# Columbia University
# Department of Physics
# QUALIFYING EXAMINATION

Wednesday, January 10, 2018
10:00AM to 12:00PM
Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Let $|0\rangle$ be the ground state of the harmonic oscillator with angular frequency $\omega$, with Hamiltonian $H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$. The ground state wave function in position representation is $\psi_0(x) = \langle x|0\rangle = A\, e^{-x^2/2\ell^2}$ with $\ell = \sqrt{\frac{\hbar}{m\omega}}$ and $A = \ell^{-\frac{1}{2}}\pi^{-\frac{1}{4}}$

   (a) Express the operator $\hat{x}$ in the *momentum* representation and verify the canonical commutation relation.

   (b) Using the above, express the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ in the momentum representation, and verify that the canonical commutation relation between them holds.

   (c) Using (b), calculate the ground state wave function $\psi_0(p) = \langle p|0\rangle$ in the momentum representation.

   (d) Check your answer by direction calculation starting from the position space representation $\psi_0(x)$.

2. For the infinite square well with walls located at $x = a$ and $x = -a$, the ground state energy is $E_1 = \pi^2\hbar^2/8ma^2$ and the ground state wavefunction is $\psi_1(x) = \frac{1}{\sqrt{a}}\cos(\pi x/2a)$. The position momentum uncertainty relationship for this state is $\Delta x \Delta p = \kappa\hbar/2$. Find $\kappa$.

   (A potentially useful formula is $\int dx\, x^2\cos^2(bx) = \frac{x^3}{6} + \left(\frac{x^2}{4b} - \frac{1}{8b^3}\right)\sin(2bx) + \frac{x\cos(2bx)}{4b^2}$.)

3. Consider the two potential energy functions:

$$\begin{aligned} V_1(x) &= \frac{m\omega^2}{2}x^2, \qquad x > 0 \\ &= \infty, \qquad x \leq 0 \\ V_2(x) &= \frac{m\omega^2}{2}x^2, \qquad -\infty \leq x \leq \infty \end{aligned}$$

A particle with mass $m$, subject to the potential $V_1$, is initially in its quantum ground state.

(a) Write the normalized wave function, $\phi(x)$, of this state. (Hint: This wave function can be determined easily from simple harmonic oscillator solutions you already know.)

(b) At a certain time (say, $t = 0$), the impenetrable barrier at $x = 0$ is *very suddenly* removed so that for all $t > 0$ the same system is subject to the potential $V_2$ instead of $V_1$. If the energy is then measured, what is the probability of finding the system in its new ground state, $\psi_0(x)$?

4. In this problem we consider a one-dimensional quantum harmonic oscillator with frequency $\omega$ and mass $m$. The Hamiltonian of the system is

$$\hat{H} = \hbar\omega(\hat{n} + 1/2).$$

The orthonormal eigenstates of the Hamiltonian are denoted by $|n\rangle$ and have the property that $\hat{n}|n\rangle = n|n\rangle$ where $\hat{n} = \hat{a}^\dagger \hat{a}$.

The eigenstates of the operator $\hat{a}$ are called *quasi-classical* states for reasons that we examine in this problem. Consider an arbitrary complex number $\alpha$.

(a) Show that the state

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

is an eigenstate of $\hat{a}$ with eigenvalue $\alpha$, that is $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Also show that it is properly normalized.

(b) Calculate the expectation value of energy $\langle \hat{H} \rangle = \langle \alpha | \hat{H} | \alpha \rangle$ for a quasi-classical state $|\alpha\rangle$.

(c) Also calculate the expectation values of position $\langle \hat{x} \rangle$ and momentum $\langle \hat{p} \rangle$. Recall that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \quad \text{and} \quad \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger).$$

(d) Calculate the root mean square deviations $\Delta x$ and $\Delta p$ for the position and the momentum in this state. Show that $\Delta x \Delta p = \hbar/2$.

(e) Now, suppose that at time $t = 0$ the oscillator is in a quasi-classical state $|\alpha\rangle$ with $\alpha = |\alpha| e^{i\phi}$, where $|\alpha|$ is a real positive number. Show that at any later time $t$ the oscillator is also in a quasi-classical state that can be written as $e^{-i\omega t/2}|\alpha(t)\rangle$. Determine the value of $\alpha(t)$ in terms of $|\alpha|$, $\phi$, $\omega$, and $t$.

