

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 9, 2017

3:00PM to 5:00PM

Classical Physics

Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}'' \times 11''$ paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

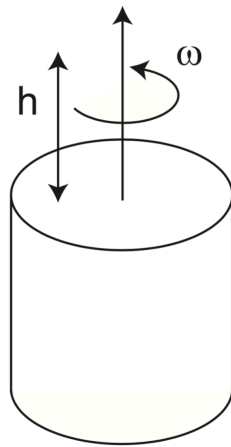
Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. An infinite straight wire carries zero current at all times except for a sharp pulse at $t = 0$, i.e. $I(t) = I_0\tau\delta(t)$, where $\delta(t)$ is the Dirac Delta function and $I_0\tau$ is a given constant.
 - (a) Calculate (in Lorentz gauge) the potentials everywhere in space and time
 - (b) Calculate the electric and magnetic fields everywhere in space and time

2. Consider a cylinder of radius R and length L that is uniformly charged with charge density ρ . The cylinder rotates about its axis (the z -axis) with angular velocity ω , as shown in the figure. The top of the cylinder lies in the xy -plane $z = 0$.
- (a) Compute the magnitude of the current density, J , as a function of distance, r , from the center of the cylinder.
 - (b) Compute the magnetic induction, \vec{B} , along the z -axis at the position $\vec{r} = (0, 0, h)$, i.e. at height h above the top of the cylinder.
 - (c) Find the expression for $B_z(h)$ in the limit of $h \gg R, L$, i.e. very far away from the cylinder. You can solve this directly using your solution to part (b), or by alternative means. Even an order-of-magnitude estimate, with the correct units, will result in partial credit.

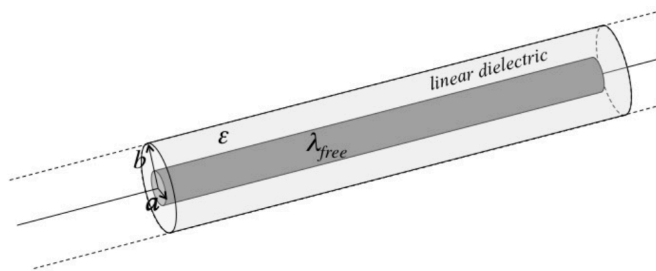


3. A point particle of charge q and position $\vec{x}(t)$ moves according to

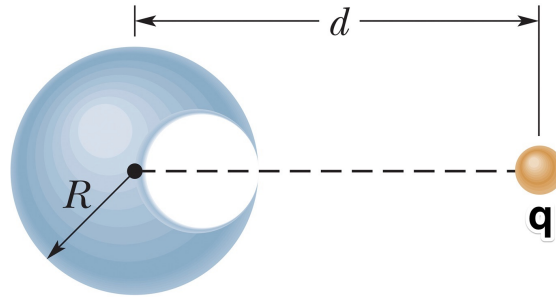
$$\begin{aligned}\vec{x}(t) &= 0 && \text{when } t < 0 \\ \vec{x}(t) &= \vec{v}_0 \left[t - \frac{r_0}{v_0} (1 - e^{-v_0 t/r_0}) \right] && \text{when } t > 0\end{aligned}\tag{1}$$

where $v_0 = |\vec{v}_0|$ with \vec{v}_0 a fixed velocity and r_0 a fixed length. (The above formulas represent the emission of a charged particle during the decay of a state at $t = 0$.) Evaluate the total electromagnetic energy radiated by the particle in the dipole approximation. Under what conditions, on v_0 and r_0 , is the dipole approximation valid?

4. Consider a very long coaxial cable. The inner conductor is a cylinder of radius a and carries free charge with linear charge density λ_{free} . It is surrounded by a cylindrical linear dielectric with permittivity ϵ and radius b .
- (a) Find the following quantities for all three regions (i) $s < a$, (ii) $a < s < b$, and (iii) $b < s$:
1. Electric displacement \vec{D} ;
 2. Electric field \vec{E} ;
 3. Polarization \vec{P} .
- (b) Calculate the potential V on the axis ($s = 0$) of the conductor.
- (c) Calculate the bound charges, i.e. give the bound volume charge density ρ_b in the bulk of the dielectric and the bound surface charge density σ_b on the inner ($s = a$) and outer ($s = b$) surfaces of the linear dielectric.



