2017 Solution Explanations
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1. There is a theoretical limit to the luminosity that an object bound gravitationally can emit. This is called the Eddington luminosity L_{edd} and it is reached when the inward gravitational force on a piece of material is balanced by the outgoing radiation pressure. Consider a cloud of hydrogen surrounding a central object of mass M and luminosity L. Let σ_T be the Thomson cross-section (the cross-section for elastic scattering of photons on hydrogen) and m_H be the mass of a proton. Show that the maximum luminosity L_{edd} that the central object can have and still not expel the cloud of hydrogen by radiation pressure is given by

$$L_{edd} = 4\pi c G M m_H / \sigma_T$$

Calculate L_{edd} for the Sun.

This limit is reached when the object is in Hydrostatic Equilibrium, $F_{Grav} = F_{rad}$. If the radiation pressure gets any higher, the photons emitted from the object will impart enough force on the hydrogen molecules in the cloud and push them away.

$$F_{Grav} = \frac{GMm_H}{r^2}$$

r is the distance between the object and the hydrogen molecules.

To find the pressure force, we can use some dimensional analysis. We need to get it in terms of luminosity, which is energy / time.

$$L = \frac{E}{t} = \frac{kgm^2}{s^3}$$

The units of force are $\frac{kgm^1}{s^2}$, so we can get L into units of F by multiplying by $\frac{kgm^2}{s^3} \cdot \frac{s}{m}$, so we can divide L by the velocity. A simpler way of thinking about this:

$$L = \frac{E}{t} \approx \frac{Fd}{t} = Fv$$

The radiation pressure is carried by photons, so the speed is c.

$$F_{press} = \frac{L}{c} = \frac{GMm_H}{r^2}$$

Now the tough part is introducing the Thomson cross section. The way that I think about this is the force on the hydrogen by the photons is increased by a factor of σ_T, because the probability of interaction increases with the cross section. However, the power decreases according to the inverse square law, so by a factor of $\frac{1}{4\pi r^2}$.

$$\boxed{\frac{GMm_H}{r^2} = \frac{L\sigma_T}{4\pi r^2 c} \rightarrow L_{edd} = \frac{4\pi c G M m_H}{\sigma_T}}$$

For the sun,

$$\boxed{L_{edd} = \frac{4\pi(3e8)(6.67e-11)(1.98e30)(1.67e-27)}{6.65e-29} = 1.25e31\,W}$$

2.

A. A quantum mechanical particle of mass m, charge q, and zero spin moves in two dimensions in a potential given, as a function of the polar coordinates r and θ, by $V(r,\theta) = \frac{1}{2}kr^2 + \frac{3}{2}kr^2\sin^2\theta$. Calculate the energies and degeneracies of the lowest 6 energy levels.

B. Now an electric field E is added along the $\theta = 45°$ direction. Calculate the resulting shift in the energy levels from part A.

A. Whenever we want to find the energy levels, we have to solve the Schrödinger equation.

$$\left(-\frac{h^2}{2m}\nabla^2 + \frac{kr^2}{2} + \frac{3}{2}kr^2\sin^2\theta\right)\psi = E\psi$$

On the outside this looks kind of impossible, like how can we even solve this. The key is to realize we can write it in terms of cartesian coordinates.

$$x = rcos\theta, y = rsin\theta$$

$$\frac{kr^2}{2} + \frac{3}{2}kr^2\sin^2\theta = \frac{k}{2}(r^2 + 3r^2\sin^2\theta) = \frac{k}{2}(x^2 + y^2 + 3y^2) = \frac{k}{2}(x^2 + 4y^2)$$

Now it is just a 2-dimensional simple harmonic oscillator in x and y, with $V_x = \frac{1}{2}kx^2, V_y = \frac{4}{2}ky^2$.

Therefore $\omega = \sqrt{\frac{k}{m}}, \omega_y = \sqrt{\frac{4k}{m}} = 2\sqrt{\frac{k}{m}} = 2\omega$

Following normal SHO protocol, this gives us energies $E_x = \left(n_x + \frac{1}{2}\right)h\omega, E_y = 2\left(n_y + \frac{1}{2}\right)h\omega$.

To make it easiest I like to make a table to put in all of the energies and degeneracies as I go.

n_x	n_y	E_{tot}	Degeneracy
0	0	$\left(\frac{1}{2} + \frac{2}{2}\right)h\omega = \frac{3}{2}h\omega$	1
0	1	$\left(\frac{1}{2} + \frac{6}{2}\right)h\omega = \frac{7}{2}h\omega$	2

1	0	$\left(\boldsymbol{\frac{3}{2}+\frac{2}{2}}\right)\boldsymbol{h\omega}=\boldsymbol{\frac{5}{2}h\omega}$	1
1	1	$\left(\boldsymbol{\frac{3}{2}+\frac{6}{2}}\right)\boldsymbol{h\omega}=\boldsymbol{\frac{9}{2}h\omega}$	2
2	0	$\left(\frac{5}{2}+\frac{2}{2}\right)h\omega=\frac{7}{2}h\omega$	Not new!
0	2	$\left(\boldsymbol{\frac{1}{2}+\frac{10}{2}}\right)\boldsymbol{h\omega}=\boldsymbol{\frac{11}{2}h\omega}$	3
2	1	$\left(\frac{5}{2}+\frac{6}{2}\right)h\omega=\frac{11}{2}h\omega$	Not new!
1	2	$\left(\boldsymbol{\frac{3}{2}+\frac{10}{2}}\right)\boldsymbol{h\omega}=\boldsymbol{\frac{13}{2}h\omega}$	3

We've hit the 6 lowest energy levels, with the 2 lowest having degeneracy 1, the next 2 having degeneracy 2, and the next 2 having degeneracy 3. I found the degeneracies just by making linear equations and seeing what $n_x$was possible for each given n_y.

