PHYS 509C Assignment 2

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Code for this assignment is here:
`https://github.com/callum-mccracken/PHYS-509C-A2`

1 Three easy applications of Bayes's theorem:

A. Suppose supernovae follow a poisson distribution with an unknown rate R. Calculate and plot the posterior distribution for R, given an observation of 4 supernovae in 10 centuries, using (a) a prior uniform in R and (b) a prior uniform in $\log_{10}(R)$.

Remember Bayes's theorem:

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{probability of data}}$$

The priors:

- Uniform in R, but we're not given bounds... Suppose it goes from some n to m.
$$P(R) = \frac{1}{m-n}.$$
- Uniform in $\log_{10}(R)$: use $P(x)dx = P(f(x))df(x)$ to get
$$\begin{aligned} P(R) &= P(\log_{10}(R))\frac{d\log_{10}(R)}{dR} \\ &= \frac{\ln(10)}{\ln(m) - \ln(n)}\frac{1}{\ln(10)R} \\ &= \frac{1}{\ln(m)-\ln(n)}\frac{1}{R} \end{aligned}$$

Likelihood: Poisson distribution ($T = 1000$ years)

$$P(k|R) = \frac{e^{-RT}(RT)^k}{k!}$$

Probability of Data:

$$P(k) = \int P(R)P(k|R)dR$$

for uniform R,

$$\begin{aligned}
P(k) &= \int_0^\infty \frac{1}{m-n}\frac{e^{-RT}(RT)^k}{k!}dR \\
&= \frac{1}{m-n}\frac{1}{k!}\int_0^\infty e^{-RT}(RT)^k dR \\
&= \frac{1}{m-n}\frac{1}{k!}\frac{1}{T}\int_0^\infty e^{-\lambda}(\lambda)^k d\lambda \\
&= \frac{1}{m-n}\frac{1}{k!}\frac{1}{T}k! \\
&= \frac{1}{m-n}\frac{1}{T}
\end{aligned}$$

or for uniform $\log_{10}(R)$

$$\begin{aligned}
P(k) &= \int_0^\infty \frac{1}{\ln(m)-\ln(n)}\frac{1}{R}\frac{e^{-RT}(RT)^k}{k!}dR \\
&= \frac{1}{\ln(m)-\ln(n)}\int_0^\infty T\frac{1}{RT}\frac{e^{-RT}(RT)^k}{k!}dR \\
&= \frac{1}{\ln(m)-\ln(n)}\int_0^\infty \frac{e^{-\lambda}(\lambda)^{k-1}}{k!}d\lambda && [\lambda = RT] \\
&= \frac{1}{\ln(m)-\ln(n)}\frac{1}{k}
\end{aligned}$$

Now calculate posteriors:

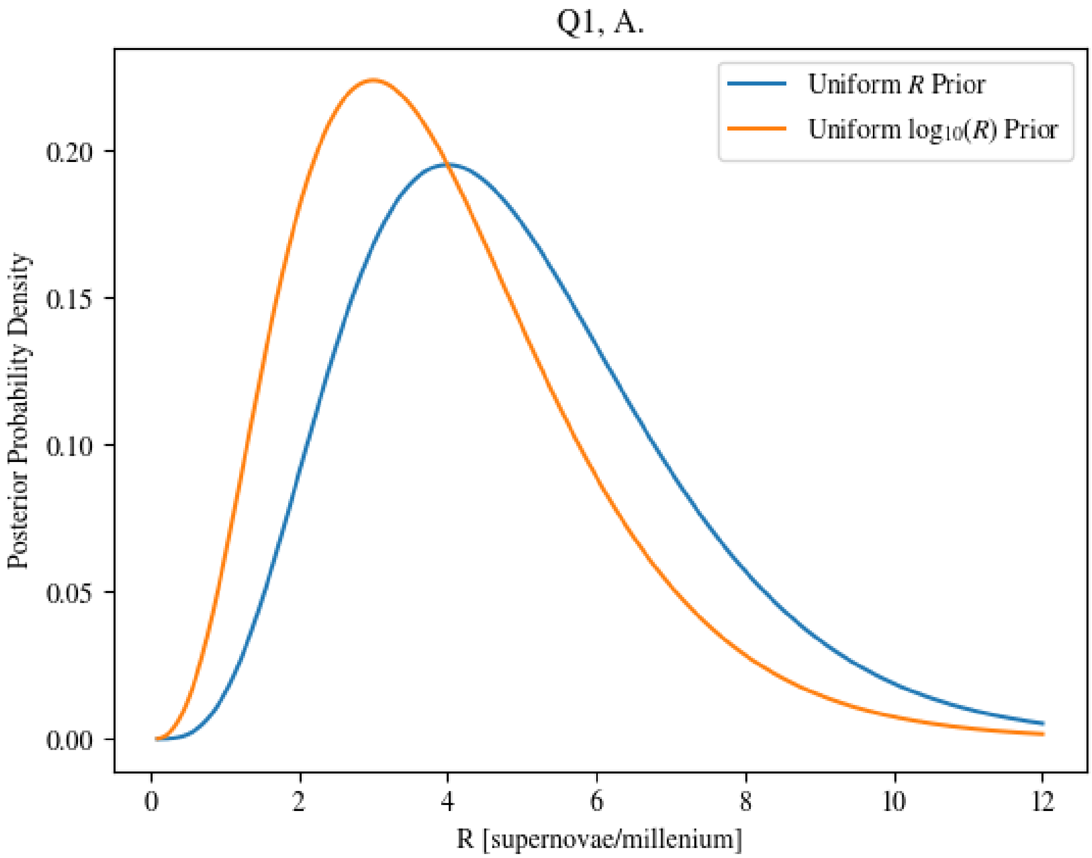
Uniform R:

$$\begin{aligned}
P(R|k) &= \frac{P(k|R)P(R)}{P(k)} \\
&= \frac{\frac{e^{-RT}(RT)^k}{k!}\frac{1}{m-n}}{\frac{1}{m-n}\frac{1}{T}} \\
&= \frac{\frac{e^{-RT}(RT)^k}{k!}}{\frac{1}{T}} \\
&= \frac{Te^{-RT}(RT)^k}{k!}
\end{aligned}$$

Uniform $\log_{10}(R)$

$$\begin{aligned}
P(R|k) &= \frac{P(k|R)P(R)}{P(k)} \\
&= \frac{\frac{e^{-RT}(RT)^k}{k!}\frac{1}{\ln(m)-\ln(n)}\frac{1}{R}}{\frac{1}{\ln(m)-\ln(n)}\frac{1}{k}} \\
&= \frac{Te^{-RT}(RT)^{k-1}}{(k-1)!}
\end{aligned}$$

Plot of these distributions:



B. Measurements are drawn from a uniform distribution spanning the interval (0, m). The probability of getting a measurement outside of this range is zero. The endpoint m is not well-known, but a prior experiment yields a Gaussian prior of m = 3 +/- 1. You take three measurements, getting values of 2.5, 3.1, and 2.9. Use Bayes' theorem to calculate and plot the new probability distribution for m.

Likelihood: if our model is a uniform distribution on $[0, m]$ and we have independent measurements (which I think we can assume for this question, right?), then $P(D|m) = \frac{1}{m^3}$ if $m \geq 3.1$, $= 0$ otherwise. That's the product of three uniform probabilities.

Prior (Gaussian, $\mu = 3, \sigma = 1$): $P(m) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(m-3)^2}{2}}$.

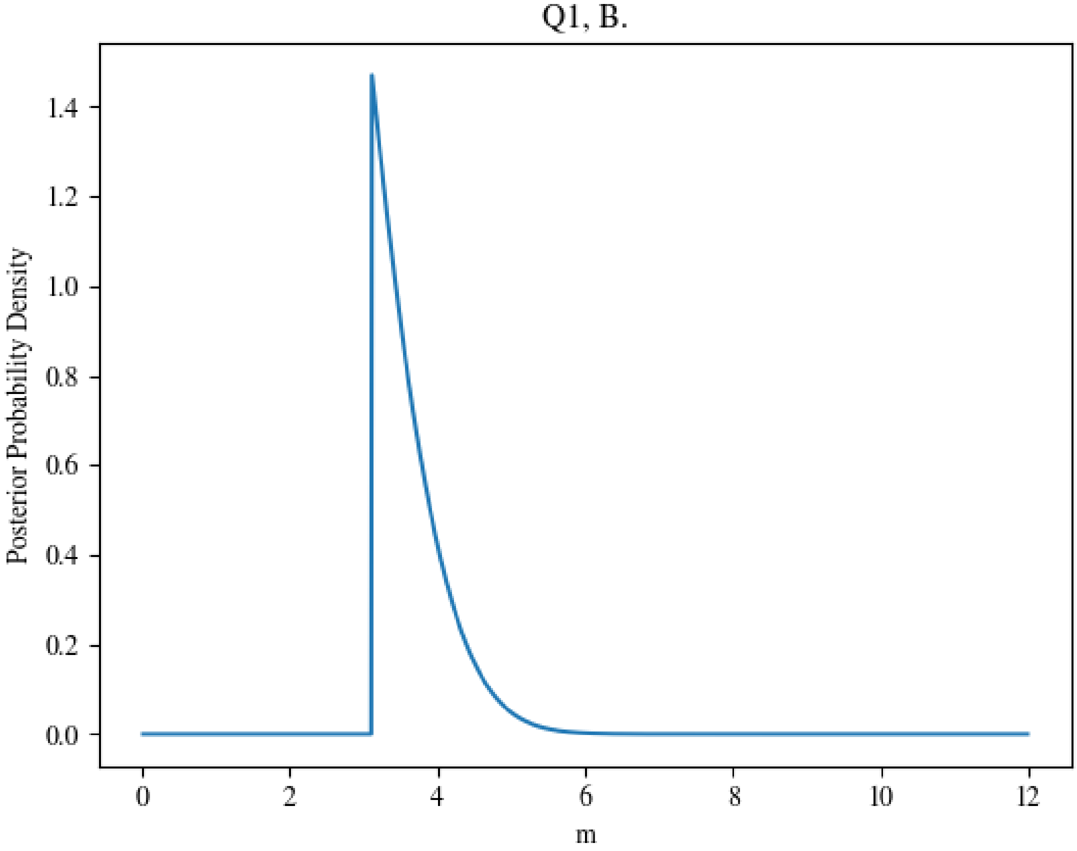
Probability of Data:

$$\begin{aligned} P(D) &= \int_0^\infty P(m)P(D|m)dm && (1) \\ &= \int_{3.1}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(m-3)^2}{2}} \frac{1}{m^3} dm && (2) \\ & && (3) \end{aligned}$$

Posterior:

$$\begin{aligned} P(m|D) &= \frac{P(D|m)P(m)}{P(D)} \\ &= \frac{\frac{1}{m^3}\frac{1}{\sqrt{2\pi}}e^{-\frac{(m-3)^2}{2}}}{\int_{3.1}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{(m'-3)^2}{2}}\frac{1}{m'^3}dm'} && [m \geq 3.1,\ \text{else } 0] \\ &= \frac{\frac{1}{m^3}e^{-\frac{(m-3)^2}{2}}}{\int_{3.1}^\infty e^{-\frac{(m'-3)^2}{2}}\frac{1}{m'^3}dm'} && [m \geq 3.1,\ \text{else } 0] \end{aligned}$$

Plot:



C. Consider a person considering whether or not to launch a rocket with a possible malfunctioning component. In the control centre there is a warning light that is not completely reliable. During launch the warning light doesn't go on. From a costs standpoint, should she abort the mission or not? Compute and compare the expected cost of launching to the expected cost of aborting, given that the light didn't go on.

- $P(\text{light on}|\text{malfunction}) = 1/2$, $P(\text{light on}|\text{no malfunction}) = 1/3$
- $C(\text{no launch, no malfunction}) = 2M$, $C(\text{launch, malfunction}) = 5M$
- $C(\text{no launch, malfunction}) = C(\text{launch, no malfunction}) = 0$
- $\text{Prior}(\text{malfunction}) = 2/5$

Let F denote component failure, L denote the light being on, and A denote a launch (since L was taken).

Priors: $P(F) = \frac{2}{5}, P(\neg F) = \frac{3}{5}$.
Likelihood of observing data (no light): $P(\neg L|F) = \frac{1}{2}, P(\neg L|\neg F) = \frac{2}{3}$.
Probability of seeing our data (no light):

$$P(\neg L) = P(\neg L|F)P(F) + P(\neg L|\neg F)P(\neg F) = \frac{3}{5}$$

Posteriors:

$$\begin{aligned} P(F|\neg L) &= \frac{P(F)P(\neg L|F)}{P(\neg L)} = \frac{\frac{2}{5}\frac{1}{2}}{\frac{3}{5}} = \frac{1}{3} \\ P(\neg F|\neg L) &= 1 - P(F|\neg L) = \frac{2}{3} \end{aligned}$$

Costs (recall that I used A to denote a launch, which may have been a mistake because upon re-reading this it looks like A should mean abort...):

$$\begin{aligned} C(A) &= P(F|\neg L)C(F|A) + P(\neg F|\neg L)C(\neg F|A) \\ &= \frac{1}{3}(5M) + \frac{2}{3}(0) \\ &= \frac{5}{3}M \end{aligned}$$

$$\begin{aligned} C(\neg A) &= P(F|\neg L)C(F|\neg A) + P(\neg F|\neg L)C(\neg F|\neg A) \\ &= \frac{1}{3}(0) + \frac{2}{3}(2M) \\ &= \frac{4}{3}M \end{aligned}$$

The expected cost of not launching is less, so from a cost standpoint, the mission should be aborted, and the rocket makers should probably invest in a better warning light for next time.

