PHYS 509C Assignment 1

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September 22, 2022

Code for this assignment is here:
`https://github.com/callum-mccracken/PHYS-509C-A2`
(Many of the questions this time were quite long so I've paraphrased.)

1 Three easy applications of Bayes's theorem:

A. Suppose supernovae follow a poisson distribution with an unknown rate R. Calculate and plot the posterior distribution for R, given an observation of 4 supernovae in 10 centuries, using (a) a prior uniform in R and (b) a prior uniform in $\log_{10}(R)$.

Remember Bayes's theorem:

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{probability of data}}$$

The priors:

- Uniform in R, but we're not given bounds... Suppose it goes from some n to m.
$$P(R) = \frac{1}{m-n}.$$
- Uniform in $\log_{10}(R)$: use $P(x)dx = P(f(x))df(x)$ to get
$$\begin{aligned} P(R) &= P(\log_{10}(R))\frac{d\log_{10}(R)}{dR} \\ &= \frac{\ln(10)}{\ln(m)-\ln(n)}\frac{1}{\ln(10)R} \\ &= \frac{1}{\ln(m)-\ln(n)}\frac{1}{R} \end{aligned}$$

Likelihood: Poisson distribution ($T = 1000$ years)

$$P(k|R) = \frac{e^{-RT}(RT)^k}{k!}$$

Probability of Data:

$$P(k) = \int P(R)P(k|R)dR$$

for uniform R,

$$
\begin{aligned}
P(k) &= \int_0^\infty \frac{1}{m-n} \frac{e^{-RT}(RT)^k}{k!} dR \\
&= \frac{1}{m-n} \frac{1}{k!} \int_0^\infty e^{-RT}(RT)^k dR \\
&= \frac{1}{m-n} \frac{1}{k!} \frac{1}{T} \int_0^\infty e^{-\lambda}(\lambda)^k d\lambda \\
&= \frac{1}{m-n} \frac{1}{k!} \frac{1}{T} k! \\
&= \frac{1}{m-n} \frac{1}{T}
\end{aligned}
$$

or for uniform $\log_{10}(R)$

$$
\begin{aligned}
P(k) &= \int_0^\infty \frac{1}{\ln(m)-\ln(n)} \frac{1}{R} \frac{e^{-RT}(RT)^k}{k!} dR \\
&= \frac{1}{\ln(m)-\ln(n)} \int_0^\infty T \frac{1}{RT} \frac{e^{-RT}(RT)^k}{k!} dR \\
&= \frac{1}{\ln(m)-\ln(n)} \int_0^\infty \frac{e^{-\lambda}(\lambda)^{k-1}}{k!} d\lambda && [\lambda = RT] \\
&= \frac{1}{\ln(m)-\ln(n)} \frac{1}{k}
\end{aligned}
$$

Now calculate posteriors:

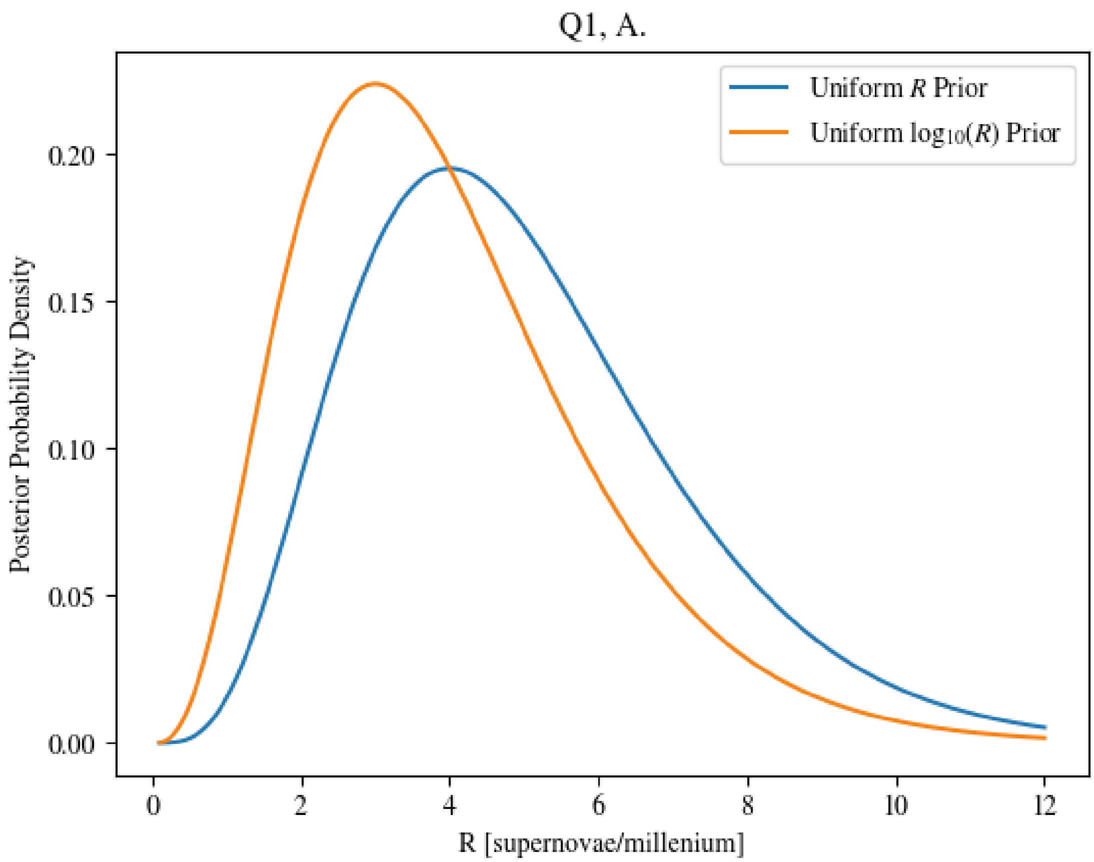
Uniform R:

$$
\begin{aligned}
P(R|k) &= \frac{P(k|R)P(R)}{P(k)} \\
&= \frac{\frac{e^{-RT}(RT)^k}{k!}\frac{1}{m-n}}{\frac{1}{m-n}\frac{1}{T}} \\
&= \frac{\frac{e^{-RT}(RT)^k}{k!}}{\frac{1}{T}} \\
&= \frac{Te^{-RT}(RT)^k}{k!}
\end{aligned}
$$

Uniform $\log_{10}(R)$

$$
\begin{aligned}
P(R|k) &= \frac{P(k|R)P(R)}{P(k)} \\
&= \frac{\frac{e^{-RT}(RT)^k}{k!}\frac{1}{\ln(m)-\ln(n)}\frac{1}{R}}{\frac{1}{\ln(m)-\ln(n)}\frac{1}{k}} \\
&= \frac{Te^{-RT}(RT)^{k-1}}{(k-1)!}
\end{aligned}
$$

Plot of these distributions:



B. Measurements are drawn from a uniform distribution spanning the interval (0, m). The probability of getting a measurement outside of this range is zero. The endpoint m is not well-known, but a prior experiment yields a Gaussian prior of m = 3 +/- 1. You take three measurements, getting values of 2.5, 3.1, and 2.9. Use Bayes' theorem to calculate and plot the new probability distribution for m.

Likelihood: if our model is a uniform distribution on $[0, m]$ and we have independent measurements (which I think we can assume for this question, right?), then $P(D|m) = \frac{1}{m^3}$ if $m \geq 3.1$, $= 0$ otherwise. That's the product of three uniform probabilities.

Prior (Gaussian, $\mu = 3, \sigma = 1$): $P(m) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(m-3)^2}{2}}$.

