PHYS 506 Assignment 5

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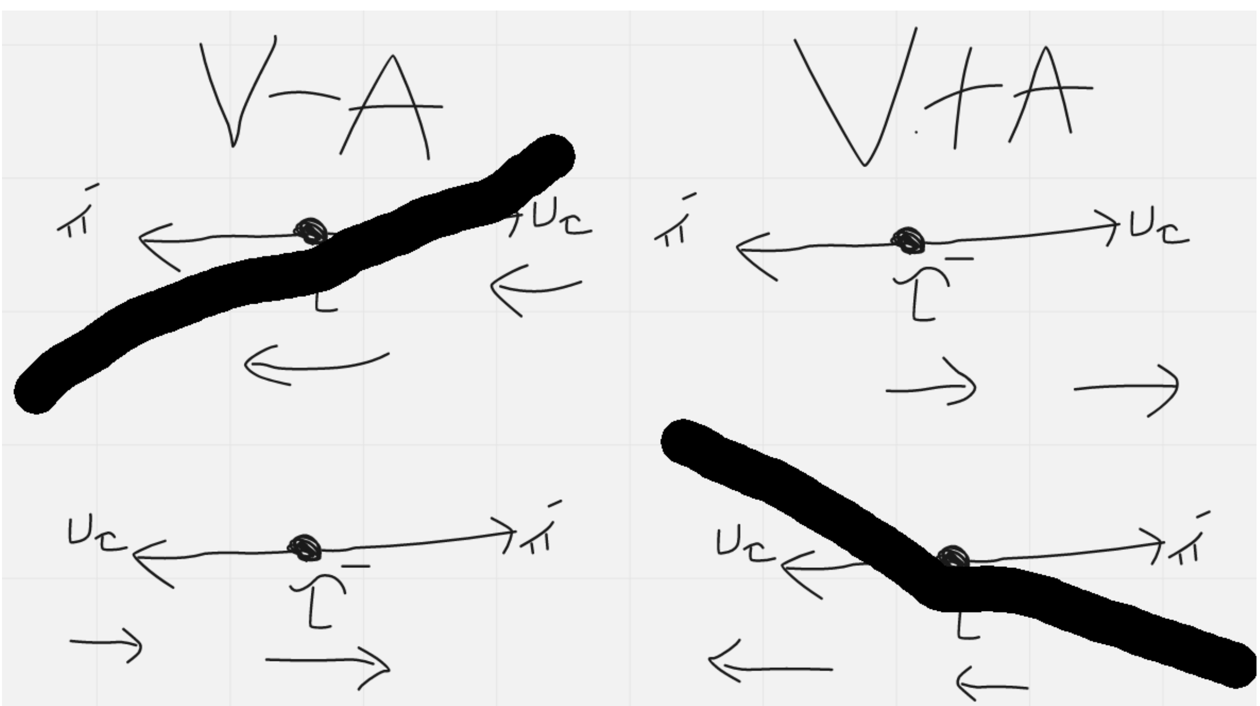
1 Consider the decay at rest $\tau^- \to \pi^- \nu_\tau$, where the spin of the tau is in the positive z-direction and the ν_τ and π^- travel in the z-directions. Sketch the allowed spin configurations assuming that the form of the weak charged-current interaction is i) V-A and ii) V+A.

i) In this case, V-A $\implies$ only LH chiral particle states / RH chiral anti-particle states have non-zero projections (the RH particle / LH anti-particle states go to zero). So in our interaction $\tau^- \to \pi^- \nu_\tau$, the neutrino is produced in a LH chiral state. Using the fact that neutrinos are nearly massless, we can say LH chiral $\approx$ LH helicity for the neutrino, which gives us our spin anti-parallel to momentum, as shown on the left.

ii) In this case, we have the opposite thing where only RH particle / LH anti-particle states have non-zero projections $\implies$ RH chiral neutrino $\approx$ RH helicity neutrino $\implies$ spin parallel to momentum.

The question says to make the spin of τ positive, so that's why 2/4 are crossed out.

