PHYS 509C Assignment 1

Callum McCracken, 20334298

September 22, 2022

Code for this assignment is here:

`https://github.com/callum-mccracken/PHYS-509C-A1`

It's in a bit of a strange format since I make it write the LaTeX file that I use for making the document you're reading, but here are the highlights:

- Open the file with `numpy.loadtxt()`
- Get the mean with `numpy.mean()`
- Get the standard deviation with `numpy.std()`
- Get the correlation coefficient with `numpy.corrcoef()`
- Get the skew with `scipy.stats.skew()`
- Use `scipy.stats.chi2.pdf()` for the chi-squared PDF
- Integrate using `scipy.integrate.quad()`

1 `fakedata.out` **contains 200 observations of three random variables: X, Y, and Z (each variable in its own column, listed in that order). Calculate the following for this data:**

A. The mean values of X, Y, and Z.

$\overline{X} = 49.85, \overline{Y} = -1.56, \overline{Z} = -19.38.$

B. The standard deviations for all thre variables.

$\sigma_X = 12.75, \sigma_Y = 13.63, \sigma_Z = 11.06.$

C. The three correlation coefficients between the three variables.

$C_{X,Y} = 0.30, C_{X,Z} = 0.72, C_{Y,Z} = -0.30.$

D. The skew for X, Y, and Z.

$\mathrm{Skew}(X) = -0.10, \mathrm{Skew}(Y) = 0.05, \mathrm{Skew}(Z) = -0.31.$

2 Numerically calculate the probability that a number drawn from a χ^2 distribution with $n = 5$ degrees of freedom will be larger than $\chi^2 = 5$. Do the same for $n = 10$. Do not use a lookup table or a pre-existing function to evaluate the answer, but calculate it for yourself as if you had just discovered the χ^2 distribution for the first time.

- $\mathrm{P}(\chi^2_5 < 5) = 0.42$.
- $\mathrm{P}(\chi^2_{10} < 5) = 0.89$.

3 Three independent random numbers X_1, X_2, X_3 are drawn from uniform distributions with means of 0 and variances of 1/3. Let Z be the sum of these three numbers. Derive the normalized probability distribution for Z.

A uniform distribution's PDF is

$$P(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases}$$

with mean $\frac{a+b}{2}$ and variance $\frac{(b-a)^2}{12}$.
Here we have a mean of 0 $\implies b = -a \implies \frac{(b-a)^2}{12} = \frac{a^2}{3}$.
And a variance of $\frac{1}{3} \implies a = -1, b = 1$.

$$P(x) = \begin{cases} \frac{1}{2}, & x \in [-1,1] \\ 0, & x \notin [-1,1] \end{cases}$$

Or in terms of heaviside step functions:

$$P(x) = \frac{1}{2}(H(x+1) - H(x-1))$$

If we have two independent variables of this type, we have:

$$Y = X_1 + X_2$$
$$P(y) = \int_{x_1, x_2 | x_1 + x_2 = y} P(x_1, x_2)$$

It's not super obvious to me what to do here, after some googling it seems this can be done with CDFs:

The CDF of Y is found using all possibile combinations of $x_1 + x_2 < y$. At this point note $y \in [-2, 2]$.

$$\begin{aligned}F(y) &= \iint_{x_1+x_2<y} P(x_1, x_2) dx_1 dx_2 \\ &= \int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{y-x_1} P(x_1, x_2) dx_1 dx_2\end{aligned}$$

And if we take the derivative we'll get $P(y)$:

$$\begin{aligned}P(y) &= \frac{d}{dy} \int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{y-x_1} P(x_1, x_2) dx_1 dx_2 \\ &= \int_{x_1=-\infty}^{\infty} \frac{d}{dy} \int_{x_2=-\infty}^{y-x_1} P(x_1, x_2) dx_1 dx_2 \\ &= \int_{x_1=-\infty}^{\infty} P(x_1, y - x_1) dx_1\end{aligned}$$

