## Introduction

This coursework covers material in the first three weeks of the module. It is divided into three parts which roughly will correspond to the material in each week; however you can carry out the assignment in the time that suits you, with all parts being handed in as a single submission. The assignments will be closely related to the support in the lab classes which you should attend to get help with completing them.

## Tasks

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**Week 1:**   15. January

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**1. Solving Underdetermined Problems (30%)**

As explained in lectures the linear equation $x_1 + 2x_2 = 5$ is used to introduce the concept of minimum norm solutions of underdetermined problems:

$$\text{minimize} \quad \Phi = \sum_i |x_i|^p, \qquad \text{subject to} \quad Ax = b,$$

where in our case, $A = (1\ 2)$, $b = 5$.

   a.) Write a function of two variables, $x$ and $p$, where $x$ is a vector of length 2 and $p$ is a scalar; this function will compute the value of $\Phi$ as given above.

   b.) Use library functions to compute solutions of the above optimization problem, for $p = 1,\ 1.5,\ 2,\ 2.5,\ 3,\ 3.5,\ 4$.

   c.) Plot the solutions you have obtained as points on a 2D graph together with the line representing the constraint equation $x_1 + 2x_2 = 5$. The result should look like Figure 1.

   d.) Another solution method is to use the *Moore-Penrose generalised inverse*

$$A^\dagger := A^\mathsf{T}(AA^\mathsf{T})^{-1},$$

   and then apply it to get $x_{\mathsf{MP}} = A^\dagger b$. Implement this solution and plot it on the same graph as in the previous question. What value of $p$ does this correspond to and why?
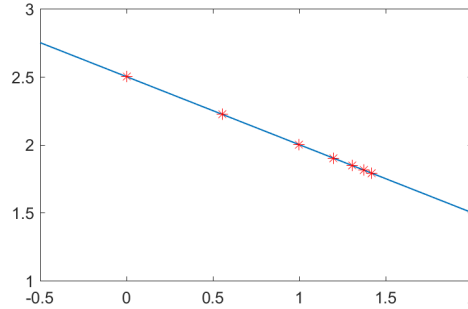
Figure 1: Example for a line plot for Exercise 1

In parts 2 and 3 of the assignment we will implement discrete convolution of a 1D function by an explict matrix vector multiplcation, analyse its behaviour and compare to other methods for performing convolution.

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**Week 2:** 22. January

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**2. Singular Value Decomposition (30%)**

In this part we analyse the matrix representation of convolution using Singular Value Decomposition (SVD).

a.) Set up a spatial grid on the interval $[-1, 1]$ in $n$ equally spaced steps of size $\delta n$. The grid represents the values $[x_1, \ldots x_n]$ with $x_i = -1 + (i-1)\delta n$ for $i = 1, \ldots, n$ and $\delta n = 2/(n-1)$. Make sure that $x_1 = -1$ and $x_n = 1$.

b.) Create a vector of values of the Gaussian function centred at $\mu = 0$ with $\sigma = 0.2$, given by

$$G(x) = \frac{\delta n}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

You should evaluate this function at the grid points you create in part a).

c.) Create the convolution matrix of size $n \times n$ with entries

$$A_{i,j} = G(x_i - x_j) = \frac{\delta n}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x_i - x_j)^2}{2\sigma^2}\right) \tag{1}$$

d.) Plot the matrix $A$ as an image, where $A$ is considered as a 2D-array for the case $n = 100$.

e.) Compute the SVD of matrix $A$ using library functions. You should obtain three matrices $U$, $W$, $V$, where $W$ is a diagonal matrix of same size as $A$ containing the singular values. Verify that the equation $A = UWV^\mathsf{T}$ is satisfied.

f.) Compute the pseudoinverse $A^\dagger$ of $A$ by using the formula $A^\dagger = VW^\dagger U^\mathsf{T}$ as given in lectures. This requires you to create a method for constructing $W^\dagger$. For the case $n = 10$, check that this has the property $WW^\dagger = W^\dagger W = Id_n$ where $Id_n$ is the $n \times n$ Identity matrix. Check also that $AA^\dagger = A^\dagger A = Id_n$.

g.) Repeat the last two steps for $n = 20$. What do you observe? Choose $n = 100$ again and plot the first 9 columns of $V$, the last 9 columns of $V$, and the singular values on a logarithmic scale, i.e. $\log(\mathrm{diag}(W))$.

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**Week 3:** 29. January

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### 3. Convolutions and Fourier transform (40%)

In the following we want to examine the convolution of a 1D signal; we define a step function on the interval $[-1, 1]$ by

$$f(x) = \chi_{(-0.95,\,-0.6]}(x) + 0.2\chi_{(-0.6,\,-0.2]}(x) - 0.5\chi_{(-0.2,0.2]}(x) + 0.7\chi_{(0.4,0.6]}(x) - 0.7\chi_{(0.6,1]}(x),$$

where the characteristic function of an interval $(a, b]$ is defined as

$$\chi_{(a,b]}(x) = \begin{cases} 1 & \text{for} \quad a < x \le b \\ 0 & \text{otherwise.} \end{cases}$$
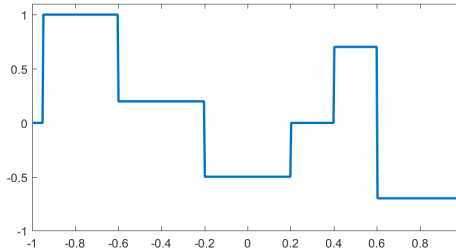
Here's a plot of $f$ in Figure 2.



Figure 2: The step function $f$

a.) Create a function for $f$ as above and plot it on a grid on the interval $[-1, 1]$. Choose a sufficiently large number of grid points $n$ to resolve the jumps.

b.) Compute the matrix $A$ as in part 2 for $\sigma = 0.05,\ 0.1,\ 0.2$ and plot the singular values.

c.) Verify that the plot of the singular values follows (half) a Gaussian function and determine the variance of this Gaussian in each case.

d.) Perform the convolution of the function $f$ with the the matrix $A$ (by matrix multiplication) for all three choices of $\sigma$ and plot the result.

e.) Since convolution is equivalent to multiplication in Fourier space, perform convolution by multiplication in Fourier space for the three choices of $\sigma$ and plot the result (remember to take the inverse Fourier transform); comment on any differences that you observe.

f.) Repeat the convolution with the matrix $A$ using periodic boundary conditions when assembling $A$.

## Report

Write one report for all 3 parts. Explain your method and present your results and figures. Make sure that you provide an answer to all questions. The total length of the report would normally be between 6-10 pages. Submit your report using Moodle. Code can be uploaded separately.