Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 11, 2016
1:00PM to 3:00PM
Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

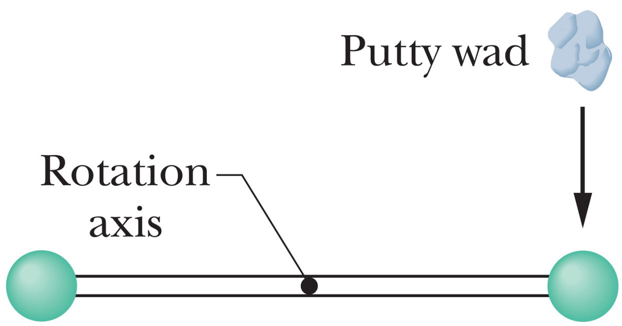
Good Luck!

1. Two balls, each of mass $m = 2.00$ kg and negligible radius are attached to a thin rod of length $L = 50.0$ cm of negligible mass. The rod is free to rotate in a vertical plane without friction about a horizontal axis through its center in the gravitational field at the surface of the Earth (assumed to be uniform). With the rod initially stationary and horizontal (see Figure), a wad of wet putty of mass $M = 50.0$ g drops onto one of the balls, hitting it with a speed $v_0 = 3.00$ m/s and then sticking to it.

   (a) What is the angular speed of the system just after the putty wad hits?

   (b) What is the ratio of the kinetic energy of the system after the collision to that of the putty wad just before?

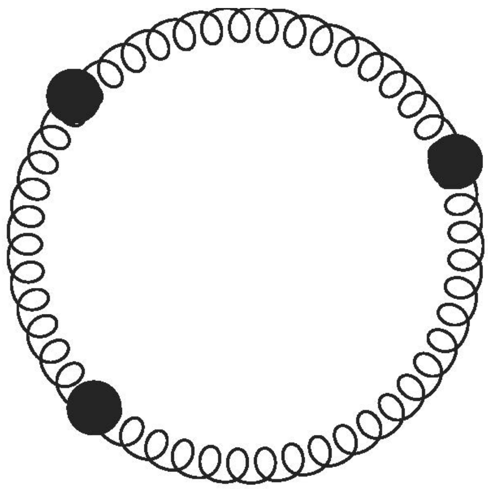
   (c) Through what angle will the system rotate before it momentarily stops?



2. Consider the general problem of N beads of mass m that slide frictionlessly around a circular hoop. Adjacent pairs of beads are connected by springs with identical spring constants and equilibrium lengths. This system of springs and beads forms a closed circle around the hoop. For any given N, the spring constant is chosen such that in equilibrium, the springs are under tension T. Answer the following questions:

   (a) Suppose $N = 2$. For $t < 0$ bead #1 is held fixed at a reference position, $\theta_1 = 0$, and bead #2 is held fixed at $\theta_2 = \pi + \Delta$ where $\Delta \ll \pi$. At $t = 0$ the beads are released. Find the subsequent motion of the two beads, *i.e.* $\theta_1(t)$ and $\theta_2(t)$.

   (b) Suppose $N = 3$. Find the normal modes of the system and their corresponding frequencies. The figure below shows this three-bead configuration.



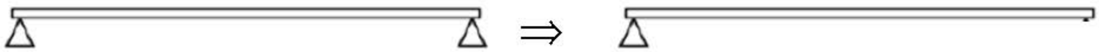
3. A mass m can slide without friction in two dimensions on a horizontal surface. It is attached by a massless rope of length L to a second mass M through a hole in the surface. The mass M hangs vertically in a uniform vertical gravitational field g.

   (a) If the mass m moves in a circle of radius r_0 centered on the hole with constant angular velocity ω_0, what is value of ω_0?

   (b) Show that this motion is stable against small perturbations.

   (c) Find the frequency of small oscillations about this circular motion.

4. As shown in the figure, a uniform thin rod of weight W and length L is supported horizontally by two supports, one at each end of the rod. At $t = 0$, one of the supports is removed. Find the force on the remaining support in terms of W immediately thereafter (at $t = 0$). At this instant, what is the angular acceleration around the remaining support in terms of L and g?



5. A block of mass $m = 200$ g is attached to a horizontal spring with spring constant $k = 0.85$ N/m. The other end of the spring is fixed. When in motion, the system is damped by a force proportional to the velocity, with proportionality constant $-b = -0.2$ kg/s.

   (a) Write the differential equation of motion for the system.

