Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 15, 2016
1:00PM to 3:00PM
General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Suppose the earth's lower atmosphere is approximated as a classical ideal gas with mean molecular weight M in a uniform gravitational field which produces an acceleration g.

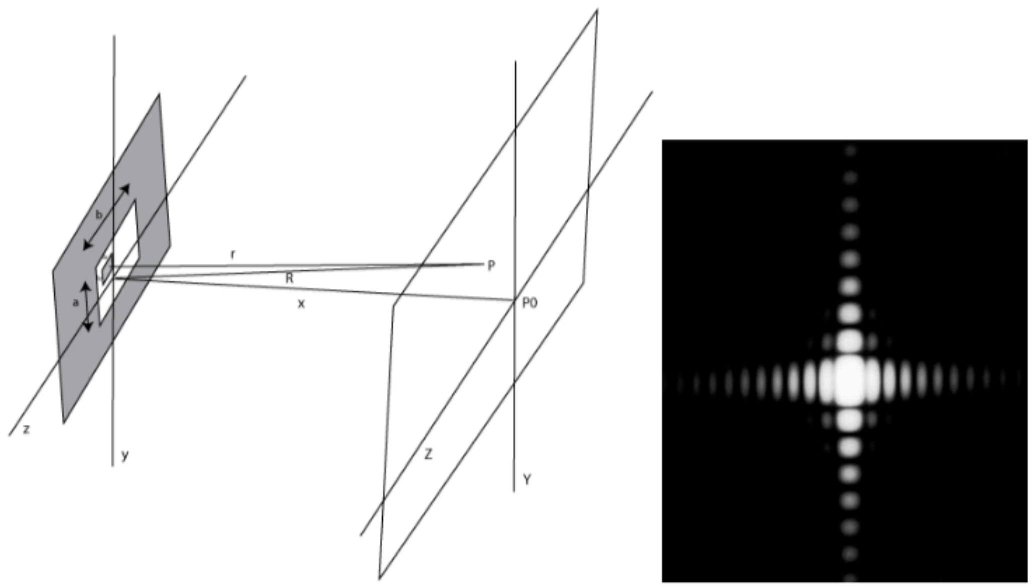
   (a) Assume a small volume of air is in hydrostatic equilibrium with its surrounding atmosphere at temperature T and pressure P. Find an expression for the pressure gradient dP/dz around it in terms of P, T, g and M. (z is the altitude above the earth surface.)

   Our lower atmosphere is very slightly convective. Large blobs of it have zero buoyancy. They move up or move down while adjusting their density to be the same as that of the surrounding air without significant exchange of heat with it. Assume air is composed of diatomic molecules, each of which can be modeled classically as two point masses at the ends of a rigid, massless stick.

   (b) Find an expression for the temperature T at any point in the lower atmosphere in terms of the pressure P at the same point and T_0 and P_0, the temperature and pressure at the surface of the earth respectively, ie., find $T(P, T_0, P_0)$.

   (c) Combine your answers from parts (a) and (b) to determine the temperature and pressure as a function of height z above the earth's surface, ie., find $T(z, T_0)$ and $P(z, P_0, T_0)$.

2. A laser beam is incident from the left (pointed in the $+\hat{x}$ direction) upon a rectangular aperture with length a in the y-direction and length b in the z-direction, where $b > a$ (see Figure). The laser is monochromatic with a wavelength λ and its intensity is uniform across the opening. Light passing through the aperture is collected on a screen at a very large distance $x(\gg a, b, \lambda)$ away from both the aperture and the laser. Coordinates on the distant screen are denoted by their physical coordinates (Y, Z), or by the angles $(\theta_Y \approx Y/x, \theta_Z \approx Z/x)$ measured with respect to the $\hat{x}$ axis.



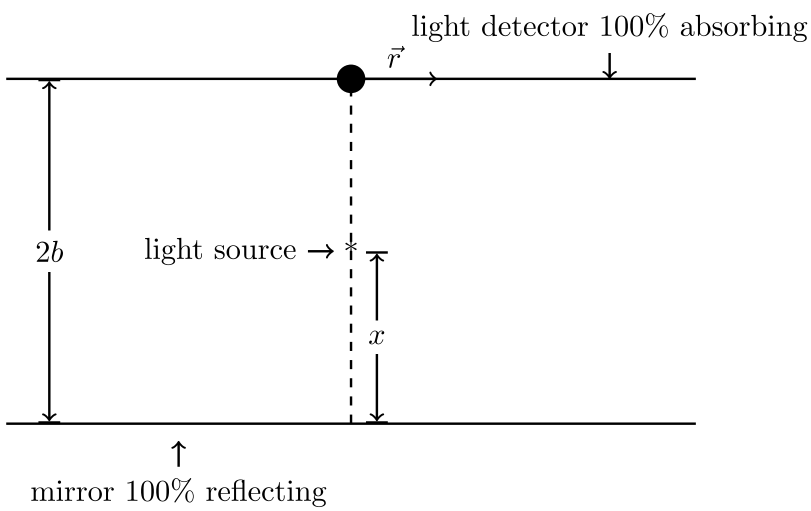
(a) The intensity pattern $I(\theta_Y, \theta_Z)$ as measured on the distant screen is shown in the Figure on the right. Which directions in this figure correspond to the Y axis, and which correspond to the Z axis? (example answer: the vertical direction corresponds to the Y axis) Explain your reasoning.

