2015 Solution Explanations
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1. Order of magnitude calculation: Imagine that a tea cup could be made impervious to neutrons (spin = 1/2). How many neutrons could the cup hold without running over, if the cup and its neutrons were cooled as close to absolute zero as possible? Assume that the cup is in vacuum in a constant gravitational field $g = 9.8$ m/s^2. Don't forget to consider the difference in gravitational potential energy between the top and bottom of the cup.

Let's assume our cup is 30cm by 30cm, in the shape of a half-sphere. The volume is

$$V = \frac{1}{2}\left(\frac{4}{3}\pi(0.3m)^3\right) = 0.0565m^3$$

The Fermi energy is the energy difference between the highest and lowest occupied single-particle states in a quantum system of non-interacting fermions at *absolute zero temperature,* like we have here. We assume that the only relevant energy is the gravitational potential energy, mgL. We can set these equal.

$$E_F \sim E_{pot}$$

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} = mgL$$

Now simply isolate for N and solve, where L is 0.3m, V is 0.0565m^3, m is the mass of a neutron. $\boxed{\text{This gives N} \sim 1.08\text{x}10^{20}}$.

Why Fermi energy? It's the energy diff between the highest and lowest states – in our case, the most significant energy difference is the difference between potential energy at the top and bottom of the cup. The fermi energy is defined at absolute 0, and because we actually are at absolute 0, it provides a good representation of the real energy diff between states.

2. A solid superconducting sphere of radius a is placed in a uniform external magnetic field $\vec{B} = B_0\hat{z}$. Because it is superconducting, the sphere expels magnetic fields, and $\vec{B} = 0$ everywhere inside the sphere. The superconductor achieves this by acquiring a uniform magnetization (magnetic moment density) $\vec{M} = M\hat{z}$ inside its volume. It can be shown that the magnetic field due to a uniformly magnetized sphere is a perfect dipole field in the region outside of the sphere.

A. Calculate the *total* magnetic field everywhere outside of this superconducting sphere. Hint: the $\vec{B}$ field never penetrates the sphere's surface.

B. Calculate the surface current density on the sphere as a function of the polar angle θ, measured from the $+z$ axis.

A. Our conducting sphere in inside an external magnetic field. We also know that the sphere is uniformly magnetized, and uniformly magnetized spheres creates a magnetic field that looks like a perfect dipole. So, our total B field is

$$B = \vec{B}_{dip} + B_0\hat{z}$$

We need B$_{dipole}$. From formula sheet/Griffiths:
$B_{dip} = \frac{\mu_0}{4\pi r^3}((3\vec{m}\cdot\hat{r})\hat{r} - \vec{m})$

Where m is the magnetic moment. We shouldn't just leave it m, we should find this in terms of variables we know. We can say our total magnetic moment is M. The magnetization acts to cancel out the magnetic field so it is zero inside the sphere, so $M \propto -\frac{B_0}{\mu_0}\hat{z}$ (get the μ_0 from unit analysis). But we also know (from formula sheet…) that $M_{uniformsphere} = \frac{3}{2\mu_0}B \rightarrow$
$B_{uniformsphere} = \frac{2\mu_0}{3}M = -B_0\hat{z} \rightarrow$

$$M = \frac{-3}{2\mu_0}B_0\hat{z}$$

(would I be supposed to derive this?) On an exam I would probably just say $M = -\frac{B_0}{\mu_0}\hat{z}$ and be done with it.

$$B_{dip} = \frac{\mu_0}{4\pi r^3}\big((3M\hat{z}\cdot\hat{r})\hat{r} - M\hat{z}\big)$$

We should convert the z-hat to spherical coordinates.

$$B_{dip} = \frac{\mu_0}{4\pi r^3}\Big(\big(3M(cos\theta\hat{r} - sin\theta\hat{\theta})\cdot\hat{r}\big)\hat{r} - M\hat{z}\Big) = \frac{\mu_0}{4\pi r^3}\big((3Mcos\theta)\hat{r} - M\hat{z}\big)$$

$$B_{tot} = \frac{\mu_0}{4\pi r^3}\left(\left(3\frac{-3}{2\mu_0}B_0cos\theta\right)\hat{r} - \frac{-3}{2\mu_0}B_0\hat{z}\right) + B_0\hat{z} = \frac{-3B_0}{8\pi r^3}(3cos\theta\hat{r} - \hat{z}) + B_0\hat{z}$$

Test: goes to B_0 at infinity. True!

Note from future: Looking back, I plugged in big M here, but the magnetization is $M = \frac{m}{V}$.
So maybe I should be multiplying my dipole field by $\frac{4}{3}\pi a^3$?

B. Surface charge density as a function of polar angle theta.

$$K_b = M\times\hat{n} = M\times\hat{r} = \frac{-3}{2\mu_0}B_0\hat{z}\times\hat{r} = \frac{-3}{2\mu_0}B_0\big(cos\theta\hat{r} - sin\theta\hat{\theta}\big)\times\hat{r} = \frac{+3}{2\mu_0}B_0sin\theta\hat{\theta}\times\hat{r}$$

$$\boxed{= \frac{-3}{2\mu_0}B_0sin\theta\hat{\phi}}$$

3. Consider a beam of neutrons of mass m_N, momentum p and magnetic moment $\vec{\mu} = \gamma\vec{s}$. The beam's polarization P is defined by

$$P = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow}$$

where $n_\uparrow$ and $n_\downarrow$ are the numbers of neutrons with spin up and spin down in the z direction, respectively. Note that $\vec{s}$ is the spin angular momentum.

An unpolarized beam is incident from $x = -\infty$ and it encounters a constant magnetic field which ramps up in magnitude from $B = 0$ when $x < 0$ to field strength $B > 0$ when $x > 0$. Assume that the magnetic field profile can be approximated by a step function, that is,

$$\vec{B}(\vec{x}) = B\hat{z}\theta(x)$$

A. For what range of momentum will the transmitted beam be 100% polarized?

B. Calculate the polarization of the reflected beam as a function of p.

A.
We have an unpolarized beam coming in from $x = -\infty$. Once it hits x = 0, there is a constant magnetic field in the z direction, $B = B\hat{z}$. This is a scattering problem – as there is an additional Hamiltonian when x > 0 , some of the beam will be reflected/transmitted depending on it's energy.