(f) Evaluate $\langle \hat{x} \rangle(t)$ and $\langle \hat{p} \rangle(t)$. Justify briefly why these states are called *quasi-classical*.

5. Consider a particle in one dimension subject to the double-delta-function potential energy

$$V(x) = -g\,\delta(x-a) - g\,\delta(x+a)$$

where $g$ is a positive constant.

(a) Find an equation from which the energy of the lowest bound state can be determined.

(b) Using this equation, find approximate expressions for the ground state energy in the limits where $a$ is very large and where $a$ approaches zero.

# 1 Harmonic oscillator in momentum representation (Greene)

## 1.1 Problem

Let $|0\rangle$ be the ground state of the harmonic oscillator with angular frequency $\omega$, with Hamiltonian $H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$. The ground state wave function in position representation is $\psi_0(x) = \langle x|0\rangle = A\,e^{-x^2/2\ell^2}$ with $\ell = \sqrt{\frac{\hbar}{m\omega}}$ and $A = \ell^{-\frac{1}{2}}\pi^{-\frac{1}{4}}$

(a) Express the operator $\hat{x}$ in the *momentum* representation and verify the canonical commutation relation.

(b) Using the above, express the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ in the momentum representation, and verify that the canonical commutation relation between them holds.

(c) Using (b), calculate the ground state wave function $\psi_0(p) = \langle p|0\rangle$ in the momentum representation.

(d) Check your answer by direction calculation starting from the position space representation $\psi_0(x)$.

## 1.2 solution

(a) On wave functions $\psi(p)$, $\hat{x}$ acts as $\hat{x} = i\hbar\partial_p$. The canonical commutation relations are $[\hat{x},\hat{p}] = [i\hbar\partial_p, p] = i\hbar$.

(b) $\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{1}{\ell}\hat{x} + i\frac{\ell}{\hbar}\hat{p}\right)$, $\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{\ell}\hat{x} - i\frac{\ell}{\hbar}\hat{p}\right)$, $[\hat{a},\hat{a}^\dagger] = \frac{1}{2}(1+1) = 1$.

(c) $\hat{a}|0\rangle = 0$ in momentum representation becomes $(\partial_p + \frac{\ell^2}{\hbar^2}p)\psi_0(p) = 0$, which is solved by $\psi_0(p) = B\,e^{-\ell^2p^2/2\hbar^2}$, $B = \hbar^{-\frac{1}{2}}\ell^{+\frac{1}{2}}\pi^{-\frac{1}{4}}$.

(d) $\psi_0(p) = \langle p|0\rangle = \int dx\,\langle p|x\rangle\langle x|0\rangle = \int dx\,\frac{1}{\sqrt{2\pi\hbar}}\,e^{-ipx/\hbar}\,A\,e^{-x^2/2\ell^2} = \frac{1}{\sqrt{2\pi\hbar}}A\sqrt{2\pi}\ell\,e^{-\ell^2p^2/2\hbar^2} = B\,e^{-\ell^2p^2/2\hbar^2}$.

# 2 Uncertainty in potential well (Hughes)

## 2.1 Problem

For the infinite square well with walls located at $x = a$ and $x = -a$, the ground state energy is $E_1 = \pi^2\hbar^2/8ma^2$ and the ground state wavefunction is $\psi_1(x) = \frac{1}{\sqrt{a}}\cos(\pi x/2a)$. The position momentum uncertainty relationship for this state is $\Delta x \Delta p = \kappa\hbar/2$. Find $\kappa$. (A potentially useful formula is $\int dx\, x^2\cos^2(bx) = \frac{x^3}{6} + \left(\frac{x^2}{4b} - \frac{1}{8b^3}\right)\sin(2bx) + \frac{x\cos(2bx)}{4b^2}$.)

## 2.2 Solution

The uncertainties are $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$, $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$. By symmetry, it is clear that $\langle x\rangle = 0$ and $\langle p\rangle = 0$. Moreover since $H = \frac{p^2}{2m}$, we immediately get $\langle p^2\rangle = 2m\langle H\rangle = 2mE_1 = \pi^2\hbar^2/4a^2$. Finally $\langle x^2\rangle = \frac{1}{a}\int_{-a}^{a} dx\, x^2\, \cos^2(\pi x/2a)$, which with the help of the useful formula is readily computed to give $\langle x^2\rangle = a^2(\frac{1}{3} - \frac{2}{\pi^2})$. Thus $\Delta x\Delta p = \kappa\,\hbar/2$ with $\kappa = \sqrt{\frac{\pi^2}{3} - 2}$. (Note that $\kappa > 1$, consistent with the general Heisenberg uncertainty relation $\Delta x\Delta p \geq \hbar/2$.)

