

Columbia University  
Department of Physics  
QUALIFYING EXAMINATION

Monday, January 8, 2018  
10:00AM to 12:00PM  
Classical Physics  
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

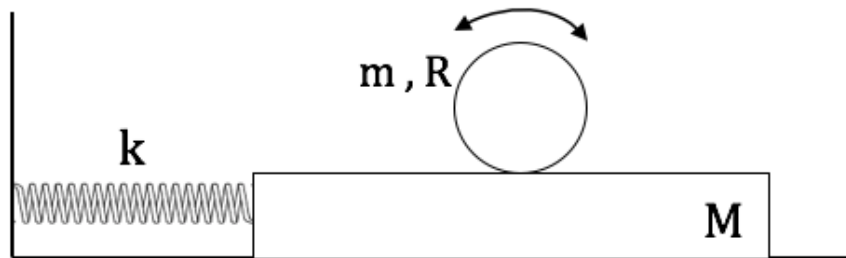
You may refer to the single handwritten note sheet on  $8\frac{1}{2}$ "  $\times$  11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. A block of mass  $M$  slides along a frictionless surface. The block is connected to the wall with a spring of spring constant  $k$ . A cylinder of mass  $m$ , radius  $R$ , and rotational inertia  $\frac{1}{2}mR^2$  rolls without slipping on the block.
- (a) What is the frequency for small oscillations of the system around the starting position?
  - (b) Describe the motion associated with this oscillation.
  - (c) What is the maximum oscillation amplitude that the block can have before the cylinder starts slipping if the friction coefficient is  $\mu$ ?



2. A particle of mass  $m$  moves in a potential energy

$$U = -\frac{k}{r} - \frac{\alpha}{2r^2}$$

where  $k > 0$ . The particle has angular momentum  $\ell$ .

- (a) For what values of  $\alpha$  are circular orbits possible? For what values of  $\alpha$  are these stable?
- (b) Consider an orbit that is almost circular, with  $r$  varying between two values  $r_{\min}$  and  $r_{\max}$ . Let  $T$  be the time interval required for  $r$  to move from  $r_{\min}$  to  $r_{\max}$ . For which values of  $\alpha$  is  $T$  greater than the period of revolution about the origin (i.e., the time required for  $\phi$  to range from 0 to  $2\pi$ )? For which values of  $\alpha$  is  $T$  less than the period of revolution about the origin?

3. A particle of mass  $m$  moves in the potential

$$V = \lambda_0 xy + \frac{1}{4}\lambda_1(x^4 + y^4)$$

with  $\lambda_0$  and  $\lambda_1$  positive.

- (a) When are the stable equilibria for the particle?
- (b) Give the Lagrangian appropriate for small oscillations about one of the equilibrium positions.
- (c) Give the normal frequencies and modes of vibration corresponding to the equilibrium position in (b).

4. A circular platform with radius  $R$  and moment of inertia  $I_P$  rotates in the horizontal plane on a frictionless bearing. A bead of mass  $m$  and negligible size is free to slide without friction on a wire of length  $2R$  along a diameter of the platform.
- (a) At  $t = 0$ , the bead is released at rest from  $r_0 \equiv r(t = 0) = 0.10R$ . An external mechanism maintains the platform's angular velocity at a fixed value  $\omega_0 = 6.0\text{s}^{-1}$ . Find the time in seconds for the bead to reach the edge of the platform. It is useful to recall that  $e^3 \approx 20$ .
- (b) Now assume that at the moment the bead is released the external mechanism is switched off. Find an algebraic expression for the radial velocity of the bead at  $r = R$ , again assuming it is released at rest from  $r_0$ .

5. Consider a particle of charge  $e$  and mass  $m$  moving under the action of an isotropic harmonic oscillator of potential  $U = \frac{1}{2}K(x^2 + y^2 + z^2)$ , with  $K$  a positive constant. The particle is in a magnetic field aligned along the  $z$ -axis  $\vec{B} = B\hat{z}$ . Assume that the magnetic field is weak, so that  $\frac{eB}{2m} \ll \sqrt{\frac{K}{m}}$ .
- (a) With the above approximation, find the eigenmodes or normal modes for the motion of the particle and show your solutions correspond to circular motion. What is the angular frequency in your solutions?
  - (b) Show explicitly from your solutions that when  $B \neq 0$  the particle's motion will generally exhibit precession. What is the precessional frequency?