B. Now we're seeing the Stark effect – adding a perturbation to our system will cause the energy levels and to split and therefore break the degeneracy. We need to either solve the Schrödinger equation with the new potential $V=\frac{1}{2}kx^2+2ky^2-qEx-qEy$, or use perturbation theory. However, we would have to use degenerate perturbation theory…. NO THANKS! Furthermore, the wavefunctions are kind of complicated in the SHO when not in the ground state:

$$\psi(x,y)=\psi(x)\psi(y)=\left(\frac{1}{\sqrt{n!}}(a_+)^n\right)^2\left(\frac{\alpha}{\pi}\right)^{1/2}e^{-\frac{\alpha x^2}{2}}e^{-\frac{\alpha y^2}{2}}$$

Having to find this for like, 6 states? I don't think so…

$$\left(-\frac{h^2}{2m}\nabla^2+\frac{1}{2}kx^2+2ky^2-qE(x+y)\right)\psi=E\psi$$

Break this into

$$\left(-\frac{h^2}{2m}\nabla^2+\frac{1}{2}kx^2-qEx\right)\psi=E\psi$$

And

$$\left(-\frac{h^2}{2m}\nabla^2+2ky^2-qEy\right)\psi=E\psi$$

3. "Tumour treating fields" is a novel medical treatment for brain cancer that relies on applying low-intensity alternating electrical fields to the brain. It is hypothesized that this therapy may act on tubulin protein assemblies, which have a permanent electric dipole moment, by preventing them from orienting properly during cell division and so slowing tumour growth. The treatment is only effective if the frequency of the applied field is lower than the lowest frequency at which the tubulin molecules rotate, since otherwise the field will average to zero during the time it takes the dipole moment to reorient itself.

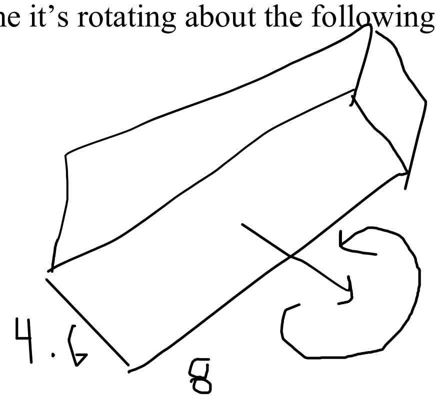
The tubulin complex has a mass of 110,000 amu, an electric dipole moment of 40 e·nm, and dimensions of $4.6 \times 8.0 \times 6.5$ nm. Do an order of magnitude calculation of the approximate maximum frequency at which the electric field should be applied.

There is an incredibly simple way to do this, but the problem is it doesn't exactly come to mind – I keep thinking of dipoles, and the way to solve this using dipoles is quite difficult. But, we could use kinetic energy…

For a rotating object, $KE = \frac{1}{2}I\omega^2$. However, equipartition theory says at thermal equilibrium the kinetic energy is also equal to $KE = \frac{1}{2}kT$.

$$\frac{1}{2}I\omega^2 = \frac{1}{2}kT \rightarrow \omega = \sqrt{\frac{kT}{I}}$$

I assume it's rotating about the following axis:



I'm not sure which axis makes the most sense to rotate around, but I figure it's a magnitude calculation so no big deal.

$$I = \int_0^V \rho r^2 dV$$

Where r is the distance from the volume element to the rotation axis.

$$I = \rho(\int_0^w \int_0^l \int_0^{\frac{h}{2}} h^2 dhdldw + \int_0^w \int_0^{\frac{l}{2}} \int_0^h l^2 dhdldw = \frac{\rho w(l^3 h + l h^3)}{12} = \frac{mw(l^3 h + l h^3)}{whl\ 12}$$
$$= \frac{m(l^2 + h^2)}{12}$$

Maybe calculated this wrong... but whatever.

$$\omega = \sqrt{\frac{kT}{\frac{m(l^2 + h^2)}{12}}}$$

Assume normal body temperature, 37 degrees = 310 K.

$$\omega = \sqrt{\frac{1.38e - 23 * 310}{\frac{(110000 * 1.67e - 27)((8e - 9)^2 + (4.6e - 9)^2)}{12}}} = 1.8e9 \frac{rad}{s} = 286\ GHz$$

Therefore, the frequency of the applied electric field cannot exceed 286 GHz.

4. A reversible heat engine is built using a photon gas as the working medium. At Point 1, the temperature of the gas and surrounding chamber is T_h and its volume is zero. The gas is slowly expanded isothermally until it reaches volume V_2. It then expands adiabatically to volume V_3, at which point its temperature is T_c. Next it contracts isothermally to volume zero. Finally the system is heated at constant volume to return it to Point 1.

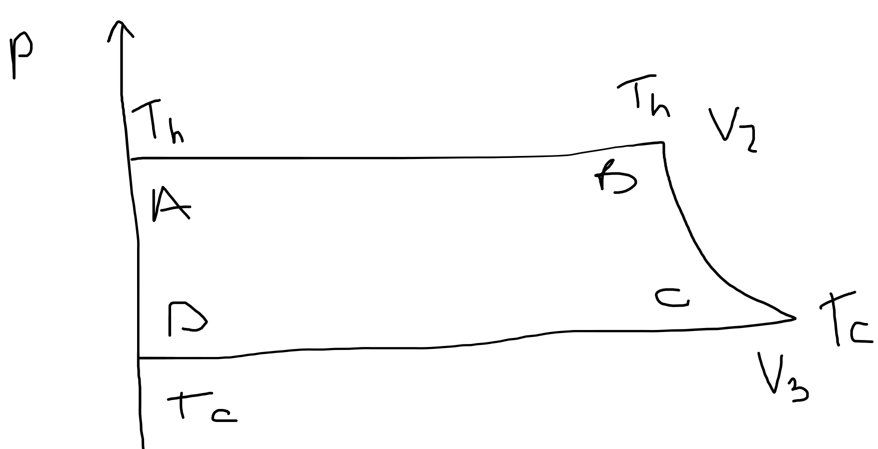
Draw a diagram of of this cycle in the PV plane. Then calculate the work done, heat absorbed, and change in entropy of the gas during each of the four steps of the cycle.