# 3 Potential jump (Humensky)

## 3.1 Problem

Consider the two potential energy functions:

$$
\begin{aligned}
V_1(x) &= \frac{m\omega^2}{2}x^2, \qquad x > 0 \\
&= \infty, \qquad x \leq 0 \\
V_2(x) &= \frac{m\omega^2}{2}x^2, \qquad -\infty \leq x \leq \infty
\end{aligned}
$$

A particle with mass $m$, subject to the potential $V_1$, is initially in its quantum ground state.

(a) Write the normalized wave function, $\phi(x)$, of this state.

(b) At a certain time (say, $t = 0$), the impenetrable barrier at $x = 0$ is *very suddenly* removed so that for all $t > 0$ the same system is subject to the potential $V_2$ instead of $V_1$. If the energy is then measured, what is the probability of finding the system in its new ground state, $\psi_0(x)$?

## 3.2 Solution

(a) Find $\phi(x)$ for $V_1(t = 0)$.

$$H|E\rangle = E|E\rangle$$

$$x > 0 : \left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2\right)|E\rangle = E|E\rangle$$

For $x > 0$, the differential equation is identical to that of the harmonic oscillator, so for $x > 0$ the wavefunctions take the form of harmonic oscillator energy eigenstate wavefunctions, i.e. Hermite polynomials times a Gaussian. Due to the infinite potential for $x \leq 0$, we must in addition require $\phi(0) = 0$. This is the case for the eigenfunctions antisymmetric under $x \to -x$. The lowest energy state satisfying this is the $n = 1$ excited state of the harmonic oscillator, which takes the form

$$\phi(x) = A\, x\, e^{-ax^2/2}\,, \qquad a = \frac{m\omega}{\hbar}\,,$$

where $A$ is fixed by requiring $1 = \int_0^\infty dx\, \phi(x)^2 = \frac{1}{2}A^2 \int_{-\infty}^\infty dx\, x^2 e^{-ax^2}$. The integral can be evaluated by observing it equals $(-\partial_a)\int dx\, e^{-ax^2} = (-\partial_a)\frac{\sqrt{\pi}}{\sqrt{a}} = \frac{\sqrt{\pi}}{2\, a^{3/2}}$. Thus $A = 2\left(\frac{m\omega}{\hbar}\right)^{3/4}\pi^{-1/4}$.

(b) Probability to be in ground state after barrier removed: now the ground state is the usual ground state for a harmonic oscillator:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\left(\frac{m\omega x^2}{2\hbar}\right)}$$

The probability to be in this state for $t > 0$ is time-independent and given by

$$P(\psi_0) = |\langle\psi_0|\phi\rangle|^2$$

where

$$\langle\psi_0|\phi\rangle = \frac{2}{\sqrt{\pi}}\left(\frac{m\omega}{\hbar}\right)\int_0^\infty x\, e^{-\frac{m\omega x^2}{\hbar}}\, dx$$

To solve this integral, change variables to $y = (m\omega/\hbar)\, x^2$; then $dy = 2\,(m\omega/\hbar)\, xdx$,

$$\langle\psi_0|\phi_1\rangle = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-y} dy = \frac{1}{\sqrt{\pi}},$$

and

$$P(\psi_0) = |\langle\psi_0|\phi\rangle|^2 = \frac{1}{\pi}\,.$$

# 4 Quasi-classical states (Will)

## 4.1 Problem

In this problem we consider a one-dimensional quantum harmonic oscillator with frequency $\omega$ and mass $m$. The Hamiltonian of the system is

$$\hat{H} = \hbar\omega(\hat{n} + 1/2).$$

The orthonormal eigenstates of the Hamiltonian are denoted by $|n\rangle$ and have the property that $\hat{n}|n\rangle = n|n\rangle$ where $\hat{n} = \hat{a}^\dagger\hat{a}$.

