

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 9, 2012

3:10PM to 5:10PM

Classical Physics

Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

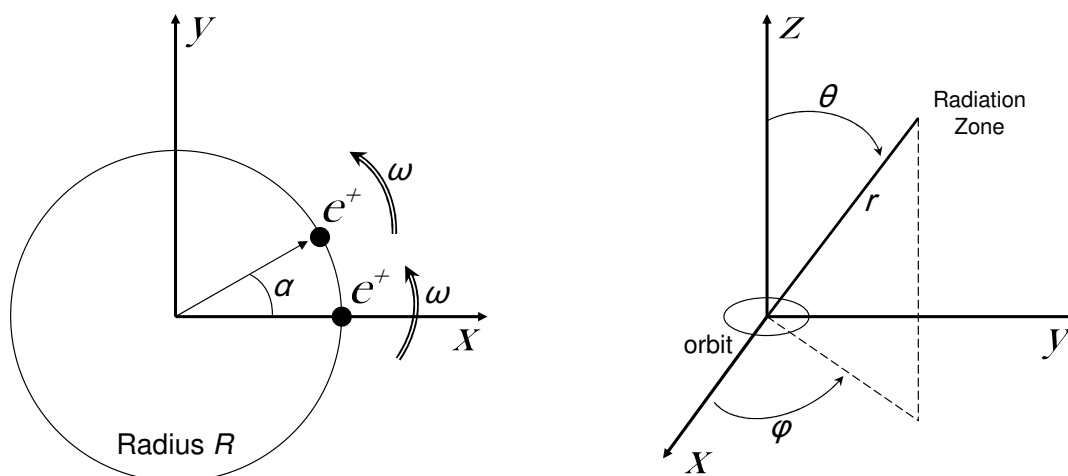
You may refer to the single handwritten note sheet on $8\frac{1}{2}$ " \times 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

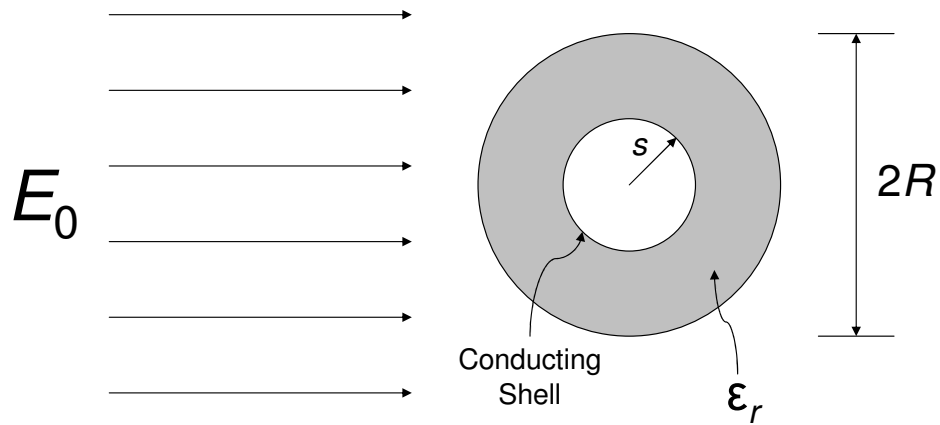
Questions should be directed to the proctor.

Good Luck!

- Two point particles (each having the same electric charge $+e$) travel in the xy -plane around the circumference of a circle (having radius R). Both charges travel at the same constant angular velocity ω but maintain a fixed angular separation α throughout the motion. Assume that the motion of the particles is non-relativistic ($v/c = \omega R/c \ll 1$). The particles radiate energy at distances r far from the circular orbit. Find the electric and magnetic fields produced in the radiation zone ($r \gg R$). Then, find the time-averaged power radiated per unit solid angle in the (θ, ϕ) direction shown in the diagram.



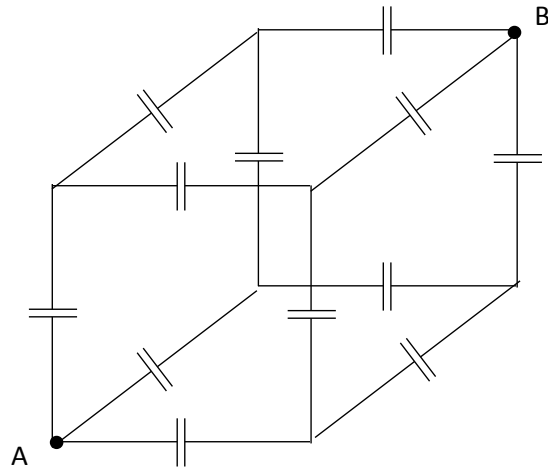
2. A dielectric sphere of radius R is hollowed-out in the region $0 \leq r \leq s$ and a thin, grounded, conducting shell inserted at $r = s$. The sphere is placed in a uniform, external E -field $E = E_0 \hat{z}$ along the z -axis. The dielectric constant is ϵ_r .
- (a) Calculate the potential in the region $r \geq R$.
- (b) Roughly sketch the polarization and induced charge in the region $r \leq R$.