   (b) Show that the system is underdamped. Calculate the oscillation period and compare it to the natural period.

   (c) How long does it take for the oscillating block to lose 99.9% of its total mechanical energy? How many cycles does this correspond to? By what factor does the amplitude decrease during this time?

   The system is now subject to a harmonic external force, $F(t) = F_0 \cos(\omega_e t)$, with a fixed amplitude $F_0 = 1.96$ N.

   (d) Calculate the driving frequency $\omega_{e,max}$ at which amplitude resonance (*i.e.* when the amplitude is maximized) occurs, and find the steady-state maximum amplitude to which this corresponds.

1 Falling putty (Johnson)

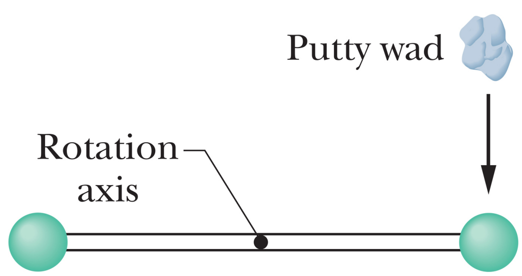
1.1 Problem

Two balls, each of mass $m = 2.00$ kg and negligible radius are attached to a thin rod of length $L = 50.0$ cm of negligible mass. The rod is free to rotate in a vertical plane without friction about a horizontal axis through its center in the gravitational field at the surface of the Earth (assumed to be uniform). With the rod initially stationary and horizontal (see Figure), a wad of wet putty of mass $M = 50.0$ g drops onto one of the balls, hitting it with a speed $v_0 = 3.00$ m/s and then sticking to it.

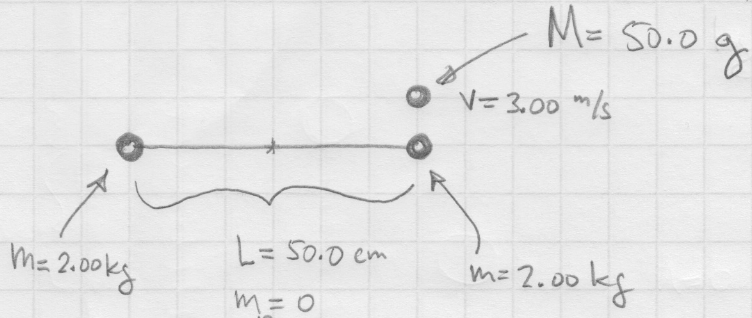
a. What is the angular speed of the system just after the putty wad hits?

b. What is the ratio of the kinetic energy of the system after the collision to that of the putty wad just before?

c. Through what angle will the system rotate before it momentarily stops?



1.2 Solution



(a) Conserve angular momentum: $L_i = L_f$, $L = I\omega$

$$L_i = I_p \omega_p = \left[M\left(\frac{L}{2}\right)^2\right]\left[\frac{V}{(L/2)}\right] = M\frac{L}{2}V$$

$$L_f = \left(I_p + I\right)\omega = \left[M\left(\frac{L}{2}\right)^2 + 2m\left(\frac{L}{2}\right)^2\right]\omega = (M+2m)\left(\frac{L}{2}\right)^2\omega$$

$$M\frac{L}{2}V = (M+2m)\left(\frac{L}{2}\right)^2\omega$$

solve for ω

$$\omega = \frac{M}{(M+2m)}\frac{2}{L}V = \frac{50.0\times10^{-3}\,kg}{(50.0\times10^{-3}\,kg + 2\cdot2.00\,kg)}\frac{2(3.00\,m/s)}{(50.0\times10^{-2}\,m)}$$

$$\boxed{\omega = 0.148\ \frac{rad}{sec}}$$

(b) $K_i = \frac{1}{2}Mv^2$

$K_f = \frac{1}{2}(I_p + I)\omega^2$

$$\frac{K_f}{K_i} = \frac{(I_p+I)\omega^2}{Mv^2} = \frac{(M+2m)\left(\frac{L}{2}\right)^2\omega^2}{Mv^2}$$

$$\frac{K_f}{K_i} = \frac{(50.0\times10^{-3}\,kg + 2(2.00\,kg))}{(50.0\times10^{-3}\,kg)(3.00\,m/s)^2}\left(\frac{50.0\times10^{-2}\,m}{2}\right)^2\left(0.148\,\frac{rad}{s}\right)^2$$

$$\boxed{\frac{K_f}{K_i} = 1.23\times10^{-2}}$$

(c) after the collision, energy is conserved

$K_i = \alpha\,\frac{1}{2}Mv^2$, $\alpha = 1.23\times10^{-2}$ (from part b)