Since we had two independent variables,

$$\begin{aligned}P(y) &= \int_{x_1=-\infty}^{\infty} P(x_1) P(y - x_1) dx_1 \\ &= \int_{x_1=-\infty}^{\infty} \frac{1}{2}(H(x_1 + 1) - H(x_1 - 1)) \frac{1}{2}(H(y - x_1 + 1) - H(y - x_1 - 1)) dx_1 \\ &= \frac{1}{4} \int_{x_1=-\infty}^{\infty} H(x_1 + 1) H(y - x_1 + 1) - H(x_1 + 1) H(y - x_1 - 1) \\ &- H(x_1 - 1) H(y - x_1 + 1) + H(x_1 - 1) H(y - x_1 - 1) dx_1\end{aligned}$$

Consider the products of steps we have:

- $H(x_1 + 1) H(y - x_1 + 1)$
  To be non-zero: $x_1 + 1 > 0$ and $y - x_1 + 1 > 0$.
- $H(x_1 + 1) H(y - x_1 - 1)$
  To be non-zero: $x_1 + 1 > 0$ and $y - x_1 - 1 > 0$.

- $H(x_1 - 1)H(y - x_1 + 1)$

  To be non-zero: $x_1 - 1 > 0$ and $y - x_1 + 1 > 0$.
- $H(x_1 - 1)H(y - x_1 - 1)$

  To be non-zero: $x_1 - 1 > 0$ and $y - x_1 - 1 > 0$.

So we have points of interest at $x_1 = -1, y - 1, y + 1, 1$. How these relate to each other depends on y.

Consider if the conditions above can be met simultaneously for $y \in [-2, 2]$, i.e. whether the products will be zero.

- $H(x_1 + 1)H(y - x_1 + 1)$ can be non-zero for $y \in [-2, 2]$
- $H(x_1 + 1)H(y - x_1 - 1)$ can be non-zero for $y \in [0, 2]$
- $H(x_1 - 1)H(y - x_1 + 1)$ can be non-zero for $y \in [0, 2]$
- $H(x_1 - 1)H(y - x_1 - 1)$ is always zero for $y \in [-2, 2]$

So for $y \in [-2, 0]$:

$$\begin{aligned} P(y) &= \frac{1}{4} \int_{x_1=-1}^{y+1} 1 - 0 - 0 + 0 dx_1 \\ &= \frac{1}{4}(y + 2) \end{aligned}$$

And for $y \in [0, 2]$:

$$\begin{aligned} P(y) &= \frac{1}{4} \int_{x_1=y-1}^{1} 1 - 1 - 1 + 0 dx_1 \\ &= \frac{1}{4}(-y) \end{aligned}$$

All together,

$$P(y) = \frac{1}{4}(y + 2)H(y + 2) - \frac{1}{2}yH(y) + (\frac{1}{4}y - \frac{1}{2})H(y - 2)$$

Then take another convolution to get $P(z)$ for $Z = Y + X_3$

$$\begin{aligned}
P(z) &= \int_{y=-\infty}^{\infty} P_y(y)P_{x_3}(z-y)dy \\
&= \int_{y=-\infty}^{\infty} \left(\frac{1}{4}(y+2)H(y+2) - \frac{1}{2}yH(y) + (\frac{1}{4}y - \frac{1}{2})H(y-2)\right) \\
&\qquad \times \left(\frac{1}{2}(H(z-y+1) - H(z-y-1))\right) dy \\
&= \int_{y=-\infty}^{\infty} \left(\frac{1}{4}(y+2)H(y+2) - \frac{1}{2}yH(y) + (\frac{1}{4}y - \frac{1}{2})H(y-2)\right) \frac{1}{2}H(z-y+1) \\
&\qquad - \left(\frac{1}{4}(y+2)H(y+2) - \frac{1}{2}yH(y) + (\frac{1}{4}y - \frac{1}{2})H(y-2)\right) \frac{1}{2}H(z-y-1)dy \\
&= \int_{y=-\infty}^{\infty} \frac{1}{8}(y+2)H(y+2)H(z-y+1) - \frac{1}{4}yH(y)H(z-y+1) \\
&\qquad + \frac{1}{8}(y-2)H(y-2)H(z-y+1) - \frac{1}{8}(y+2)H(y+2)H(z-y-1) \\
&\qquad + \frac{1}{4}yH(y)H(z-y-1) - \frac{1}{8}(y-2)H(y-2)H(z-y-1)dy
\end{aligned}$$

