PHYS 509C Assignment 3

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Code for this assignment is here:
`https://github.com/callum-mccracken/PHYS-509C-A3`

1 S&P 500

A. Fit a Gaussian with ML.

For a Gaussian, $P(R) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(R-\mu)^2}{2\sigma^2}}$.

Calculate the negative log likelihood:

$$
\begin{aligned}
L &= \prod P(R_i) \\
-\ln(L) &= -\sum \ln P(R_i) \\
-\ln(L) &= -\sum \ln\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(R_i-\mu)^2}{2\sigma^2}}\right) \\
-\ln(L) &= \sum \left(-\ln\left(\frac{1}{\sqrt{2\pi}\sigma}\right) - \ln e^{-\frac{(R_i-\mu)^2}{2\sigma^2}}\right) \\
-\ln(L) &= \sum \left(\ln\left(\sqrt{2\pi}\sigma\right) + \frac{(R_i-\mu)^2}{2\sigma^2}\right)
\end{aligned}
$$

This can be minimized computationally (see the code). Doing so gives:

$$\mu_0 = 0.0, \sigma_0 = 0.0129$$

B. Fit a Laplace distribution, also with ML.

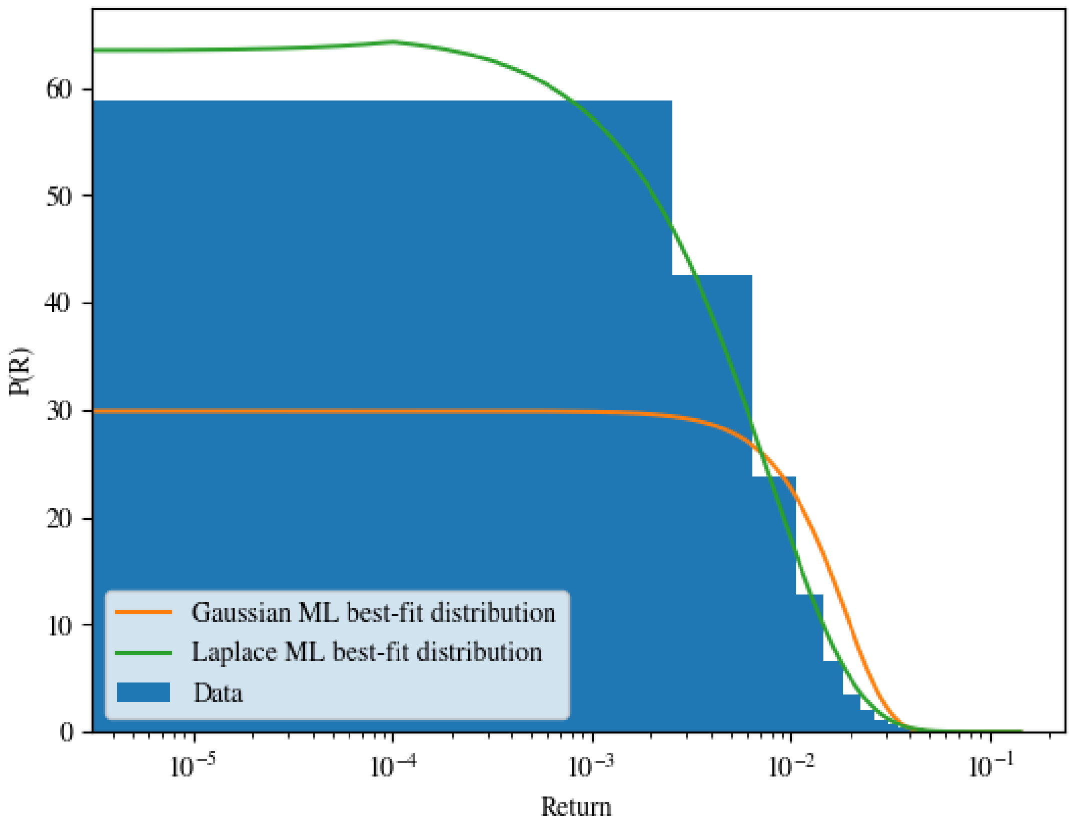
Our new distribution, $f(R) = \frac{1}{2B} e^{-\frac{|R-A|}{B}}$. Calculate the negative log likelihood:

$$
\begin{aligned}
L &= \prod f(R_i) \\
-\ln(L) &= -\sum \ln f(R_i) \\
-\ln(L) &= -\sum \ln\left(\frac{1}{2B} e^{-\frac{|R-A|}{B}}\right) \\
-\ln(L) &= \sum \left(-\ln\left(\frac{1}{2B}\right) - \ln e^{-\frac{|R-A|}{B}}\right) \\
-\ln(L) &= \sum \left(\ln(2B) + \frac{|R-A|}{B}\right)
\end{aligned}
$$

Again, minimize computationally, which gives:

$$A_0 = 0.0, B_0 = 0.00694$$

C. Plot a histogram on a log scale of the R_i and overlay each best fit. Which one looks better?



The Laplace distribution looks better.