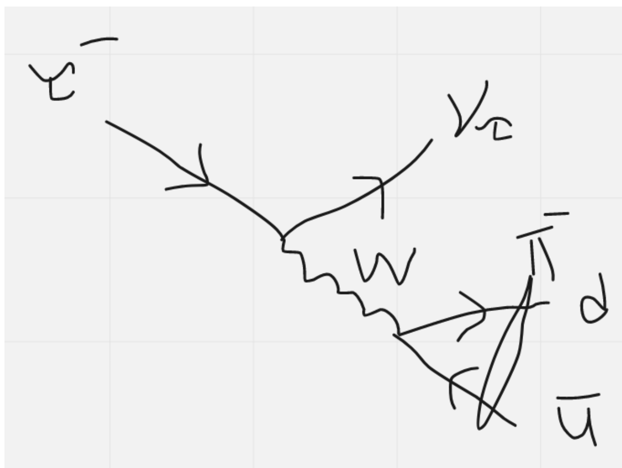


2 Calculate the partial decay width for the decay $\tau^- \to \pi^- \nu_\tau$ in the following steps.

a): Draw the Feynman diagram and snow that the corresponding matrix element is

$$\mathcal{M} = \sqrt{2} G_F f_\pi \bar{u}(p_\nu) \gamma^\mu \frac{1}{2}(1-\gamma^5) u(p_\tau) g_{\mu\nu} p_\pi^\nu$$

The Feynman diagram:



The parts:

- Incoming τ^-: $u(p_\tau)$
- Outgoing ν_τ: $\bar{u}(p_\nu)$
- W^- propagator: $\frac{-i[g_{\mu\nu} - q_\mu q_\nu / m_W^2]}{q^2 - m_W^2}$ (or appx $\frac{-i g_{\mu\nu}}{m_W^2}$)
- $\tau^- \to W^- \nu_\tau$ vertex: $\frac{-i g_W}{\sqrt{2}} \frac{1}{2} \gamma^\mu (1-\gamma^5)$
- Outgoing pion with the vertex: $\frac{-i g_W}{\sqrt{2}} \frac{1}{2} f_\pi p_\pi^\nu$

Put those all together:

$$
\begin{aligned}
-iM_{fi} &= \bar{u}(p_\nu)\frac{-ig_W}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5)u(p_\tau)\frac{-ig_{\mu\nu}}{m_W^2}\frac{g_W}{\sqrt{2}}\frac{1}{2}f_\pi p_\pi^\nu \\
M_{fi} &= \frac{1}{m_W^2}\frac{g_W}{\sqrt{2}}\frac{1}{2}f_\pi\frac{g_W}{\sqrt{2}}\bar{u}(p_\nu)\gamma^\mu\frac{1}{2}(1-\gamma^5)u(p_\tau)g_{\mu\nu}p_\pi^\nu \\
&= \frac{g_W^2}{4m_W^2}f_\pi\bar{u}(p_\nu)\gamma^\mu\frac{1}{2}(1-\gamma^5)u(p_\tau)g_{\mu\nu}p_\pi^\nu \\
&= \sqrt{2}G_F f_\pi\bar{u}(p_\nu)\gamma^\mu\frac{1}{2}(1-\gamma^5)u(p_\tau)g_{\mu\nu}p_\pi^\nu
\end{aligned}
$$

That last line came from $\frac{G_F}{\sqrt{2}} = \frac{g_W^2}{8m_W^2}$ (for low q^2).

b): Taking the τ^- spin in the z-direction and the four-momentum of the neturino to be $p_\nu = p^*(1,\sin(\theta),0,\cos(\theta))$, show that the leptonic current is $j^\mu = \sqrt{2m_\tau p^*}(-s,-c,-ic,s)$, where $s=\sin(\theta/2), c=\cos(\theta/2)$. For this configuration, the spinor for the τ^- can be taken to be u_1 for a particle at rest.

The leptonic current: $\bar{u}(p_\nu)\gamma^\mu\frac{1}{2}(1-\gamma^5)u(p_\tau)$. Make the approximation that chirality = helicity for high-energy particles like this.

We have $p_\nu = p^*(1,\sin(\theta),0,\cos(\theta)), p_\tau = (m_\tau,0,0,0)$, i.e. $s_\tau = 0, c_\tau = 1$

$$
\begin{aligned}
\bar{u}(p_\nu)\gamma^\mu\frac{1}{2}(1-\gamma^5)u(p_\tau) &= \bar{u}(p_\nu)\gamma^\mu\frac{1}{2}(1-\gamma^5)u_1(p_\tau) \\
&= \bar{u}(p_\nu)\gamma^\mu\frac{1}{2}(1-\gamma^5)\sqrt{E_\tau+m_\tau}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \\
&= \bar{u}(p_\nu)\gamma^\mu\frac{1}{2}\begin{pmatrix}1&0&-1&0\\0&1&0&-1\\-1&0&1&0\\0&-1&0&1\end{pmatrix}\sqrt{2m_\tau}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \\
&= \sqrt{2m_\tau}\frac{1}{2}\bar{u}(p_\nu)\gamma^\mu\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}
\end{aligned}
$$