2 COVID-19 Study

- Sample size: 3330 people in Santa Clara County, California
- $N_{+,t} = 50$ positive test results in test group
- Control group 1: 3324 people, definitely negative
- $N_{+,c1} = 16$ positives in control 1
- Control group 2: 157 people, definitely negative
- $N_{+,c2} = 130$ positives in control 2 (27 false negatives)

Calculate the Bayesian 95% central interval on the fraction of people in Santa Clara County who actually had antibodies for COVID-19, marginalizing over the false positive and false negative rates. Assume flat priors on all parameters. Submit a plot of the posterior distribution for the true incidence rate as well as your code or calculation.

We want to find the distribution of the true disease in the population and then get the 95% interval.

Relevant variables, $R_{TP}, R_{FP}, R_{TN}, R_{FN}, R_P, R_N$ (P=Positive, N=Negative, T=True, F=False).

Use Bayes to get probability distributions for 3 independent ones of these (just three since $R_P = 1 - R_N, R_{TP} = 1 - R_{FN}, R_{TN} = 1 - R_{FP}$). Each one 3 follows a binomial distribution, so

Distribution for $P(R_{FP}|$**data from control group 1**$)$:

Prior: $P(R_{FP}) = 1$

Data: in our control group 1, $N_1 = 3324, N_{FP,1} = 16$.

Likelihood (binomial):

$$P(N_1 = 3324, N_{FP,1} = 16|R_{FP}) = \binom{N_1}{N_{FP,1}} R_{FP}^{N_{FP,1}}(1 - R_{FP,1})^{N_1 - N_{FP,1}}$$

Normalization:

$$P(N_1 = 3324, N_{FP,1} = 16) = \int_0^1 dR_{FP} \binom{N_1}{N_{FP,1}} R_{FP}^{N_{FP,1}} (1 - R_{FP})^{N_1 - N_{FP,1}}$$

Posterior:

$$\begin{aligned} P(R_{FP}|N_1 = 3324, N_{FP,1} = 16) &= \frac{P(R_{FP})P(N_1 = 3324, N_{FP,1} = 16|R_{FP})}{P(N_1 = 3324, N_{FP,1} = 16)} \\ &= \frac{R_{FP}^{N_{FP,1}}(1 - R_{FP,1})^{N_1 - N_{FP,1}}}{\int_0^1 dR_{FP} R_{FP}^{N_{FP,1}}(1 - R_{FP})^{N_1 - N_{FP,1}}} \end{aligned}$$

Similarly, the other distributions:

$$P(R_{FN}|N_2 = 157, N_{FN,2} = 27) = \frac{R_{FN}^{N_{FN,2}}(1 - R_{FN,2})^{N_2 - N_{FN,2}}}{\int_0^1 dR_{FN} R_{FN}^{N_{FN,2}}(1 - R_{FN})^{N_2 - N_{FN,2}}}$$

$$P(R_P|N_t = 3330, N_{P,t} = 50) = \frac{R_P^{N_{P,t}}(1 - R_{P,t})^{N_t - N_{P,t}}}{\int_0^1 dR_P R_P^{N_{P,t}}(1 - R_P)^{N_t - N_{P,t}}}$$

What we really want is the PDF for the probability of having antibodies $P(A)$. Let's find that by relating the things we already have, make it a variable, say R_A.

