PHYS 509C Assignment 3

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November 4, 2022

Code for this assignment is here:

`https://github.com/callum-mccracken/PHYS-509C-A3`

1 S&P 500

A. Fit a Gaussian with ML.

For a Gaussian, $P(R) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(R-\mu)^2}{2\sigma^2}}$.

Calculate the negative log likelihood:

$$
\begin{aligned}
L &= \prod P(R_i) \\
-\ln(L) &= -\sum \ln P(R_i) \\
-\ln(L) &= -\sum \ln\left(\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(R_i-\mu)^2}{2\sigma^2}}\right) \\
-\ln(L) &= \sum\left(-\ln\left(\frac{1}{\sqrt{2\pi}\sigma}\right) - \ln e^{-\frac{(R_i-\mu)^2}{2\sigma^2}}\right) \\
-\ln(L) &= \sum\left(\ln\left(\sqrt{2\pi}\sigma\right) + \frac{(R_i-\mu)^2}{2\sigma^2}\right)
\end{aligned}
$$

This can be minimized computationally (see the code). Doing so gives:

$$\mu_0 = 0.000108, \sigma_0 = 0.0129$$

B. Fit a Laplace distribution, also with ML.

Our new distribution, $f(R) = \frac{1}{2B} e^{-\frac{|R-A|}{B}}$. Calculate the negative log likelihood:

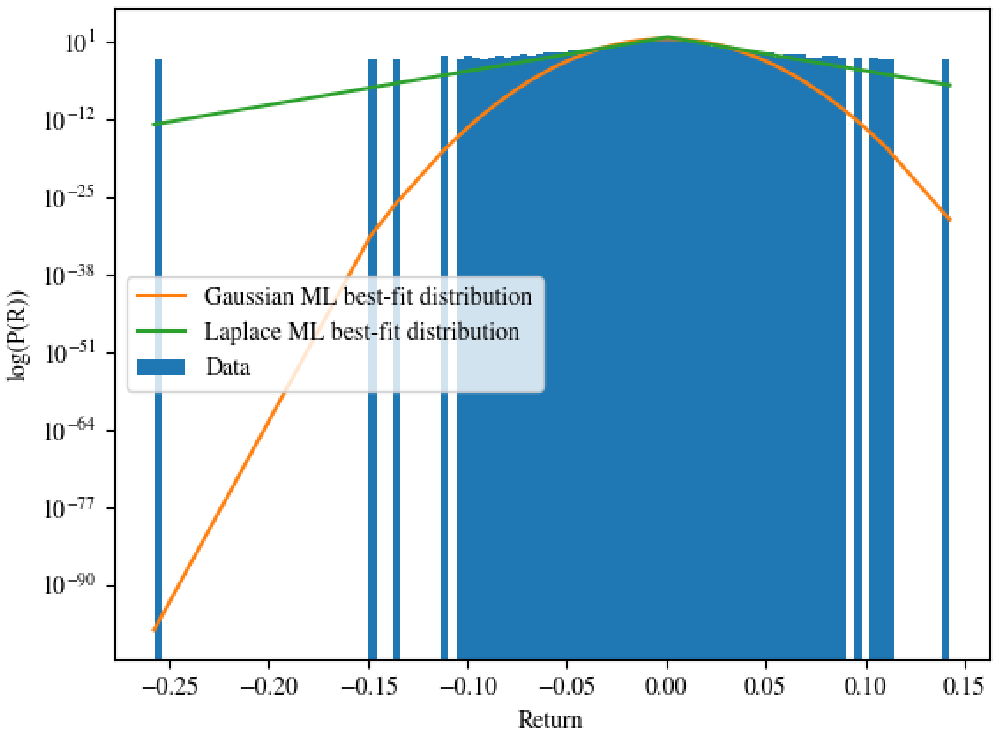
$$
\begin{aligned}
L &= \prod f(R_i) \\
-\ln(L) &= -\sum \ln f(R_i) \\
-\ln(L) &= -\sum \ln\left(\frac{1}{2B}e^{-\frac{|R-A|}{B}}\right) \\
-\ln(L) &= \sum\left(-\ln\left(\frac{1}{2B}\right) - \ln e^{-\frac{|R-A|}{B}}\right) \\
-\ln(L) &= \sum\left(\ln\left(2B\right) + \frac{|R-A|}{B}\right)
\end{aligned}
$$