Probability of Data:

$$P(D) = \int_0^\infty P(m)P(D|m)dm \tag{1}$$

$$= \int_{3.1}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(m-3)^2}{2}} \frac{1}{m^3} dm \tag{2}$$

$$\tag{3}$$

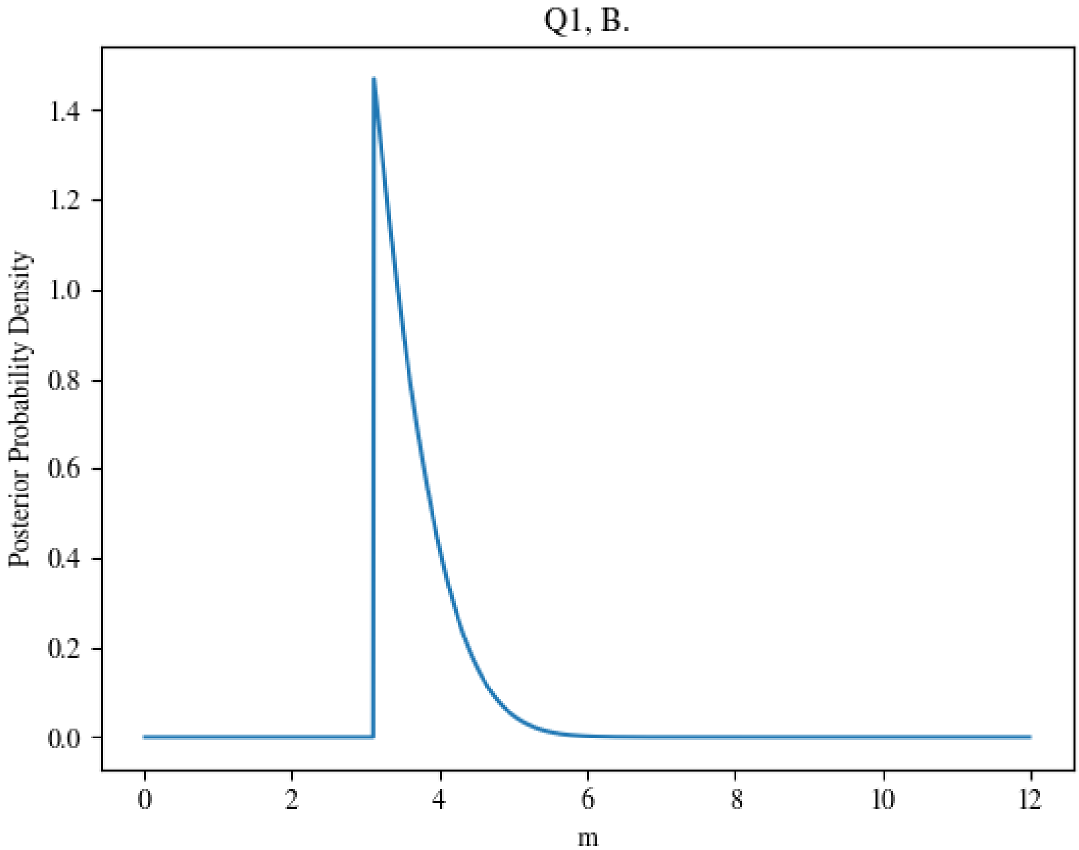
Posterior:

$$P(m|D) = \frac{P(D|m)P(m)}{P(D)}$$

$$= \frac{\frac{1}{m^3}\frac{1}{\sqrt{2\pi}}e^{-\frac{(m-3)^2}{2}}}{\int_{3.1}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{(m'-3)^2}{2}}\frac{1}{m'^3}dm'} \qquad [m \geq 3.1,\ \text{else } 0]$$

$$= \frac{\frac{1}{m^3}e^{-\frac{(m-3)^2}{2}}}{\int_{3.1}^\infty e^{-\frac{(m'-3)^2}{2}}\frac{1}{m'^3}dm'} \qquad [m \geq 3.1,\ \text{else } 0]$$

Plot:



C. Consider a person considering whether or not to launch a rocket with a possible malfunctioning component. In the control centre there is a warning light that is not completely reliable. During launch the warning light doesn't go on. From a costs standpoint, should she abort the mission or not? Compute and compare the expected cost of launching to the expected cost of aborting, given that the light didn't go on.

