2016 Solution Explanations
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1. DNA is a long flexible molecule. The DNA from one cell of your body, if stretched out in a line, would extend 2 meters in length but a DNA double helix is just 2 nm across. The persistence length of a DNA molecule is ~ 50 nm, meaning that we can model a random tangle of DNA as a 3D random walk with a step size of 50 nm.

A. Provide an estimate of the diameter of a ball of DNA strand, 2 m in length, using the random walk model. How does this compare to the cell nucleus diameter of 10 μm?

B. In reality the DNA is neatly coiled in chromosomes. In the approximation that the entropy of this state is much lower than that of the "balled up" state in part A, do an order of magnitude estimate of the change in entropy when the DNA changes from the "balled up" state to the chromosomal state, and estimate the minimum amount of energy the cell must spend to accomplish this.

A. We're modeling the DNA as a random walk, which is a walk that starts at (x,y,z)=(0,0,0) and moves one step in either direction with equal probability. So, we expect our DNA strand to be all coiled up randomly.
For a random walk with a Gaussian distribution, the root mean square translation distance after N steps is $d_{rms} = \sqrt{N}\lambda$, where lambda is the step size. We will assume this distance is the diameter of the molecule.

$$d_{rms} = \sqrt{\left(\frac{2m}{50 \times 10^{-9}m}\right)} 50nm = 316\mu m$$

Way too big!

B.

$$\Delta S = Nkln\left(\frac{V_i}{V_f}\right) = Nkln\left(\frac{(316)^3}{(10)^3}\right) = 1.77 \times 10^{-12} J/K$$

Should be Vf / Vi.
To get the minimum amount of energy, I multiply this by temperature (where I assumed the temperature was that of an average human, 37 degrees):

$$\Delta \mathrm{E} = \mathrm{T}\Delta S = 310K * 1.77 \times \frac{10^{-12}J}{K} = 5.5 \times 10^{-10}J$$

This seems reasonable.

2. C3PO told Han Solo, "Sir, the possibilty of successfully navigating an asteroid field is approximately 3,720 to 1." In this problem you will check C3PO's math.

Suppose that the asteroid belt was formed by an Earth-sized planet that was broken into 100 m wide chunks, perhaps by the Death Star. Our solar system's asteroid belt has an inner radius of 2.1 AU, an outer radius of 3.3 AU, and a height of 2 AU.

Suppose Han Solo flies the Millenium Falcon from Earth to Jupiter at close to light speed, passing through the asteroid belt in the process. Calculate the probability that the Falcon (diameter = 30 m) will collide with an asteroid, assuming that it takes no evasive action. Neglect gravitational interactions.

The total volume of the asteroid field is:

$$V_{field} = \pi r^2 h = \pi(2AU)(3.3^2 - 2.1^2) * (1.5e11m)^3 = 1.4 \times 10^{35} m^3$$

The total volume of the chunks matches the earth:

$$V_{chunks} = \frac{4}{3}\pi(6.4e6m)^3 = 1 \times 10^{21} m^3$$

Because each chunk has a radius of 50m, the number of chunks is $N_{chunks} = \frac{1\times10^{21}m^3}{\frac{4}{3}\pi(50^3)} = 1.9 \times 10^{15}$.

So, the chunks take up a total of $\frac{1\times10^{21}m^3}{1.4\times10^{35}m^3} \times 100\% = 7 \times 10^{-13}\%$ of the total asteroid field.

Han Solo will traverse through a cylinder sized portion of the belt, assuming he is travelling in the radial direction. This volume is $V_{Han} = \pi h r^2 = \pi * 1.2AU * (35m)^2 = 7 \times 10^{14} m^3$, where I added 20m radius just to make sure he is safe.

What is the probability that there is a chunk anywhere in this volume?

$$P = N_{chunks}\frac{V_{Han}}{V_{field}} = 1.9 \times 10^{15} \frac{7 \times 10^{14} m^3}{1.4 \times 10^{35} m^3} = 5 \times 10^{-6}$$

So, the probability hitting an asteroid is $5 \times 10^{-6}: \rightarrow\ 1: 200000$. This doesn't account for the volume of the asteroid, though…

3. A 1D *relativistic* harmonic oscillator obeys Hooke's law (F=-kx). If A is the maximum amplitude of oscillation and m is the mass, calculate:

A. The total energy of the oscillator

B. The maximum speed of the mass

C. The magnitude of the acceleration of the mass when $x = A/2$

Hint: the relativistic Lagrangian in this problem is

$$L = mc^2(1 - \sqrt{1-\beta^2}) - \frac{1}{2}kx^2$$

where $\beta \equiv v/c$.

A. I know what you're thinking: L = T-V, so we can just say E = T+V. Not quite. Use the conserved quantity:

$$E = \frac{dL}{d\dot{q}}\dot{q} - L = p\dot{q} - L$$

This can be derived by finding $\frac{dL}{dt}$, plugging in the E-L equation, and finding the conserved quantity.

Because this is relativistic, we must use relativistic momentum, $p = \gamma m v = \frac{m\dot{x}}{\sqrt{1-\dot{x}^2}}$

$$E = \frac{m\dot{x}^2}{\sqrt{1-\dot{x}^2}} - m\left(1 - \sqrt{1-\dot{x}^2}\right) + \frac{1}{2}kx^2 = m\left(\frac{\dot{x}^2 - \sqrt{1-\dot{x}^2} + 1 - \dot{x}^2}{\sqrt{1-\dot{x}^2}}\right) + \frac{1}{2}kx^2$$

$$E = m\left(\frac{1}{\sqrt{1-\dot{x}^2}} - 1\right) + \frac{1}{2}kx^2$$

We don't want variables, so let's plug something in. Energy is always conserved, so for simplicity we can find the energy when the spring is at its maximum amplitude – this is a turning point, so velocity = 0 here.

$$\boxed{E = m\left(\frac{1}{1} - 1\right) + \frac{1}{2}kA^2 = \frac{1}{2}kA^2 = E_{tot}}$$

B. Maximum speed – this is when x = 0. To find velocity, I'll use the fact that energy is conserved again.

$$\frac{1}{2}kA^2 = E_{tot} = m\left(\frac{1}{\sqrt{1 - \dot{x_{max}}^2}} - 1\right)$$

Isolate vmax, this gives

$$\frac{1}{2m}kA^2+1=\frac{1}{\sqrt{1-\dot{x_{max}}^2}} \to \dot{x}_{max} = \sqrt{\frac{-1}{\left(\frac{1}{2m}kA^2+1\right)^2}+1}$$