For $x < 0$, the beam has energy $H = \frac{p^2}{2m}$.

For $x > 0$, now there is an additional potential, $H' = -\mu \cdot B = -\gamma S_z B_z. \rightarrow H_{tot} = \frac{p^2}{2m} - \gamma S_z B_z$.

What potential the neutrons see depend on their spins: spin up neutrons see an attractive (negative) potential, $V = -\frac{\gamma hB}{2}$, while spin down neutrons will see a repulsive potential, $V = +\frac{\gamma hB}{2}$. So spin down neutrons do not want to be here.

Because the potential is infinite in height (see diagram) there is no tunneling. So if the energy of the beam is not high enough to cross the barrier, it will be entirely reflected. Therefore, to be transmitted, the energy of the beam must be

$$\boxed{E_{beam} = \frac{p^2}{2m} > \frac{\gamma hB}{2} \rightarrow p = \sqrt{\gamma hBm}}$$

B. We need to find the reflection coefficients for the spin up and spin down states. For this, we need the wavefunction. There will be a different wavefunction for the spin up and down neutrons:

For x < 0:

$$\psi_I(x) = A_\pm e^{ik_1 x} + B_\pm e^{-ik_1 x} = A_\pm e^{\frac{ip}{h}x} + B_\pm e^{-\frac{ip}{h}x}$$

Because $k_1 = \sqrt{\frac{2mE}{h^2}} = \sqrt{\frac{2m(p^2/2m)}{h^2}} = \frac{p}{h}$

For x > 0:

$$\psi_I(x) = C_\pm e^{ik_2 x} + D_\pm e^{-ik_2 x} = C_\pm e^{ik_2 x}$$

The D term disappears because that represents a wave coming in from the right, which is not happening here.

$$k_2 = \sqrt{\frac{2m(E-V)}{h^2}} = \sqrt{\frac{2m(\frac{p^2}{2m} \mp \frac{\gamma hB}{2})}{h^2}} = \sqrt{\frac{p^2 \mp \gamma hBm}{h^2}}$$

We can use boundary conditions to find the constants. The BCs are:

1. $\psi(t)$ constant at x = 0
2. $d\psi(t)/dx$ constant at x = 0

1. $A_\pm + B_\pm = C_\pm$

2. $\frac{ip}{h} A_\pm - \frac{ip}{h} B_\pm = ik_\pm C_\pm$

We need a third condition: $|A|^2 + |B|^2 = 1$

I don't really want to do the algebra here, but I think we just need to solve for the reflection coefficient, B. Then we can just find:

$$P = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow} \rightarrow P = \frac{|B_+|^2 - |B_-|^2}{|B_+|^2 + |B_-|^2}$$

4. Consider two electrons (spin-$\frac{1}{2}$ fermions) placed in a three-dimensional, isotropic harmonic potential given by $U(r) = (1/2)m\omega^2 r^2$.

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A. What are the energies and degeneracies of the two lowest energy levels of the two-particle system?

B. Suppose that the electrons have a magnetic moment and interact via a weak spin-spin interaction of the form $\hat{V} = \alpha\hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2$. How are the energies and degeneracies of the states in Part A changed by this spin-spin interaction?

Here we have a harmonic oscillator with potential $U(x, y, z) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2)$, with $E = E_x + E_y + E_z = \left(n_x + n_y + n_z + 3\ \times \frac{1}{2}\right)h\omega$ and $\psi = \psi(x)\psi(y)\psi(z)$. We have two particles, which can either be spin up or spin down.

The lowest energy level is the ground state, when n = 0, so $E = 2\left(\frac{3}{2}\right)h\omega = 3h\omega$ (the factor of 2 is from the sum of both particle's energies). Because of the Pauli exclusion principle, the electrons cannot occupy the same state, so one must be spin up and the other spin down. The overall wavefunction must be anti-symmetric, so they can only be in the s = 0 singlet state, so degeneracy 1.

$$\boxed{E_0 = 3h\omega;\ d_n = 1}$$

The next lowest energy level is $3h\omega + 1h\omega = 4h\omega$. We can achieve this by having one particle is in the ground state and one is in the first excited state: $E = \frac{3}{2}h\,\omega + \frac{5}{2}h\omega = 4h\omega$. They are not in the same state, so we don't have to worry about Pauli.

The degeneracy of the second particle in the first excited state is 3, because it can either be (nx ny nz) = (1 0 0), (0 1 0), or (0 0 1). When considering the combined spins of the particles, they either be $\uparrow\uparrow, \downarrow\downarrow, \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), or\ \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$. So, the total degeneracy is 3x4=12.

$$\boxed{E_1 = 4h\omega;\ d_n = 12}$$

Just for fun let's do the next lowest energy level, which would be $E = 5h\omega$. This can be achieved as long as we have two n's equalling one, which is either

(0 0 0) + (2 0 0), (0 2 0), (0 0 2)

(0 0 0) + (1 1 0), (1 0 1), (0 1 1)

(1 0 0), (0 1 0), (0 0 1) + (1 0 0), (0 1 0) (0 0 1)

The degeneracy is 12 for those first two because of the reasons above, so d = 24. For the bottom one, we have to consider Pauli exclusion again, where if they are in the same state the spins must be anti-symmetric, so degeneracy = 1 when they're in the same state. This adds a total of 3 to our degeneracy, for the states (1 0 0) (1 0 0), (0 1 0) (0 1 0), and (0 0 1), (0 0 1).