Hint: the internal energy of a photon gas is $U = bVT^4$, where b is a constant, and for any ultra-relativistic gas $U = 3PV$.

First, create a new PV relation for this gas, before drawing a PV diagram.

$$P = \frac{U}{3V} = \frac{bVT^4}{3V} = \frac{bT^4}{3}$$

So, pressure only depends on temperature!



This gives us something like this, kind of (maybe adiabatic portion should be steeper).

Equations we'll need:

$$\Delta U = W + Q; W = -\int PdV; \Delta S = \frac{Q}{T}$$

A → B

This part is isothermal (no temperature change), so $\Delta T = 0$.

$$\Delta U = W + Q = 0, W = -Q$$

$$W = -\int PdV = -P(V_2 - V_1) = -PV_2 = -\frac{bT_H^4 V_2}{3}$$

$$Q = +\frac{bT_H^4 V_2}{3} \rightarrow S = \frac{bT_H^3 V_2}{3}$$

B → C

This part is adiabatic so $Q = 0$.

$$\Delta U = b(V_3 T_C^4 - V_2 T_H^4) = W$$

Note: What I did initially is derive a new adiabatic relation (I got $VT^6 = constant, or\ VP^{3/2} = constant$ and then found $W = -\int PdV = -\int \left(\frac{c}{V}\right)^{\frac{2}{3}} dV = -c^{\frac{2}{3}}(V_f^{\frac{1}{3}} - V_i^{\frac{1}{3}})$

NOTE from future: I did some Googling and I think this WAS the correct method, except the relation was supposed to be VT^3 = constant. Oops!

Since c is constant, we can use any $VP^{3/2} = c = V(\frac{bT^4}{3})^{3/2} = V(\frac{b}{3})^{3/2} T^6 \rightarrow c = (\frac{b}{3})^{3/2} V_2 T_H^6$

$$W = \left(\left(\frac{b}{3}\right)^{\frac{3}{2}} V_2 T_H^6\right)^{\mathbf{\frac{2}{3}}} (V_2^{\frac{1}{3}} - V_3^{\frac{1}{3}}) = \frac{b}{3} V_2^{\frac{2}{3}} T_H^4 \, (V_2^{\frac{1}{3}} - V_3^{\frac{1}{3}})$$

Using the adiabatic relation, we can write V3 in terms of V2…

$$V_2 T_H^6 = V_3 T_C^6 \rightarrow V_3 = \frac{V_2 T_H^6}{T_c^6}$$

$$W = \frac{b}{3} V_2^{\frac{2}{3}} T_H^4 \left(V_2^{\frac{1}{3}} - \left(\frac{V_2 T_H^6}{T_c^6}\right)^{\frac{1}{3}}\right) = \frac{b}{3} V_2^{\frac{2}{3}} T_H^4 V_2^{\frac{1}{3}} \left(1 - \left(\frac{T_H}{T_c}\right)^2\right) = \frac{b}{3} T_H^4 V_2 \left(1 - \left(\frac{T_H}{T_c}\right)^2\right)$$

This is obviously different, but I wonder if it would reduce to the same relation with some playing around… doesn't seem like it... Why though? Maybe actually it would be the same. Could test this numerically?

SECOND NOTE from future: Maybe now they would be the same, if I used the correct new adiabatic relation. Maybe this would be fun to test sometime.

For entropy, $\Delta S = 0$, as usual when adiabatic (and reversible).

C → D

Isothermal again, so

$$W = -P(V_2 - V_1) = -P(0 - V_{3)} = +\frac{bT_C^4 V_3}{3}$$

$$Q = -\frac{bT_C^4 V_3}{3} \to S = -\frac{bT_C^3 V_3}{3}$$

D → A

The volume is constant, so $W = 0$.

$\Delta U = Q = bV_2 T_2^4 - bV_1 T_1^4 = 0$ because V2 and V1 are zero. So I guess $W = Q = \Delta S = 0$.

This process is reversible so the total entropy must be 0. $\Delta S = \frac{bT_H^3 V_2}{3} - \frac{bT_C^3 V_3}{3}$. Does $T_H^3 V_2 = T_C^3 V_3$?

What about total energy... This should also = 0.

$$\Delta U = 0 + b(V_3 T_C^4 - V_2 T_H^4) + 0 + 0 \to V_3 T_C^4 = V_2 T_H^4$$

Are both these relations true? Shouldn't be… Hm….

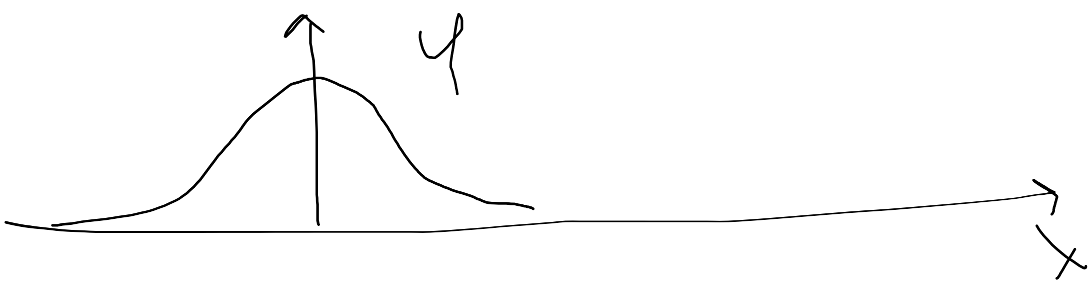
$$V_2 = \frac{T_C^3 V_3}{T_H^3} = \frac{T_C^4 V_3}{T_H^4} \;??$$

5.

A. A quantum particle in 1D is prepared in an state localized at $x = 0$, with an initial position distribution given by a Gaussian of RMS width σ. The potential is $V(x) = 0$ everywhere. Estimate the probability density of such a particle at a later time t as a function of x.

B. The particle is then prepared in an equal amplitude superposition of two spatially strongly localized states with widths $\sigma \ll d$ peaked at $-d/2$ and $+d/2$ at time $t = 0$. Estimate the probability density of such a particle at a later time t, and comment explicitly on any interference effects that are present.