5. The figure below shows a spherical hollow inside a uniformly charged sphere of radius R ; the surface of the hollow passes through the center of the sphere and “touches” the right side of the sphere. The charge of the sphere before hollowing was Q . With what electrostatic force does the hollowed-out charged sphere attract charge q that lies at a distance d from the center of the charged sphere, on the straight line connecting the centers of the sphere and of the hollow?



(1) Solution ENM

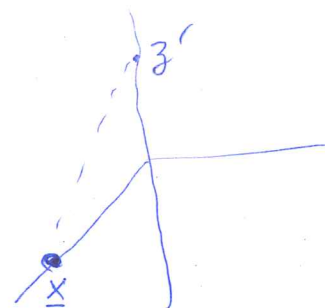
$$\vec{j} = \vec{e}_z I_0 \tau \delta(t) \delta(x) \delta(y)$$

$$\square^2 A_\mu = -\frac{4\pi j_\mu}{c} \quad \text{only } A_z \neq 0 \quad \phi = 0$$

$$A_z = \frac{1}{c} \int \frac{d^3x' dt' \delta(t - t' - |\vec{x} - \vec{x}'|/c)}{|\vec{x} - \vec{x}'|} I_0 \tau \delta(t') \delta(x') \delta(y')$$

$$= \frac{I_0 \tau}{c} \int_{-\infty}^{\infty} \frac{dz' \delta(t - \sqrt{x^2 + (z - z')^2}/c)}{\sqrt{x^2 + (z - z')^2}}$$

$$d\sqrt{x^2 + (z - z')^2} = \frac{(z' - z) dz'}{\sqrt{x^2 + (z - z')^2}}$$



$$\frac{dz'}{\sqrt{x^2 + (z - z')^2}} = \frac{d\sqrt{x^2 + (z - z')^2}}{z' - z}$$

$$A_z = \frac{I_0 \tau}{c} \int \frac{d\sqrt{x^2 + (z - z')^2}}{z' - z} \delta(t - \frac{\sqrt{x^2 + (z - z')^2}}{c}) = \frac{I_0 \tau}{z' - z}$$

$$t^2 c^2 = x^2 + (z - z')^2$$

$$\vec{B} = \nabla \times \vec{A} \quad \vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}$$

$$A_z = \frac{I_0 \tau}{\sqrt{t^2 c^2 - x^2}}$$

$$B_x = \frac{\partial A_z}{\partial y} \quad B_y = -\frac{\partial A_z}{\partial x} \quad E_z = -\frac{1}{c} \frac{\partial A_z}{\partial t}$$

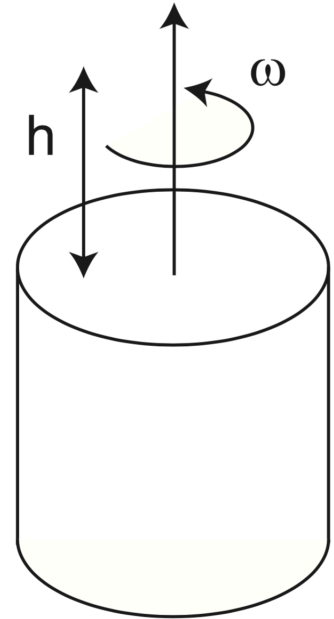
$$B_x = \frac{I_0 \tau y}{(t^2 c^2 - x^2)^{3/2}}$$

$$B_y = \frac{-I_0 \tau x}{(t^2 c^2 - x^2)^{3/2}}$$

$$E_z = \frac{I_0 \tau t c}{(t^2 c^2 - x^2)^{3/2}}$$

Electricity & Magnetism (Metzger)

Consider a cylinder of radius R and length L that is uniformly charged with charge density ρ . The cylinder rotates with a uniform angular velocity ω through its center around the z-axis, as shown in the figure. The x-y axis ($z = 0$) lies parallel to the top of the cylinder.