For $u(p_\nu)$ we can use $u_\downarrow$ since we had P_L, and $\phi=0$ wlog since it's along the z axis:

$$
\begin{aligned}
\bar{u}(p_\nu) &= u_\downarrow(p_\nu)^\dagger \gamma^0 \\
&= \sqrt{E_\nu} \begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}^\dagger \gamma^0 \\
&= \sqrt{p^*} \begin{pmatrix} -s & c & -s & c \end{pmatrix}
\end{aligned}
$$

Calculate some matrix products:

$$
\begin{aligned}
\gamma^0 u_1(p_\tau) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -s & c & -s & c \end{pmatrix} \gamma^0 u_\downarrow(p_\tau) &= \begin{pmatrix} -s & c & -s & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= -2s
\end{aligned}
$$

$$
\begin{aligned}
\gamma^1 u_1(p_\tau) &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \end{pmatrix} \\
\begin{pmatrix} -s & c & -s & c \end{pmatrix} \gamma^1 u_\downarrow(p_\tau) &= \begin{pmatrix} -s & c & -s & c \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \\
&= -2c \\
\gamma^2 u_\downarrow(p_\tau) &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -i \\ 0 \\ -i \end{pmatrix} \\
\begin{pmatrix} -s & c & -s & c \end{pmatrix} \gamma^2 u_\downarrow(p_\tau) &= \begin{pmatrix} -s & c & -s & c \end{pmatrix} \begin{pmatrix} 0 \\ -i \\ 0 \\ -i \end{pmatrix} \\
&= -2ic
\end{aligned}
$$

$$\gamma^3 u_\downarrow(p_\tau) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -s & c & -s & c \end{pmatrix} \gamma^2 u_\downarrow(p_\tau) = \begin{pmatrix} -s & c & -s & c \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$
$$= 2s$$

Now put it all toether:

$$\bar{u}(p_\nu)\gamma^\mu \frac{1}{2}(1-\gamma^5)u(p_\tau) = \sqrt{2m_\tau}\frac{1}{2}\bar{u}(p_\nu)\gamma^\mu \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$
$$= \sqrt{2m_\tau}\frac{1}{2}\sqrt{p^*}\begin{pmatrix} -2s & -2c & -2ic & 2s \end{pmatrix}$$
$$= \sqrt{2m_\tau p^*}(-s, -c, -ic, s)$$

c): Write down the 4-momentum of the π^- and show that $|\mathcal{M}|^2 = 4G_F^2 f_\pi^2 m_\tau^3 p^* \sin^2(\theta/2)$. We have $p_\nu = p^*(1, \sin(\theta), 0, \cos(\theta)), p_\tau = (m_\tau, 0, 0, 0)$.Conservation of momentum:

$$p_\tau = p_\nu + p_\pi$$
$$(m_\tau, 0, 0, 0) = p^*(1, \sin(\theta), 0, \cos(\theta)) + p_\pi$$
$$(m_\tau - p^*, -p^*\sin(\theta), 0, -p^*\cos(\theta)) = p_\pi$$

So then we have

$$\begin{aligned}
M_{fi} &= \sqrt{2}G_F f_\pi \bar{u}(p_\nu)\gamma^\mu \frac{1}{2}(1-\gamma^5)u(p_\tau)g_{\mu\nu}p_\pi^\mu \\
&= \sqrt{2}G_F f_\pi \sqrt{2m_\tau p^*}(-s,-c,-ic,s)g_{\mu\nu}(m_\tau - p^*, -p^*\sin(\theta), 0, -p^*\cos(\theta)) \\
&= \sqrt{2}G_F f_\pi \sqrt{2m_\tau p^*}(-s(m_\tau - p^*) - c(p^*\sin(\theta)) + 0 + sp^*\cos(\theta)) \\
&= \sqrt{2}G_F f_\pi \sqrt{2m_\tau p^*}(-sm_\tau + p^*(s - c\sin(\theta) + s\cos(\theta))) \\
&= 2G_F f_\pi \sqrt{m_\tau p^*}(-sm_\tau + p^*(0)) \\
&= -2G_F f_\pi \sqrt{p^*} m_\tau^{\frac{3}{2}} \sin\left(\frac{\theta}{2}\right) \\
|M_{fi}|^2 &= 4G_F^2 f_\pi^2 m_\tau^3 p^* \sin^2\left(\frac{\theta}{2}\right)
\end{aligned}$$

d): Hence show that $\Gamma(\tau^- \to \pi^-\nu_\tau) = \frac{G_F^2 f_\pi^2}{16\pi} m_\tau^3 \left(\frac{m_\tau^2 - m_\pi^2}{m_\tau^2}\right)^2$

