $x \uparrow \infty$ , and  $\tau^\pm(x)$  are nonincreasing. Then, for arbitrary  $t > 0$ ,  $T - t \downarrow 0$ ,  $K_1 \uparrow \infty$ , and  $K_2 \uparrow \infty$ , with  $K_2/K_1 > 1$  constant,

$$\log \left( \frac{RT_t^{\mathbb{Q}}(K_1/F_{t-})}{RT_t^{\mathbb{Q}}(K_2/F_{t-})} \right) \xrightarrow{\mathbb{P}} (p_\mathbb{Q}^+ - 1) \log \left( \frac{K_2 - F_{t-}}{K_1 - F_{t-}} \right), \quad (20)$$

while for  $K_1 \downarrow 0$  and  $K_2 \downarrow 0$  with  $K_1/K_2 > 1$  constant,

$$\log \left( \frac{LT_t^{\mathbb{Q}}(K_1/F_{t-})}{LT_t^{\mathbb{Q}}(K_2/F_{t-})} \right) \xrightarrow{\mathbb{P}} (p_\mathbb{Q}^- + 1) \log \left( \frac{K_1}{K_2} \right). \quad (21)$$

The risk premia and the  $ERP_t(k)$  and  $VRP_t(k)$  measures, of course, also depend on the jump tails in the statistical distribution. We next discuss the high-frequency realized variation measures and reduced-form forecasting models that we rely on in estimating the tails under  $\mathbb{P}$ .

### III. Objective Measures

Our estimates of the relevant expectations under the objective, or statistical, probability measure  $\mathbb{P}$  are based on high-frequency intraday data and corresponding realized variation measures for the diffusive and discontinuous components of the total variation in equation (2). In addition, we rely on newly developed EVT-based approximations and nonparametric reduced form modeling procedures for translating the ex-post measurements into forward-looking  $\mathbb{P}$  counterparts to the  $QV_t^{\mathbb{Q}}$ ,  $RT_t^{\mathbb{Q}}$ , and  $LT_t^{\mathbb{Q}}$  expectations defined above. The foundations for the EVT approximations are formally based on the general theoretical results in Bollerslev and Todorov (2011), and we provide details on the calculations involved in these approximations in the Appendix. We continue the discussion here with a description of our high-frequency-based realized variation measures.

#### A. Realized Jumps and Volatility

For ease of notation, we normalize the unit time interval to represent a day, where, by convention, the day starts with the close of trading on the previous day. The resulting daily  $[t, t + 1]$  time interval is naturally broken into the  $[t, t + \pi_t]$  overnight time period and the  $[t + \pi_t, t + 1]$  active part of the trading day. The opening time of the market  $t + \pi_t$  is obviously fixed in calendar time. Conceptually, however, it is useful to treat  $\{\pi_t\}_{t=1,2,\dots}$  as a latent discrete-time stochastic process that implicitly defines the proportion of the total day  $t$  variation due to the overnight price change. With  $n$  equally spaced high-frequency price observations, this leaves us with a total of  $n - 1$  intraday

increments for day  $t + 1$ , each spanning a time interval of length  $\Delta_{n,t} \equiv \frac{1-\pi_t}{n}$ , say  $\Delta_i^{n,t} f \equiv f_{t+\pi_t+i\Delta_{n,t}} - f_{t+\pi_t+(i-1)\Delta_{n,t}}$  for  $i = 1, \dots, n - 1$ .<sup>23</sup>

The realized variation for day  $t$ , denoted  $RV_t$ , is simply defined as the summation of the squared high-frequency increments over the active part of the trading day. The realized variation concept was first introduced into the economics and finance literatures by Andersen and Bollerslev (1998), Andersen et al. (2003), and Barndorff-Nielsen and Shephard (2001). The key insight stems from the basic result that, for increasingly finer-sampled increments, or  $n \uparrow \infty$ ,  $RV_t$  consistently estimates the total ex-post variation of the price process. Formally,

$$RV_t \equiv \sum_{i=1}^{n-1} (\Delta_i^{n,t} f)^2 \xrightarrow{\mathbb{P}} \int_{t+\pi_t}^{t+1} \sigma_s^2 ds + \int_{t+\pi_t}^{t+1} \int_{\mathbb{R}} x^2 \mu(ds, dx), \quad (22)$$

where the right-hand side equals the quadratic variation of the price over the  $[t + \pi_t, t + 1]$  time interval, as defined by the corresponding increment to the total variation in equation (2).

As discussed above, the quadratic variation obviously consists of two distinct components: the variation due to the continuous sample price path and the variation coming from jumps. It is possible to separately estimate these two components by decomposing the summation of the squared increments into separate summations of the small and large price changes, respectively. In particular, it follows that, for the continuous variation,

$$CV_t \equiv \sum_{i=1}^{n-1} (\Delta_i^{n,t} f)^2 \mathbf{1}_{\{|\Delta_i^{n,t} f| \leq \alpha \Delta_{n,t}^{\varpi}\}} \xrightarrow{\mathbb{P}} \int_{t+\pi_t}^{t+1} \sigma_s^2 ds, \quad (23)$$

while, for the jump variation,

$$JV_t \equiv \sum_{i=1}^{n-1} (\Delta_i^{n,t} f)^2 \mathbf{1}_{\{|\Delta_i^{n,t} f| > \alpha \Delta_{n,t}^{\varpi}\}} \xrightarrow{\mathbb{P}} \int_{t+\pi_t}^{t+1} \int_{\mathbb{R}} x^2 \mu(ds, dx), \quad (24)$$

where  $\alpha > 0$  and  $\varpi \in (0, 0.5)$  denote positive constants.<sup>24</sup> These measures for the continuous and jump variation were first implemented in the context of high-frequency financial data by Mancini (2001). They are based on the intuitive idea that, over short time intervals, the continuous component of the price will be normally distributed with variance proportional to the length of the time interval. Thus, if the high-frequency price increment exceeds a certain threshold, it must be associated with a jump in the price.

<sup>23</sup> Even though it is convenient to treat the sampling times  $(t + \pi_t, t + \pi_t + \Delta_{n,t}, \dots, t + 1)$  as stochastic, they are obviously fixed in calendar time. As discussed further in the Appendix, all we need from a theoretical perspective is for the value of  $\pi_t$  to be (conditionally) independent of the  $f_t$  logarithmic price process.

<sup>24</sup> The actual choice of  $\alpha$  and  $\varpi$ , together with our normalization procedure designed to account for the strong intraday pattern in the volatility, are discussed in Subsection B of the Appendix.

Building on this same idea, the  $JV_t$  jump component may be further split into the variation coming from positive, or right tail, jumps,

$$RJV_t \equiv \sum_{i=1}^{n-1} |\Delta_i^{n,t} f|^2 1_{\{\Delta_i^{n,t} f > \alpha \Delta_{n,t}^m\}} \xrightarrow{\mathbb{P}} \int_{t+\pi_t}^{t+1} \int_{\mathbb{R}} x^2 1_{\{x>0\}} \mu(ds, dx), \quad (25)$$

and the variation coming from negative, or left tail, jumps,

$$LJV_t \equiv \sum_{i=1}^{n-1} |\Delta_i^{n,t} f|^2 1_{\{\Delta_i^{n,t} f < -\alpha \Delta_{n,t}^m\}} \xrightarrow{\mathbb{P}} \int_{t+\pi_t}^{t+1} \int_{\mathbb{R}} x^2 1_{\{x<0\}} \mu(ds, dx). \quad (26)$$

It follows, of course, by definition that  $RV_t = CV_t + RV_t + JV_t$ .

The different ex-post variation measures discussed above all pertain to the actual realized price process. By contrast, the option-implied diffusive and jump tail measures discussed in Section II provide ex-ante estimates of the risk-neutral expectations of the relevant sources of risks. We next discuss our translation of the actual realized risks into comparable measures of objective, or statistical, ex-ante expectations of these risks.

## B. Expected Jumps and Tails

The most challenging problem in quantifying the statistical expectations in the equity and variance risk premia in equations (6) and (9), respectively, concerns the  $\mathbb{P}$  equivalent to the previously discussed  $RT_t^{\mathbb{Q}}(k)$  and  $LT_t^{\mathbb{Q}}(k)$  risk-neutral tail measures, say  $RT_t^{\mathbb{P}}(k)$  and  $LT_t^{\mathbb{P}}(k)$ , respectively. These measures pertain to the truly large jumps. However, these types of tail events are invariably rare, and possibly even nonexistent, over a limited calendar time span. This is akin to a peso type problem, and as such straightforward estimation of the tail expectations based on sample averages generally will not be reliable. Instead, we rely on EVT and the more frequent medium-sized jumps implicit in the high-frequency intraday data to meaningfully extrapolate the behavior of the large jumps, and more precisely estimate the extreme parts of the jump tails.

Compared to conventional fully parametric procedures hitherto used in the literature for estimating the expected impact of jumps, our approach is much more flexible and has the distinct advantage of not relying on a tight connection between the jumps of all different sizes. Indeed, as discussed further below, the statistical properties of the small and large jumps differ quite dramatically, as indirectly evidenced by the markedly different estimates for the tails obtained across competing parametric specifications that use *all* of the jumps for the tail estimation.

A completely general nonparametric approach allowing jumps of different sizes to exhibit arbitrary temporal variation is obviously impossible. Instead,

we merely assume that the temporal variation in the jump intensity is a linear function of the stochastic volatility, as in<sup>25</sup>

$$v_t^{\mathbb{P}}(dx) = (\alpha_0^- \mathbf{1}_{\{x < 0\}} + \alpha_0^+ \mathbf{1}_{\{x > 0\}} + (\alpha_1^- \mathbf{1}_{\{x < 0\}} + \alpha_1^+ \mathbf{1}_{\{x > 0\}})\sigma_t^2) v^{\mathbb{P}}(x) dx, \quad (27)$$

where  $v^{\mathbb{P}}(x)$  is a valid Lévy density, and  $\alpha_0^{\pm}$  and  $\alpha_1^{\pm}$  are nonnegative free parameters. This semiparametric structure does admittedly restrict the form of the temporal dependencies somewhat. Importantly, however, the assumption in equation (27) is only needed to pin down the dynamics of the risk premia. All of the unconditional results reported in the paper, that is, the level of the risk premia, continue to hold under the analogous weaker assumption for the  $\mathbb{Q}$  jump tails given in equation (19).<sup>26</sup> The assumption in equation (27) still provides a very flexible specification, and we provide direct support for its empirical validity in Section VI below. Virtually all parametric models hitherto estimated in the literature are based on simplified versions of this  $v_t^{\mathbb{P}}(dx)$  measure, typically restricting  $\alpha_1^{\pm} = 0$ . By contrast, the more general specification in equation (27) explicitly allows for time-varying jump intensities as a function of the latent volatility process  $\sigma_t^2$ . This includes so-called “self-affecting” jumps in which  $\alpha_1^{\pm} > 0$ , and  $\sigma_t^2$  depends on past realized jumps, as suggested by the data. Of course, the specification also allows for the relative magnitudes of the constant and time-varying intensities to differ across positive and negative jumps.

Relying on (27) to describe the temporal dependencies in the jump intensity, the equivalents to the  $RT_t^{\mathbb{Q}}(k)$  and  $LT_t^{\mathbb{Q}}(k)$  measures under the statistical distribution may be expressed as

$$RT_t^{\mathbb{P}}(k) = \left( \alpha_0^+ + \frac{\alpha_1^+}{T-t} \mathbb{E}_t^{\mathbb{P}} \left( \int_t^T \sigma_s^2 ds \right) \right) \int_{\mathbb{R}} (e^x - e^k)^+ v^{\mathbb{P}}(x) dx, \quad (28)$$

and

$$LT_t^{\mathbb{P}}(k) = \left( \alpha_0^- + \frac{\alpha_1^-}{T-t} \mathbb{E}_t^{\mathbb{P}} \left( \int_t^T \sigma_s^2 ds \right) \right) \int_{\mathbb{R}} (e^k - e^x)^+ v^{\mathbb{P}}(x) dx. \quad (29)$$

To estimate  $RT_t^{\mathbb{P}}(k)$  and  $LT_t^{\mathbb{P}}(k)$ , and in turn  $QV_t^{\mathbb{P}}$ , we therefore only need to quantify the integrals with respect to the Lévy density  $v^{\mathbb{P}}(x)$  and the conditional expectation of the integrated volatility  $\mathbb{E}_t^{\mathbb{P}}(\int_t^T \sigma_s^2 ds)$ .

Similar to our estimation of the  $\mathbb{Q}$  jump tails, our estimates of the integrals with respect to  $v^{\mathbb{P}}(x)$  are again based on the medium-sized jumps coupled with EVT for meaningfully extrapolating the behavior of the tails. The separation of the jump intensity into the constant and time-varying part, that is,  $\alpha_0^{\pm}$  and  $\alpha_1^{\pm}$ , respectively, is effectively achieved through the correlation between the

<sup>25</sup> This parallels the significantly weaker assumption for the risk-neutral jump intensity in equation (19). Extensions to allow for nonlinear functions of the volatility, or other directly observable state variables, are straightforward.

<sup>26</sup> Also, given the differences in orders of magnitude of the  $\mathbb{P}$  and  $\mathbb{Q}$  tails reported in the empirical part, the practical import of assumption (27) for the actual empirical results is rather limited.

occurrence of jumps and past volatility, as measured directly by the realized continuous variation  $CV_t$  in equation (23). A more detailed description of the relevant calculations is again given in the Appendix.