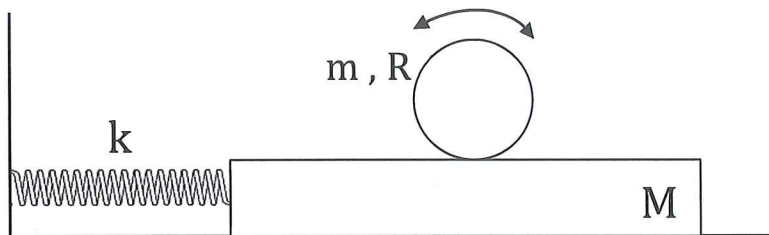
Mechanics Quals Problem

M. Shaevitz  
Dec, 2017

1.

A block of mass  $M$  slides along a frictionless surface. The block is connected to the wall with a spring of spring constant  $k$ . A cylinder of mass  $m$ , radius  $R$ , and rotational inertia  $\frac{1}{2}mR^2$  rolls without slipping on the block.

- What is the frequency for small oscillations of the system around the starting position?
- Describe the motion associated with this oscillation.
- What is the maximum oscillation amplitude that the block can have before the cylinder starts slipping if the friction coefficient is  $\mu$ ?



Solution:

Let  $X$  ( $x$ ) be the position of the block (cylinder) with respect to the wall. Let  $\theta$  be the clockwise angle of the cylinder.

$$I = \frac{1}{2}mR^2$$

The rolling condition requires that:

$$\dot{x} = \dot{X} + R\dot{\theta} \quad \ddot{x} = \ddot{X} + R\ddot{\theta} \Rightarrow \frac{\partial \dot{x}}{\partial \dot{X}} = 1 \quad \frac{\partial \dot{x}}{\partial \dot{\theta}} = R$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 \quad U = \frac{1}{2}kX^2 \quad L = T - U$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} = \frac{d}{dt}\left(M\dot{X} + m\dot{x}\frac{\partial \dot{x}}{\partial \dot{X}}\right) + kX = M\ddot{X} + m\ddot{x} + kx = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(m\dot{x}\frac{\partial \dot{x}}{\partial \dot{\theta}} + I\dot{\theta}\right) = mR\ddot{x} + I\ddot{\theta} = mR\ddot{x} + \frac{1}{2}mR^2\ddot{\theta}$$

Substitute  $\ddot{x} = \ddot{X} + R\ddot{\theta}$  gives

$$(m + M)\ddot{X} + mR\ddot{\theta} + kX = 0$$

$$m\ddot{X} + \frac{3}{2}mR\ddot{\theta} = 0 \Rightarrow mR\ddot{\theta} = -\frac{2}{3}m\ddot{X}$$

$$\Rightarrow (m + M)\ddot{X} - \frac{2}{3}m\ddot{X} + kX = 0$$

a)

$$\ddot{X} + \frac{k}{\left(M + \frac{m}{3}\right)} X = 0 \Rightarrow \omega = \sqrt{\frac{k}{\left(M + \frac{m}{3}\right)}}$$

b) Since  $R\ddot{\theta} = -\frac{2}{3}\ddot{X}$ , the motion is such that when block goes to the right, the cylinder rotates counter clockwise.

$$c) X = A \cos(\omega t) \Rightarrow \ddot{X} = -\omega^2 A \cos(\omega t) \Rightarrow \ddot{X}_{Max} = \omega^2 A$$

$$mR\ddot{\theta} = -\frac{2}{3}m\ddot{X} \Rightarrow \ddot{\theta}_{Max} = \frac{3}{2} \frac{\ddot{X}_{Max}}{R} = \frac{3\omega^2 A}{2R}$$

Friction provides the torque for the cylinder rotation so:

$$\ddot{\theta}_{Max} = \frac{3\omega^2 A}{2R} < \frac{Rf_{max}}{I} = \frac{2(\mu mg)}{mR} = \frac{2\mu g}{R} \Rightarrow A < \frac{4\mu g}{3\omega^2}$$



2.

