PHYS 506 Assignment 6

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1 Kaon Mixing

The $K_S - K_L$ mass difference can be expressed as

$$\Delta m = m(K_L) - m(K_S) \approx \sum_{q,q'} \frac{G_F^2}{3\pi^2} f_K^2 m_K |V_{qd}V_{qs}^* V_{q'd} V_{q's}^*| m_q m_{q'},$$

where q and q' are the quark flavours appearing in the box diagram. Using the values for the CKM matrix elemnts given in (14.8), obtain expressions for the relative contributions to Δm arising from the different combinations of quarks in the box diagrams.

Equation 14.8:

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.973 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}$$

Since W interactions must interact $u, c, t \to d, s, b$, the only (q, q') combinations with non-zero contributions will be $(u/c/t, u/c/t)$.

The term in the sum:

$$\frac{G_F^2}{3\pi^2} f_K^2 m_K |V_{qd}V_{qs}^* V_{q'd} V_{q's}^*| m_q m_{q'}.$$

Since we're only worried about relative contributions, we can get rid of the constant factors.

$$|V_{qd}V_{qs}^* V_{q'd} V_{q's}^*| m_q m_{q'}$$

$$\begin{aligned} (u,u) &: |V_{ud}V_{us}^* V_{ud} V_{us}^*| m_u m_u \\ &= |V_{ud}|^2 |V_{us}|^2 m_u^2 \\ &= |0.974|^2 |0.225|^2 (2\,\text{MeV})^2 \\ &= 0.192\,\text{MeV}^2 \end{aligned}$$

$$\begin{aligned} (u,c) &: |V_{ud}V_{us}^* V_{cd} V_{cs}^*| m_u m_c \\ &= |V_{ud}||V_{us}||V_{cd}||V_{cs}| m_u m_c \\ &= |0.974||0.225||0.225||0.973| (2\,\text{MeV})(1270\,\text{MeV}) \\ &= 122\,\text{MeV}^2 \end{aligned}$$

$$\begin{aligned}(u,t) &: |V_{ud}V_{us}^*V_{td}V_{ts}^*|m_u m_t \\ &= |V_{ud}||V_{us}||V_{td}||V_{ts}|m_u m_t \\ &= |0.974||0.225||0.009||0.040|(2\,\mathrm{MeV})(172\,000\,\mathrm{MeV}) \\ &= 27.1\,\mathrm{MeV}^2\end{aligned}$$

$$(c,u) = (u,c) = 122\,\mathrm{MeV}^2$$

$$\begin{aligned}(c,c) &: |V_{cd}|^2|V_{cs}|^2 m_c^2 \\ &= |0.225|^2|0.973|^2(1270\,\mathrm{MeV})^2 \\ &= 77\,300\,\mathrm{MeV}^2\end{aligned}$$

$$\begin{aligned}(c,t) &: |V_{cd}V_{cs}^*V_{td}V_{ts}^*|m_c m_t \\ &= |V_{cd}||V_{cs}||V_{td}||V_{ts}|m_c m_t \\ &= |0.225||0.973||0.009||0.040|(1270\,\mathrm{MeV})(172\,000\,\mathrm{MeV}) \\ &= 17\,200\,\mathrm{MeV}^2\end{aligned}$$

$$(t,u) = (u,t) = 27.1\,\mathrm{MeV}^2$$

$$(t,c) = (c,t) = 17\,200\,\mathrm{MeV}^2$$

$$\begin{aligned}(t,t) &: |V_{td}V_{ts}^*V_{td}V_{ts}^*|m_t m_t \\ &= |V_{td}|^2|V_{ts}|^2 m_t^2 \\ &= |0.009|^2|0.040|^2(172\,000\,\mathrm{MeV})^2 \\ &= 3830\,\mathrm{MeV}^2\end{aligned}$$

The relative contributions:

(u,u)	$(u,c)=(c,u)$	(u,t)	(c,c)	$(c,t)=(t,c)$	(t,t)
0.192	122	27.1	77300	17200	3830

2 B mixing.

Show that the $B^0 - \bar{B}^0$ mass difference is dominated by the exchange of two top quarks in the box diagram.

Essentially repeat the previous problem for kaons, but now using B and its box diagram / quarks (d, b):

$$\Delta m = m(B_H) - m(B_L) \approx \sum_{q,q'} \frac{G_F^2}{3\pi^2} f_B^2 m_B |V_{qd} V_{qb}^* V_{q'd} V_{q'b}^*| m_q m_{q'}$$