Then we have remaining:
(1 0 0), (0 1 0)
(1 0 0), (0 0 1)
(0 1 0), (0 0 1)
And each of these have 4 spin states. So dn = 24+3+3x4 = 39.
(Note: Why is it that the states directly above can't repeat with the different particles? For example, shouldn't (1 0 0), (0 1 0) and (0 1 0) (1 0 0) be considered different states? No – because the particles are indistinguishable, so we can't tell those apart)

B. Now our Hamiltonian has a perturbation:

$$H = H + H' = \frac{p^2}{2m} + U(r) + \alpha s_1 \cdot s_2$$

According to first order perturbation theory:

$$E_n^1 = \langle n|H'|n\rangle = \alpha\langle n|s_1 \cdot s_2|n\rangle$$

Recall $s_1 \cdot s_2 = \frac{1}{2}(S^2 - s_1^2 - s_2^2)$

Our general wavefunction has two parts, the wavefunction dictating the position of the particle (for SHO, the Hermite polynomials) as well as the wavefunction describing the spins. This perturbation only acts on the spins, so we can ignore the Hermite polynomials.

$$E_n^1 = \frac{\alpha}{2}\langle n|S^2 - s_1^2 - s_2^2|n\rangle$$

S depends on the total spin of the 2 particles, and s_1 and s_2 are always the same, because they depend on the intrinsic spin of each one, which is ½. So that means there are two options.
Recall

$$S^2|s\ m> = h^2 s(s+1)|l\ m>$$

When S = 0

$$\frac{\alpha}{2}\langle n|S^2 - s_1^2 - s_2^2|0\ 0\rangle = \frac{\alpha}{2}\langle n|S^2 - s_1^2 - s_2^2\ (|\frac{1}{2}\frac{1}{2}\rangle\left|\frac{1}{2}\frac{-1}{2}\right\rangle - \left|\frac{1}{2}\frac{-1}{2}\right\rangle\left|\frac{1}{2}\frac{1}{2}\right\rangle)$$

Where of course n is the same on the left, but I am lazy.

$\frac{\alpha}{2}\langle 0\ 0|S^2 - s_1^2 - s_2^2|0\ 0\rangle = \frac{\alpha}{2}\left\langle 0\ 0\middle|0|0\ 0\rangle - h^2(\frac{1}{2})(\frac{3}{2})|0\ 0\rangle - h^2(\frac{1}{2})(\frac{3}{2})|0\ 0\rangle\middle|\quad\right\rangle =$
$\langle 0\ 0|\alpha h^2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)|0\ 0\rangle = \frac{-3h^2\alpha}{4}$

When S =1 (note: notation is kind of wrong below, whatever)
NOTE FROM FUTURE: Should I be doing something different because this is non-degenerate perturbation theory?)
$\frac{\alpha}{2}\langle n|S^2 - s_1^2 - s_2^2|n\rangle = \frac{\alpha}{2}\left\langle n\middle|2h^2 - h^2(\frac{1}{2})(\frac{3}{2}) - h^2(\frac{1}{2})(\frac{3}{2})\middle|n\right\rangle = \alpha h^2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right) = \frac{+h^2\alpha}{4}$

The lowest energy level (because it is the singlet state) changes to

$$\boxed{E_0 = 3h\omega - \frac{3h^2\alpha}{4};\ d_n = 1}$$

And the second lowest energy level changes to

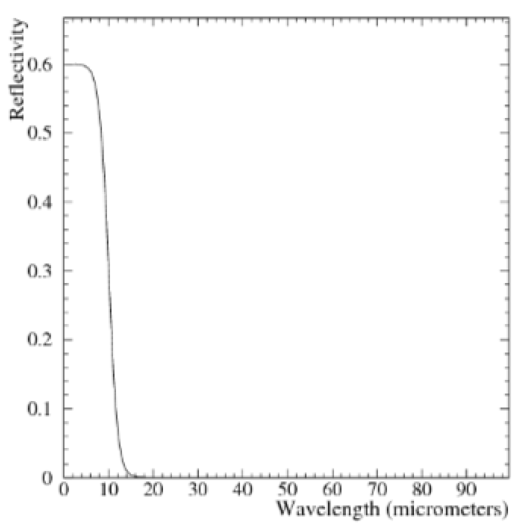
$$\boxed{E_1 = 4h\omega - \frac{3h^2\alpha}{4}, d_n = 3}$$

For the singlet states, and

$$\boxed{E_1 = 4h\omega + \frac{h^2\alpha}{4}, d_n = 9}$$

For the states with S=1 (there were 3 singlet states, and the rest were triplets).

5. Assuming that the Sun is a perfect black body radiator and that all of Pluto's thermal energy comes from the Sun, estimate the expected average temperature of the surface of Pluto given that the surface of the Sun is at 5778 K.
Required constants: the Pluto-to-Sun distance is about 5.4 light-hours and the Sun's radius is about 2.3 light seconds. The surface reflectivity of Pluto is measured to be:



The power radiated by the sun, according to the S B law, is:

$$P_{sun} = A_{sun}\sigma T_{sun}^4$$

This power drops off according to the inverse square law. The flux (power per area) Pluto will receive is:

$$f_{@p} = \frac{P_{sun}}{4\pi d_{s2p}^2}$$

This flux hits Pluto, and some is absorbed by the planet, raising the temperature. We can find the amount absorbed using the plot. Wiens law:

$$\lambda = \frac{b}{T} = 2.89 \times \frac{10^{-3} mK}{5778K} = 0.5\mu m$$

The reflectivity at this wavelength is 0.6, so 40% of the power from the sun is absorbed and contributing to raising its temperature. Now, only one side of Pluto is facing the sun. So the area in our Power equation (SB law) should not be the area of the whole planet, but the area of the disk facing the sun.

$$P_{pluto} = 0.4 \times f_{@p} \pi r_p^2 = A_p \sigma T_p^4$$

$$T_p = \left(\frac{0.4 \times \frac{P_{sun}}{4\pi d_{s2p}^2} \pi r_p^2}{4\pi r_p^2 \sigma} \right)^{\frac{1}{4}} \sim 50\ K$$

So, the temp of Pluto is around 50K or -223 degrees Celsius! ☺

6. A white dwarf is a stellar remnant supported by electron degeneracy pressure. Electrons become degenerate when they are packed closely enough that the Pauli exclusion principle produces an additional form of pressure to keep them apart. The electron degeneracy pressure is a consequence of the Heisenberg uncertainty principle. If you call N_e the number density of electrons, what is the uncertainty Δx on the electron location in a white dwarf? For non-relativistic electrons, what is the uncertainty in velocity Δv? Explain why it makes sense to write the degeneracy pressure as $P_{degen} \sim N_e m_e (\Delta v)^2$, where m_e is the electron mass. At the center of a white dwarf of mass M and radius R, the gravitational pressure is $P_{grav} \sim GM^2/R^4$. Derive the relation between R and M for a stable white dwarf (i.e. when $P_{degen} \sim P_{grav}$). How does R scale with mass M?