A. The particle begins in a distribution given by a Gaussian:



So, we can say that

$$\psi(x,0) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

We need to find how this evolves with time. We basically have a free particle here, which, utilizing Fourier transforms, has the following relations:

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,0) e^{-ikx}\, dx$$

And

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{hk^2 t}{2m})}\, dk$$

Let's start with phi.

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} e^{-ikx}\, dx = \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-ikx}\, dx$$

Thanks to my handy formula sheet, I know that

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}\,dx = \sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}+c}$$

Where here, $a = \frac{1}{2\sigma^2}, b = -ik, c = 0$

$$\phi(k) = \frac{1}{2\pi\sigma}\sqrt{2\pi\sigma^2}\,e^{\frac{\sigma^2 k^2}{2}} = \frac{e^{\frac{\sigma^2 k^2}{2}}}{\sqrt{2\pi}}$$

Now time evolution:

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{\frac{\sigma^2 k^2}{2}}}{\sqrt{2\pi}}\,e^{i\left(kx-\frac{hk^2 t}{2m}\right)}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{\sigma^2 k^2}{2}}\,e^{i\left(kx-\frac{hk^2 t}{2m}\right)}dk =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2\left(-\frac{\sigma^2}{2}+\frac{iht}{2m}\right)+ikx}dk$$

Now the same thing, with $= -\frac{\sigma^2}{2}+\frac{iht}{2m}, b = +ik, c = 0$

$$\psi(x,t) = \frac{1}{2\pi}\sqrt{\frac{\pi}{-\frac{\sigma^2}{2}+\frac{iht}{2m}}}\,e^{-\frac{k^2}{4\left(-\frac{\sigma^2}{2}+\frac{iht}{2m}\right)}} = \frac{1}{2\pi}\sqrt{\frac{\pi}{-\frac{\sigma^2}{2}+\frac{iht}{2m}}}\,e^{-\frac{mk^2}{2(-\sigma^2+iht)}}$$

This is weird but... ok. The probability density is this squared.

$$\boxed{|\psi(x,t)|^2 = \frac{1}{4\pi^2}\sqrt{\frac{\pi}{-\frac{\sigma^2}{2}-\frac{iht}{2m}}}\sqrt{\frac{\pi}{-\frac{\sigma^2}{2}+\frac{iht}{2m}}}\,e^{-\frac{mk^2}{2(-\sigma^2-iht)}}e^{-\frac{mk^2}{2(-\sigma^2+iht)}} = \frac{1}{4\pi}\sqrt{\frac{4m^2}{m^2\sigma^4+h^2t^2}}\,e^{\frac{mk^2\sigma^2}{2(\sigma^4+h^2t^2)}}}$$

A math error is likely.

B. Hmm…. Starts in super position like this?

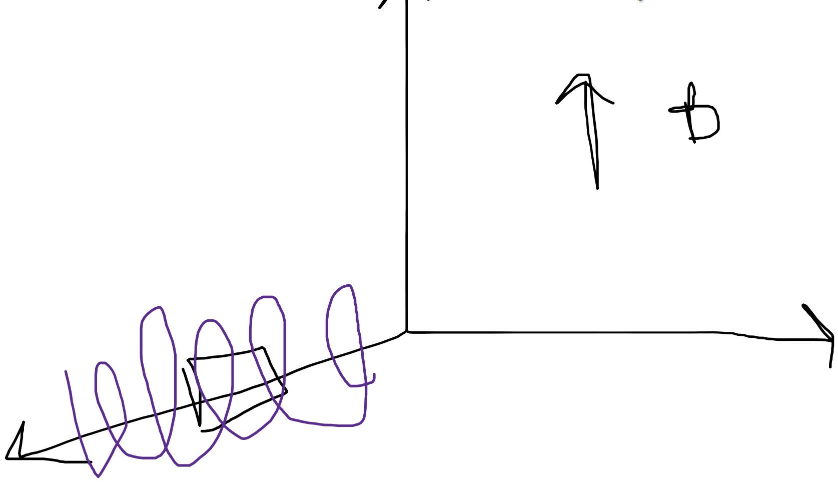
$$\boxed{\psi(x,0) = \frac{A}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{\left(x-\frac{d}{2}\right)^2}{2\sigma^2}} + \frac{B}{\sqrt{2\pi\sigma^2}}\,e^{-\frac{\left(x+\frac{d}{2}\right)^2}{2\sigma^2}}}$$

Do I just time evolve the same as before? Although now the math is likely trickier because the mean isn't zero. Come back.

6. A tall, thin, vertical chimney of length h begins to topple over, as a rigid unit, by pivoting from its base. When it has tipped by some angle θ_{break}, it breaks, due to the internal torque exceeding the strength of the material which is constant along its length. Determine the position x along the chimney, as measured from its base, where it will break.

8. In an NMR apparatus, a magnetic field of 10 T along the z-axis is used to (partially) polarize the hydrogen nuclear spins in a sample of room temperature water that is 1×1×1 cm^3. The spins are quickly rotated into the x-y plane using an AC field along the y-axis, and then the AC field is turned off and the nuclear spins precess around the primary 10 T field. A 1000 turn coil with radius 2 cm, with its axis also pointing along the y axis, surrounds the water sample. Estimate the amplitude of the AC voltage induced in the coil due to the precessing spins. Note that the proton's magnetic moment is 1.4×10^{-26} J/T, and that the magnetic field of a dipole is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{3\vec{r}(\vec{m}\cdot\vec{r})}{r^5} - \frac{\vec{m}}{r^3}\right)$$



My drawing is sad, but whatever. Basically, what we have here is our constant B field of 10 T in the z direction. Then we have a temporary electric field in the y-direction, which causes the spins to rotate into the x-y plane. Once the electric field is off, the spins are free to precess around the z axis. This precession induces a magnetic dipole field, and a changing magnetic field (well, really a changing flux) induces a voltage (emf):

$$V = -\frac{d\phi}{dt}; \phi = \int B \cdot dA$$

So, all we need is B really, which we're given, and then we can find the voltage induced by the changing dipole flux.

$$B_{dipole} = \frac{\mu_0}{4\pi}(\frac{3r(m \cdot r)}{r^5} - \frac{m}{r^3})$$

Each of the proton spins induces this B-field, which creates a flux through the solenoid. If we can find the flux through the solenoid, we can find the induced voltage.

