Assignment 3

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Inverse Problems

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1 Critical Analysis

[1] E. J. Candes, J. Romberg and T. Tao. Stable signal recovery from incomplete and inaccurate measurements. Comm. Pure Appl. Math., **59** 1207-1223. (2005)

The paper has 7002 citations and has clearly been highly influential. The main result of the paper that if a signal $\boldsymbol{x}_0$ is sufficiently sparse, then it may be recovered with fewer samples than is required by the Nyquist-Shannon theorem, which has essentially kickstarted the field of 'Compressed Sensing'. One such area of research which this paper has had a significant impact on is MRI, with 269 articles published between 2007 and 2014 [1] concerning CS applied to MR applications. An example paper that cites [1] is "Sparse MRI: The application of compressed sensing for rapid MR imaging." (2007) by Lustig M, Donoho D, Pauly JM. They note that in their abstract that imaging speed is important in many MRI applications, but the speed at which data can be collected is fundamentally limited. Therefore methods to reduce the amount of required data without degrading the image quality are highly important. In the paper they seek to exploit the sparsity implicit in MR images to develop a sampling method which will lead to reduced scan time and more accurate image reconstruction. The paper itself has 5438 citations and has contributed massively to research into CS MRI. The application of CS into MRI has already been introduced into a clinical setting, demonstrating the impact of both the aforementioned papers.

We now turn to answer the question "How and under what circumstances can a signal f be exactly reconstructed from a discrete set of samples?". Given known test signals $a_k \in \mathbb{R}^m$ and n (noisy) measurements y_k where $n \ll m$:

$$y_k = \langle f, a_k \rangle + e_k \quad \text{or} \quad y = Af + e \tag{1}$$

then we can reconstuct $f^\sharp$ via *basis pursuit denoising*

$$\min ||f^\sharp||_{\ell_1} \quad \text{subject to} \quad ||Af^\sharp - y||_{\ell_2} < \epsilon \tag{2}$$

[1] https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4984938/

This inverts the classic paradigm of measuring everything (all pixels, wavelet coefficients, etc..) and then compressing - e.g. taking the d largest coefficients. Instead by sampling the signal f so that measurements are "incoherent", and then *reconstructing* via ℓ_1-minimisation - enforcing sparsity in the solution space through convex relaxation, far fewer samples of the signal are required. This procedure must therefore satisfy a few of conditions:

1. Firstly the matrix A must satisfy a *uniform uncertainty principle*, which essentially states that the entries must be sufficiently random. Examples of such matrices include Gaussian random matrices (which is used in the first numerical experiment) or Fourier ensembles.

2. The second condition is that the signal f is sufficiently sparse - roughly that the number of zeros in f is of the same order or greater than the number of observations m.

3. Thirdly that the pertubation noise e satisfies $||e||_{\ell_2} \leq \epsilon$. The main result of the paper is that in such a scenario then the solution found by the above procedure satisfies

$$||f^\sharp - f|| \leq C_S \cdot \epsilon \tag{3}$$

where C_S is a some proportionality constant depending on the "S-restricted isometry constant δ_S" of A.

2 Numerical Experiments

We repeated the experiments in the paper by forming a 300×1024 Gaussian ensemble matrix A. In the paper the matrix A is constructed by a simple i.i.d Gaussian $(\mathsf{A})_{ij} \sim \mathcal{N}(0, e^2)$ draw for every element in the matrix (it is unclear exactly what value for e they used in the paper). Note that the non-zero singular value spectrum shown below -

