Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 13, 2017
11:00AM to 1:00PM
General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. The range of the strong nuclear force is $R \sim 1$ fm. Assuming that virtual mesons are produced in strong interactions, estimate the mass of the lightest meson (the "π-meson").

2. For a system consisting of two thin lenses, the image distance after the second lens is

$$s_{i_2} = \frac{f_2 d - (f_1 f_2 s_{o_1})/(s_{o_1} - f_1)}{d - f_2 - (f_1 s_{o_1})/(s_{o_1} - f_1)}.$$

Here, f_1 and f_2 are the focal lengths of the two lenses, d is the distance between the two lenses, and s_{o_1} is the object distance in front of first lens. Derive expressions for (i) the back focal length, (ii) the front focal length, and (iii) the effective focal length of the two-lens system as $d \to 0$.

3. Consider a Polyacetylene chain $-CH = CH - CH = CH - \cdots$, and represent its longitudinal stretching vibrations by an infinite linear chain of identical masses M connected by alternating springs of force constants K_1 and K_2. K_1 is for the $CH = CH$ pair and K_2 is for the $= CH - CH$ pair. The separation of identical $CH - CH$ units and of identical $CH = CH$ units are both a.

   (a) Write the equations of motion for the stretching modes.

   (b) The frequencies ω of these modes as a function of wave vector k are

$$\omega^2 = \tfrac{K_1+K_2}{M} \left(1 \pm \sqrt{1 - \tfrac{4\,K_1 K_2}{(K_1+K_2)^2} \sin^2 \tfrac{ka}{2}}\,\right) . \tag{1}$$

   Sketch the dispersions $\omega(k)$ by obtaining the values of ω for the wave vectors $k \to 0$ and $k \to \pi/a$.

   (c) In a Debye model the dispersion of the lowest (acoustical) branch ω is approximated by a linear dispersion $\omega(k) = c_s k$, where c_s is the speed of sound in the chain:

   1. Find c_s.
   2. Find the contribution from these modes to the chain's heat capacity per unit length, in the low temperature limit.
      Hint: You can use the following approximation

$$\int_0^{x_D} \frac{x}{e^x - 1}\, dx \approx \frac{\pi^2}{6} \qquad \text{for } x_D \gg 1 \tag{2}$$

4. Cosmological observations suggest that an otherwise unidentified form of energy density fills our universe. In the simplest model for it, an otherwise empty box (volume V and temperature T) filled only with it in thermal equilibrium with the container walls would contain a total energy

$$U(V,T) = u(T)V \tag{3}$$

and have a pressure exerted on the container walls

$$P = \frac{u}{\hat{n}}\,, \tag{4}$$

where $\hat{n}$ is a constant. Use the laws of thermodynamics to derive the most general form of $u(T)$. The constant $\hat{n}$ best fit to present cosmological data is close to minus one.

5. Upon death, a human body takes 10-12 hours to reach room temperature. Estimate the daily caloric intake to sustain a human life.

6. Make a rough estimate of how many neutrons you can pour in a coffee cup, treating the neutrons as a degenerate fermi gas of noninteracting spin-1/2 particles at zero temperature. For this purpose, consider a cup, modeled as a cylinder with base area A and height L, placed on a table somewhere on earth. There are N neutrons inside the cup. Taking into account gravitational potential energy, the classical Hamiltonian (energy) for a single particle is

$$H = \frac{p^2}{2m} + mgz \tag{5}$$

where z is the vertical position of the neutron relative to the bottom of the cup. You can take the temperature to be zero, and assume the height L of the cup to be sufficiently large so no electrons spill out of the cup.

(a) Compute the Fermi energy for this system.

Hint: Sums over states $|s\rangle$ with energies ϵ_s of a point particle living in a large three-dimensional space can be approximated by a phase space integral:

$$\sum_s f(\epsilon_s) = \frac{1}{h^3} \int f(H(x,p))\, d^3x\, d^3p\,, \tag{6}$$

where $H(x,p)$ is the Hamiltonian of the particle and h is Planck's constant. (If the particle has internal degrees of freedom such as spin, there in addition a sum over internal states.)

(b) By requiring that none of the occupied neutron states at $T = 0$ has more energy than the minimum energy required to escape from the cup, derive an expression estimating the maximum number of neutrons N you can put in the cup.

