Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 15, 2016
3:10PM to 5:10PM
General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

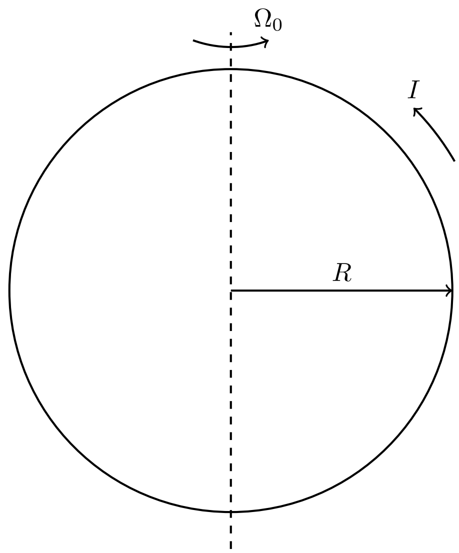
You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Electric current I is circulating along an ideally conducting thin hoop of mass M and radius R. The hoop is placed in vacuum and infinite volume and constrained to rotate on an axis that passes through the conductor and center of the hoop as shown in the figure. Initially the hoop rotates about this axis with angular velocity $\vec{\Omega}_0$. How will its angular velocity evolve with time? [Partial credit will be given for a solution based on dimensional analysis.]



2. In high energy nuclear collisions between nucleus A and B, it is conventional to specify the center-of-mass collision energy in terms of $\sqrt{S_{NN}}$, which is the energy of the collision between one nucleon from A and one nucleon from B, assuming that both nucleons are motionless in their parent nucleus's rest frame. (Here nucleon denotes either a proton or a neutron.)

   In reality the nucleons are not motionless when viewed from the rest frame of the nucleus, due to their Fermi momentum. The density of nucleons in a large nucleus is about $0.16\ fm^3$, where $1 fm = 10^{-15} m$.

   (a) Find the Fermi momentum p_F for a nucleus with the above density, assuming zero temperature and an equal density of protons and neutrons.
   (b) At RHIC, the nominal value of $\sqrt{S_{NN}}$ is $200\ GeV$ when colliding beams of nuclei having equal energies but opposite directions. Find the range of energies about this central value due to Fermi momenta within each nucleus aligned and anti-aligned with the collision direction.
   (c) Repeat part (b) for the LHC, where $\sqrt{S_{NN}} = 5000\ GeV$.

   For this problem, you can take the rest energy of a nucleon to be $M_N c^2 \approx 1000\ GeV$. You may find it convenient to use $\hbar c \approx 0.2\ GeV \cdot fm$.

3.

(a) An experiment requires a flat electrode surface to stay free of adsorbed molecules for a duration τ (the maximum allowed adsorption coverage is $f < 10\%$). Assuming that each incident molecule sticks to the surface, estimate the maximum allowed background air pressure P in terms of τ, f, the temperature T, and the typical mass M and diatemeter d of an adsorbed molecule.

(b) Estimate the order-of-magnitude numerical value of P from part (a) at room temperature, for $\tau \sim 1$ hour.

4.

(a) In solid metals the effective mass of a conduction electron can be different from that of a bare mass electron. Explain the concept of effective mass, and describe why the effective and bare electron masses may be different.

(b) Describe an experiment to determine the effective mass of an electron in metals. It can either be a direct experiment or a combination of a few measurements of other properties which allows a derivation of the effective electron mass

(c) Suppose you had access to a beam of neutrons at a neutron scattering facility, and all its relevant equipment. Explain how you would go about measuring the mass of the neutron.

5. We have an ideal gas composed of N He atoms contained in a vessel of volume V. The vessel is a cube of volume $V = L^3$, where L is the length of the cube. Consider the limit of low temperature ($T \to 0$) and assume that the system is an ideal gas at all temperatures.

   (a) Consider cyclic boundary conditions for the wavefunctions of momentum and energy to obtain the energy states E. [In cyclic boundary conditions the wave functions can be viewed as defined in an infinite volume, but are required to be unchanged by translation through a distance L in the x or y directions]. Calculate the density of states as a function of energy.

   (b) The He^3 isotope has two protons and one neutron in its nucleus. He^3 atoms have spin one-half.

   (i) What is the value of the chemical potential μ at $T = 0$?

