Basics / Notation

Signature $(1, -1, -1, -1)$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$d' = d/\gamma$$
$$E^2 = \vec{p}^2 c^2 + m^2 c^4$$
$$x^\mu = (t, x, y, z)$$
$$x_\mu = (t, -x, -y, -z)$$
$$p = (E, -p_1, -p_2, -p_3)$$
$$p^2 = p^\mu p_\mu = E^2 - \vec{p}^2$$

Don't fuck up boosts:

$$x = \gamma(x^* - vt)$$
$$t = \gamma(t^* - vx^*)$$
$$x^* = \gamma(x + vt)$$
$$t^* = \gamma(t + vx)$$
$$p_x = \gamma(p_x^* - vE^*)$$
$$E = \gamma(E^* - vp_x^*)$$
$$p_x^* = \gamma(p_x + vE)$$
$$E^* = \gamma(E + vp_x)$$

Rel. K: $K = E - mc^2$

Mandelstam: $1 + 2 \to 3 + 4$

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - p_3)^2$$
$$u = (p_1 - p_4)^2$$

Conservation Laws

Conservation of energy, momentum, charge, spin, baryon number, lepton number, colour charge, charge conjugation $C\psi = i\gamma^2\psi^\dagger$

Lepton flavour conserved at γ, Z but not W vertices.

Symm. $\to$ conserved:

- transform under symm, Taylor expand, write as $\psi' = (1 + i\epsilon\hat{A})\psi, \implies \hat{A}$ conserved
- Translation $\leftrightarrow$ momentum
- Flavour symm. in strong interaction

Spin Algebra Stuff

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$S_- = S_x - iS_y, S_+ = S_x + iS_y$$

Isospin, $\hat{T} = \frac{1}{2}\sigma$, same algebra, $T_+ d = u$

$T_3 u = \frac{1}{2}u, T_3 d = -\frac{1}{2}d$

$T_3 \bar{u} = -\frac{1}{2}\bar{u}, T\bar{d} = \frac{1}{2}\bar{d}$

$T_3 s = 0$

$T^2 = \frac{4}{3}I$

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Fermi's Golden Rule

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i)$$

Lifetime: $\tau = \frac{\hbar}{\Gamma}$

Lor. Inv. T: M

$$M_{fi}^{ab} = 4(E_a E_b E_c E_d)^{\frac{1}{2}} T_{fi}^{ab}$$

If we have an $a \to 1 + 2$ decay,

$$\Gamma_{fi} = \frac{p^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 d\Omega$$

Cross Sections

For $1 + 2 \to 3 + 4$:

$$\sigma = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} \int |M_{fi}|^2 d\Omega^*$$
$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s} \frac{p_f^*}{p_i^*} |M_{fi}|^2$$

For elastic $e^- p \to e^- p$ scattering:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{m_p E_1}\right)^2 |M_{fi}|^2$$

$(E_3 = E_3(\theta))$

Relativistic QM

Klein-Gordon: $(\partial^\mu \partial_\mu + m^2)\psi = 0$

Dirac: $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$

$$\gamma^0 = \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$
$$\gamma^i = \beta\alpha_i = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

Adjoints:

$$\begin{aligned} \bar{\psi} &= \psi^\dagger \gamma^0 \\ &= (\psi_0^*, \psi_1^*, -\psi_2^*, -\psi_3^*) \end{aligned}$$

Positive E solutions:

$$u_1(p) = \sqrt{E + m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}$$
$$u_2(p) = \sqrt{E + m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

Negative E solutions: antiparticles

$$v_1(p) = \sqrt{E + m} \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}$$
$$v_2(p) = \sqrt{E + m} \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Current:

$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi = \bar{\psi}\gamma^\mu \psi$$

Helicity

$$h = \frac{\mathbf{S} \cdot \mathbf{p}}{p} = \begin{pmatrix} \sigma \cdot \hat{p} & 0 \\ 0 & \sigma \cdot \hat{p} \end{pmatrix}$$
$$s = \sin(\theta/2), c = \cos(\theta/2)$$
$$u^\uparrow = \sqrt{E + m} \begin{pmatrix} c \\ se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix}$$
$$u^\downarrow = \sqrt{E + m} \begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{p}{E+m}s \\ -\frac{p}{E+m}ce^{i\phi} \end{pmatrix}$$
$$v^\uparrow = \sqrt{E + m} \begin{pmatrix} \frac{p}{E+m}s \\ -\frac{p}{E+m}ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}$$
$$u^\downarrow = \sqrt{E + m} \begin{pmatrix} \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

Rel.Lim.: fraction=1, $E + m = E$

Bethe-Bloch

$$\frac{dE}{dx} \approx -4\pi\hbar^2 c^2 \alpha^2 \frac{nZ}{m_2 v^2}\left(\ln\left(\frac{2\beta^2\gamma^2 c^2 m_e}{I_e}\right) - \beta^2\right)$$

Parity

$$\hat{P} = \gamma^0$$
$$\hat{P}u(m,0) = +u(m,0)$$
$$\hat{P}v(m,0) = -v(m,0)$$
$$\hat{P}u_\uparrow(p,\theta,\phi) = u_\downarrow(p,\pi-\theta,\pi+\phi)$$

(not conserved in weak interatcions)