Because we're dealing with Δx and Δv, it's clear we will have some Heisenberg uncertainly principles in here. First:

$$N_e = \frac{N}{V} = \frac{N}{\frac{4}{3}\pi r^3} \sim \frac{N}{\frac{4}{3}\pi \Delta x^3} \rightarrow \Delta x \sim N_e^{-\frac{1}{3}}$$

$$\Delta x \Delta p = \Delta x\ m\Delta v > \frac{h}{2} \rightarrow \Delta v > \frac{h}{2m_e \Delta x} \sim \frac{N_e^{\frac{1}{3}} h}{2m_e}$$

As usual, everything should be hbar.

Why does it make sense to write the degeneracy pressure as $P_{degen} \sim N_e m_e (\Delta v)^2$?

Think of ideal gas law: $P = \frac{NkT}{V} = \left(\frac{N}{V}\right) kT \sim N_e kT$

kT is our energy -> $kT \sim E \sim \frac{1}{2} m v^2 \sim m_e (\Delta v)^2$!!!!!!! $P \sim N_e kT \sim N_e m_e (\Delta v)^2$.

Now, for the white dwarf to be in hydrostatic equilibrium, the pressures must balance.

$$P_{degen} \sim P_{grav}$$

$$\frac{GM^2}{R^4} = N_e m_e (\Delta v)^2 = N_e m_e \left(\frac{hN_e^{\frac{1}{3}}}{2m_e}\right)^2 \sim N_e^{\frac{5}{3}}$$

I'm ignoring constants. As we know, Ne ~ N/V.

$$N_e^{\frac{5}{3}} = \left(\frac{N}{V}\right)^{5/3} \sim \left(\frac{N}{R^3}\right)^{5/3}$$

N? We can write this in terms of mass too. $M = Nm_e \rightarrow N \sim M$

$$N_e^{\frac{5}{3}} \sim \left(\frac{M}{R^3}\right)^{5/3} = \frac{M^{\frac{5}{3}}}{R^5} = \frac{M^2}{R^4}$$

$$M^{\frac{5}{3}-\frac{6}{3}} \sim R \rightarrow \; M^{-\frac{1}{3}} \sim R$$

So, as the mass increases, the radius decreases!

8. The speed of ocean waves on the surface is found to depend on the density ρ of the fluid, the gravitational acceleration g, and the wavelength λ according to $v = K\rho^a g^b \lambda^c$, provided that the water is very deep. Here K is a dimensionless constant.

A. Determine the exponents a, b, and c.

B. Consider two sinusoidal ocean waves with frequencies ω and $\omega + \Delta\omega$, where $\Delta\omega \ll \omega$. If superimposed these will form a beat pattern. If the velocity at frequency ω is v, calculate the effective group velocity of the beat pattern.

A. According to simple dimensional analysis, the formula for velocity is

$$v = K\sqrt{g\lambda}$$

B.

When superimposed, these two waves form a beat pattern with frequency $\Delta\omega$ ($\omega_2 - \omega_1$). A beat pattern is when two waves with different frequencies constructively and deconstructively interfere continuously. When this happen with sound waves, we hear the beats at a frequency of $\Delta\omega$.

We want to find the group velocity of the beat pattern. The phase velocity is the velocity of the peaks of each wave, $v_{phase} = \frac{\omega}{k}$. This equation is like the standard $v = \frac{x}{t}$ equation (note $k = \frac{2\pi}{\lambda}$) so is in units of 1/m or radians/meter. The group velocity is the velocity with which the entire group of waves, wave 1 + wave 2, could travel, , $v_{group} = \frac{d\omega}{dk}$.

We know $d\omega = \Delta\omega$; $dk = k_2 - k1 = \frac{2\pi}{\lambda_1} - \frac{2\pi}{\lambda_2}$. To find an equation for k, let's plug this into our equation for v early so we can get it in terms of variables we know.

$$v = K\sqrt{g\lambda} = \frac{\omega}{k} = K\sqrt{\frac{g2\pi}{k}} \rightarrow \frac{k^2 2\pi g}{k} = \frac{\omega^2}{K^2} \rightarrow k = \frac{\omega^2}{2\pi g K^2}$$

$$k_2 - k_1 = \frac{(\omega + \Delta\omega)^2}{2\pi g K^2} - \frac{(\omega)^2}{2\pi g K^2} = \frac{\omega^2 + 2\omega\Delta\omega + \Delta\omega^2 - \omega^2}{2\pi g K^2} = \frac{2\omega\Delta\omega}{2\pi g K^2}$$

So therefore, the group velocity is

$$\boxed{v_{group} = \frac{d\omega}{dk} = \frac{\Delta\omega 2\pi g K^2}{2\omega\Delta\omega} = \frac{\pi g K^2}{2\omega}}$$

9. A supercapacitor is an energy storage device with a very high capacitance C in series with an internal resistance R_i. It can be charged to a maximum voltage V_{max}. You desire to use one to power a load that requires a constant power input P. The load automatically varies its resistance R_L in order to keep the delivered power in R_L constant.

A. How long could the capacitor provide constant power P, if it is initially charged to V_{max}?

B. What fraction of the capacitor's total stored energy is still in the capacitor at the point when it can no longer deliver the requested power?