Our estimates for the expected integrated volatility, and the time-varying parts of  $RT_t^P$ ,  $LT_t^P$ , and  $QV_t^P$ , are based on reduced-form time series modeling procedures, as originally advocated by Andersen et al. (2003). This provides a particular convenient and parsimonious approach for capturing the dynamic dependencies in the integrated volatility process and corresponding forecasts. From a statistical perspective, this is also a much easier problem than that of directly estimating the tails, as we have plenty of reliable data and previous empirical insights to go by.<sup>27</sup>

In particular, it is well established that integrated volatility is a stationary, but highly persistent, process; see, for example, Bollerslev et al. (2009) and the reduced form model estimates reported therein. It has also previously been documented that jumps in the price tend to be accompanied by jumps in the spot volatility  $\sigma_t$ , and that negative and positive jumps impact volatility differently, that is, a leverage effect through jumps. To accommodate all of these nonlinear dependencies in the construction of our estimates for the expected integrated volatility, we formulate a vector autoregression (VAR) of order 22, or 1 month, for the four-dimensional vector consisting of  $CV_t$ ,  $RJV_t$ ,  $LJV_t$ , and the squared overnight increments  $(f_t - f_{t+\pi_t})^2$ . To keep the model tractable, we follow the HAR-RV modeling approach of Corsi (2009) and Andersen, Bollerslev, and Diebold (2007) and restrict all but the daily, weekly, and monthly lags, that is lags 1, 5, and 22, respectively, to be equal to zero. The forecasts for  $\mathbb{E}_t^P(\int_t^T \sigma_s^2 ds)$  is then simply obtained by recursively iterating the VAR to the relevant horizon. Further details concerning the model estimates and forecast construction are given in Subsection C of the Appendix.

Before turning to a discussion of the actual empirical results related to the  $P$  and  $Q$  tail measures and risk premia, we briefly discuss the data underlying our estimates.

## IV. Data and Preliminary Estimates

### A. High-Frequency Data and Statistical Measures

Our high-frequency data for the S&P 500 futures contract come from Tick Data Inc. and span the period from January 1990 to December 2008. The prices are recorded at 5-minute intervals, with the first price for the day at 8:35 (CST) and the last price at 15:15. All in all, this leaves us with a total of 81 intraday price observations for each of the 4,751 trading days in the sample.

The resulting realized variation measures all exhibit the well-known volatility clustering effects, with a sharp increase in their levels toward the end of

<sup>27</sup> The theoretical results in Andersen, Bollerslev, and Meddahi (2004) and Sizova (2009) also indicate that the loss in forecast efficiency from reduced-form procedures is typically small relative to fully efficient parametric procedures, and that the reduced-form procedures have the added advantage of a built-in robustness to model misspecification.

the sample and the Fall 2008 financial market crises. The realized variation measures in turn form the basis for our nonparametric estimates of the statistical expectations of the total quadratic variation and the left and right tail measures,  $QV_t^{\mathbb{P}}$ ,  $LT_t^{\mathbb{P}}$ , and  $RT_t^{\mathbb{P}}$ . We do not discuss the details of the estimation results here, instead referring interested readers to the Appendix and the actual estimates reported therein. Nonetheless, it is instructive to highlight a few of the key empirical findings.

Directly in line with the discussion in the previous section, the stochastic volatility process is highly persistent. In addition, negative “realized” jumps significantly impact future volatility. In contrast to the strong own dynamic dependencies in the volatility, the effect of negative jumps is relatively short lived, however. Our results also indicate that positive jumps do not affect future volatility. These findings are consistent with the empirical results in Barndorff-Nielsen, Kinnebrock, and Shephard (2010) and Todorov and Tauchen (2011). They also support a two-factor stochastic volatility model, as in Todorov (2010), in which one of the factors is highly persistent while the other factor is short lived and mainly triggered by (negative) jumps.

Our estimation results also point to the existence of nontrivial temporal variation in the tail jump intensities, as manifested by the highly statistically significant estimates for  $\alpha_1^{\pm}$ . Interestingly, the right and left jump tails behave quite similar under the statistical distribution with almost identical tail decays. Taken as a whole, however, our empirical results suggest much more complicated dependencies than those allowed by the parametric models hitherto estimated in the literature, further underscoring the advantages of our flexible nonparametric approach for characterizing the behavior of the  $\mathbb{P}$  jump tails.

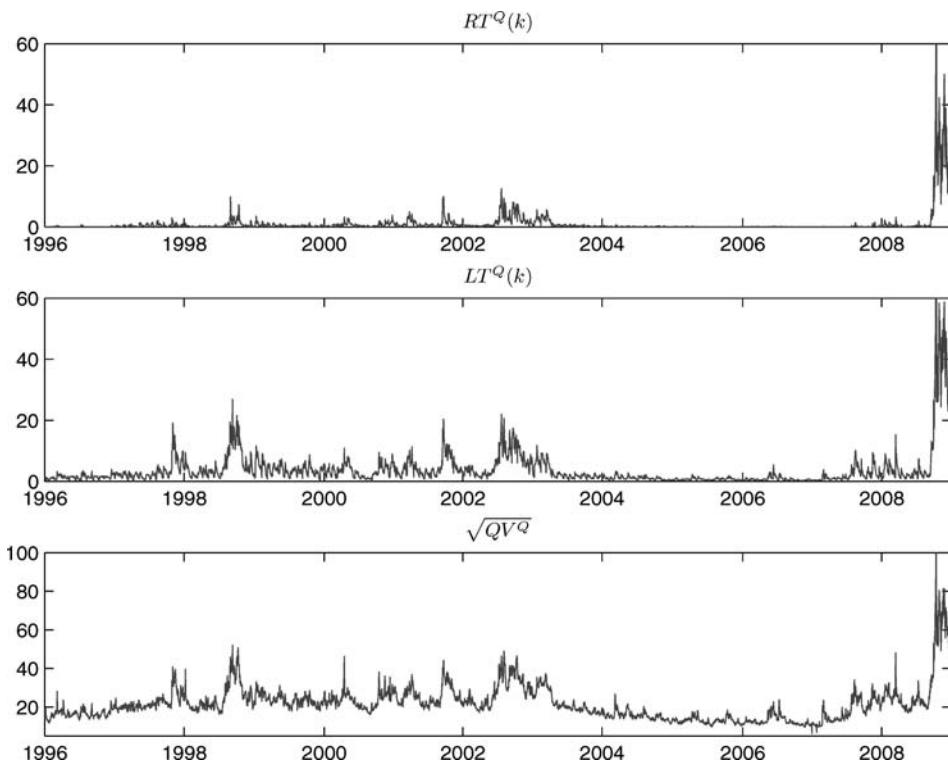
### *B. Options Data and Risk-Neutral Measures*

Our options data come from OptionMetrics. The data consist of closing bid and ask quotes for all S&P 500 options traded on the CBOE. The data span the period from January 1996 to the end of December 2008, for a total of 3,237 trading days. To avoid obvious inconsistencies and problems with the price quotes, we apply similar filters to those used by Carr and Wu (2003, 2009) for “cleaning” the data.

The estimates for the two tail measures  $RT_t^{\mathbb{Q}}(k)$  and  $LT_t^{\mathbb{Q}}(k)$  are based on the closest-to-maturity options for day  $t$  with at least 8 calendar days to expiration.<sup>28</sup> More specifically, we first calculate Black–Scholes implied volatilities for all out-of-the-money options with moneyness  $k$  using linear interpolation of options that bracket the targeted strike.<sup>29</sup> From these implied volatilities we

<sup>28</sup> The options always expire on a Saturday with the settlement at the opening of the preceding Friday. The median time-to-expiration of the options in our sample is 14 working days.

<sup>29</sup> These prices are taken directly from OptionMetrics and are based on the corresponding mid quotes. In the absence of any deeper out-of-the-money option than the desired moneyness  $k$ , we use the implied volatility for the deepest out-of-the-money option that is available as a proxy.



**Figure 1. Implied measures.** The estimates are based on S&P 500 options spanning the period from January 1996 to December 2008, for a total of 3,237 trading days. The log-moneyness of the options used for the left and right tail measures is fixed at  $k = 0.9$  and  $k = 1.1$ , respectively.

then compute an out-of-the money option price with moneyness  $k$ , and in turn the  $RT_t^Q(k)$  and  $LT_t^Q(k)$  measures as described in Section II above.

Our calculation of the implied total variance measure  $QV_t^Q$  follows Carr and Wu (2009) and is similar to the approach used by the CBOE in its calculation of the VIX index. There are two differences, however, between our construction of  $QV_t^Q$  and the calculation of the VIX index, as described in the white paper available on the CBOE website. First, we rely on trading, or business, time, while the VIX index is constructed using a calendar-time convention. Second, for compatibility with the tail measures  $RT_t^Q(k)$  and  $LT_t^Q(k)$ , we use the same closest-to-maturity options in the calculation of  $QV_t^Q$ , whereas the VIX is based on linear interpolation of options that bracket a 1-month maturity.

The three resulting risk-neutral variation measures are plotted in Figure 1. All of the measures are reported in annualized percentage form. Guided by the results from the Monte Carlo simulation study designed to investigate the accuracy of the approximations in equations (17) and (18) and discussed in more detail in Section VI below, here and throughout the rest of the analysis, the moneyness for the left and right tails is fixed at 0.9 and 1.1, respectively. As

**Table I**  
**Mean Jump Intensities**

The jump intensities are reported in annualized units. The jump sizes are percentage changes in the price level. The estimates under  $\mathbb{Q}$  are based on S&P 500 options data from January 1996 through June 2007, while the estimates under  $\mathbb{P}$  rely on high-frequency 5-minute S&P futures prices from January 1990 through June 2007. Standard errors for the estimates are reported in parentheses.

Jump Size	Under $\mathbb{Q}$	Under $\mathbb{P}$
>7.5%	0.5551(0.0443)	0.0098(0.0142)
>10%	0.2026(0.0206)	0.0050(0.0083)
>20%	0.0069(0.0014)	0.0010(0.0022)
<-7.5%	0.9888(0.0525)	0.0036(0.0048)
<-10%	0.5640(0.0346)	0.0017(0.0026)
<-20%	0.0862(0.0084)	0.0002(0.0005)

is immediately evident from Figure 1, the magnitude of the left tail measure substantially exceeds that of the right tail. This asymmetry in the tails under  $\mathbb{Q}$  contrasts sharply with the previously discussed approximately symmetric tail behavior under  $\mathbb{P}$ . This result, of course, is also consistent with the well-known fact that out-of-the-money puts tend to be systematically “overpriced” when assessed by standard pricing models; see, for example, the discussion in Bates (2009), Bondarenko (2003), Broadie, Chernov, and Johannes (2009b), and Foresi and Wu (2005). Moreover, the three risk-neutral variation measures all clearly vary over time, reaching unprecedented levels in Fall 2008. We study these dynamic patterns more closely below.

## V. Tails, Fears, and Risk Premia

We begin our discussion of the equity and variance risk premia and the relationship between the  $\mathbb{P}$  and  $\mathbb{Q}$  measures in Table I with a direct comparison of the average intensity for large jumps under the two different measures. In an effort to focus on “normal” times and prevent our estimates of the risk premia and investor fears to be unduly influenced by the recent financial market crises, we explicitly exclude the July 2007 through December 2008 part of the sample from this estimation.

As is immediately evident from Table I, the estimated frequencies for the occurrences of large jumps are orders of magnitude smaller than the implied jump intensities under the risk-neutral measure.<sup>30</sup> Since we explicitly rely on the frequently occurring medium-sized jumps and EVT for meaningfully uncovering the  $\mathbb{P}$  jump tails, this marked difference cannot simply be attributed to a standard peso type problem and a nonrepresentative sample. The table also reveals a strong sense of investor fears, as manifested by the much larger intensities for the left jump tails under  $\mathbb{Q}$  and the willingness to protect against

<sup>30</sup> This is consistent with the aforementioned earlier empirical evidence reported in Aït-Sahalia, Wang, and Yared (2001); see their Section 3.5.

negative jumps. For instance, the estimated risk-neutral jump intensity for negative jumps less than  $-20\%$  is more than 12 times larger than that for positive jumps in excess of  $20\%$ , and the difference is highly statistically significant. The differences in the left and right  $\mathbb{P}$  jump tails, on the other hand, are not nearly as dramatic and also statistically insignificant.