$U_i = 0$

$K_f = 0$

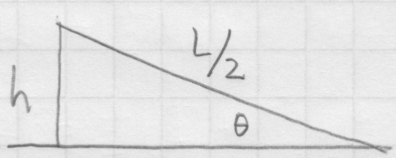
$U_f = Mgh + \underbrace{mgh - mgh}$

Note: h must be positive, so the system rotates through $180^\circ + \theta$

the two balls cancel each other because one is above the zero point and the other is below

$$\alpha\,\frac{1}{2}Mv^2 = Mgh \rightarrow h = \frac{\alpha v^2}{2g} = \frac{L}{2}\sin\theta$$

$$\sin\theta = \frac{\alpha v^2}{Lg},\quad \theta = \sin^{-1}\left[\frac{\alpha v^2}{Lg}\right]$$



$\sin\theta = \frac{h}{L/2}$, $h = \frac{L}{2}\sin\theta$

$\theta = 1.3^\circ$

$\boxed{\text{system rotates } 181^\circ}$

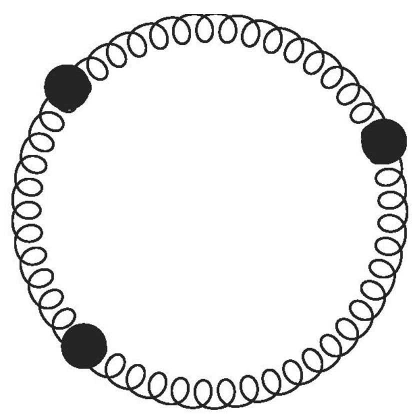
2 Beads on a hoop (Cole)

2.1 Problem

Consider the general problem of N beads of mass m that slide frictionlessly around a circular hoop. Adjacent pairs of beads are connected by springs with identical spring constants and equilibrium lengths. This system of springs and beads forms a closed circle around the hoop. For any given N, the spring constant is chosen such that in equilibrium, the springs are under tension T. Answer the following questions:

a. Suppose $N = 2$. For $t < 0$ bead #1 is held fixed at a reference position, $\theta_1 = 0$, and bead #2 is held fixed at $\theta_2 = \pi + \Delta$ where $\Delta \ll \pi$. At $t = 0$ the beads are released. Find the subsequent motion of the two beads, *i.e.* $\theta_1(t)$ and $\theta_2(t)$.

b. Suppose $N = 3$. Find the normal modes of the system and their corresponding frequencies. The figure below shows this three-bead configuration.



2.2 Solution

N Beads on a hoop – solutions

The displacement of bead with index i from its equilibrium position will be written ξ_i. Since the net force on a bead is zero with all of the beads at their equilibrium positions we can write the equations of motion purely in terms of the displacements ξ.

a. The spring constant k can be expressed in terms of the tension using $T = 2\pi R k$ or $k = T/2\pi R$. The equations of motion for the two beads can be written

$$\begin{aligned} mR\,\ddot{\xi}_1 &= kR\,(\xi_2 - \xi_1) - kR\,(\xi_1 - \xi_2) = 2kR\,(\xi_2 - \xi_1) \\ mR\,\ddot{\xi}_2 &= kR\,(\xi_1 - \xi_2) - kR\,(\xi_2 - \xi_1) = 2kR\,(\xi_1 - \xi_2) \end{aligned}$$

If we add and subtract the two equations of motion we obtain,

$$\begin{aligned} mR\left(\ddot{\xi}_1 + \ddot{\xi}_2\right) &= 0. \\ mR\left(\ddot{\xi}_2 - \ddot{\xi}_1\right) &= 4kR(\xi_2 - \xi_1) \end{aligned}$$

If we define $\xi_s = \xi_1 + \xi_2$ and $\xi_d = \xi_2 - \xi_1$ and simplify we obtain the two equations

$$\begin{aligned} \ddot{\xi}_2 &= 0. \\ \ddot{\xi}_d &= 4\left(\frac{k}{m}\right)\xi_d \end{aligned}$$