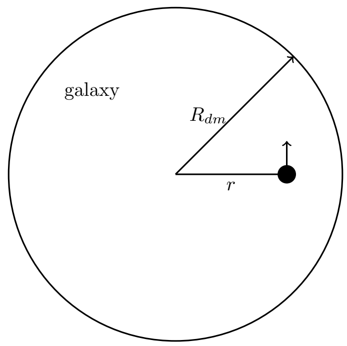
   (ii) Assume that the temperature is raised slightly so that T remains small. The total energy of the He^3 ideal gas is written as $U(T) = U(0) + F(T)$. Use phenomenology (qualitative) considerations to show that the leading term in $F(T)$ is proportional to T^2.

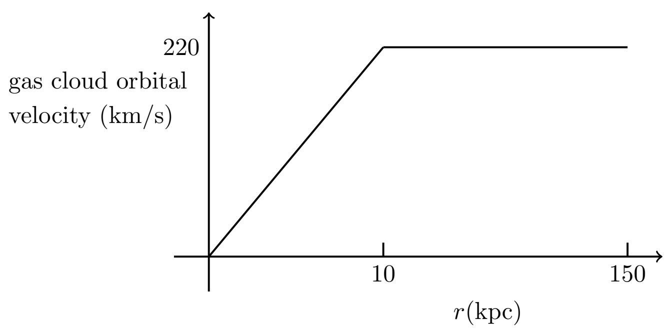
   (c) The He^4 isotope has two protons and two neutrons in its nucleus. He^4 atoms have zero spin.

   (iii) Show that the chemical potential μ is negative.

   (iv) Obtain an expression for the temperature at which the value of the chemical potential comes very close to zero ($\mu \to 0$).

6. Assume a toy model for a spherical galaxy in which the mass of the dark matter is much larger than that of the visible matter $M_{dm} \gg M_{visible}$. Assume the dark matter is spherically symmetrically distributed (with unknown density distribution) in a sphere of radius $R_{dm} = 150\ kpc$ ($1\ kpc = 3 \times 10^{19}\ m$). Gas clouds are observed to orbit inside the galaxy at various radii $r < R_{dm}$. The orbital velocity of these gas clouds is observed to be roughly constant for $r \gtrsim 10\ kpc$ with $v \sim 220\ km/s$, as shown by the solid line in the second figure.





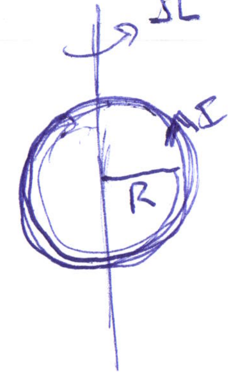
(a) Use the observation of constant gas cloud orbital velocities at $r \gtrsim 10\ kpc$ to deduce the density distribution of dark matter as a function of radius $\rho(r)$.

(b) Estimate the total amount of dark matter in this galaxy. Express your answer in solar masses ($1\ M_{sol} = 2 \times 10^{30}\ kg$).

(c) The visible matter in this galaxy, $M_{visible} \sim 5 \times 10^{11}\ M_{sol}$, is concentrated at $r \lesssim 10\ kpc$. Explain qualitatively how this accounts for the rotation curve behavior at $r \lesssim 10\ kpc$.

(d) It is believed that the dark matter in galaxies cannot be dominantly composed of particles of the Standard Model. Why?

G2-1 Beloborodov



General:

Electric current I is circulating along an ideally conducting thin hoop of mass M and radius R. The hoop is placed in infinite vacuum space and set in rotation with angular velocity $\vec{\Omega}_0$ that is perpendicular to the hoop axis. How will its angular velocity evolve with time?

Solution:

The magnetic dipole moment of the hoop $\mu = \pi R^2 I/c$ rotates about $\vec{\Omega}$ and hence its second time derivative is $|\ddot{\vec{\mu}}| = \Omega^2 \mu$. This generates magneto-dipole radiation with power $\dot{E} = 2\ddot{\vec{\mu}}^2/3c^3$, and hence the hoop kinetic energy $E = MR^2\Omega^2/4$ is gradually decreasing,

$$\frac{d}{dt}\left(\frac{MR^2\Omega^2}{4}\right) = -\frac{2}{3c^3}\left(\frac{\Omega^2 \pi R^2 I}{c}\right)^2 \qquad \Rightarrow \qquad \Omega(t) = \left(\frac{1}{\Omega_0^2} + \frac{8\pi^2 R^2 I^2\, t}{3Mc^5}\right)^{-2}.$$