C. Now we can use our handy E L equations.

$$L = m\left(1-\sqrt{1-\dot{x}^2}\right)-\frac{1}{2}kx^2$$
$$\frac{d}{dt}\left(m\left(-\frac{1}{2}\right)(1-\dot{x}^2)^{-\frac{1}{2}}(-2\dot{x})\right)=-kx$$
$$= \frac{d}{dt}\left(m(1-\dot{x}^2)^{-\frac{1}{2}}(\dot{x})\right)=m\ddot{x}(1-\dot{x}^2)^{-\frac{1}{2}}+mx\left(-\frac{1}{2}\right)(1-\dot{x}^2)^{-\frac{3}{2}}(-2\dot{x}\ddot{x})$$

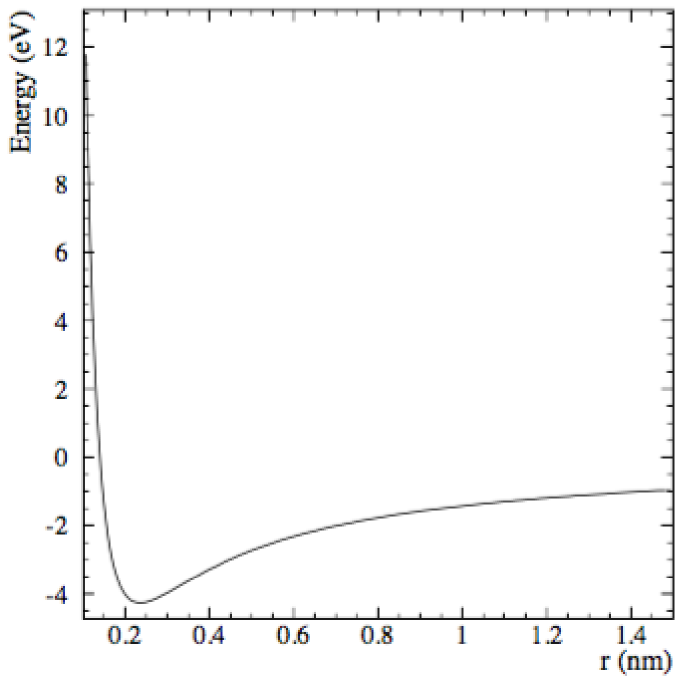
After some simplifying…

$\ddot{x}=-\frac{kx(1-\dot{x}^2)^{-\frac{3}{2}}}{m}=-\frac{kx}{m\gamma^3}$

Where I used gamma for simplicity. Now, we can find gamma – simply go back to our energy, and find when x = A/2.

$$\frac{1}{2}kA^2=m(\gamma-1)-\frac{1}{2}k\left(\frac{A}{2}\right)^2 \to \frac{3}{8}kA^2=m(\gamma-1)\to\gamma=\frac{\frac{3}{8}kA^2}{m}+1$$

$$\boxed{\ddot{x}=-\frac{kA}{2m}\left(\frac{3kA^2}{8m}+1\right)^{-3}}$$



4. A NaCl molecule has an ionic bond between a Na^+ and a Cl^- ion. The potential for this system as a function of r, the distance between the ions, may be described by:

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} + \frac{A}{r^k}$$

where A and k are constants that characterize the repulsive potential between the ions at small r.

Measurements show that this potential has a minimum of -4.26 eV at $r =$ 0.236 nm. The atomic masses of Na and Cl are 22 and 37. Using this information and the above potential, estimate the energy gap between two lowest vibrational modes of the molecule.

For really any potential, we can model it as a simple harmonic oscillator about the minimum point, assuming small oscillations. Let's do that here. We want to recover something that looks like $V = \frac{1}{2}Kx^2$, which we can adapt to the quantum problem $V = \frac{1}{2}m\omega^2x^2$ using $\omega = \sqrt{\frac{K}{m}}$.

Let's Taylor expand the potential about the minimum, r_0.

$$V(r) = V(r_0) + V'(r_0)(r - r_0) + \frac{V''(r_0)}{2}(r - r_0)^2$$

Because $V(r_0)$ is the minimum, $V'(r_0) = 0$. We can also ignore the constant because we don't care about constants.

$$V(r) = \frac{V''(r_0)}{2}(r - r_0)^2$$

Bingo! This is like $= \frac{1}{2}Kx^2$ with K = $V''(r_0)$.

$$V' = \frac{e^2}{4\pi\epsilon_0 r^2} - \frac{Ak}{r^{k+1}} = 0$$

$$V'' = \frac{-2e^2}{4\pi\epsilon_0 r^3} + \frac{A(k+1)k}{r^{k+2}}$$

We can maybe eliminate A... from the first equation above, with V':

$$\frac{e^2 r^{k+1}}{k4\pi\epsilon_0 r^2} = A = \frac{e^2 r^{k-1}}{k4\pi\epsilon_0}$$

$$V'' = \frac{-2e^2}{4\pi\epsilon_0 r^3} + \frac{\frac{e^2 r^{k-1}}{k4\pi\epsilon_0}(k+1)k}{r^{k+2}} = \frac{-2e^2}{4\pi\epsilon_0 r^3} + \frac{e^2(k+1)}{r^3 4\pi\epsilon_0} = \frac{e^2(k-1)}{4\pi\epsilon_0 r^3}$$

Wow that truly simplifies!!!
So $K = \frac{e^2(k-1)}{4\pi\epsilon_0 r_0^3}$
We can get K now using V_0 and r_0. I did a quick calculation and got k = 3.35, this might be wrong though, I didn't double check. I'll just use it.
To get omega, we need the reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{22 \times 37}{59} \times 1.66e^{-27}kg = 2.29e^{-26}kg$

$$\omega = \sqrt{\frac{K}{\mu}}$$

For a SHO, the energy levels are $E = \left(n + \frac{1}{2}\right)h\omega$, so the difference between the lowest energy levels are $\Delta E = \frac{3}{2}h\omega - \frac{1}{2}h\omega = h\omega$.
So just plug in omega and calculate.

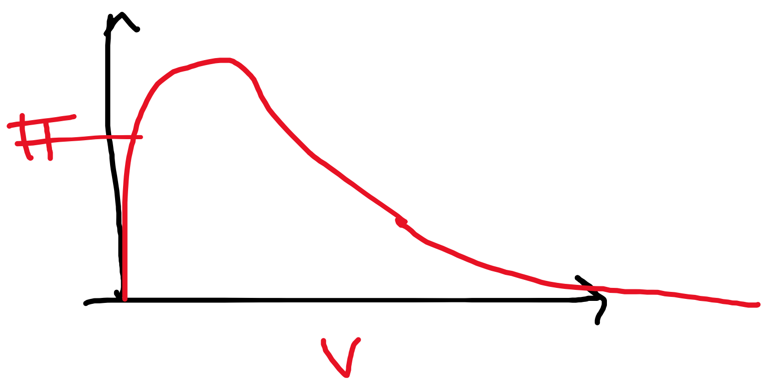
5. Mars has a mass of $M = 6.4 \times 10^{23}$ kg and a radius of 3400 km. Suppose that Mars were given an Earthlike atmosphere: pressure = 100 kPa, temperature = 10°C, made of 78% nitrogen (atomic mass 28) and 22% oxygen (atomic mass 32).