(c) Using $m_N \sim 1$ GeV/c^2, $h = 4.14 \times 10^{-15}$ eV·s, how many neutrons can you pour in a cup with $A \sim 100$ cm^2, $L \sim 10$ cm, according to this estimate? Compare to the actual number of neutrons in a cup of coffee, and comment on the possible origin of any wild discrepancies you might observe.

GENERAL PHYSICS

Meson mass. SOLUTION.

The virtual process has a duration τ. The maximum range of the force is then $R \sim c\tau$.

According to the uncertainty principle, $\Delta E = Mc^2 \sim \frac{\hbar}{2\tau}$. Therefore, $R \sim \frac{\hbar}{2Mc}$, and thus the minimum mass is

$$M \sim \frac{\hbar}{2Rc}. \tag{1}$$

The numerical value is $M \sim 200\ m_e$, which reasonably well estimates the mass of the π-meson.

Problem #4

Johnson
Section 5-2

$$S_{i2} = \frac{f_2 d - (f_1 f_2 S_{o1})/(S_{o1} - f_1)}{d - f_2 - (f_1 S_{o1})/(S_{o1} - f_1)}$$

(i) back focal length: $S_{o1} \to \infty$

$S_{o1} \gg f_1$ so $S_{o1} - f_1 \to S_{o1}$

$$\text{b.f.l.} = \frac{f_2 d - (f_1 f_2 S_{o1})/(S_{o1})}{d - f_2 - (S_{o1} f_1)/(S_{o1})} = \boxed{\frac{f_2 d - f_2 f_1}{d - (f_1 + f_2)}}$$

(ii) front focal length: $S_{i2} \to \infty$

$S_{o2} \to f_2$. since $S_{o2} = d - S_{i1}$,

$f_2 = d - S_{i1} \to S_{i1} = d - f_2$

$$\frac{1}{S_{o1}} = \frac{1}{f_1} - \frac{1}{S_{o2}} \to \frac{1}{S_{o1}} = \frac{1}{f_1} - \frac{1}{d - f_2} = \frac{(d - f_2) - f_1}{(d - f_2) f_1}$$

$$\boxed{\text{f.f.l} = \frac{(d - f_2) f_1}{d - (f_1 + f_2)}}$$

Problem #4 Continued

Johnson
Section 5-2

(iii) if $d \to 0$ (lenses are touching)

$$f.f.l. = \frac{f_1 f_2}{f_1 + f_2} \; ; \; b.f.l = \frac{f_1 f_2}{f_1 + f_2}$$

$$\boxed{f_e = \frac{f_1 f_2}{f_1 + f_2}} \longrightarrow \frac{1}{f_1} + \frac{1}{f_2} = \frac{1}{f_e}$$

General Section 5: condensed matter

Solution

(a) the equations of motion for stretching modes are

$$\mathrm{M}\ddot{u}_n = \mathrm{K}_1 u_{n+1} - (\mathrm{K}_1 + \mathrm{K}_2)u_n + \mathrm{K}_2 u_{n-1}$$

$$\mathrm{M}\ddot{u}_{n-1} = \mathrm{K}_2 u_n - (\mathrm{K}_1 + \mathrm{K}_2)u_{n-1} + \mathrm{K}_1 u_{n-2}$$

(b) the dispersions $\omega(k)$ are sketched by obtaining the values of ω for the wave vectors k→0 and k→π/a from the dispersion relationship given in the problem.

(c) In a Debye model the dispersion of the lowest (acoustical) branch w is approximated by a linear dispersion ω(k)=c_sk, where c_s is the speed of sound in the chain, that is obtained from the k→0 limit:

(c.1) $c_s = [K_1K_2 / M(K_1+K_2)] (a/2^{1/2})$

(c.2) Heat capacity at constant volume is given by $C_V = \left(\frac{\partial U}{\partial T}\right)_V$

We can evaluate U from

$$U = U_0 + \int_0^{\nu_D} \frac{h\nu}{e^{h\nu/kT} - 1} g(\nu)d\nu$$

Where $g(\nu)$ is the density of states, $\nu_D = Nc_s/L$ is the Debye frequency, and $U_0$is the zero point energy.