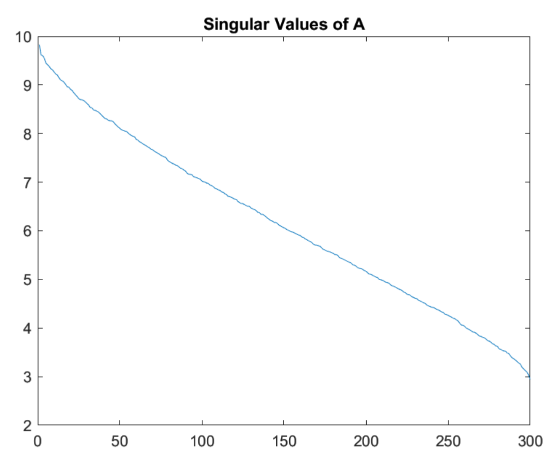
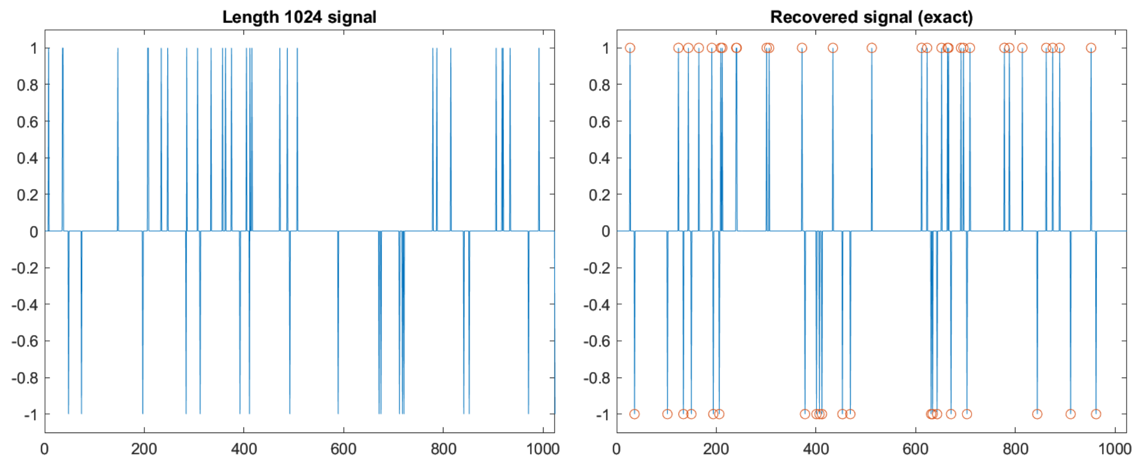


Figure 1: Singular values of A

which has roughly decay rate $\propto 1/k$. This matrix satisfies the 'uniform uncertainty principle' under which the stable recovery condition holds. A 1024×1 sparse signal $\boldsymbol{f}_0$ is convolved with the matrix A to produce an (observed) 300×1 sample $\boldsymbol{y}$ with which white Gaussian noise $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \boldsymbol{I})$ is added -

$$\boldsymbol{y} = \mathsf{A}\boldsymbol{f_0} + \boldsymbol{\epsilon} \tag{4}$$

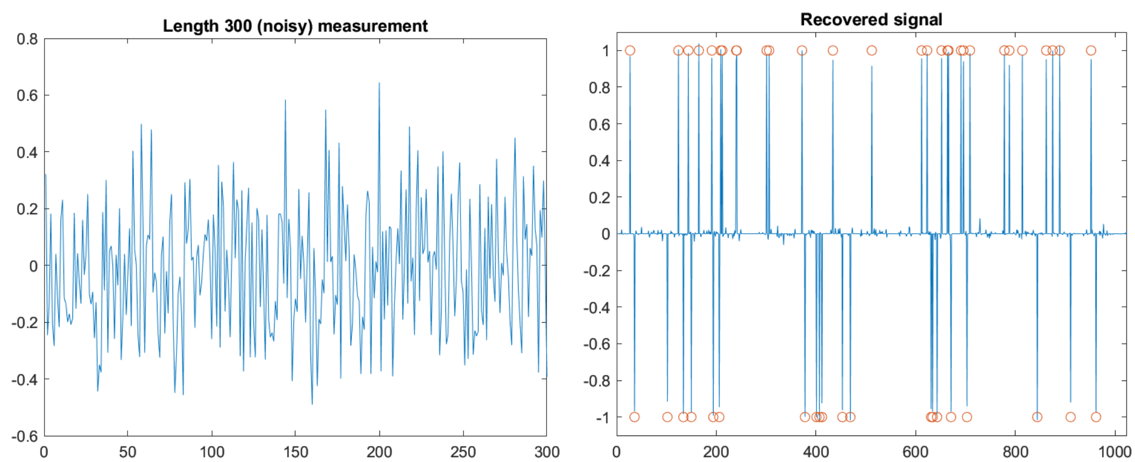
the task is to reconstruct $\boldsymbol{x}_0$ with an approximation $\boldsymbol{x}^\sharp$ as accurately as possible. By using the ℓ_1 norm as a regulariser, we enforce sparsity in the solution space and can recover highly accurate signals.