Equation 14.8:

$$\begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.974 & 0.225 & 0.004 \\ 0.225 & 0.973 & 0.041 \\ 0.009 & 0.040 & 0.999 \end{pmatrix}$$

$$\begin{aligned} (u,u) : &|V_{ud} V_{ub}^* V_{ud} V_{ub}^*| m_u m_u \\ &= |V_{ud}|^2 |V_{ub}|^2 m_u^2 \\ &= |0.974|^2 |0.004|^2 (2\,\mathrm{MeV})^2 \\ &= 6.07 \times 10^{-5}\,\mathrm{MeV}^2 \end{aligned}$$

$$\begin{aligned} (u,c) : &|V_{ud} V_{ub}^* V_{cd} V_{cb}^*| m_u m_c \\ &= |V_{ud}||V_{ub}||V_{cd}||V_{cb}| m_u m_c \\ &= |0.974||0.004||0.225||0.041| (2\,\mathrm{MeV})(1270\,\mathrm{MeV}) \\ &= 0.0913\,\mathrm{MeV}^2 \end{aligned}$$

$$\begin{aligned} (u,t) : &|V_{ud} V_{ub}^* V_{td} V_{tb}^*| m_u m_t \\ &= |V_{ud}||V_{ub}||V_{td}||V_{tb}| m_u m_t \\ &= |0.974||0.004||0.009||0.999| (2\,\mathrm{MeV})(172\,000\,\mathrm{MeV}) \\ &= 12.0\,\mathrm{MeV}^2 \end{aligned}$$

$$(c,u) = (u,c) = 0.0913\,\mathrm{MeV}^2$$

$$\begin{aligned}(c,c) &: |V_{cd}|^2|V_{cs}|^2 m_c^2 \\ &= |0.225|^2|0.041|^2(1270\,\mathrm{MeV})^2 \\ &= 137\,\mathrm{MeV}^2\end{aligned}$$

$$\begin{aligned}(c,t) &: |V_{cd}V_{cb}^*V_{td}V_{tb}^*|m_c m_t \\ &= |V_{cd}||V_{cb}||V_{td}||V_{tb}|m_c m_t \\ &= |0.225||0.041||0.009||0.999|(1270\,\mathrm{MeV})(172\,000\,\mathrm{MeV}) \\ &= 18\,100\,\mathrm{MeV}^2\end{aligned}$$

$$(t,u) = (u,t) = 12.0\,\mathrm{MeV}^2$$

$$(t,c) = (c,t) = 18\,100\,\mathrm{MeV}^2$$

$$\begin{aligned}(t,t) &: |V_{td}V_{tb}^*V_{td}V_{tb}^*|m_t m_t \\ &= |V_{td}|^2|V_{tb}|^2 m_t^2 \\ &= |0.009|^2|0.999|^2(172\,000\,\mathrm{MeV})^2 \\ &= 2\,390\,000\,\mathrm{MeV}^2\end{aligned}$$

The relative contributions:

(u,u)	$(u,c)=(c,u)$	(u,t)	(c,c)	$(c,t)=(t,c)$	(t,t)
6.07×10^{-5}	0.0913	12.0	137	18100	2390000

The two-top term is by far the largest, i.e. it dominates.

3 B lifetime.

Given the lifetimes of the neutral B-mesons are $\tau = 1.53\,\text{ps}$, calculate the mean distance they travel when produced at the KEKB collider in collisions of 8 GeV electrons and 3.5 GeV positrons.

This collision will look something like this:

$$e^+e^- \to X \to B\bar{B}$$

In the rest frame of the X,

$$\begin{aligned} p_{e^-} &= (E_{e^-}, 0, 0, p_{z,e^-}) \\ p_{e^+} &= (E_{e^+}, 0, 0, p_{z,e^+}) \\ p_X &= (E_X, 0, 0, 0) \end{aligned}$$

$$\implies m_X = 8\,\text{GeV} + 3.5\,\text{GeV} = 11.5\,\text{GeV}$$

Then we can use the $a \to 1 + 2$ formula to get the momentum of the B in the X's rest frame:

$$\begin{aligned} p &= \frac{1}{2m_X}\sqrt{[m_X^2 - (m_B + m_B)^2]\,[m_X^2 - (m_B - m_B)^2]} \\ &= \frac{1}{2m_X}\sqrt{[m_X^2 - 4m_B^2]\,[m_A^2]} \\ &= \frac{1}{2}\sqrt{[m_X^2 - 4m_B^2]} \\ &\approx 2.23\,\text{GeV} \end{aligned}$$

(using 5.3 GeV for m_B.)

Deduct marks here for not understanding why this doesn't depend on the angle that the Bs come out, but apparently it is possible to show that that's the case (is what friends say).

hopefully get some clarity tomorrow?

If we take that for granted, then we can just use good ol fashioned time dilation, using d as the distance it travels in the lab frame.

$$
\begin{aligned}
\tau\gamma &= t \\
\tau\gamma &= \frac{d}{v} \\
d &= v\tau\gamma
\end{aligned}
$$

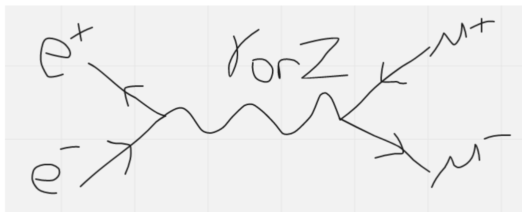
We know this is in the high-energy regime, so $v \approx c$ and (as shown on a previous assignment) $\gamma \approx \frac{p}{m}$