- $P(\text{light on}|\text{malfunction}) = 1/2$, $P(\text{light on}|\text{no malfunction}) = 1/3$
- $C(\text{no launch, no malfunction}) = 2M$, $C(\text{launch, malfunction}) = 5M$
- $C(\text{no launch, malfunction}) = C(\text{launch, no malfunction}) = 0$
- $\text{Prior}(\text{malfunction}) = 2/5$

Let F denote component failure, L denote the light being on, and A denote a launch (since L was taken).

Priors: $P(F) = \frac{2}{5}, P(\neg F) = \frac{3}{5}$.
Likelihood of observing data (no light): $P(\neg L|F) = \frac{1}{2}, P(\neg L|\neg F) = \frac{2}{3}$.
Probability of seeing our data (no light):

$$P(\neg L) = P(\neg L|F)P(F) + P(\neg L|\neg F)P(\neg F) = \frac{3}{5}$$

Posteriors:

$$\begin{aligned} P(F|\neg L) &= \frac{P(F)P(\neg L|F)}{P(\neg L)} = \frac{\frac{2}{5}\frac{1}{2}}{\frac{3}{5}} = \frac{1}{3} \\ P(\neg F|\neg L) &= 1 - P(F|\neg L) = \frac{2}{3} \end{aligned}$$

Costs:

$$\begin{aligned} C(A) &= P(F|\neg L)C(F|A) + P(\neg F|\neg L)C(\neg F|A) \\ &= \frac{1}{3}(5M) + \frac{2}{3}(0) \\ &= \frac{5}{3}M \end{aligned}$$

$$\begin{aligned} C(\neg A) &= P(F|\neg L)C(F|\neg A) + P(\neg F|\neg L)C(\neg F|\neg A) \\ &= \frac{1}{3}(0) + \frac{2}{3}(2M) \\ &= \frac{4}{3}M \end{aligned}$$

The expected cost of not launching is less, so from a cost standpoint, the mission should be aborted, and the rocket makers should probably invest in a better warning light for next time.

2 COVID-19 Study

- Sample size: 3330 people in Santa Clara County, California
- $N_{+,t} = 50$ positive test results in test group
- Control group 1: 3324 people, definitely negative
- $N_{+,c1} = 16$ positives in control 1
- Control group 2: 157 people, definitely negative
- $N_{+,c2} = 130$ positives in control 2 (27 false negatives)

Calculate the Bayesian 95% central interval on the fraction of people in Santa Clara County who actually had antibodies for COVID-19, marginalizing over the false positive and false negative rates. Assume flat priors on all parameters. Submit a plot of the posterior distribution for the true incidence rate as well as your code or calculation.

We want to find the distribution of the true positive rate in the population (and then get the 95% interval), $P(\text{real pos})$.

We have positive test probabilities.

A positive test could have come from a true positive or a false positive

$$\begin{aligned} P(\text{test, pos}) &= P(\text{test, true pos})P(\text{real, pos}) + P(\text{test, false pos})P(\text{real, neg}) \\ &= P(\text{test, true pos})P(\text{real, pos}) + P(\text{test, false pos})(1 - P(\text{real, pos})) \end{aligned}$$

$$\frac{P(\text{test, pos}) - P(\text{test, false pos})}{P(\text{test, true pos}) - P(\text{test, false pos})} = P(\text{real, pos})$$

The question says to marginalize over false positive and false negative rates, write it in terms of those

$$\frac{P(\text{test, pos}) - P(\text{test, false pos})}{1 - P(\text{test, false neg}) - P(\text{test, false pos})} = P(\text{real, pos})$$

And rewrite using shorter variable names (most variable names should be clear, I used A for antibodies, F=False, N=negative, P=positive):

$$\frac{P(P) - P(FP)}{1 - P(FN) - P(FP)} = P(A)$$

The priors:
$P(A) = P(P) = P(FP) = P(FN) = 1$ (uniform between 0 and 1)
Likelihood of seeing our data:
(Here our data is $P(FP) = \frac{16}{3324}, P(FN) = \frac{27}{157}$)

$$P(D|R_P, R_{FP}, R_{FN}) = P(D_1|R_P, R_{FP}, R_{FN})P(D_2|R_P, R_{FP}, R_{FN}) = \frac{R_P - R_{FP}}{1 - R_{FN} - R_{FP}}$$