(b) How does the angular distribution of intensity $I(\theta_Y, \theta_Z)$ change if the screen is moved a factor of 2 further away from the aperture? Explain your answer.

(c) How would the measured intensity distribution $I(\theta_Y, \theta_Z)$ change if the wavelength of the laser light λ is doubled? Briefly explain your answer.

(d) Calculate the intensity distribution $I(\theta_Y, \theta_Z)$ in terms of the angles θ_Y, θ_Z and normalized to the intensity $I(0)$ at the center of the screen. *Hint: Do not worry about the constant amplitude in front of the electric field or intensity until the end of the derivation.*

3. A point dipole source of monochromatic light (wavelength λ) is suspended a height x above a perfectly reflecting mirror. Emitted light directed down towards the mirror below has the same amplitude and polarization as light emitted upwards in the opposite direction. A planar, fully-absorbing light detector, oriented exactly parallel to the mirror is placed a height $2b$ above the mirror. The detected light intensity depends on the source position x and on r, the distance from an axis through the source and normal to the mirror and detector, as shown in the figure. Assume the light emitter is small enough that it intercepts none of the reflected light. Assume in what follows that $\lambda \ll b$ and $r \ll 2b - x$. There are many local maxima at $r = 0$ as x is varied, and at any fixed x as r is varied.



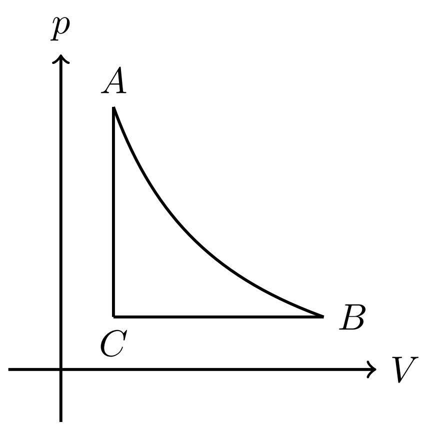
(a) At what values of x are intensity maxima observed on the light detector at $r = 0$?

(b) The source is placed at $x = b$ and the wavelength λ is such that an intensity maximum is observed on the light detector at $r = 0$. At what other values of r will intensity maxima be observed on the light detector?

(c) How is the answer to part (a) changed if the mirror moves with velocity v parallel to the detector plane? How is the answer to part (b) changed?

4. One mole of a diatomic ideal gas is driven around the cycle ABCA shown on the pV diagram below. Step AB is isothermic, with a temperature $T_A = 500\ K$; step BC is isobaric; and step CA is isochoric. The volume of the gas at point A is $V_A = 1.00\ L$, and at point B is $V_B = 4.00\ L$. Treat a diatomic gas molecule as two point masses at the ends of a rigid, massless rod.

   The ideal gas constant is $R = 8.31\ J/mol \cdot K$.



   (a) What is the pressure p_B at point B?

   (b) What is the total work done in completing one cycle (ABCA)?

   (c) What is the entropy change $S_c - S_B$?

5. A rope of uniform linear density μ and total length L is suspended from one end and hangs vertically under its own weight. It is lightly tapped at the lower end.

   How long does it take for the perturbation to reach the top of the rope?

6. For a non-relativistic ideal gas the partition function for N particles (of mass m) in a volume V is

$$Z_N = \frac{Z_1^N}{N!} \tag{1}$$

where

$$Z_1 = \int \frac{d^3k d^3p}{h^3} \exp[-\frac{\vec{p}^2}{2mkT}] = \frac{V}{\lambda^3} \tag{2}$$

with

$$\lambda = \frac{h}{\sqrt{2\pi mkT}} \tag{3}$$

Now consider an extreme-relativistic ideal gas of N particles in a volume V. Give the following:

(a) The partition function Z_N at a temperature T

(b) $E(N,T)$ the energy of the gas

(c) P the pressure of the gas

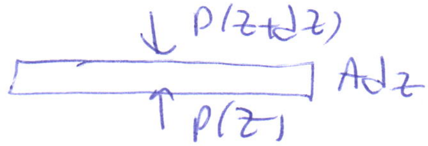
If m is the mass of the particle when do you expect the extreme-relativistic approximation to be good?

