PHYS 509C Assignment 4

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Code for this assignment is here:

`https://github.com/callum-mccracken/PHYS-509C-A4`

1 Medical Trials

A. A medical study tests a treatment on 100 patients. It is known that there is a 50% chance that a patient, if untreated, will get better naturally. (Else the patient dies!) The researcher wants to see if the treatment increases the recovery rate. She designs a study to test the treatment: she will test it on 100 patients, and will reject the null hypothesis at the 95% confidence level if enough patients recover. How many of the 100 patients must survive at the end of the study in order for her to reject the null hypothesis under these conditions?

B. Her hospital's medical ethics board advises her that if the treatment proves to be very effective, then it would be unethical to continue the study. Instead, she should end the study early and publish the results so that other patients can benefit from the treatment. Therefore she modifies the study. Starting after the first 25 patients are treated, she counts up how many patients have recovered, and calculates the probability that at least that many would have recovered just by chance. If this probability is less than 1%, she will end the study immediately and reject the null hypothesis, concluding that the treatment is effective. She continues to calculate this probability after each additional patient is treated until the treatment has proven effective or until she has treated 100 patients. The treatment is deemed successful if either the study ended early due to its apparent effectiveness, or if after 100 patients the number of recovered patients is greater than that calculated in Part A. In these two cases she will either write a paper saying that the treatment proved effective at the 99% CL or at the 95% CL, depending on whether the trial ended early or not. Suppose that in reality the treatment has no effect on patient outcomes. What is the probability that the null hypothesis is rejected anyway? What is the probability that researcher publishes a paper rejecting the null hypothesis at the 99% CL?

2 Chi-squared fits with systematics.

A. A theory predicts that a variable y depends on a variable x according to: $y = 3x^2 - 1$. A dataset is obtained. The resolution on each y measurement is 0.02. Use a χ^2 statistic to test whether the data are consistent with the theory. Quote a p-value.

The dataset: x y

0.1 -0.951

0.2 -0.842

0.3 -0.741

0.4 -0.492

0.5 -0.229

0.6 0.118

0.7 0.494

0.8 0.957

0.9 1.449

1.0 2.055

B. Your graduate student now comes to you with worries about a possible systematic on the measured y values. She suspects that each y value could be shifted by an amount $dy = ax$, where a is some constant. Through diligent work she has determined that $a = 0 \pm 0.05$. Repeat the calculation of Part A, this time including the effects of this systematic uncertainty.

3 Consider flipping an unfair coin ten times, and getting 10 heads. Calculate the Feldman-Cousins 90% confidence interval for p, the probability of getting heads on the coin. Submit a copy of your code or equivalent.

Since we're told this is an unfair coin but nothing else, a uniform prior $P(p) = 1, p \in [0, 1]$ seems like a reasonable choice.

Find our maximum likelihood p_{best}, using d to refer to the data of getting 10 heads in 10 tries:

$$\begin{aligned} L(n|p, N) &= \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} \\ L(d|p) &= \frac{10!}{10!(10-10)!} p^{10} (1-p)^{10-10} \\ &= p^{10} \end{aligned}$$

Normalize:

$$L(d|p) = 11p^{10}$$

This is clearly maximized (over $[0, 1]$) when $p = 1$.

$$\implies p_{\text{best}} = 1$$

Now calculate the likelihood ratio:

$$\begin{aligned} R &= \frac{L(d|p)}{L(d|p_{\text{best}})} \\ &= \frac{11p^{10}}{111^{10}} \\ &= p^{10} \end{aligned}$$

Now start from the highest R and go to lower R until the integral of L is 0.9. Conveniently $R = L$ here, and the highest R is at $p = 1$. Call the lower bound on the interval a.

$$\begin{aligned}
0.9 &= \int_a^1 L dp \\
&= 11 \int_a^1 p^{10} dp \\
&= 11 \left. \frac{p^{11}}{11} \right|_a^1 \\
&= 1 - a^{11} \\
0.1 &= a^{11} \\
a &= \sqrt[11]{0.1} \approx 0.811
\end{aligned}$$

So the Feldman-Cousins confidence interval is $[\sqrt[11]{0.1}, 1]$.

4 Fitting with correlated noise in the time domain.

Suppose that we measure a time-varying noise signal $n(t)$ that is sampled at time intervals Δt. We record the value at times $t_k = k\Delta t$, where k ranges from 0 to $N-1$, and N is the number of samples recorded. Assume that N is an even number. The function $n(t)$ can be written as a discrete Fourier transform:

$$n(t) = \sum_{m=0}^{N-1} [A_m \cos(m\omega_0 t) + B_m \sin(m\omega_0 t)]$$

where $\omega_0 = \frac{2\pi}{N\Delta t}$. (Note that one may set $B_0 = B_{N/2} = 0$ without loss of generality here, since the sine function will equal zero if $m = 0$, and at any discrete time $t_k = k\Delta t$ if $m = N/2$.) We may collectively refer to the set of N coefficients $A_0 ... A_{N/2}, B_1 ... BN/2 - 1$ as $\tilde{n}$.

A common case is Gaussian stationary noise. This can be defined as noise for which, at every frequency $f_m = \frac{m\omega_0}{2\pi}$, A_m and B_m are two indepen-dent random variables distributed as Gaussians with mean of zero and standard deviation of σ_m. (Of course $B_0 = B_{N/2} = 0$ still.) The quantity σ_m^2 is proportional to a quantity called the "power spectral density" of the noise. If σ_m has a constant value σ for all frequencies (that is, $\sigma_m = \sigma$ for all values of m), then $n(t)$ is called "white noise".

this question has a note!

A. Consider the measurements $n(t_1)$ and $n(t_2)$ taken at two possibly different times $t_1 = k_1\Delta t$ and $t_2 = k_2\Delta t$. Derive a formula for the covariance $\mathrm{cov}(n(t_1), n(t_2))$. Calculate the mean and variance of $n(t_k)$.

B. Suppose we are trying to fit a function $C_s(t)$ to some measured time series, where $s(t)$ is a known shape and C is an unknown normalization we would like to fit for. Our model for the measured data $g(t)$ is $g(t) = C_s(t) + n(t)$, where $n(t)$ is the randomly generated noise from our stationary noise model described above. If we write down a least squares fit directly using the N data points $g(t_k)$, we would find that they have a non-trivial covariance matrix (see Part A). But suppose that we take a discrete Fourier transform of $g(t)$ and $s(t)$ to get some sets of coefficients $\tilde{g}$ and $\tilde{s}$, analogous to $\tilde{n}$. Show that using these you can now write down a much simpler expression for the least squares formula. Do this, and taking its derivative with respect to C and setting it equal to zero, derive a formula for the best fit $\hat{C}$ in terms of $g(t)$, $s(t)$, and σ_m.