We know that for $a \to 1+2$,

$$\Gamma = \frac{p^*}{32\pi^2 m_a^2}\int |M_{fi}|^2 d\Omega$$

Where

$$p^* = \frac{1}{2m_a}\sqrt{\left[m_a^2 - (m_1+m_2)^2\right]\left[m_a^2 - (m_1-m_2)^2\right]}$$

So in our case,

$$\begin{aligned}
p^* &= \frac{1}{2m_\tau}\sqrt{\left[m_\tau^2 - (m_\nu+m_\pi)^2\right]\left[m_\tau^2 - (m_\nu-m_\tau)^2\right]} \\
&\approx \frac{1}{2m_\tau}\sqrt{\left[m_\tau^2 - m_\pi^2\right]\left[m_\tau^2 - m_\tau^2\right]} \\
&= \frac{m_\tau^2 - m_\pi^2}{2m_\tau}
\end{aligned}$$

and

$$
\begin{aligned}
\Gamma(\tau^- \to \pi^- \nu_\tau) &= \frac{p^*}{32\pi^2 m_\tau^2} \int |M_{fi}|^2 d\Omega \\
&= \frac{p^*}{32\pi^2 m_\tau^2} \int 4G_F^2 f_\pi^2 m_\tau^3 p^* \sin^2\left(\frac{\theta}{2}\right) d\Omega \\
&= \frac{G_F^2 f_\pi^2}{8\pi^2 m_\tau^2} (p^*)^2 m_\tau^3 \int \sin^2\left(\frac{\theta}{2}\right) d\Omega \\
&= \frac{G_F^2 f_\pi^2}{8\pi^2 m_\tau^2} \left(\frac{m_\tau^2 - m_\pi^2}{2m_\tau}\right)^2 m_\tau^3 \int \sin^2\left(\frac{\theta}{2}\right) d\Omega \\
&= \frac{G_F^2 f_\pi^2}{32\pi^2} m_\tau^3 \left(\frac{m_\tau^2 - m_\pi^2}{m_\tau^2}\right)^2 \int \sin^2\left(\frac{\theta}{2}\right) d\Omega \\
&= \frac{G_F^2 f_\pi^2}{32\pi^2} m_\tau^3 \left(\frac{m_\tau^2 - m_\pi^2}{m_\tau^2}\right)^2 (2\pi) \\
&= \frac{G_F^2 f_\pi^2}{16\pi} m_\tau^3 \left(\frac{m_\tau^2 - m_\pi^2}{m_\tau^2}\right)^2
\end{aligned}
$$

e): Using the value of f_π obtained in the previous problem, find a numerical value for $\Gamma(\tau^- \to \pi^- \nu_\tau)$.

Well first let's do the previous problem: from the prediction of 11.25 and the measured value of the charged pion lifetime $\tau_\pi = 2.6033 \times 10^{-8}\,\mathrm{s}$ find f_π. Note charged pions decay almost always to muons + muon neutrinos.

11.25: $\Gamma_\pi = \frac{1}{\tau_\pi} = \frac{G_F^2}{8\pi m_\pi^3} f_\pi^2 [m_\ell (m_\pi^2 - m_\ell^2)]^2$

$$
\begin{aligned}
\frac{1}{\tau_\pi} &= \frac{G_F^2}{8\pi m_\pi^3} f_\pi^2 [m_\mu (m_\pi^2 - m_\mu^2)]^2 \\
\sqrt{\frac{8\pi m_\pi^3}{\tau_\pi [m_\mu (m_\pi^2 - m_\mu^2)]^2 G_F^2}} &= f_\pi
\end{aligned}
$$

using these:

- $m_\pi = 139.6\,\mathrm{MeV}$
- $\tau_\pi = 2.6033 \times 10^{-8}\,\mathrm{s} = 3.955\,180\,8 \times 10^{16}\,\mathrm{GeV}^{-1}$
- $m_\mu = 105.7\,\mathrm{MeV}$

- $G_F = 1.166\,38 \times 10^{-5}\,\text{GeV}^{-2}$

we get $f_\pi = 0.128\,\text{GeV} \approx m_\pi$.
Then using $f_\pi = m_\pi$ and $m_\tau = 1.777\,\text{GeV}$, we find

$$\begin{aligned}\Gamma(\tau^- \to \pi^- \nu_\tau) &= \frac{G_F^2 m_\pi^2}{16\pi} m_\tau^3 \left(\frac{m_\tau^2 - m_\pi^2}{m_\tau^2}\right)^2 \\ &\approx 2.922 \times 10^{-13}\,\text{GeV}\end{aligned}$$

f): Given that the lifetime of the τ^- lepton is measured to be $\tau_\tau = 2.906 \times 10^{-13}\,\text{s}$, find an approximate value for the $\tau^- \to \pi^- \nu_\tau$ branching ratio.
Convert that time to a decay rate in GeV: $\tau_\tau = 2.906 \times 10^{-13}\,\text{s} = 4.415 \times 10^{11}\,\text{GeV}^{-1}$ $\implies \Gamma_\tau = \frac{1}{\tau_\tau} = 2.265 \times 10^{-12}\,\text{GeV}$

$$\begin{aligned}BR &= \frac{\Gamma(\tau^- \to \pi^- \nu_\tau)}{\Gamma_\tau} \\ &= \frac{2.922 \times 10^{-13}\,\text{GeV}}{2.265 \times 10^{-12}\,\text{GeV}} \\ &= 0.129\end{aligned}$$