This one is a bit mysterious, because according to this thread: https://math.stackexchange.com/questions/183292/is-this-a-non-linear-differential-equation-and-is-there-a-solution#comment423835_183292

If the resistance is included, we have an incredibly complicated equation to solve. If resistance is not included: we need to find a differential equation for voltage involving power and capacitance.

$$Q = CV \rightarrow \frac{dQ}{dt} = C\frac{dV}{dt} = I$$

But also $I = \frac{P}{V} \rightarrow \; C\frac{dV}{dt} = \frac{P}{V}$

$$\int V dV = \frac{\int P}{C} dt$$

$$\frac{V^2}{2}\Big|_{V_{max}}^{V} = \frac{P}{C}t \rightarrow t = \frac{CV_{max}^2}{2P}$$

Constant power until V hits 0. Also note, maybe should have flipped a and b around in the integral. The thing is, I don't know how this is correct without including the resistance, because the delivered power is constant *because* of varying the resistance.

B. Not sure about this one. Maybe as a way to at least get some part marks, I can say the total energy is

$$W = \int P dt = \frac{C}{2}(V_{max}^2 - V^2)\int \frac{dt}{t} = \frac{C}{2}(V_{max}^2 - V^2)\ln\left(\frac{t_{max}}{t}\right) = \frac{C}{2}(V_{max}^2 - V^2)\ln\left(\frac{\frac{CV_{max}^2}{2P}}{t}\right)$$

What is the initial t though? It can't be zero. I feel like this also just tells us the total energy, not the stored fraction…

10. Do an order of magnitude calculation of the atomic number Z at which atomic electrons begin to move at relativistic speeds (defined here as $pc > mc^2/2$, where p is the electron's momentum and m is its rest mass).

Our threshold for the speeds being relativistic is $pc > \frac{mc^2}{2} \rightarrow mvc > \frac{mc^2}{2} \rightarrow v > \frac{c}{2}$

Here we have atomic electrons orbiting around the nucleus. The sum of the net forces should be zero, and the forces we have acting here are the Coulomb force, drawing the electrons to the protons, and the centripetal force.

$$-F_c = F_{cent}$$

$$-\frac{kq_1q_2}{r^2} = \frac{mv^2}{r}$$

$q_1 = -q_2$, where q2 is the electron/proton charge. Now, we also need a factor of Z somewhere – luckily, the proton charge should be multiplied by a factor of Z, since Z = the number of protons in the nucleus.

$$\frac{Zkq^2}{r^2} = \frac{mv^2}{r} \rightarrow v^2 = \frac{Zkq^2}{ma_0} > \frac{c^2}{4}$$

a_0 is the Bohr radius, which is the most probable distance between the electron and the nucleus.

$$Z > \frac{ma_0c^2}{4kq^2} = 1311$$

Seems kind of reasonable… I assume no elements that exist have relativistic electrons, so it would have to be something kind of large.
Note: a google says this should actually occur around Z = 137 – 178. Hm….

We could use the fact that angular momentum of the electron is quantized according to the Bohr model; $L = mvr = \frac{nh}{2pi} = hbar$ for the ground state…

$$Z > \frac{mrv^2}{kq^2} = \frac{Lv}{kq^2} = \frac{hbar \times c}{2kq^2} = 67$$

This is maybe better, but now seems too low!

11. Consider two entangled identical spin-1/2 particles, with magnetic moments related to their spin by $\vec{m} = \mu\vec{s}$, separated by a large distance. The four Bell states are given as,

$$|\Phi_{\pm}^{(AB)}\rangle = \frac{1}{\sqrt{2}}\left[|z_+, z_+\rangle \pm |z_-, z_-\rangle\right]$$

$$|\Psi_{\pm}^{(AB)}\rangle = \frac{1}{\sqrt{2}}\left[|z_+, z_-\rangle \pm |z_-, z_+\rangle\right]$$

where $|z_\pm\rangle$ are the two spin states along or against the z-axis, the first $z_\pm$ refers to particle A which is nearby, and the second $z_\pm$ refers to particle B which is far away. Consider two operations, both of which entail applying a magnetic field only at location A. In the first operation, the magnetic field in in the x-direction. In the second operation, the magnetic field is in the z- direction. In both cases, the fields act for a time t.

This problem can be solved by writing down the Hamiltonians H_z and H_x for the two operations above, along with the time evolution operators, $U_z(t,0)$ and $U_x(t,0)$.
Hint: $U(t,0)$ obeys the equations $i\hbar\frac{d}{dt}U(t,0) = HU(t,0)$, $U(t,0)^\dagger = U(t,0)^{-1}$, $U(0,0) = \mathcal{I}$.

A. Find a time duration, t_A, which results in $U_z(t_A,0)|z_+\rangle = -i|z_+\rangle$.

B. Using time, t_A, find $U_z(t_A,0)|z_-\rangle$, and $U_x(t_A,0)|z_\pm\rangle$ (Hint: relate the $|z_\pm\rangle$ basis, which are eigenstates of the z-components of spin, to the $|x_\pm\rangle$ basis which are eigenstates of the x-components of spin.)

C. Using the two operations above show that any one of the Bell states can be transformed into any other, operating only at location A.

Hint: the Pauli spin matrices are:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

A. We want to find a t_A such that $U_z(t_A, 0)|z_+\rangle = -i|z_+\rangle$.

This isn't even quantum, we just want to find an eigenvector that gives this eigenvalue when it acts on z+. Because we have to spin ½ particles in a magnetic field, the Hamiltonian is predictable:

$$H = -m \cdot B$$

Which will give energies $E_\pm = \mp\frac{\mu B h}{2}$, with the usual spin eigenvectors.