1. Compute the magnitude of the current density, J , as a function of distance, r , from the center of the cylinder.
2. Compute the magnetic induction, \vec{B} , along the z-axis at the position $\vec{r} = (0, 0, h)$, i.e. a high h above the top of the cylinder.
3. Simplify the expression for the value of $B_z(h)$ in the limit that $h \gg R, L$, i.e. very far away from the cylinder. You can solve this directly using your solution to part 2, or by alternative means. Even an order-of-magnitude estimate here with the correct units will result in partial credit.

Solution:

1.

$$\vec{J}(r) = \rho \vec{v}(r) = \rho(\omega \times \vec{r}) \Rightarrow |\vec{J}| = \rho\omega r$$

2.

$$\vec{B} = \frac{\mu_0}{4\pi} \int d^3\vec{r}' \vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

and hence, defining an angle α between the source point $\vec{r}' = (r', \phi')$ and the location on the axis $\vec{r} = z\hat{z}$,

$$\begin{aligned} B_z(\vec{r}) &= \frac{\mu_0}{4\pi} \int d^3\vec{r}' J(\vec{r}') \frac{\sin \alpha}{|\vec{r} - \vec{r}'|^2} = \frac{\mu_0}{4\pi} \int_0^L dz' \int_0^{2\pi} d\phi' \int_0^R r' dr' \rho\omega r' \frac{r'}{[(h+z')^2 + (r')^2]^{3/2}} \\ &= \frac{\mu_0}{2} \rho\omega \int_0^L dz' \int_0^R dr' \frac{(r')^3}{[(h+z')^2 + (r')^2]^{3/2}} = \frac{\mu_0}{2} \rho\omega \int_0^L dz' \left[\frac{2(h+z')^2 + R^2}{[(h+z')^2 + R^2]^{1/2}} - 2(h+z') \right] \\ &= \frac{\mu_0}{2} \rho\omega \left[(h+L) \left(\sqrt{(h+L)^2 + R^2} - (h+L) \right) - h \left(\sqrt{h^2 + R^2} - h \right) \right] \end{aligned}$$

3. There is an easy and a hard way to solve this problem.

First, we can expand B_z in terms of small parameters h/L and h/R ,

$$B_z = \frac{\mu_0}{2} \rho\omega h^2 \left[(1 + L/h) \left(\sqrt{1 + 2L/h + (L/h)^2 + (R/h)^2} - (1 + L/h) \right) - \left(\sqrt{1 + (R/h)^2} - 1 \right) \right]$$

Using the expansion $(1 + \epsilon)^{1/2} \simeq 1 + \epsilon/2 - \epsilon^2/8 + \epsilon^3/16 - \frac{5}{128}\epsilon^4$ for $\epsilon = R/h, L/h \ll 1$, after much simplification, we obtain

$$B_z \simeq \frac{\mu_0}{2} \rho\omega h^2 \times \frac{LR^5}{4h^5} = \frac{\mu_0 LR^4}{8h^3} \rho\omega$$

As a second, easier approach, we note that far away from a magnetic dipole $\vec{m} = m\hat{z}$, the B-field along the direction \hat{z} is

$$B_z \simeq \frac{\mu_0}{2\pi} \frac{m}{z^3} \Big|_{z=h} = \frac{\mu_0 LR^4}{8h^3} \rho\omega,$$

where

$$|\vec{m}| = \frac{1}{2} \int |\vec{r}' \times \vec{J}| d^3\vec{r}' = \frac{1}{2} \int_0^{2\pi} d\phi \int_0^R r \cdot (\rho r\omega) \cdot r dr = \pi L \rho\omega \int_0^R r^3 dr = \frac{\pi LR^4}{4} \rho\omega$$

is the magnetic moment.