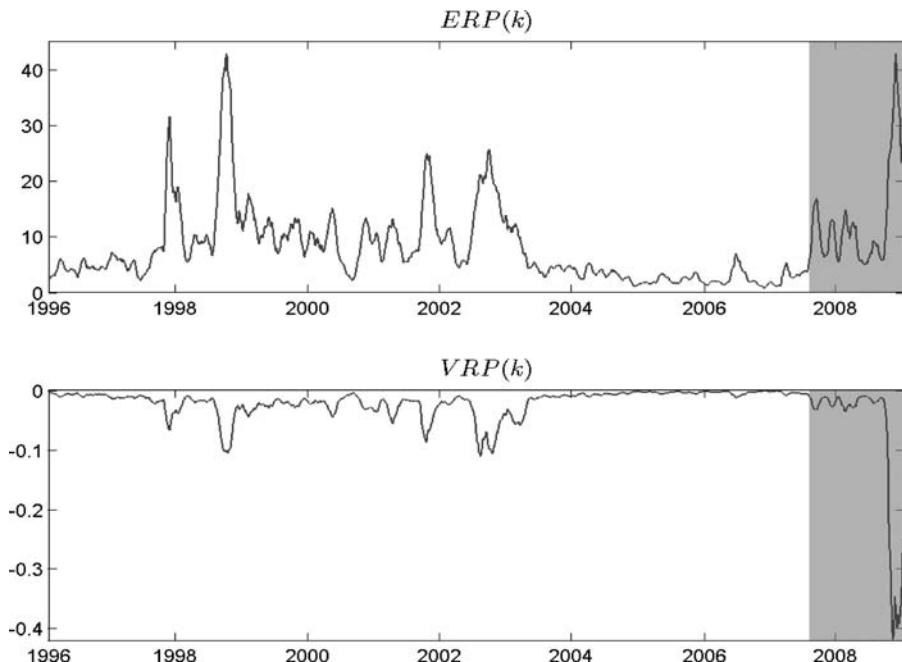
Although the option-implied risk-neutral intensities for positive jumps are much smaller than the intensities for negative jumps, they are still much higher than the estimated intensities for positive jumps under the statistical measure. This contrasts with previous estimates from parametric models, which typically imply trivial and even negative premia for positive jumps. Most of these findings, however, are based on tightly parameterized models that constrain the overall jump intensity, that is, the intensity for jumps of any size, to be the same under the risk-neutral and statistical measures. Thus, if negative jumps are priced, in the sense that their risk-neutral intensity exceeds their statistical counterpart, this automatically implies that positive jumps will carry no or even a negative risk premium. Meanwhile, our nonparametric approach, which does not restrict the shape and relation between the  $\mathbb{P}$  and  $\mathbb{Q}$  jumps, suggests that the large positive jumps do in fact carry a premia, albeit of a much smaller magnitude than the negative jumps.<sup>31</sup>

Turning next to the actual risk premia, Figure 2 plots the components of the equity and variance risk premia that are due to compensation for rare events, that is,  $ERP(k)$  and  $VRP(k)$ , respectively. As mentioned before, we explicitly exclude the last 18 months of the sample in the estimation of the volatility forecasting model and different jump tail measures, so that the premia in the shaded parts of the figure, coinciding with the advent of the recent financial market crises, are based on the “in-sample” estimates with data through June 2007 only. Also, for ease of interpretation, both of the premia are reported at a monthly frequency based on 22-day moving averages of the corresponding daily estimates.<sup>32</sup>

There are obvious similarities in the general dynamic dependencies in the two jump risk premia, with many of the peaks in  $ERP(k)$  coinciding with the troughs in  $VRP(k)$ . There are also important differences, however, in the way

<sup>31</sup> As discussed further below, the temporal variation in the jump intensities means that positive jumps may indeed carry a risk premium. This is reminiscent of the U-shaped pattern in the projection of the pricing kernel on the space of market returns previously documented in Ait-Sahalia and Lo (2000) and Rosenberg and Engle (2002), among others.

<sup>32</sup> The time-to-expiration of the options used in the calculation of the risk premia changes systematically over the month, inducing monthly periodicity in the day-by-day estimates. To illustrate, consider the popular jump-diffusion model of Duffie, Pan, and Singleton (2000). For simplicity, suppose there is only a single continuous volatility factor. Let  $\kappa^{\mathbb{P}}$  and  $\theta^{\mathbb{P}}$  denote the mean-reversion parameter and the mean of  $c_t^2$  under  $\mathbb{P}$ , with the corresponding parameters under  $\mathbb{Q}$  denoted by  $\kappa^{\mathbb{Q}}$  and  $\theta^{\mathbb{Q}}$ , respectively. The unconditional mean of the risk premium may then be expressed as  $K_0 + K_1(\theta^{\mathbb{P}} - \theta^{\mathbb{Q}})(1 - e^{-\kappa^{\mathbb{Q}}(T-t)})(\kappa^{\mathbb{Q}}(T-t))^{-1}$ , where the two constants  $K_0$  and  $K_1$  are determined by the risk premia parameters. This expected premium obviously depends on the horizon and the relative import of the continuous stochastic volatility component, in turn inducing monthly periodicity in the daily measures.



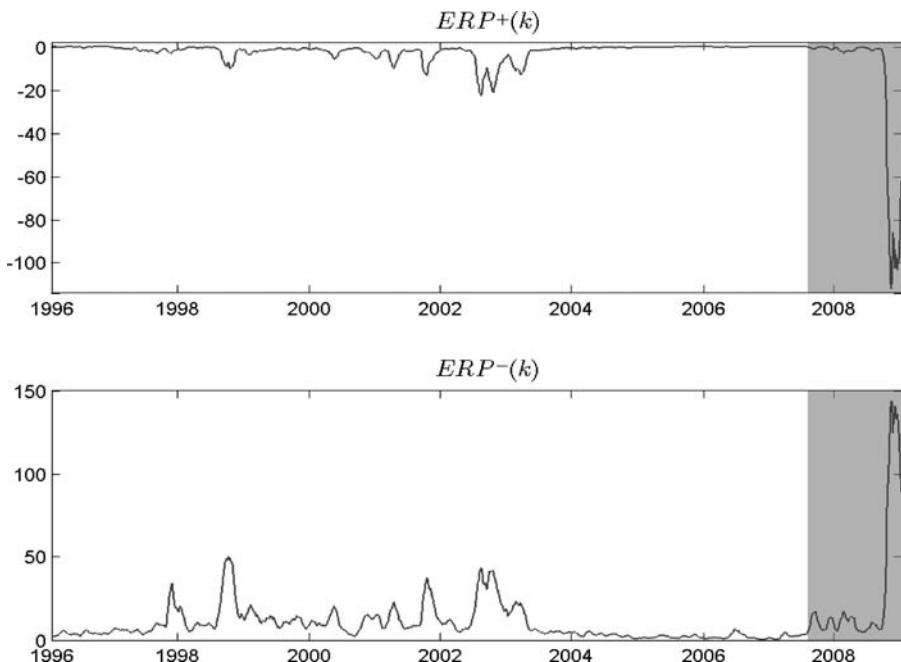
**Figure 2. Equity and variance risk premia due to large jumps.** The estimates for the risk premia are based on 5-minute S&P 500 futures prices and options. The values for the premia in the shaded area, corresponding to July 2007 through December 2008, are based on the parameter estimates for the  $\mathbb{P}$  and  $\mathbb{Q}$  measures obtained using data through June 2007. The log-moneyness of the options used for the left and right tails is fixed at  $k = 0.9$  and  $k = 1.1$ , respectively.

in which the compensation for the tail events manifest in the two premia. We begin with a more detailed discussion of the equity jump risk premium depicted in the top panel.

#### A. Equity Jump Risk Premium

Most of the peaks in the equity jump risk premium are readily associated with specific economic events. In particular, the October 1997 “mini-crash” and the August to September 1998 turmoil associated with the Russian default and the long-term capital management (LTCM) debacle both resulted in sharp, but relatively short lived, increases in the required compensation for tail risks. A similar albeit much smaller increase is observed in connection with the September 11, 2001 attacks. Interestingly, the magnitude of the equity jump risk premium in October 2008 is about the same as the premium observed in August to September 1998.

To better understand where this compensation for tail risk is coming from, it is instructive to decompose the total jump risk premium into the parts associated with negative and positive jumps, say  $ERP_t^+(k)$  and  $ERP_t^-(k)$ , respectively.



**Figure 3. Decomposition of equity jump premia.** The estimates for the equity risk premium components are based on 5-minute S&P 500 futures prices and options. The values for the premia in the shaded area, corresponding to July 2007 through December 2008, are based on the parameter estimates for the  $\mathbb{P}$  and  $\mathbb{Q}$  measures obtained using data through June 2007. The log-moneyness of the options used for the left and right tails is fixed at  $k = 0.9$  and  $k = 1.1$ , respectively.

This decomposition, depicted in Figure 3, shows that, although the behavior of the two tails is clearly related, the contributions to the overall risk premium are far from symmetric. Most noticeably, the October 1997 “mini-crash” seems to be associated with an increased fear among investors of additional sharp market declines and a corresponding peak in  $ERP_t^-(k)$ , while there is hardly any effect on  $ERP_t^+(k)$ . Similarly, the August to September 1998 market turmoil had a much bigger impact on  $ERP_t^-(k)$  than it did on  $ERP_t^+(k)$ . Conversely, the dramatic stock market declines observed in July and September 2002 in connection with the burst of the dot-com “bubble” resulted in peaks and troughs in both  $ERP_t^+(k)$  and  $ERP_t^-(k)$ , with less of a discernable impact on the total jump risk premium  $ERP_t(k)$ . Likewise, the March 2003 start of the Second Gulf War is hardly visible in  $ERP_t(k)$ , while it clearly influenced both  $ERP_t^+(k)$  and  $ERP_t^-(k)$ . Most dramatic, however, the sharp increase in  $ERP_t(k)$  associated with the Fall 2008 financial crises masks even larger offsetting changes in  $ERP_t^+(k)$  and  $ERP_t^-(k)$  that dwarf the separate jump risk premia observed during the rest of the sample. As such, this directly underscores the notion that investors simply did not know what the “right” price was at the time.

Looking at the typical sample values, again excluding the July 2007 to December 2008 part of the sample, the median of the estimated equity risk

premia due to rare events equals 5.2%.<sup>33</sup> This is quite high, and, compared to the prototypical estimate of 8% for the equity risk premia in postwar U.S. data (see, for example, Cochrane (2005)), our results imply that fears of rare events account for roughly two-thirds of the total expected excess return. These numbers contrast with the estimation results reported in Broadie, Chernov, and Johannes (2009a), which imply a mean  $ERP_t(k)$  of 1.85%, and the estimates in Eraker (2004), which imply a premium for rare events of only 0.65%.<sup>34</sup>

At a more general level, our findings suggest that excluding fears and tail events, the magnitude of the equity risk premium is quite compatible with the implications from standard consumption-based asset pricing models with “reasonable” levels of risk aversion. Of course, this raises the question of why investors price tail risk so high. Recent pricing models that pay special attention to tail risks include Pan, Liu, and Wang (2005) and Bates (2008); the work by Bansal and Shaliastovich (2009) based on the concept of confidence risk; and Drechsler (2011), who emphasizes the role of time-varying model uncertainty in amplifying the impact of jumps.

Further corroborating these ideas, we show next that fears of rare events account for an even larger fraction of the historically large and difficult to explain variance risk premium.

### B. Variance Jump Risk Premium

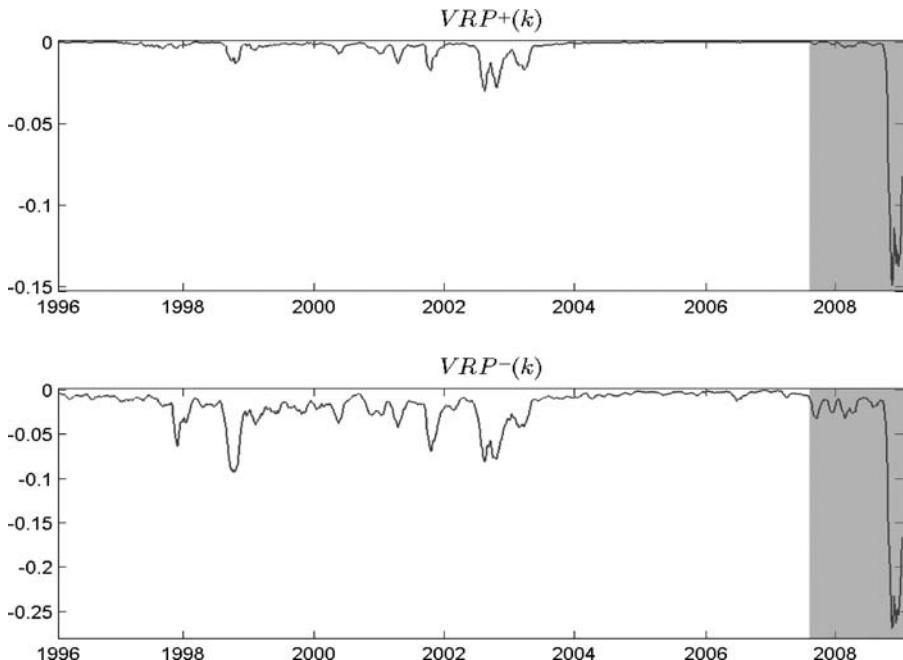
The general dynamic dependencies in the jump variance risk premium depicted in the second panel in Figure 2 fairly closely mirror those of the jump equity premium in the first panel. Similarly, decomposing  $VRP_t(k)$  into the parts associated with negative and positive jumps, say  $VRP_t^+(k)$  and  $VRP_t^-(k)$ , respectively, reveals a rather close coherence between the two tail measures.<sup>35</sup> With the notable exception of the aforementioned October 1997 “mini-crash” and the August to September 1998 Russian default and LTCM debacle, most of the spikes coincide between the two time series plotted in Figure 4. The magnitude of  $VRP_t^-(k)$ , of course, typically exceeds that of  $VRP_t^+(k)$  by a factor of two to four.<sup>36</sup> Do these findings of large tail risk premia imply special treatment, or compensation, for rare events? If large (or jump-related) risks were treated the same as small (or diffusive) risks, what would the magnitudes of the tail risk premia be?

<sup>33</sup> Including the more recent period, the median increases to 5.6%.

<sup>34</sup> Both of these numbers are based on a specific stochastic volatility model with correlated jumps, referred to in the literature as a stochastic volatility with correlated jumps (SVCJ) type model, in which the parametric form of the jumps effectively implies an exponential decay.

<sup>35</sup> This result is not simply an artifact of the assumption for the  $\mathbb{P}$  jump intensity in equation (27). The  $\mathbb{P}$  tail moments are orders of magnitude smaller than the moments under  $\mathbb{Q}$ , and as such the dynamic dependencies seen in the figure are largely driven by the dependencies in the  $\mathbb{Q}$  jump intensity estimated under the less restrictive assumption in (19).

<sup>36</sup> This is also in line with the recent empirical findings in Andersen and Bondarenko (2010) related to their so-called up and down implied variance measures.



**Figure 4. Decomposition of variance jump premia.** The estimates for the variance risk premia components are based on 5-minute S&P 500 futures prices and options. The values for the premia in the shaded area, corresponding to July 2007 through December 2008, are based on the parameter estimates for the  $\mathbb{P}$  and  $\mathbb{Q}$  measures obtained using data through June 2007. The log-moneyness of the options used for the left and right tails is fixed at  $k = 0.9$  and  $k = 1.1$ , respectively.