The magnetic dipole lines up with the 10 T B field (z), and, if we're inside our solenoid, the radial direction points in the z-direction as well. So, the dot product $m \cdot r = mr$.

$$B_{dipole} = \frac{\mu_0}{4\pi}\left(\frac{3mr^2}{r^5} - \frac{m}{r^3}\right) = \frac{2m\mu_0}{4\pi r^3}$$

The area in our flux integral is the area of the solenoid. Note there is a dot product in here too!

$$\phi = \int B \cdot dA = \int BdAcos\theta = \int \frac{2m\mu_0}{4\pi r^3} 2\pi lcos\theta dr$$

What about length? I'm not 100% sure about this. I think we can maybe use N.
Normally, N is given as the number of turns per length. But, if we multiply that by the length, we'd just get the total number of turns, which we're given here as 1000. So maybe that is like $Nl = 1000$?

$$\phi = \int \frac{2m\mu_0}{4\pi r^3} cos\theta * 2\pi Nrdr = \int \frac{Nm\mu_0}{r^2} cos\theta dr = -N\frac{m\mu_0}{r} cos\theta$$

We need to get a t in here, since emf is the change of flux with time. Can we make our usual replacement, $\theta = \omega t$?

$$\frac{d\phi}{dt} = \frac{d}{dt}\left(-N\frac{m\mu_0}{r} cos\omega t\right) = +\frac{Nm\mu_0\omega}{r} sin\omega t = V$$

What is omega though. I believe we can say omega is the Larmor frequency, $\omega = -\gamma B = \frac{geB}{2m_p}$?
This is the rate the spins are precessing, so it would make sense that this is the rate that the flux is changing. I think. Or is it? The Larmor frequency is the frequency that the spins precess around the external magnetic field. So, once the AC field is turned off, they will start precessing at this frequency about z. I can buy this for being the omega of interest (I'm slightly confused for some reason).

What about m, the magnetic moment? We're given the magnetic moment for one proton, but we have a container of water in which only the hydrogen is relevant. For one water molecule:

$$m_{water} = 18amu * 1.66e - 27kg = 3e - 26kg$$
$$m_{hydrogen} = \frac{2 * 3e - 26kg}{18} = 3.3e - 27\ kg$$

Because for each water molecule, there are 2 hydrogen atoms and 1 oxygen (2/18 protons are H).

In a 1cm/1cm/1cm cube of water:

$$1l = 1kg; 1cm^3 = 0.001L \rightarrow 1cm^3 = 0.001kg$$

So, there are $\frac{0.001}{3e-26kg} = 3.33e22$ water molecules. For each one there are 2 protons from hydrogen, so the total number of hydrogen protons are $3.33e22 * 2 = 6.67e22$. The total magnetic moment Is $6.67e22 * 1.4e - 26\frac{J}{T} = 9.3e - 4\, J/T$.

$$\omega = -\gamma B = \frac{geB}{2m_p} = \frac{1.6e - 19 * 10T}{2 * 1.66e - 27kg} = 4.8e8$$

I just assumed g is 1, I don't care to Google.

Now, to find V. Let's take it at the max, so $sin\omega t = 1$. I'm only doing this to get rid of it.
I'm taking r as the radius of the solenoid.

$$V = \frac{(1000)(9.3e - 4\frac{J}{T})(1.26 \times 10^{-6})(4.8e8)}{(0.02m)} = 28.1V!$$

Seems reasonable! Actually cool. I should maybe check the units but w/e.

9. The wavefunction of the ground state of the hydrogen atom is

$$\Psi(r, \theta, \phi) = \frac{1}{\sqrt{4\pi}} \frac{2}{a_0^{3/2}} e^{-r/a_0}$$

where a_0 is the Bohr radius. This wavefunction was calculated in the approximation that the proton is a point particle. Suppose instead that the proton were a sphere with uniform charge density and radius $R \ll a_0$. How much would the binding energy of the ground state shift compared to the point particle case? You may leave your answer in the form of an integral.

We want to determine how the ground state binding energy will shift, which is a hint to us that this is a perturbation theory question. So, we need to find the new potential energy, now that the proton has a finite size. We can find the new potential, and then just multiply by q to get the potential energy (recall $U = qV$)

Model the proton as an insulator with radius R (there are obviously no electrons moving around inside it, so it is not a conductor). I'll use Gauss's law to find the electric field, then $V = -\int Edl$ to get the potential. Outside the sphere, E-field and therefore potential just looks like a point charge: $V = \frac{kq}{r}$. Also recall charge density $\rho = \frac{q}{V}$.

Inside the sphere:

$$\int E \cdot dA = \frac{q_{enc}}{\epsilon_0} = \frac{\rho V_{enc}}{\epsilon_0} = E4\pi r^2 = \frac{\rho \frac{4\pi r^3}{3}}{\epsilon_0} \rightarrow E = \frac{\rho r}{3\epsilon_0} = \frac{qr}{3(\frac{4}{3}\pi R^3)\ \epsilon_0}$$

$$E = \frac{qr}{4\pi\epsilon_0 R^3}$$

$$E(r) = \begin{cases} \frac{qr}{4\pi\epsilon_0 R^3}, & r < R \\ \frac{q}{4\pi\epsilon_0 r^2}\ , & r > R \end{cases}$$

Now integrate to find potential. Don't forget we need to break it up into two integrals!