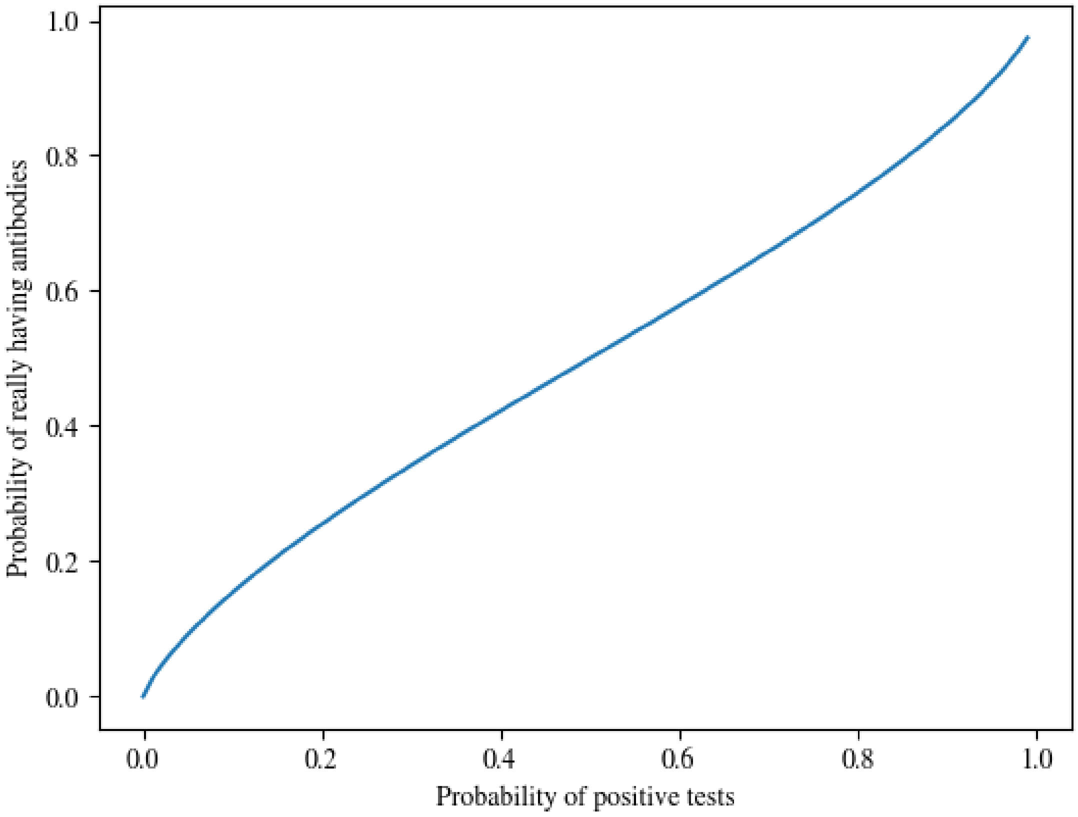
A positive test could have come from a true positive or a false positive, so

$$\begin{aligned} R_P &= R_{TP} R_A + R_{FP}(1 - R_A) \\ \frac{R_P - R_{FP}}{R_{TP} - R_{FP}} &= R_A \\ \frac{R_P - R_{FP}}{1 - R_{FN} - R_{FP}} &= R_A \end{aligned}$$

We have PDFs for all those variables, use those to get the PDF for R_A. At this point I moved to numerical work since the distributions are getting a little gross.

At this point I also wasn't quite sure how to do a transformation with multiple variables, but it's getting pretty late... The idea here though would be to get a new PDF that depends on R_P, then integrate over R_{FN} and R_{FP}, and find the 95% interval graphically, expanding outwards from the max value. Sorry I couldn't quite figure that out...

Here's a plot of $P(R_A)$ as a function of $P(R_P)$:



3 CO_2 Meter

If we assume that the data follows an exponential plateau, approaching some steady state value C, then that means for any t our expected y (concentration in ppm) is