(a) $\boldsymbol{f}_0$ (b) $\boldsymbol{f}^\sharp$

Figure 2: Signal distortion and recovery in the noiseless case

Observe that the recovered signal is essentially exact. In the presence of noise the solution $\boldsymbol{x}^\sharp$ is still highly accurate; the experiment was repeated in the presence of noise with $\sigma = 0.05$ as was used in the paper-



(a) $\boldsymbol{y} = \mathsf{A}\boldsymbol{f_0} + \boldsymbol{\epsilon}$ (b) $\boldsymbol{f}^\sharp$

Figure 3: Signal distortion and recovery in the noisy case

Another way of constructing A is to first form two arbitrarily orthonomal basis U and V of size 300×300 and 1024×300 respectively and then a designed singular value matrix W.

Then A can be constructed as -

$$\mathsf{A} = \mathsf{U}\mathsf{W}\mathsf{V}^T \tag{5}$$

By altering the singular value spectrum of A we can show how this procedure can yield much more unsatisfactory results. If the singular value matrix W is instead designed to have values $\exp(-k/100)$ on the k'th diagonal, a decaying spectrum, then repeating the experiments gave the following unstable results -

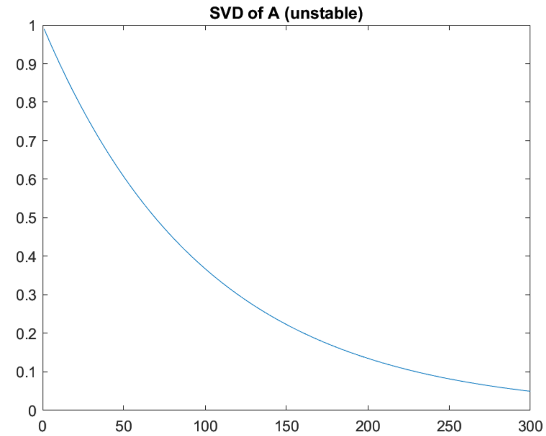
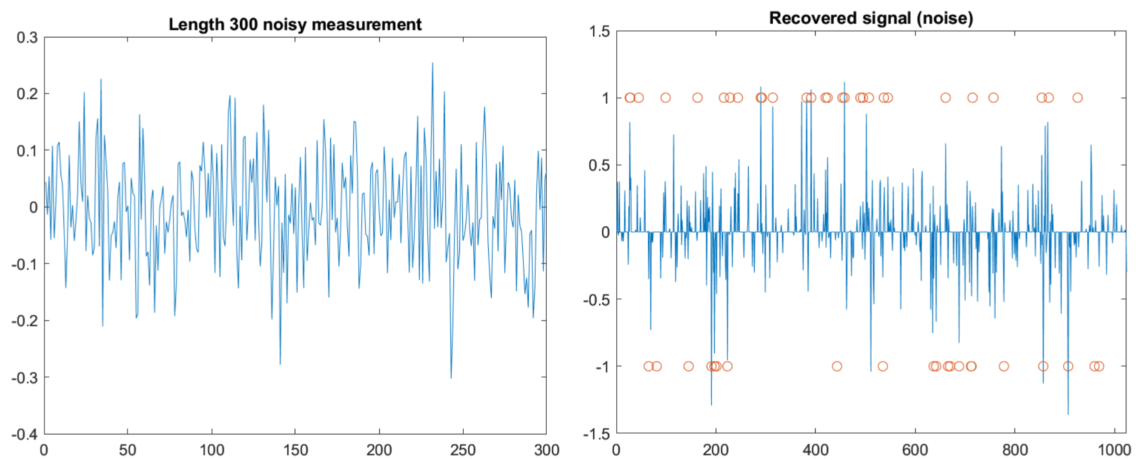


Figure 4: Singular values of A



(a) $\boldsymbol{y} = \mathsf{A}\boldsymbol{f_0} + \boldsymbol{\epsilon}$ (b) $\boldsymbol{f}^\sharp$

Figure 5: Unstable recovery

this shows clearly that the resulting recovery is now *unstable*.