To begin answering these questions, it is instructive to further decompose the variance jump tail premia. In particular, from the formal definition of  $VRP_t^+(k)$ , it follows that

$$\begin{aligned} VRP_t^+(k) &= \frac{1}{T-t} \left( \mathbb{E}_t^{\mathbb{P}} \int_t^T \int_{x>k} x^2 ds v_s^{\mathbb{P}}(dx) - \mathbb{E}_t^{\mathbb{Q}} \int_t^T \int_{x>k} x^2 ds v_s^{\mathbb{P}}(dx) \right) \\ &\quad + \frac{1}{T-t} \mathbb{E}_t^{\mathbb{Q}} \left( \int_t^T \int_{x>k} x^2 ds v_s^{\mathbb{P}}(dx) - \int_t^T \int_{x>k} x^2 ds v_s^{\mathbb{Q}}(dx) \right), \end{aligned} \quad (30)$$

with a similar decomposition available for  $VRP_t^-(k)$ .

The first term on the right-hand side in equation (30) reflects the compensation for time-varying jump intensity risk. It mirrors the compensation demanded by investors for the continuous part of the quadratic variation, that is,  $VRP_t^c(k)$  in equation (10). As such, it is naturally associated with investors' willingness to hedge against changes in the investment opportunity set.

Specifically, for the jump intensity process in equation (27), the relevant difference between the  $\mathbb{P}$  and  $\mathbb{Q}$  expectations takes the form

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{P}} \int_t^T \int_{x>k} x^2 ds v_s^{\mathbb{P}}(dx) - \mathbb{E}_t^{\mathbb{Q}} \int_t^T \int_{x>k} x^2 ds v_s^{\mathbb{P}}(dx) \\ &= \alpha_1^+ \int_{x>k} x^2 v^{\mathbb{P}}(x) dx \left( \mathbb{E}_t^{\mathbb{P}} \int_t^T \sigma_s^2 ds - \mathbb{E}_t^{\mathbb{Q}} \int_t^T \sigma_s^2 ds \right). \end{aligned} \quad (31)$$

For Lévy type jumps, that is, jumps with constant jump intensity, this part of the variance jump risk premium will be identically equal to zero.

By contrast, the second term on the right-hand side in equation (30) involves the wedge between the risk-neutral and objective jump intensities. But this difference is evaluated under the same probability measure, and as such it is effectively purged of the premia due to temporal variation in the jump intensities. If jumps were treated the same as diffusive risks, this component would be zero, since the predictable quadratic variation of the continuous price does not change when changing the measure from  $\mathbb{P}$  to  $\mathbb{Q}$ .

In more technical terms, the quadratic variation of the log-price process may be split into its predictable component and a residual martingale component solely due to jumps,

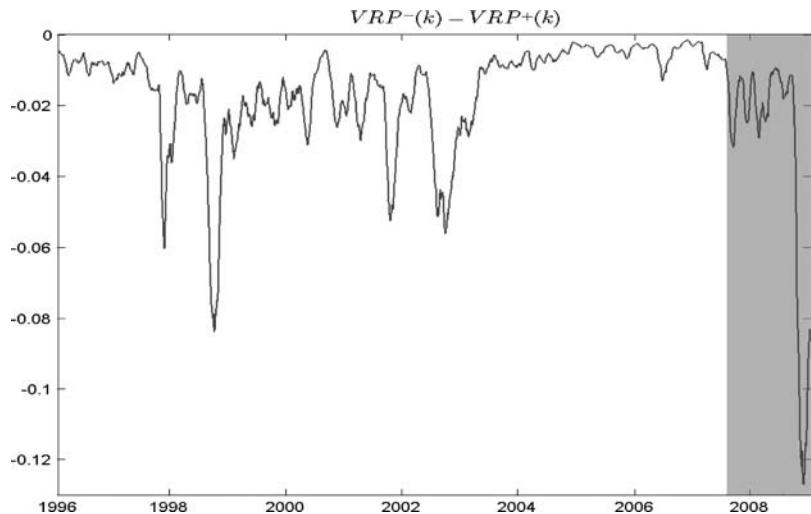
$$\begin{aligned} QV_{[t,T]} &= \int_t^T \sigma_s^2 ds + \int_t^T \int_{\mathbb{R}} x^2 ds v_s^{\mathbb{P}}(dx) + \int_t^T \int_{\mathbb{R}} x^2 \tilde{\mu}(ds, dx) \\ &\equiv \langle f_s, f_s \rangle_{[t,T]} + \int_t^T \int_{\mathbb{R}} x^2 \tilde{\mu}(ds, dx), \end{aligned}$$

where  $\langle f_s, f_s \rangle_{[t,T]}$  denotes the predictable quadratic variation; see, for example, Jacod and Shiryaev (2003). The pricing of the first term,  $\langle f_s, f_s \rangle_{[t,T]}$ , is naturally associated with changes in the investment opportunity set, while the pricing of the martingale part reflects any “special” treatment of jump risk.

The high-frequency-based estimation results discussed in the previous section indicate significant temporal variation in the tail jump intensities. Part of the tail risk premia therefore comes from the first term in equation (30). However, our estimation results, see Table A.I in the Appendix, also suggest that the left and right tails behave quite similarly under  $\mathbb{P}$ , so that in particular<sup>37</sup>

$$\alpha_1^+ \int_{x>k} x^2 v^{\mathbb{P}}(x) dx \approx \alpha_1^- \int_{x<-k} x^2 v^{\mathbb{P}}(x) dx.$$

<sup>37</sup> Note that this result is not “by construction” as assumption (27) allows for asymmetric tails since the left and right tails are allowed to have different loadings on the constant and time-varying parts of the jump intensity.



**Figure 5. Investor fears.** The estimates for the  $FI_t(k)$  Investor Fears index are based on 5-minute S&P 500 futures prices and options. The values in the shaded area, corresponding to July 2007 through December 2008, are based on the parameter estimates for the  $\mathbb{P}$  and  $\mathbb{Q}$  measures obtained using data through June 2007. The log-moneyness of the options used for the left and right tails is fixed at  $k = 0.9$  and  $k = 1.1$ , respectively.

Combining this approximation with (31) and the comparable expression for the left tail measure, it follows that the difference

$$FI_t(k) = VRP_t^-(k) - VRP_t^+(k) \quad (32)$$

will be largely void of the risk premia due to temporal variation in the jump intensities. Consequently,  $FI_t(k)$  may be interpreted as a direct measure of investor fears, or “crash-o-phobia.”<sup>38</sup>

The corresponding plot in Figure 5 shows that the sharpest increase in investor fears over the sample did indeed occur during the recent financial crises. However, the Russian default and long-term capital management collapse in August to September 1998 resulted in a spike in the Investor Fears index of almost two-thirds of its recent peak. Slightly less dramatic increases are manifest in connection with the October 1997 “mini-crash”; September 11, 2001; and the summer of 2002; and the burst of the dot-com “bubble.” The figure also reveals systematically low investor fears from 2003 through mid-2007, corresponding to the general run-up in market values.

The average sample values reported in Table II further corroborate the idea that much of the variance risk premium comes from “crash-o-phobia,” or special

<sup>38</sup> Rubinstein (1994) first attributed the smirk-like pattern in post October 1987 Black–Scholes implied volatilities for the aggregate market portfolio when plotted against the degree of moneyness as evidence of “crash-o-phobia”; see also Foresi and Wu (2005) for more recent related international empirical evidence.

**Table II**  
**Variance Risk Decomposition**

The average sample estimates are reported in annualized form, with standard errors in parentheses. The estimates under  $\mathbb{Q}$  are based on S&P 500 options data from January 1996 through June 2007, while the estimates under  $\mathbb{P}$  rely on 5-minute high-frequency S&P 500 futures data from January 1990 through June 2007.

Under $\mathbb{Q}$	
$\mathbb{E}\left(\frac{1}{T-t} \int_t^T \int_{x > \ln 1.1} x^2 \mathbb{E}_t^{\mathbb{Q}} v_s^{\mathbb{Q}}(dx) ds\right)$	0.0032(0.0004)
$\mathbb{E}\left(\frac{1}{T-t} \int_t^T \int_{x < \ln 0.9} x^2 \mathbb{E}_t^{\mathbb{Q}} v_s^{\mathbb{Q}}(dx) ds\right)$	0.0180(0.0015)
$\mathbb{E}(QV_t^{\mathbb{Q}})$	0.0472(0.0046)
Under $\mathbb{P}$	
$\mathbb{E}\left(\frac{1}{T-t} \int_t^T \int_{x > \ln 1.1} x^2 \mathbb{E}_t^{\mathbb{P}} v_s^{\mathbb{P}}(dx) ds\right)$	0.000196(0.000483)
$\mathbb{E}\left(\frac{1}{T-t} \int_t^T \int_{x < \ln 0.9} x^2 \mathbb{E}_t^{\mathbb{P}} v_s^{\mathbb{P}}(dx) ds\right)$	0.000062(0.000115)
$\mathbb{E}(QV_t^{\mathbb{P}})$	0.0269(0.0031)

compensation for tail risks. Excluding the post-July 2007 crises period, the average sample variance risk premium equals  $\mathbb{E}(VRP_t) = 0.020$ . Of that total premium,  $\mathbb{E}(VRP_t^+(k))/\mathbb{E}(VRP_t) = 14.8\%$  is due to compensation for right tail risks, while  $\mathbb{E}(VRP_t^-(k))/\mathbb{E}(VRP_t) = 88.4\%$  comes from left tail risks. Looking at the difference, our results therefore suggest that close to three-quarters of the variance risk premium may be attributed to investor fears as opposed to more standard rational-based pricing arguments.<sup>39</sup>

Comparing these numbers to the implications from the aforementioned parametric model estimates reported in the literature, the study by Broadie, Chernov, and Johannes (2009a) implies that  $\mathbb{E}(VRP_t(k))/\mathbb{E}(VRP_t) = 24.4\%$ , while the results in Eraker (2004) suggest that  $\mathbb{E}(VRP_t(k))/\mathbb{E}(VRP_t) = 20.0\%$ .<sup>40</sup> These numbers obviously differ quite dramatically from the results obtained with our nonparametric approach, which effectively suggests that all or most of the variance risk premium may be attributed to the compensation for jump tail

<sup>39</sup> Note that these proportions are maturity specific, and based on a median maturity of 14 days. To check the sensitivity of our results with respect to the maturity and statistical uncertainty associated with our nonparametric inference procedure, we redid the estimation using monthly data comprised of options on the fifth business day of each month. The resulting monthly estimates for  $\mathbb{E}(VRP_t^+(k))/\mathbb{E}(VRP_t) = 8.0\%$  and  $\mathbb{E}(VRP_t^-(k))/\mathbb{E}(VRP_t) = 60.8\%$  still imply that the fear component accounts for more than half of the variance risk premium. Thus, even though the relative importance of  $VRP_t(k)$  drops somewhat, the general findings remain intact. We note that these differences are also consistent with the biases observed in the Monte Carlo simulations reported in Section VI below.

<sup>40</sup> These numbers are calculated on the basis of the specific parameter estimates for the SVCJ models reported in the two studies, with the horizon fixed at 14 days, corresponding to the median time-to-expiration of the options used in our analysis, along with  $k = 10\%$ .

risk. Importantly, however, existing parametric studies generally constrain the wedge between the risk-neutral and statistical jump distributions to a negative shift in the mean jump size;<sup>41</sup> see, for example, the discussion in Singleton (2006). The results reported here indicate that the difference in the jump behavior under the two probability measures is much more intricate than that, and that time-varying jump intensities play an especially important role for properly understanding the risk premia and the inherent fear component.

We next check the robustness of our findings with respect to the general assumptions concerning the  $\mathbb{P}$  and  $\mathbb{Q}$  dynamic tail dependencies, and the specific asymptotic approximations underlying our jump tail estimates.

## VI. Robustness Checks

Our approach for estimating the tail risk premia and the Investor Fears index is extremely flexible, with the only parametric assumptions relating to the very general form of the time-varying jump intensities for the  $\mathbb{P}$  and  $\mathbb{Q}$  jump tails in equations (27) and (19), respectively. We begin our discussion below with a series of formal rank tests for the empirical validity of these assumptions that underlie our estimation of the  $RT_t$  and  $LT_t$  tail measures. We then present summary results from a detailed Monte Carlo simulation study designed to investigate the accuracy of our  $\mathbb{Q}$  jump tail estimation. We also briefly comment on some additional robustness checks related to the use of lower frequency daily data and shorter-lived options.

### A. Time Variation in $\mathbb{P}$ Jump Tails

The relevant assumption in equation (27) restricts the intensities for the positive and negative jumps to be affine functions, possibly with different coefficients, of the latent instantaneous volatility  $\sigma_t^2$ . This in turn implies that the left and right realized jump variances,  $\int_{t+\pi_t}^{t+1} \int_{x>0} x^2 \mu(ds, dx)$  and  $\int_{t+\pi_t}^{t+1} \int_{x<0} x^2 \mu(ds, dx)$ , depend upon the past only through the temporal variation in the integrated stochastic volatility. Consequently, the unconditional (auto)covariances equal

$$\text{cov} \left( \int_{t+\pi_t}^{t+1} \int_{x>0} x^2 \mu(ds, dx), \int_{t-1+\pi_t}^t \sigma_s^2 ds \right) = K_1 \text{cov} \left( \int_{t+\pi_t}^{t+1} \sigma_s^2 ds, \int_{t-1+\pi_t}^t \sigma_s^2 ds \right),$$

and

$$\begin{aligned} & \text{cov} \left( \int_{t+\pi_t}^{t+1} \int_{x>0} x^2 \mu(ds, dx), \int_{t-1+\pi_t}^t \int_{x>0} x^2 \mu(ds, dx) \right) \\ &= K_1 \text{cov} \left( \int_{t+\pi_t}^{t+1} \sigma_s^2 ds, \int_{t-1+\pi_t}^t \int_{x>0} x^2 \mu(ds, dx) \right), \end{aligned}$$

<sup>41</sup> Broadie, Chernov, and Johannes (2009) also consider changing the variance of the jumps.