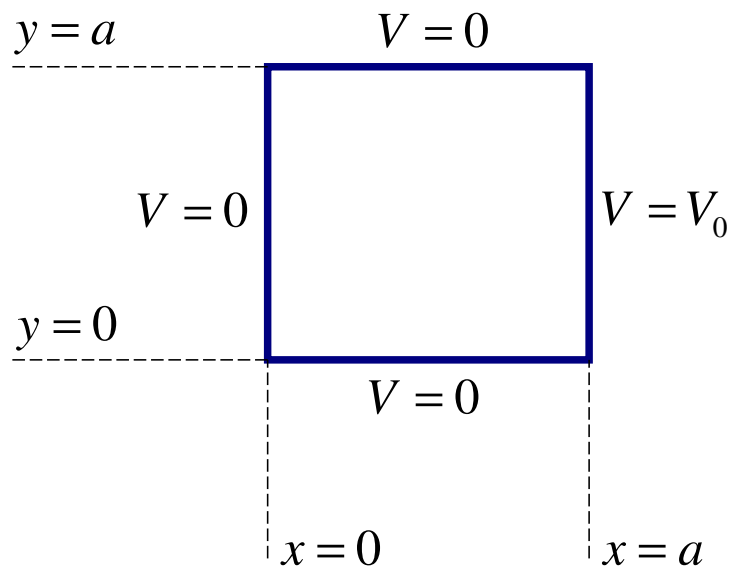
3.

- (a) Consider a long, straight cylindrical wire, along the z -direction, of electrical conductivity σ and radius a carrying a uniform axial current J . Calculate the magnitude and direction of the Poynting vector at the surface of the wire.
- (b) Consider a thick conducting slab (conductivity σ , and oriented in the xy -plane with the surface at $z = 0$) exposed to a normally incident plane EM wave with peak amplitudes E_0 and B_0 . Calculate the Poynting vector within the slab, averaged in time over one wave period. Consider σ large, namely $\sigma \gg \omega\epsilon_0$. (Hint: For large σ , the wave vector in the conductor, $K = (\alpha + i\beta)\hat{z}$ has coefficients $\alpha = \beta = \sqrt{\frac{\omega\mu_0\sigma}{2}}$). Assume that the conductor is nonferromagnetic.
- (c) In part (b), if σ is infinite, what is the average value of the Poynting vector everywhere in space?

4. Consider a set of 12 identical capacitors, each of capacitance C . As shown in the figure below, they are connected together such that they form the geometry of a cube. Find the equivalent total capacitance of this arrangement, as measured between points diagonally opposite one another (*e.g.* measured between points A and B in the figure).



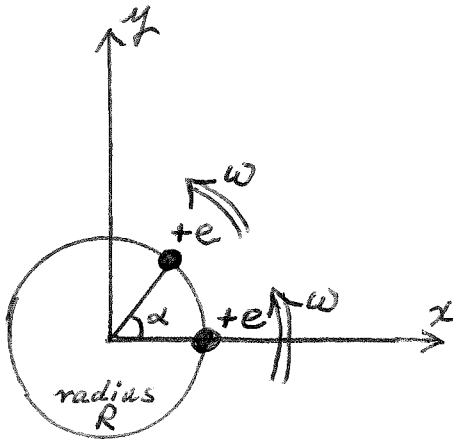
5. Consider an infinite pipe with a square cross section as drawn, with three sides grounded and one side at potential V_0 .
- Calculate the potential everywhere inside the pipe.
 - Calculate the capacitance per unit length between the side at potential V_0 and the remaining three sides.
 - If one of the sides adjacent to the side at potential V_0 is also brought to the same potential (with the other two sides kept grounded), what is the new potential inside the cube?



