Linear Algebra Review 1

MAT 442

Colin Moore

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Vector Spaces

A vector space V over a field F has the following properties:

1. $\forall x, y \in V$, $x + y = y + x$
2. $\forall x, y, z \in V$, $(x + y) + z = x + (y + z)$
3. *There exists an element in* V, *denoted by* 0, *such that* $\forall x \in V, x + 0 = x$
4. $\forall x \in V$, $\exists y \in V$ *such that* $x + y = 0$
5. $\forall x \in V$, $1 \cdot x = x$
6. $\forall \alpha, \beta \in F$ *and* $\forall x \in V, (\alpha\beta)x = \alpha(\beta x)$
7. $\forall \alpha \in F$ *and* $\forall x, y \in V$, $\alpha(x + y) = \alpha x + \alpha y$
8. $\forall \alpha, \beta \in F$, *and* $\forall x \in V$, $(\alpha + \beta)x = \alpha x + \beta x$

Remark

Vector multiplication need not be defined. Only scalar multiplication and vector addition are required.

Vector Spaces

Thereom

Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if:

1. $0 \in W$
2. $x + y \in W$ *whenever* $x, y \in W$
3. $\alpha x \in W$ *whenever* $\alpha \in F$ *and* $x \in W$

- Is the intersection of subspaces of a vector space V a subspace of V?
- Is the union of subspaces of a vector space V a subspace of V?

Linear Transformations

Definiton: Let V and W be vector spaces over F. We call a function $T : V \to W$ a **linear transformation from V to W** if, $\forall x, y \in V$ and $\forall \alpha \in F$, we have:

1. $\mathrm{T}(x + y) = \mathrm{T}(x) + \mathrm{T}(y)$
2. $\mathrm{T}(\alpha x) = \alpha \mathrm{T}(x)$

• If T is linear, then $\mathrm{T}(0) = 0$
• To show that a given transformation is linear, it is enough to prove that $\mathrm{T}(\alpha x + y) = \alpha \mathrm{T}(x) + \mathrm{T}(y)$

Problem: Is the following transformation T: $V \to V$ linear?

$$T(A) = A^t \; with \; F = \mathbb{C}, V = Mat_2(\mathbb{C})$$

Linear Transformations

Definition: Let V and W be vector spaces and let T: V $\rightarrow$ V be linear. We define the **null space** (or **kernel**) of T as:

$$N(T) = \{x \in V : T(x) = 0\}$$

We define the **range** (or **image**) of T as:

$$R(T) = \{T(x) : x \in V\}$$

or in other words:

$$R(T) = \{y \in W : T(x) = y\}$$

Important note: N(T) is a subspace of **V** while R(T) is a subspace of **W**

Problem: Find N(T) and R(T) for a linear transformation T: $R^3 \rightarrow R^2$ defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$$

Linear Transformations

Theorem

Let V and W be vectors spaces, and let T: V $\rightarrow$ W be linear. If $\beta = \{v_1, v_2, ..., v_n\}$ is a basis for V, then

$$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2), ..., T(v_n)\})$$

Example: Let T: $R^3 \rightarrow R^3$ be defined by T(a, b, c) = (a+b, 2b, 0). Find a basis for R(T).

Using the standard basis for R^3, we see

$$\begin{aligned} R(T) &= \text{span}(\{T(1,0,0), T(0,1,0), T(0,0,1)\}) \\ &= \text{span}(\{(1,0,0), (1,2,0), (0,0,0)\}) \\ &= \text{span}(\{(1,0,0), (1,2,0)\}) \end{aligned}$$

Linear Transformations

Dimension Theorem

Let V and W be vector spaces, and let T: V $\rightarrow$ W be linear. If V is finite-dimensional, then

$$\text{nullity(T)} + \text{rank(T)} = \dim(\text{V})$$

Proof:

- Suppose dim(V) = n, dim(N(T)) = k, define a basis for N(T) as $\{v_1, ..., v_k\}$
- Extend to a basis for V, so $\beta = \{v_1, ..., v_k, v_{k+1}, ..., v_n\}$
- Claim $S = \{T(v_{k+1}), T(v_{k+2}), ..., T(v_n)\}$ is a basis for R(T)
- $T(v_i) = 0$ for $1 \leq i \leq k$ since v_i comes from the null space of T, thus

$$\begin{aligned} R(T) &= \text{span}(T(\beta)) = \text{span}(\{T(v_1), ..., T(v_n)\}) \\ &= \text{span}(\{0, ..., T(v_{k+1}), T(v_{k+2}), ..., T(v_n)\}) \\ &= \text{span}(\{T(v_{k+1}), T(v_{k+2}), ..., T(v_n)\}) \end{aligned}$$