G1-1 Ruderman

Section 5 - 1
Ruderman

a.) $PV = \nu RT \Rightarrow$

$P = \frac{M_{Tot}}{VM} RT = \frac{n\bar{m}}{M} RT$



$-P(z+dz)A + P(z)A - n\bar{m}Ag\,dz = 0$

$-dP/dz = \frac{PMg}{RT}$ Ans 2

b.) $dS = 0 = \frac{\nu C_v dT}{T} + \frac{P}{T}dV$ $\quad PdV + Vdp = \nu R dT$

$dS = 0 = \nu C_v \frac{dT}{T} + \frac{\nu R\,dT}{T} - \frac{Vdp}{T}$

$0 = \nu C_v dT/T + \frac{\nu R dT}{T} - \frac{\nu R\,dp}{P}$

$(C_v + R)dT/T = R\,dP/P$ $\quad dT/T = \frac{R}{C_v+R}\frac{dP}{P}$

$T = T_0 \left(P/P_0\right)^{R/R+C_v}$ The diatomic molecule has 5 dof $\Rightarrow C_v = 5/2 R$

$T = T_0 (P/P_0)^{2/7}$ Ans 2

c.) $dT/T = \frac{2}{7}\frac{dP}{P}$; $\frac{dP}{P} = -\frac{Mg\,dz}{RT}$

$7/2\ dT/T = -\frac{Mg\,dz}{RT} \Rightarrow T = T_0 - \frac{2}{7}\frac{Mgz}{R}$ Ans 2

$P = P_0 (T/T_0)^{7/2} \Rightarrow P = P_0\left(1 - \frac{2}{7}\frac{Mgz}{R}\right)^{7/2}$ Ans 2

Solution:

(a) The angles Θ_Y and Θ_Z are Fourier Transform pairs with the aperture geometry measured in units of the wavelength λ. Therefore, the **vertical direction on the image corresponds to the Y direction** because direction with the narrowest width corresponds to the widest angular scale.

(b) The angular pattern is unchanged in the limit that the distance to the screen is much greater than λ or the aperture size, i.e. we are in the Fraunhofer Diffraction limit.

(c) If λ is doubled, then the angular size of the intensity pattern **doubles**. For instance, the 1st zeros of intensity occur at the angles $\Theta_Y = \lambda/a$, $\Theta_z = \lambda/b$. The intensity pattern $I(\Theta_Y, \Theta_Z)$ is the Fourier Transform of the aperature in units of the physical size $\Rightarrow$ if λ increases, then dimensionless aperture size decreases $\Rightarrow \Theta_Y, \Theta_Z$ increase.

(d) The amplitude of the electric field on the image screen is proportional to the Fourier Transform of the opening aperture,

$$E(\Theta_Y, \Theta_Z) \propto \frac{e^{i(kr-\omega t)}}{r} \int \exp^{ik[\Theta_Y y' + \Theta_Z z']} dy' dz' \propto \int_{-a/2}^{a/2} e^{ik\theta_Y y'} dy' \int_{-b/2}^{b/2} e^{ik\theta_Z z'} dz' \tag{1}$$

Let $\alpha = ka\theta_Y/2$ and $\beta = kb\theta_Z/2$ and we get

$$E(\Theta_Y, \Theta_Z) \propto \left(\frac{\sin\alpha}{\alpha}\right)\left(\frac{\sin\beta}{\beta}\right) \tag{2}$$

Therefore, the intensity on the image screen is given by

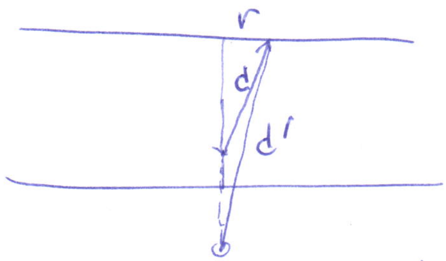
$$I \propto |E|^2 \propto \left(\frac{\sin\alpha}{\alpha}\right)^2 \left(\frac{\sin\beta}{\beta}\right)^2, \tag{3}$$

or

$$I = I(0)\left(\frac{\sin\alpha}{\alpha}\right)^2 \left(\frac{\sin\beta}{\beta}\right)^2, \tag{4}$$

where $I(0)$ is the central intensity.