The first equation which describes the first normal mode of the system and has the solution $\xi_s = \xi_0 + \omega_s\,t$, describes the simultaneous motion of the beads around the circle at constant separation. The second equation which describes the second n normal mode corresponds to simple harmonic oscillation of the ξ_d coordinate with frequency $\omega_d = 2\sqrt{k/m}$. We can write the general solution of that equation, $\xi_d = A\cos\omega_d\,t + B\sin\omega_d\,t$. We can obtain the constants ξ_0, ω_s, A, and B from the initial conditions. we have

$$\begin{aligned} \xi_s(t=0) &= \xi_0 = \xi_1(t=0) + \xi_2(t=0) = \Delta/2 \\ \xi_d(t=0) &= A = \xi_2(t=0) - \xi_1(t=0) = \Delta/2 \\ \dot{\xi}_s(t=0) &= \omega_s = \dot{\xi}_1(t=0) + \dot{\xi}_2(t=0) = 0 \\ \dot{\xi}_d(t=0) &= B\omega_d = \dot{\xi}_2(t=0) - \dot{\xi}_1(t=0) = 0 \end{aligned}$$

Or more succinctly, $\xi_0 = \Delta$, $A = \Delta$, $\omega_s = 0$, $B = 0$. Now we express ξ_1 and ξ_2 in terms of ξ_s and ξ_d,

$$\xi_1 = \frac{1}{2}\left(\xi_s - \xi_d\right), \xi_2 = \frac{1}{2}\left(\xi_s + \xi_d\right)$$

with the results for $\xi_1(t)$ and $\xi_2(t)$,

$$\begin{aligned} \xi_1(t) &= \frac{\Delta}{2}\left[1 - \cos\left(\omega_d\,t\right)\right] \\ \xi_2(t) &= \frac{\Delta}{2}\left[1 + \cos\left(\omega_d\,t\right)\right] \end{aligned}$$

c. The equations of motion take the form

$$\ddot{\xi}_i = \frac{k}{m}\left(2\xi_i - \xi_{i-1} - \xi_{i+1}\xi\right) \equiv \omega_0^2 \left(2\xi_i - \xi_{i-1} - \xi_{i+1}\xi\right)$$

with i cyclic: $i = 0 \rightarrow i = 3$and $i = 4 \rightarrow i = 1$. hewre we have defined with $\omega_0 \equiv \sqrt{k/m}$. If we assume the existence of normal mode solutions to the motion of the form
$U = A\cos\left(\omega\, t - \phi\right)$ with $\xi_i = C_i U$ and substitute into the equations of motion we obtain an eigenvalue equation

$$\begin{bmatrix} \omega^2 - 2\omega_0^2 & \omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega^2 - 2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega_0^2 & \omega^2 - 2\omega_0^2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \omega_0^2 \begin{bmatrix} r^2 - 2 & 1 & 1 \\ 1 & r^2 - 2 & r^2 \\ 1 & 1 & r^2 - 2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = 0$$

$r = \omega/\omega_0$. Applying the usual requirement on the determinant (zero)

$$Det \begin{bmatrix} r^2 - 2 & 1 & 1 \\ 1 & r^2 - 2 & 1 \\ 1 & 1 & r^2 - 2 \end{bmatrix} = 0$$

we obtain the characteristic equation

$$\left(r^2 - 2\right)\left(\left(r^2 - 2\right)^2 - 1\right) - \left(r^2 - 2 - 1\right) + \left(1 - \left(r^2 - 2\right)\right) = 0$$

Simplifying, we can write the characteristics equation

$$\left(r^2 - 2\right)^3 - 3\left(r^2 - 2\right) + 2 = 0$$

Expanding out all the terms and cancelling where appropriate we obtain

$$r^6 - 6r^4 + 9r^2 = r^2\left(r^2 - 3\right)^2 = 0$$

with the solutions $r^2 = 0$ and (degenerate) $r^2 = 3$ (taking only the positive root for solutions to normal mode motion. The $r^2 = 0$ solution corresponds to no oscillation. The resulting equation(s) for the (unnormalized) eigenvector taking $C_1 = 1$ are

$$C_2 + C_3 = 2, -2C_2 + C_3 = -1$$

which give as solutions, $C_2 = 1$ and $C_3 = 1$ for a normalized eigenvector,

$$C = \sqrt{\frac{1}{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

Clearly this solution corresponds to the simultaneous motion of the beads around the hoop. Now consider the degenerate solution $r^2 = 3$ which means $\omega = \sqrt{3}\omega_0$. We obtain a redundant equation for the eigenvectors, $C_1 + C_2 + C_3 = 0$. The redundancy (due to the degeneracy which, in turn results from the symmetry of the problem under cyclic permutation of the indices) means that we have freedom in choosing the remaining two eigenvectors as long as they are orthogonal. One valid choice based on intuition about how normal modes work is to have one bead fixed and the others to oscillate with opposite phase, namely

