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① a) changing B flux induces a voltage (aka an electric field around ring)

$$\oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint \vec{B}\cdot d\vec{A}$$

let Θ indicate the angle of the loop, then $A_x(\Theta) = A_0\cos(\Theta)$ and we know $\Theta(t) = \omega_0 t$, $A_0 = 4\pi a^2$

in general the area vector is $\vec{A} = (A_x, A_y, A_z)^T$, but in this case we only care about the component in the direction of $\vec{B}$ (which I choose to label $\hat{x}$) since we look at the dot product $\vec{B}\cdot d\vec{A}$

the total flux as a function of time is then

$$\Phi(t) = \iint \vec{B}\cdot d\vec{A} = 4\pi a^2 B\cos(\omega_0 t)$$

taking a time derivative:

$$\frac{\partial}{\partial t}\iint \vec{B}\cdot d\vec{A} = -4\pi a^2 B\omega_0 \sin(\omega_0 t)$$

And here we note that $\oint \vec{E}\cdot d\vec{l}$ around the loop is just the induced voltage, so: $\underline{V = 4\pi a^2 \omega B\sin(\omega_0 t)}$

With $P = IV$ and $V = IR$ we have the power dissipated:

$$P = \frac{V^2}{R} = \frac{16\pi^2 a^4 \omega_0^2 B^2 \sin^2(\omega_0 t)}{R}$$

— THIS IS ENERGY LOST TO JOULE HEATING PER SECOND —

thus since we assume all kinetic energy dissipation is due to joule heating:

$$\boxed{\frac{dK}{dt} = -\frac{16\pi^2 a^4 \omega_0^2 B^2 \sin^2(\omega_0 t)}{R}} \quad 1a)$$

(at least in the beginning when $\omega \approx \omega_0$)

1b) In this case kinetic energy is due to rotational motion

$$K = \frac{1}{2} I \omega^2$$

taking a time derivative: $\frac{dK}{dt} = I\omega\dot{\omega}$

we want to set this equal to our previous expression to find $\omega(t)$

However, since ΔK for a cycle is small, we can assume two things:

- the rate of change of K is approximately constant, and I will use $\langle \sin^2 \rangle$ to average it
- ω is basically constant throughout a cycle, so the above approx is OK and our earlier expression for dK/dt is ok too

this gives: $\left\langle \frac{dK}{dt} \right\rangle = \frac{-8\pi^2 \omega^2 B^2 a^4}{R}$

We can now solve the diff. eqn. for $\omega(t)$:

$$I \cancel{\omega} \dot{\omega} = -\frac{8\pi^2 B^2 a^4}{R} \omega^{\cancel{2}}$$

$$\dot{\omega} = -\frac{8\pi^2 B^2 a^4}{IR} \omega$$

$$\boxed{\omega(t) \approx \omega_0 \exp\left(\frac{-8\pi^2 B^2 a^4}{IR} t\right)}$$

I have already applied boundary cond s.t. $\omega(0) = \omega_0$

ω is 1/e of its original value at $\boxed{t_0 = \frac{IR}{8\pi^2 a^4 B^2}}$ 1b)

② In general, I would like to think of the potentials. If $\vec{\nabla} U = 0$ and $\vec{\nabla} \cdot \vec{\nabla} U > 0$, then there is a stable equilibrium. So (in theory) we should be able to find a contradiction between those statements if φ_e (electric potential) is not time-dependent (and thus so is $\vec{E}$).

(turns out this wasn't needed)

The wave equation for φ_e is $\nabla^2 \varphi_e - \overset{0}{\cancel{\frac{\partial^2 \varphi_e}{\partial t^2}}} = -\frac{\rho}{\varepsilon_0} \Rightarrow \vec{\nabla} \cdot (\vec{\nabla} \varphi_e) = -\frac{\rho}{\varepsilon_0}$

and since grav. fields have similar form we know $\vec{\nabla} \cdot (\vec{\nabla} \varphi_g) = -4\pi G \mu$ where μ represents mass density (also assuming static grav. field).