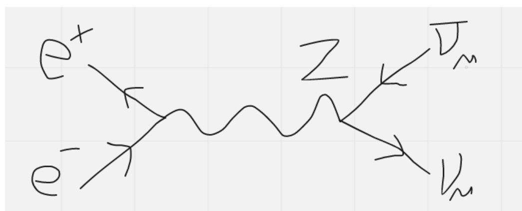
$$
\begin{aligned}
d &= c\tau\frac{p}{m} \\
&= 299\,792\,458\,\mathrm{m/s}(1.53\,\mathrm{ps})\frac{2.23\,\mathrm{GeV}}{5.3\,\mathrm{GeV}} \\
&= 0.193\,\mathrm{mm}
\end{aligned}
$$

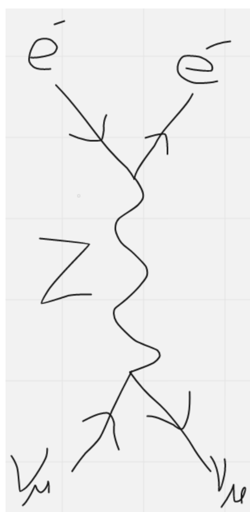
4 More diagrams.

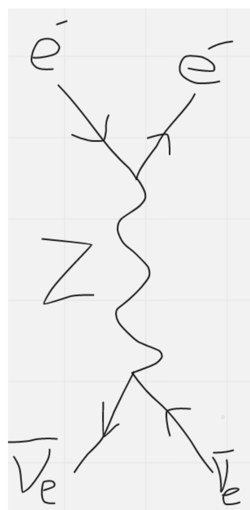
Draw all possible lowest-order Feynman diagrams for the processes

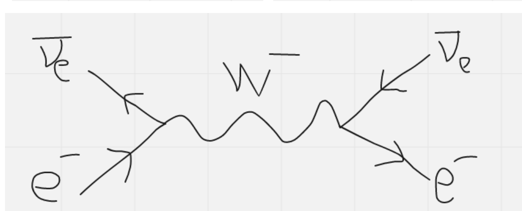
$$e^+e^- \to \mu^+\mu^-, e^+e^- \to \nu_\mu\bar{\nu}_\mu, \nu_\mu e^- \to \nu_\mu e^-, \bar{\nu}_e e^- \to \bar{\nu}_e e^-.$$









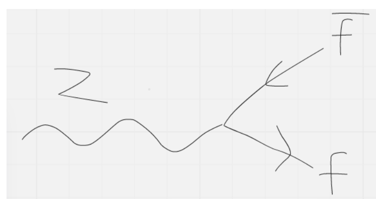


5 Z branching ratios.

Starting from the matrix element, work through the calculation of the $Z \to f\bar{f}$ partial decay rate, expressing the answer in terms of the vector and axial-vector couplings of Z. Taking $\sin^2(\theta_W) = 0.2315$, show that

$$R_\mu = \frac{\Gamma(Z \to \mu^+\mu^-)}{\Gamma(Z \to \text{hadrons})} \approx \frac{1}{20}$$

Let's begin with a Feynman diagram:



And now construct a matrix element with all the terms:

- Incoming Z: $\epsilon_\mu(p_Z)$
- $Z \to f\bar{f}$ vertex: $-i\frac{1}{2}g_Z\gamma^\mu[c_V - c_A\gamma^5]$
- Outgoing f: $\bar{u}(p_f)$
- Outgoing $\bar{f}$: $v(p_{\bar{f}})$

$$\begin{aligned}
-i\mathcal{M} &= \epsilon_\mu(p_Z)\bar{u}(p_f)\left(-i\frac{1}{2}g_Z\gamma^\mu[c_V - c_A\gamma^5]\right)v(p_{\bar{f}}) \\
&= \epsilon_\mu(p_Z)\bar{u}(p_f)\left(-i\frac{1}{2}g_Z\gamma^\mu[c_V - c_A\gamma^5]\right)v(p_{\bar{f}}) \\
\mathcal{M} &= \epsilon_\mu(p_Z)\bar{u}(p_f)\frac{1}{2}g_Z\gamma^\mu[c_V - c_A\gamma^5]v(p_{\bar{f}})
\end{aligned}$$

This is a chiral interaction, take the relativistic limit where chirality $\approx$ helicity, and see which currents are zero:

$$\begin{aligned}
\bar{u}_R\gamma^\mu[c_V - c_A\gamma^5]v_R &= \overline{P_R u}\gamma^\mu[c_V - c_A\gamma^5]v_R \\
&= u^\dagger P_R\gamma^0\gamma^\mu[c_V - c_A\gamma^5]v_R \\
&= u^\dagger\frac{1}{2}(1+\gamma^5)\gamma^0\gamma^\mu[c_V - c_A\gamma^5]v_R \\
&= u^\dagger\gamma^0\gamma^\mu[c_V - c_A\gamma^5]\frac{1}{2}(1+\gamma^5)v_R \\
&= u^\dagger\gamma^0\gamma^\mu[c_V - c_A\gamma^5]P_R v_R \\
&= u^\dagger\gamma^0\gamma^\mu[c_V - c_A\gamma^5](0) \\
&= 0
\end{aligned}$$