Again, minimize computationally, which gives:

$$A_0 = 0.000488, B_0 = 0.00764$$

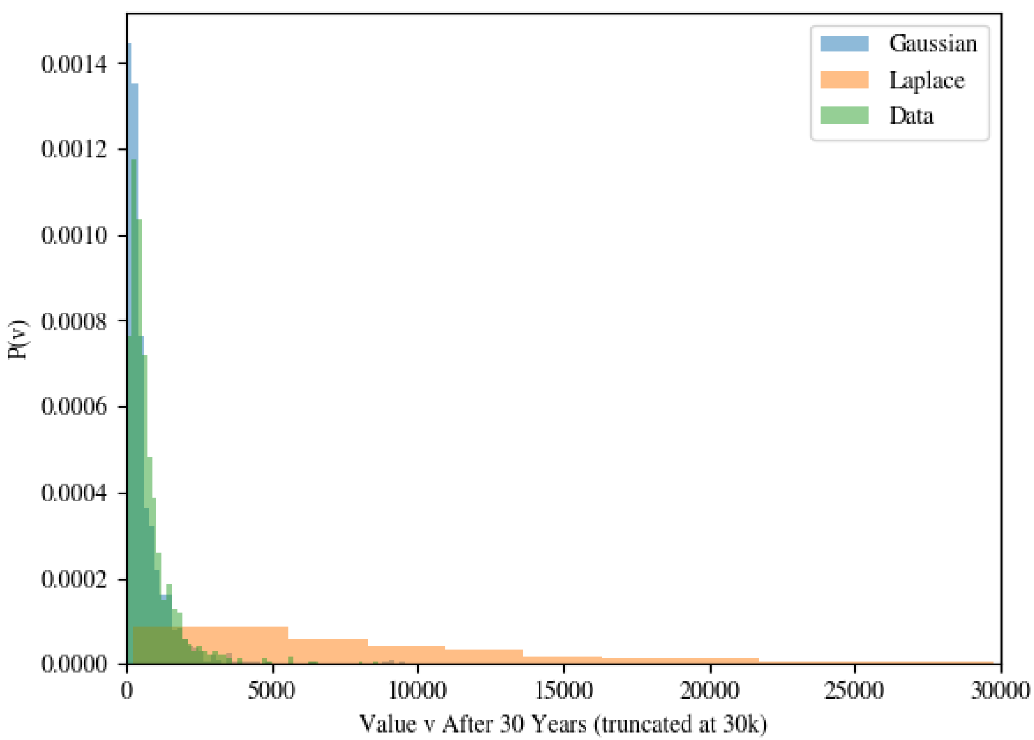
C. Plot a histogram on a log scale of the R_i and overlay each best fit. Which one looks better?

I'd say the Laplace distribution looks better.



D. Simulate the growth of a \$100 investment using the best-fit Laplacian, Gaussian, and drawing returns from the data.

See code for how the simulation was done. Here, it looks like the Gaussian might be a better fit.



2 Lambda CDM Cosmology

Given:

$$
\begin{aligned}
a(t) &= \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{t}{t_\Lambda}\right) \\
t_\Lambda &= \frac{2}{\left(3H_0\sqrt{\Omega_\Lambda}\right)} \\
a(t_{\text{universe}}) &= 1 \\
H_0 &= 67.27 \pm 0.60\,\text{km/s/Mpc} \\
\Omega_m &= 0.3166 \pm 0.0084 \\
\Omega_m H_0^3 &= 96433 \pm 290 \\
\Omega_m + \Omega_\Lambda &= 1
\end{aligned}
$$

Find the age of the universe in terms of H_0, Ω_Λ:

$$
\begin{aligned}
a(t_u) &= \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{t_u}{t_\Lambda}\right) \\
1 &= \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{\frac{1}{3}} \sinh^{\frac{2}{3}}\left(\frac{t_u}{t_\Lambda}\right) \\
\left(\frac{\Omega_\Lambda}{\Omega_m}\right)^{\frac{1}{3}} &= \sinh^{\frac{2}{3}}\left(\frac{t_u}{t_\Lambda}\right) \\
\sqrt{\frac{\Omega_\Lambda}{\Omega_m}} &= \sinh\left(\frac{t_u}{t_\Lambda}\right) \\
t_\Lambda \sinh^{-1}\left(\sqrt{\frac{\Omega_m}{\Omega_m}}\right) &= t_u \\
\frac{2}{3H_0\sqrt{1-\Omega_m}} \sinh^{-1}\left(\sqrt{\frac{1-\Omega_m}{\Omega_m}}\right) &= t_u
\end{aligned}
$$

Recall the error propagation equation with correlation, for a function $f(x, y)$:

$$\sigma_f^2 = \left(\frac{df}{dx}\right)^2 \sigma_x^2 + \left(\frac{df}{dy}\right)^2 \sigma_y^2 + 2\left(\frac{df}{dx}\right)\left(\frac{df}{dy}\right)\rho\sigma_x\sigma_y$$

We have uncertainties on H_0 and Ω_m, and also the uncertainty on $f(H_0, \Omega_m) = \Omega_m H_0^3$. Use that to get ρ, then plug that value into the same equation for t_u to get the uncertainty on t_u.

The computation is done with sympy (see `q2.py`), but result is

$$t_u = 13.803 \pm 0.024 \text{ Billion years.}$$

3 Parameter Estimation With Supernovae

A. Some telescope measures luminosity at various redshifts. The redshift z is measured with negligible uncertainty. The distance D depends on redshift according to: $D = \frac{1}{H_0}(z + 0.5z^2(1 - q_0))$. H_0 = Hubble, q_0 = acc/deceleration, and depends on the densities of matter and dark energy in the universe according to $q0 = \Omega_M/2 - \Omega_\Lambda$. Assume $\Omega_M + \Omega_\Lambda = 1, \Omega_i \geq 0$. Apparent luminosity: $L = L_0/D^2$, where L_0 is its intrinsic brightness. The astronomical magnitude of each supernova is given by $m = -2.5\log_{10}(L)$. From studies of nearby supernovae, $\sigma_m = 0.1$, presumably due to some intrinsic random variation in the intrinsic brightness. Using the data file, determine the best-fit and "1 sigma" uncertainty for Ω_Λ from this data.