Probability of Data (we have two):

$$P(D) = P()$$

for uniform R,

$$\begin{aligned}
P(k) &= \int_0^\infty \frac{1}{m-n}\frac{e^{-RT}(RT)^k}{k!}dR \\
&= \frac{1}{m-n}\frac{1}{k!}\int_0^\infty e^{-RT}(RT)^k dR \\
&= \frac{1}{m-n}\frac{1}{k!}\frac{1}{T}\int_0^\infty e^{-\lambda}(\lambda)^k d\lambda \\
&= \frac{1}{m-n}\frac{1}{k!}\frac{1}{T}k! \\
&= \frac{1}{m-n}\frac{1}{T}
\end{aligned}$$

or for uniform $\log_{10}(R)$

$$\begin{aligned}
P(k) &= \int_0^\infty \frac{1}{\ln(m)-\ln(n)}\frac{1}{R}\frac{e^{-RT}(RT)^k}{k!}dR \\
&= \frac{1}{\ln(m)-\ln(n)}\int_0^\infty T\frac{1}{RT}\frac{e^{-RT}(RT)^k}{k!}dR \\
&= \frac{1}{\ln(m)-\ln(n)}\int_0^\infty \frac{e^{-\lambda}(\lambda)^{k-1}}{k!}d\lambda \qquad [\lambda = RT] \\
&= \frac{1}{\ln(m)-\ln(n)}\frac{1}{k}
\end{aligned}$$

Now calculate posteriors:

Uniform R:

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P(R|k) &= \frac{P(k|R)P(R)}{P(k)} \\
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&= \frac{\frac{e^{-RT}(RT)^k}{k!}}{\frac{1}{T}} \\
&= \frac{Te^{-RT}(RT)^k}{k!}
\end{aligned}
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Uniform $\log_{10}(R)$

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\begin{aligned}
P(R|k) &= \frac{P(k|R)P(R)}{P(k)} \\
&= \frac{\frac{e^{-RT}(RT)^k}{k!}\frac{1}{\ln(m)-\ln(n)}\frac{1}{R}}{\frac{1}{\ln(m)-\ln(n)}\frac{1}{k}} \\
&= \frac{Te^{-RT}(RT)^{k-1}}{(k-1)!}
\end{aligned}
$$

3 CO_2 Meter

Assume that the data follows an exponential plateau, approaching some steady state value C.

Do a maximum likelihood fit for this level, and determine the uncertainty on it.

Note that you have not been given the uncertainties on the measured values—instead, assume that all measurements have the same uncertainty level, and fit for it as one of the parameters in your fit.

Submit your code, the functional form you fit, and your result for the steady state value (with uncertainty).

What uncertainty value per data point did you get?

How much of that uncertainty can be attributed to the time binning (measurements are only reported to the nearest minute)?

