Linear Algebra Review 2

MAT 442

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Warm Up

Given that the matrix

$$A = \begin{pmatrix} 2 & 4 & 6 & 2 & 4 \\ 1 & 2 & 3 & 1 & 1 \\ 2 & 4 & 8 & 0 & 0 \\ 3 & 6 & 7 & 5 & 9 \end{pmatrix}$$

can be row reduced to

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

give a basis for the column space of A.
What is the rank of the row space of A?

Elementary Matrix Operations

Thereom

Let A be an *m x n* matrix. Any of the following operations on the rows [columns] of A is called an **elementary row [column] operation**

1. interchanging any two rows [columns] of A (type 1)
2. multiplying any row [column] of A by a nonzero scalar (type 2)
3. adding any scalar multiple of a row [column] of A to another row [column] (type 3)

Note that if a matrix Q can be obtained from a matrix P by means of elementary row operations, then P can be obtained from Q by elementary row operations of the same type.

Elementary Matrix Operations

Definition

An n x n **elementary matrix** is a matrix obtained by performing a single elementary operation on the identity matrix, I_n. The elementary matrix is said to be of **type 1, 2, or 3** depending upon which elementary operation was performed.

Theorem 3.1. Let A be an m x n matrix and suppose that B is obtained from A by performing an elementary row [column] operation. Then there exists an m x m (n x n) elementary matrix E such that B = EA [B = AE]. In fact, E is obtained from I_m [I_n] by performing the same elementary row [column] operation that was performed on A to obtain B.

Example

Let

$$A = \begin{pmatrix} 5 & 1 & 2 & 3 & 4 & 0 \\ 1 & 0 & 4 & 3 & 3 & 2 \\ 2 & 6 & 0 & 0 & 2 & 1 \end{pmatrix}$$

By performing a type 3 operation on A (multiplying row 3 by 2 and adding to row 1) we obtain the matrix

$$B = \begin{pmatrix} 9 & 13 & 2 & 3 & 8 & 2 \\ 1 & 0 & 4 & 3 & 3 & 2 \\ 2 & 6 & 0 & 0 & 2 & 1 \end{pmatrix}$$

Then by Theorem 3.1, B = EA, where E is obtained by applying the same type 3 operation to I_3

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 & 2 & 3 & 4 & 0 \\ 1 & 0 & 4 & 3 & 3 & 2 \\ 2 & 6 & 0 & 0 & 2 & 1 \end{pmatrix}$$

Elementary Matrix Operations

Theorem

Elementary matrices are invertible, and the inverse of an elementary matrix is an elementary matrix of the same type.

Note that elementary row and column operations on a matrix are **rank preserving**.

Thereom: Let A be an m x n matrix. If P and Q are invertible m x m and n x n matrices, respectively, then

- rank(AQ) $=$ rank(A)
- rank(PA) $=$ rank(A)
- rank(PAQ) $=$ rank(A)

Elementary Matrix Operations

If A is invertible, than A is a product of elementary matrices.

Proof: Suppose A is invertible. Then we can obtain the identity matrix from I using a finite sequence of elementary row operations.
In terms of elementary matrices, this looks like

$$E_n E_{n-1} \ldots E_2 E_1 A = I$$

elementary matrices are invertible, so

$$A = E_1^{-1} E_2^{-1} \ldots E_{n-1}^{-1} E_n^{-1} I$$

Thus A is a product of elementary matrices.
(pay attention to the order of the elementary matrices compared to their inverses)

Rank of a Matrix Properties

Recall that for an m x n matrix A, we define L_A to be the mapping $L_A : F^n \rightarrow F^m$ defined by $L_A(x) = Ax$. We call L_A a **left multiplication transformation**.

We define the rank of a matrix A to be the rank of the linear transformation L_A

The rank of any matrix equals the maximum number of its linearly independent columns.

The rows and columns of any matrix generate subspaces of the same dimensiom

Determinants

Some basic properties of determinants:

- If A has a row consisting of only zeroes, then det(A) $=0$
- If A has two identical rows, then det(A) $=0$
- if A_{nxn} has rank less than n, then det(A) $=0$

Theorem

Let E be an elementary matrix. Then

- If E is type 1, then det(E) $=$ -1
- If E is type 2, then det(E) $=\alpha$
- If E is type 3, then det(E) $=1$

Theorem: Let A,B be nxn matrices. Then det(AB) $=$ det(A)$\cdot$det(B)

Corollary: A matrix A is invertible if and only if det(A) $!=0$

Determinants

Theorem: For any nxn matrix A, $\det(A^t) = \det(A)$

Proof:
If A is not invertible, then neither is A^t, thus $\det(A) = 0 = \det(A^t)$
If A is invertible, then A is the product of a finite number of elementary matrices. So

$$A^t = (E_k ... E_2 E_1)^t = E_k^t ... E_2^t E_1^t$$

By the previous theorem, the determinant of A^t is equal to the product of determinants of elementary matrices. From here it is sufficient to show that if E is an elementary matrix, then $\det(E) = \det(E^t)$

Type 1: E is obtained from I by swapping two rows, i and j. E^t is obtained by swapping two columns, j and i. So $det(E^t) = -1 = det(E)$
Type 2: If E is type 2, then $E^t = E$
Type 3: If E is type 3, then so is E^t. Hence $det(E) = 1 = det(E^t)$

Diagonalization

- **eigenvector**: v is an eigenvector of A if there exists a scalar λ such that $Av = \lambda v$
- **eigenvalue:** The scalar λ is called the eigenvalue of A corresponding to the eigenvector v

Theorem

A linear operator T on a vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T. Further, $D = [T]_\beta$ is a diagonal matrix, and the diagonals are the eigenvalues of T.