We have these:

$$\begin{aligned}
D &= \frac{1}{H_0}\left(z + \frac{1}{2}z^2(1 - q_0)\right) \\
q_0 &= \frac{\Omega_M}{2} - \Omega_\Lambda \\
\Omega_M + \Omega_\Lambda &= 1 : \Omega_M, \Omega_\Lambda > 0 \\
L &= \frac{L_0}{D^2} \\
m &= -2.5\log_{10}(L) \\
\sigma_m &= \pm 0.1
\end{aligned}$$

Write m as a function of the other variables:

$$
\begin{aligned}
m &= -2.5\log_{10}(L) \\
&= -2.5\log_{10}\left(\frac{L_0}{D^2}\right) \\
&= -2.5\log_{10}\left(\frac{L_0}{\frac{1}{H_0^2}\left(z+\frac{1}{2}z^2(1-q_0)\right)^2}\right) \\
&= -2.5\log_{10}\left(\frac{L_0H_0^2}{\left(z+\frac{1}{2}z^2\left(1-\left(\frac{\Omega_M}{2}-\Omega_\Lambda\right)\right)\right)^2}\right) \\
&= -2.5\log_{10}\left(\frac{L_0H_0^2}{\left(z+\frac{1}{2}z^2\left(1-\left(\frac{1-\Omega_\Lambda}{2}-\Omega_\Lambda\right)\right)\right)^2}\right) \\
&= -2.5\log_{10}\left(\frac{L_0H_0^2}{\left(z+\frac{1}{2}z^2\left(1-\frac{1-\Omega_\Lambda}{2}+\Omega_\Lambda\right)\right)^2}\right) \\
&= -2.5\log_{10}\left(\frac{L_0H_0^2}{\left(z+\frac{1}{2}z^2\left(\frac{1+3\Omega_\Lambda}{2}\right)\right)^2}\right)
\end{aligned}
$$

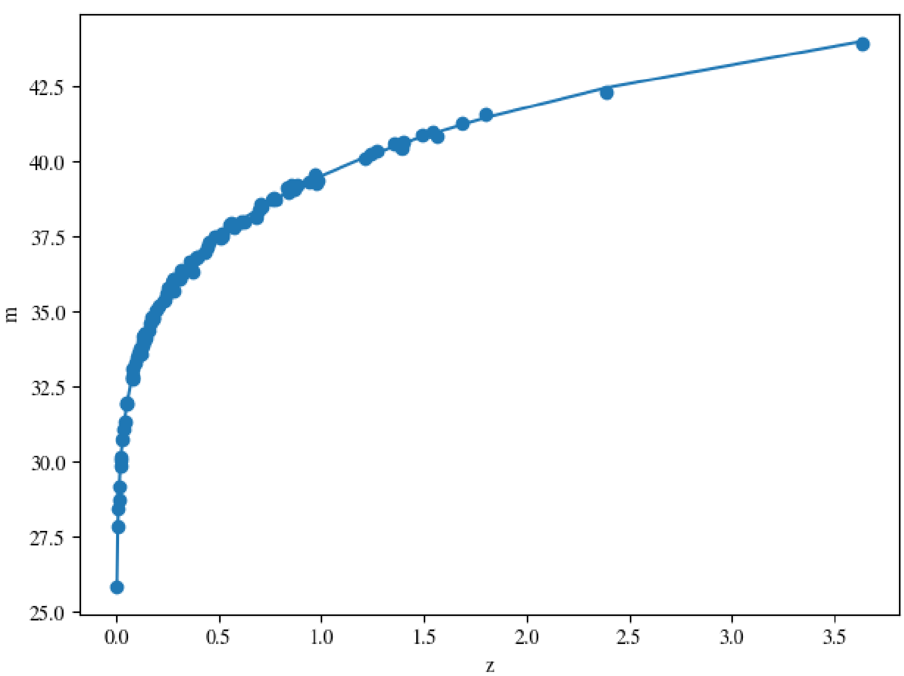
Since we're given the uncertainty on m, let's model this as a Gaussian a mean using the equation above, and SD σ_m:

$$
P(m) = \frac{1}{\sqrt{2\pi}\sigma_m}\exp\left(\frac{-\left(m+2.5\log_{10}\left(\frac{L_0H_0^2}{\left(z+\frac{1}{2}z^2\left(\frac{1+3\Omega_\Lambda}{2}\right)\right)^2}\right)\right)^2}{2\sigma_m}\right)
$$

$$L(\Omega_\Lambda, L_0H_0^2) = \prod_i \frac{1}{\sqrt{2\pi}\sigma_m} \exp\left(- \frac{\left(m_i + 2.5 \log_{10}\left(\frac{L_0H_0^2}{\left(z_i + \frac{1}{2}z_i^2\left(\frac{1+3\Omega_\Lambda}{2}\right)\right)^2} \right) \right)^2}{2\sigma_m} \right)$$

$$-\ln L(\Omega_\Lambda, L_0H_0^2) = \sum_i \ln\left(\sqrt{2\pi}\sigma_m\right) + \left(\frac{\left(m_i + 2.5 \log_{10}\left(\frac{L_0H_0^2}{\left(z_i + \frac{1}{2}z_i^2\left(\frac{1+3\Omega_\Lambda}{2}\right)\right)^2} \right) \right)^2}{2\sigma_m} \right)$$

