Let's define a vector space V over the field $\mathbb{R}$ to include all the polynomials of degree 2 or less. Then $V = P_2$. This means that V contains all of the polynomials of the form $ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$.

For this example we will define a vector $u \in V$ as $u(x) = 5 + 3x + 2x^2$. Now how we represent this vector depends entirely on the basis we want to work with. Since a vector space can have many different bases, the coordinate vector that we use to reference $u(x)$ will be different for each one.

For example, lets consider the standard basis for $P_2$, which we will call $\beta$. Then $\beta = \{1, x, x^2\}$. Finding the coordinate vector for $u(x)$ with respect to $\beta$ is straightforward.

$$[u(x)]_\beta = (5, 3, 2)$$

Clearly $u(x) = 5(1) + 3(x) + 2(x^2) = 5 + 3x + 2x^2$. But now lets find the coordinate vector for $u(x)$ with respect to the ordered basis $\gamma$, defined as $\gamma = \{1 - x, 1 + x, x^2 + 1\}$. In this case

$$[u(x)]_\gamma = (0, 3, 2)$$

Now again see that $u(x) = 0(1 - x) + 3(1 + x) + 2(x^2 + 1) = 5 + 3x + 2x^2$, a linear combination of the basis vectors. You can calculate this using brute force, row reduction, or remembering that from $Ax = b$ you can solve for $x$ as $x = A^{-1}b$. In this case you can use $b = (5, 3, 2)$ and A can be represented by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Next we define the linear transformation T: V $\to$ V as $T(v(x)) = v('x)$. The matrix representations of T, $[T]_\beta^\gamma$ and $[T]_\gamma^\beta$, are calculated below

$$T(1) = 0 = (0, 0, 0)$$
$$T(x) = 1 = (\frac{1}{2}, \frac{1}{2}, 0)$$
$$T(x^2) = 2x = (-1, 1, 0)$$

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1/2 & -1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$
\begin{aligned}
T(1-x) &= -1 = (-1,0,0)\\
T(1+x) &= 1 = (1,0,0)\\
T(x^2+1) &= 2x = (0,2,0)
\end{aligned}
$$

$$
[T]_\gamma^\beta = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}
$$

Finally,

$$
[T(u(x))]_\gamma = T(5+3x+2x^2) = 3+4x = \begin{pmatrix} -1/2 \\ 7/2 \\ 0 \end{pmatrix}
$$

$$
[T]_\beta^\gamma \, [u(x)]_\beta = \begin{pmatrix} 0 & 1/2 & -1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 7/2 \\ 0 \end{pmatrix}
$$

And going in the other direction

$$
[T(u(x))]_\beta = T(5+3x+2x^2) = 3+4x = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}
$$

$$
[T]_\gamma^\beta \, [u(x)]_\gamma = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}
$$

In all cases we are pointing to the polynomial 3 + 4x, but the coordinate vector changes depending on the basis we choose to use. Hope this helps!