The eigenstates of the operator $\hat{a}$ are called *quasi-classical* states for reasons that we examine in this problem. Consider an arbitrary complex number $\alpha$.

(a) Show that the state

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

is an eigenstate of $\hat{a}$ with eigenvalue $\alpha$, that is $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Also show that it is properly normalized.

(b) Calculate the expectation value of energy $\langle\hat{H}\rangle = \langle\alpha|\hat{H}|\alpha\rangle$ for a quasi-classical state $|\alpha\rangle$.

(c) Also calculate the expectation values of position $\langle\hat{x}\rangle$ and momentum $\langle\hat{p}\rangle$. Recall that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \quad \text{and} \quad \hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a} - \hat{a}^\dagger).$$

(d) Calculate the root mean square deviations $\Delta x$ and $\Delta p$ for the position and the momentum in this state. Show that $\Delta x \Delta p = \hbar/2$.

(e) Now, suppose that at time $t = 0$ the oscillator is in a quasi-classical state $|\alpha\rangle$ with $\alpha = |\alpha|e^{i\phi}$, where $|\alpha|$ is a real positive number. Show that at any later time $t$ the oscillator is also in a quasi-classical state that can be written as $e^{-i\omega t/2}|\alpha(t)\rangle$. Determine the value of $\alpha(t)$ in terms of $|\alpha|$, $\phi$, $\omega$, and $t$.

(f) Evaluate $\langle\hat{x}\rangle(t)$ and $\langle\hat{p}\rangle(t)$. Justify briefly why these states are called *quasi-classical.*

## 4.2 Solution

Solution:

(a) Direct check whether eigenvalue equation holds:

$$\rightsquigarrow \hat{a}|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n{}' \frac{\alpha^n}{\sqrt{n!}} \underbrace{\hat{a}|n\rangle}_{=\sqrt{n}\,|n-1\rangle}$$

$$= e^{-|\alpha|^2/2} \sum_n{}' \frac{\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle$$

$$= \alpha e^{-|\alpha|^2/2} \sum_n{}' \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \alpha|\alpha\rangle \quad \checkmark$$

Normalization:

$$\rightsquigarrow \langle\alpha|\alpha\rangle = e^{-|\alpha|^2} \sum_n{}' \frac{|\alpha|^{2n}}{n!} = 1$$

(b) Energy expectation value:

$$\langle\hat{H}\rangle = \langle\alpha|\hat{H}|\alpha\rangle = \hbar\omega \langle\alpha|\hat{n} + \tfrac{1}{2}|\alpha\rangle$$

with $\langle\alpha|\hat{n}|\alpha\rangle = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2$

$$= \hbar\omega\left(|\alpha|^2 + \tfrac{1}{2}\right)$$

(c) Position & momentum:

$$\langle\hat{x}\rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle\alpha|\hat{a} + \hat{a}^\dagger|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}} (\alpha + \alpha^*)$$

$$\langle\hat{p}\rangle = -i\sqrt{\frac{m\hbar\omega}{2}} \langle\alpha|\hat{a} - \hat{a}^\dagger|\alpha\rangle = i\sqrt{\frac{m\hbar\omega}{2}} (\alpha^* - \alpha)$$

(d) Root mean-square deviations for position & momentum:

$$\Delta x^2 = \frac{\hbar}{2m\omega} \langle\alpha|(\hat{a} + \hat{a}^\dagger)^2|\alpha\rangle - \langle x\rangle^2 \qquad \text{Note: } [a, a^\dagger] = 1$$

$$= \frac{\hbar}{2m\omega}\left((\alpha+\alpha^*)^2+1\right) - \langle\hat{x}\rangle^2$$

$$\rightsquigarrow \Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Delta p^2 = -\frac{m\hbar\omega}{2}\langle\alpha|(\hat{a}-\hat{a}^+)^2|\alpha\rangle - \langle\hat{p}\rangle^2$$

$$= -\frac{m\hbar\omega}{2}\left((\alpha-\alpha^*)^2-1\right) - \langle\hat{p}\rangle^2$$

$$\rightsquigarrow \Delta p = \sqrt{\frac{m\hbar\omega}{2}}$$

Product of uncertainties:

$$\Delta x \cdot \Delta p = \frac{\hbar}{2} \quad \rightsquigarrow \text{minimal uncertainty state}$$