2 Solution: Fermi Motion in Nuclei

a) The density of states is

$$dN = g\,V\,\frac{d^3p}{(2\pi\hbar)^3} \Rightarrow \frac{N}{V} = \rho_0 = \frac{g}{(2\pi\hbar)^3}\int_0^{p_F} 4\pi p^2 dp = \frac{g}{(2\pi\hbar)^3}\frac{4\pi}{3}p_F^3$$

$$\text{or}\quad p_F = (2\pi\hbar)\left[\frac{3}{4\pi g}\rho_0\right]^{1/3} \quad .$$

Taking $g = 4$ (2 spin states x 2 nucleon 'states'), we find $p_F = 0.267$ GeV.

b) The velocity of a nucleon with p_F is approximately $\beta_F \approx p_F/M_n \approx 0.27$ (the velocity is still small enough that we can use this form to calculate the velocity rather than the correct $\beta_F = p_F/\sqrt{M_N^2 + p_F^2}$). Velocities don't add, but rapidities do, so it is convenient to work with the rapidities of the two colliding nuclei:

$\sqrt{s_{NN}} = 200$ GeV results from a 100 GeV + 100 GeV nucleon-nucleon collision. Using $E = m\cosh y$, we find that the rapidity of each beam is

$$y_B = \pm\cosh^{-1}(100) \approx 5.3 \quad .$$

Taking $0.27 \approx 0.3$, the spread in center-of-mass energies is then

$$\begin{array}{ccc} 2m_N\cosh(5.3-0.3) & \text{to} & 2m_N\cosh(5.3+0.3) \\ 148\text{ GeV} & \text{to} & 270\text{ GeV} \quad . \end{array}$$

c) Calculation is the same as part b), but now with

$$y_B = \pm\cosh^{-1}(2500) \approx 8.5 \quad \Rightarrow$$

the spread in center-of-mass energies is now

$$\begin{array}{ccc} 2m_N\cosh(8.5-0.3) & \text{to} & 2m_N\cosh(8.5+0.3) \\ 3640\text{ GeV} & \text{to} & 6630\text{ GeV} \quad . \end{array}$$

GENERAL PHYSICS

Clean surface. SOLUTION.

a) If the typical air molecule velocity is v, and the air density in the vacuum chamber is n, then

$$nvd^2\tau < f. \tag{1}$$

Furthermore, $n \approx P/(k_B T)$ and $v \approx \sqrt{k_B T/M}$. Therefore,

$$P < \frac{f\sqrt{k_B T M}}{d^2\tau}. \tag{2}$$

b) Using $T \sim 300$ K, $M \sim 30 \times 10^{-27}$ kg, and $d \sim 3 \times 10^{-10}$ m, we find

$$P < 10^{-8} \text{ Pa}. \tag{3}$$

Section 6-4
Uemura

Subject: Example answer for my problem in the general section
From: "Yasutomo J. Uemura" <tomo@lorentz.phys.columbia.edu>
Date: 1/14/2016 7:54 PM
To: Norman Christ <nhc@phys.columbia.edu>, "'Randy Torres'" <rtorres@phys.columbia.edu>

Dear Randy and Norman:

Here is an example answer for my problem.

Sincerely yours,

Tomo Uemura

This problem is related to mass, momentum and energy.

In vacuum, a classical particle has the kinetic energy of
E = (1/2) (mv)^{2}/m = (1/2) p^{2}/ m with p = mv being
momentum of the particle.

The second derivative of E with respect to p d^{2} E / d(p)^{2} = 1 / m
gives the inverse of the mass which is a measure
of how much energy you can gain for a given momentum if the
disperson in quadratic.

In solid state physics, the dispersion of energy versus momentum for
conduction electrons in metals deviates from the quadratic relationship
because of the formation of energy band. The band dispersion of
an electron in solids is usually given as E(k) with (hbar)k being the momenum
p, and k denoting the wavenumber. So, the inverse of the effective mass m*
of an electron in the solid can be obtained by
[d^{2} E(k) / d(k)^{2}] * 1/(hbar)^{2} = 1/m*.

There are various ways to measure the effective mass m*. Most of the
observables of metals are given as combinations of the effective mass m*
and the carrier density n. So, if the system is composed of single
spieces of metallic carriers, one can obtain n by the measurements of
the Hall effect, and then derive m* from the Pauli paramagnetic
susceptibility or the electronic specific heat. Pauli susceptibility
and the specific heat become very large for "heavy mass" electrons
because they are proportional to the effective mass m*. Square of
the Plasma frequency obtained in optical conductivity studies is
proportional to (n/m*). So, determination of n and the Plasma frequency
also allows derivation of m*.