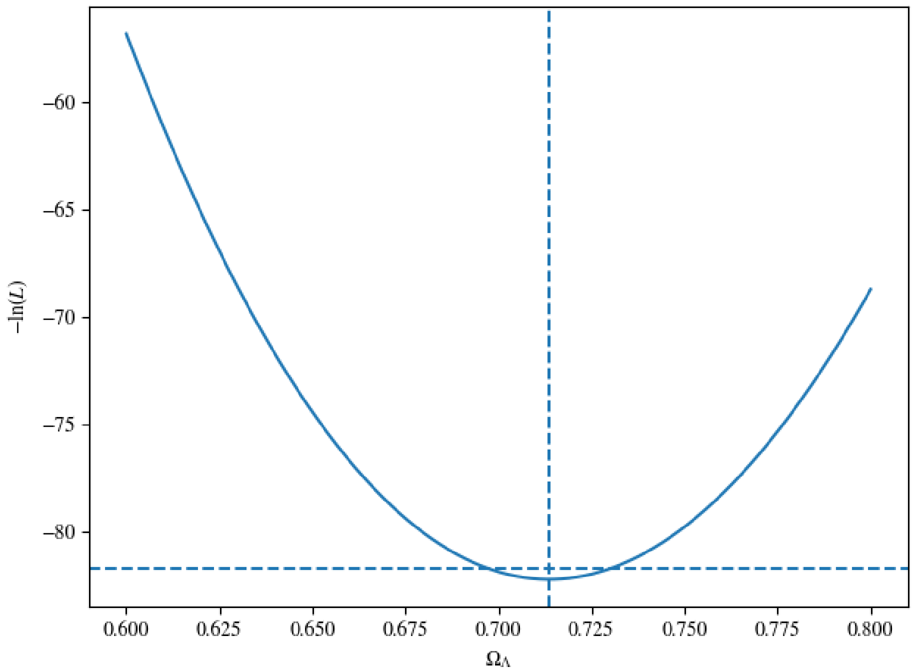
We can minimize this, to find the best-fit $H_0L_0^2$ and Ω_Λ.
The best-fit values: $\Omega_\Lambda = 0.714, H_0L_0^2 = 4.87 \times 10^{-16}$. A plot to make sure the fit actually worked:



We can sub in our best-fit $H_0L_0^2$ and find the points where $-\Delta \ln L = 0.5$ for "1σ" uncertainties.

See the code for details, but we end up with almost symmetric errors, $+0.0164$ and -0.0161:

$$\Omega_\Lambda = 0.714^{+0.0164}_{-0.0161}.$$



B. A possible systematic uncertainty in this measurement: a such that $L_0(z) = L_0(1 + az)$. $a = 0 \pm 0.2$. Incorporating this as a new systematic to the calculation in Part A, calculate the total uncertainty on Ω_Λ.

Proceeding in essentially the same way, except making the substitution $L_0 \to L_0(1+az)$ in the likelihood function, and adding an extra term into the likelihood function to deal with the prior on a – we took a Gaussian prior with mean 0, SD 0.2 and found a ML value of $a = 0.0247$, see code for details – we find:

$$\Omega_\Lambda = 0.753^{+0.0166}_{-0.0164}.$$

4 Magnesium has three stable isotopes with atomic weights of 24, 25, and 26. You are given one mole of enriched magnesium. The block weighs 25.2 grams. You do not know the fractions of Mg-24, Mg-25, and Mg-26 in the block, only the total mass.

A. Let p_1, p_2, and p_3 be the fractions of Mg-24, Mg-25, and Mg-26 atoms in your sample. Obviously $p_1 + p_2 + p_3 = 1$. You also have the constraint that the total mass is 25.2g. Use maximum entropy principles to derive the joint probability distribution $P(p_1, p_2)$ that has the largest entropy given the constraints. (Hint: assume that the measure function $m(x)$ is constant when calculating the entropy of this continuous distribution – see the formula for the entropy of a continuous probability distribution in Gregory's book. Also, think carefully about the allowed ranges for each variable. The PDF won't depend upon p_3 because $p_1 + p_2 + p_3 = 1$ determines p_3.)

The formula from Gregory for a continuous distribution with m constant is:

$$S_c = -\int P(y)\ln(P(y))dy + \text{constant}$$

Or in our case,

$$S_c = -\iiint P(p_1, p_2, p_3)\ln(P(p_1, p_2, p_3))dp_1dp_2dp_3 + \text{constant}$$

And here we have the constraint $p_1 + p_2 + p_3 = 1$, as well as $24p_1 + 25p_2 + 26p_3 = 25.2$.

