PHYS 509C Assignment 1

Callum McCracken, 20334298

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Code for this assignment is here:

`https://github.com/callum-mccracken/PHYS-509C-A1`

1 `fakedata.out` contains 200 observations of three random variables: X, Y, and Z (each variable in its own column, listed in that order). Calculate the following for this data:

A. The mean values of X, Y, and Z.

$\overline{X} = 49.85, \overline{Y} = -1.56, \overline{Z} = -19.38.$

B. The standard deviations for all thre variables.

$\sigma_X = 12.75, \sigma_Y = 13.63, \sigma_Z = 11.06.$

C. The three correlation coefficients between the three variables.

$C_{X,Y} = 0.30, C_{X,Z} = 0.72, C_{Y,Z} = -0.30.$

D. The skew for X, Y, and Z.

$\mathrm{Skew}(X) = -0.10, \mathrm{Skew}(Y) = 0.05, \mathrm{Skew}(Z) = -0.31.$

2 Numerically calculate the probability that a number drawn from a χ^2 distribution with $n = 5$ degrees of freedom will be larger than $\chi^2 = 5$. Do the same for $n = 10$. Do not use a lookup table or a pre-existing function to evaluate the answer, but calculate it for yourself as if you had just discovered the χ^2 distribution for the first time.

- $\mathrm{P}(\chi^2_5 < 5) = 0.42$.
- $\mathrm{P}(\chi^2_{10} < 5) = 0.89$.

3 Three independent random numbers X_1, X_2, X_3 are drawn from uniform distributions with means of 0 and variances of 1/3. Let Z be the sum of these three numbers. Derive the normalized probability distribution for Z.

$$
\begin{aligned}
\nabla^c \mathcal{M}_{cab}(\vec{x}, t) &= T_{ab}(\vec{x}, t) + x_a \nabla^c T_{cb}(\vec{x}, t) - T_{ba}(\vec{x}, t) - x \\
&= 0 + x_a \nabla^c T_{cb}(\vec{x}, t) - x_b \nabla^c T_{ca}(\vec{x}, t) \\
&= -\frac{\partial}{\partial t} \mathcal{M}_{ab}(\vec{x}, t)
\end{aligned}
$$

$$
\Longrightarrow \frac{\partial}{\partial t} \mathcal{M}_{ab}(\vec{x}, t) + \nabla^c \mathcal{M}_{cab}(\vec{x}, t) = 0
$$

4 Confirm that for an infinitesimal rotation in the $a-b$ plane,

$$-(x_a\nabla_b - x_b\nabla_a)\psi_\sigma(\vec{x},t) = \frac{1}{i\hbar}[\psi_\sigma(\vec{x},t), M_{ab}]$$

and

$$-(x_a\nabla_b - x_b\nabla_a)\psi^{\dagger\sigma}(\vec{x},t) = \frac{1}{i\hbar}[\psi^{\dagger\sigma}(\vec{x},t), M_{ab}]$$

where M_{ab} is given in (3.25).

(3.25)

$$M_{ab} = -\frac{i\hbar}{2}\int d^3x[\psi^{\dagger\sigma}(\vec{x},t)(x_a\nabla_b - x_b\nabla_a)\psi_\sigma(\vec{x},t) \\ - (x_a\nabla_b - x_b\nabla_a)\psi^{\dagger\sigma}(\vec{x},t)\psi_\sigma(\vec{x},t)]$$

Start with the commutators:

$$\begin{aligned}
\frac{1}{i\hbar}[\psi_\sigma(\vec{x},t), M_{ab}] &= \frac{1}{i\hbar}\psi_\sigma(\vec{x},t)M_{ab} - \frac{1}{i\hbar}M_{ab}\psi_\sigma(\vec{x},t) \\
&= -\frac{1}{2}\psi_\sigma(\vec{x},t)\int d^3y[\psi^{\dagger\sigma}(\vec{y},t)(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
&\quad - (y_a\nabla_b - y_b\nabla_a)\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{y},t)] \\
&\quad + \frac{1}{2}\int d^3y[\psi^{\dagger\sigma}(\vec{y},t)(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
&\quad - (y_a\nabla_b - y_b\nabla_a)\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{y},t)]\psi_\sigma(\vec{x},t) \\
&= \frac{1}{2}\int d^3y - \psi_\sigma(\vec{x},t)\psi^{\dagger\sigma}(\vec{y},t)(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
&\quad + \psi_\sigma(\vec{x},t)(y_a\nabla_b - y_b\nabla_a)\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{y},t) \\
&\quad + [\psi^{\dagger\sigma}(\vec{y},t)(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
&\quad - (y_a\nabla_b - y_b\nabla_a)\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{y},t)]\psi_\sigma(\vec{x},t)
\end{aligned}$$

Recall equal-time anti-commutators:

$$\begin{aligned}
\psi_\sigma(\vec{x},t)\psi^{\rho\dagger}(\vec{y},t) + \psi^{\rho\dagger}(\vec{y},t)\psi_\sigma(\vec{x},t) &= \delta^\rho_\sigma\delta(\vec{x}-\vec{y}) \\
\psi_\sigma(\vec{x},t)\psi_\rho(\vec{y},t) + \psi_\rho(\vec{y},t)\psi_\sigma(\vec{x},t) &= 0
\end{aligned}$$

Also note that the ∇_is are w.r.t. y_i for now, so we can pass the $\vec{x}$ field operators through the $(y_a\nabla_b - y_b\nabla_a)$ terms:

$$
\begin{aligned}
\frac{1}{i\hbar}[\psi_\sigma(\vec{x},t), M_{ab}] = \frac{1}{2}\int & d^3y - [-\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{x},t) + \delta^\sigma_\sigma \delta(\vec{x}-\vec{y})](y_a\nabla_b - y_b\nabla_a \\
& + (y_a\nabla_b - y_b\nabla_a)[-\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{x},t) + \delta^\sigma_\sigma\delta(\vec{x}-\vec{y})]\psi_\sigma(\vec{y},t \\
& + [\psi^{\dagger\sigma}(\vec{y},t)(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
& - (y_a\nabla_b - y_b\nabla_a)\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{y},t)]\psi_\sigma(\vec{x},t) \\
= \frac{1}{2}\int & d^3y\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{x},t)(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
& - \delta^\sigma_\sigma\delta(\vec{x}-\vec{y})(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
& - (y_a\nabla_b - y_b\nabla_a)\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{x},t) \\
& + (y_a\nabla_b - y_b\nabla_a)\delta^\sigma_\sigma\delta(\vec{x}-\vec{y})\psi_\sigma(\vec{y},t) \\
& + [\psi^{\dagger\sigma}(\vec{y},t)(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
& - (y_a\nabla_b - y_b\nabla_a)\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{y},t)]\psi_\sigma(\vec{x},t) \\
= \frac{1}{2}\int & d^3y - \psi^{\dagger\sigma}(\vec{y},t)(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t)\psi_\sigma(\vec{x},t) \\
& - \delta^\sigma_\sigma\delta(\vec{x}-\vec{y})(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
& + (y_a\nabla_b - y_b\nabla_a)\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{y},t)\psi_\sigma(\vec{x},t) \\
& - \delta^\sigma_\sigma\delta(\vec{x}-\vec{y})(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
& + [\psi^{\dagger\sigma}(\vec{y},t)(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t) \\
& - (y_a\nabla_b - y_b\nabla_a)\psi^{\dagger\sigma}(\vec{y},t)\psi_\sigma(\vec{y},t)]\psi_\sigma(\vec{x},t) \\
= \int & d^3y 0 + 0 - \delta^\sigma_\sigma\delta(\vec{x}-\vec{y})(y_a\nabla_b - y_b\nabla_a)\psi_\sigma(\vec{y},t)
\end{aligned}
$$

Note that σ is not summed, so $\delta^\sigma_\sigma = 1$. Integrate the $\delta(\vec{x}-\vec{y})$ and we get the desired result:

$$\frac{1}{i\hbar}[\psi_\sigma(\vec{x},t), M_{ab}] = -(x_a\nabla_b - x_b\nabla_a)\psi_\sigma(\vec{x},t)$$