Solution



G1-3 Ruderman

Section 5 - 3
Ruderman

$$I \propto \left| \frac{e^{ikd}}{d} - \frac{e^{ikd'}}{d'} \right|^2$$

minus sign due to phase change at mirrored surface $|\vec{E}| = 0$

$$I \propto \frac{1}{d^2} + \frac{1}{d'^2} - 2\frac{\cos k(d-d')}{dd'}$$

a.) $d = 2b - x \; ; \; d' = 2b + x$ $\quad (r = 0)$

For a maximum $\cos 2kx = -1$

$$x = \left(\frac{N}{2} - \frac{1}{4}\right)\lambda \quad \underset{\approx}{\text{Ans}} \quad N \text{ any integer } N \geq 1$$

b.) $x = b, \; r \neq 0 \quad d = \sqrt{r^2 + b^2} = b + r^2/2b$

$$d' = \sqrt{(3b)^2 + r^2} = 3b + r^2/6b$$

$$d - d' = -2b + \frac{r^2}{3b}$$

$$k(d-d') = \left(\underbrace{-2bk}_{\text{odd integer}} + r^2/3b \, k\right) = \text{odd integer} \times \pi$$

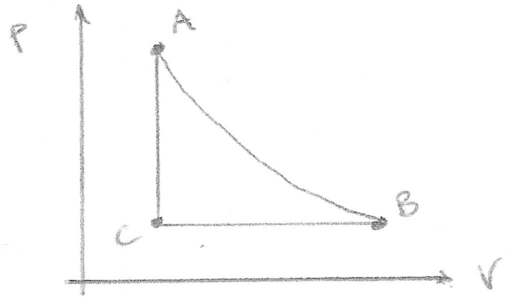
$$\Rightarrow \quad \frac{r^2}{3b}k = 2p\pi \quad p = 0, 1, 2 \ldots$$

$$r_p = \sqrt{3b\lambda p} \quad \underset{\approx}{\text{Ans}}$$

c.) The are no changes in the optical paths and wavelengths so the results of a + b hold.

Quals Question: General/Thermodynamics (Dodd)

One mole of a diatomic ideal gas is driven around the cycle ABCA shown on the pV diagram below. Step AB is isothermic, with a temperature T_A = 500 K; step BC is isobaric; and step CA is isochoric. The volume of the gas at point A is V_A = 1.00 L, and at point B is V_B = 4.00 L.



The ideal gas constant R = 8.31 J/mol·K, and $\gamma = 1 + \frac{2}{f}$ where f is the number of degrees of freedom for the molecular gas.

a). What is the pressure p_B at point B?

b). What is the total work done in completing one cycle (ABCA)?

c). What is the entropy change $S_C - S_B$?

Solution:

a). The ideal gas equation is:

$$pV = nRT$$

so:

$$p_B = \frac{nRT_B}{V_B}.$$

But $T_B = T_A$ since AB is isothermic, and so:

$$p_B = \frac{nRT_A}{V_B} = \frac{(1\ mol)\left(8.31\frac{J}{mol \cdot K}\right)(500\ K)}{(4.00 \times 10^{-3}\ m^3)} = \underline{1.04 \times 10^6\ Pa}$$

b). Determine the work done in each step, and sum:

For AB (isothermic: $T_B = T_A$):

$$W_{AB} = \int_{V_A}^{V_B} pdV = nRT_A \int_{V_A}^{V_B} \frac{dV}{V} = nRT_A ln\left(\frac{V_B}{V_A}\right)$$

For BC (isobaric: $p_C = p_B$, and $V_A = V_C$):

$$W_{BC} = \int_{V_B}^{V_C} pdV = p_B(V_C - V_B) = p_B(V_A - V_B)$$

For CA (isochoric: $V_A = V_C$):

$$W_{CA} = 0$$

So the total work is:

$$W = nRT_A ln\left(\frac{V_B}{V_A}\right) + p_B(V_A - V_B)$$
$$= (1\ mol)\left(8.31 \frac{J}{mol \cdot K}\right)(500\ K) ln\left(\frac{4.00 \times 10^{-3}\ m^3}{1.00 \times 10^{-3}\ m^3}\right)$$
$$+ (1.04 \times 10^6\ Pa)(1.00 \times 10^{-3}\ m^3 - 4.00 \times 10^{-3}\ m^3) = \underline{2{,}640\ J}$$

c). The 1st law tells us:

$$dU = dQ - dW$$

and since:

$$dS = \frac{dQ}{T}$$

then:

$$TdS = dU + pdV$$

i.e.