4 Retirement investments.

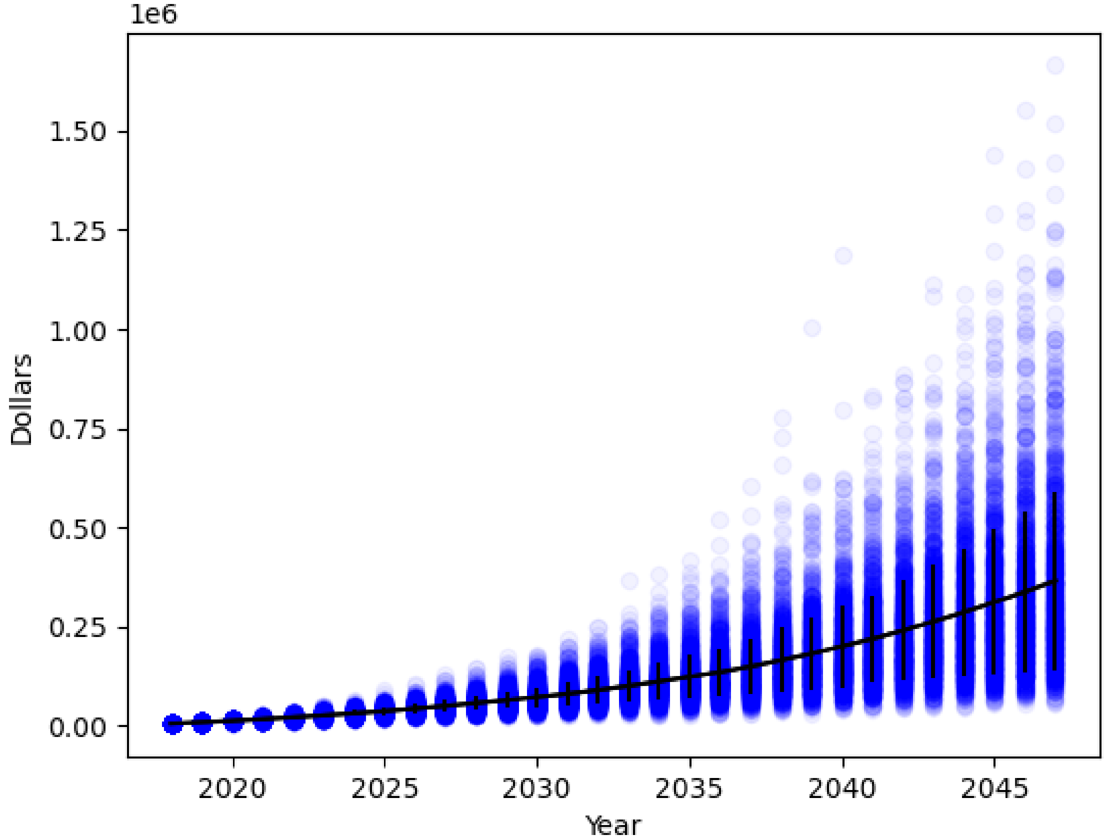
A. The percentage yield on an investment has a Gaussian distribution with mean of 8% and standard deviation (SD) of 15%. (A yield of 8% would mean the amount of money increases by a factor of 1.08 in a year. A yield of -8% would mean multiplying by 0.92 instead.) Suppose that you put $3000 into a retirement account investing in this item on January 1st of every year, starting in 2018. What is the mean amount of money you will have in the account on Dec 31, 2047? Show a plot of the distribution of the amount of money on that date for 1000 trials of the "experiment". What is the SD? Hand in your code or equivalent documentation.

I did this computationally, here are the results:

Value on Dec 31, 2047: 370000

Standard deviation: 228000

how to deal with dec 31st vs jan 1?



B. Suppose now that the retirement account contains three classes of investments: Canadian stocks, foreign stocks, and bonds. The yields on these three investments each vary randomly but with some correlation. Here is the yield information for each investment: $\mu_C = 8\%, \sigma_C = 15\%, \mu_F = 8\%, \sigma_F = 15\%, \mu_B = 5\%, \sigma_B = 7\%, \rho_{CF} = 0.50, \rho_{CB} = 0.20, \rho_{FB} = 0.05$. On January 1 of each year you put \$1000 into each class of investment. Show the distribution of the total amount of money in your account on Dec 31, 2047. What are the mean and SD?

Let's do this computationally again, but first let's explain some mystery numbers that will appear in the code, using the hint in the question.

$$C = \mathcal{N}(\mu = 0.08, \sigma = 0.15)$$

Then define a linear combination of C and F in terms of another Gaussian:

$$Y = C - F$$

$Y = \mathcal{N}(\mu = 0, \sigma_Y = \sigma_C)$.
Check this works out:

$$\begin{aligned}
\operatorname{Var}(Y) &= 1^2 \operatorname{Var}(C) + 1^2 \operatorname{Var}(F) - 2 \operatorname{Cov}(C, F) \\
\operatorname{Var}(Y) &= \sigma_C^2 + \sigma_F^2 - 2\rho_{CF}\sigma_C\sigma_F \\
\sigma_C^2 &= \sigma_C^2 + \sigma_C^2 - 2\frac{1}{2}\sigma_C^2 \\
\sigma_C^2 &= \sigma_C^2
\end{aligned}$$