$$V = -\int_r^{\infty} E \cdot dr = -\int_R^{\infty} \frac{q}{4\pi\epsilon_0 r^2} dr - \int_r^R \frac{qr}{4\pi\epsilon_0 R^3} dr$$

$$V = \frac{q}{4\pi\epsilon_0 r}\Big|_R^{\infty} - \frac{qr^2}{8\pi\epsilon_0 R^3}\Big|_r^R = -\frac{q}{4\pi\epsilon_0 R} - \left(\frac{q}{8\pi\epsilon_0 R} - \frac{qr^2}{8\pi\epsilon_0 R^3}\right) = -\frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

We still need to include the term from infinity to R, because we always come from infinity to our point with potential.

$$V(r) = \begin{cases} -\frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right), & r < R \\ -\frac{q}{4\pi\epsilon_0 R}\ , & r > R \end{cases}$$

So, our potential energy is

$$U(r) = \begin{cases} -\frac{e^2}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right), & r < R \\ -\frac{e^2}{4\pi\epsilon_0 R}\ , & r > R \end{cases}$$

Now we can use perturbation theory on the ground state of hydrogen. The ground state is

$$\psi_{100}(r,\theta,\phi\ ) = \frac{1}{\sqrt{4\pi a^3}} e^{-\frac{r}{a}}$$

And the shift in the ground state energy is:

$$E_o^1 = \langle \psi_{100}^1 | H' | \psi_{100}^1 \rangle$$

Because $H = H_0 + H', H' = H - H_0 = -\frac{e^2}{4\pi\epsilon_0 R} - \left(-\frac{e^2}{4\pi\epsilon_0 R} - \frac{e^2}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)\right) = \frac{e^2}{8\pi\epsilon_0 R}(3 - \frac{r^2}{R^2})$

I'm a bit confused about this though, because similar problems online include the $-\frac{e^2}{4\pi\epsilon_0 R}$ in the perturbation, but wasn't that already there? So isn't it not a perturbation but the original U. Ignoring that:

$$E_o^1 = \int_0^{2\pi}\int_0^{\pi}\int_{-\infty}^{\infty} \frac{1}{4\pi a^3} e^{-\frac{2r}{a}} \frac{e^2}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) r^2 sin\theta dr d\theta d\phi$$

10. Consider two spatially localized spin 1/2 particles, coupled by an exchange interaction, and immersed in an inhomogeneous magnetic field. They can be described by the Hamiltonian

$$H = -J\left[S_+^1 S_-^2 + S_-^1 S_+^2\right] - h_1 S_z^1 - h_2 S_z^2$$

where the spin operators are normalized by

$$\left[S_i^a, S_j^b\right] = i\delta^{ab}\epsilon_{ijk} S_k^a$$

In this expression S_i^a is the spin operator for spin a in the i^{th} Cartesian direction.

Find all of the eigenvalues of H.

Useful facts:

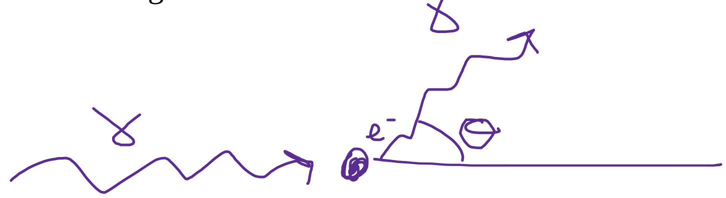
$$S_\pm = S_x \pm i S_y$$

$$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad S_z = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This question is not too bad although I didn't get around to writing it out.

12. Consider a relativistic elastic collision between a photon with initial energy E_i and an electron at rest. As a result of the collision the photon is now moving at an angle θ relative to its initial direction. Calculate the final energy of the photon.

This is Compton scattering:



We can use conservation of four-momentum.

Recall $p^{\mu} = \begin{bmatrix} E \\ p_x \\ p_y \\ p_z \end{bmatrix}$ (I may get some notation wrong, I'm not double-checking specifics)

$$p_i^{\gamma} + p_i^{e} = p_f^{\gamma} + p_f^{e}$$

We're not told anything about the final state of the electron, I presume it doesn't move but to remove any ambiguity, we will just move it to one side of the equation so we can square it and just be left with its mass (four momentum squared is just the -mass squared). Also I'll let c = 1.

$$(p_i^{\gamma} + p_i^{e} - p_f^{\gamma})\hat{}2 \quad = \left(p_f^{e}\right)^2$$

$$p_i^{\gamma 2} + p_i^{e 2} + p_f^{\gamma 2} + 2p_i^{\gamma} p_i^{e} - 2p_i^{e} p_f^{\gamma} - 2p_f^{\gamma} p_i^{\gamma} = -m_e^2$$

Because photons have no mass, the four-momentum squared of a photon is just 0. Because the electron was initially not moving, its initial energy is just $E = m_e c^2$.

$$-m_e^2 + 2\begin{bmatrix} -E_i \\ p_i \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} m_e \\ 0 \\ 0 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -m_e \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} E_f \\ p_f cos\theta \\ p_f sin\theta \\ 0 \end{bmatrix} - 2\begin{bmatrix} -E_f \\ p_f cos\theta \\ p_f sin\theta \\ 0 \end{bmatrix}\begin{bmatrix} E_i \\ p_i \\ 0 \\ 0 \end{bmatrix} = -m_e^2$$

Because photons are massless, the momentums are just equal to the initial and final energy. We can cancel out the electron masses and take the dot products of these vectors.

$$0 = -E_i m_e + E_f m_e - (-E_f E_i + E_f E_i cos\theta) = E_f(m_e + E_i - E_i cos\theta) - E_i m_e$$

$$E_f = \frac{E_i m_e}{m_e + E_i - E_i cos\theta} = \frac{E_i m_e}{m_e + E_i(1 - cos\theta)} = \frac{E_i}{1 + \frac{E_i}{m_e c^2}(1 - cos\theta)}$$