D. The

2 Lambda CDM Cosmology

Given:

$$
\begin{aligned}
a(t) &= \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{t}{t_\Lambda}\right) \\
t_\Lambda &= \frac{2}{\left(3H_0\sqrt{\Omega_\Lambda}\right)} \\
a(t_{\text{universe}}) &= 1 \\
H_0 &= 67.27 \pm 0.60\,\text{km/s/Mpc} \\
\Omega_m &= 0.3166 \pm 0.0084 \\
\Omega_m H_0^3 &= 96433 \pm 290 \\
\Omega_m + \Omega_\Lambda &= 1
\end{aligned}
$$

Calculate the correlation between the uncertainties on H_0 and Ω_m:

How to deal with the correlation thing? Have we dealt with that in class?

Then use error propagation to estimate the age of the universe, with uncertainty:

3 Parameter Estimation With Supernovae

A. Some telescope measures luminosity at various redshifts. The redshift z is measured with negligible uncertainty. The distance D depends on redshift according to: $D = \frac{1}{H_0}(z + 0.5z^2(1 - q_0))$. H_0 = Hubble, q_0 = acc/deceleration, and depends on the densities of matter and dark energy in the universe according to $q0 = \Omega_M/2 - \Omega_\Lambda$. Assume $\Omega_M + \Omega_\Lambda = 1, \Omega_i \geq 0$. Apparent luminosity: $L = L_0/D^2$, where L_0 is its intrinsic brightness. The astronomical magnitude of each supernova is given by $m = -2.5\log_{10}(L)$. From studies of nearby supernovae, $\sigma_m = 0.1$, presumably due to some intrinsic random variation in the intrinsic brightness. Using the data file, determine the best-fit and "1 sigma" uncertainty for Ω_Λ from this data.

$$
\begin{aligned}
D &= \frac{1}{H_0}\left(z + \frac{1}{2}z^2(1 - q_0)\right) \\
q_0 &= \frac{\Omega_M}{2} - \Omega_\Lambda \\
\Omega_M + \Omega_\Lambda &= 1 : \Omega_M, \Omega_\Lambda > 0 \\
L &= \frac{L_0}{D^2} \\
m &= -2.5\log_{10}(L) \\
\sigma_m &= \pm 0.1
\end{aligned}
$$

Using data file, (col1 = z, col2 = m), find the best-fit and 1σ uncertainty for Ω_Λ.

$$
\begin{aligned}
D &= \frac{1}{H_0}\left(z + \frac{1}{2}z^2(1-q_0)\right)\\
q_0 &= \frac{\Omega_M}{2} - \Omega_\Lambda\\
\Omega_M + \Omega_\Lambda &= 1 : \Omega_M, \Omega_\Lambda > 0\\
L &= \frac{L_0}{D^2}\\
m &= -2.5\log_{10}(L)\\
\sigma_m &= \pm 0.1
\end{aligned}
$$

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B. A possible systematic uncertainty in this measurement: a such that $L_0(z) = L_0(1 + az)$. $a = 0 \pm 0.2$. Incorporating this as a new systematic to the calculation in Part A, calculate the total uncertainty on Ω_Λ.

4 **Magnesium has three stable isotopes with atomic weights of 24, 25, and 26. You are given one mole of enriched magnesium. The block weighs 25.2 grams. You do not know the fractions of Mg-24, Mg-25, and Mg-26 in the block, only the total mass.**

A. Let p_1, p_2, and p_3 be the fractions of Mg-24, Mg-25, and Mg-26 atoms in your sample. Obviously $p_1 + p_2 + p_3 = 1$. You also have the constraint that the total mass is 25.2g. Use maximum entropy principles to derive the joint probability distribution $P(p_1, p_2)$ that has the largest entropy given the constraints. (Hint: assume that the measure function $m(x)$ is constant when calculating the entropy of this continuous distribution – see the formula for the entropy of a continuous probability distribution in Gregory's book. Also, think carefully about the allowed ranges for each variable. The PDF won't depend upon p_3 because $p_1 + p_2 + p_3 = 1$ determines p_3.)

The formula from Gregory for a continuous distribution with m constant is:

$$S_c = -\int P(y)\ln(P(y))dy + \text{constant}$$

Or in our case,

$$S_c = -\int_0^1 \int_0^{1-p_1} P(p_1', p_2', 1-p_1-p_2)\ln(P(p_1', p_2', 1-p_1-p_2))dp_2'dp_1' + \text{constant}$$

And here we have the constraint $C = p_1 + p_2 + p_3 - 1 = 0$.