**Table III**  
**Rank Tests for Affine  $\mathbb{P}$  Jump Intensities**

The table reports the results of the Gill and Lewbel (1992) rank test, as detailed in Appendix Subsection D. The realized variation measures are based on 5-minute high-frequency S&P 500 futures data from January 1990 through December 2008. The test has a limiting  $\chi_d^2$  distribution under the null hypothesis that the autocovariance matrix has a reduced rank of 1. The calculation of the asymptotic variance for the autocovariance matrices are based on a Parzen kernel with a lag length of 70.

Autocovariance Matrix	Test Stat.	d	p-value
$(CV_t, RJV_t, LJV_t)$	3.7937	2	0.1500
$(CV_t, RJV_t)$	1.2616	1	0.2613
$(CV_t, LJV_t)$	0.2590	1	0.6108
$(RJV_t, LJV_t)$	3.5299	1	0.0603

where  $K_1 = (\alpha_1^+)^2 \int_{x>0} x^2 v^{\mathbb{P}}(x) dx$ . Now, defining the vector

$$\mathbf{Z}_t \equiv \left( \int_{t+\pi_t}^{t+1} \sigma_s^2 ds, \int_{t+\pi_t}^{t+1} \int_{x>0} x^2 \mu(ds, dx), \int_{t+\pi_t}^{t+1} \int_{x<0} x^2 \mu(ds, dx) \right)',$$

it follows that the autocovariance matrix for  $\mathbf{Z}_t$  should be of reduced rank.<sup>42</sup> We state this testable implication in the following proposition.

**PROPOSITION 2:** *If  $\sigma_t^2$  is a covariance stationary and ergodic process, and  $\int_{\mathbb{R}} x^2 v^{\mathbb{P}}(x) dx < \infty$ , the assumption in equation (27) implies that  $\text{rank}(\text{cov}(\mathbf{Z}_t, \mathbf{Z}_{t-1})) = 1$ .*

In practice, of course, the vector  $\mathbf{Z}_t$  is not directly observable. However, as discussed in Section III above, it may be accurately estimated by  $(CV_t, RJV_t, LJV_t)$ . Even so, this vector will invariably include some estimation error, which will render the resulting autocovariance matrix of full rank even if  $\text{cov}(\mathbf{Z}_t, \mathbf{Z}_{t-1})$  is not. Hence, in order to formally test whether the deviations from reduced rank in the actually observed sample autocovariance matrices for  $(CV_t, RJV_t, LJV_t)$ , and subsets thereof, are statistically significant, we rely on the test developed by Gill and Lewbel (1992); further details concerning the implementation are given in Subsection D of the Appendix.<sup>43</sup>

None of the resulting test statistics reported in Table III reject the reduced rank hypothesis. As such, the results do not provide any evidence against the affine restriction in equation (27) related to the actual jump intensities.<sup>44</sup>

<sup>42</sup> The possibility of jumps in  $\sigma_t^2$  means that the covariance matrix of  $\mathbf{Z}_t$  will not necessarily be of reduced rank.

<sup>43</sup> This test is relatively easy to implement, and it overcomes some of the computational difficulties associated with other tests for rank deficiency based on the smallest eigenvalues of the matrix. The Monte Carlo simulation evidence discussed in Gill and Lewbel (1992) also suggests that the test exhibits desirable size and power properties.

<sup>44</sup> The test statistics reported in the table are based on the realized variation measures over the full sample. However, very similar test statistics are obtained for the shorter January 1990

### B. Time Variation in $\mathbb{Q}$ Jump Tails

In contrast to the affine dependencies for the  $\mathbb{P}$  jump intensities underlying our estimates of the statistical tail measures, our estimates for the risk-neutral tail measures rely on the less restrictive condition in equation (19). This assumption merely requires the temporal variation to be the same across different jump sizes, and as such is trivially satisfied by all parametric models hitherto estimated in the literature. This level of generality for the  $\mathbb{Q}$  jump measures may be particularly important to properly accommodate the seemingly complex dynamic dependencies in investor fears.

To more formally investigate this hypothesis, suppose that the  $\mathbb{Q}$  jump intensities satisfied the analogous affine dependencies assumed for the  $\mathbb{P}$  jump measure,

$$\nu_t^{\mathbb{Q}}(dx) = \left( \alpha_0^{+\mathbb{Q}} \mathbf{1}_{\{x>0\}} + \alpha_0^{-\mathbb{Q}} \mathbf{1}_{\{x<0\}} + (\alpha_1^{+\mathbb{Q}} \mathbf{1}_{\{x>0\}} + \alpha_1^{-\mathbb{Q}} \mathbf{1}_{\{x<0\}}) \sigma_t^2 \right) v^{\mathbb{Q}}(x) dx, \quad (33)$$

where  $v^{\mathbb{Q}}(x)$  denotes a valid Lévy density, and  $\alpha_0^{\pm\mathbb{Q}}$  and  $\alpha_1^{\pm\mathbb{Q}}$  are nonnegative constants.<sup>45</sup> Similar to the definition of  $\mathbf{Z}_t$  in Proposition 2 above, define the vector of risk-neutral measures  $\mathbf{Y}_t \equiv (RT_t^{\mathbb{Q}}(k), LT_t^{\mathbb{Q}}(k), QV_t^{\mathbb{Q}})'$ . If (33) holds, then  $RT_t^{\mathbb{Q}}(k)$ ,  $LT_t^{\mathbb{Q}}(k)$ , and  $QV_t^{\mathbb{Q}}$  should all be linear functions of the risk-neutral expected integrated volatility  $\mathbb{E}_t^{\mathbb{Q}}(\int_t^T \sigma_s^2 ds)$ . Consequently, the covariance matrix, and equivalently the lag-one autocovariance matrix, for  $\mathbf{Y}_t$  should be of reduced rank one. If (33) does not hold,  $QV_t^{\mathbb{Q}}$  may still depend linearly on the integrated volatility  $\mathbb{E}_t^{\mathbb{Q}}(\int_t^T \sigma_s^2 ds)$ . However, in this situation additional time-varying factors will typically affect  $\mathbf{Y}_t$ , so that  $\text{rank}(\text{var}(\mathbf{Y}_t)) \equiv \text{rank}(\text{cov}(\mathbf{Y}_t, \mathbf{Y}_{t-1})) > 1$ . The following testable proposition summarizes this implication.

**PROPOSITION 3:** *If  $\mathbf{Y}_t$  is a covariance stationary and ergodic process, and  $\int_{|x|>\epsilon} e^x v^{\mathbb{Q}}(x) dx < \infty$  for some  $\epsilon > 0$ , the assumption in equation (33) implies that  $\text{rank}(\text{cov}(\mathbf{Y}_t, \mathbf{Y}_{t-1})) = 1$ .*

The results from the rank tests for the  $\mathbb{Q}$  tail measures are reported in Table IV. As seen from the table, all of the tests clearly reject the affine structure for the risk-neutral jump intensities.<sup>46</sup> Of course, this contrasts sharply with the comparable test results for the  $\mathbb{P}$  jump intensities in Table III, which generally support the affine structure.

Taken as a whole, the robustness tests in Tables III and IV pertaining to the temporal variation in the jump intensities are consistent with the key minimal

through June 2007 sample employed in the tail estimation. We also implemented the same set of tests with all of the variables replaced by their square roots, that is,  $(CV_t^{1/2}, RJV_t^{1/2}, LJV_t^{1/2})$ . All of these tests strongly rejected the reduced rank hypothesis, highlighting the power of the procedure.

<sup>45</sup> This assumption, together with equation (27), implies that the temporal variation in all risk premia would be spanned by the level of the stochastic volatility.

<sup>46</sup> The tests reported in the table are based on the full sample through the end of 2008. As before, very similar results obtain when excluding the July 2007 through December 2008 part of the sample.

**Table IV**  
**Rank Tests for Affine  $\mathbb{Q}$  Jump Intensities**

The table reports the results of the Gill and Lewbel (1992) rank test, as detailed in Appendix Subsection D. The variation measures are based on S&P 500 options data from January 1996 through December 2008. The test has a limiting  $\chi_d^2$  distribution under the null hypothesis that the autocovariance matrix has a reduced rank of 1. The calculation of the asymptotic variance for the autocovariance matrices are based on a Parzen kernel with a lag length of 70.

Autocovariance Matrix	Test Stat.	d	p-value
$(QV_t^{\mathbb{Q}}, RT_t^{\mathbb{Q}}, LT_t^{\mathbb{Q}})$	65.8861	2	0.0000
$(QV_t^{\mathbb{Q}}, RT_t^{\mathbb{Q}})$	29.3789	1	0.0000
$(QV_t^{\mathbb{Q}}, LT_t^{\mathbb{Q}})$	31.7486	1	0.0000
$(RT_t^{\mathbb{Q}}, LT_t^{\mathbb{Q}})$	12.4262	1	0.0000

assumptions underlying our nonparametric inference procedures. Importantly, the test results clearly point to the existence of much more complex dynamic dependencies in the risk-neutral jump tails than hitherto entertained in the literature, and in the form of dependencies that cannot simply be spanned by the level of the stochastic volatility. As such, this also calls into question the validity of parametric procedures that restrict the form of the tails to be the same across the  $\mathbb{P}$  and  $\mathbb{Q}$  distributions when trying to understand notions of investor fears, or “crash-o-phobia,” embedded in market prices.

### C. Estimation of $\mathbb{Q}$ Tail Measures

To gauge the magnitude of the approximation errors in Proposition 1 and the accuracy of the estimates for the  $LT_t^{\mathbb{Q}}$  and  $RT_t^{\mathbb{Q}}$  tail measures, we simulate artificial option prices from four different parametric models. Model A refers to the classical Merton model with constant volatility and normally distributed compound Poisson jumps. Following Carr and Wu (2003), we fix the continuous volatility at 14%, the jump intensity at two, the mean log jump size at -5%, and the volatility of the log jump size at 13%, all in annualized units.<sup>47</sup>

The remaining three models, Models B, C, and D, all assume that the instantaneous volatility follows a square-root diffusion process,

$$d\sigma_t^2 = \kappa_i(\theta_i - \sigma_t^2)dt + \eta\sigma_t dW_t, i = p, q, \quad (34)$$

where  $\eta = 0.1$ , and the rest of the parameters under the  $\mathbb{P}$  and  $\mathbb{Q}$  probability measures are fixed at  $\theta_p = 0.13^2$ ,  $\kappa_p = 5.04$ ,  $\theta_q = 0.14^2$ , and  $\kappa_q = 4.3457$ , respectively. Following Bates (2000), Model B further assumes normally distributed

<sup>47</sup> The only difference from Carr and Wu (2003) is the reduction in the mean log-jump size from -10% to -5% in order to more closely resemble the sample averages in our data.

**Table V**  
**Estimation of Risk-Neutral Tail Measures**

The table reports summary statistics for the estimation of the  $LT_t$  and  $RT_t$  measures based on out-of-the-money options computed with simulated data from Models A-D as described in the text. Time-to-maturity  $T - t$  is fixed at 15 days, and the log-moneyness  $k$  for the left and right tails are set at 0.9 and 1.1, respectively. The length of the simulated series is 5,000 days. MAD denotes the median absolute deviation.

	$LT_t^{\mathbb{Q}}$			$RT_t^{\mathbb{Q}}$		
	True	Median	MAD	True	Median	MAD
Model A	0.0484	0.0496	—	0.0201	0.0238	—
Model B	0.0418	0.0437	0.0072	0.0173	0.0205	0.0039
Model C	0.0407	0.0406	0.0056	0.0112	0.0128	0.0021
Model D	0.0380	0.0408	0.0070	0.0101	0.0139	0.0032

jumps with jump intensity proportional to the instantaneous variance,

$$\nu_t(dx) = \lambda_i \sigma_t^2 \frac{e^{-0.5(x-\mu_i)^2/\eta_i^2}}{\sqrt{2\pi\eta_i^2}}, i = p, q, \quad (35)$$

with the  $\mathbb{P}$  and  $\mathbb{Q}$  parameters set at  $\lambda_p = 2/\theta_p$ ,  $\mu_p = 0$ ,  $\eta_p = 0.0561$ ,  $\lambda_q = 2/\theta_q$ ,  $\mu_q = -0.05$ , and  $\eta_q = 0.13$ , respectively. These values for the  $\mathbb{Q}$  jump parameters ensure that, conditional on the mean level of the volatility, the jumps have the same distribution and intensity as in Model A.

Finally, Models C and D postulate a generalized tempered stable process for the jump compensator, as in, for example, Bates (2009) and Carr et al. (2003),

$$\nu_t(dx) = \frac{\sigma_t^2}{|x|^{\alpha+1}} (c^- e^{-M_i|x|} \mathbf{1}_{\{x<0\}} + c^+ e^{-N_i|x|} \mathbf{1}_{\{x>0\}}), i = p, q, \quad (36)$$

where for Model C the parameters are fixed at  $M_p = N_p = 13.0543$ ,  $M_q = 3$ ,  $N_q = 8.1313$ ,  $\alpha = 0.0$ , and  $c^- = c^+ = 0.3600/\theta_p$ , while for Model D we set  $M_p = N_p = 19.5026$ ,  $M_q = 3$ ,  $N_q = 9.1990$ ,  $\alpha = 0.5$ , and  $c^- = c^+ = 0.2053/\theta_p$ . The  $\mathbb{Q}$  jump parameters are chosen so that the expected values of  $LT_t^{\mathbb{Q}}$  and  $RT_t^{\mathbb{Q}}$  implied by  $\nu_t^{\mathbb{Q}}(dx)$  are in line with their corresponding sample means from our data. The  $\mathbb{P}$  jump parameters imply that the mean value of the daily  $P$  jump variation equals 0.25 in daily percentage units, which is generally consistent with recent nonparametric empirical evidence reported in the literature.