$$y(t) = C - Be^{-At}$$

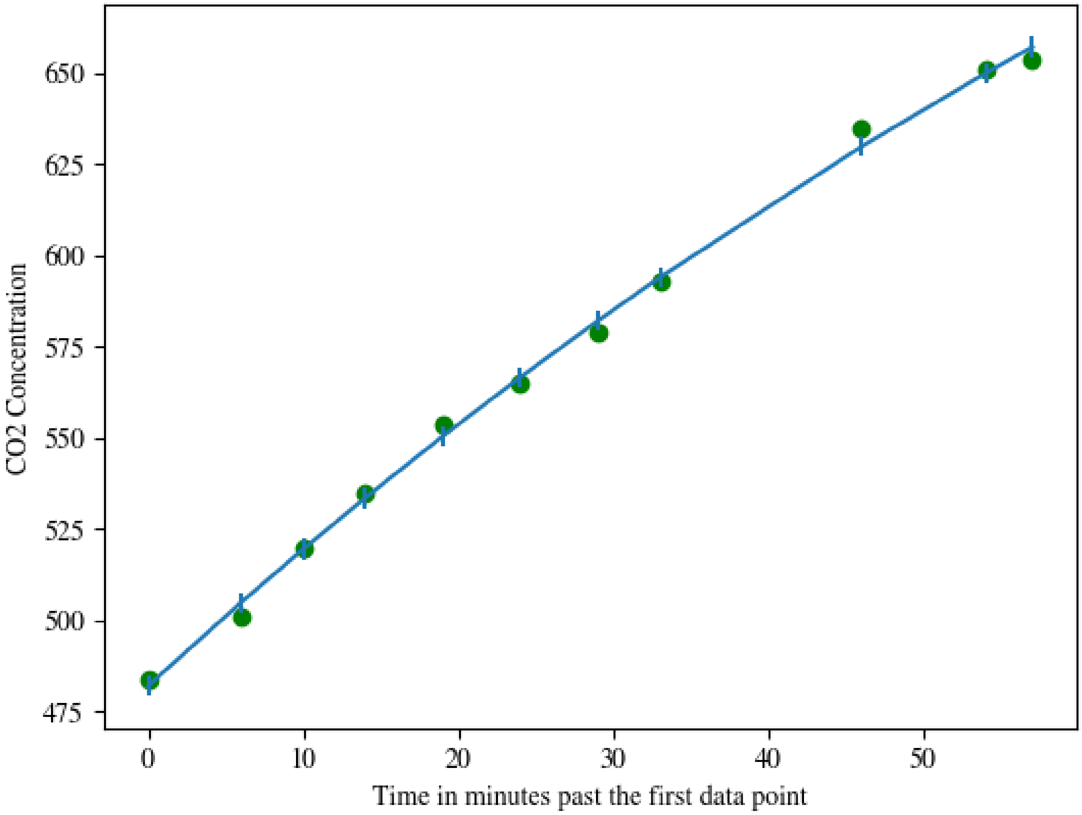
To include the error as a fit parameter, let's say (since we have no other reason to pick a different distribution) that for every t we have a Gaussian distribution with mean $\mu = C - Be^{-At}$ and standard deviation σ_y.
With that, we can do a maximum likelihood fit.

$$\begin{aligned} L &= \prod_i P(y(t_i) = y_i) \\ -\ln(L) &= -\ln\left(\prod_i P(y(t_i) = y_i)\right) \\ -\ln(L) &= \sum_i -\ln(P(y(t_i) = y_i)) \end{aligned}$$

I fit this numerically, and got the results here:

$$\begin{aligned} C &= 923 \\ B &= 440 \\ A &= 0.008875 \\ \sigma_y &= 2.77 \end{aligned}$$

A plot just to make sure the fit is reasonable:



Then how much of that is due to error in t?

Well if the error in t is 1 minute, then the error in the exponential is given as follows (using error propagation equations):

$$E = e^{-At}$$
$$\sigma_E = Ae^{-At}\sigma_t$$

If we use $\sigma_t = 1$ (minute) and $A = 0.008875$ from before, evaluated at the minimum measured t to maximize uncertainty, i.e. $t = 0$, we get the uncertainty from t is $A = 0.008875$.

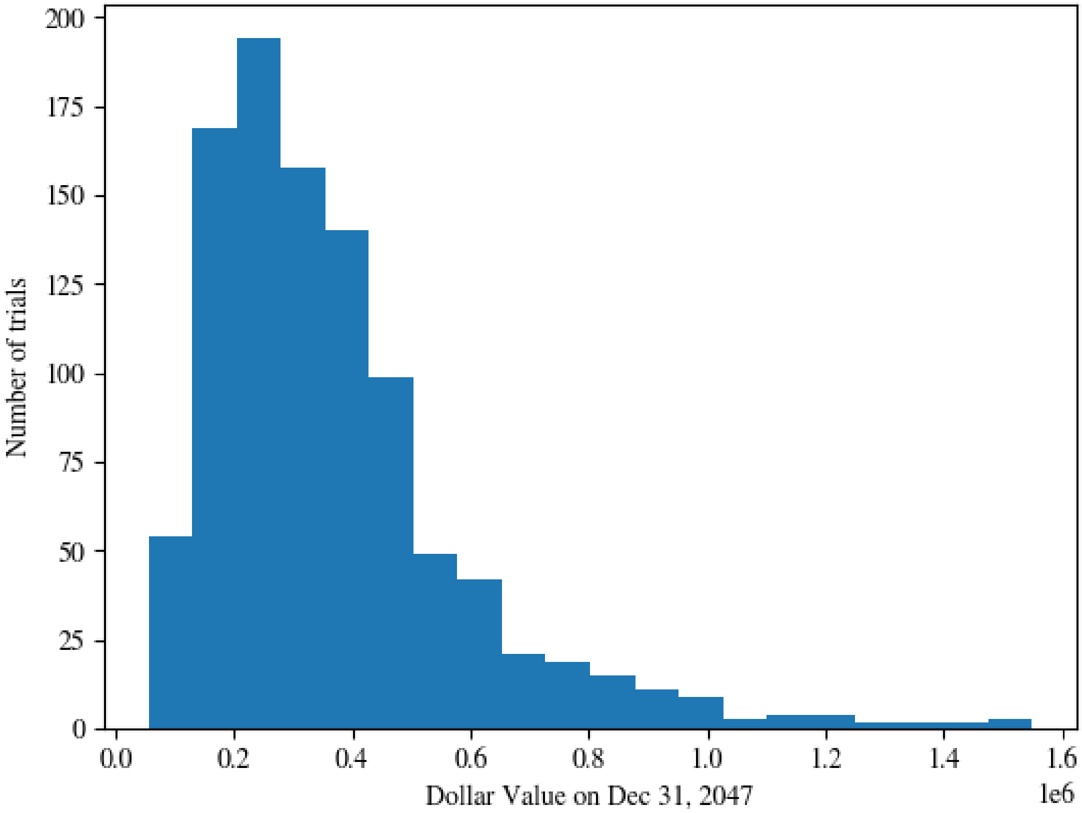
4 Retirement investments.