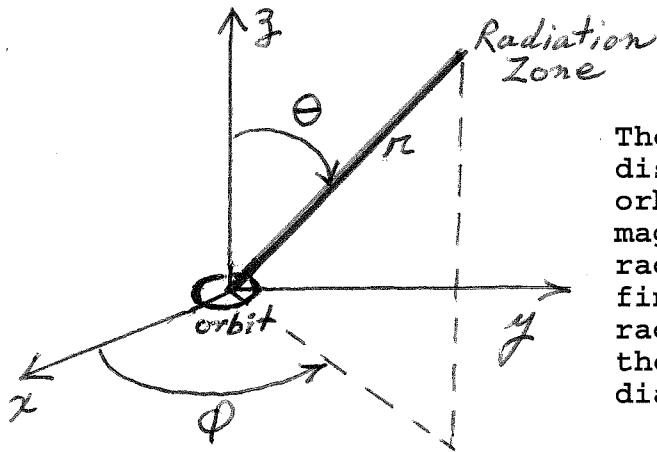
QUALIFYING EXAMINATION

E&M PROBLEM

ALLAN BLAER



Two point particles (each having the same electric charge $+e$) travel in the $x-y$ plane around the circumference of a circle (having radius R). Both charges travel at the same constant angular velocity ω but maintain a fixed angular separation α throughout the motion. Assume that the motion of the particles is non-relativistic ($v/c = \omega R/c \ll 1$).



The particles radiate energy at distances r far from the circular orbit. Find the electric and magnetic fields produced in the radiation zone ($r \gg R$). Then, find the time-averaged power radiated per unit solid angle in the (θ, ϕ) direction shown in the diagram.

E+M Problem Solution

This non-relativistic problem can be done either using the electric dipole approximation or using the non-relativistic Liénard-Wiechert fields (Larmor formula). Here is the electric dipole approach.

Electric dipole moment $\equiv \vec{D}(t)$.

$$\vec{D}(t) = eR[\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)] + eR[\hat{x}\cos(\omega t + \alpha) + \hat{y}\sin(\omega t + \alpha)]$$

Using complex notation: $\vec{D}(t) = eR(\hat{x} + i\hat{y})e^{-i\omega t} + eR(\hat{x} + i\hat{y})e^{-i(\omega t + \alpha)}$
(take Real part to get $\vec{D}_{\text{physical}}$)

$$\therefore \vec{D}(t) = eR(\hat{x} + i\hat{y})e^{-i\omega t}(1 + e^{-i\alpha})$$

In the radiation zone: $\vec{E} = \frac{1}{\pi} \frac{1}{c^2} [\hat{r} \times (\hat{r} \times \ddot{\vec{D}})]_{t' = t - \frac{r}{c}}$
(for electric dipole radiation)

$$\therefore \vec{E} = \frac{1}{\pi} \frac{1}{c^2} eR(-\omega^2) e^{-i\omega(t - \frac{r}{c})} [1 + e^{-i\alpha}] \left\{ \hat{r} \times [\hat{r} \times (\hat{x} + i\hat{y})] \right\}$$

$$\therefore \vec{E} = \frac{e^{i(kr - \omega t)}}{\pi} \left(\frac{-eR\omega^2}{c^2} \right) [1 + e^{-i\alpha}] \left\{ \hat{r} \times (\hat{r} \times \hat{x}) + i \hat{r} \times (\hat{r} \times \hat{y}) \right\}$$

In the radiation zone: $\vec{B} = \hat{r} \times \vec{E}$. ($k \equiv \frac{\omega}{c}$)

$$\overline{\frac{dP}{d\Omega}}^{\text{time avg}} = \frac{1}{2} \frac{c}{4\pi} (\vec{E}^* \cdot \vec{E}) r^2$$