$$C = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

Giving a final eigenvector

$$C = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

3 Two masses and a rope (Humensky)

3.1 Problem

A mass m can slide without friction in two dimensions on a horizontal surface. It is attached by a massless rope of length L to a second mass M through a hole in the surface. The mass M hangs vertically in a uniform vertical gravitational field g.

a. If the mass m moves in a circle of radius r_0 centered on the hole with constant angular velocity ω_0, what is value of ω_0?

b. Show that this motion is stable against small perturbations.

c. Find the frequency of small oscillations about this circular motion.

3.2 Solution

a. Balance the gravitational force on M and the centrifugal force on m

$$m r_0 \omega_0^2 = Mg \quad \text{or} \quad \omega_0 = \sqrt{\frac{Mg}{m r_0}} \tag{1}$$

b. Describe the motion of m with the polar coordinates:

$$(r(t), \phi(t)) = (r_0 + \delta r(t), \omega_0 t + \delta\phi(t)) \tag{2}$$

and write the equation for the radial acceleration:

$$\begin{aligned} (m+M)\ddot{r} &= -Mg + mr(\omega_0 + \delta\dot{\phi})^2 && (3) \\ (m+M)\delta\ddot{r} &\approx m(\delta r \omega_0^2 + 2 r_0 \omega_0 \delta\dot{\phi}) && (4) \\ &\approx m(\delta r \omega_0^2 - 4\delta r \omega_0^2) && (5) \\ &\approx -3m\omega_0^2 \delta r && (6) \end{aligned}$$

where in Eq. (5) we have used the conservation of angular momentum $L = mr^2(\omega_0 + \delta\dot{\phi})$ allowing us to equate

$$0 = 2\delta r r_0 \omega_0 + r_0^2 \delta\dot{\phi} \quad \text{or} \quad \delta\dot{\phi} = -2\frac{\delta r}{r_0}\omega_0. \tag{7}$$

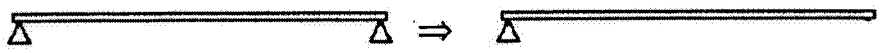
The negative sign in Eq. (6) implies that this perturbed motion is stable and the frequency of small oscillations is also determined by that equation:

$$\omega = \sqrt{\frac{3m}{m+M}}\,\omega_0 \tag{8}$$

4 Two supports and a rod (Shaevitz)

4.1 Problem

As shown in the figure, a uniform thin rod of weight W and length L is supported horizontally by two supports, one at each end of the rod. At $t = 0$, one of the supports is removed. Find the force on the remaining support in terms of W immediately thereafter (at $t = 0$). At this instant, what is the angular acceleration around the remaining support in terms of L and g?



4.2 Solution

Let $N = Force$ of the remaining support.

Acceleration of CM: $Ma = W - N$

Angular acceleration around CM: $I\alpha = N(L/2)$

$I = ML^2/12$

Find N in order to keep point at rest at the support

$$a - \alpha(L/2) = 0 = \frac{W-N}{M} - \frac{NL^2}{4I} = \frac{W-N}{M} - \left(\frac{NL^2}{4}\right)\left(\frac{12}{ML^2}\right)$$

$$= \frac{W-N-3N}{M} = 0 \Rightarrow N = \frac{W}{4}$$

Find the angular acceleration around the remaining support:

$$\tau = I_{end}\alpha_{end} = \frac{ML^2}{3}\alpha_{end} = \left(\frac{L}{2}\right)W \Rightarrow \alpha_{end} = \frac{3W}{2ML} = \frac{3}{2}\frac{g}{L}$$

or

$$a_{CM} = \frac{W-N}{M} = \frac{3W}{4M} = \frac{3}{4}g = \left(\frac{L}{2}\right)\alpha_{end}$$

$$\Rightarrow \alpha_{end} = \left(\frac{3}{4}g\right)\left(\frac{2}{L}\right) = \frac{3}{2}\frac{g}{L}$$

5 Damped Harmonic Motion (Dodd)

5.1 Problem

A block of mass $m = 200$ g is attached to a horizontal spring with spring constant $k = 0.85$ N/m. The other end of the spring is fixed. When in motion, the system is damped by a force proportion to the velocity, with proportionality constant $-b = -0.2$ kg/s.

a. Write the differential equation of motion for the system.

b. Show that the system is underdamped. Calculate the oscillation period and compare it to the natural period.

c. How long does it take for the oscillating block to lose 99.9% of its total mechanical energy? How many cycles does this correspond to? By what factor does the amplitude decrease during this time?