(where now ∇_a is w.r.t. x_a)

5 Assume that

$$P_a \left|\mathcal{O}\right\rangle = 0, \left\langle\mathcal{O}\right| P_a = 0, M_{ab} \left|\mathcal{O}\right\rangle = 0, \left\langle\mathcal{O}\right| M_{ab} = 0.$$

Prove that $\left\langle\mathcal{O}\right| \psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t') \left|\mathcal{O}\right\rangle$ can only be a function of the coordinates $\vec{x}$ and $\vec{y}$ through the combination $|\vec{x}-\vec{y}| = \sqrt{(\vec{x}-\vec{y})^2}$.

Consider that expetation value of a commutator:

$$\begin{aligned}
&\frac{1}{i\hbar} \left\langle\mathcal{O}\right| \left[\psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t'), P_a\right] \left|\mathcal{O}\right\rangle \\
&= \frac{1}{i\hbar} \left\langle\mathcal{O}\right| \psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t') P_a \left|\mathcal{O}\right\rangle - \left\langle\mathcal{O}\right| P_a \psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t') \left|\mathcal{O}\right\rangle \\
&= \frac{1}{i\hbar} \left\langle\mathcal{O}\right| \psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t')(0) - (0)\psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t') \left|\mathcal{O}\right\rangle \qquad [\text{use } \left\langle\mathcal{O}\right| P_a = \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

Then consider $\left(\frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a}\right) \left\langle\mathcal{O}\right| \psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t') \left|\mathcal{O}\right\rangle$:

$$\begin{aligned}
&\left(\frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a}\right) \left\langle\mathcal{O}\right| \psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t') \left|\mathcal{O}\right\rangle \\
&= \left\langle\mathcal{O}\right| \frac{\partial}{\partial x^a}\psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t') \left|\mathcal{O}\right\rangle + \left\langle\mathcal{O}\right| \psi_\sigma(\vec{x},t)\frac{\partial}{\partial y^a}\psi^{\dagger\rho}(\vec{y},t') \left|\mathcal{O}\right\rangle \\
&= \frac{1}{i\hbar} \left\langle\mathcal{O}\right| [\psi_\sigma(\vec{x},t), \mathcal{Q}]\psi^{\dagger\rho}(\vec{y},t') \left|\mathcal{O}\right\rangle + \frac{1}{i\hbar} \left\langle\mathcal{O}\right| \psi_\sigma(\vec{x},t)[\psi^{\dagger\rho}(\vec{y},t'), \mathcal{Q}] \left|\mathcal{O}\right\rangle \\
&= \frac{1}{i\hbar} \left\langle\mathcal{O}\right| [\psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t')\mathcal{Q}] \left|\mathcal{O}\right\rangle
\end{aligned}$$

So in particular for P_a,

$$\begin{aligned} 0 &= \frac{1}{i\hbar} \langle \mathcal{O}| \left[\psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t'), P_a \right] |\mathcal{O}\rangle \\ &= \left(\frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a} \right) \langle \mathcal{O}| \psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t') |\mathcal{O}\rangle \end{aligned}$$

And if we let $\vec{r} = \vec{x} + \vec{y}$,

$$\frac{\partial}{\partial r^a} = \frac{\partial}{\partial x^a} + \frac{\partial}{\partial y^a}$$

$$\implies \frac{\partial}{\partial r^a} \langle \mathcal{O}| \psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t') |\mathcal{O}\rangle = 0$$

This implies $\langle \mathcal{O}| \psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t') |\mathcal{O}\rangle$ depends on $\vec{x}$ and $\vec{y}$ through the combination $\vec{x} - \vec{y}$ rather than $\vec{x} + \vec{y}$.