Turning to the results, Table V reports the median and median absolute deviation (MAD) estimates for the two tail measures obtained across 5,000 simulated “days” for each of the four models for options with  $T - t$  set to 15 days to maturity and  $k$  equal to 0.9 and 1.1 for the left and right tails, respectively.<sup>48</sup> These characteristics correspond directly to the average maturity time and moneyness for the options used in the actual estimation. As is evident from

<sup>48</sup> The results for the Merton Model A are exact, so the MADs are identically equal to zero.

the table, the left skew in the risk-neutral jump distributions renders the left tail measures two to three and a half times larger than the corresponding right tail measures and generally also easier to estimate. The slight upward bias observed in both tails is directly attributable to the continuous variation invariably “clouding” the identification of the true jump tails. All in all, however, the results clearly underscore the accuracy of the theoretical approximations underlying our estimation of the jump tails.<sup>49</sup>

#### *D. Daily Data and Shorter Options*

To check the robustness of our main empirical findings with respect to the use of intraday data for the extraction of the  $\mathbb{P}$  jump tails, we redid the estimation of the tail parameters using only daily data. Restricting the analysis to a coarser daily frequency invariably handicaps the detection of jumps and the corresponding jump tail estimation. Nonetheless, the key features pertaining to the behavior of the jump tails remain intact, albeit not as precisely estimated.<sup>50</sup>

In a similar vein, to corroborate the simulation results discussed above and check the robustness of our empirical results with respect to the options used in extracting the  $\mathbb{Q}$  jump tails, we re-estimated the tails with only monthly data. Specifically, by restricting our sample to the 5<sup>th</sup> business day of each month, it is possible to reduce the median time-to-maturity of the options from 14 to only 7 business days. Nonetheless, the resulting monthly Investor Fears index looks almost identical to the plot discussed in Section IV above. Further details concerning these and other robustness checks are again available in the Internet Appendix.

## **VII. Conclusion**

The recent market turmoil has spurred renewed interest in the study and economic impact of rare disaster type events. Most of these studies resort to peso type explanations for the occurrence of rare events along with specific parametric modeling assumptions for carrying out statistical inference. In contrast, the new methodologies developed here are based on flexible nonparametric procedures, which rely on high-frequency data and EVT for effectively measuring the actual expected tail events together with a rich set of options for uncovering the pricing of tail events.

Allowing the data to “speak for themselves,” our empirical findings suggest that much of the historically large equity and variance risk premia may be

<sup>49</sup> Much more detailed simulation results and further discussion for a range of different maturity times and moneyness are available in the supplementary Internet Appendix (available online in the “Supplements and Datasets” section at <http://www.afajof.org/supplements.asp>). Not surprisingly, these results reveal that the quality of the tail estimates generally deteriorates the closer to the money and the longer lived are the options used in the estimation.

<sup>50</sup> By the basic statistical principle that more data are always better, see, for example, the pertinent discussion in Zhang, Mykland, and Aït-Sahalia (2005); the high-frequency data, if used correctly, will *always* result in more accurate estimates.

ascribed to compensation for jump tail risk. These types of risks are poorly described by traditional mean-variance type frameworks. Meanwhile, excluding the part of the risk premia directly attributable to tail events, the average equity and variance premia observed in the data are in line with the general implications from standard equilibrium-based asset pricing models with reasonable preference parameters. Our decomposition of the variance risk premia further suggests that investors' fears, or "crash-o-phobia," play a crucial role in explaining the overall large magnitude of the tail risk premia. At the same time, the perception of jump tail risks appears to vary quite dramatically over time, and changes therein may help explain some of the recent sharp movements in market valuations.

## Appendix

### A. Estimation of Jump Tails

We start with some general remarks that apply for both the  $\mathbb{P}$  and the  $\mathbb{Q}$  distributions, and then subsequently specialize the discussion to the statistical and risk-neutral jump tail estimation. For ease of notation, we suppress the dependence on  $\mathbb{P}$  and  $\mathbb{Q}$  in the Lévy densities  $v^{\mathbb{P}}(x)$  and  $v^{\mathbb{Q}}(x)$  in this initial discussion.

To begin, define  $\psi^+(x) = e^x - 1$  for  $x \in \mathbb{R}_+$  and  $\psi^-(x) = e^{-x}$  for  $x \in \mathbb{R}_-$ , where  $\psi^+(x)$  maps  $\mathbb{R}_+$  into  $\mathbb{R}_+$  and  $\psi^-(x)$  maps  $\mathbb{R}_-$  into  $(1, +\infty)$ . Both mappings are one-to-one. Further, denote  $v_{\psi}^+(y) = \frac{v(\ln(y+1))}{y+1}$  for  $y \in (0, \infty)$  and  $v_{\psi}^-(y) = \frac{v(-\ln y)}{y}$ . Finally, denote the tail jump measures

$$\bar{v}_{\psi}^{\pm}(x) = \int_x^{\infty} v_{\psi}^{\pm}(u) du, \quad (\text{A1})$$

with  $x > 0$  for  $\bar{v}_{\psi}^+(x)$  and  $x > 1$  for  $\bar{v}_{\psi}^-(x)$ .

Our estimation under both  $\mathbb{P}$  and  $\mathbb{Q}$  is nonparametric and based on the assumption that the tail Lévy densities  $\bar{v}_{\psi}^{\pm}$  belong to the domain of attraction of an *extreme value distribution*; see, for example, the general discussion in Embrechts, Kluppelberg, and Mikosch (2001), along with the recent empirical analysis related to extreme market moves in Bakshi, Madan, and Panayotov (2009). This is a very wide class of jump measures, and it includes most parametric specifications previously used in the literature.

The key implication of the assumption (see, for example, Smith (1987)), is that

$$1 - \frac{\bar{v}_{\psi}^{\pm}(u+x)}{\bar{v}_{\psi}^{\pm}(x)} \underset{\text{appr}}{\sim} G(u; \sigma^{\pm}, \xi^{\pm}), \quad u > 0, x > 0, \quad (\text{A2})$$

where

$$G(u; \sigma^{\pm}, \xi^{\pm}) = \begin{cases} 1 - (1 + \xi^{\pm} u / \sigma^{\pm})^{-1/\xi^{\pm}}, & \xi^{\pm} \neq 0, \sigma^{\pm} > 0 \\ 1 - e^{-u/\sigma^{\pm}}, & \xi^{\pm} = 0, \sigma^{\pm} > 0 \end{cases} \quad (\text{A3})$$

with  $G(u; \sigma^\pm, \xi^\pm)$  the cdf of a *generalized Pareto distribution*. In what follows, we assume that the approximation in (A2) is exact when  $x$  is large enough; see the formal proofs in Bollerslev and Todorov (2011) for further details concerning the approximation error and the exact conditions under which the error is indeed asymptotically negligible.

Before we turn to our discussion of the estimation of the parameters for the generalized Pareto distribution approximation to the tail Lévy densities, we present several integrals with respect to the Lévy density  $v(x)$  that appear in  $ERP_t(k)$ ,  $VRP_t(k)$ ,  $RT_t^{\mathbb{P}}(k)$ ,  $LT_t^{\mathbb{P}}(k)$ ,  $RT_t^{\mathbb{Q}}(k)$ , and  $LT_t^{\mathbb{Q}}(k)$ . In particular, assuming that the approximation in (A2) holds, it follows that, for  $k > 0$  and some  $tr^+ < e^k - 1$ ,

$$\begin{aligned} \int_{x>k} (e^x - 1)v(x)dx &= (e^k - 1)\bar{v}_\psi^+(e^k - 1) + \int_{\mathbb{R}} (e^x - e^k)^+ v(x)dx, \\ \int_{\mathbb{R}} (e^x - e^k)^+ v(x)dx &= \int_{(0,\infty)} (x - e^k + 1)^+ v_\psi^+(x)dx = \int_{e^k-1}^{\infty} \bar{v}_\psi^+(x)dx \\ &= \bar{v}_\psi^+(tr^+) \int_{e^k-1}^{\infty} \left(1 - \left(1 - \frac{\bar{v}_\psi^+(x)}{\bar{v}_\psi^+(tr^+)}\right)\right) dx \\ &= \bar{v}_\psi^+(tr^+) \left(\frac{\sigma^+}{1 - \xi^+}\right) \left(1 + \frac{\xi^+}{\sigma^+}(e^k - 1 - tr^+)\right)^{1-1/\xi^+}, \\ \int_{x>k} (e^x - 1 - x)v(x)dx &= \int_{x>e^k-1} (x - \ln(x+1))v_\psi^+(x)dx \\ &= \bar{v}_\psi^+(e^k - 1)(e^k - 1 - k) + \bar{v}_\psi^+(tr^+) \frac{\sigma^+}{1 - \xi^+} \left(1 + \frac{\xi^+}{\sigma^+}(e^k - 1 - tr^+)\right)^{1-1/\xi^+} \\ &\quad - \frac{\bar{v}_\psi^+(tr^+)(\sigma^+)^{1/\xi^+}\xi^+}{(e^k\xi^+)^{1/\xi^+}} {}_2F_1\left(\frac{1}{\xi^+}, \frac{1}{\xi^+}; 1 + \frac{1}{\xi^+}; \frac{\frac{\xi^+}{\sigma^+}(tr^+ + 1) - 1}{\frac{\xi^+}{\sigma^+}e^k}\right), \\ \int_{x>k} x^2 v(x)dx &= k^2 \bar{v}_\psi^+(e^k - 1) + 2 \int_{x>e^k-1} \frac{\ln(x+1)}{x+1} \bar{v}_\psi^+(x)dx \\ &= k^2 \bar{v}_\psi^+(e^k - 1) + 2\bar{v}_\psi^+(tr^+) \times K_1, \end{aligned}$$

and

$$\begin{aligned} K_1 &= e^{-k/\xi^+} \xi^+ \left(\frac{\xi^+}{\sigma^+}\right)^{-1/\xi^+} \left[ \xi^+ {}_3F_2\left(\frac{1}{\xi^+}, \frac{1}{\xi^+}, \frac{1}{\xi^+}; 1 + \frac{1}{\xi^+}, 1 + \frac{1}{\xi^+}; \frac{\frac{\xi^+}{\sigma^+}(tr^+ + 1) - 1}{e^k \frac{\xi^+}{\sigma^+}}\right) \right. \\ &\quad \left. + k {}_2F_1\left(\frac{1}{\xi^+}, \frac{1}{\xi^+}, 1 + \frac{1}{\xi^+}; \frac{\frac{\xi^+}{\sigma^+}(tr^+ + 1) - 1}{e^k \frac{\xi^+}{\sigma^+}}\right) \right], \end{aligned}$$

where  ${}_3F_2(\cdot, \cdot, \cdot; \cdot, \cdot)$  and  ${}_2F_1(\cdot, \cdot; \cdot)$  are the hypergeometric functions. The above integrals are all well defined provided that  $\int_{x>\epsilon} x v_\psi^+(x) dx < \infty$  for some  $\epsilon > 0$ , that is,  $\xi^+ < 1$ . Similar calculations for  $k < 0$  and  $1 < tr^- < e^{-k}$  yield,

$$\int_{x<k} (e^x - 1) v(x) dx = (e^k - 1) \bar{v}_\psi^-(e^{-k}) - \int_{\mathbb{R}} (e^k - e^x)^+ v(x) dx,$$

$$\begin{aligned} \int_{\mathbb{R}} (e^k - e^x)^+ v(x) dx &= \int_{(0, \infty)} \left( e^k - \frac{1}{x} \right)^+ v_\psi^-(x) dx = \int_{e^{-k}}^\infty \frac{\bar{v}_\psi^-(x)}{x^2} dx \\ &= \bar{v}_\psi^-(tr^-) \left( \frac{\xi^-}{\xi^- + 1} \right) (e^k)^{1+1/\xi^-} \left( \frac{\xi^-}{\sigma^-} \right)^{-1/\xi^-} \\ &\quad \times {}_2F_1 \left( 1 + \frac{1}{\xi^-}; \frac{1}{\xi^-}; 2 + \frac{1}{\xi^-}; -\frac{1 - tr^- \frac{\xi^-}{\sigma^-}}{e^{-k} \frac{\xi^-}{\sigma^-}} \right), \end{aligned}$$

$$\begin{aligned} \int_{x<k} (e^x - 1 - x) v(x) dx &= \int_{x>e^{-k}} \left( \frac{1}{x} - 1 + \ln x \right) v_\psi^-(x) dx \\ &= \bar{v}_\psi^-(e^{-k})(e^k - 1 - k) + \bar{v}_\psi^-(tr^-) \left( e^k \frac{\sigma^-}{\xi^-} \right)^{1/\xi^-} \xi^- \times K_2, \end{aligned}$$

$$\begin{aligned} K_2 &= {}_2F_1 \left( \frac{1}{\xi^-}, \frac{1}{\xi^-}, 1 + \frac{1}{\xi^-}; \frac{tr^- \frac{\xi^-}{\sigma^-} - 1}{e^{-k} \frac{\xi^-}{\sigma^-}} \right) \\ &\quad - \frac{e^k}{\xi^- + 1} {}_2F_1 \left( 1 + \frac{1}{\xi^-}; \frac{1}{\xi^-}; 2 + \frac{1}{\xi^-}; \frac{tr^- \frac{\xi^-}{\sigma^-} - 1}{e^{-k} \frac{\xi^-}{\sigma^-}} \right), \end{aligned}$$

$$\int_{x<k} x^2 v(x) dx = k^2 \bar{v}_\psi^-(e^{-k}) + 2 \int_{x>e^{-k}} \frac{\ln(x)}{x} \bar{v}_\psi^-(x) dx = k^2 \bar{v}_\psi^-(e^{-k}) + 2 \bar{v}_\psi^-(tr^-) \times K_3,$$

and

$$\begin{aligned} K_3 &= e^{k/\xi^-} \xi^- \left( \frac{\xi^-}{\sigma^-} \right)^{-1/\xi^-} \left[ \xi^- {}_3F_2 \left( \frac{1}{\xi^-}, \frac{1}{\xi^-}, \frac{1}{\xi^-}; 1 + \frac{1}{\xi^-}, 1 + \frac{1}{\xi^-}; \frac{\frac{\xi^-}{\sigma^-} tr^- - 1}{e^{-k} \frac{\xi^-}{\sigma^-}} \right) \right. \\ &\quad \left. - k {}_2F_1 \left( \frac{1}{\xi^-}, \frac{1}{\xi^-}, 1 + \frac{1}{\xi^-}; \frac{\frac{\xi^-}{\sigma^-} tr^- - 1}{e^{-k} \frac{\xi^-}{\sigma^-}} \right) \right]. \end{aligned}$$