Is Mars's gravity strong enough to retain this atmosphere? Do an order of magnitude estimation of how long it would take for the atmosphere to escape into space. Neglect the effects of external factors such as solar wind or meteors.

Whether or not the molecules in the atmosphere are retained or not depends on their escape velocity, which we can determine by setting KE = PE (we're essentially saying KEf and PEf are 0, because it's travelled off to infinity):

$$0 = U_i + K_i = \frac{1}{2}mv^2 - \frac{GMm}{r} \rightarrow v_{esc} = \sqrt{\frac{2GM}{r}} = 5011\, m/s$$

So, if molecules are traveling faster than this velocity, they will escape. The air molecules are following the Maxwell Boltzmann speed distribution:



Let's go with the most probable velocity.

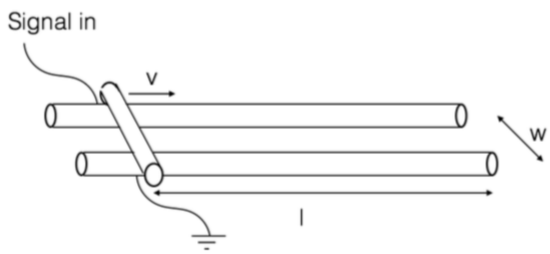
$$v_{mp} = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2k(283\,K)}{(0.78 * 28 + 0.22 * 32)}} = 402\, m/s$$

For the mass, I used the average of nitrogen and oxygen.

Most particles are under the escape velocity, but there will still be a small tail above the escape velocity! The particles will also "repopulate" the tails of the distribution – after the particles with speeds above the escape velocity have essentially fucked off, the some of the slower ones will increase their speed to maintain the Maxwell B dist. So the atmosphere will be slowly leaking.

$$P(v_1 \rightarrow v_2) = \int_{v_{esc}}^{\infty} \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-\frac{mv^2}{2kT}}\, dv$$

Not sure where to go from here…



6. A rail gun is an instrument that uses magnetic forces to accelerate a conducting object that spans two conducting rails. The schematic above shows two long rails, with a shorter rod that spans the distance $w = 4$ cm between them and extends an extra 1 cm on either side. The rods slide without rolling along the rails; assume here that the points of contact between rod and rails have zero resistance and zero friction. Driving a current from "Signal in" to ground creates a magnetic field that interacts with the rod and causes it to accelerate down the rails, in the direction marked "v". Assume that the rod starts 10 cm from the connection point. You should make appropriate simplifying assumptions about the $\vec{B}$ field from the rails.

The rods and rails are each 1 cm diameter. They are made of copper (density $= 9000$ kg/m^3, resistivity $= 2 \times 10^{-8}$ Ω·m). From the starting point (at rest, 10 cm from where the signal and ground are connected), to the ends of the rails, is a distance $l = 1.9$ m. Calculate the velocity of the rod when it flies off the rails, assuming it is powered by an ideal current source that supplies 10^4 A.

The magnetic field goes into the first rail, down the rod, and into the second rail. We want to find the force on the rod based on the B field from the rails. We assume the B-field of the rails is, according to Ampere's law:

$\oint B \cdot dl = \mu_0 I_{enc} \rightarrow B = \frac{\mu_0 I}{2\pi s}$, where s is the radial distance from the rail. There are two rails so $B_{tot} = \frac{\mu_0 I}{\pi s}$. Using the right-hand rule for current and B, the B field from both rails is pointing downwards onto the rod. Then, using the right hand rule again (this time $F \propto dl \times B$), when the current is traveling down the rod, and the B field from the rails is pointing down on the rod, F is point right, which is what we expect. Therefore, the rod travels to the right.

$$F = \int_{1cm}^{5cm} I\, dl \times B = \int_{1cm}^{5cm} IBdl = \int_{1cm}^{5cm} \frac{I^2 \mu_0}{\pi l} dl = \frac{I^2 \mu_0}{\pi} \ln(5)$$

I'm not sure if my integral bounds are right… Should it be 0 to 4? Considering we are looking at the force on the rod, I assume dl is from the bottom to the rod to the top, where the current is? And there is no current on the edges. Now, how can we relate velocity and force? It's actually simple but I always forget… work! Work is equal to the change in kinetic energy.

$$W = Fdx = \frac{1}{2} m v^2 \rightarrow v^2 = \frac{2Fdx}{m} = 2\frac{I^2 \mu_0}{m\pi} \ln(5)\, dx$$

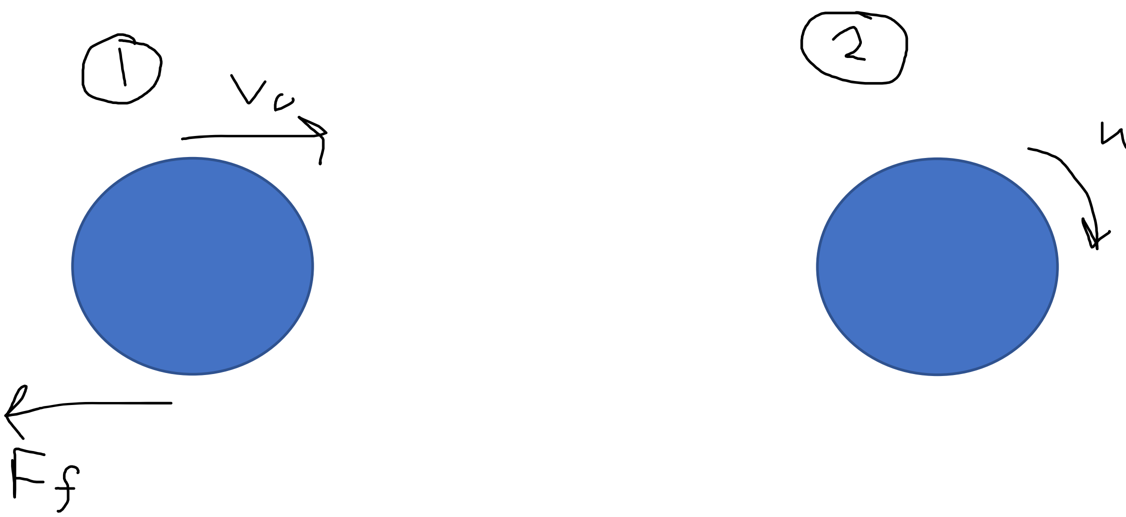
Here, we can use the density and volume of the rod to find m, and dx is simply the length of the rails, dx = 1.9m. This gave me

$$\boxed{v = 76\frac{m}{s}}$$

Instead of using work, the same result could be gathered from kinetic formulas:

$v^2 = v_0^2 + 2ax = 0 + \frac{2Fx}{m}$!!