- $H(y+2)H(z-y+1)$ can be non-zero for $z \in [-3, 3]$

  For $z \in [-3, 1]$:

$$\begin{aligned}
&\int_{y=-\infty}^{\infty} \frac{1}{8}(y+2)H(y+2)H(z-y+1)dy \\
&= \int_{y=-2}^{z+1} \frac{1}{8}(y+2)dy \\
&= \frac{z^2+6z+9}{16}
\end{aligned}$$

For $z \in [1, 3]$:

$$\int_{y=-\infty}^{\infty} \frac{1}{8}(y+2)H(y+2)H(z-y+1)dy$$
$$= \int_{y=-2}^{2} \frac{1}{8}(y+2)dy$$
$$= \left[\frac{1}{4}y^2 + \frac{1}{4}y\right]_{-2}^{2}$$
$$= \frac{1}{4}(2)^2 + \frac{1}{4}(2) - \frac{1}{4}(-2)^2 - \frac{1}{4}(-2)$$
$$= 1$$

- $H(y)H(z-y+1)$ can be non-zero for $z \in [-1, 3]$

  For $z \in [-1, 1]$

$$\int_{y=-\infty}^{\infty} -\frac{1}{4}yH(y)H(z-y+1)dy$$
$$= \int_{y=0}^{z+1} -\frac{1}{4}ydy$$
$$= -\frac{z^2+2z+1}{8}$$

  For $z \in [1, 3]$

$$\int_{y=-\infty}^{\infty} -\frac{1}{4}yH(y)H(z-y+1)dy$$
$$= \int_{y=0}^{2} -\frac{1}{4}ydy$$
$$= -\frac{1}{2}$$

- $H(y-2)H(z-y+1)$ is always zero within the possible range of y.

- $H(y+2)H(z-y-1)$ can be non-zero for $z \in [-1,3]$
  For $z \in [-1,3]$
$$\int_{y=-\infty}^{\infty} -\frac{1}{8}(y+2)H(y+2)H(z-y-1)dy$$
$$= \int_{y=-2}^{z-1} -\frac{1}{8}(y+2)dy$$
$$= -\frac{z^2+2z+1}{16}$$

- $H(y)H(z-y-1)$ can be non-zero for $z \in [1,3]$
  For $z \in [1,3]$
$$\int_{y=-\infty}^{\infty} +\frac{1}{4}yH(y)H(z-y-1)dy$$
$$= \int_{y=0}^{z-1} -\frac{1}{4}ydy$$
$$= -\frac{z^2-2z+1}{8}$$

- $H(y-2)H(z-y-1)$ is always zero for $y \in [-2,2]$.

Let's put this together in sections:

- For $z \in [-3,-1]$:

$$P(z) = \frac{z^2+6z+9}{16}$$

- For $z \in [-1,1]$:

$$
\begin{aligned}
P(z) &= \frac{z^2+6z+9}{16} - \frac{z^2+2z+1}{8} - \frac{z^2+2z+1}{16} \\
&= -\frac{z^2+3}{8}
\end{aligned}
$$

- For $z \in [1, 3]$:

$$
\begin{aligned}
P(z) &= 1 - \frac{1}{2} + 0 - \frac{z^2+2z+1}{16} + \frac{z^2-2z+1}{8} \\
&= \frac{z^2-6z+9}{16}
\end{aligned}
$$

- zero elsewhere.

So all together:

$$
\begin{aligned}
P(z) &= \frac{z^2+6z+9}{16}(H(z+3) - H(z+1)) \\
&- \frac{z^2+3}{8}(H(z+1) - H(z-1)) \\
&+ \frac{z^2-6z+9}{16}(H(z-1) - H(z-3))
\end{aligned}
$$

I see online there's a simpler version of this that's more general for higher numbers of uniform variables. Was there a better way to approach this? Seems like this approach is valid though, and the function we have is normalized.

4 Suppose that two random variables X_1 and X_2 have a continuous joint distribution for which the joint PDF is as follows: $f(x_1, x_2) = 4x_1x_2$ for $0 < x_1 < 1$ and $0 < x_2 < 1$, $= 0$ otherwise. Now consider the change of variables $Y_1 = X_1/X_2, Y_2 = X_1X_2$, and let $g(y_1, y_2)$ be the joint PDF of these two variables. Sketch the region in the y_1, y_2 plane for which g is non-zero, and calculate $g(y_1, y_2)$.