(we used the commutation property $[\gamma^5, \gamma^\mu] = 0$ here)

$$\begin{aligned}
\bar{u}_L\gamma^\mu[c_V - c_A\gamma^5]v_L &= \overline{P_L u}\gamma^\mu[c_V - c_A\gamma^5]v_L \\
&= u^\dagger P_L\gamma^0\gamma^\mu[c_V - c_A\gamma^5]v_L \\
&= u^\dagger\frac{1}{2}(1-\gamma^5)\gamma^0\gamma^\mu[c_V - c_A\gamma^5]v_L \\
&= u^\dagger\gamma^0\gamma^\mu[c_V - c_A\gamma^5]\frac{1}{2}(1-\gamma^5)v_L \\
&= u^\dagger\gamma^0\gamma^\mu[c_V - c_A\gamma^5]P_L v_L \\
&= u^\dagger\gamma^0\gamma^\mu[c_V - c_A\gamma^5](0) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\bar{u}_R\gamma^\mu[c_V - c_A\gamma^5]v_L &= \overline{P_R u}\gamma^\mu[c_V - c_A\gamma^5]v_L \\
&= u^\dagger P_R\gamma^0\gamma^\mu[c_V - c_A\gamma^5]v_L \\
&= u^\dagger\frac{1}{2}(1+\gamma^5)\gamma^0\gamma^\mu[c_V - c_A\gamma^5]v_L \\
&= u^\dagger\gamma^0\gamma^\mu[c_V - c_A\gamma^5]\frac{1}{2}(1+\gamma^5)v_L \\
&= u^\dagger\gamma^0\gamma^\mu[c_V - c_A\gamma^5]P_R v_L \\
&= u^\dagger\gamma^0\gamma^\mu[c_V - c_A\gamma^5]v_L \\
&\neq 0
\end{aligned}$$

$$
\begin{aligned}
\epsilon_\mu \bar{u}_R \gamma^\mu [c_V - c_A\gamma^5] v_L &= (E_f + m_f) \begin{pmatrix} c & s & -c & -s \end{pmatrix} \gamma^3 (c_V - c_A) \begin{pmatrix} c \\ -s \\ c \\ -s \end{pmatrix} \\
&= (E_f + m_f)(c_V - c_A) \begin{pmatrix} c & s & -c & -s \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ -s \\ c \\ -s \end{pmatrix} \\
&= (E_f + m_f)(c_V - c_A) \begin{pmatrix} c & s & -c & -s \end{pmatrix} \begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix} \\
&= (E_f + m_f)(c_V - c_A) 2(c^2 + s^2) \\
&= 2(E_f + m_f)(c_V - c_A)
\end{aligned}
$$

(Used the polarization $\epsilon_\mu = (0, 0, 0, 1)$)

$$
\begin{aligned}
\bar{u}_L \gamma^\mu [c_V - c_A\gamma^5] v_R &= \overline{P_L u} \gamma^\mu [c_V - c_A\gamma^5] v_R \\
&= u^\dagger P_L \gamma^0 \gamma^\mu [c_V - c_A\gamma^5] v_R \\
&= u^\dagger \frac{1}{2}(1 - \gamma^5) \gamma^0 \gamma^\mu [c_V - c_A\gamma^5] v_R \\
&= u^\dagger \gamma^0 \gamma^\mu [c_V - c_A\gamma^5] \frac{1}{2}(1 - \gamma^5) v_R \\
&= u^\dagger \gamma^0 \gamma^\mu [c_V - c_A\gamma^5] P_L v_R \\
&= u^\dagger \gamma^0 \gamma^\mu [c_V - c_A\gamma^5] v_R \\
&\neq 0
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_\mu \bar{u}_L \gamma^\mu [c_V - c_A\gamma^5] v_R &= (E_f + m_f) \begin{pmatrix} -s & c & -s & c \end{pmatrix} \gamma^3 (c_V + c_A) \begin{pmatrix} -s \\ -c \\ s \\ c \end{pmatrix} \\
&= (E_f + m_f)(c_V + c_A) \begin{pmatrix} -s & c & -s & c \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -s \\ -c \\ s \\ c \end{pmatrix} \\
&= (E_f + m_f)(c_V + c_A) \begin{pmatrix} -s & c & -s & c \end{pmatrix} \begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix} \\
&= (E_f + m_f)(c_V + c_A) 2(s^2 + c^2) \\
&= 2(E_f + m_f)(c_V + c_A)
\end{aligned}
$$

$$
\begin{aligned}
|\mathcal{M}|^2 &= |\mathcal{M}_{LR}|^2 + |\mathcal{M}_{RL}|^2 \\
&= 4g_Z^2 (E_f + m_f)^2 [(c_V + c_A)^2 + (c_V - c_A)^2] \\
&= g_Z^2 (2E_f)^2 [(c_V + c_A)^2 + (c_V - c_A)^2] \\
&= g_Z^2 E_Z^2 [c_V^2 + c_A^2] \\
&= g_Z^2 m_Z^2 (c_V^2 + c_A^2)
\end{aligned}
$$

$$
\implies \left\langle |\mathcal{M}|^2 \right\rangle = \frac{1}{3} g_Z^2 m_Z^2 (c_V^2 + c_A^2)
$$