7. A bowling ball is thrown with an initial horizontal translational velocity of $v_0 = 7$ m/s but a rotational speed of 0. It initially skids until it rolls without slipping. Given a coefficient of friction $\mu = 0.05$, mass $M = 7$ kg, and ball radius $R = 11$ cm, at what time t in seconds after being released does the ball stop slipping?



Initially, the ball only has translation velocity. The forces are:

$$F_y = 0 = mg - N \rightarrow N = mg$$
$$F_X = F_F = -\mu N = ma_y \rightarrow a_y = -\mu g$$

Using conservation of energy, the ball will stop splitting when all of its kinetic energy has been converted to rotational energy (ignoring losses from friction).

NOTE from future: think I should have used $KE_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{2}{5}MR^2\omega^2$? Although then I get a relation $\omega = \frac{\sqrt{\frac{2}{5}}v}{r}$. But why would rotational KE not involve the correct moment of inertia? Maybe a better way to think of it is:

When the ball has rotational motion and is slipping, it's total velocity at the contact point with the ground is $v_{tot} = v_{rot} + v_{trans}$. When it stops slipping, v_{trans}= 0, so $v_{tot} = v_{rot} + 0 = \omega r$?

$$\frac{1}{2}mv^2 = \frac{1}{2}mR^2\omega^2 \rightarrow \omega = \frac{v}{r}$$

So, the ball will stop slipping when $\omega = \frac{v}{R} = \frac{\dot{x}}{R}$

Kinetic formulas: $v = v_0 + at \rightarrow t = \frac{v-v_0}{a} = \frac{\dot{x}-v_0}{-\mu g} = \frac{\omega R-v_0}{-\mu g} = \frac{\dot{\theta}R-v_0}{-\mu g}$

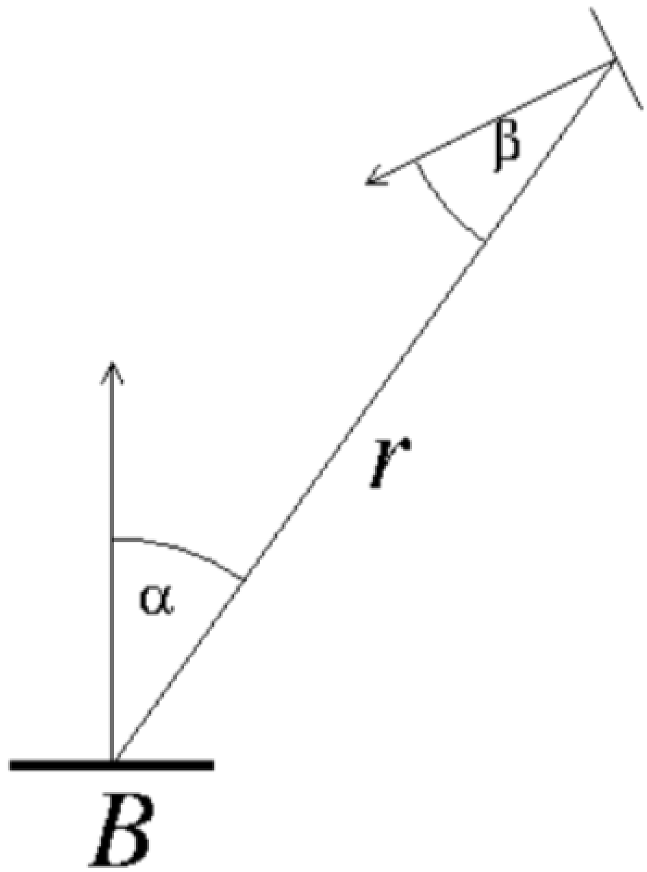
So, we're good if we find theta dot. Could use torque?

$$\tau = r \times F = I\ddot{\theta} = \frac{2}{5}MR^2\ddot{\theta} = RF = \frac{R\mu g}{M} \rightarrow \ddot{\theta} = \frac{5\mu g}{2R} \rightarrow \dot{\theta} = \frac{5\mu g t}{2R}$$

Plug in.

$$\boxed{t = \frac{-\dot{\theta}R + v_0}{\mu g} = \frac{-\frac{5\mu g t}{2R}R + v_0}{\mu g} = -\frac{5}{2}t + \frac{v_0}{\mu g} \rightarrow t = \frac{2v_0}{7\mu g} = 4.08s}$$

8. Consider a disk of radius R emitting radiation, and a detector B. Every point on the surface of the disk emits radiation isotropically into 2π steradians. The disk is located at distance r from detector B and makes an angle α from the perpendicular to the detector. The surface of the emitting disk makes an angle β with respect to the line of sight to the detector.

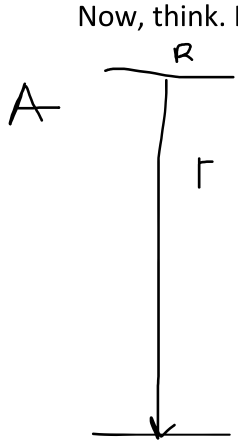


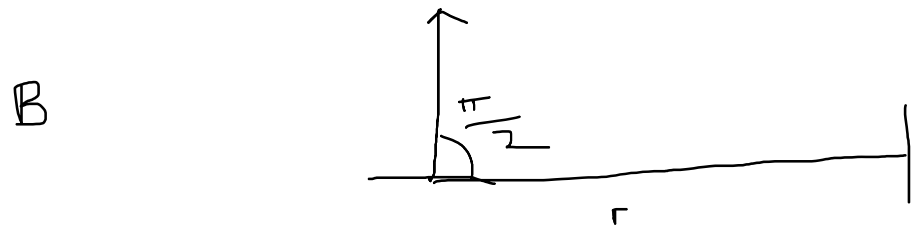
Assuming $R \ll r$, calculate the flux measured at detector B as a function of α, β, R, r and the intensity I of the emitted radiation. (Dimensionally, intensity is power per area, W/m^2.)