BEM Solution 1

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3} |\ddot{\vec{p}}(t - r/c) \times \vec{m}|^2$$

$$P = \frac{2\pi}{3c^3} |\ddot{\vec{p}}(t - r/c)|^2$$

For integrated power can let $t - r/c \rightarrow t$.

$$\vec{p} = q \vec{x}(t) = q \vec{r}_0 \left[t - \frac{r_0}{v_0} \left(1 - e^{-\frac{v_0 t}{r_0}} \right) \right] \Theta(t)$$

$$\dot{\vec{p}} = q \vec{r}_0 \left[1 - e^{-\frac{v_0 t}{r_0}} \right] \Theta(t)$$

$$\ddot{\vec{p}} = q \vec{r}_0 \frac{v_0}{r_0} e^{-\frac{v_0 t}{r_0}} \Theta(t)$$

$$P = \frac{2\pi}{3c^3} \frac{v_0^4}{r_0^2} e^{-\frac{2v_0 t}{r_0}} \quad t > 0$$

$$\mathcal{E} = \int_{-\infty}^{\infty} P dt = \frac{2\pi}{3c^3} \frac{v_0^4}{r_0^2} \frac{r_0}{2v_0} = \left[\frac{\pi}{3c^3} \frac{v_0^3}{r_0} = \mathcal{E} \right]$$

Dipole approximation good when $v_0/c \ll 1$

1. (a) $\vec{D} = ?$

• $s < a$: $\oint_S \vec{D} \cdot d\vec{a} = Q_{\text{encl.}}$

$Q_{\text{encl.}} = 0$ since it's a conductor (λ_{free} is on surface)

$\Rightarrow \boxed{\vec{D} = 0}$

• $a < s < b$: $\oint_S \vec{D} \cdot d\vec{a} = Q_{\text{encl.}} \Leftrightarrow D 2\pi s L = \lambda_{\text{free}} L$

$\Leftrightarrow D = \frac{\lambda_{\text{free}}}{2\pi s}$

$\Rightarrow \boxed{\vec{D} = \frac{\lambda_{\text{free}} \hat{s}}{2\pi s}}$

• $s > b$: $\boxed{\vec{D} = \frac{\lambda_{\text{free}} \hat{s}}{2\pi s}}$ (same as before)

(b) $\vec{E} = ?$

• $s < a$: $\oint \vec{E} \cdot d\vec{a} = \frac{q_{\text{encl.}}}{\epsilon_0} = \frac{\cancel{q_{\text{free}}} + \cancel{q_b}}{\epsilon_0} = 0$

$\Rightarrow \boxed{\vec{E} = 0}$

• $a < s < b$: $\vec{E} = \frac{\vec{D}}{\epsilon}$ (linear dielectric)

$\Rightarrow \boxed{\vec{E} = \frac{\lambda_{\text{free}} \hat{s}}{2\pi \epsilon s}}$

• $s > b$: $\vec{P} = 0$ outside $\Rightarrow \vec{E} = \frac{\vec{D} - \vec{P}}{\epsilon_0} = \boxed{\frac{\lambda_{free} \hat{s}}{2\pi\epsilon_0 s}}$

(c) $\vec{P} = ?$

Karagiorgi
Section 2-4

• $s < a$: $\boxed{\vec{P} = 0}$ (no dielectric)

• $a < s < b$: $\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \frac{\lambda_{free} \hat{s}}{2\pi s} - \epsilon_0 \frac{\lambda_{free} \hat{s}}{2\pi \epsilon s}$
 $= \frac{\lambda_{free} \hat{s}}{2\pi s} \left(1 - \frac{\epsilon_0}{\epsilon}\right)$

$\Rightarrow \boxed{\vec{P} = \left(1 - \frac{\epsilon_0}{\epsilon}\right) \frac{\lambda_{free} \hat{s}}{2\pi s}}$

• $s > b$: $\vec{P} = 0$ (outside of dielectric)

2. $V(s=0) = ?$

Conductor is an equipotential object, $V(s=0) = V$

$$V = - \int_{ref.}^0 \vec{E} \cdot d\vec{l} = - \int_{ref}^b \vec{E} \cdot d\vec{l} - \int_b^a \vec{E} \cdot d\vec{l} - \int_a^0 \vec{E} \cdot d\vec{l}$$

$$= - \int_{ref}^b \frac{\lambda_{free}}{2\pi\epsilon_0 s} \hat{s} \cdot d\vec{s} - \int_b^a \frac{\lambda_{free}}{2\pi \epsilon s} \hat{s} \cdot d\vec{s}$$

$$= + \frac{\lambda_{free}}{2\pi} \left[\frac{1}{\epsilon_0} \ln\left(\frac{ref}{b}\right) + \frac{1}{\epsilon} \ln\left(\frac{b}{a}\right) \right]$$