The system is now subject to a harmonic external force, $F(t) = F_0 \cos(\omega_e t)$, with a fixed amplitude $F_0 = 1.96$ N.

d. Calculate the driving frequency $\omega_{e,max}$ at which amplitude resonance (*i.e.* when the amplitude is maximized) occurs, and find the steady-state maximum amplitude to which this corresponds.

5.2 Solution

Q2 (20 pts.)

Block of mass 200g attached to horizontal spring with $k = 0.85N/m$

In motion, the system is damped by force $F_d = -bv$ with $b = 0.2kg/s$.

a). eom is:

$$m\frac{d^2x}{dt^2} = -kx - bv$$

i.e. $$\frac{d^2x}{dt^2} + \frac{kx}{m} + \frac{bv}{m} = 0$$

or $$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \quad \text{with } \gamma = \frac{b}{m}\,;\ \omega_0^2 = \frac{k}{m}$$

(we do know the needed coefficients, viz.

$$\gamma = \frac{b}{m} = \frac{0.2kg/s}{0.2kg} = 1s^{-1}$$

$$\omega_0^2 = \frac{k}{m} = \frac{0.85N/m}{0.2kg} = 4.25s^{-2}$$

and could include these in our equation above.)

b). The damping condition (underdamped, overdamped, ...) is governed by the sign of the term:

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4} = \frac{k}{m} - \frac{b^2}{4m^2}$$

where:

$$\omega^2 = (4.25s^{-2}) - \frac{(1s^{-1})^2}{4} = 4.0s^{-2}$$

$\Rightarrow$ since this is positive, the system is underdamped.

The period is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \frac{\gamma^2}{4}}} = \frac{2\pi}{\sqrt{4.0s^{-2}}} = \pi \text{ seconds}$$

The natural period is:

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\sqrt{4.25s^{-2}}} = \underline{3.05s}$$

i.e. the period with damping is slightly greater than without damping, as expected.

c). To lose 99.9% of the total mechanical energy:

$$E(t) = E_0 e^{-\gamma t}$$

$$\Rightarrow \quad 0.001E_0 = E_0 e^{-\gamma t}$$

$$\Rightarrow \quad -\gamma t = \ln(0.001)$$

$$\text{i.e.} \quad t = -\frac{\ln(0.001)}{1s^{-1}} = \underline{6.9s}$$

In terms of cycles (n):

$$n = \frac{t}{T} = \frac{6.9s}{\pi s} = \underline{2.2}$$

And the amplitude change is:

$$A(t) = A_0 e^{-\frac{\gamma t}{2}} = A_0 e^{-\frac{(1s^{-1})(6.9s)}{2}} = \underline{0.032A_0}$$

i.e. the amplitude decreases to $\underline{3.2\%}$ of initial.

d). Now driving the system with $F(t) = F_0\cos(\omega_E t)$ and $F_0 = 1.96N$, we want to find $\omega_{E,max}$:

$$\omega_{E,max} = \omega_0\sqrt{1 - \frac{1}{2Q^2}} \quad \text{where } Q = \frac{\omega_0}{\gamma}$$

$$\Rightarrow \omega_{E,max} = \sqrt{4.25s^{-2}}\sqrt{1 - \frac{1}{2\left(\frac{4.25s^{-2}}{(1s^{-1})^2}\right)}} = \underline{1.94s^{-1}}$$

$$\text{and } A_{max} = A_0\frac{Q}{\sqrt{1-\frac{1}{4Q^2}}} = \frac{F_0}{k}\frac{Q}{\sqrt{1-\frac{1}{4Q^2}}}$$

$$\Rightarrow A_{max} = \frac{1.96N}{0.85N/m} \frac{\frac{\sqrt{4.26s^{-2}}}{1s^{-1}}}{\sqrt{1-\frac{1}{4\left(\frac{4.26s^{-2}}{(1s^{-1})^2}\right)}}} = \underline{4.9\,m}$$

Section 1-5
Dodd