Linear Transformations

Proof (cont):

- Correct number of vectors, need to check for linear independence. Suppose we have $b_{k+1}T(v_{k+1}) + b_{k+2}T(v_{k+2}) + ... + b_nT(v_n) = 0$
- T is linear, thus $T\left(b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + ...b_nv_n\right) = 0$
- Hence $b_{k+1}v_{k+1} + b_{k+2}v_{k+2} + ...b_nv_n \in N(T)$
- $\{v_1, v_2, ..., v_k\}$ is a basis for N(T), so $\exists \mu_1, ...\mu_k \in F$ such that $b_{k+1}v_{k+1} + ... + b_nv_n = \mu_1 v_1 + ... + \mu_k v_k$
- Rewrite as $\mu_1 v_1 + ... + \mu_k v_k - b_{k+1}v_{k+1} - ... - b_nv_n = 0$
- But notice that $\{v_1, ..., v_k, v_{k+1}, ..., v_n\} = \beta$, a basis for V. Thus $b_i = 0$ for all i. Thus S is linearly independent.
- S spans R(T) and is linearly independent, thus it is a basis for R(T)
- Therefore rank T = k - n

Linear Transformations

Definition: Let $\beta = \{u_1, u_2, ..., u_n\}$ be an ordered basis for a finite dimensional vector space V. For $x \in V$ we can represent x as a linear combination of β with $a_i \in F$:

$$x = a_1 u_1 + a_2 u_2 + ... + a_n u_n$$

We define the **coordinate vector of** x **relative to** β, denoted $[x]_\beta$, by

$$[x]_\beta = (a_1, a_2, ..., a_n)^t$$

Definition

We call the m×n matrix A defined by $A_{ij} = a_{ij}$ the **matrix representation of T in the ordered bases** β **and** γ and write A $= [T]_\beta^\gamma$

Note that the jth column of A is $[T(v_j)]_\beta$

Example: Let T: $R^2 \to R^3$ be the linear transformation defined by

$$T(a_1, a_2) = (0,\ 2a_1 + 2a_2,\ a_1 - 3a_2)$$

Let β and γ be the standard ordered bases for R^2 and R^3 respectively

$$T(1,0) = (0,2,1)$$

$$T(0,1) = (0,2,-3)$$

Thus the matrix representation of T is

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 0 \\ 2 & 2 \\ 1 & -3 \end{pmatrix}$$

Linear Transformation

Problem: Let T: $R^2 \to R^3$ be given by $T(x, y) = (x - y,\ y,\ 2x + y)$
Let $A = \{(1, 2), (2, 3)\}$ and $B = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Find

- $[T]_A^B$
- a basis for ker(T)
- a basis for range(T)

Linear Transformation

Theorem

Let V and W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let T: V $\rightarrow$ W be linear. Then, for each $u \in V$, we have

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$$

Linear Transformation

Example: Let T: $P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation defined by $T(f(x)) = f'(x)$. Let $\beta = \{1, x, x^2, x^3\}$ and let $\gamma = \{1, x, x^2\}$. Then

$$A = [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Let $u(x) = 4 + x + 3x^2 + x^3$. Then

$$[T(u)]_\gamma = \left[1 + 6x + 3x^2\right]_\gamma = (1, 6, 3)^t$$

But also note

$$[T]_\beta^\gamma [u]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$$

Invertibility

Theorem

Let V and W be *finite-dimensional* vector spaces with ordered bases β and γ, respectively. Let T: V $\rightarrow$ W be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$

Problem: Let $T : P_1(R) \rightarrow R^2$ be the linear transformation defined by $T(a + bx) = (a, a + b)$. Given β, γ to be the standard ordered bases for $P_1(R)$ *and* R^2 respectively, find $T^{-1}, [T]_\beta^\gamma$, and $[T^{-1}]_\gamma^\beta$.

Quotient Spaces

Definition

Let W be a subspace of a vector space V. For any $v \in V$, the set $\{v\} + W = \{v + w : w \in W\}$ is called the **coset** of W **containing** v. We define the **quotient space** of V modulo W using the following operations:

$$(x + W) + (y + W) = (x + y) + W$$
$$\alpha(x + W) = (\alpha x) + W$$

Problem: Show that $x + W = y + W$ if and only if $x - y \in W$

Problem: Let V $= \{a_0 + a_1 X + ... + a_3 X^3 : a_j \in \mathbb{Q}\}$, a vector space over $\mathbb{Q}$ under the usual polynomial addition and scalar multiplication. Let W $=$ span$(\{x + 1, 2x - 1, x^2 + x\})$. Find a spanning set for V/W.