Flux = Power over area. The flux units match the units of intensity, which means any units we add need to be cancelled out. What is the solid angle of radiation emitted from the disk at the detector?

$$\Omega = \frac{A_{disk}}{r^2} = \frac{\pi R^2}{r^2}$$

Now, think. If we had something that looked like this (image A):



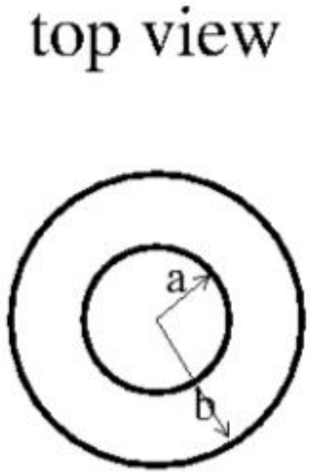


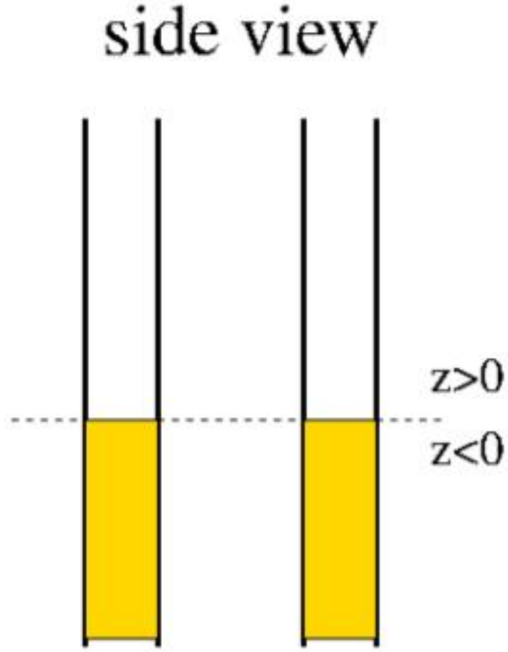
Then our flux would just be $\phi = I\frac{\pi R^2}{r^2}$, because the intensity is increased by having more surface area emitting it, and decreased with distance according to the inverse square law. However, the emitter and the detector are at angles to each other, which means that will absorb *less* than if they were directly across from each other.

If we had image B, in which alpha = pi/2, then the flux would be zero because the detector would not absorb anything. The same is true if beta was pi/2. Because of that, we can be reasonably sure our flux is:

$$\phi = I\frac{\pi R^2 cos\alpha cos\beta}{r^2}$$

(Should it be over $4\pi r^2$ instead of just r^2?)





10. A dielectric fluid ($\epsilon > \epsilon_0$) fills the space between two infinitely long concentric, perfectly conducting cylinders for $z < 0$. The region $z > 0$ between the cylinders is open to the atmosphere ($\epsilon = \epsilon_0$). The cylinders have radii a and b and are sufficiently long that edge effects can be neglected. The inner cylinder is charged to potential $\Phi = V > 0$, while the outer cylinder is grounded at $\Phi = 0$.

A. Solve for the electric field as a function of the cylindrical radius coordinate ρ for $z > 0$ and for $z < 0$.

B. Find the charge per unit length at the inner surface of the dielectric. Do this both for free charge on the conducting cylinder, and bound charge in the dielectric, for both $z < 0$ and $z > 0$.

As we know, a dielectric is an insulator.

A.
We're not told about any external E field, but we know that there is a potential difference between the cylinders, and $E = -\nabla V$, so there must be an induced E field.

We can use Gauss's law for dielectrics, which is basically just replacing ϵ_0 with ϵ. I'm not sure about the electric field inside the cylinder, but we can find it for outside.

$$\oint E \cdot dA = \frac{q}{\epsilon} = E * 2\pi\rho h = \frac{\lambda h}{\epsilon}$$

Where h is the height of the cylinder.

Isolate E, we get:

$$E = \begin{cases} \dfrac{\lambda}{2\pi\rho\epsilon_0}, & z > 0 \\ \dfrac{\lambda}{2\pi\rho\epsilon}, & z < 0 \end{cases}$$

Is this too simple? Should I be finding E inside the dielectric?
Also, we can write this in terms of variables we know (we don't know λ) and we will find this in part B.

B. Now we want the line charge, λ (which is the free charge on the conducting cylinder), and the bound surface charge in the dielectric, σ_b.

We have an equation with λ, but we need to equate it to something to get it in terms of something we know. What do we know? V! At least it's given in the question, so we can safely use it in an equation (I think).

$$V = -\int_a^b E \cdot d\rho = -\int_a^b \frac{\lambda}{2\pi\rho\epsilon_0} \cdot d\rho = \frac{\lambda \ln\left(\frac{a}{b}\right)}{2\pi\epsilon_0}$$

$$\lambda = \begin{cases} \dfrac{2\pi V\epsilon_0}{\ln\left(\frac{a}{b}\right)}, & z > 0 \\ \dfrac{2\pi V\epsilon}{\ln\left(\frac{a}{b}\right)}, & z < 0 \end{cases}$$

Bound charges? Our handy formula sheet tells us $\sigma_b = P \cdot \hat{n} = (\epsilon - \epsilon_0)E \cdot \hat{\rho} = (\epsilon - \epsilon_0)\frac{\lambda}{2\pi\rho\epsilon}$ (E is also in the rho direction so the dot product = 1).

$$\sigma_b = (\epsilon - \epsilon_0)\frac{\lambda}{2\pi\rho\epsilon} = (\epsilon - \epsilon_0)\frac{V}{\rho \ln\left(\frac{a}{b}\right)}$$

11. A charged pion (mass = 140 MeV/c^2) travels with velocity v in the $+z$ direction in the lab frame. It decays in flight to a muon (mass = 106 MeV/c^2) and a neutrino (mass = 0). In the pion's rest frame, this decay is isotropic, with the muon equally likely to go in all directions, and the neutrino going in the opposite direction from the muon.

A. Calculate the energy of the muon in the pion's rest frame.

B. The average velocity vector of the muon in the pion's rest frame is (0, 0, 0), by isotropy. What is the average velocity vector of the muon as measured in the lab frame? Hint: it is *not* simply (0,0,v)!

A. This is a simple conservation of 4 momentum question.

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right) = \left(\frac{E}{c}, \vec{p}\right)$$

We have the decay: $\pi \to \mu + \nu_\mu$, where the charge of the muon depends on that of the pion.
4 momentum is always conserved:

$$p_\pi = p_\mu + p_\nu$$

To make things easier for us, move around and square.

$$p_\nu^2 = (p_\pi - p_\mu)\hat{}2$$

My reasons for moving it around like this will be obvious later.
4 momentum squared is always the -mass squared:

$$p^2 = \left(\frac{-E}{c}, \vec{p}\right)\left(\frac{E}{c}, \vec{p}\right) = -\frac{E^2}{c^2} + p^2 = -m^2c^2$$

$$p_\nu^2 = (p_\pi - p_\mu)^2 = -m_\pi^2 - m_\mu^2 - 2(-E_\pi, 0)(E_\mu, p_\mu) = m_\nu^2 = 0$$

Isolate E_μ.

$$0 = -m_\pi^2 + m_\mu^2 + 2E_\pi E_\mu = -m_\pi^2 - m_\mu^2 + 2\sqrt{m_\pi^2 + 0}E_\mu = -m_\pi^2 - m_\mu^2 + 2m_\pi E_\mu$$

$$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = \frac{140^2 + 106^2}{2 * 140} = \frac{110MeV}{c^2}$$

B.