(e) Time evolution of the quasi-classical state $|\alpha\rangle$:

$$e^{-i\hat{H}t/\hbar}|\alpha\rangle = e^{-|\alpha|^2/2}\cdot\sum_n \frac{\alpha^n}{\sqrt{n!}} e^{iE_nt/\hbar}|n\rangle$$

$$= e^{-|\alpha|^2/2} e^{-i\omega t/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} e^{-in\omega t}|n\rangle$$

$$= e^{-i\omega t/2}|\alpha(t)\rangle$$

$$\text{with } \alpha(t) = \alpha e^{-i\omega t} = |\alpha| e^{-i(\omega t-\phi)}$$

(f) Time evolution of position and momentum:

$$\langle\hat{x}\rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(\alpha(t)+\alpha^*(t)\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}|\alpha|\cdot 2\cos(\omega t-\phi)$$

$$\langle \hat{p} \rangle(t) = i\sqrt{\frac{m\hbar\omega}{2}}\left(\alpha^*(t) - \alpha(t)\right)$$

$$= -\sqrt{\frac{m\hbar\omega}{2}}\,|\alpha|\cdot 2\sin(\omega t - \phi)$$

$$= -p_0 \sin(\omega t - \phi) \qquad \text{with } p_0 = |\alpha|\sqrt{2m\hbar\omega}$$

⇝ These are the equations of motion of a classical harmonic oscillator.

[Not asked for: In particular in the limit $|\alpha| \gg 1$ the relative uncertainties of position and mom. are small:

$$\frac{\Delta x}{x_0} = \frac{1}{2|\alpha|} \ll 1 \quad \text{and} \quad \frac{\Delta p}{p_0} = \frac{1}{2|\alpha|} \ll 1$$

⇝ comparable to motion of a classical pendulum! ]

# 5 Ground state of double delta potential (Weinberg)

## 5.1 Problem

Consider a particle in one dimension subject to the double-delta-function potential energy

$$V(x) = -g\,\delta(x-a) - g\,\delta(x+a)$$

where $g$ is a positive constant.

(a) Find an equation from which the energy of the lowest bound state can be determined.

(b) Using this equation, find approximate expressions for the ground state energy in the limits where $a$ is very large and where $a$ approaches zero.

## 5.2 Solution

Solution to Weinberg 1-2016

$V = -g\delta(x-a) - g\delta(x+a)$

Ground state is symmetric, so it is sufficient to solve in the region $x \geq 0$

Region I : $0 \leq x < a$
Region II : $a < x < \infty$

Region I : $-\frac{\hbar^2}{2m}\psi'' = -\frac{\hbar\kappa^2}{2m}\psi$

$$\Rightarrow \psi_I = A\cosh\kappa x + B\sinh\kappa x$$

symmetry $\Rightarrow B=0$

Region II : $-\frac{\hbar^2}{2m}\psi'' = -\frac{\hbar^2\kappa^2}{2m}\psi \quad , \psi(\infty)=0$

$$\Rightarrow \psi_{II} = Ce^{-\kappa x}$$

Matching conditions

(1) $\psi(a_-) = \psi(a_+)$

$$\Rightarrow A\cosh\kappa a = Ce^{-\kappa a}$$

(2)

$$\psi'(a_+) - \psi'(a_-) = \int_{a-\epsilon}^{a+\epsilon} dx\, \psi'' = \int_{a-\epsilon}^{a+\epsilon} dx \left(\frac{2m}{\hbar^2}\right) V\psi$$

$$= -\frac{2mg}{\hbar^2}\psi(a)$$

$$\Rightarrow \kappa A\sinh\kappa a = Ce^{-\kappa a}\left(-\kappa + \frac{2mg}{\hbar^2}\right)$$

Dividing $\Rightarrow$ $\boxed{\tanh\kappa a = \frac{2mg}{\kappa\hbar^2} - 1}$

a very large $\Rightarrow \tanh\kappa a \approx 1 \Rightarrow \kappa \approx \frac{mg}{\hbar^2}, \; E \approx -\frac{mg^2}{2\hbar^2}$

a very small $\Rightarrow \tanh\kappa a \ll 1 \Rightarrow \kappa \approx \frac{2mg}{\hbar^2}, \; E \approx -\frac{2mg^2}{\hbar^2}$