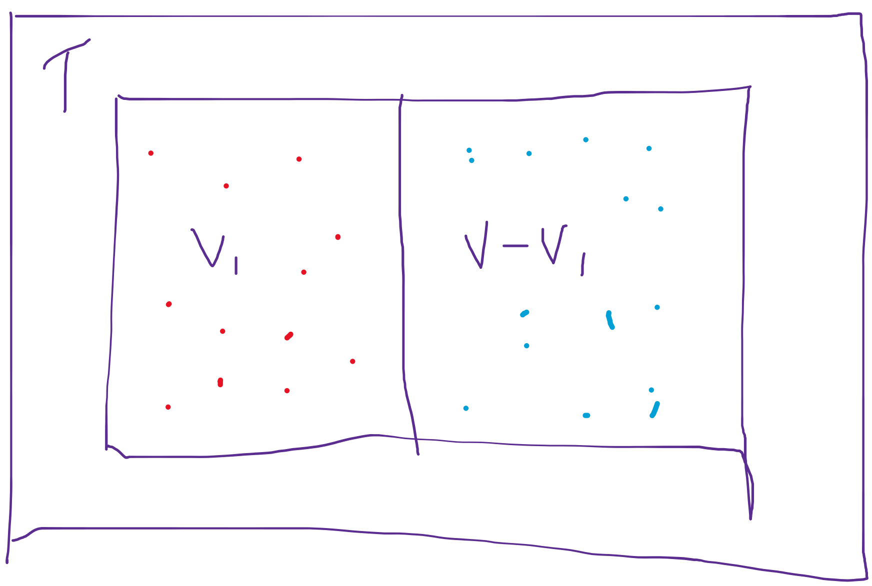
Don't forget, when multiplying four momenta, $p_{\mu} p^{\mu}$, we have to put a negative in front of one of the Es.

13. Consider a cylindrical container of volume V separated into two volumes, v_1 and $v_2 = V - v_1$, by a freely movable partition. The left volume v_1 contains N monatomic molecules of mass m_L and the right volume v_2 contains M monatomic molecules of mass m_R. The partition allows heat to flow between the two volumes. The molecules are dilute enough that you can ignore interactions and the system can be treated classically.

A. Assume that the system is in contact with a heat bath at temperature T, and that the external walls conduct heat. Compute the entropy of the system as a function of v_1. What is the equilibrium value of v_1?

B. Now assume that the external walls but not the partition are completely heat-insulating and that the total energy in the cylinder is E. What is the equilibrium value of v_1 in this case?

C. If the mass of the partition is 0.1 kg and the temperature is 300 K, what is the mean square velocity of the partition in equilibrium?



A. Let's use the SK equation for entropy. The total entropy is equal to the sum of the entropies of each volume.

$$S_{tot} = Nk(\ln \frac{v_i}{N}\left(\frac{4\pi m_l U_1}{3Nh^2}\right)^{\frac{3}{2}} + \frac{5}{2}) + Mk(\ln \frac{V-v_1}{M}\left(\frac{4\pi m_r U_2}{3Mh^2}\right)^{\frac{3}{2}} + \frac{5}{2})$$

Because it's a monatomic gas, $U_1 = \frac{3}{2}NkT$

$$S_{tot} = Nk(\ln\frac{v_i}{N}\left(\frac{2\pi m_l kT}{h^2}\right)^{\frac{3}{2}} + \frac{5}{2}) + Mk(\ln\frac{V-v_1}{M}\left(\frac{2\pi m_r kT}{h^2}\right)^{\frac{3}{2}} + \frac{5}{2})$$

I'm not sure if I should be simplying more.

To find the equilibrium volume, we can take the derivative of the entropy with respect to v1.

$$\boxed{\frac{dS_{tot}}{dv_i} = \frac{Nk}{v_i} + \frac{Mk(-1)}{V - v_i} = 0 \rightarrow V_i = \frac{NV}{N+M}}$$

Test: if M = N, $V_i = \frac{V}{2}$ which checks out.

B. I don't see how the answer would change. However, I'll use this opportunity to do a different method, which is that in equilibrium the pressures should be equal.

Note $E = \frac{3}{2}NkT + \frac{3}{2}MkT \rightarrow T = \frac{2}{3}\frac{E}{k(N+M)}$

$$P_1 = P_2 \rightarrow \frac{NkT}{v_i} = \frac{MkT}{V - v_i} \rightarrow (V - v_i)N = Mv_i \rightarrow v_i = \frac{VN}{N+M}$$

I'm not sure what should change. Is the answer supposed to be different?

C. The mean square velocity is $v = \sqrt{\frac{3kT}{M}}$. However, this is usually for the molecular speed. Can we assume it works for the partition as well? It is the moving molecules that moves the partition after all. So maybe the speed just scales by root m?

$$\boxed{v = \sqrt{\frac{3(1.38e-23)(300K)}{(0.1kg)}} = 3.5 \times 10^{-10} m/s}$$

However, in equilibrium, wouldn't the partition not be moving?

14. When the universe was very young, protons and neutrons could be converted into each other by these reactions:

$$n + e^+ \to \bar{\nu}_e + p$$

$$n + \nu_e \to p + e^-.$$

At times much earlier than one second after the Big Bang, these reactions were fast and maintained the $n:p$ ratio at close to 1 : 1. As the universe cooled, the equilibrium shifted to favour protons due to their lower mass. The rate of these reactions dropped precipitously when the universe reached a temperature of 8×10^9 K, causing both reactions to effectively cease. Over the next few minutes, about 20% of the neutrons decayed ($n \to p + e^- + \bar{\nu}_e$), while the rest were used to form helium-4 atoms. Estimate the ratio of the number of hydrogen nuclei to helium nuclei in the universe at the end of this process.

We know it's initially 1:1, and then the number of protons increase until T = 8x109 K. We need to find the ratio of n : p at this time. We can use the Maxwell Boltzmann distribution, to find the most probable # of particles with energy E at temp T. Let's ignore any kinetic motion, and say the energy of the particles is equal to the rest mass.

$$\frac{n(p)}{n(n)} = \frac{e^{-\frac{m_p c^2}{kT}}}{e^{-\frac{m_n c^2}{kT}}} = e^{-\frac{m_p c^2 - m_n c^2}{kT}} = e^{\frac{(-938.27+939.56)*1.79e-30*(3e8)^2}{1.38e-23\times 8e9}} = e^{1.88} = 6.56$$

So, the ratio of protons to neutrons is 6.56 : 1. For simplicity, let's say 656 : 100.