When are both constraints satisfied?

$$
\begin{aligned}
p_1 + p_2 + p_3 &= 1 \\
\implies p_3 &= 1 - p_1 - p_2 \\
24p_1 + 25p_2 + 26(p_3) &= 25.2 \\
\implies 24p_1 + 25p_2 + 26(1 - p_1 - p_2) &= 25.2 \\
24p_1 + 25p_2 - 26p_1 - 26p_2 &= 25.2 - 26 \\
-2p_1 - p_2 &= -0.8 \\
p_2 &= 0.8 - 2p_1 \\
p_3 &= 1 - p_1 - p_2 \\
\implies p_3 &= 1 - p_1 - (0.8 - 2p_1) \\
p_3 &= 0.2 + p_1
\end{aligned}
$$

Use the constraints to get the triple integral down to a single integral:

$$
\begin{aligned}
S_c = -\iiint &\delta(p_1 + p_2 + p_3 - 1)\delta(p_2 = 0.8 - 2p_1) \\
&P(p_1, p_2, p_3)\ln(P(p_1, p_2, p_3))dp_1 dp_2 dp_3 + \text{constant} \\
= -\int_0^{p_1} &P(p_1')\ln(P(p_1'))dp_1' + \text{constant}
\end{aligned}
$$

Maximize the entropy:

$$
\begin{aligned}
\frac{dS_c}{dp_1} &= 0 \\
-\frac{d}{dp_1}\int_0^{p_1} P(p_1')\ln(P(p_1'))dp_1' + 0 &= 0 \\
\implies P(p_1)\ln(P(p_1)) &= 0
\end{aligned}
$$

This means either $P = 0$ (not possible for a probability distribution), or

$\ln(P) = 0$.

$$\begin{aligned} \ln(P(p_1)) &= 0 \\ \implies P(p_1) &= 1 \end{aligned}$$

To normalize, find the bounds on p_1:

$$\begin{aligned} 0 \leq p_2 &\leq 1 \\ 0 \leq 0.8 - 2p_1 &\leq 1 \\ -0.8 \leq -2p_1 &\leq 0.2 \\ 0.8 \geq 2p_1 &\geq -0.2 \\ 0.4 \geq p_1 &\geq -0.1 \\ \implies 0.4 \geq p_1 &\geq 0 \end{aligned}$$

$$\begin{aligned} 0 \leq p_3 &\leq 1 \\ 0 \leq 0.2 + p_1 &\leq 1 \\ -0.2 \leq p_1 &\leq 0.8 \\ \implies 0 \leq p_1 &\leq 0.8 \end{aligned}$$

Combining those, we see, $p_1 \in [0, 0.4]$. Normalize:

$$\int_0^{0.4} P(p_1) dp_1 = 0.4$$

So our probability distribution is:

$$P(p_1) = 2.5, p_1 \in [0, 0.4]$$

We could also write this with delta functions if we didn't want to specify a range:

$$P(p_1, p_2, p_3) = 2.5\delta(p_1 + p_2 + p_3 - 1)\delta(24p_1 + 25p_2 + 26p_3 - 25.2)$$

B. Suppose we measure $p_1 = \frac{12}{20}, p_2 = \frac{3}{20}, p_3 = \frac{5}{20}$.

Prior: $P(p_1) = 2.5, p_1 \in [0, 0.4]$

Likelihood: multinomial with $p_1, p_2 = 0.8 - 2p_1, p_3 = 0.2 + p_1$

$$P(n_1, n_2, n_3|p_1, p_2, p_3) = \frac{n!}{n_1!n_2!n_3!} p_1^{n_1} p_2^{n_2} p_3^{n_3}$$

$$P(n_1, n_2, n_3|p_1) = \frac{n!}{n_1!n_2!n_3!} p_1^{n_1} (0.8 - 2p_1)^{n_2} (0.2 + p_1)^{n_3}$$