A diagonal matrix has the form

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Characteristic Polynomial

Definition

The **characteristic polynomial** of a matrix A $\in M_{nxn}$ is defined as

$$f(t) = det(A - tI_{\mathrm{n}})$$

The eigenvalues of a matrix A are the roots of its characteristic polynomial. That is, λ is an eigenvalue of A if and only if $det(A - \lambda I_n) = 0$.

Example: Given the following matrix

$$B = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 6 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

we can find the eigenvalues by computing the roots of its characteristic polynomial

$$det(B - \lambda I) = \begin{pmatrix} 1-\lambda & 2 & 5 \\ 0 & 6-\lambda & 4 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)(6-\lambda)(1-\lambda)$$

Characteristic Polynomials and Eigenvalues

Exercise: Find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

- V $= P_1(R)$ and T$(ax+b) = (-6a+2b)x + (-6a+b)$
- V $= M_{2x2}(R)$ and T$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$

Diagonalizability

Defintion

A polynomial f(t) **splits over** F if there are scalars (not necessarily unique) such that

$$f(t) = c(t - a_1)(t - a_2)\dots(t - a_n)$$

A polynomial splits if it factors completely into linear polynomials. Note that $x^2 + 1$ does not split over $\mathbb{R}$, but it does split over $\mathbb{C}$

Theorem: The characteristic polynomial of any diagonalizable linear operator splits.

Definition: The eigenspace of T corresponding to the eigenvalue λ is defined as $E_\lambda = \{v \in V : T(v) = \lambda v\} = N(T-\lambda I)$

T is diagonalizable if and only if the multiplicity of λ_i equals the dimension of its eigenspace for all i

To check if a linear operator T on an n-dimensional vector space is diagonalizable the following two properties must hold

- The characteristic polynomial of T splits
- For each eigenvalue λ of T, the multiplicity of λ equals n - rank(T-λI)

Invariant Subspaces

Definition

Let T be a linear operator on a vector space V. A subspace W of V is called a **T-invariant subspace** if $T(v) \in W$ for all $v \in W$

Let T be a linear operator on a vector space V, and let x be a nonzero vector in V. The subspace

$$W = \text{span}(\{x, T(x), T(x)^2, ...\})$$

is called the **T-cyclic subspace of V generated by x**. This subspace is the smallest possible T-invariant subspace of V containing x.

T-Cyclic Subspaces

Example: Let T be the linear operator on R^3 defined by

$$T(a, b, c) = (a^2, b - a, c + b)$$

Take the vector $v_1 = (1, 1, 0)$

$$\begin{aligned} T(v_1) &= T(1, 1, 0) = (1, 0, 0) \\ T^2(v_1) &= T(1, 0, 0) = (1, 0, 0) = T(v_1) \end{aligned}$$

So the T-Cyclic Subspace of v_1 is

$$span(\{v_1, T(v_1)\}) = span(\{(1, 1, 0), (1, 0, 0)\})$$

T-Cyclic Subspaces

Example:

Let T be the linear operator on $P(R)$ defined by $\mathrm{T}(f(x)) = f'(x)$
The T-cyclic subspace generated by $x^3 + 2x^2 + 1$ is

$$span(\{x^3 + 2x^2 + 1,\ 3x^2 + 4x,\ 6x + 4,\ 6\})$$

T-Cyclic Subspaces

Theorem

Let T be a linear operator on a finite dimensional vector space V. Let W denote the T-cyclic subspace of V generated by a nonzero vector v $\in$ W. Let k = dim(W). Then

- $\{v, T(v), T^2(v), ..., T^{k-1}(v)\}$ is a basis for W
- if α_1v + α_2T(v) + ... + $\alpha_{k-1}T^{k-1}$(v) + T^k(v) = 0, then the characteristic polynomial of T_W is
  $f(t) = (-1)^k(\alpha_1 + \alpha_2 t + ... + \alpha_{k-1} t^{k-1} + t^k)$

T-Cyclic Subspaces

Example: Let T be the linear operator on R^3 defined by

$$T(a, b, c) = (-b + c, a + c, 3c)$$

Let $v = (1, 0, 0)$. Then the T-cyclic subspace generated by v is

$$W = span(\{(1, 0, 0), (0, 1, 0)\}) = \{(s, t, 0) : s, t \in R\}$$

To calculate the characteristic polynomial, see that

$$T^2(v) = T(T(1, 0, 0)) = T(0, 1, 0) = (-1, 0, 0) = -v$$

$$T^2(v) = -v \implies v + 0T(v) + T^2(v) = 0$$

By the previous theorem, the characteristic polynomial of the T-cyclic subspace generated by v is

$$f(t) = (-1)^2(1 + 0t + t^2) = t^2 + 1$$

T-Invariant and T-Cyclic Subspaces

Exercise:

Determine whether the given subspace W is a T-invariant subspace of V.

- $V = P_3(R)$, $T(f(x)) = f'(x)$ and $W = P_2(R)$
- $V = P_3(R)$, $T(f(x)) = xf(x)$ and $W = P_2(R)$

Find an ordered basis for the T-cyclic subspace generated by the vector z.

- $V = P_3(R)$, $T(f(x))) = f''(x)$, and $z = x^3$
- $V = M_{2x2}(R)$, $T(A) = A^t$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$