3 Predict the ratio of $K^- \to e^-\bar{\nu}_e$ and $K^- \to \mu^-\bar{\nu}_\mu$ weak interaction decay rates and compare your answer to the measured value of $\frac{\Gamma(K^-\to e^-\bar{\nu}_e)}{\Gamma(K^-\to\mu^-\bar{\nu}_\mu)} = (2.488 \pm 0.012) \times 10^{-5}$.

Recall Thomson 11.25:

$$\Gamma(\pi^- \to \ell^-\bar{\nu}_\ell) = \frac{G_F^2}{8\pi m_\pi^3} f_\pi^2 \left[m_\ell(m_\pi^2 - m_\ell^2)\right]^2$$

This is for pions, but let's see if we can derive something similar for kaons.

Start the same way, in the CoM frame:

$$p_K = (m_K, 0, 0, 0), \qquad p_\ell = p_3 = (E_\ell, 0, 0, p), \qquad p_{\bar{\nu}} = p_4 = (p, 0, 0, -p)$$

(where p is the magnitude of the momentum fo both the charge lepton and the antineutrino)

The weak leptonic current associated with the $\ell^-\bar{\nu}_\ell$ vertex is $j_\ell^\nu = \frac{g_W}{\sqrt{2}}\bar{u}(p_3)\frac{1}{2}\gamma^\nu(1-\gamma^5)\nu(p_4)$

And again since the kaon is a bound $q\bar{q}$ state, the corresponding hadronic current cannot be expressed in terms of free particle Dirac spinors. However, the kaon current has to be a four-vector such that the four-vector scalar product with the leptonic current vibes a Lorentz-invariant expression for the matrix element.

There might be a subtlety here since kaons don't have zero spin, but I'm pretty sure to some approximation we can use the same expression for pions, making a constant replacement and ending up with a similar expression as for pions but with the kaon mass:

$$\frac{\Gamma(K^- \to e^-\bar{\nu}_e)}{\Gamma(K^- \to \mu^-\bar{\nu}_\mu)} = \left[\frac{m_e(m_K^2 - m_e^2)}{m_\mu(m_K^2 - m_\mu^2)}\right]^2$$

Using $m_e = 0.511\,\text{MeV}, m_K = 494\,\text{MeV}, m_\nu = 106\,\text{MeV}$, we get

$$\frac{\Gamma(K^- \to e^-\bar{\nu}_e)}{\Gamma(K^- \to \mu^-\bar{\nu}_\mu)} \approx 2.55 \times 10^{-5}$$

The measured value:

$$\frac{\Gamma(K^- \to e^-\bar{\nu}_e)}{\Gamma(K^- \to \mu^-\bar{\nu}_\mu)} = (2.488 \pm 0.012) \times 10^{-5}$$

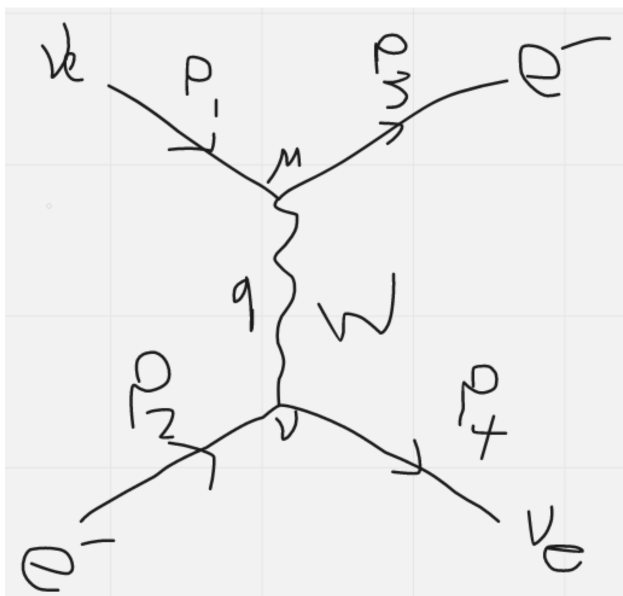
These are relatively close to each other – our calculation isn't within the the uncertainty of the measured value, but it's the same order of magnitude at least, and we've made some approximations, especially with the spin thing.