A particle of mass  $m$  moves in a potential energy

$$U = -\frac{k}{r} - \frac{\alpha}{2r^2}$$

where  $k > 0$ . The particle has angular momentum  $\ell$ .

a) For what values of  $\alpha$  are circular orbits possible? For what values of  $\alpha$  are these stable?

b) Consider an orbit that is almost circular, with  $r$  varying between two values  $r_{\min}$  and  $r_{\max}$ . Let  $T$  be the time interval required for  $r$  to move from  $r_{\min}$  to  $r_{\max}$ . For which values of  $\alpha$  is  $T$  greater than the period of revolution about the origin (i.e., the time required for  $\phi$  to range from 0 to  $2\pi$ )? For which values of  $\alpha$  is  $T$  less than the period of revolution about the origin?

Solution

$$a) \quad U = -\frac{k}{r} - \frac{\alpha}{2r^2}$$

$$U_{\text{eff}} = -\frac{k}{r} - \frac{\alpha}{2r^2} + \frac{l^2}{2mr^2}$$

$$U'_{\text{eff}} = \frac{k}{r^2} + \left(\alpha - \frac{l^2}{m}\right) \frac{1}{r^3}$$

$$U''_{\text{eff}} = -\frac{2k}{r^3} \pm 3\left(\alpha - \frac{l^2}{m}\right) \frac{1}{r^4}$$

$$\text{Circular orbit?} \Rightarrow U'_{\text{eff}} = 0 \Rightarrow k = \left(\frac{l^2}{m} - \alpha\right) \frac{1}{r} \Rightarrow \text{any } \alpha$$

$$\text{stable?} \Rightarrow U''_{\text{eff}} \Big|_r > 0 \Rightarrow \frac{l^2}{m} - \alpha = kr > 0 \Rightarrow \text{any } \alpha$$

$$b) \quad T = \frac{2\pi}{\omega_{\text{osc}}}, \quad \omega_{\text{osc}}^2 = U''/m = \left(\frac{l^2}{m} - \alpha\right) \frac{1}{mr^4}$$

$$\text{orbital period} = \frac{2\pi}{\omega_{\text{orb}}}, \quad \omega_{\text{orb}}^2 = \left(\frac{l}{mr^2}\right)^2$$

$$\omega_{\text{orb}}^2 - \omega_{\text{osc}}^2 = \frac{l^2}{m^2 r^4} - \left(\frac{l^2}{m^2 r^4} - \frac{\alpha}{mr^4}\right) = \frac{\alpha}{mr^4}$$

$$\Rightarrow \alpha > 0 \Rightarrow \omega_{\text{orb}} > \omega_{\text{osc}} \Rightarrow \text{orbital period} < T$$

$$\alpha < 0 \Rightarrow \omega_{\text{orb}} < \omega_{\text{osc}} \Rightarrow \text{orbital period} > T$$

3. A particle of mass  $m$  moves in the potential

$$V = \lambda_0 xy + \frac{1}{4} \lambda_1 (x^4 + y^4)$$

with  $\lambda_0$  and  $\lambda_1$  positive.

(i) Where are the stable equilibria for the particle?

(ii) Give the Lagrangian appropriate for small oscillations about one of the equilibrium positions.

(iii) Give the normal frequencies and modes of vibration corresponding to the equilibrium position in (ii).

Solution:

$$V = \lambda_0 xy + \frac{1}{4} \lambda_1 (x^4 + y^4)$$

$$(i) \text{ Require } 0 = \frac{\partial V}{\partial x} = \lambda_0 y + \lambda_1 x^3 \Rightarrow y = -\frac{\lambda_1}{\lambda_0} x^3$$

$$0 = \frac{\partial V}{\partial y} = \lambda_0 x + \lambda_1 y^3 \Rightarrow x = -\frac{\lambda_1}{\lambda_0} y^3$$

So possible equilibria are

$$x = \left(\frac{\lambda_1}{\lambda_0}\right)^{1/4} x^9 \quad \text{or} \quad x_0 = \pm \sqrt[4]{\lambda_0/\lambda_1}$$

$$y_0 = \mp \sqrt[4]{\lambda_0/\lambda_1}$$

$$\det \left( \frac{\partial^2 V}{\partial x \partial y} \right)_{x_0 y_0} = \det \begin{pmatrix} 3\lambda_0 & \lambda_0 \\ \lambda_0 & 3\lambda_0 \end{pmatrix} = 8\lambda_0^2 > 0 \quad \text{so both are stable.}$$

$$(ii) L = \frac{m}{2} (\dot{\delta x}^2 + \dot{\delta y}^2) - \frac{3\lambda_0}{2} (\delta x^2 + \delta y^2 + \frac{2}{3} \delta x \delta y)$$

$$\delta x = x - x_0$$

$$\delta y = y - y_0$$

$$(ii) \frac{d}{dt} \frac{\partial L}{\partial \dot{\delta x}} - \frac{\partial L}{\partial \delta x} = 0 \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\delta y}} - \frac{\partial L}{\partial \delta y} = 0$$