For the 1-D problem here, $g(\nu)$ can be written as

$$g(\nu) = \frac{2L}{c_s}$$

Where L is the length of the molecular chain. The factor 2 comes from positive and negative values of the wave vector. We can now express U as follows:

$$U = U_0 + \frac{2L}{c_s}\int_0^{v_D} \frac{hv}{e^{hv/kT} - 1} dv$$

Using the following substitution: $x = hv/kT$, we can simplify the integral as

$$U = U_0 + \frac{2(kT)^2 L}{hc_s}\int_0^{x_D} \frac{x}{e^x - 1} dx$$

for large x_D the integral is equal to $\pi^2/6$, so we can write

$$U = U_0 + \frac{\pi^2 (kT)^2 L}{3hc_s}$$

Heat capacity is then given by the temperature derivative of the above formula:

$$C_V = \frac{2\pi^2 k^2 LT}{3hc_s}$$

Upon substituting Debye temperature and frequency: $\theta_D = hv_D/k,\ v_D = \ Nc_s/L$

$$C_v = \frac{2\pi^2}{3} Nk \frac{T}{\theta_D}$$

General Physics part I
(Thermodynamics)

M. Ruderman

Cosmological observations suggest that an otherwised unidentified form of energy density fills our Universe. In the simplest models for it an otherwise empty box (volume V and temperture T) filled only with it in thermal equilibrium with the container walls would contain a total energy

$$U(V, T) = u(T)\, V$$

and have a pressure exerted on the container walls

$$P = \frac{u}{\hat{n}}$$

what are the thermodynamic constraints on the form of $u(T)$ for any $\hat{n}$

(the constant $\hat{n}$ best fit to present cosmological data is close to minus one.)

Gen Phys Parts (M. Ruderman)

Answer

$$T dS = DQ = dU \; p dV = u' V dT + u dV + \frac{u}{\hat{n}} dV$$

(+)

2nd Law $\qquad \frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 S}{\partial T \partial V}$

$$\frac{u'}{T} = \left(1 + \frac{1}{\hat{n}}\right)\left(\frac{u'}{T} - \frac{u}{T^2}\right)$$

$$\Rightarrow \quad \frac{u'}{T} = [\hat{n} + 1] \frac{u}{T^2}$$

or $\quad \boxed{u(T) = K T^{\hat{n}+1}}$

W.A. Zajc

1 Problem: Black Rate

Upon death, a human body takes 10-12 hours to reach room temperature. Estimate the daily caloric intake to sustain a human life.

2 Solution: Basal Rate

Say a human mass is the equivalent of 70 kG of water. The amount of heat exchanged in reaching room temperature is

$$Q \sim mc_W \Delta T = (70 \text{ kg})\ (1 \text{ cal/gm-K})\ (37°\text{C} - 22°\text{C}) \approx 1050 \text{ kcal} \quad .$$

This is lost in 10-12 hours, implies basal metabolic rate is 2-2.4 times this value, that is, 2100-2520 kcal = 2100-2520 food calories per day.

This number is slightly high, but not crazy. For example, rule of thumb that humans radiate at about 100 watts would lead to 2064 food calories per day.

The real point of this problem is that no other information should be supplies, i.e., students should know the specific heat of water, that people are mostly water, and that a food calorie is not a physics calorie.

Solutions to Problem Set 9

December 13, 2016

1. **Neutrons in a coffee mug**

(a)Suppose the Fermi energy is ϵ_F. So the number of states under this energy level is

$$N = \frac{2}{h^3}\int d^3x d^3p\,\Theta(\epsilon_F - p^2/2m - mgx_3) \tag{1}$$

For a fixed x_3, the integral over momentum gives the volume of a ball of radius $\sqrt{2m(\epsilon_F - mgx_3)}$. So the integral is

$$N = \frac{2A}{h^3}\frac{4\pi}{3}\int_0^{\epsilon_F/mg} dx_3\,[2m(\epsilon_F - mgx_3)]^{3/2} \tag{2}$$

Note the L is large enough to satisfy the condition $L > \epsilon_F/mg$. The integral over x_3 gives

$$N = \frac{32\sqrt{2}\pi A m^{1/2}\epsilon_F^{5/2}}{15h^3 g} \tag{3}$$

Thus we can express the Fermi energy in terms of N:

$$\epsilon_F = \left(\frac{15h^3 gN}{32\sqrt{2}\pi A m^{1/2}}\right)^{2/5} \tag{4}$$

(b) In order to escape from the mug, neutrons must gain energy larger than mgL. When the Fermi energy is mgL, the max number of neutron we can put in the mug is

$$N = \frac{32\sqrt{2}\pi A m^{1/2}(mgL)^{5/2}}{15h^3 g} = \frac{32\sqrt{2}\pi A m^3 g^{3/2} L^{5/2}}{15h^3} \tag{5}$$

(c)If neutrons are replaced by bosons, the answer is infinity because we can put any number of bosons and the same state.