A. The percentage yield on an investment has a Gaussian distribution with mean of 8% and standard deviation (SD) of 15%. (A yield of 8% would mean the amount of money increases by a factor of 1.08 in a year. A yield of -8% would mean multiplying by 0.92 instead.) Suppose that you put $3000 into a retirement account investing in this item on January 1st of every year, starting in 2018. What is the mean amount of money you will have in the account on Dec 31, 2047? Show a plot of the distribution of the amount of money on that date for 1000 trials of the "experiment". What is the SD? Hand in your code or equivalent documentation.

I did this computationally, here are the results:

Mean value on Dec 31, 2047: 370000

Standard deviation: 220000



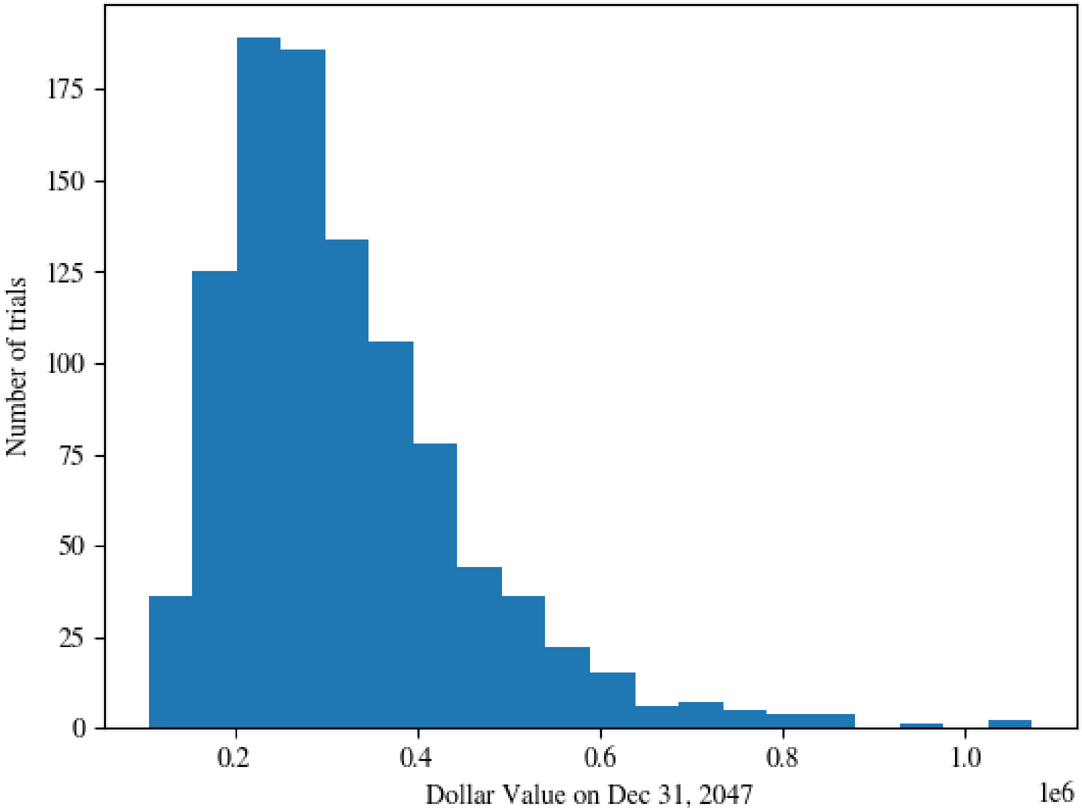
B. Suppose now that the retirement account contains three classes of investments: Canadian stocks, foreign stocks, and bonds. The yields on these three investments each vary randomly but with some correlation. Here is the yield information for each investment: $\mu_C = 8\%, \sigma_C = 15\%, \mu_F = 8\%, \sigma_F = 15\%, \mu_B = 5\%, \sigma_B = 7\%, \rho_{CF} = 0.50, \rho_{CB} = 0.20, \rho_{FB} = 0.05$. On January 1 of each year you put $1000 into each class of investment. Show the distribution of the total amount of money in your account on Dec 31, 2047. What are the mean and SD?