### A.1. Estimation of $\mathbb{P}$ Jump Tails

The estimation builds on the general theoretical results in Bollerslev and Todorov (2011). In particular, let

$$\theta \equiv (\sigma^\pm, \xi^\pm, \alpha_0^\pm \bar{v}_\psi^\pm(tr^\pm), \alpha_1^\pm \bar{v}_\psi^\pm(tr^\pm), \pi), \pi \equiv \mathbb{E}(\pi_t)$$

denote the parameter vector that we seek to estimate. Our estimation is then based on the scores associated with the log-likelihood function of the generalized Pareto distribution,

$$\phi_1^\pm(u) = -\frac{1}{\sigma^\pm} + \frac{\xi^\pm u}{(\sigma^\pm)^2} \left(1 + \frac{1}{\xi^\pm}\right) \left(1 + \frac{\xi^\pm u}{\sigma^\pm}\right)^{-1},$$

$$\phi_2^\pm(u) = \frac{1}{(\xi^\pm)^2} \ln \left(1 + \frac{\xi^\pm u}{\sigma^\pm}\right) - \frac{u}{\sigma^\pm} \left(1 + \frac{1}{\xi^\pm}\right) \left(1 + \frac{\xi^\pm u}{\sigma^\pm}\right)^{-1}.$$

The specific estimation equations that we use are

$$\frac{1}{N} \sum_{t=1}^N \sum_{j=1}^{n-1} \phi_i^\pm(\psi^\pm(\Delta_j^{n,t} p) - tr^\pm) \mathbf{1}_{\{\psi^\pm(\Delta_j^{n,t} p) > tr^\pm\}} = 0, \quad i = 1, 2,$$

$$\frac{1}{N} \sum_{t=1}^N \sum_{j=1}^{n-1} \mathbf{1}_{\{\psi^\pm(\Delta_j^{n,t} p) > tr^\pm\}} - (1 - \pi) \alpha_0^\pm \bar{v}_\psi^\pm(tr^\pm) - \alpha_1^\pm \bar{v}_\psi^\pm(tr^\pm) CV_t = 0,$$

$$\frac{1}{N} \sum_{t=2}^N \left( \sum_{j=1}^{n-1} \mathbf{1}_{\{\psi^\pm(\Delta_j^{n,t} p) > tr^\pm\}} - (1 - \pi) \alpha_0^\pm \bar{v}_\psi^\pm(tr^\pm) - \alpha_1^\pm \bar{v}_\psi^\pm(tr^\pm) CV_t \right) CV_{t-1} = 0,$$

and

$$(1 - \pi) \frac{1}{N} \sum_{t=1}^N (p_{t+\pi_t} - p_t)^2 - \pi \frac{1}{N} \sum_{t=1}^N RV_t = 0.$$

The only remaining issue relates to the choice of  $tr^\pm$ . On the one hand, high thresholds are preferred for the asymptotic extreme value approximation in (A2) to work the best. On the other hand, we also need sufficiently many jumps above the thresholds to make the estimation reliable. Faced with this tradeoff, we set the values for  $tr^\pm$  so that they correspond to jumps in the log-prices of 0.6%. In the sample, we have 229 such jumps, 127 of them negative and 102 of them positive. This choice for  $tr^\pm$  exceeds the maximum threshold value for  $\alpha_{t,i}$ .

**Table A.I**  
**Estimates for  $\mathbb{P}$  Tail Parameters**

The estimates are based on 5-minute high-frequency S&P 500 futures data from January 1990 through June 2007. The truncation levels are fixed at  $tr^- = e^{0.006} \approx 1.006$  and  $tr^+ = e^{0.006} - 1 \approx 0.006$  for the left and right tails, respectively.

Left Tail			Right Tail		
Parameter	Estimate	St. Error	Parameter	Estimate	St. Error
$\xi_{\mathbb{P}}^-$	0.3865	0.1050	$\xi_{\mathbb{P}}^+$	0.4361	0.1558
$100\sigma_{\mathbb{P}}^-$	0.1709	0.0222	$100\sigma_{\mathbb{P}}^+$	0.1740	0.0272
$\alpha_0^- v_{\psi}^{\mathbb{P}^-}(tr^-)$	0.0084	0.0030	$\alpha_0^+ v_{\psi}^{\mathbb{P}^+}(tr^+)$	0.0002	0.0021
$\alpha_1^- v_{\psi}^{\mathbb{P}^-}(tr^-)$	0.0333	0.0050	$\alpha_1^+ v_{\psi}^{\mathbb{P}^+}(tr^+)$	0.0361	0.0041

used in the construction of  $CV_t$  and  $JV_t$  (see equation (A10) below), so that the continuous component in the price will have negligible effect in our estimation.

The actual results from the estimation are reported in Table A.I. As discussed in the main text, the  $\mathbb{P}$  tails are clearly time varying, but otherwise appear close to symmetric. Not reported in the table, our estimate for  $\pi \equiv \mathbb{E}(\pi_t)$  equal to 0.2076 (0.0448) implies that on average roughly 20% of the total variation is due to changes in the price during nontrading hours.

## A.2. Estimation of $\mathbb{Q}$ Jump Tails

We start our discussion of the  $\mathbb{Q}$  jump tail estimation with a formal proof of Proposition 1.

*Proof of Proposition 1:* Using the Meyer-Ito formula (Theorem 68 in Protter (2004)) for the payoff function of out-of-the-money calls, the definition of local time (and the behavior of the transitional distribution of a Lévy process over a small time interval), together with the fact that futures prices are martingales under the risk-neutral measure, we can write for arbitrary  $K > F_{t-}$  and  $T \downarrow t$

$$e^{r(t,T)} C_t(K) \sim \int_t^T \mathbb{E}_t^{\mathbb{Q}} \left( \int_{\mathbb{R}} \left[ 1_{\{F_{s-} < K\}} (F_{s-} e^x - K)^+ + 1_{\{F_{s-} > K\}} (K - F_{s-} e^x)^+ \right] v_s^{\mathbb{Q}}(dx) \right) ds, \quad (\text{A4})$$

where  $f(x) \sim g(x)$  is to be interpreted as  $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = 1$ . A similar result holds for close-to-maturity out-of-the-money puts. This result was first derived in Carr and Wu (2003) in the proof of their Theorem 1.

Next, note that since  $F_t$  is a martingale under  $\mathbb{Q}$ , using Doob's inequality for  $K > F_{t-}$ , that is, for out-of-the-money call options, it follows that

$$\mathbb{Q}_t \left( \sup_{u \in [t, T]} |F_u - F_{t-}| \geq K - F_{t-} \right) \leq C \times (T - t), \quad (\text{A5})$$

where the constant  $C$  is adapted to the information set at time  $t$ , and is otherwise proportional to the risk-neutral (conditional to time  $t$ ) expectation of the

quadratic variation of the  $\mathbb{Q}$ -martingale process  $\{F_t\}_t$ . With this, we may write

$$\begin{aligned} e^{r(t,T)} C_t(K) &= \int_t^T \mathbb{E}_t^\mathbb{Q} \int_{\mathbb{R}} 1_{\{F_{s-} < K\}} (F_{s-} e^x - K)^+ v_s^\mathbb{Q}(dx) ds + O((T-t)^2) \\ &= \int_t^T \mathbb{E}_t^\mathbb{Q} 1_{\{F_{t-} < K\}} \int_{\mathbb{R}} (F_{s-} e^x - K)^+ v_s^\mathbb{Q}(dx) ds + O((T-t)^2). \end{aligned} \quad (\text{A6})$$

To continue further, note that for every  $x \in \mathbb{R}$ ,

$$|(F_{s-} e^x - K)^+ - (F_{t-} e^x - K)^+| \leq e^x |F_{s-} - F_{t-}|. \quad (\text{A7})$$

Now, making use of the Burkholder-Davis-Gundy inequality, it follows that

$$\mathbb{E}_t^\mathbb{Q} \left( \sup_{u \in [t, T]} |F_u - F_{t-}| \right) \leq C \sqrt{T-t}, \quad (\text{A8})$$

where the constant  $C$  satisfies the same restrictions spelled out immediately above. Combining these results, we have

$$e^{r(t,T)} C_t(K) = (T-t) F_{t-} R T_t^\mathbb{Q}(k) + O((T-t)^{3/2}), \quad (\text{A9})$$

where  $R T_t^\mathbb{Q}(k)$  is formally defined in equation (17).

Using the decomposition of the risk-neutral Lévy density in (19), it follows further that

$$\frac{R T_t^\mathbb{Q}(k_1)}{R T_t^\mathbb{Q}(k_2)} = \frac{\int_{\mathbb{R}} (e^x - e^{k_1})^+ v_\mathbb{Q}(x) dx}{\int_{\mathbb{R}} (e^x - e^{k_2})^+ v_\mathbb{Q}(x) dx},$$

where  $k_1 = \frac{K_1}{F_{t-}}$  and  $k_2 = \frac{K_2}{F_{t-}}$ . Finally, applying the above formulas with  $tr^+ = K_1/F_{t-} - 1$ , together with the result in Smith (1987) concerning the residual function  $\tau^+(x)$ , concludes the proof for the right tail. The proof for the puts and the left tail proceeds in a completely analogous fashion.  $\square$

In the proof immediately above, we used a somewhat stronger version of the approximation in (A2), and the fact that the true value of  $\sigma^\pm$  is  $\xi^\pm/x$ . A similar restriction underlies the popular Hill estimator of the tail index. However, this stronger version of (A2) is not scale invariant. Therefore, while Proposition 1 conveys the main idea, in our actual estimation we only make use of (A2).

In particular, recall that our estimation of the jump tail probabilities under  $\mathbb{Q}$  only requires the weak separability condition for the time dependence and jump sizes implicit in the Lévy density in equation (19). The parameter vector that we seek to estimate is

$$\theta \equiv (\sigma_\mathbb{Q}^\pm, \xi_\mathbb{Q}^\pm, \alpha_{\mathbb{Q}}^\pm v_\psi^{\mathbb{Q}\pm}(tr^\pm)).$$

**Table A.II**  
**Estimates for  $\mathbb{Q}$  Tail Parameters**

The estimates are based on S&P 500 options data from January 1996 through June 2007. The log-moneyness of the options used for the left and right tails is fixed at  $k = 0.9$  and  $k = 1.1$ , respectively.

Left Tail			Right Tail		
Parameter	Estimate	St. Error	Parameter	Estimate	St. Error
$\xi_{\mathbb{Q}}^-$	0.2581	0.02816	$\xi_{\mathbb{Q}}^+$	0.0793	0.0147
$\sigma_{\mathbb{Q}}^-$	0.0497	0.0021	$\sigma_{\mathbb{Q}}^+$	0.0238	0.0010
$\alpha_{\mathbb{Q}}^+ \bar{v}_{\psi}^{\mathbb{Q}-}(\frac{1}{1-0.075})$	0.9888	0.0525	$\alpha_{\mathbb{Q}}^+ \bar{v}_{\psi}^{\mathbb{Q}+}(0.075)$	0.5551	0.0443

From the calculations above, we have that for  $0 < tr^+ \leq e^k - 1$ ,

$$\mathbb{E}(RT_t^{\mathbb{Q}}(k)) = \alpha_{\mathbb{Q}}^+ \bar{v}_{\psi}^{\mathbb{Q}+}(tr^+) \frac{\sigma_{\mathbb{Q}}^+}{1 - \xi_{\mathbb{Q}}^+} \left( 1 + \frac{\xi_{\mathbb{Q}}^+}{\sigma_{\mathbb{Q}}^+} (e^k - 1 - tr^+) \right)^{1-1/\xi_{\mathbb{Q}}^+},$$

while for  $1 < tr^- \leq e^{-k}$ ,

$$\begin{aligned} \mathbb{E}(LT_t^{\mathbb{Q}}(k)) &= \alpha_{\mathbb{Q}}^- \bar{v}_{\psi}^{\mathbb{Q}-}(tr^-) \frac{\xi_{\mathbb{Q}}^-}{\xi_{\mathbb{Q}}^- + 1} (e^k)^{1+1/\xi_{\mathbb{Q}}^-} \left( \frac{\xi_{\mathbb{Q}}^-}{\sigma_{\mathbb{Q}}^-} \right)^{-1/\xi_{\mathbb{Q}}^-} \\ &\times {}_2F_1 \left( 1 + \frac{1}{\xi_{\mathbb{Q}}^-}; \frac{1}{\xi_{\mathbb{Q}}^-}; 2 + \frac{1}{\xi_{\mathbb{Q}}^-}; \frac{tr^- \frac{\xi_{\mathbb{Q}}^-}{\sigma_{\mathbb{Q}}^-} - 1}{e^{-k} \frac{\xi_{\mathbb{Q}}^-}{\sigma_{\mathbb{Q}}^-}} \right), \end{aligned}$$

where  $\alpha_{\mathbb{Q}}^+ \equiv \mathbb{E}(\frac{1}{T-t} \mathbb{E}_t^{\mathbb{Q}}(\int_t^T \varphi_s^+ ds))$  and  $\alpha_{\mathbb{Q}}^- \equiv \mathbb{E}(\frac{1}{T-t} \mathbb{E}_t^{\mathbb{Q}}(\int_t^T \varphi_s^- ds))$ .