$$dS = \frac{dU}{T} + \frac{p}{T} dV$$

We know that:

$$U = \frac{1}{2} fnRT = \frac{nRT}{\gamma - 1}$$

so:

$$dS = \left(\frac{nR}{\gamma - 1}\right)\frac{dT}{T} + nR\frac{dV}{V}$$

Integrating to find the entropy change:

$$S_C - S_B = \left(\frac{nR}{\gamma - 1}\right)\int_{T_B}^{T_C}\frac{dT}{T} + nR\int_{V_B}^{V_C}\frac{dV}{V} = \left(\frac{nR}{\gamma - 1}\right)ln\left(\frac{T_C}{T_B}\right) + (nR)ln\left(\frac{V_C}{V_B}\right)$$

Since $T_B = T_A$, $p_C = p_B$, and $V_A = V_C$, then:

$$\frac{T_C}{T_B} = \frac{V_A}{V_B}$$

so for this diatomic gas, for which $f = 5$ (and $\gamma = \frac{7}{5}$):

$$S_C - S_B = (nR)ln\left(\frac{V_A}{V_B}\right)\left(\frac{1}{\gamma - 1} + 1\right) = (1\ mol)\left(8.31\frac{J}{mol \cdot K}\right)ln\left(\frac{1.00\times 10^{-3}\ m^3}{4.00\times 10^{-3}\ m^3}\right)\left(\frac{1}{\left(\frac{7}{5} - 1\right)} + 1\right)$$

$$= \underline{-40.3\ J/K}$$

G1-5

General Quals Problem

M. Shaevitz
Nov, 2015

A rope of uniform linear density μ and total length L is suspended from one end and hangs vertically under its own weight. It is lightly tapped at the lower end.

How long does it take for the perturbation to reach the top of the rope?

Solution:

For waves on a string $\Rightarrow \; v = \sqrt{\dfrac{T}{\mu}}$

The tension at a distance y above the lower end of the rope is: $T = \mu g y$

The wave velocity as a function of y above the end is: $v = \sqrt{\dfrac{T}{\mu}} = \sqrt{gy}$

Integrating the time over the length of the rope gives:

$$t = \int_0^L \frac{1}{v(y)}\,dy = \int_0^L \frac{1}{\sqrt{gy}}\,dy = 2\sqrt{\frac{L}{g}}$$

A. Mueller
G1-6

Problem in Statistical Physics

For a nonrelativistic ideal gas the partition function for N particles (of mass m) in a volume V is

$$Z_N = \frac{(Z_1)^N}{N!}$$

where $$Z_1 = \int \frac{d^3x\, d^3p}{h^3} e^{-\frac{\vec{p}^{\,2}}{2mkT}} = V/\lambda^3 \text{ with } \lambda = \frac{h}{\sqrt{2\pi m k T}}.$$

Now consider an ultrarelativistic ideal gas of N particles in a volume V. Give the following:

(i) The partition function Z_N at a temperature T;

(ii) $E(N,T)$, the energy of the gas;

(iii) P, the pressure of the gas.

If m is the mass of the particle when do you expect the ultrarelativistic approximation to be good

Solution: (i) $$Z_N = \frac{1}{N!}\left[\int \frac{d^3x\, d^3p}{h^3} e^{-\frac{pc}{kT}}\right]^N = \frac{1}{N!}\left(4\pi\left(\frac{kT}{hc}\right)^3 V\right)^N \left(\underbrace{\int_0^\infty z^2 dz\, e^{-z}}_{2}\right)^N$$

$$Z_N = \frac{V^N}{N!}\frac{1}{\lambda_R^{3N}} \qquad \lambda_R = \frac{hc}{2kT\,\pi^{1/3}}$$

(ii) $F = -kT \ln Z \qquad E = F + TS$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N} = +\frac{kT}{Z}\frac{\partial Z}{\partial T} + k\ln Z$$

$$E = \underbrace{-kT\ln Z}_{F} + kT\ln Z + \frac{kT}{Z}\frac{\partial Z}{\partial T} = kT^2 \overbrace{\frac{1}{Z}\frac{\partial Z}{\partial T}}^{3/T\, N} = \underline{3NkT = E}$$

(iii) $$P = -\frac{\partial F}{\partial V} = kT\frac{1}{Z}\frac{\partial Z}{\partial V} = kT\frac{N}{V}\ ; \ \boxed{PV = NkT}$$

Ultrarelativistic approximation should be good when

E/N = energy per particle $\gg mc^2$ or $\underline{3kT \gg mc^2}$