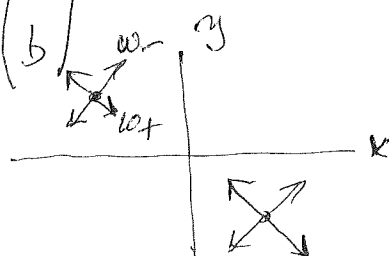
$$m \begin{pmatrix} \ddot{\delta x} \\ \ddot{\delta y} \end{pmatrix} + \begin{pmatrix} 3\lambda_0 & \lambda_0 \\ \lambda_0 & 3\lambda_0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = 0$$

normal freq

$$\det \begin{pmatrix} 3\lambda_0 - m\omega^2 & \lambda_0 \\ \lambda_0 & 3\lambda_0 - m\omega^2 \end{pmatrix} = 0 \Rightarrow \omega_{\pm}^2 = \frac{2\lambda_0}{m}$$

$$\omega_{\pm}^2 = \frac{4\lambda_0}{m}; \quad \begin{pmatrix} \pm \lambda_0 & \lambda_0 \\ \lambda_0 & \pm \lambda_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\omega_+, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \quad \omega_-, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



W.A. Zajc

## ~~1~~ Problem: Rotating Platform

4.

A circular platform with radius  $R$  and moment of inertia  $I_P$  rotates in the horizontal plane on a frictionless bearing. A bead of mass  $m$  and negligible size is free to slide without friction on a wire of length  $2R$  along a diameter of the platform.

a) At  $t = 0$ , the bead is released at rest from  $r_0 \equiv r(t = 0) = 0.10R$ . An external mechanism maintains the platform's angular velocity at a fixed value  $\omega_0 = 6.0 \text{ s}^{-1}$ . Find the time in seconds for the bead to reach the edge of the platform. It is useful to recall that  $e^3 \approx 20$ .

b) Now assume that at the moment the bead is released the external mechanism is switched off. Find an algebraic expression for the radial velocity of the bead at  $r = R$ , again assuming it is released at rest from  $r_0$ .



ROTATING  
PLATFORM

Zajc

ZAJZ

Solution

a.) There is a tangential Coriolis force which can be ignored;  
the only relevant component is the radial force  $F_r = +mr\omega^2$

so EOM is  $F_r = mr\omega^2 = m\ddot{r}$

$$\Rightarrow r(t) = A \cosh \omega t + B \sinh \omega t$$

IC's  $\Rightarrow r(t) = 0.10R \cosh \omega t = 0.10R \left\{ \frac{e^{\omega t} + e^{-\omega t}}{2} \right\}$

so if  $r(t_R) = R$  have  $R = 0.05R \{ e^{\omega t_R} + e^{-\omega t_R} \}$

which is solved to reasonable accuracy when  $e^{\omega t_R} = 20$

$$\Rightarrow \omega t_R = 3 \Rightarrow t_R = \frac{3}{\omega} = \frac{3}{6.0 \text{ s}^{-1}} = \underline{\underline{0.5 \text{ s}}}$$

Note: Can also derive EOM from  $L = \frac{1}{2} m r^2 \omega^2$

b.) This can also be done in Lagrangian formalism using

$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} I_p \dot{\theta}^2$  and recognizing centrifugal potential,  
but elementary methods are easier:

frictionless bearings  $\Rightarrow$  no external torques  $\Rightarrow L_i = L_f$

frictionless motion on wire  $\Rightarrow$  no dissipation  $\Rightarrow KE_i = KE_f$

$$\text{so } \frac{L_i^2}{2I_i} = \frac{L_f^2}{2I_f} + \frac{1}{2} m \dot{r}^2 = \frac{L_i^2}{2I_f} + \frac{1}{2} m \dot{r}^2$$

$$\begin{aligned} \text{or } \dot{r}^2 &= \frac{L_i^2}{m} \left( \frac{1}{I_i} - \frac{1}{I_f} \right) = \frac{[(I_p + m r_0^2) \omega_0]^2}{m} \left\{ \frac{1}{(I_p + m r_0^2)} - \frac{1}{I_p + m R^2} \right\} \\ &= \frac{I_p + m r_0^2}{I_p + m R^2} (R^2 - r_0^2) \omega_0^2 \end{aligned}$$