Of course, in the pion's rest frame because the decay is isotropic, the average velocity in every direction is just 0, because it cancels out.

In the lab frame, the pion is traveling with velocity v in the z direction. The decay is still isotropic. Because it's traveling in +z, we can still assume the average velocities $< v_x > = < v_y > = 0$. But what about v_z? To find this…. Can I just boost to the lab frame and find the momentum in the pion's frame?

Let's boost the four-momentum vector of the muon in the pion's frame (indicated with primes) to the lab frame.

$$\begin{bmatrix} E_\mu \\ p_\mu \end{bmatrix} = \Lambda \begin{bmatrix} E'_\mu \\ p'_\mu \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} E'_\mu \\ p'_\mu \end{bmatrix}$$

I just ignored the other elements of the transformation matrix, because only the z term is relevant here.

$$\begin{bmatrix} E_\mu \\ p_\mu \end{bmatrix} = \begin{bmatrix} - \quad - \\ -\beta\gamma E'_\mu + \gamma p'_\mu \end{bmatrix}$$

$$p_\mu = -\beta\gamma E'_\mu + \gamma p'_\mu \rightarrow \langle p_\mu \rangle = -\beta\gamma \langle E'_\mu \rangle + \gamma \langle p'_\mu \rangle = -\beta\gamma \langle E'_\mu \rangle + 0$$

Where the last term is 0 because the average velocity (and therefore momentum) in the pion's frame is 0. We also know what E'_μ is because we calculated it in A.

$p_\mu = -\beta\gamma \left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi} \right)$

Now, we kind of want to not write this in terms of beta and gamma, because there are velocity terms in there and it's probably… not ideal. Can we get rid of this?
Let's also boost the four velocity (?) of the pion.

We know, because it's at rest:

$$\begin{bmatrix} E'_\pi \\ v' \end{bmatrix} = \begin{bmatrix} m_\pi \\ 0 \end{bmatrix}$$

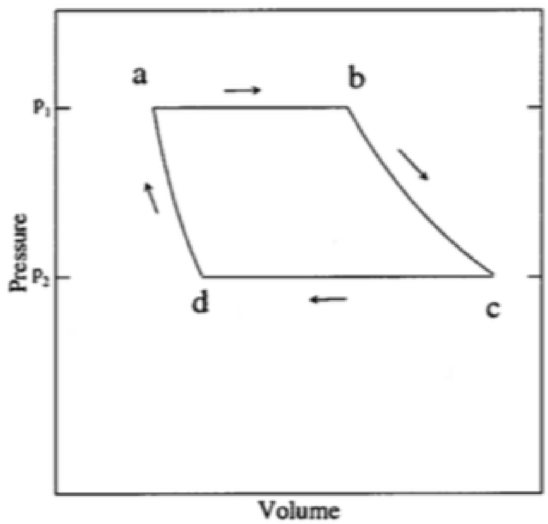
We want:

$$\Lambda \begin{bmatrix} m_\pi \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda m_\pi \\ v' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} m_\pi \\ 0 \end{bmatrix} \rightarrow -\beta\gamma m_\pi = v' \rightarrow -\beta\gamma = \frac{v'}{m_\pi}$$

$$\boxed{p_\mu = -\beta\gamma \left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi} \right) = v' \left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi^2} \right)}$$

Note: maybe could have done differently… beta and gamma are in terms of the pion's velocity, because that's the velocity of this frame, which we're given (v). So maybe I could write:

$p_\mu = \gamma m_\mu v_\mu = -\beta\gamma \left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi} \right) \rightarrow v_\mu = -\beta \left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi m_\mu} \right) = -\frac{\mathrm{v}}{\mathrm{c}} \left(\frac{m_\pi^2 + m_\mu^2}{2m_\pi m_\mu} \right)$



12. An ideal gas of N molecules, each with g degrees of freedom, is carried around the reversible closed cycle shown above. Two of the segments are adiabatic, and two are isobaric (constant pressure).

A. What is the value of g for a monatomic gas? What is g for a gas of diatomic molecules that can be treated as rigid rotators?

B. Show that when an ideal gas undergoes an adiabatic process, $PV^\gamma =$ constant, where γ is related to g. Determine the relation between γ and g.

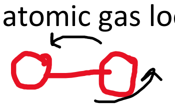
C. Of the states labelled a, b, c, and d in the diagram, which has the highest temperature? Which has the lowest?

D. For each segment of the loop, state whether the entropy of the gas is increasing, decreasing, or staying constant. Explain how you can tell.

E. For each segment of the loop, state whether the gas is doing positive work, negative work, or no work.

F. When cycled in this way, does the system act as a heat engine or a refrigerator? Explain.

A. A monatomic gas only has translational degrees of freedom, so g = 3 (x, y and z directions). A diatomic gas looks like:



And also has 3 translational degrees of freedom, + 2 rotational. The third rotational is frozen out except at very high temps due to quantum effects.

B. When adiabatic, $Q = 0 \rightarrow \Delta U = W \rightarrow \frac{g}{2}\mathrm{NkdT} = -\mathrm{PdV} = -\frac{\mathrm{NkT}}{\mathrm{V}}dV$

$$\frac{g}{2}\frac{dT}{T} = -\frac{dV}{V} \rightarrow \left(\frac{T_f}{T_i}\right)^{\frac{g}{2}} = \frac{V_i}{V_f}$$

We want this in terms of pressure and volume, so use the ideal gas law to say $T = \frac{PV}{Nk}$

$$\left(\frac{P_f V_f}{P_i T_i}\right)^{\frac{g}{2}} = \frac{V_i}{V_f}$$

Now just rearrange! We get that

$$\boxed{P_f V_f^{\gamma} = P_i V_i^{\gamma} = constant, \gamma = \frac{g+2}{g}}$$

C. For a to b and c to d, we can simply use ideal gas law, where T increases as V increases. Therefore

$$T_B > T_A, T_c > T_d$$

For the adiabatic portions, use the constants we determined earlier:

$$\left(\frac{T_c}{T_b}\right)^{\frac{g}{2}} = \frac{V_b}{V_c} \rightarrow T_C = (\frac{V_b}{V_c})^{\frac{2}{g}} T_b$$