Let's do this computationally again, rather than using the hint (thanks though!) we can just use `numpy.random.multivariate_normal()` which takes a covariance matrix and means, to generate correlated Gaussians.

Mean value on Dec 31, 2047: 320000

Standard deviation: 140000

Plot of our results:

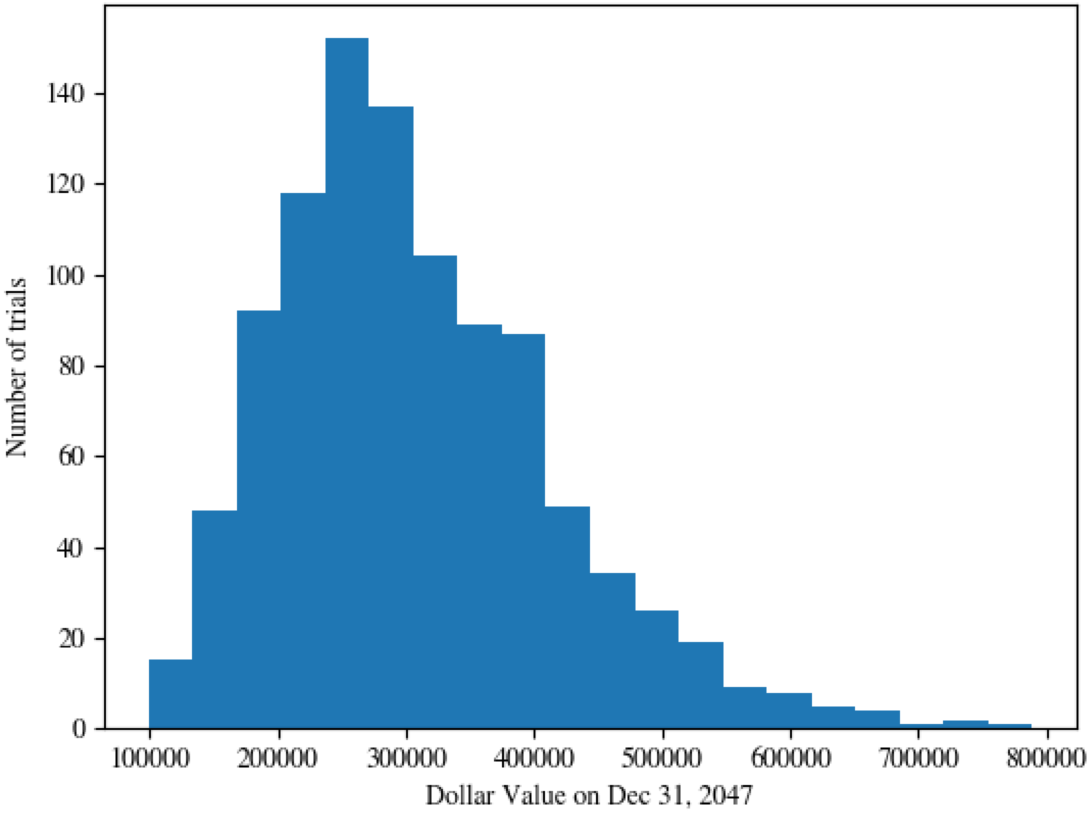


C. Now suppose we add a procedure called "rebalancing". On January 1 of each year we contribute a total of $3000 to the account, but at the same time we redistribute the total amount of money in the account evenly between the three investments. How does this change the total amount on Dec 31, 2047? Show a plot of the distribution, and report the mean and SD as well.

Mean value on Dec 31, 2047: 310000

Standard deviation: 110000

Plot of results:


Number of trials
0
20
40
60
80
100
120
140
100000
200000
300000
400000
500000
600000
700000
800000
Dollar Value on Dec 31, 2047