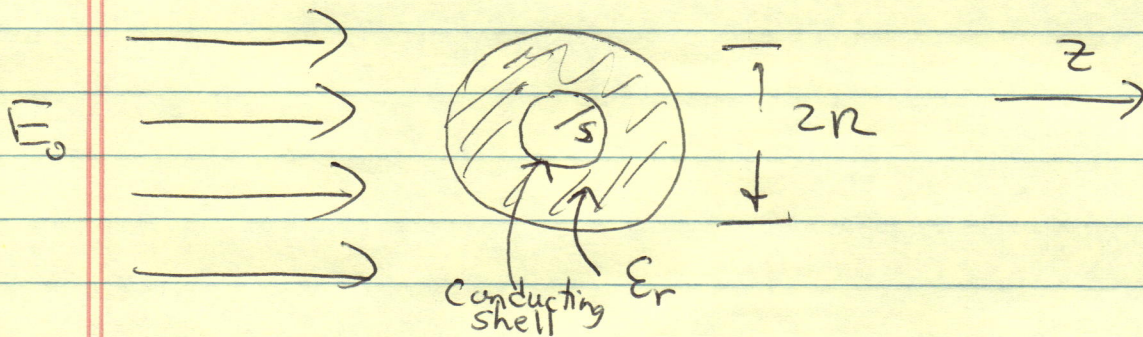
$$= \frac{c}{8\pi} \left(\frac{e^2 R^2 \omega^4}{c^4} \right) \underbrace{[2 + 2\cos\alpha]}_{4\cos^2(\frac{\alpha}{2})} \left\{ \underbrace{|\hat{r} \times (\hat{r} \times \hat{x})|^2}_{\underbrace{|\hat{r} \times \hat{x}|^2}_{\underbrace{\sin^2\theta_{\hat{r}, \hat{x}}}_{\underbrace{1 - (\hat{r} \cdot \hat{x})^2}_{1 - \sin^2\theta \cos^2\phi}}}} + \underbrace{|\hat{r} \times (\hat{r} \times \hat{y})|^2}_{\underbrace{|\hat{r} \times \hat{y}|^2}_{\underbrace{\sin^2\theta_{\hat{r}, \hat{y}}}_{\underbrace{1 - (\hat{r} \cdot \hat{y})^2}_{1 - \sin^2\theta \sin^2\phi}}}} \right\}$$

$$2 - \sin^2\theta = 1 + \cos^2\theta$$

$$\therefore \overline{\frac{dP}{d\Omega}} = \frac{e^2 R^2 \omega^4}{2\pi c^3} \cos^2\left(\frac{\alpha}{2}\right) [1 + \cos^2\theta]$$

Hailey E-M 1

A dielectric sphere of radius R is hollowed out in the region $0 \leq r \leq s$ and a thin, grounded conducting shell inserted at $r = s$. The sphere is placed in a uniform, external E-field $E = E_0 \hat{z}$ along the z -axis. The dielectric constant is ϵ_r .



a.) Calculate the potential in the region $r \geq R$

b.) roughly sketch the polarization and induced charge in the region $r \leq R$.

Solution

$$a) \quad r > R \quad \phi_o(r, \theta) = -E_o r \cos \theta + \sum_0^{\infty} a_n r^{-(n+1)} P_n(\cos \theta)$$

$$s \leq r \leq R \quad \phi_i(r, \theta) = \sum_0^{\infty} (b_n r^n + c_n r^{-(n+1)}) P_n$$

You can see only terms 0, 1 are required to match boundary conditions.

$$\phi_i(s, \theta) = 0 \Rightarrow b_0 + \frac{c_0}{s} = 0 \quad c_0 = -b_0 s \quad P_0$$

$$b_1 s + \frac{c_1}{s^2} = 0 \quad c_1 = -b_1 s^3 \quad P_1$$

$$\Rightarrow \phi_i(r, \theta) = b_0 \left(1 - \frac{s}{r}\right) P_0 + b_1 r \left(1 - \frac{s^3}{r^3}\right) P_1$$

$$\phi_o(R) = \phi_i(R)$$

$$b_0 \left(1 - \frac{s}{R}\right) = a_0 / R \quad P_0$$

$$\textcircled{1} \quad \boxed{b_1 R \left(1 - \frac{s^3}{R^3}\right) = -E_o R + \frac{a_1}{R^2}} \quad P_1$$

$$-\frac{\partial \phi_o}{\partial r} \Big|_R = -\epsilon_r \frac{\partial \phi_i}{\partial r} \Big|_R \quad (\text{D-field normal component continuous})$$

$$-s/R^2 b_0 \epsilon_r = a_0 / R^2 \quad P_0$$

$$\textcircled{2} \quad \boxed{-b_1 \epsilon_r - \frac{2s^3}{R^3} b_1 \epsilon_r = E_o + \frac{2a_1}{R^3}} \quad P_1$$