Then using Fermi's golden rule and assuming the masses of the decay products are negligible compared to their energies (also noting M has no spatial dependence),

$$\begin{aligned}
\Gamma &= \frac{p^*}{32\pi^2 m_Z^2} \int \left\langle |M|^2 \right\rangle d\Omega^* \\
&= \frac{p^*}{32\pi^2 m_Z^2} 4\pi \left\langle |M|^2 \right\rangle \\
&= \frac{p^*}{8\pi m_Z^2} \left\langle |M|^2 \right\rangle \\
&= \frac{p^*}{8\pi} \frac{1}{3} (c_V^2 + c_A^2) g_Z^2 \\
&\approx \frac{m_Z/2}{8\pi} \frac{1}{3} (c_V^2 + c_A^2) g_Z^2 \\
&= \frac{g_Z^2 m_Z}{48\pi} (c_V^2 + c_A^2)
\end{aligned}$$

In the ratio later the constants will cancel, so consider only the $c_V^2 + c_A^2$ terms. Using table 15.1, for μ we have:

$$\begin{aligned}
c_V &= -0.04 \\
c_A &= -\frac{1}{2} \\
c_V^2 + c_A^2 &= 0.2516
\end{aligned}$$

For $u/c/t$:

$$\begin{aligned}
c_V &= +0.19 \\
c_A &= +\frac{1}{2} \\
c_V^2 + c_A^2 &= 0.2861
\end{aligned}$$

and $d/s/b$:

$$\begin{aligned}
c_V &= -0.35 \\
c_A &= -\frac{1}{2} \\
c_V^2 + c_A^2 &= 0.3725
\end{aligned}$$

Note that Z cannot actually decay into tauons due to mass constraints, so in the following equation we exclude t. Also since there are three quark colours, we have factors of 3 below.

$$
\begin{aligned}
R_\mu &= \frac{\Gamma(Z \to \mu^+\mu^-)}{\Gamma(Z \to \text{hadrons})} \\
&= \frac{\Gamma(Z \to \mu^+\mu^-)}{\Gamma(Z \to \text{quarks except t})} \\
&= \frac{\Gamma(Z \to \mu^+\mu^-)}{3\Gamma(Z \to u\bar{u}) + 3\Gamma(Z \to c\bar{c}) + 3\Gamma(Z \to d\bar{d}) + 3\Gamma(Z \to s\bar{s}) + 3\Gamma(Z \to b\bar{b})} \\
&= \frac{\Gamma(Z \to \mu^+\mu^-)}{6\Gamma(Z \to u\bar{u}) + 9\Gamma(Z \to d\bar{d})} \\
&= \frac{0.2516}{6(0.2861) + 9(0.3725)} \\
&= 0.0496340573 \\
&\approx 0.05 \\
&= \frac{1}{20}
\end{aligned}
$$

6 Asymmetry in weak decays

From the measurement of the muon asymmetry parameter, $\mathcal{A}_\mu = 0.1456 \pm 0.0091$, determine the corresponding value of $\sin^2(\theta_W)$.

We have an expression for $\mathcal{A}$ in terms of $\frac{c_V}{c_A}$, and another for $\frac{c_V}{c_A}$ in terms of $\sin^2(\theta_W)$:

$$\begin{aligned}\mathcal{A}_\mu &= \frac{2\frac{c_V}{c_A}}{1+\left(\frac{c_V}{c_A}\right)^2} \\ \frac{c_V}{c_A} &= 1 - 4\sin^2(\theta_W)\end{aligned}$$

Call the fraction $\frac{c_V}{c_A} = C$.

$$\begin{aligned}\mathcal{A}_\mu &= \frac{2C}{1+C^2} \\ \mathcal{A}_\mu C^2 - 2C + \mathcal{A}_\mu &= 0 \\ \mathcal{A}_\mu C^2 - 2C + \mathcal{A}_\mu &= 0\end{aligned}$$

$$\begin{aligned}C &= \frac{2 \pm \sqrt{4 - 4\mathcal{A}_\mu^2}}{2\mathcal{A}_\mu} \\ &= \frac{1 \pm \sqrt{1 - \mathcal{A}_\mu^2}}{\mathcal{A}_\mu} \\ 1 - 4\sin^2(\theta_W) &= \frac{1 \pm \sqrt{1 - \mathcal{A}_\mu^2}}{\mathcal{A}_\mu} \\ \sin^2(\theta_W) &= \frac{1}{4} - \frac{1 \pm \sqrt{1 - \mathcal{A}_\mu^2}}{4\mathcal{A}_\mu}\end{aligned}$$

Plugging in $\mathcal{A}_\mu = 0.1456$, we get $\sin^2(\theta_W) = 0.2317$ (using the minus in $\pm$ otherwise we get a nonsense answer).