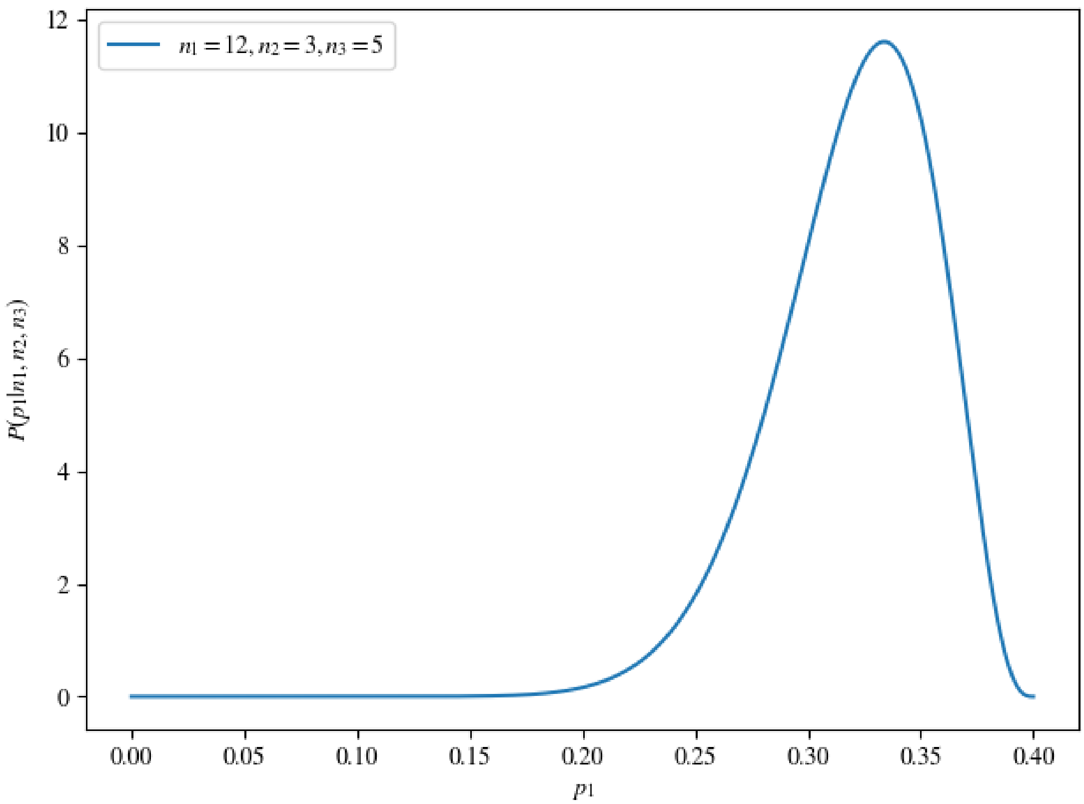
Prob of data:

$$\begin{aligned} P(n_1, n_2, n_3) &= \int_0^{0.4} P(p_1) P(n_1, n_2, n_3|p_1) dp_1 \\ &= \int_0^{0.4} P(p_1) \frac{n!}{n_1!n_2!n_3!} p_1^{n_1} (0.8 - 2p_1)^{n_2} (0.2 + p_1)^{n_3} dp_1 \end{aligned}$$

Posterior:

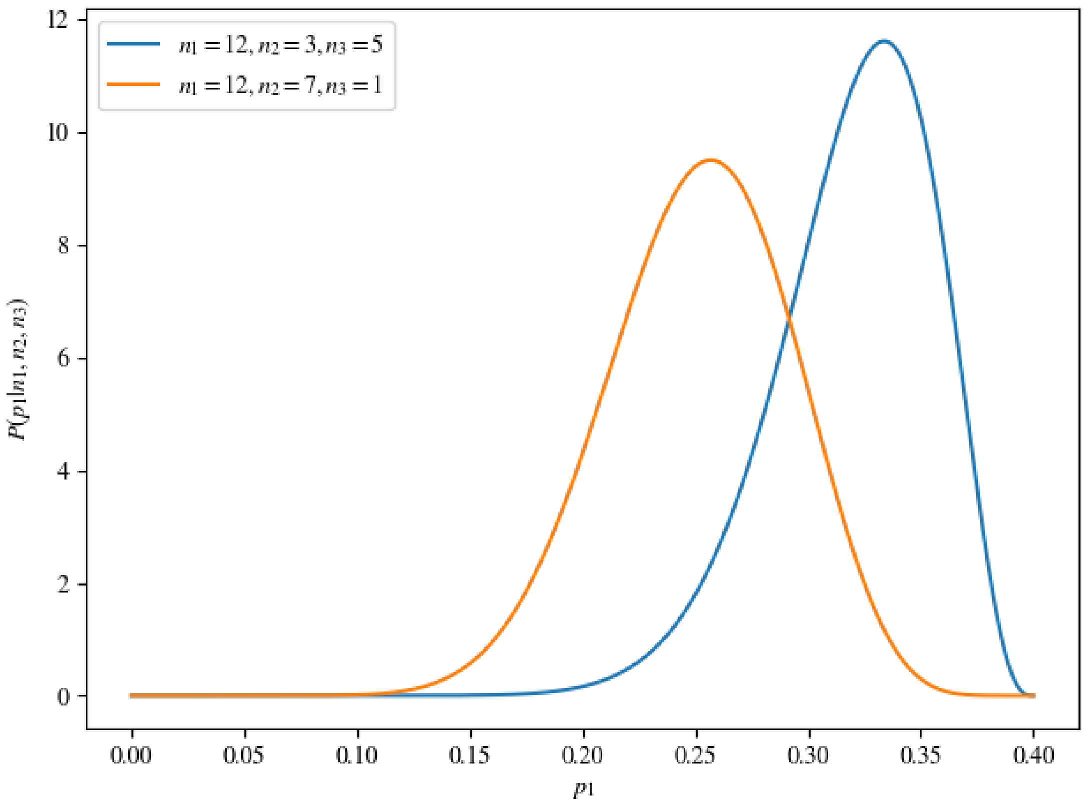
$$\begin{aligned} P(p_1|n_1, n_2, n_3) &= \frac{P(n_1, n_2, n_3|p_1) P(p_1)}{P(n_1, n_2, n_3)} \\ &= \frac{P(p_1) \frac{n!}{n_1!n_2!n_3!} p_1^{n_1} (0.8 - 2p_1)^{n_2} (0.2 + p_1)^{n_3}}{\int_0^{0.4} P(p_1) \frac{n!}{n_1!n_2!n_3!} p_1^{n_1} (0.8 - 2p_1)^{n_2} (0.2 + p_1)^{n_3} dp_1} \\ &= \frac{p_1^{n_1} (0.8 - 2p_1)^{n_2} (0.2 + p_1)^{n_3}}{\int_0^{0.4} p_1^{n_1} (0.8 - 2p_1)^{n_2} (0.2 + p_1)^{n_3} dp_1} \end{aligned}$$