For a stable minimum, then, $\underline{-4\pi G \mu m - \frac{\rho q}{\varepsilon_0} > 0}$ (bcuz $U = m\varphi_g + q\varphi_e$)

At this point, though, we don't even know what a minimum looks like, so we need to solve $\vec{\nabla} U = 0$.

Given some particle of charge q and mass m, the potential is

$$U = m\varphi_g + q\varphi_e$$

At this point I assume that φ_g is from a spherical body $\varphi_g = -\frac{GM}{r}$.

Then $\vec{\nabla} U = -\frac{GMm}{r^2}\hat{r} + q\vec{E}$ and thus, for a minimum,

$$\vec{E} = \frac{m}{q}\frac{GM}{r^2}\hat{r} \Rightarrow \underline{\varphi_e = \frac{m}{q}\frac{GM}{r} = -\frac{m}{q}\varphi_g}$$

Now try to see if this is a stable minimum

$$\vec{\nabla} \cdot (\vec{\nabla} U) = \vec{\nabla} \cdot \left[\vec{\nabla}\left(m\varphi_g - q\frac{m}{q}\varphi_g\right)\right]$$

$$= \vec{\nabla} \cdot \left[\vec{\nabla}\left(m\varphi_g - m\varphi_g\right)\right] \Rightarrow \underline{\nabla^2 U = 0}$$

This is not > 0 and thus is <u>not a stable minimum!</u>

Thus assuming static electric fields (and thus static electric potential) and a standard radial gravitational field (i.e. point mass) acting on a point mass/charge, there is no stable minimum in the total potential, and thus no stable equilibrium.

It is, however, possible to have a meta-stable equilibrium w/ $\vec{\nabla} U = 0$, $\nabla^2 U = 0$

4) In this case the velocity distribution of particles is maxwellian:

$$f(v) = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right)$$

Given pressure P and temperature T, the number density is known:

$$n = \frac{N}{V} = \frac{P}{k_B T}$$

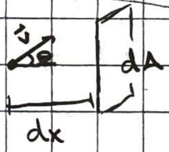
Since we are considering a classical ideal gas.

Consider a box with face area dA and width dx.
There are $dN = n\,dA\,dx$ particles inside this box, with their velocities as given above.

- 1-dimensional constant velocity: particle enters box, takes time $dt = 2dx/v$ to hit plate and exit $\Rightarrow \frac{dN}{dt} = \frac{n\,dA\,dx}{2dx/v} = \frac{nv}{2}dA$
- 3-dimensional constant velocity: really only need to consider x-component $v_x = v\cos\theta$

all angles equally likely so $f(\theta)\,d\theta = d\theta/2\pi$

note $\theta \in [-\pi, \pi]$ is all we care abt



$$\Rightarrow \frac{dN/dt}{d\theta\,dA} = \frac{nv\cos\theta}{4\pi}$$

- 1-dimensional non-const velocity: $v \to f(v)\,dv$

Combining all of this gives: $\frac{dN/dt}{d\theta\,dA\,dv} = \frac{nv\cos\theta f(v)}{4\pi}$

We can integrate:

$$\frac{dN/dt}{dA} = \frac{n}{4\pi}\int_0^\infty dv \int_{-\pi}^{\pi} d\theta\, 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2k_B T}\right) v\cos\theta$$

$$= \frac{P}{k_B T}\left[\frac{m}{2\pi k_B T}\right]^{3/2} \int_{-\pi}^{\pi}\cos\theta\, d\theta \int_0^\infty v^3 \exp\left(-\frac{mv^2}{2k_B T}\right) dv$$

$$= \frac{2P}{k_B T}\left[\frac{m}{2\pi k_B T}\right]^{3/2} \int_0^\infty v^3 \exp\left(-\frac{mv^2}{2k_B T}\right) dv$$

at the moment I don't remember the formula for this integral so I will leave it as is. Note, though, that all we need to do after is integrate over dA (giving A) so:

$$\boxed{\frac{dN}{dt} = \frac{2PA}{k_B T}\left[\frac{m}{2\pi k_B T}\right]^{3/2} \int_0^\infty v^3 \exp\left(\frac{-mv^2}{2k_B T}\right) dv} \quad 4)$$