Then to get the error (assuming we can just use the good ol error propagation formula):

$$
\begin{aligned}
y &= f(x) \\
\implies \sigma_y^2 &= \left(\frac{df}{dx}\right)^2 \sigma_x^2 \\
\implies \sigma_{\sin^2(\theta_w)}^2 &= \left(\frac{d\sin^2(\theta_w)}{d\mathcal{A}_\mu}\right)^2 \sigma_{\mathcal{A}_\mu}^2 \\
&= \left(\frac{d}{d\mathcal{A}_\mu}\frac{1}{4} - \frac{1-\sqrt{1-\mathcal{A}_\mu^2}}{4\mathcal{A}_\mu}\right)^2 \sigma_{\mathcal{A}_\mu}^2 \\
&= \left(\frac{\sqrt{1-\mathcal{A}_\mu^2}-1}{4\mathcal{A}_\mu^2\sqrt{1-\mathcal{A}_\mu^2}}\right)^2 \sigma_{\mathcal{A}_\mu}^2 \\
&\approx 0.0012
\end{aligned}
$$

So we have $\sin^2(\theta_W) = 0.2317 \pm 0.0012$.

7 Higgs Mechanism and Lagrangians.

Show that the Lagrangian for a complex scalar field ϕ, $\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi)$, with the covariant derivative $D_\mu = \partial_\mu + igB_\mu$, is invariant under local $U(1)$ gauge transformations, $\phi(x) \to \phi'(x) = e^{ig\chi(x)}\phi(x)$, provided the gauge field transforms as $B_\mu \to B'_\mu = B_\mu - \partial_\mu\chi(x)$.

Start with the Lagrangian and expand a little:

$$
\begin{aligned}
\mathcal{L} &= (D_\mu\phi)^*(D^\mu\phi) \\
&= ([\partial_\mu + igB_\mu]\phi)^*([\partial^\mu + igB^\mu]\phi) \\
&= (\partial_\mu\phi + igB_\mu\phi)^*(\partial^\mu\phi + igB^\mu\phi) \\
&= (\partial_\mu\phi^* - igB_\mu\phi^*)(\partial^\mu\phi + igB^\mu\phi) \\
&= (\partial_\mu\phi^*)(\partial^\mu\phi) - (igB_\mu\phi^*)(\partial^\mu\phi) + (\partial_\mu\phi^*)(igB^\mu\phi) - (igB_\mu\phi^*)(igB^\mu\phi) \\
&= (\partial_\mu\phi^*)(\partial^\mu\phi) - ig(B_\mu\phi^*)(\partial^\mu\phi) + ig(\partial_\mu\phi^*)(B^\mu\phi) + g^2(B_\mu\phi^*)(B^\mu\phi)
\end{aligned}
$$

Now apply these transformations:

$$
\begin{aligned}
\phi'(x) &= e^{ig\chi(x)}\phi(x) \\
B'_\mu &= B_\mu - \partial_\mu\chi(x)
\end{aligned}
$$

The transformed Lagrangian:

$$
\mathcal{L}' = (\partial_\mu\phi'^*)(\partial^\mu\phi') - ig(B'_\mu\phi'^*)(\partial^\mu\phi') + ig(\partial_\mu\phi'^*)(B'^\mu\phi') + g^2(B'_\mu\phi'^*)(B'^\mu\phi')
$$

Bits of this:

$$\begin{aligned}
(\partial_\mu \phi'^*)(\partial^\mu \phi') &= (\partial_\mu (e^{ig\chi(x)}\phi(x))^*)(\partial^\mu e^{ig\chi(x)}\phi(x)) \\
&= (\partial_\mu (e^{-ig\chi(x)}\phi^*(x)))(\partial^\mu e^{ig\chi(x)}\phi(x)) \\
&= (\partial_\mu (e^{-ig\chi(x)})\phi^*(x) + e^{-ig\chi(x)}\partial_\mu(\phi^*(x)))(\partial^\mu (e^{ig\chi(x)})\phi(x) + e^{ig\chi(x)}\partial^\mu(\phi(x))) \\
&= (-ig(\partial_\mu \chi)e^{-ig\chi}\phi^* + e^{-ig\chi}(\partial_\mu \phi^*))(ig(\partial^\mu \chi)e^{ig\chi}\phi + e^{ig\chi}(\partial^\mu \phi)) \\
&= (-ig(\partial_\mu \chi)\phi^* + (\partial_\mu \phi^*))(ig(\partial^\mu \chi)\phi + (\partial^\mu \phi)) \\
&= g^2(\partial_\mu \chi)(\partial^\mu \chi)\phi^*\phi + ig(\partial^\mu \chi)(\partial_\mu \phi^*)\phi - ig(\partial_\mu \chi)\phi^*(\partial^\mu \phi) + (\partial_\mu \phi^*)(\partial^\mu \phi)
\end{aligned}$$