Chirality

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$
$$P_R = \frac{1}{2}(1+\gamma^5), P_R u_R = u_R$$
$$P_L = \frac{1}{2}(1-\gamma^5), P_L u_L = u_L$$
$$(\gamma^5)^2 = 1, \gamma^5\gamma^\mu = \gamma^\mu\gamma^5$$

In rel. limit (but not generally!),

$$u_R = u_\uparrow, u_L = u_\downarrow, v_R = v_\uparrow, v_L = v_\downarrow$$

I.e. $u_R = \lim_{rel} u_\uparrow$

For massless particles, helicity = chirality

Matrix Elements

$$\langle |M_{fi}|^2 \rangle = \frac{1}{4}\sum_{\text{spins}} |M|^2$$

$e^+e^- \to \mu^+\mu^-$

$$\langle |M_{fi}|^2 \rangle = 2e^4\left(\frac{t^2+u^2}{s^2}\right)$$

$e^- q \to \mu^- q$

$$\langle |M_{fi}|^2 \rangle = 2Q_q^2 e^4\left(\frac{s^2+u^2}{t^2}\right)$$

Deep Inelastic Scattering Rutherford: non-rel. e^-, no proton recoil

$$\langle |M_{fi}|^2 \rangle = \frac{m_p^2 m_e^2 e^4}{p^4 \sin^4(\theta/2)}$$
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E_K^2 \sin^4(\theta/2)}$$

Mott: rel. e^-, no proton recoil

$$\langle |M_{fi}|^2 \rangle = \frac{m_p^2 e^4}{E^2 \sin^4(\theta/2)}\cos^2(\theta/2)$$
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4(\theta/2)}\cos^2(\theta/2)$$

Form factors: p finite size, some recoil

$$M_{fi} = M_{fi}^{pt} F(q^2)$$
$$F(q^2) = \int \rho(r) e^{iq\cdot r} d^3r$$
$$\frac{d\sigma}{d\Omega} \to \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} |F(q^2)|^2$$
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4(\theta/2)}\frac{E_3}{E_1}$$
$$\left(\frac{G_e^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)$$

(last one = Rosenbluth)

$$\langle R^2 \rangle = -6\left[\frac{dF(q^2)}{dq^2}\right]_{q^2=0}$$

For high Q^2 elastics

$$\frac{d\sigma}{d\Omega}_{\text{elastic}} \propto \frac{1}{Q^6}\frac{d\sigma}{d\Omega}_{\text{Mott}}$$

Bjorken x: fraction of momentum transferred to struck quark

$$x = \frac{Q^2}{2p_2 \cdot q}$$

Elasticity (fract. energy loss): $y = \frac{p_2\cdot q}{p_2\cdot p_1}$

$$v = \frac{p_2 \cdot q}{m_p}$$

Any 2 of 4 variables are indep, x, y, v, Q^2

Deep inelastic scattering $Q^2 >> m_p^2 y^2$:

$$\frac{d^2\sigma}{dx dQ^2} \approx \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right]$$
$$Q^2 = 4E_1E_3\sin^2\frac{\theta}{2}$$
$$x = \frac{Q^2}{2m_p(E_1-E_3)}$$
$$y = 1 - \frac{E_3}{E_1}$$

F_i = structure functions.

Bjorken scaling: $F_i \approx$ flat over Q^2 and $F_2(x) = 2xF_1(x)$.

Can rewrite σ in terms of q^2:

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2 Q_q^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s}\right)^2\right]$$

Feynman Rules

$-iM = \prod[\text{terms}]$

- Put adjoints first
- Put polarizations out front
- Same index on all things involved in a vertex

QED

In lepton: $u(p)$

Out lepton: $\bar{u}(p)$

In anti-lepton: $\bar{v}(p)$

Out anti-lepton: $v(p)$

In photon: $\epsilon_\mu(p)$

Out photon: $\epsilon_\mu(p)^*$

Lepton prop: $\frac{i\gamma^\mu q_\mu + m}{q^2 - m^2}$

Photon prop: $\frac{-ig_{\mu\nu}}{q^2}$

QCD

Quarks: same as QED leptons

Gluons: same as QED photons

Gluon prop: $-\frac{ig_{\mu\nu}}{q^2}\delta^{ab}$

Vertices (qqg, ggg, gggg): $-ig_s\frac{1}{2}\lambda_{ji}^a\gamma^\mu$

$$\psi = \psi_{\text{flavor}}\chi_{\text{spin}}\xi_{\text{colour}}\eta_{\text{space}}$$

Colour confinement: colour neutral final states

Colour factors $C(ik \to jl) = \frac{1}{4}\sum_{a=1}^8 \lambda_{ji}^a\lambda_{lk}^a$ (r=1, g=2, b=3). Remember colour conservation when applicable!

$$\langle |C|^2 \rangle = \frac{1}{9}\sum_{ijkl=1}^{3} |C(ij \to kl)|^2$$

$M_{QCD} = CM_{QED}$ in some cases

Weak

Fermions: same as before

W prop: $-\frac{i[g_{\mu\nu} - q_\mu q_\nu/m_W^2]}{q^2 - m_W^2}$

If q^2 small: $\frac{ig_{\mu\nu}}{m_W^2}$

Vertex: $\frac{-ig_w}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5)$