Then, it is also true that

$$0 = \frac{1}{i\hbar} \langle \mathcal{O}| \left[\psi_\sigma(\vec{x},t)\psi^{\dagger\rho}(\vec{y},t'), M_{ab} \right] |\mathcal{O}\rangle$$

And finally to show this function only depends on the magnitude $|\vec{x} - \vec{y}|$, let's show that $(x_a\nabla_b - x_b\nabla_a)f(\vec{x}) = 0$ for every pair (a,b) $\implies f(\vec{x})$ depends only on $|\vec{x}|$.

Consider just x^1, x^2 for now.

$$(x^1\nabla_2 - x^2\nabla_1)f(x^1, x^2, \ldots) = 0 \implies f = f((x^1)^2 + (x^2)^2, \ldots)$$

Then x^1, x^2, x^3 (or equivalently for f, $(x^1)^2 + (x^2)^2, x^3$):

$(x^2\nabla_3 - x^3\nabla_2)f((x^1)^2 + (x^2)^2, x^3, \ldots) = 0 \implies f = f((x^1)^2 + (x^2)^2 + (x^3)^2,$

And if we continue in the same way for however many coordinates we have, we find:

$$f = f(\sum_i (x^i)^2)$$

Or equivalently,

$$f = f(\sqrt{\sum_i (x^i)^2}) = f(|\vec{x}|)$$

Since we have a function of this type above, which satisfies $(x_a\nabla_b - x_b\nabla_a)f(\vec{x}) = 0$ for every pair (a, b), we can say $\langle\mathcal{O}|\, \psi_\sigma(\vec{x}, t)\psi^{\dagger\rho}(\vec{y}, t')\, |\mathcal{O}\rangle$ is only a function of $|\vec{x} - \vec{y}| = \sqrt{\sum_i (x_i - y_i)^2}$.

6 Show that the Galilean transformations defined above are a symetry of the theory defined by the action and Lagrangian density (3.5).

The Galilean transformations defined above:

$$\delta\psi_\sigma(\vec{x},t) = \left(-\vec{v}t\cdot\vec{\nabla} + \frac{i}{\hbar}m\vec{v}\cdot\vec{x}\right)\psi_\sigma(\vec{x},t)$$
$$\delta\psi^{\sigma\dagger}(\vec{x},t) = \left(-\vec{v}t\cdot\vec{\nabla} - \frac{i}{\hbar}m\vec{v}\cdot\vec{x}\right)\psi^{\sigma\dagger}(\vec{x},t)$$

To show this is a symmetry, we want to show $\delta\mathcal{L}(\vec{x},t) = \frac{\partial}{\partial t}R(\vec{x},t) + \vec{\nabla}\cdot\vec{J}(\vec{x},t)$ for some $R, \vec{J}$.

The Lagrangian:

$$\mathcal{L} = \frac{i\hbar}{2}\psi^{\sigma\dagger}\frac{\partial}{\partial t}\psi_\sigma - \frac{i\hbar}{2}\frac{\partial}{\partial t}\psi^{\sigma\dagger}\psi_\sigma - \frac{\hbar^2}{2m}\nabla\psi^{\sigma\dagger}\cdot\nabla\psi_\sigma - \frac{\lambda}{2}\left(\psi^{\sigma\dagger}\psi_\sigma\right)^2$$

This is does not explicitly depend on $\vec{x}$ or t, so we have

$$\begin{aligned}
\delta\mathcal{L} &= \left(-\vec{v}t\cdot\vec{\nabla} + \frac{i}{\hbar}m\vec{v}\cdot\vec{x}\right)\mathcal{L} \\
&= \left(-v_a t\nabla_a\mathcal{L} + \frac{i}{\hbar}mv_a x_a\mathcal{L}\right) \\
&= 0 + \frac{\partial}{\partial t}\left(t\frac{i}{\hbar}mv_a x_a\mathcal{L}\right) \\
&= \frac{\partial}{\partial t}\left(t\frac{i}{\hbar}m\vec{v}\cdot\vec{x}\mathcal{L}\right) + \nabla\cdot\vec{0}
\end{aligned}$$

So these transformations are a symmetry, with $R = \frac{it}{\hbar} m\vec{v}\cdot\vec{x}\mathcal{L}$, $\vec{J} = 0$.