4 Electron neutrino scattering cross section.

a. Assuming that the process $\nu_e e^- \to e^- \nu_e$ only occurs by the weak charged-current interaction (ignore Z), show that $\sigma_{CC}^{\nu_e e^-} \approx \frac{2m_e E_\nu G_F^2}{\pi}$, where E_ν is in the lab frame where the struck e^- is at rest.

Remember in chapter 12.2 when we derived the charged-current cross section of neutrino scattering off a nucleus? Let's try to do something similar here.

We have a feynman diagram like the middle bit of Thomson figure 12.4, just with $d = \mu^-, u = \nu_\mu$ and $\mu = e$.



For a neutrino interacting with an electron at rest, the CoM energy squared is $s = (p_1 + p_2)^2 = (E_\nu + m_e)^2 - E_\nu^2 = 2m_e E_\nu + m_e^2$. For high-energy neutrinos we can ignore the m_e^2 term and just say $s = 2m_e E_\nu$.

Calculate the matrix element here, ignoring q^2 dependence as in 12.2.1:

- Incoming ν_e: $u(p_1)$
- Outgoing e^-: $\bar{u}(p_3)$
- W^- propagator: appx $\frac{-ig_{\mu\nu}}{m_W^2}$
- $e^- \to W^- \nu_e$ vertex 1: $\frac{-ig_w}{\sqrt{2}} \frac{1}{2} \gamma^\mu (1 - \gamma^5)$
- $e^- \to W^- \nu_e$ vertex 2: $\frac{-ig_w}{\sqrt{2}} \frac{1}{2} \gamma^\nu (1 - \gamma^5)$
- Incoming e^-: $u(p_2)$

- Outgoing ν_e: $\bar{u}(p_4)$

$$-iM_{fi} = \left[\bar{u}(p_3)\frac{-ig_w}{\sqrt{2}}\frac{1}{2}\gamma^\mu(1-\gamma^5)u(p_1)\right]\frac{-ig_{\mu\nu}}{m_W^2}\left[\bar{u}(p_4)\frac{-ig_w}{\sqrt{2}}\frac{1}{2}\gamma^\nu(1-\gamma^5)u(p_2)\right]$$
$$M_{fi} = \frac{g_w^2}{2m_W^2}g_{\mu\nu}\left[\bar{u}(p_3)\gamma^\mu\frac{1}{2}(1-\gamma^5)u(p_1)\right]\left[\bar{u}(p_4)\gamma^\nu\frac{1}{2}(1-\gamma^5)u(p_2)\right]$$

For high-energy neutrino scattering, let's approximate chiral states = helicity states, and from here the calculation of σ is exactly the same as in 12.2.1, yielding

$$\sigma_{CC}(\nu_e e^- \to \nu_e e^-) = \frac{G_F^2 s}{\pi}.$$

Using our $s = 2m_e E_\nu$, we find what the question wanted us to show:

$$\sigma_{CC}^{\nu_e e^-} \approx \frac{2m_e E_\nu G_F^2}{\pi}$$

b. Using the above result, estimate the probability that a 10 MeV solar electron neutrino will undergo a charged-current weak interaction with an electron in the earth if it travels along a trajectory passing through the centre of the earth. Take the earth to be a sphere of radius $R = 6400\,\text{km}$ and uniform density $\rho = 5520\,\text{kgm}^{-3}$.

To get the probability of an interaction (which we know will be relatively small), we can use $\sigma = \frac{P}{n_e}$, where n_e is the surface area number density of electrons in our path, i.e. $n_e = N_e/A$.

Consider the cross section first:

$$\sigma_{CC}^{\nu_e e^-} = \frac{2m_e E_\nu G_F^2}{\pi}$$

$$G_F = 1.166\,38 \times 10^{-5}\,\text{GeV}^{-2}$$

Mass volume density is ρ, get mass are density:

$$\begin{aligned} M/A &= \rho d \\ &= \rho 2R \end{aligned}$$

Some fraction of the mass of the earth is due to electrons, say $\frac{1}{2000}$ since the earth is mostly compoased of atoms with equal amounts of neutrons and protons, and $m_e \approx \frac{1}{1000} m_p$.

$$M_e/A = \frac{1}{1000}\rho R$$

The electron mass here can be written as $M_e = N_e m_e$,

$$\begin{aligned}
N_e m_e/A &= \frac{1}{1000}\rho R \\
n_e m_e &= \frac{1}{1000}\rho R \\
n_e &= \frac{1}{m_e}\frac{1}{1000}\rho R \\
&= 3.878\,198\,7 \times 10^{37}\,\mathrm{m}^{-2}
\end{aligned}$$