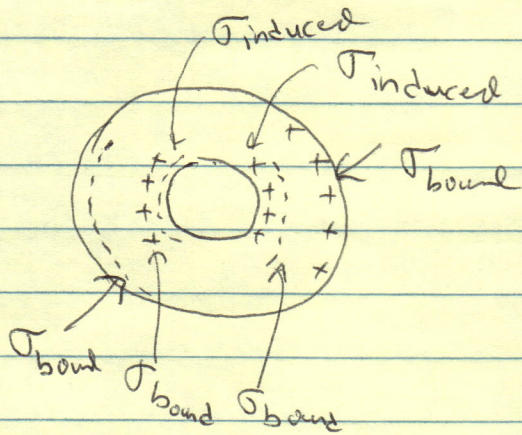
$$\Rightarrow a_0 = b_0 = 0$$

$$\text{From } \textcircled{1} + \textcircled{2} \quad b_1 = \frac{-E_o}{1 - \frac{s^3}{R^3} - \frac{1}{3} \left[(1 - \epsilon_r) - \frac{s^3}{R^3} (1 + 2\epsilon_r) \right]}$$

$$\text{Ans } \phi_o(r, \theta) = -E_o r \cos \theta \left\{ 1 + \frac{R^3}{3r^3} \frac{[(1 - \epsilon_r) - \frac{s^3}{R^3} (1 + 2\epsilon_r)]}{1 - \frac{s^3}{R^3} - \frac{1}{3} [(1 - \epsilon_r) - \frac{s^3}{R^3} (1 + 2\epsilon_r)]} \right\}$$

A quick check shows that for $S \rightarrow 0$ this approaches the dielectric sphere in uniform \vec{E} and for $R \rightarrow S$, $\epsilon_r \rightarrow 1$ it approaches the ~~the~~ conducting shell in uniform \vec{E} ,

b.)



All charges
decreasing towards
 $\theta = \pi/2$

Solution

(a) cylindrical coord. notes

$$\vec{E} = \frac{\vec{J}}{\sigma} = \frac{J}{\sigma} \hat{z}$$

Ampere's Law

Near surface

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi a} \hat{\theta}$$

$$B = \frac{\mu_0 J a}{2} \hat{\theta} \quad \text{since} \quad I = \pi a^2 J$$

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{J}{\sigma} \hat{z} \times \frac{J a}{2} \hat{\theta} \Rightarrow \boxed{\vec{S} = -\frac{J^2 a}{2\sigma} \hat{r}}$$

(b) $\vec{K} = (\beta + i\alpha) \hat{z}$

$$\vec{E}(\vec{r}, t) = E_0(r) e^{-(\beta z - \omega t)} e^{-\alpha z} \quad \text{Inside Conductor}$$

$$\vec{H} = \frac{1}{\omega \mu_0} \vec{K} \times \vec{E} = \frac{1}{\omega \mu_0} (\beta + i\alpha) \hat{z} \times \vec{E}$$

$$= \frac{1}{\omega \mu_0} \left(\frac{\omega \mu_0 \sigma}{2} \right)^{1/2} (1 + i) \hat{z} \times \vec{E}$$

$$= \sqrt{\frac{\sigma}{\omega \mu_0}} e^{\frac{\pi}{4} i} \hat{z} \times \vec{E}$$

$$(b) \quad \vec{S} = \vec{E} \times \vec{H}$$

$$= \sqrt{\frac{\sigma}{\omega \mu_0}} e^{i\frac{\pi}{4}} (\vec{E} \times (\hat{z} \times \vec{E}))$$

$$= \sqrt{\frac{\sigma}{\omega \mu_0}} e^{i\frac{\pi}{4}} E^2 \hat{z}$$

Averaged over one period,

$$\overline{S} = \frac{1}{2} \text{Re} (E^* \times H)$$

$$= \frac{1}{2} \sqrt{\frac{\sigma}{\omega \mu_0}} \cos\left(\frac{\pi}{4}\right) E_0^2 e^{-2\alpha z} \hat{z}$$

$$\overline{S} = \frac{\sqrt{2}}{4} \sqrt{\frac{\sigma}{\omega \mu_0}} E_0^2 e^{-2\alpha z} \hat{z}$$

(c) As $\sigma \rightarrow \infty$, $d \rightarrow \infty$

$$\Rightarrow \sqrt{\sigma} e^{-2\alpha z} \rightarrow 0$$

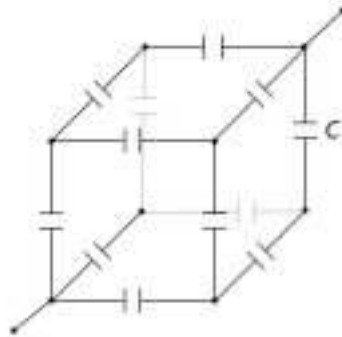
So, $\overline{S} = 0$ inside

Total reflection at surface gives stationary waves.

$$\overline{S} = \phi$$

E&M Question

Consider a set of 12 identical capacitors, each of capacitance C . As shown in the figure below, they are connected together such that they form the geometry of a cube. Find the equivalent total capacitance of this arrangement, as measured between points diagonally opposite one another (eg. measured between the lower left point on the figure and the upper right point).