To sketch the region, first notice the possible ranges of the variables.

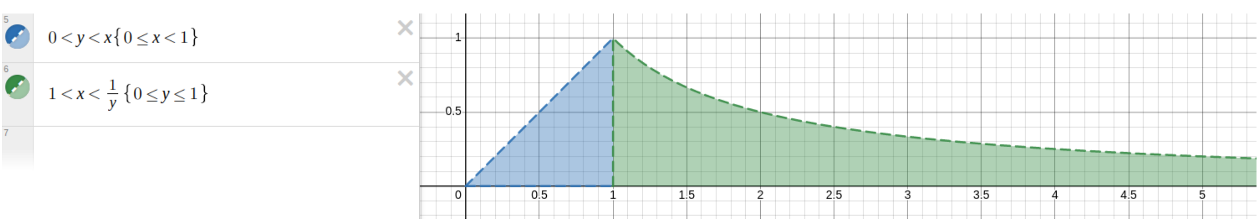
y_1 can take any value between 0 and infinity. y_2 has a lower bound of zero and a global upper bound of 1.

But consider minimum and maximum values of y_2 for a given y_1.

If $y_1 \leq 1$, our maximal value will be found by taking $x_2 = 1$ (well arbitrarily close to 1), which means $y_1 = x_1$ which in turn means $y_2 = y_1$ (again in the same arbitrarily close way).

On the other hand if $y_1 > 1$, we can find the value by taking $x_1 = 1 \implies y_2 = x_2 \implies y_2 = \frac{1}{y_1}$.

A sketch of the region where $g \neq 0$ is as follows:



To find $g(y_1, y_2)$ use the Jacobian:

$$g(y_1, y_2) = f(x_1, x_2) \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

To find these, we'll need $x_1(y_1, y_2), x_2(y_1, y_2)$:

$$\begin{aligned}
y_1 &= \frac{x_1}{x_2} \\
y_2 &= x_1 x_2 \implies x_2 = \frac{y_2}{x_1} \\
y_1 &= \frac{x_1}{\frac{y_2}{x_1}} \\
\implies x_1^2 &= y_1 y_2 \\
x_1 &= \sqrt{y_1 y_2} \\
x_2 &= \sqrt{\frac{y_2}{y_1}}
\end{aligned}$$

$$\begin{aligned}
g(y_1, y_2) &= 4x_1x_2 \begin{vmatrix} \frac{1}{2}\sqrt{\frac{y_2}{y_1}} & \frac{1}{2}\sqrt{\frac{y_1}{y_2}} \\ -\frac{1}{2}\sqrt{\frac{y_2}{y_1}}\frac{1}{y_1} & \frac{1}{2}\sqrt{\frac{1}{y_1y_2}} \end{vmatrix} \\
&= 4y_2 \left| \frac{1}{2}\sqrt{\frac{y_2}{y_1}}\frac{1}{2}\sqrt{\frac{1}{y_1y_2}} + \frac{1}{2}\sqrt{\frac{y_1}{y_2}}\frac{1}{2}\sqrt{\frac{y_2}{y_1}}\frac{1}{y_1} \right| \\
&= 4y_2 \left(\frac{1}{4y_1} + \frac{1}{4y_1} \right) \\
&= \frac{2y_2}{y_1}
\end{aligned}$$

5 Suppose that galactic supernovae obey Poissonian statistics. The mean number of supernovae per century is 1/3.

- What is the most likely date for the next supernova?

  Poissonian statistics means we have a pdf of the form $P(k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$, where k is the number of supernova observed in time t (measured in centuries), and $\lambda = 1/3$ is the mean number of supernovae in a century.

  To find the probability that a supernova happens at a particular time, we'll consider a small interval around that particular time and find the probability of one supernova happening inside that interval and nothing happening before.

  The probability of having one supernova after time t is given by

  $$P(k = 1; t) = \lambda t e^{-\lambda t}$$

  .