If we go any further analytically it might be messy, but here's a plot:



C. If the data were instead $n_1 = 12, n_2 = 7, n_3 = 1$, what is the posterior? Argue why it makes sense.

It's the posterior from above, with new n values, here's a plot of both:



They're not same, which makes sense since although the posterior doesn't depend on p_2 and p_3, it does depend on n_2 and n_3.

5 Jeffreys Priors.

$$g(\theta) \propto \sqrt{I(\theta)}$$
$$I(\theta) = \left\langle \left[\frac{\partial}{\partial\theta} \ln(L(x|\theta)) \right]^2 \right\rangle = \int dx L(x|\theta) \left[\frac{\partial}{\partial\theta} \ln(L(x|\theta)) \right]^2$$

A. Consider a measurement in which we flip a single coin once, and want to estimate the probability p for the coin coming up heads. Derive the Jeffreys prior $g(p)$ in this case.

Use the formula, where the likelihood of each hypothesis is binomial, with heads = "success", i.e. $L(n \text{ heads in } N \text{ trials}|p) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$.

$L(n|N,p) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$

$$\begin{aligned}
\frac{\partial}{\partial p} \ln(L(n|N,p)) &= \frac{\partial}{\partial p} \ln\left(\frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} \right) \\
&= \frac{\partial}{\partial p} \ln\left(\frac{N!}{n!(N-n)!} \right) + n\ln(p) + (N-n)\ln(1-p) \\
&= 0 + \frac{n}{p} - \frac{N-n}{1-p}
\end{aligned}$$

Here we flip a single coin once, so $N = 1, n \in \{0, 1\}$

$$\begin{aligned}
L(n|p) &= p^n (1-p)^{1-n} \\
\frac{\partial}{\partial p} \ln(L(n|p)) &= \frac{n}{p} - \frac{1-n}{1-p}
\end{aligned}$$

$$
\begin{aligned}
I(p) &= \left\langle \left[\frac{\partial}{\partial p} \ln(L(n|p)) \right]^2 \right\rangle \\
&= \left\langle \left[\frac{n}{p} - \frac{1-n}{1-p} \right]^2 \right\rangle \\
&= \left\langle \frac{(n-p)^2}{p^2(1-p)^2} \right\rangle \\
&= \sum_n L(n|p) \frac{(n-p)^2}{p^2(1-p)^2} \\
&= L(0|p) \frac{(-p)^2}{p^2(1-p)^2} + L(1|p) \frac{(1-p)^2}{p^2(1-p)^2} \\
&= (1-p) \frac{1}{(1-p)^2} + p \frac{1}{p^2} \\
&= \frac{1}{1-p} + \frac{1}{p} \\
&= \frac{1}{p(1-p)} \\
g(p) &\propto \frac{1}{\sqrt{p(1-p)}}
\end{aligned}
$$

Find the constant of proportionality by normalizing:

$$
\begin{aligned}
\int_0^1 g(p) dp &= 1 \\
\int_0^1 A \frac{1}{\sqrt{p(1-p)}} dp &= 1 \\
A\pi &= 1 \\
A &= \frac{1}{\pi}
\end{aligned}
$$

$$g(p) = \frac{1}{\pi\sqrt{p(1-p)}}$$

B. Suppose that you start with this prior, then flip the coin three times, yielding three heads. What is the probability that $p < 0.5$?

Prior: $g(p) = \frac{1}{\pi\sqrt{p(1-p)}}$

Likelihood: $P(3 \text{ heads}|p) = \prod_{i=1}^{3} P(\text{heads}|p) = p^3$

Probability of data:

$$\begin{aligned} P(3 \text{ heads}) &= \int P(p)P(3 \text{ heads}|p)dp \\ &= \int \frac{1}{\pi\sqrt{p(1-p)}} p^3 dp \\ &= \int_0^1 \frac{p^3}{\pi\sqrt{p(1-p)}} dp \\ &= \frac{5}{16} \end{aligned}$$

Bayes's Theorem:

$$\begin{aligned} P(p|3 \text{ heads}) &= \frac{P(p)P(3 \text{ heads}|p)}{P(3 \text{ heads})} \\ &= \frac{\frac{1}{\pi\sqrt{p(1-p)}} p^3}{\frac{5}{16}} \\ &= \frac{16p^3}{5\pi\sqrt{p(1-p)}} \end{aligned}$$

So the probability that $p < 0.5$ is

$$\begin{aligned} P &= \int_0^{0.5} P(p|3\text{ heads})dp \\ &= \int_0^{0.5} \frac{16p^3}{5\pi\sqrt{p(1-p)}}dp \\ &= \frac{15\pi - 44}{30\pi} \\ &\approx 0.033 \end{aligned}$$

C. Suppose $p = \psi^4$. Derive the Jeffreys prior for ψ, starting with the likelihood for a single coin flip expressed as a function of ψ.