Because V_c is larger, T_C is smaller than T_b.

$$T_B > T_c, T_a > T_d$$

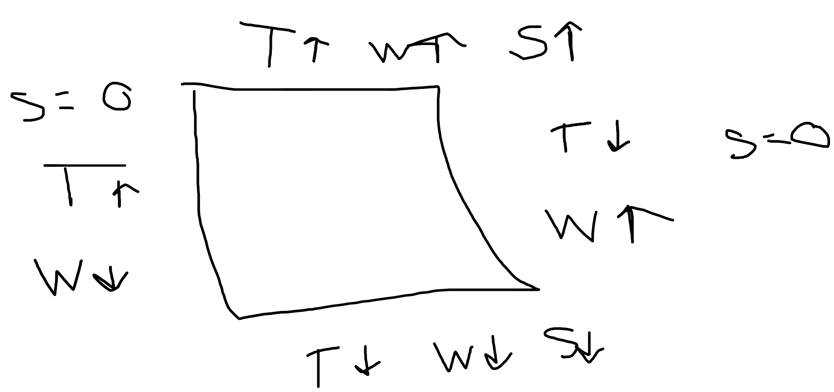
Combining these together, we get the highest temperature is at point B and the lowest is point D.

D. For the adiabatic portions, because $dS = \frac{dQ}{T}$ we know the change in entropy is zero, because by definition there is 0 heat exchange.

For the isobaric portions, I believe I can just go by volume – entropy increases as the volume increases because there are more possible states. So, from a->b, entropy increases, and decreases from c -> d.

E. This is asking about the work done BY the gas. Note that we have to pay attention to the sign because it changes whether we're thinking about work done by or on the gas.
Work done BY the gas is $W = +PdV$. So work is **positive from a -> b** and **negative from c -> d**, because of the changes in volume.

What about the adiabatic portions? $W = -\Delta U \propto -\Delta T$, so it depends whether the temp increases or decreases! So work is **positive from b -> c** and **negative from d -> a.**



First the gas does work to increase the temp. Then we cool it. Then we compress it (T goes down again) by doing work on the gas. Then we heat the gas again (Work done on the gas) and the cycle starts again. Therefore, this is a heat engine.

13. A plate of cross-sectional area A and small thickness h is made of metal with electrical resistivity ρ. The plate is suddenly placed in an external electric field E_0 normal to its surface. Find the rate of heat production by Joule heating in the plate, in watts, as a function of time, and calculate the total energy lost in the Joule heating.

As we know, E field is 0 in a conductor. This is because the charges will align on the surface of the field to cancel out the external electric field. Once this occurs, there will be no more current – so, in this question, we're thinking about *before* equilibrium is reached, when the charges are moving around in attempt to cancel out the field. As the charges are moving, they create an electric field:

$\oint E \cdot dA = \frac{q}{\epsilon_0} \rightarrow E_{ind} = \frac{-q}{A\epsilon_0}\hat{z}$ which is negative, because I am assuming the external field is $+E_0\hat{z}$.

Watts is energy/time, which is the power. We can use equations to get this in a form that makes sense for our question

$$P = I^2R = (J^2A^2)\left(\frac{\rho L}{A}\right) = \left(\left(\frac{E^2}{\rho^2}\right)A^2\right)\left(\frac{\rho L}{A}\right) = \frac{E^2Ah}{\rho}$$

This is the power. However, we want the total energy lost, so I think we need to multiply this by dt and integrate. But we need it in terms of a time. Because our E is not in equilibrium yet, it should be changing:

$E_{tot} = E_0 - \frac{q(t)}{A\epsilon_0} = E_0 - \frac{Idt}{A\epsilon_0}$

We want to integrate the change of charge (aka the current) over time…

$$\dot{q} = I = JA = \frac{AE}{\rho} = \frac{A}{\rho}(E_0 - \frac{q}{A\epsilon_0})$$

Now we have a differential equation for charge! We can integrate by using the integrative factor method.

$$\frac{dq}{dt} + \frac{q}{\rho\epsilon_0} = \frac{AE_0}{\rho} \rightarrow \frac{dq}{dt}e^{t/\rho\epsilon_0} + \frac{qe^{t/\rho\epsilon_0}}{\rho\epsilon_0} = \frac{AE_0}{\rho}e^{t/\rho\epsilon_0}$$
$$\int (qe^{\frac{t}{\rho\epsilon_0}})' = \int \frac{AE_0}{\rho}e^{\frac{t}{\rho\epsilon_0}}$$
$$q(t)e^{\frac{t}{\rho\epsilon_0}} = AE_0\epsilon_0 e^{\frac{t}{\rho\epsilon_0}} |_0^t = AE_0\epsilon_0\left(e^{\frac{t}{\rho\epsilon_0}} - 1\right)$$

$$q(t) = AE_0\epsilon_0(1 - e^{\frac{-t}{\rho\epsilon_0}})$$

Now plug this all back into the power equation.

$$E_{tot} = E_0 - \frac{AE_0\epsilon_0\left(1 - e^{\frac{-t}{\rho\epsilon_0}}\right)}{A\epsilon_0} = E_0\left(1 - 1 + e^{\frac{-t}{\rho\epsilon_0}}\right) = E_0 e^{\frac{-t}{\rho\epsilon_0}}$$

$$P = \frac{E^2Ah}{\rho} = \frac{E_0^2 e^{\frac{-2t}{\rho\epsilon_0}}Ah}{\rho}$$

This is the rate of Joule heating, in watts, as a function of time. Now integrate over time to get the total energy lost.

$$\boxed{P = \frac{dE}{dt} \rightarrow \int dE = \int \frac{E_0^2 e^{\frac{-2t}{\rho\epsilon_0}}Ahdt}{\rho} = E = \frac{E_0^2 e^{\frac{-2t}{\rho\epsilon_0}}Ah\epsilon_0}{-2}|_0^\infty = \frac{E_0^2 Ah\epsilon_0}{2}}$$

14. A quantum mechanical particle of mass m and charge Q is confined to move in a circle of radius R in the xy plane. There is no potential.

A. Calculate the energy eigenstates and the degeneracy of each energy level.

B. Now a weak electric field $\vec{E} = F\hat{x}$ is applied. Calculate the electric dipole moment of the system in its ground state, to lowest non-zero order in F.