Our time evolution operator: $U_z(t_A,0) = e^{\frac{-i\mu B}{2}t_A}$

$$e^x = \cos(x) + isin(x)$$

This equals –1 when x = 3pi/2.

$$\boxed{\frac{-\mu B_z}{2}t_A = \frac{3\pi}{2} \rightarrow t_A = \frac{3\pi}{\mu B_z}}$$

B. Now since we're acting on the z- state, instead of z+, this adds a – sign to the exponent of our time evolution operator. Because $e^{-x} = \cos(x) - isin(x)$, we know

$$\boxed{U_z(t_A,0)|z_-\rangle = +i|z_-\rangle.}$$

$U_x(t_A,0)\left|z_\pm\right\rangle$ is more complicated. Ux can only act on the x states, so if we write z in terms of x+ and x-, we can see how Ux acts on that.

$$x_+\rangle = \frac{1}{\sqrt{2}}(1\ 1), x_-\rangle = \frac{1}{\sqrt{2}}(1 - 1)$$

Therefore

$\left|z_\pm\right\rangle = \frac{1}{2}|x_+\rangle \pm \frac{1}{2}|x_-\rangle$

$U_x(t_A,0)|z_+\rangle = \frac{1}{2}(\ |x_+\rangle + |x_-\rangle) = \frac{i}{2}(-|x_+\rangle + |x_-\rangle)$

We want to write this in terms of z again though.

$$U_x(t_A,0)|z_+\rangle = \frac{i}{2}\frac{1}{\sqrt{2}}\big(-(1\ 1) + (1 - 1)\big) = \frac{1}{2}(0 - 2) = (0 - 1) = -i|z_-\rangle$$

Similarly, we find

$$U_x(t_A,0)|z_-\rangle = -i|z_+\rangle$$

It changes spin down to spin up and vice versa!

C. I'm not going to write this out, but I think it isn't too bad. We can use Ux to change the direction of particle A (up to down or vice versa) and we can use Uz to change z+ to -z+ if need be. For example (ignoring constants)

$$U_x(\phi_+\rangle = -|z_-, z_+ > -|z_+, z_- >$$

Now put Uz on this state

$$U_z(\phi_+\rangle = -|z_-, z_+ > +|z_+, z_- >$$

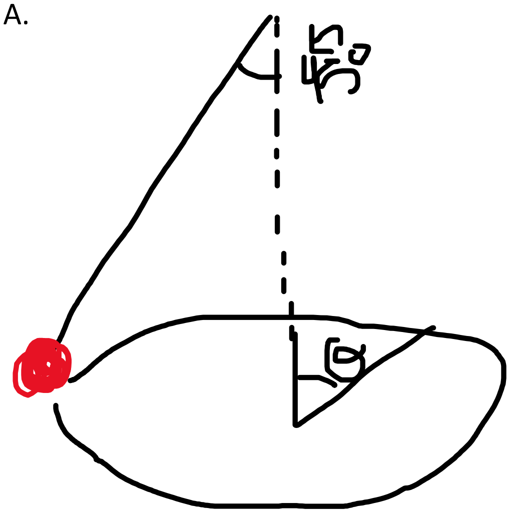
Now this is equal to $(\psi_+\rangle$!

12. A point mass with mass m hangs from a massless string with length a. The top end of the string is tied to the ceiling. The mass is set in motion so that it orbits smoothly in a horizontal circle with the string making an angle of 45° to vertical.

A. Calculate the velocity of the mass.

B. Suppose now that the mass is started in a horizontal orbit at a string angle of $45° + \epsilon$, with the same initial velocity as in Part A. Calculate the frequency of oscillation of the string's angle about 45°.

A.



My drawing kind of sucks. To find the velocity of the mass we can just balance the forces in the x and y direction.

$$F_y = mg - T cos\theta \rightarrow T = \frac{mg}{cos\theta}$$

$$F_x = \frac{mv^2}{r} - T sin\theta \rightarrow v = \sqrt{\frac{T sin\theta r}{m}} = \sqrt{\frac{\frac{mg}{cos\theta} sin\theta r}{m}} = \sqrt{gr}$$

To find r, use trig: $\sin 45 = \frac{r}{a} \rightarrow r = \frac{a}{\sqrt{2}}$

$$v = \sqrt{\frac{ga}{\sqrt{2}}}$$

B. We want the frequency of oscillations, which means we will need to find the equations of motion and find omega. So, let's find the Lagrangian!

$L\ = T - V$

$V = -mgh$ as usual. But we want to write h in terms of variables we know, $V = -mga cos\theta$

Kinetic energy is dependent on both theta and phi. Ignore my drawing above, theta is the 45 degree angle and phi is horizontal. Note v = omega*r.

$$T = \frac{1}{2}mv_\theta^2 + \frac{1}{2}mv_\phi^2 = \frac{1}{2}m\omega_\theta^2 r^2 + \frac{1}{2}m\omega_\phi^2 r^2$$

For theta, r is just the length of the rope a. For phi, r is the radius of the horizontal circle, which we can find with trig: $sin\theta = \frac{r}{a}$

$$T = \frac{1}{2}m\dot{\theta}^2 a^2 + \frac{1}{2}ma^2\dot{\phi}^2 \sin^2\theta$$

$$\mathcal{L} = \frac{1}{2}m\dot{\theta}^2 a^2 + \frac{1}{2}ma^2\dot{\phi}^2 \sin^2\theta + \ mgacos\theta$$

Now EL equations. For theta:

$$\left(m\ddot{\theta}a^2\right) = ma^2\dot{\phi}^2 2sin\theta cos\theta - mgasin\theta$$

Phi:

$$\frac{d}{dt}\left(ma^2\dot{\phi}\sin^2\theta\right) = 0$$

So it appears this term $ma^2\dot{\phi}\sin^2\theta$ does not change with time, so $ma^2\dot{\phi}\sin^2\theta = C \rightarrow$

$$\dot{\phi}(t) = \frac{C}{ma^2\sin^2\theta}$$

I am tempted to plug in phi(0)… but I think it is not useful because I don't know theta(0), since we don't know epsilon(0).

$$\dot{\phi}(0) = \frac{C}{ma^2\sin^2\theta_0} = \sqrt{\frac{ga}{\sqrt{2}}} \rightarrow C = \sqrt{\frac{ga}{\sqrt{2}}}ma^2\sin^2(\theta_0+\epsilon) \rightarrow \ \dot{\phi}(t) = \frac{\sqrt{\frac{ga}{\sqrt{2}}}\sin^2(\theta_0+\epsilon_0)}{\sin^2\theta} =$$