(d)Plugging in the numbers yields $N_n = 1.8\times 10^{17}$. On the other hand, the number of water molecules in the mug is roughly $N_w = \frac{10^3}{18}N_A$ (the mole mass of water is $18g/mol$) and each water molecule contains 8 neutrons. So the number of neutron is $N_n' = 2.7\times 10^{26} \sim 10^9 N_n$. There is a wild discrepancy between these two approximations. Neutrons in coffee do not behave like noninteracting particles in a weak uniform gravitational field at all. They are very tightly bound to other neutrons and protons in the nuclei, and those nuclei are tightly bound together with electrons into H_2O molecules and the H_2O molecules are also bound to each other. So neutrons gain a larger effective mass. Since that $N \sim m^3$, there are more neutrons than 1.8×10^{17}. Alternatively, we can use a square potential well of binding energy $-U_b$ to model those interactions. It will allow for many more one-particle states confined inside this well than confined inside the gravitational potential because U_b is much larger than the the gravitational potential energy mgL. This would also resolve the discrepancy.

2.**The density of states**

This problem is quite straightforward-just follow the definition of all the formulae listed.
(a)Since the Hamiltonian is x independent, we can first integrate over x.

$$\begin{aligned}\Omega(\epsilon) &= \sum_k \Theta(\epsilon - \epsilon_k) = \frac{L}{h^3}\int d^3p\,\Theta(\epsilon - p^2/2m) \\ &= \frac{L}{h^3}\frac{4\pi}{3}(2m\epsilon)^{3/2}\end{aligned} \tag{6}$$

So the number of states is

$$g(\epsilon) = \frac{d\Omega(\epsilon)}{d\epsilon} = \frac{4\sqrt{2}\pi m^{3/2}L}{h^3}\sqrt{\epsilon} \tag{7}$$

(b)Essentially, I've calculated $\Omega(\epsilon)$ in the first problem where it is denoted by N. Use that result here:

$$\Omega(\epsilon) = \frac{32\sqrt{2}\pi A m^{1/2}\epsilon^{5/2}}{15h^3 g} \tag{8}$$

So the number of states is

$$g(\epsilon) = \frac{d\Omega(\epsilon)}{d\epsilon} = \frac{16\sqrt{2}\pi A m^{1/2}}{3h^3 g}\epsilon^{3/2} \tag{9}$$

Note the infinity height of the mug guarantees $\epsilon/mg < L$ for any fixed energy ϵ.

3.**Quantum fields and particles**

The vanishing of electric field at the boundary requires that the wave vector $\vec{k}$ should be represented by 3 nonnegative integers (n_x, n_y, n_z) denoted by $\vec{n}$. Precisely, the relation between $\vec{k}$ and $\vec{n}$ is

$$\vec{k} = \frac{\pi}{L}\vec{n} \tag{10}$$

For every such $\vec{k}$, the "amplitude" $\vec{E}_{\vec{k}}$ satisfies the eq(4) in the problem set which is also the equation of motion of a simple harmonic oscillator of frequency $\omega_{\vec{k}} = c|\vec{k}| = \pi c/L\sqrt{n_x^2 + n_y^2 + n_z^2}$. Then we get a series of simple harmonic oscillators parameterized by $(\vec{k}, \alpha)$ or equivalently $(\vec{n}, \alpha)$. The basic idea of quantum field theory is that we quantize all of these simple harmonic oscillators. As we know, for each simple harmonic oscillator $(\vec{k}, \alpha)$, its quantum states are labeled by nonnegative integers $\{N_{\vec{k},\alpha}\}$. Thus a complete quantum state of the electric field is specified by given the quantum number $N_{\vec{k},\alpha}$ for all the quantum oscillators $(\vec{k}, \alpha)$. (Mathematically, the complete Hilbert space is the tensor product of the Hilbert spaces of ∞ quantum simple oscillators). So in the wave picture, the energy of the state specified by the list $N_{\vec{k},\alpha}$ is then

$$U_w(\{N_{\vec{k},\alpha}\}) = \sum_\alpha \sum_{\vec{k}} \hbar N_{\vec{k},\alpha}\omega_{\vec{k}} = \sum_\alpha \sum_{\vec{k}} N_{\vec{k},\alpha}\hbar k c \tag{11}$$