Use this to get the probability we're after:

$$\begin{aligned}
P &= \sigma n_e \\
&= \frac{2 m_e E_\nu G_F^2}{\pi} n_e \\
&= \frac{2 E_\nu G_F^2}{\pi}\frac{1}{1000}\rho R \\
&= \frac{2(10\,\mathrm{MeV})(1.166\,38 \times 10^{-5}\,\mathrm{GeV}^{-2})^2}{\pi}\frac{1}{1000} 5520\,\mathrm{kgm}^{-3} 6400\,\mathrm{km} \\
&= 6.68 \times 10^{-10}
\end{aligned}$$

(There were some extra factors in there from implicit $\hbar$s and cs. In the spirit of natural units, I've left those out.)

5 T2K

The T2K experiment uses an off-axis ν_μ beam produced from $\pi^+ \to \mu^+\nu_\mu$ decays. Consider the case where the pion has velocity β along the z-direction in the laboratory frame and a neutrino with energy E^* is produced at an angle θ^* with respect to the z'-axis in the π^+ rest frame.

a. Show that the neutrino energy in the pion rest frame is $p^* = (m_\pi^2 - m_\mu^2)/2m_\pi$

We know that for any decay of the form $a \to 1 + 2$, we have

$$p^* = \frac{1}{2m_a}\sqrt{\left[m_a^2 - (m_1+m_2)^2\right]\left[m_a^2 - (m_1-m_2)^2\right]}$$

In our case, $\pi^+ \to \mu^+\nu_\mu$

$$\begin{aligned}
p^* &= \frac{1}{2m_\pi}\sqrt{\left[m_\pi^2 - (m_\mu+m_\nu)^2\right]\left[m_\pi^2 - (m_\nu-m_\mu)^2\right]} \\
&\approx \frac{1}{2m_\pi}\sqrt{\left[m_\pi^2 - (0+m_\nu)^2\right]\left[m_\pi^2 - (0-m_\mu)^2\right]} \\
&= \frac{m_\pi^2 - m_\mu^2}{2m_\pi}
\end{aligned}$$

b. Using a Lorentz transformation, show that the energy E and angle of production θ of the neutrino in the lab frame are $E = \gamma E^*(1 + \beta\cos(\theta^*))$, $E\cos(\theta) = \gamma E^*(\cos(\theta^*) + \beta)$, where $\gamma = E_\pi/m_\pi$.

In the rest frame, we have

$$\begin{aligned}
p_\pi^* &= (m_\pi, 0, 0, 0) \\
p_\nu^* &= p^*(1, \sin(\theta^*), 0, \cos(\theta^*)) \\
p_\mu^* &= (m_\pi - p^*, -p^*\sin(\theta^*), 0, -p^*\cos(\theta^*))
\end{aligned}$$

And in the lab frame we'll have

$$\begin{aligned}
p_\pi &= (E_\pi, 0, 0, m_\pi\beta) \\
p_\nu &= p(1, \sin(\theta), 0, \cos(\theta)) \\
p_\mu &= (E_\pi - p, -p\sin(\theta), 0, -p\cos(\theta))
\end{aligned}$$

First show $\gamma = \frac{E_\pi}{m_\pi}$ using the Lorentz transformations for momentum/energy (boost with $-\beta$ to get into the lab frame):

$$\begin{aligned} E_\pi &= \gamma(E^*_\pi + \beta p^*_{z,\pi}) \\ E_\pi &= \gamma((m_\pi) + \beta(0)) \\ E_\pi &= \gamma m_\pi \\ \gamma &= \frac{E_\pi}{m_\pi} \end{aligned}$$

Then on use the same transformations on the neutrino, and also use $p^* = E^*$ for the neutrino since it's essentially massless:

$$\begin{aligned} E &= \gamma(E^* + \beta p^*_{z,\nu}) \\ &= \gamma(E^* + \beta p^* \cos(\theta^*)) \\ &= \gamma(E^* + \beta E^* \cos(\theta^*)) \\ &= \gamma E^*(1 + \beta \cos(\theta^*)) \\ p_{z,\nu} &= \gamma(p^*_{z,\nu} + \beta E^*) \\ E\cos(\theta) &= \gamma(p^* \cos(\theta^*) + \beta E^*) \\ &= \gamma(E^* \cos(\theta^*) + \beta E^*) \\ &= \gamma E^*(\cos(\theta^*) + \beta) \end{aligned}$$

c. Using the expressions for E^* and θ^* in terms of E and θ, show that $\gamma^2(1-\beta\cos(\theta))(1+\beta\cos(\theta^*)) = 1$