Regardless, phi is constant so we can plug this into our equation for theta.

$$\ddot{\theta} = \frac{2C^2}{m^2a^4\sin^3\theta}cos\theta - \frac{g}{a}sin\theta$$

Generally, we cannot solve this at all. So, consider our $\theta = \theta_0 + \epsilon \rightarrow \ddot{\theta} = \ddot{\epsilon}$

$$\ddot{\epsilon} = \frac{2C^2}{m^2a^4\sin^3(\theta_0+\epsilon)}\cos\left(\theta_0+\epsilon\right) - \frac{g}{a}\sin\left(\theta_0+\epsilon\right)$$

We can Taylor expand the sin and cos. Epsilons multiplied by each other go to zero because they're so small.

$$\sin(\theta_0+\epsilon)=\sin(\theta_0)\cos(\epsilon)+\cos(\theta_0)\sin(\epsilon)=\frac{1}{\sqrt{2}}(\cos(\epsilon)+\sin(\epsilon))=\frac{1}{\sqrt{2}}(1+\epsilon)$$

$$\sin(\theta_0+\epsilon)\approx\frac{1+\epsilon}{\sqrt{2}},\cos(\theta_0+\epsilon)\approx\frac{1-\epsilon}{\sqrt{2}}$$

$$\ddot{\epsilon}=\frac{2C^2}{m^2a^4\left(\frac{1+\epsilon}{\sqrt{2}}\right)^3}\left(\frac{1-\epsilon}{\sqrt{2}}\right)-\frac{g}{a}\left(\frac{1+\epsilon}{\sqrt{2}}\right)=\frac{4C^2}{m^2a^4(1+3\epsilon)}\left(\frac{1-\epsilon}{}\right)-\frac{g}{a}\left(\frac{1+\epsilon}{\sqrt{2}}\right)$$

$$\ddot{\epsilon}=\frac{4C^2(1-\epsilon)}{m^2a^4(1+3\epsilon)}-\frac{g}{a}\left(\frac{1+\epsilon}{\sqrt{2}}\right)$$

Hmm... I feel like this should simplify more…

$$\frac{4C^2(1-\epsilon)}{m^2a^4(1+3\epsilon)}-\frac{g}{a}\left(\frac{1+\epsilon}{\sqrt{2}}\right)=\frac{4C^2(1-\epsilon)a\sqrt{2}-g(1+\epsilon)m^2a^4(1+3\epsilon)}{m^2a^4(1+3\epsilon)a\sqrt{2}}=\frac{4C^2(1-\epsilon)a\sqrt{2}-g(1+4\epsilon)m^2a^4}{m^2a^4(1+3\epsilon)a\sqrt{2}}$$

What about plugging in C.

$$\frac{2C^2}{m^2a^4\left(\frac{1+\epsilon}{\sqrt{2}}\right)^3}\left(\frac{1-\epsilon}{\sqrt{2}}\right)=\frac{2\left(\sqrt{\frac{ga}{\sqrt{2}}}ma^2\sin^2(\theta_0+\epsilon)\right)^2}{m^2a^4\left(\frac{1+\epsilon}{\sqrt{2}}\right)^3}\left(\frac{1-\epsilon}{\sqrt{2}}\right)$$

$$=\frac{2\frac{ga}{\sqrt{2}}m^2a^4\quad\left(\frac{1+\epsilon}{\sqrt{2}}\right)^2}{m^2a^4\left(\frac{1+\epsilon}{\sqrt{2}}\right)^3}\left(\frac{1-\epsilon}{\sqrt{2}}\right)=\frac{2\frac{ga}{\sqrt{2}}}{\left(\frac{1+\epsilon}{\sqrt{2}}\right)}\left(\frac{1-\epsilon}{\sqrt{2}}\right)=\frac{2\frac{ga}{\sqrt{2}}\quad(1-\epsilon)}{1+\epsilon}$$

Okay well, I'm not sure what's up here, but if we could get epsilon out of the denominator, we could get a term that's multiplied by epsilon in the numerator plus a constant. Our frequency would be the term multiplied by epsilon, similar to when $\ddot{x}=\omega^2x$.

13. On a dry, clear and calm day, the sea level temperature is measured to be 20°C. A group of hikers plan to hike to a summit at 3000m elevation. Estimate the temperature at the summit that day. Show details of your calculations.

We're not given any pressures, just the temperature and the height. So we want to find some equations in terms of height and temp – how does temperature change with height? We can derive how pressure changes with height:

Barometric equations:

$$\frac{dP}{dz} = -g\rho$$

Ideal gas law:

$$PV = NkT \rightarrow P = \frac{N}{V}kT = \frac{N\rho}{m}kT = \frac{\rho}{m}RT$$

Divide these by each other:

$$\frac{dP}{P} = -\frac{g\rho m}{\rho RT}dz = -\frac{gm}{RT}dz$$

Then this can be integrated to get barometric formula. But we want this in terms of temp instead. We can assume this is adiabatic (We always assume it is adiabatic pretty much).

Therefore $P^{1-\gamma}T^{\gamma} = constant.$

Take derivative.

$$(1-\gamma)P^{-\gamma}T^{\gamma}dP + P^{1-\gamma}T^{\gamma-1}\gamma dT = 0$$

$$\frac{P^{-\gamma}}{P^{1-\gamma}}dP = -\frac{\gamma}{(1-\gamma)}\,T^{\gamma-1}/\,T^{\gamma}\,dT$$

$$\frac{dP}{P} = -\frac{\gamma}{(1-\gamma)}\frac{dT}{T}$$

Plug into above formula

$$\frac{dP}{P} = -\frac{gm}{RT}dz = -\frac{\gamma}{(1-\gamma)}\frac{dT}{T}$$

Ts cancel out!

$$\frac{-gm}{R}dz = -\frac{\gamma}{(1-\gamma)}dT \rightarrow T - T_0 = \frac{1-\gamma}{\gamma}\frac{gm}{R}z$$

$\gamma = \frac{f+2}{f} = \frac{7}{5}$

$$T = 293K + \frac{1-\frac{7}{5}}{\frac{7}{5}}\frac{9.8 * 0.029}{8.31}3000 = 263K = -9.31 \text{ degrees C}$$

This question confuses me, just because I'm not sure if I would have thought to do it this way on an exam… I also don't always have the mass of air on deck.