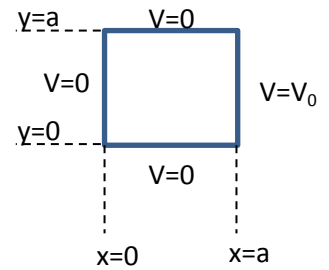
Answer

By symmetry, the 3 capacitors that connect to the bottom left corner all must have the same voltage across them. Therefore, adding wires that connected their “downstream” ends together would not change the circuit. The same statement can be made about the 3 capacitors connected to the upper right corner.

Thus the circuit can equivalently be thought of as two sets of 3 parallel capacitors each, separated by the remaining 6 capacitors, which are all in parallel with each other; ie. a series combination of 3 parallel capacitors followed by 6 parallel capacitors followed by 3 parallel capacitors. This is equivalent to a series combination of a $3C$ capacitor, a $6C$ capacitor and a $3C$ capacitor (since in series $C_{\text{tot}} = C_1 + C_2 + \dots$). The equivalent capacitance of this series combination is found by solving $1/C_{\text{eq}} = 1/(3C) + 1/(6C) + 1/(3C)$, giving $C_{\text{eq}} = 6C/5$.

E+M

Consider an infinite pipe with a square cross section as drawn, with three sides grounded and one side at potential V_0



(a) Calculate the potential everywhere inside the pipe

(b) Calculate the capacitance per unit length between the side at potential V_0 and the remaining three sides.

(d) If one of the sides adjacent to the side at potential V_0 is also brought to the same potential (with the other two sides kept grounded), what is the new potential inside the cube?

Solution

(a) Solutions are products of sines and cosines with sinhs and coshs

$$V(x, y) = X(x)Y(y)$$

Since $V=0$ for $y=0$ and $y=a$, solutions for $Y(y)$ have to be sines

$$Y_n(y) = A_n \sin(k_n y)$$

with $k_n = n\pi/a$

Since $V=0$ for $x=a$, solutions for $X(x)$ have to be sinhs

$$X_n(x) = B_n \sinh(k_n x)$$

$$V(x, y) = \sum A_n B_n \sinh(k_n x) \sin(k_n y) = \sum C_n \sinh(k_n x) \sin(k_n y)$$

Plug in the boundary condition at $x=a$

$$\sum C_n \sinh(k_n a) \sin(k_n y) = V_0$$

Solution by Fourier decomposition:

$$C_{2n+1} \sinh(k_{2n+1} a) = 4V_0 / (2n + 1)\pi a \text{ and even coefficients are zero}$$

So

$$V(x, y) = \sum \frac{4V_0}{\pi(2n + 1)a \sinh(k_{2n+1} a)} \sinh(k_{2n+1} x) \sin(k_{2n+1} y)$$

(b) To calculate capacitance we need charge on the plates. The charge density we can calculate using the electric field

$$\mathbf{E}_\perp = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$$

For the plate at V_0 , this gives:

$$\frac{\sigma}{\epsilon_0} = -E_x|_{x=a} = \frac{\partial V}{\partial x}\bigg|_{x=a} = \sum C_n k_n \cosh(k_n a) \sin(k_n y)$$

Total charge per unit length of the pipe

$$Q = \int \sigma dA = \int_0^a \sigma dy = \epsilon_0 \sum C_n k_n \cosh(k_n a) \int_0^a \sin(k_n y) dy$$

$$= 2\varepsilon_0 \sum C_n \cosh(k_n a)$$

Net charge induced on the remaining 3 plates is $-Q$, so the capacitance per unit length is

$$C = \frac{Q}{V_0} = \frac{2\varepsilon_0 \sum C_n \cosh(k_n a)}{V_0}$$

(c) Use superposition: solution is the sum of the solution of part (a) above and the solution with x and y interchanged, since it will satisfy Laplace's equation as well as all the boundary conditions.

$$V(x, y) = \sum \frac{4V_0}{\pi(2n+1)a \sinh(k_{2n+1}a)} \sinh(k_{2n+1}x) \sin(k_{2n+1}y) \\ + \sum \frac{4V_0}{\pi(2n+1)a \sinh(k_{2n+1}a)} \sinh(k_{2n+1}y) \sin(k_{2n+1}x)$$