Replacing the unconditional risk-neutral expectations of the jump tail measures with the sample analogues of  $RT_t^{\mathbb{Q}}(k)$  and  $LT_t^{\mathbb{Q}}(k)$  for three different levels of moneyness for each of the tails results in an exactly identified GMM estimator. The levels of moneyness that we use in the estimation are  $\frac{K}{F_{t-}} = 0.9250, 0.9125$ , and  $0.9000$  for the left tail, and  $\frac{K}{F_{t-}} = 1.0750, 1.0875$ , and  $1.1000$  for the right tail. These levels of moneyness are sufficiently “deep” in the tails to guarantee that the effect of the diffusive price component in measuring  $LT_t^{\mathbb{Q}}(k)$  and  $RT_t^{\mathbb{Q}}(k)$  is minimal, and that the extreme value distribution provides a good approximation to the jump tail probabilities. Also, the truncation levels are fixed at  $tr^- = (1 - 0.075)^{-1}$  and  $tr^+ = 0.075$  for the left and right tails, respectively. This corresponds directly to the moneyness for the closest-to-the-money options used in the estimation.

The results from the estimation are reported in Table A.II. Not only is the left tail under the  $\mathbb{Q}$  distribution at a higher level, it also decays at a slower rate. Although not directly comparable, the standard errors for the  $\mathbb{Q}$  tail parameters are much smaller than the ones for the  $\mathbb{P}$  tail parameters reported in Table

A.I above. This is quite intuitive as our estimation for the risk-neutral distribution is based on expectations for the large jumps, while the estimates for the statistical distribution rely on actual jump realizations in the high-frequency data for inferring the relevant expectations, thus involving an extra layer of estimation error uncertainty.

### B. Estimation of Realized Measures

This section of the Appendix provides further details concerning the estimation of  $CV_t$ ,  $RJV_t$ , and  $LJV_t$  in equations (23) to (26) and the relevant truncation levels. Theoretically, any  $\alpha > 0$  and  $\varpi \in (0, 0.5)$  will work. In the actual estimation, however, we fix  $\varpi = 0.49$ . Our choice of  $\alpha$  is a bit more involved and takes into account the fact that volatility changes across days and also has a strong diurnal pattern over the trading day.

To account for the latter feature, we estimate nonparametrically a time-of-day factor  $TOD_i$ ,  $i = 1, 2, \dots, n$ ,

$$TOD_i = NOI_i \frac{\sum_{t=1}^N (f_{(t-1+\pi_t)+i\Delta_{n,t}} - f_{(t-1+\pi_t)+(i-1)\Delta_{n,t}})^2 \mathbf{1}_{\{|f_{(t-1+\pi_t)+i\Delta_{n,t}} - f_{(t-1+\pi_t)+(i-1)\Delta_{n,t}}| \leq \bar{\alpha} \Delta_n^{0.49}\}}}{\sum_{t=1}^N \sum_{i=1}^M (f_{(t-1+\pi_t)+i\Delta_{n,t}} - f_{(t-1+\pi_t)+(i-1)\Delta_{n,t}})^2}, \quad (\text{A10})$$

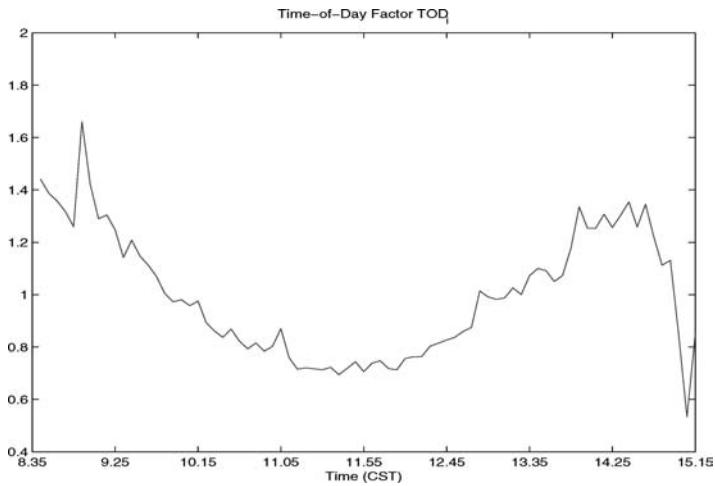
where

$$NOI_i = \frac{\sum_{t=1}^N \sum_{i=1}^{n-1} \mathbf{1}_{\{|f_{(t-1+\pi_t)+i\Delta_{n,t}} - f_{(t-1+\pi_t)+(i-1)\Delta_{n,t}}| \leq \bar{\alpha} \Delta_n^{0.49}\}}}{\sum_{t=1}^N \mathbf{1}_{\{|f_{(t-1+\pi_t)+i\Delta_{n,t}} - f_{(t-1+\pi_t)+(i-1)\Delta_{n,t}}| \leq \bar{\alpha} \Delta_n^{0.49}\}}},$$

$$\bar{\alpha} = 3\sqrt{\frac{\pi}{2}}$$

$$\sqrt{\frac{1}{N} \sum_{t=1}^N \sum_{i=2}^{n-1} |f_{(t-1+\pi_t)+i\Delta_{n,t}} - f_{(t-1+\pi_t)+(i-1)\Delta_{n,t}}| |f_{(t-1+\pi_t)+(i-1)\Delta_{n,t}} - f_{(t-1+\pi_t)+(i-2)\Delta_{n,t}}|},$$

and  $\Delta_n \equiv \frac{1}{n}$ . The truncation level  $\bar{\alpha}$  is based on the average volatility in the sample, as measured by the so-called bipower variation measure defined in Barndorff-Nielsen and Shephard (2004). Consequently,  $\bar{\alpha} \approx \sqrt{(1 - \mathbb{E}(\pi_t))\mathbb{E}(\sigma_t^2)}$ . The  $NOI_i$  factor puts the numerator and denominator in equation (A10) in the same units. This factor is approximately equal to  $n - 1$ . This calculation of  $TOD_i$  implicitly assumes that  $\pi_t$  and  $\sigma_t^2$  are both stationary and ergodic



**Figure A.1. Time-of-day factor.** The estimates are based on 5-minute high-frequency S&P 500 futures data from January 1990 through June 2007.

processes. The resulting  $TOD_i$  estimates are plotted in Figure A.1. The figure shows the previously well-documented U-shape in the average volatility over NYSE trading hours, with separate end effects for the futures before and after the cash market opens; see, for example, the related discussion in Andersen and Bollerslev (1997) pertaining to an earlier sample period.

To account for the time-varying volatility across days, we use the estimated continuous volatility for the previous day (for the first day in the sample we use  $\bar{\omega}$ ). Putting all of the pieces together, our time-varying threshold may be succinctly expressed as  $\alpha_{t,i} = 3\sqrt{CV_{t-1}} \times TOD_i \times \Delta_n^{0.49}$ .

### C. Estimation of Expected Integrated Volatility

Our estimation of  $\mathbb{E}_t(\int_t^T \sigma_s^2 ds)$  is based on the following restricted 22-order VAR,

$$\mathbf{X}_t = \mathbf{A}_0 + \mathbf{A}_1 \mathbf{X}_{t-1} + \mathbf{A}_5 \sum_{i=1}^5 \mathbf{X}_{t-i} / 5 + \mathbf{A}_{22} \sum_{i=1}^{22} \mathbf{X}_{t-i} / 22 + \epsilon_t, \quad (\text{A11})$$

for the four-dimensional vector

$$\mathbf{X}_t \equiv (CV_t \ RJV_t \ LJV_t \ (p_{t+\pi_t} - p_t)^2)'.$$

We approximate the multiperiod expected integrated volatility from the sum of the relevant forecasts for the first element in  $\mathbf{X}_t$ , which we in turn approximate by easy-to-calculate recursive linear projections using the Kalman filter. Similar reduced-form forecasting procedures for realized variation measures

**Table A.III**  
**Estimation Results for the Restricted VAR(22) Model in (A.11)**

The realized variation measures underlying the estimates are based on 5-minute high-frequency S&P 500 futures data from January 1990 through June 2007. Standard errors are reported in parentheses.

<b>A'</b> <sub>0</sub>	0.0368 (0.0190)	0.0010 (0.0216)	0.0430 (0.0188)	-0.0153 (0.0456)
<b>A</b> <sub>1</sub>	0.6956 (0.0316) -0.0449 (0.0360) -0.0028 (0.0313) 0.0831 (0.0758)	-0.0728 (0.0324) -0.0348 (0.0369) -0.0175 (0.0321) -0.0250 (0.0778)	0.2022 (0.0380) 0.1145 (0.0432) 0.0367 (0.0376) 0.1788 (0.0910)	-0.0618 (0.0146) -0.0299 (0.0167) -0.0211 (0.0145) -0.0613 (0.0351)
<b>A</b> <sub>5</sub>	0.0672 (0.0462) 0.0734 (0.0525) -0.0001 (0.0457) 0.2325 (0.1107)	0.0325 (0.0841) -0.0515 (0.0957) 0.0018 (0.0833) 0.0495 (0.2016)	-0.0974 (0.0947) 0.0018 (0.1077) 0.0063 (0.0938) -0.1477 (0.2270)	0.0709 (0.0341) 0.0532 (0.0388) 0.0512 (0.0338) 0.2938 (0.0818)
<b>A</b> <sub>22</sub>	0.1669 (0.0464) 0.0399 (0.0528) 0.0308 (0.0460) -0.0097 (0.1113)	-0.2100 (0.1786) -0.2202 (0.2031) -0.0964 (0.1769) -0.0355 (0.4280)	0.0962 (0.1857) 0.3187 (0.2113) 0.1724 (0.1840) 0.6235 (0.4452)	0.0285 (0.0606) 0.0353 (0.0689) 0.0386 (0.0600) -0.1462 (0.1452)

have been demonstrated to work well empirically by Andersen et al. (2003) among many others.

The resulting parameter estimates for the VAR are reported in Table A.III. Not surprisingly, the estimates for the (1, 1) elements in the  $\mathbf{A}_1$ ,  $\mathbf{A}_5$ , and  $\mathbf{A}_{22}$  matrices point to the existence of strong own persistence in the continuous volatility component. The estimates also reveal significant asymmetries in the dynamic dependencies between the continuous volatility and both the tails and the overnight increments, underscoring the importance of including these variables in the construction of the integrated volatility forecasts.

#### D. Rank Test

We briefly sketch the basic ideas and calculations required for the rank tests that form the basis for assessing Propositions 2 and 3; additional details concerning the implementation of the test are given in Gill and Lewbel (1992). Let  $A$  denote a general  $p \times p$  matrix. The hypothesis of interest may then be expressed as

$$H_0 : \text{rank}(A) \equiv r \quad \text{vs.} \quad H_1 : \text{rank}(A) > r, \quad r < p,$$

where we have an asymptotically normally distributed estimator of  $A$ , say

$$\sqrt{N}\text{vec}(\hat{A} - A) \xrightarrow{\mathbb{L}} \mathcal{N}(0, V).$$

Also, let  $\hat{V}$  denote a consistent estimator of  $V$ . The rank test is based on LDU (lower diagonal upper triangular) decomposition with complete pivoting, that is, the elements of the diagonal matrix are decreasing in absolute value. Toward

this end, write the matrices associated with the LDU decomposition as

$$\hat{P}\hat{A}\hat{Q} = \hat{L}\hat{D}\hat{U},$$

where  $\hat{P}$  and  $\hat{Q}$  are permutation matrixes,

$$\hat{L} = \begin{pmatrix} \hat{L}_{11} & \mathbf{0} \\ \hat{L}_{21} & \mathbf{I}_{p-r} \end{pmatrix}, \quad \hat{D} = \begin{pmatrix} \hat{D}_1 & \mathbf{0} \\ \mathbf{0} & \hat{D}_2 \end{pmatrix}, \quad \hat{U} = \begin{pmatrix} \hat{U}_{11} & \hat{U}_{12} \\ \mathbf{0} & \hat{U}_{22} \end{pmatrix},$$

and  $\hat{L}_{11}$ ,  $\hat{U}_{11}$ , and  $\hat{U}_{22}$  are unit lower triangular. In addition, define

$$\hat{H} = [-\hat{L}_{22}^{-1}\hat{L}_{21}\hat{L}_{11}^{-1} \ \hat{L}_{22}^{-1} \ \mathbf{0}] \quad \hat{K} = \begin{pmatrix} -\hat{U}_{11}^{-1}\hat{U}_{12}\hat{U}_{22}^{-1} \\ \hat{U}_{22}^{-1} \end{pmatrix}.$$

Further, let  $\Xi_i$  denote the  $(p - r) \times (p - r)$  matrix with ones along the  $i^{\text{th}}$  diagonal and zeros everywhere else, and define

$$\Xi = \begin{pmatrix} \Xi_1 \\ \vdots \\ \Xi_n \end{pmatrix}$$

and

$$\hat{W} = \Xi'(\hat{K}' \otimes \hat{H})(\hat{Q}' \otimes \hat{P})\hat{V}(\hat{Q} \otimes \hat{P}')(\hat{K} \otimes \hat{H}')\Xi.$$

Finally, let  $\hat{d}_2 = \text{diag}(\hat{D}_2)$ . The rank test statistic,

$$\chi = N\hat{d}_2'\hat{W}^{-1}\hat{d}_2,$$

is then distributed as  $\chi_{p-r}^2$  under the null hypothesis that  $\text{rank}(A) \equiv r$ .

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