$$\mu_Y = \mu_C - \mu_F = 0$$

So to generate F we can use this:

$$F = C - Y, Y = \mathcal{N}(\mu = 0, \sigma = \sigma_C)$$

Then to generate B, consider another combination: $Q = aY + bB$

$$\begin{aligned}
\mu_Q &= a\mu_Y + b\mu_B \\
\mu_Q &= a(0) + b(0.05) \\
\mu_Q &= b(0.05)
\end{aligned}$$

Let $b = 1$ for simplicity.

$$
\begin{aligned}
\operatorname{Var}(Q) &= a^2 \operatorname{Var}(Y) + \operatorname{Var}(B) + 2ab \operatorname{Cov}(Y, B) \\
&= a^2(0.15)^2 + (0.07)^2 + 2a(\operatorname{Cov}(C, B) - \operatorname{Cov}(F, B)) \\
&= a^2(0.15)^2 + (0.07)^2 + 2a(\rho_{C,B}\sigma_C\sigma_B - \rho_{F,B}\sigma_F\sigma_B) \\
&= a^2(0.15)^2 + (0.07)^2 + 2a((0.20)(0.15)(0.07) - (0.05)(0.15)(0.07)) \\
&= a^2(0.15)^2 + (0.07)^2 + 0.00315a
\end{aligned}
$$

We still have a free parameter, let $a = 1$

$$
\begin{aligned}
\sigma_Q^2 &= (0.15)^2 + 0.00315 + (0.07)^2 \\
\sigma_Q &= 0.174785583
\end{aligned}
$$

(there's some round-off error here but I assume it's small enough we can ignore it)

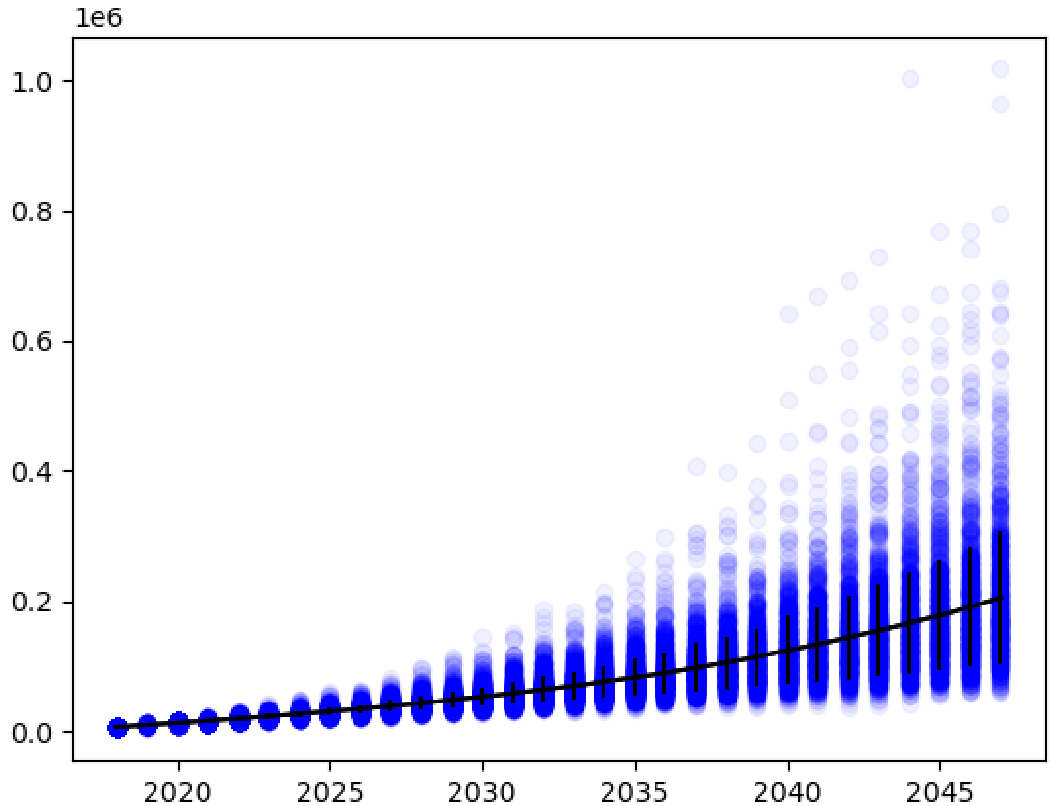
So $Q = Y + B, Q = \mathcal{N}(\mu = 0.05, \sigma = 0.174785583)$.