  If we want this to be the only supernova before time t we also need to consider the probability that there were no supernovae before t, i.e.

  $$P(k = 0; t - \Delta t) = e^{-\lambda(t-\Delta t)}$$

  These two things are independent (the probability that a supernova happens in one time interval vs another) so the joint probability function is as follows:

  $$\begin{aligned} P(\text{supernova in } [t - \Delta t, t]) &= P(k = 0; t - \Delta t) P(k = 1; t) \\ &= \lambda t e^{-\lambda t} e^{-\lambda(t-\Delta t)} \end{aligned}$$

  In the limit as $\Delta t \to 0$ we get:

  $$P(\text{supernova at } t) = \lambda t e^{-2\lambda t}$$

This has a maximum: find it.

$$
\begin{aligned}
\frac{d}{dt}P(\text{supernova at } t) &= \lambda e^{-2\lambda t} - 2\lambda^2 t e^{-2\lambda t} \\
0 &= \lambda e^{-2\lambda t} - 2\lambda^2 t e^{-2\lambda t} \\
2\lambda^2 t e^{-2\lambda t} &= \lambda e^{-2\lambda t} \\
2\lambda t &= 1 \\
t &= \frac{1}{2\lambda} \\
&= \frac{1}{2\frac{1}{3}} \\
&= \frac{3}{2}
\end{aligned}
$$

So the next supernova is most likely in 1.5 centuries. That's September 22, 2172. I wonder if humans will be around by then...

- What is the probability distribution for the length of the interval between now and the next galactic supernova?

  It's what we had above, the probability of the next supernova happening a time t_0 away from now is given by:

  $$P(t_0) = \lambda t e^{-2\lambda t_0}.$$

  (Maybe we were not supposed to derive that earlier...)

6 Consider an infinite series of random variables X_i, where each variable is generated from its predecessor according to $X_i = aX_{i-1} + B_i$. Here a is a constant and B_i is a Gaussian random variable with mean m and standard deviation s. If all of the X_i are identically distributed with mean μ and standard deviation σ, then what constraints does this place on a, m, and s? What condition will result in the X_i also being independent from each other? In the case that they are identically distributed but not necessarily independent, derive a formula for the correlation coefficient between X_i and X_{i-j}.

First recall a few things:

- the definition of correlation coefficient: $\rho_{A,B} = \frac{\mathrm{Cov}(A,B)}{\sigma_A \sigma_B}$
- the relationship between variance and covariance of summed variables $\mathrm{Var}(A+B) = \mathrm{Var}(A) + \mathrm{Var}(B) + 2\,\mathrm{Cov}(A,B)$
- How does scaling affect the mean?

$$\overline{aX} = a\overline{X}$$

- How does scaling affect Variance?

$$\begin{aligned}\mathrm{Var}(aX) &= \overline{(aX)^2} - (\overline{aX})^2 \\ &= a^2\left(\overline{X^2} - (\overline{X})^2\right) \\ &= a^2\,\mathrm{Var}(X)\end{aligned}$$

- How does scaling affect Covariance?

$$\begin{aligned}\mathrm{Cov}(aX,Y) &= \overline{aXY} - \overline{aX}\overline{Y} \\ &= a(\overline{XY} - \overline{X}\overline{Y})\end{aligned}$$

- For independent Gaussians, their sum is also a Gaussian, with mean $\mu_{A+B} = \mu_A + \mu_B$ and variance $\sigma^2_{A+B} = \sigma^2_A + \sigma^2_B$.

And find a relationship between X_i and X_{i-j}:

$$
\begin{aligned}
X_i &= aX_{i-1} + B_i \\
&= a(aX_{i-2} + B_{i-1}) + B_i \\
&= a^2 X_{i-2} + aB_{i-1} + B_i \\
&= a^3 X_{i-3} + a^2 B_{i-2} + aB_{i-1} + B_i \\
&= a^4 X_{i-4} + a^3 B_{i-3} + a^2 B_{i-2} + a^1 B_{i-1} + a^0 B_{i-0} \\
&. \\
&= a^j X_{i-j} + \sum_{k=0}^{j-1} a^k B_{i-k} \\
\implies X_i - a^j X_{i-j} &= \sum_{k=0}^{j-1} a^k B_{i-k}
\end{aligned}
$$