Let's start with solving the Schrodinger equation. (Note I'm using h but I always mean hbar)

$$-\frac{h^2}{2m}\nabla^2\psi + 0 = E\psi \rightarrow \nabla^2\psi = -\frac{2mE\psi}{h^2}$$

We can use polar coordinates, with z =0. d/dr is also 0, because the radius is constant. Therefore:

$$\nabla^2 = \frac{1}{R^2}\frac{d^2\psi}{d\phi^2} = -\frac{2mE}{h^2} \rightarrow \frac{d^2\psi}{d\phi^2} = -\frac{2mER^2\psi}{h^2}$$

Let $n^2 \frac{2mER^2}{h^2} \rightarrow E = \frac{n^2h^2}{2mR^2}$

These are our energy levels. We just need to find out the possible values of n. Let's also solve the Schrödinger equation – we can either use sin and cos or complex exponentials. I kind of prefer sins, but I think exponentials are more common.

$$\psi(\phi) = Ae^{in\phi} + Be^{-in\phi}$$

Normalize using the fact that for our ring, phi goes from 0 to 2pi, so the particle has a 100% probability of being found somewhere there:

$$\int_0^{2\pi} |\psi(\phi)|_{\pm}{}^2 d\phi = \int_0^{2\pi} A^2 e^{\pm in\phi} d\phi \rightarrow A = \frac{1}{\sqrt{2\pi}} \rightarrow \psi(\phi)_{\pm} = \frac{1}{\sqrt{2\pi}} e^{\pm in\phi}$$

Now, for n. Because we're on a circle, it must be true that $\psi(\phi) = \psi(\phi + 2\pi)$

$$\frac{1}{\sqrt{2}} e^{\pm in\phi} = \frac{1}{\sqrt{2}} e^{\pm in(\phi+2\pi)} \rightarrow e^{+in\phi} e^{-in\phi} e^{-2\pi in} = 1 \rightarrow e^{-2\pi in} = 1$$
$$= \cos(-2\pi n) + isin(2\pi n)$$

For this to be true, n needs to be any integer, $n = 0, \pm1, \pm2 \ldots$

Therefore $E = \frac{n^2h^2}{2mR^2}$, $n = 0, \pm1, \pm2 \ldots$

We see that the ground state is non-degenerate, and all the other states have degeneracy 2.

B. The electric field creates a splitting of the energy levels, called the Stark effect. We can treat this as a perturbation with potential $V = QE_{ext}x = QFx$. Using trig, it's easy to see that $V = QFRcos\theta$.

The dipole moment is defined as $p = Qd$. The dipole moment is in the same direction as the external field, so again $p = QRcos\theta$. Can I find the first order wavefunction, then the expectation value of the dipole moment? I'll try…

$$\psi_m^1 = \sum_{m!=n} \frac{< \psi_m^0 | H' | \psi_n^0 > \psi_m^0}{E_n^0 - E_m^0}$$

Let ψ_n^0 be the ground state, so $\psi_n^0 = \frac{1}{\sqrt{2\pi}}$, and $E_n^0 = 0$. We want the lowest order correction, so m = 1? $\psi_m^0 = \frac{1}{\sqrt{2\pi}} e^{\pm i\theta}, E_m^0 = \frac{h^2}{2mR^2}$. Therefore $E_n^0 - E_m^0 = -\frac{h^2}{2mR^2}$.

$$< \psi_m^0 | H' | \psi_n^0 > \psi_m^0 = \frac{1}{\sqrt{2\pi}} e^{\pm i\theta} \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{\mp i\theta} \, QFRcos\theta \frac{1}{\sqrt{2\pi}} d\theta$$
$$= \frac{QFR}{(2\pi)^{3/2}} \, e^{\pm i\theta} \int_0^{2\pi} e^{\mp i\theta} \, cos\theta$$

This integral can be solved by doing integration by parts twice. The answer is π!

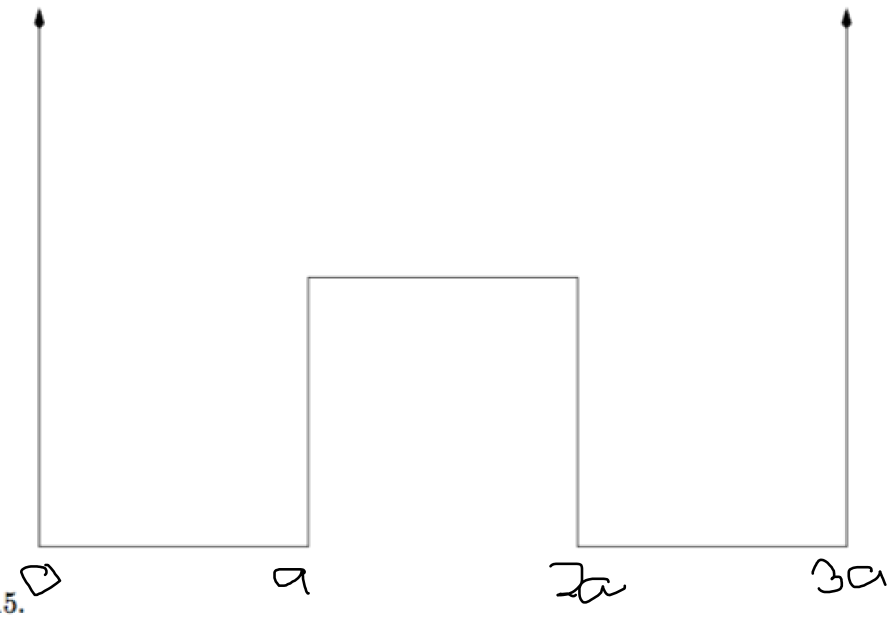
$$\psi_m^1 = \frac{QFR\pi e^{\pm i\theta}}{(2\pi)^{3/2}} \times - \frac{2mR^2}{h^2} = \frac{QFRmR^2 e^{\pm i\theta}}{\sqrt{2\pi} h^2}$$

Now the electric dipole moment is $< p > = < QRcos\theta > = \left(\frac{QFRmR^2}{\sqrt{2\pi}h^2}\right)^2 QR \int_0^{2\pi} e^{\mp i\theta} cos\theta e^{\pm i\theta} d\theta = 0$

Hm. It says first non-zero correction, so maybe I should actually continue? Let m = -1 and 1, and then m = 2, etc. When I let m = 1 and -1, ψ_m^1, the only difference becomes

$$\psi_m^1 = \frac{QFRmR^2 (e^{\pm i\theta} + e^{\mp i\theta})}{\sqrt{2\pi} h^2} = \frac{2QFRmR^2 cos\theta}{\sqrt{2\pi} h^2}$$

But this leads to a $\cos^3 \theta$ in the integral, which is also zero.



15.

A particle is initially trapped in the ground state of the left side of a double well potential with an infinite energy barrier separating left and right. Now the barrier is suddenly lowered to a finite but still high value (see the figure). Because of tunnelling, the particle appears, for the first time, completely on the right side of the double well potential at $t = 9$ s. Calculate the probability of finding the particle on the right side at $t = 12$ s.