This is consistent with part a) solution in limit  $I_p \rightarrow \infty$

~~Classical Mechanics~~

Hailey

5. Consider a particle of charge  $e$  and mass  $m$  moving under the action of an isotropic harmonic oscillator of potential  $U = \frac{1}{2} K (x^2 + y^2 + z^2)$  with  $K$  a positive constant. The particle is in a magnetic field aligned along the  $z$ -axis  $\vec{B} = B\hat{z}$ . Assume that the magnetic field is weak, so that  $\frac{eB}{2m} \ll \sqrt{K/m}$ .

eigenmodes

- a) With the above approximation, find the general solution(s) for the motion of the particle and show your solutions correspond to circular motion. What is the angular frequency in your solution(s)?
- ~~b) Show explicitly from your solutions that the unperturbed ( $B=0$ ) circular modes of the particle precess when  $B \neq 0$ . What is the magnitude of the precessional frequency in your solution(s)?~~
- b) Show explicitly from your solutions that when  $B \neq 0$  the particle's motion will generally exhibit precession. What is the precessional frequency?

# Solution Class Mech. Hailey

a) Using either a Lagrangian or just writing down the Cartesian components by inspection for  $\vec{F} = \vec{F}_B + \vec{F}_{sho}$   
And  $\vec{F}_B = e \vec{v} \times \vec{B}$  one obtains ( $\omega_0^2 = \frac{k}{m}$ )

$$\begin{aligned} m \ddot{x} &= -kx + eB\dot{y} & \ddot{x} &= -\omega_0^2 x + \frac{eB}{m} \dot{y} \\ m \ddot{y} &= -ky - eB\dot{x} & \ddot{y} &= -\omega_0^2 y - \frac{eB}{m} \dot{x} \\ m \ddot{z} &= -kz & \ddot{z} &= -\omega_0^2 z \end{aligned}$$

The z motion is just SHO and not affected by the B-field.

Let  $x_+ = x + iy$  and combine the 1<sup>st</sup> 2 equations. Then

$$\ddot{x}_+ = -\omega_0^2 x_+ - \frac{ieB}{m} \dot{x}_+$$

This has solution  $x_+ = C e^{i(\alpha t + \phi)}$

where  $\alpha$  is the root of the equation

$$-\alpha^2 - \frac{eB}{m} \alpha + \omega_0^2 = 0$$

$$\alpha = -\frac{eB}{2m} \pm \sqrt{\left(\frac{eB}{2m}\right)^2 + \omega_0^2}$$

$$eB/2m = \omega_L < \omega_0 \quad \text{so} \quad = \sqrt{k/m}$$

$$\alpha = \pm \omega_0 - \omega_L \quad \text{so the 2}$$

homogeneous solutions are



$$x_{1+} = C_1 e^{i\{\omega_0 - \omega_L\}t + \phi_1}$$

$$x_{2+} = C_2 e^{-i\{\omega_0 + \omega_L\}t + \phi_2}$$

The general solution is a linear combination  
Take real and imaginary parts

$$x_1 = C_1 \cos[(\omega_0 - \omega_L)t + \phi_1]$$

$$y_1 = C_1 \sin[(\omega_0 - \omega_L)t + \phi_1]$$

$$x_2 = C_2 \cos[(\omega_0 + \omega_L)t + \phi_2]$$

$$y_2 = -C_2 \sin[(\omega_0 + \omega_L)t + \phi_2]$$

These solutions clearly represent circular motion.

b) by inspection the first solution has  
Angular frequency  $\vec{\omega} = (\omega_0 - \omega_L) \hat{k}$   
And the second  $\vec{\omega} = -(\omega_0 + \omega_L) \hat{k}$   
so that in the plane perpendicular to  
the  $\vec{B}$ -field there is a precession  
at frequency  $\vec{\omega}_L = -\omega_L \hat{k}$