Choose $ref = b \Rightarrow \boxed{V = \frac{\lambda_{free}}{2\pi \epsilon} \ln \frac{b}{a}}$

$$3. \rho_b = -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot \left[\left(1 - \frac{\epsilon_0}{\epsilon}\right) \frac{\lambda_{free}}{2\pi} \frac{1}{s} \hat{s} \right]$$

$$\vec{\nabla} \cdot \vec{f} = \frac{1}{s} \frac{\partial}{\partial s} (s f_s) + \frac{1}{s} \frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z}$$

$$\rho_b = - \left(1 - \frac{\epsilon_0}{\epsilon}\right) \frac{\lambda_{free}}{2\pi} \frac{1}{s} \underbrace{\frac{\partial}{\partial s} \left(s \cdot \frac{1}{s}\right)}_0 + 0 + 0 = 0$$

$$\Rightarrow \boxed{\rho_b = 0}$$

$$\sigma_b = \vec{P} \cdot \hat{n}$$

For outer surface $s=b$ and $\hat{n}=\hat{s}$

$$\sigma_b = \left(1 - \frac{\epsilon_0}{\epsilon}\right) \frac{\lambda_{free}}{2\pi s} \hat{s} \cdot \hat{s} = \boxed{\left(1 - \frac{\epsilon_0}{\epsilon}\right) \frac{\lambda_{free}}{2\pi b} \text{ (outer)}}$$

For inner surface $s=a$ and $\hat{n}=-\hat{s}$

$$\sigma_b = \boxed{-\left(1 - \frac{\epsilon_0}{\epsilon}\right) \frac{\lambda_{free}}{2\pi a} \text{ (inner)}}$$

Problem #3 (10 points)

Johnson
Section 2-5

$$F = G \frac{Mm}{d^2} \quad \leftarrow \text{force with no chunk removed}$$

$$F_1 = G \frac{M_c m}{(d - \frac{R}{2})^2} \quad \leftarrow \text{force between chunk and } m$$

$$F_2 = \text{force between rest of object and } m$$

$$F = F_1 + F_2 \quad \leftarrow \text{What is } F_2?$$

$$\text{Given } \left\{ \begin{array}{l} R = 4.00 \text{ cm} \\ M = 2.95 \text{ kg} \end{array} \right\}, \quad \begin{array}{l} d = 9.00 \text{ cm} \\ m = 0.431 \text{ kg} \end{array}$$

$$V = \frac{4}{3} \pi R^3$$

$$\rho = \frac{M}{V}$$

What is the
mass of the
chunk?

$$\rightarrow M_c = \rho \frac{4}{3} \pi \left(\frac{R}{2}\right)^3 = M \frac{\frac{4}{3} \pi \left(\frac{R}{2}\right)^3}{\frac{4}{3} \pi R^3}$$

$$\boxed{M_c = \frac{M}{8}} \quad (3 \text{ points})$$

$$F_2 = F - F_1 = G \frac{Mm}{d^2} - G \frac{M_c m}{(d - \frac{R}{2})^2}$$

$$= G M m \left[\frac{1}{d^2} - \frac{1/8}{(d - \frac{R}{2})^2} \right]$$

$$F_2 = (6.67 \times 10^{-11} \frac{\text{Nm}^2}{\text{kg}^2}) (2.95 \text{ kg}) (0.431 \text{ kg}) \times \quad (7 \text{ points})$$

$$\left[\frac{1}{(9.00 \times 10^{-2} \text{ m})^2} - \frac{1}{(9.00 \times 10^{-2} \text{ m} - \frac{4.00 \times 10^{-2} \text{ m}}{2})^2} \right] = \boxed{8.31 \times 10^{-9} \text{ N}}$$