Results:

Value Dec 31, 2047: 204918.0565428472

Standard deviation: 102948.69611667504

Plot of our results:

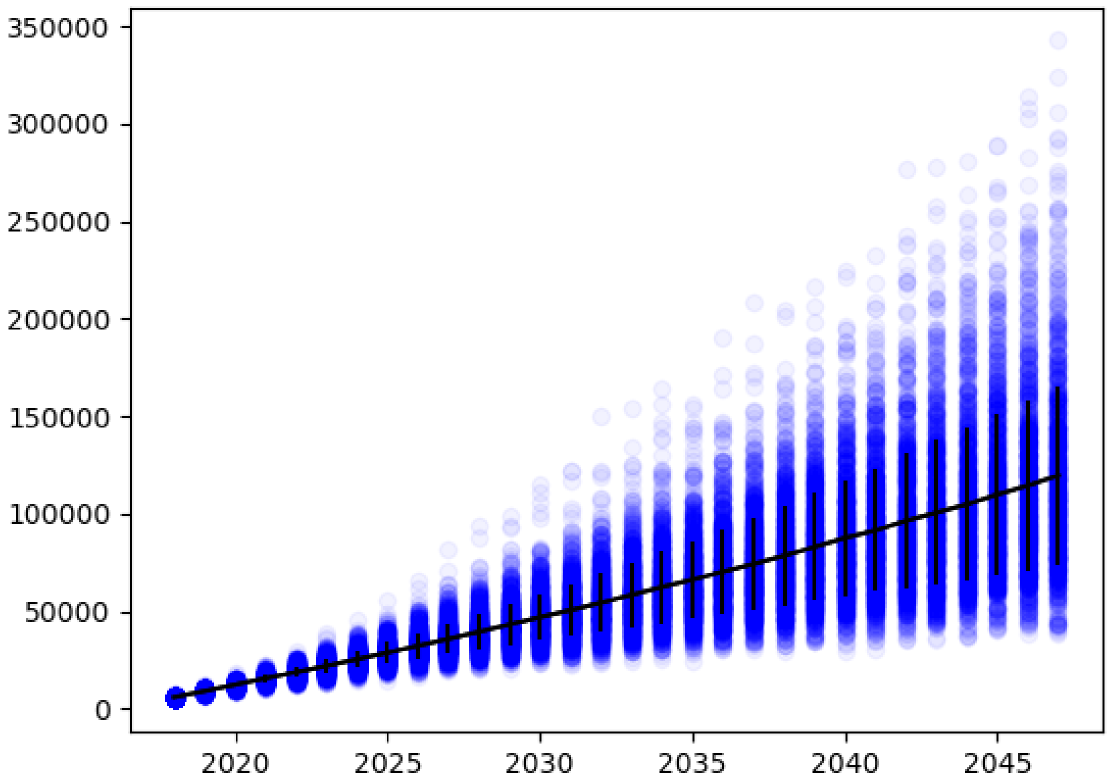


C. Now suppose we add a procedure called "rebalancing". On January 1 of each year we contribute a total of $3000 to the account, but at the same time we redistribute the total amount of money in the account evenly between the three investments. How does this change the total amount on Dec 31, 2047? Show a plot of the distribution, and report the mean and SD as well.

Value Dec 31, 2047: 119628.72535169181

Standard deviation: 45756.874877228125

Plot of results:



This feels kinda sussy, why is everything so much lower? Was there another constraint we could have used? Also this looks linear rather than exponential...