Maximize using Lagrange multipliers:

$$d(S_c - \lambda C) = 0$$
$$\sum_i \frac{\partial S_c}{\partial p_i}dp_i - \lambda\frac{\partial C}{\partial p_i}dp_i = 0$$

Equating dp_i terms:

$$\frac{\partial S_c}{\partial p_i} = \lambda \frac{\partial C}{\partial p_i}$$

$$-\frac{\partial}{\partial p_i} \int_0^1 \int_0^{1-p_1} P(p_1', p_2', 1-p_1-p_2) \ln(P(p_1', p_2', 1-p_1-p_2)) dp_2' dp_1' = \lambda(1)$$

Find λ using p_3:

$$-\frac{\partial}{\partial p_3} \int_0^1 \int_0^{1-p_1} P(p_1', p_2', 1-p_1-p_2) \ln(P(p_1', p_2', 1-p_1-p_2)) dp_2' dp_1' = \lambda$$

Then note that since $p_1 + p_2 = 1 - p_3$ and $p_i \in [0, 1]$,

To get the maximum entropy

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5 Jeffreys Priors.

$$g(\theta) \propto \sqrt{I(\theta)}$$
$$I(\theta) = \left\langle \left[\frac{\partial}{\partial\theta} \ln(L(x|\theta))\right]^2 \right\rangle = \int dx L(x|\theta) \left[\frac{\partial}{\partial\theta} \ln(L(x|\theta))\right]^2$$

A. Consider a measurement in which we flip a single coin once, and want to estimate the probability p for the coin coming up heads. Derive the Jeffreys prior $g(p)$ in this case.

Use the formula, where the likelihood of each hypothesis is $L(\text{head}|p) = p, L(\text{tail}|p) = 1 - p$. Let's define a head to have value $x = 0$ and tail value $x = 1$, so then we can write $L(x|p)$ as a sum of delta functions, $L(x|p) = \delta(x - 0)p + \delta(x - 1)(1 - p)$.

$$\begin{aligned}
\frac{\partial}{\partial p} \ln(L(x|p)) &= \frac{\partial}{\partial p} \ln(\delta(x-0)p + \delta(x-1)(1-p)) \\
&= \frac{\partial}{\partial p} \ln(\delta(x-0)p + \delta(x-1)(1-p)) \\
&= \frac{\partial}{\partial p} \ln((\delta(x-0) - \delta(x-1))p + \delta(x-1)) \\
&= \frac{\delta(x-0) - \delta(x-1)}{(\delta(x-0) - \delta(x-1))p + \delta(x-1)}
\end{aligned}$$

$$
\begin{aligned}
I(p) &= \int dx L(x|p)\left[\frac{\partial}{\partial p}\ln(L(x|p))\right]^2 \\
&= \int dx[\delta(x-0)p+\delta(x-1)(1-p)]\left[\frac{\delta(x-0)-\delta(x-1)}{(\delta(x-0)-\delta(x-1))p+\delta(x-1)}\right]^2 \\
&= \int dx \frac{[\delta(x-0)-\delta(x-1)]^2}{(\delta(x-0)-\delta(x-1))p+\delta(x-1)} \\
&= \frac{1}{p}+\frac{1}{1-p} \\
&= \frac{1}{p(1-p)} \\
g(p) &\propto \frac{1}{\sqrt{p(1-p)}}
\end{aligned}
$$

Find the constant of proportionality by normalizing:

$$
\begin{aligned}
\int_0^1 g(p)dp &= 1 \\
\int_0^1 A\frac{1}{\sqrt{p(1-p)}}dp &= 1 \\
A\pi &= 1 \\
A &= \frac{1}{\pi}
\end{aligned}
$$

$$
g(p) = \frac{1}{\pi\sqrt{p(1-p)}}
$$

B. Suppose that you start with this prior, then flip the coin three times, yielding three heads. What is the probability that $p < 0.5$?

Prior: $g(p) = \frac{1}{\pi\sqrt{p(1-p)}}$

Likelihood: $P(3 \text{ heads}|p) = \prod_i = 1^3 P(\text{heads}|p) = p^3$
Probability of data:

$$
\begin{aligned}
P(3 \text{ heads}) &= \int P(p) P(3 \text{ heads}|p) dp \\
&= \int \frac{1}{\pi\sqrt{p(1-p)}} p^3 dp \\
&= \int_0^1 \frac{p^3}{\pi\sqrt{p(1-p)}} dp \\
&= \frac{5}{16}
\end{aligned}
$$

Bayes's Theorem:

$$
\begin{aligned}
P(p|3 \text{ heads}) &= \frac{P(p)P(3 \text{ heads}|p)}{P(3 \text{ heads})} \\
&= \frac{\frac{1}{\pi\sqrt{p(1-p)}} p^3}{\frac{5}{16}} \\
&= \frac{16p^3}{5\pi\sqrt{p(1-p)}}
\end{aligned}
$$

So the probability that $p < 0.5$ is

$$
\begin{aligned}
P &= \int_0^{0.5} P(p|3 \text{ heads}) dp \\
&= \int_0^{0.5} \frac{16p^3}{5\pi\sqrt{p(1-p)}} dp \\
&= \frac{15\pi - 44}{30\pi} \\
&\approx 0.033
\end{aligned}
$$

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