Find the variance and mean of the sum of the Gaussians (assuming they're independent):

$$
\begin{aligned}
&\mathrm{Var}(B_i + a^1 B_{i-1} + \ldots + a^j B_{i-j}) \\
&= \mathrm{Var}(B_i) + \mathrm{Var}(a^1 B_{i-1} + \ldots + a^j B_{i-j}) \\
&= s^2 + a^2 \,\mathrm{Var}(B_{i-1} + aB_{i-2} + \ldots + a^{j-1} B_{i-j}) \\
&= s^2 + a^2 s^2 + a^2 \,\mathrm{Var}(aB_{i-2} + \ldots + a^{j-1} B_{i-j}) \\
&= s^2 + a^2 s^2 + a^4 s^2 + \ldots + a^{2j} s^2 \\
&= s^2 \sum_{k=0}^{j} a^{2k}
\end{aligned}
$$

$$\overline{B_i + a^1 B_{i-1} + \ldots + a^j B_{i-j}} = m \sum_{k=0}^{j} a^j$$

- How are a, m, s constrained?

  For the sum and variance of infinitely many B_i to be defined for non-zero m, s, we need $a \in (-1, 1)$.

  I'm not sure if m, s need to be constrained, I don't think they do? Clearly $s > 0$ by definition of variance.

- What's the condition such that the X_i are independent from each other?

  Well if $a = 0$ then $X_i = B_i$, just a Gaussian, and I think we can assume all the B_i are independent even though the question doesn't specifically say so.

- Find $\rho_{X_i, X_{i-j}}$ if X_i, X_{i-j} are not independent.

  Use our equation above with covariance.

$$\mathrm{Var}(\sum_{k=0}^{j-1} a^k B_{i-k}) = \mathrm{Var}(X_i) + \mathrm{Var}((-a^j) X_{i-j}) + 2\,\mathrm{Cov}(X_i, (-a^j) X_{i-j})$$

$$\begin{aligned}
\mathrm{Var}(\sum_{k=0}^{j-1} a^k B_{i-k}) &= \mathrm{Var}(X_i) + \mathrm{Var}((-a^j) X_{i-j}) \\
&\quad + 2\,\mathrm{Cov}(X_i, (-a^j) X_{i-j}) \\
s^2 \sum_{k=0}^{j} a^{2k} &= \sigma^2 + a^{2j}\sigma^2 - 2a^j\,\mathrm{Cov}(X_i, X_{i-j}) \\
\frac{\sigma^2 + a^{2j}\sigma^2 - s^2 \sum_{k=0}^{j} a^{2k}}{2a^j} &= \mathrm{Cov}(X_i, X_{i-j})
\end{aligned}$$

So we get the correlation coefficient:

$$\begin{aligned}
\rho_{X_i,X_{i-j}} &= \frac{\mathrm{Cov}(X_i, X_{i-j})}{\sigma_{X_i}\sigma_{X_{i-j}}} \\
&= \frac{\sigma^2 + a^{2j}\sigma^2 - s^2 \sum_{k=0}^{j} a^{2k}}{2a^j\sigma^2} \\
&= \frac{1}{2a^j} + \frac{1}{2}a^j - \frac{s^2}{2\sigma^2}\sum_{k=0}^{j} a^{2k-j}
\end{aligned}$$

Knowing that the correlation coefficient must be between -1 and +1 we may get an extra restriction on s.

$$\begin{aligned}
-1 < \frac{1}{2a^j} + \frac{1}{2}a^j - \frac{s^2}{2\sigma^2}\sum_{k=0}^{j} a^{2k-j} < 1 \\
-1 - \frac{1}{2a^j} - \frac{1}{2}a^j < -\frac{s^2}{2\sigma^2}\sum_{k=0}^{j} a^{2k-j} < 1 - \frac{1}{2a^j} - \frac{1}{2}a^j \\
1 + \frac{1}{2a^j} + \frac{1}{2}a^j > \frac{s^2}{2\sigma^2}\sum_{k=0}^{j} a^{2k-j} > -1 + \frac{1}{2a^j} + \frac{1}{2}a^j \\
\left(1 + \frac{1}{2a^j} + \frac{1}{2}a^j\right)\frac{2\sigma^2}{\sum_{k=0}^{j} a^{2k-j}} > s^2 > \left(-1 + \frac{1}{2a^j} + \frac{1}{2}a^j\right)\frac{2\sigma^2}{\sum_{k=0}^{j} a^{2k-j}}
\end{aligned}$$