Now 20% of the neutrons is converted to protons: 656 : 100 → 676 : 80.

There are 80 neutrons, which means 40 He can be created. 80 protons are also taken.

Protons : Helium: 676 : 80 → 596: 40. All the protons make hydrogen, we assume. Therefore, dividing by 40, the ratio is

$$\boxed{15:1}$$

15. The Breakthrough Starshot project proposes to use powerful lasers to accelerate a lightweight space probe to a final velocity of $0.2c$. Suppose the probe has a mass of 10 g and has a 99% reflective solar sail with an area of 16 m^2, and that the laser shines on it for 10 minutes. Calculate the total amount of energy that strikes the sail during the acceleration, assuming that the spacecraft starts at rest.

16. Find an approximate expression for the terminal velocity of a cylindrical magnet, with diameter D and length also D, mass M, and magnetic moment μ falling down the center of a very long vertically-oriented copper tube (resistivity of copper is ρ) with radius R and wall thickness T. Make the following assumptions:

A. the axis of the magnet stays aligned with the (vertical) axis of the tube

B. $D \ll R$ and $T \ll R$

C. the terminal velocity is slow enough that self-inductance of the tube can be ignored, as can air resistance

Note that the magnetic field of a dipole is given by

$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{3\vec{r}(\vec{m}\cdot\vec{r})}{r^5} - \frac{\vec{m}}{r^3}\right)$$

Hint: it's OK to express your answer as a function of M, μ, R, and ρ times a unitless definite integral. You don't need to evaluate the numerical value of that integral.

This one is kind of a mess to be honest. Don't trust it.

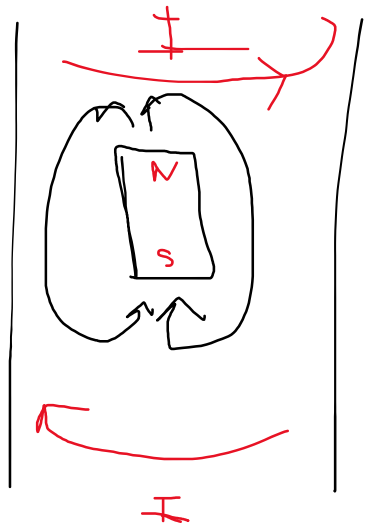
Conceptually, we have a cylindrical magnet following down a copper tube. This magnet is a dipole, and hence creates a magnetic dipole field according to the above equation. As this magnet falls down the tube, the magnetic flux through the tube is changing, which induces an emf/voltage in the copper. So now there is a current in the copper. This current is now creating its own magnetic field which the magnet feels, opposing its motion. The terminal velocity will occur when the acceleration is zero, aka the forces are balanced:

$$F_g + F_{ind} = 0$$

So, assuming we can get the induced force in terms of velocity, we can just isolate v. The flux through a given ring of the copper tube is increasing below the magnet, and decreasing above the magnet. According to Lenz's, the induced emf wants to oppose this.

$$\epsilon = -\frac{d\phi}{dt} = -\frac{d}{dt}\int B \cdot dA$$

Using the right-hand rule (fingers curl with B, and thumb points in direction of current) and then saying induced current is the *opposite* of this gives current in this direction:



Ok now. Let's start with the flux. B is in the z-hat direction. I think dA also is, because we are caring about the flux through the circular cross section. B is not changing with this area.

$$\phi = \int B \cdot dA = \int B(2\pi dl dR)$$
$$B_{dipole} = \frac{\mu_0}{4\pi}\left(\frac{3r(m \cdot r)}{r^5} - \frac{m}{r^3}\right)$$

The magnetic moment, m, is in the z-direction, along the bar magnet (at least we assume so). What about r? Since we're caring about the flux through that circular area, r should be pointing along z? So the dot product is just multiplication.

$$B_{dipole} = \frac{2\mu_0 m}{4\pi z^3}\hat{z}$$

Note I'm bad at these directional things so don't judge me if I'm wrong.

$$\epsilon = -\frac{d\phi}{dt} = -\frac{d}{dt}\left(\int B 2\pi dl dR\right)$$

The only thing changing with time is length, so we can say $\frac{dl}{dt} = v$.

$$\epsilon = -\left(\int 2\pi B v dR\right)$$

$$\epsilon = -2\pi \frac{2\mu_0 m}{4\pi z^3} vR = -\frac{\mu_0 m}{z^3} vR = V$$

Ok so, this is our induced emf in the cylinder. Note it has a resistivity $\rho = \frac{R_e L}{A} = \frac{R_e A}{L} = \frac{2\pi R L R_e}{L} = 2\pi R R_e$, R_e is resistance to avoid confusion with radius.

$$V = IR = \frac{I\rho}{2\pi R} \rightarrow I = \frac{V 2\pi R}{\rho} = \frac{\frac{\mu_0 m}{z^3} vR\, 2\pi R}{\rho} = \frac{\mu_0 m v 2\pi R^2}{\rho z^3}$$

Ok, now let's construct our new magnetic force, which is needed to balance with gravity.

$$F = \int I dl \times B$$

$$F = \int \frac{\mu_0 m v 2\pi R^2}{\rho z^3} \frac{2\mu_0 m}{4\pi z^3} dl = \int \frac{\mu_0^2 m^2 v R^2}{\rho z^6} dl$$

And dl must be going around the cylinder, like the current. However, F, I, and B should all be perpendicular – but shouldn't F be going in the z direction though, so it can balance gravity? I suppose because I am using the dipole field here… If I used the mag field induced in the cylinder, it would be perp.

$$F = \int \frac{\mu_0 m v 2\pi R^2}{\rho z^3} \quad \frac{\mu_0}{2\pi l} dl \text{ ?}$$

I think this is like 80% of the way there. Idk. Once I figure out this integral thing, I just need to set it equal to mg and isolate for v…