Use the Jeffreys prior formula again, where the likelihood is our binomial from before, with $p = \psi^4$. Note that we take $\psi \in [0, 1]$, we could also use $\psi \in [-1, 1]$ but this would mean we have to worry about absolute values, which are yucky.

$$\begin{aligned} L(n|N, \psi) &= L(n|N, p = \psi^4) \\ &= \frac{N!}{n!(N-n)!}\psi^{4n}(1-\psi^4)^{N-n} \end{aligned}$$

$$\begin{aligned} L(n|N = 1, \psi) &= \psi^{4n}(1-\psi^4)^{1-n} \\ \ln(L(n|\psi)) &= 4n\ln(\psi) + (1-n)\ln\big(1-\psi^4\big) \\ \frac{\partial}{\partial\psi}\ln(L(n|\psi)) &= 4n\frac{1}{\psi} + (1-n)\frac{1}{1-\psi^4}(-4\psi^3) \\ &= \frac{4n(1-\psi^4)}{\psi(1-\psi^4)} + \frac{(1-n)(-4\psi^4)}{\psi(1-\psi^4)} \\ &= \frac{4(n-\psi^4)}{\psi(1-\psi^4)} \end{aligned}$$

$$
\begin{aligned}
I(\psi) &= \left\langle \left[\frac{\partial}{\partial \psi} \ln(L(n|\psi)) \right]^2 \right\rangle \\
&= \left\langle \left[\frac{4(n-\psi^4)}{\psi(1-\psi^4)} \right]^2 \right\rangle \\
&= \left\langle \frac{16(n-\psi^4)^2}{\psi^2(1-\psi^4)^2} \right\rangle \\
&= \sum_n L(n|\psi) \frac{16(n-\psi^4)^2}{\psi^2(1-\psi^4)^2} \\
&= L(0|\psi) \frac{16(0-\psi^4)^2}{\psi^2(1-\psi^4)^2} + L(1|\psi) \frac{16(1-\psi^4)^2}{\psi^2(1-\psi^4)^2} \\
&= (1-\psi^4) \frac{16\psi^6}{(1-\psi^4)^2} + \psi^4 \frac{16}{\psi^2} \\
&= \frac{16\psi^6}{1-\psi^4} + \frac{16\psi^2(1-\psi^4)}{1-\psi^4} \\
&= \frac{16\psi^2}{1-\psi^4} \\
g(\psi) &\propto \frac{4\psi}{\sqrt{1-\psi^4}}
\end{aligned}
$$

Find the constant of proportionality by normalizing:

$$
\begin{aligned}
\int_0^1 g(\psi) d\psi &= 1 \\
\int_0^1 A \frac{4\psi}{\sqrt{1-\psi^4}} d\psi &= 1 \\
A\pi &= 1 \\
A &= \frac{1}{\pi}
\end{aligned}
$$

$$g(\psi) = \frac{4\psi}{\pi\sqrt{1-\psi^4}}$$

D. Demonstrate explicitly that if you take the Jeffreys prior for ψ from Part C and do a change of variables to p, you get back the Jeffreys prior for p that you found in part A. This will confirm that Jeffreys' procedure for generating priors encodes the same information for both of these parametrizations.

$$\begin{aligned}
g(\psi) &= \frac{4\psi}{\pi\sqrt{1-\psi^4}} \\
g(\psi) &= g(p)\left|\frac{dp}{d\psi}\right| \\
&= \frac{1}{\pi\sqrt{p(1-p)}}\left|\frac{d}{d\psi}\psi^4\right| \\
&= \frac{1}{\pi\sqrt{\psi^4(1-\psi^4)}}\left|4\psi^3\right| \\
&= \frac{4}{\pi\sqrt{\psi^4(1-\psi^4)}}\psi^2|\psi| \\
&= \frac{4\psi}{\pi\sqrt{(1-\psi^4)}}
\end{aligned}$$

(we can drop the absolute value since we've defined ψ to be positive)

E. Finally, show that if you started with a uniform prior for p and a uniform prior for ψ, then these priors are actually different after converting from one parametrization to another with a change of variables. Thus a uniform prior is not a Jeffreys prior for this problem.

Uniform for p: $P(p) = 1, p \in [0, 1]$.

Change of variables from $p \to \psi$:

$$\begin{aligned} P(\psi) &= P(p) \left| \frac{dp}{d\psi} \right| \\ &= (1) 4\psi^3 \\ &\neq \text{uniform distribution} \end{aligned}$$