14. An electron with non-relativistic velocity v is approaching a proton at rest at an impact parameter b. Calculate the distance of closest approach between the electron and proton. (Do not treat the proton as an immobile object—it will recoil!)

I'm a bit confused conceptually over this problem; I'm adapting my solution from another one I found but I feel like I don't 100% understand.

Firstly, we know that the distance of closest approach is when the kinetic energy is equal to the potential energy, because the electron will momentary stop.

$$\frac{1}{2}mv^2 = \frac{-kq_1q_2}{d_c} = \frac{ke^2}{d_c}$$

However, I think this solution treats the proton as an immobile object; it doesn't account for energy used by the proton in the recoil. Instead I will go to the center of mass frame:

$$\mu = \frac{m_1m_2}{m_1+m_2}$$

Going to this frame causes us to pick up an additional effective potential, $\frac{L^2}{2\mu r^2}$.

$$\frac{1}{2}\mu v^2 = \frac{ke^2}{r_c} + \frac{L^2}{2\mu r_c^2}$$

$L = \mu v b$ (why can we say this is b tho?)

This can be solved to get an equation that's quadratic in r, and then the quadratic formula can be used.

15. A charged object is at distance $L = 1$ cm away from the centre of a 1 m $\times$ 1 m surface of a metal. When released, it takes about 2 seconds to reach the surface. Calculate how long it takes when released from a point 10 cm away from the center.

This question can be done with the method of images. However, there is a trick – because we have a t1, a t2, a d1, and a d2, we could also just find the ratio between them, because all of the other constants should cancel out…

What does t depend on? Obviously, distance, charge (because more charge would induce more charge on the plate, and make it travel faster (I think), epsilon nought (because always), and mass (I didn't think of mass at first but things won't cancel out otherwise).

$$t_1 = d^\alpha q^\beta \epsilon_0^\gamma m^\omega = [m]^\alpha [C]^\beta \left[\frac{C^2}{Nm^2}\right]^\gamma [kg]^\omega = [m]^\alpha [C]^\beta \left[\frac{C^2 s^2}{kgm^3}\right]^\gamma [kg]^\omega$$

We want seconds to be the only one remaining, so gamma must be ½ and then I can change everything to fit this.

$$t_1 = [m]^{3/2} [C]^{-1} \left[\frac{Cs}{kg^{\frac{1}{2}} m^{\frac{3}{2}}}\right] [kg]^{\frac{1}{2}} \rightarrow t \propto d^{\frac{3}{2}}$$

$$\boxed{\frac{t_1}{t_2} = \left(\frac{d_1}{d_2}\right)^{3/2} \rightarrow t_2 = t_1 \left(\frac{d_2}{d_1}\right)^{3/2} = 2s \left(\frac{10cm}{1cm}\right)^{3/2} = 63.2 \text{ seconds}}$$

I wonder I this would still get full marks on an exam? Because I didn't really show any E and M knowledge.

16. The solar radiation reaching the top of the Earth's atmosphere is 1400 W/m^2. For stars of the mass of the Sun and less, the proton-proton chain (PP chain) is the main source of energy. The most important fusion branch of the PP chain is the one that converts 4 protons ($m_p = 1.6473 \times 10^{-27}$ kg) into one helium nucleus ($m_{He} = 6.6446 \times 10^{-27}$ kg). Calculate the lifetime of the Sun assuming that the emitted energy is coming from the PP chain. The lifetime of the Sun is attained when 70% of the protons in the Sun's core have been converted into helium nuclei. The Sun's core represents only 20% of the Sun mass and, at its birth, the primordial composition of the Sun's mass was approximately 75% of hydrogen and 25% of helium. The distance of the Earth from the Sun is 1.5×10^{11} m, and a year has 365 days.

First, find the mass of the sun using the details given.

$$F_g + F_c = 0 = \frac{GMm}{r^2} + \frac{mv^2}{r} \rightarrow M_{sun} = \frac{v^2 r}{G}$$

To get velocity, we can use v = x/t, where x is the circumference of the earth's orbit around the sun and t is one year.

$$M_{core} = \frac{(\frac{2\pi r}{T})^2 r}{G} \times 0.20 = 2e^{30} kg \times 0.20$$

The part of the core that is already helium will of course not transform to helium, so we only have to worry about the hydrogen in the core, which is 75% of the core mass. And when 70% of this has changed to helium, the sun will die.

$$M_{core} * 0.75 * 0.70 = M_{availableHydrogen}$$

$$\frac{M_{availableHydrogen}}{4M_{hydrogen}} = \#\ of\ possible\ reactions = \frac{2.1e30}{4*1.67e^{-27}} = 3.18e55$$

The energy from these reactions is what powers the sun. The energy created is what remains after the protons turn to helium.

$$M_{energy} = M_{helium} - 4M_{hydrogen} = 5.54e^{-29}kg$$
$$E = mc^2 = 5.54e^{-29}(3e8)^2 = 4.896e^{-12}J/reaction$$

The flux at the earth is 1400W/m^2 , so at the sun this is

$$L_{sun} = \frac{1400W}{m^2} * (4\pi(1.5e^{11}m)^2) = 3.96e^{26}W = 3.96e^{26}J/s$$

The sun will burn out when there is no more energy left to power this:

$$t = \left(\frac{E}{reaction}\right) * \frac{\#\ reactions}{rate\ of\ energy\ consumption} = 4.896e^{-12}J * \frac{3.18e^{55}}{\frac{3.96e^{26}J}{s}} = 3.93e^{17}s$$

This is around 12 billion years!