$$\begin{aligned}
-ig(B'_\mu \phi'^*)(\partial^\mu \phi') &= -ig((B_\mu - \partial_\mu \chi(x))(e^{ig\chi(x)}\phi(x))^*)(\partial^\mu (e^{ig\chi(x)}\phi(x))) \\
&= -ig((B_\mu - (\partial_\mu \chi))(e^{-ig\chi}\phi^*))(ig(\partial^\mu \chi)e^{ig\chi}\phi + e^{ig\chi}(\partial^\mu \phi)) \\
&= (-igB_\mu \phi^* + ig(\partial_\mu \chi)\phi^*)(ig(\partial^\mu \chi)\phi + (\partial^\mu \phi)) \\
&= g^2 B_\mu (\partial^\mu \chi)\phi^*\phi - g^2(\partial_\mu \chi)(\partial^\mu \chi)\phi^*\phi - igB_\mu \phi^*(\partial^\mu \phi) + ig(\partial_\mu \chi)\phi^*(\partial^\mu \phi)
\end{aligned}$$

$$\begin{aligned}
ig(\partial_\mu \phi'^*)(B'^\mu \phi') &= ig(\partial_\mu (e^{ig\chi(x)}\phi(x))^*)((B^\mu - \partial^\mu \chi(x))(e^{ig\chi(x)}\phi(x))) \\
&= ig(\partial_\mu (e^{-ig\chi(x)}\phi^*(x)))e^{ig\chi}(B^\mu \phi - (\partial^\mu \chi)\phi) \\
&= ig(-ig(\partial_\mu \chi)e^{-ig\chi}\phi^* + e^{-ig\chi}(\partial_\mu \phi^*))e^{ig\chi}(B^\mu \phi - (\partial^\mu \chi)\phi) \\
&= (g^2(\partial_\mu \chi)\phi^* + ig(\partial_\mu \phi^*))(B^\mu \phi - (\partial^\mu \chi)\phi) \\
&= g^2 B^\mu (\partial_\mu \chi)\phi^*\phi - g^2(\partial_\mu \chi)(\partial^\mu \chi)\phi^*\phi + igB^\mu(\partial_\mu \phi^*)\phi - ig(\partial^\mu \chi)(\partial_\mu \phi^*)\phi
\end{aligned}$$

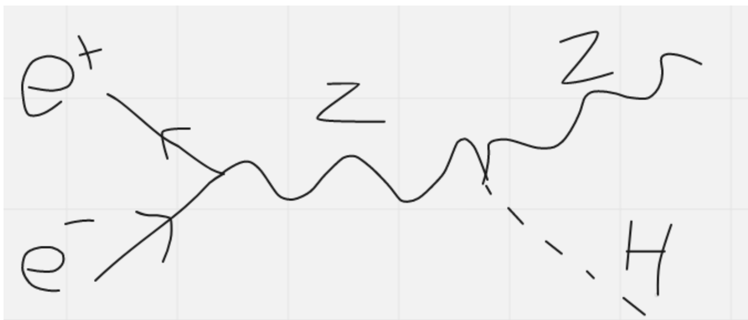
$$\begin{aligned}
g^2(B'_\mu \phi'^*)(B'^\mu \phi') &= g^2((B_\mu - (\partial_\mu \chi))e^{-ig\chi}\phi^*)((B^\mu - (\partial^\mu \chi))e^{ig\chi}\phi) \\
&= g^2((B_\mu - (\partial_\mu \chi))\phi^*)((B^\mu - (\partial^\mu \chi))phi) \\
&= (g^2 B_\mu \phi^* - g^2(\partial_\mu \chi)\phi^*)(B^\mu \phi - (\partial^\mu \chi)\phi) \\
&= g^2 B_\mu B^\mu \phi^*\phi - g^2 B_\mu(\partial^\mu \chi)\phi^*\phi - g^2 B^\mu(\partial_\mu \chi)\phi^*\phi + g^2(\partial_\mu \chi)(\partial^\mu \chi)\phi^*\phi
\end{aligned}$$

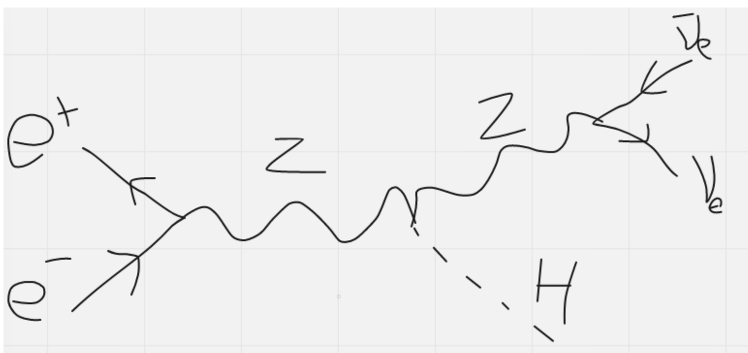
Adding those back up, we have a beautiful cancellation of most terms, leaving us with the expanded version of the old Lagrangian:

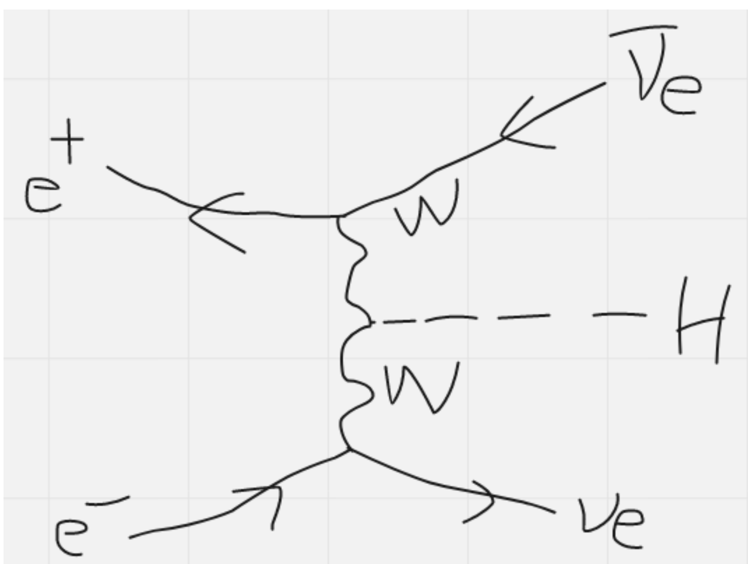
$$
\begin{aligned}
\mathcal{L}' &= g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi^*\phi + ig(\partial^\mu\chi)(\partial_\mu\phi^*)\phi - ig(\partial_\mu\chi)\phi^*(\partial^\mu\phi) + (\partial_\mu\phi^*)(\partial^\mu\phi) \\
&+ g^2B_\mu(\partial^\mu\chi)\phi^*\phi - g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi^*\phi - igB_\mu\phi^*(\partial^\mu\phi) + ig(\partial_\mu\chi)\phi^*(\partial^\mu\phi) \\
&+ g^2B^\mu(\partial_\mu\chi)\phi^*\phi - g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi^*\phi + igB^\mu(\partial_\mu\phi^*)\phi - ig(\partial^\mu\chi)(\partial_\mu\phi^*)\phi \\
&+ g^2B_\mu B^\mu\phi^*\phi - g^2B_\mu(\partial^\mu\chi)\phi^*\phi - g^2B^\mu(\partial_\mu\chi)\phi^*\phi + g^2(\partial_\mu\chi)(\partial^\mu\chi)\phi^*\phi \\
&= (\partial_\mu\phi^*)(\partial^\mu\phi) - igB_\mu\phi^*(\partial^\mu\phi) + igB^\mu(\partial_\mu\phi^*)\phi + g^2B_\mu B^\mu\phi^*\phi \\
&= (\partial_\mu\phi^*)(\partial^\mu\phi) - ig(B_\mu\phi^*)(\partial^\mu\phi) + ig(\partial_\mu\phi^*)(B^\mu\phi) + g^2(B_\mu\phi^*)(B^\mu\phi) \\
&= \mathcal{L}
\end{aligned}
$$

8 Higgs diagrams.

Draw the lowest-order Feynman diagrams for the process $e^+e^- \to HZ$ and $e^+e^- \to H\nu_e\bar{\nu}_e$, which are the main Higgs production mechanisms at a future high-energy linear collider.







9 Higgs production.

Assuming a total Higgs production cross section of 20 pb and an integrated luminosity of $10\,\mathrm{fb}^{-1}$, how many $H \to \gamma\gamma$ and $H \to \mu^+\mu^-\mu^+\mu^-$ events are expected in each of the ATLAS and CMS experiments?

Well first, how many H are expected?

$$
\begin{aligned}
N &= \sigma \int \mathcal{L}(t)dt \\
&= 20\,\mathrm{pb}10\,\mathrm{fb}^{-1} \qquad\qquad = 20\,000\,\mathrm{fb}10\,\mathrm{fb}^{-1} = 200000
\end{aligned}
$$

Then to find the number of events of a particular type we need branching ratios.
Using Thomson table 17.1, we have $BR(H \to \gamma\gamma) = 0.2\%$.

$$
\begin{aligned}
\implies N_{H\to\gamma\gamma} &= N \times BR(H \to \gamma\gamma) \\
&= 200000 \times 0.2\% \\
&= 400
\end{aligned}
$$

Then $H \to \mu^+\mu^-\mu^+\mu^-$ will come from $BR(H \to ZZ^*) = 2.7\%$ and $BR(Z \to \mu^+\mu^-) = 3.4\%$ (found this on wikipedia, must be somewhere in the book too though).

$$
\begin{aligned}
BR(H \to \mu^+\mu^-\mu^+\mu^-) &= BR(H \to ZZ^*)BR(Z \to \mu^+\mu^-)^2 \\
&= 3.1212 \times 10^{-5}
\end{aligned}
$$

(square since both Zs must decay to $\mu^+\mu^-$)

$$
\begin{aligned}
\implies N_{H\to\mu^+\mu^-\mu^+\mu^-} &= N \times BR(H \to \mu^+\mu^-\mu^+\mu^-) \\
&= 200000 \times 3.1212 \times 10^{-5} \\
&= 6.2424 \\
&\approx 6
\end{aligned}
$$

Assuming both of the experiments get the full luminosity, that's 800 and 12.