Derive a similar expression to the ones above for the unstarred E, θ, flipping the sign on β for the inverse transform:

$$\begin{aligned} E^* &= \gamma(E - \beta p_{z,\nu}) \\ &= \gamma(E - \beta p \cos(\theta)) \\ &= \gamma(E - \beta E \cos(\theta)) \\ &= \gamma E(1 - \beta \cos(\theta)) \end{aligned}$$

Use this:

$$
\begin{aligned}
\gamma^2(1-\beta\cos(\theta))(1+\beta\cos(\theta^*)) &= \frac{1}{E}\gamma E(1-\beta\cos(\theta))\frac{1}{E^*}\gamma E^*(1+\beta\cos(\theta^*)) \\
&= \frac{1}{E}E^*\frac{1}{E^*}E \\
&= \frac{EE^*}{EE^*} \\
&= 1
\end{aligned}
$$

d. Assuming that the pions have a flat energy spectrum in the range $1-5\,\mathrm{GeV}$, sketch the form of the resulting neutrino energy spectrum at the T2K far detector (SuperK), which is off-axis at $\theta = 2.5°$. Given that SuperK is 295km from the beam, explain why this angle was chosen.

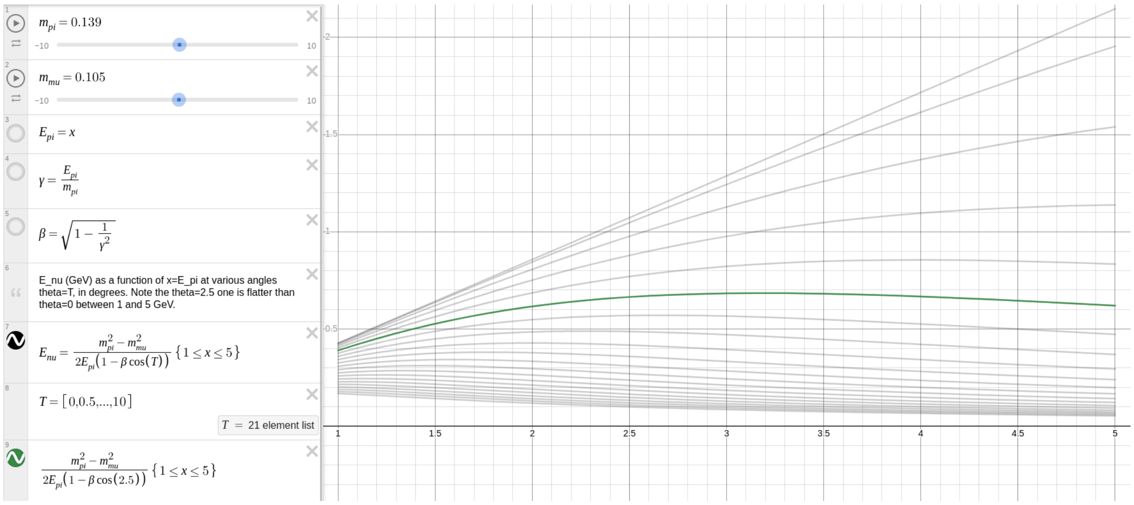
First, get an expression for β:

$$
\gamma = \frac{1}{\sqrt{1-\beta^2}} \implies \beta = \sqrt{1-\frac{1}{\gamma^2}}
$$

And use our other knowledge from above to get an expression for E_ν in terms of other variables we have:

$$
\begin{aligned}
E_\nu^* &= \gamma E_\nu(1-\beta\cos(\theta)) \\
p^* &= \gamma E_\nu(1-\beta\cos(\theta)) \\
\frac{m_\pi^2-m_\mu^2}{2m_\pi} &= \frac{E_\pi}{m_\pi}E_\nu(1-\beta\cos(\theta)) \\
\frac{m_\pi^2-m_\mu^2}{2E_\pi(1-\beta\cos(\theta))} &= E_\nu \\
\frac{m_\pi^2-m_\mu^2}{2E_\pi\left(1-\sqrt{1-\frac{m_\pi^2}{E_\pi^2}}\cos(\theta)\right)} &= E_\nu
\end{aligned}
$$

This is something we can plot for different values of θ:



E_ν (GeV) as a function of E_π at various angles θ, in degrees.

Note the $\theta = 2.5$ line is flatter than $\theta = 0$ between 1 and 5 GeV.

I suppose at a large distance like 295km, it's a good idea to have a more energy-uniform beam of neutrinos, so you can tell which ones came from the experiment vs external sources and also so you only need to design detectors to measure within a smaller range of energies, hence the choice of 2.5°.