Direct method to obtain m* is to measure the cycrotron resonance
frequency. Performing detailed measurements of ARPES (Angle Resolved Photo
Emission Spectroscopy) will also give dispersion relation of the
conduction electron, from which one can derive effective mass by the second
derivative calculation shown above.

How to determine neutron mass: By performing Bragg diffraction
from a material with known lattice constant and crystal structure,
one can measure the wave length of a neutron which is inversely
proportional to the wave number k and momentum p.
By the time-of-flight measurement, one can determine the velocity v

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of the same neutron.  Then the mass can be calculated as p/v.

There is an alternative way.  Suppose an inelastic excitation energy
is known from independent measurements of optical, Raman, or
electron diffraction studies.  Then perform inelastic neutron
scattering studies of this excitation which will allow determination of
the initial and final momentum.  The energy difference Delta E can be
given as (1/2)[(hbar k_{f})^{2} - (hbar k_{i})^{2}](1/m).  Since k_{i}
and k_{f} can be determined by the Bragg diffraction at the monochrometer
and analyzer of neutron scattering, one can derive the neutron mass m.
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General Particle Statistics/thermal Solution

(a) Under cyclic boundary conditions and no interactions

- momentum is:

$$p = \hbar \left(m \, 2\pi / L \right)$$

where m = positive integer

- Energy is:

$$E = \frac{p^2}{2M} \; ; \quad M \equiv \text{mass}$$

- The density of states is

$$g(E) = (2s+1) \frac{2^{1/2}}{\pi^2} \frac{M^{3/2}}{\hbar^3} E^{1/2}$$

(iv)

Use the density of states in (c) to obtain for $\mu = 0$:

$$n = \frac{N}{V} \int g(E) f(E,T) \, dE =$$

$$= \frac{\sqrt{2}}{\pi^2} \frac{M^{3/2}}{\hbar^3} \int_0^\infty \frac{E^{1/2} \, dE}{e^{E\beta} - 1}$$

By change in variable

$$E = \beta x$$

$$n = \frac{\sqrt{2}}{\pi} \frac{M^{3/2}}{\hbar^3} (k_B T_0)^{3/2} \int_0^\infty \frac{x^{1/2} dx}{e^x - 1}$$

Where T_0 is the temperature for $\mu = 0$

where S is spin.

For ^4He atoms $S = 0$.

(b)

(i) The chemical potential at $T = 0$ is the Fermi energy E_F

From the density of states in (a):

$$E_F = \frac{\hbar^2}{2M}\left(3\pi^2 n\right)^{2/3}$$

where $n = \frac{N}{V}$

(ii) The number of atoms that change energy is

$\Delta n \sim T$

The change in energy of these atoms is:

$$\Delta E \sim k_B T$$

where k_B is Boltzmann's constant.

$$F(T) \sim \Delta n \cdot \Delta E \sim T^2$$

(c)

(iii)

$$f(E,T) = \frac{1}{e^{(E-\mu)\beta} - 1} \; ; \; \beta = \frac{1}{k_B T}$$

for $T \to 0$, $E \to 0$. This requires that:

$$\mu < 0$$

Soln. Hailey General G2-6

a.) $\frac{mv^2}{r} = \frac{GM(<r)m}{r^2} = \frac{Gm}{r^2}\int_0^r 4\pi r^2 \rho(r)\,dr$

$v = \text{const}$ if $\rho = \frac{K}{r^2}$ $\Rightarrow$ $v^2 = 4\pi K G$

$K = v^2/4\pi G$

$\rho(r) = v^2/4\pi G \frac{1}{r^2}$ Ans 2

b.) $M_{DM} \approx \int_0^{R_{dm}} \frac{v^2}{4\pi G}\frac{1}{r^2} 4\pi r^2 dr = \frac{v^2 R_{dm}}{G}$

$\frac{M_{DM}}{M_\odot} = (220\times10^3 \,\text{m/s})^2 \frac{\times 150\times10^3 \times 3\times10^{16}}{6.67\times10^{-11}\; 2\times10^{30}\,\text{kg}} = 1.6\times10^{12} M_\odot$ Ans 2

c.) At $r \lesssim 10$ kpc The visible matter dominates the gravitational potential since $M_{dm}(<r) \propto r$ so $M_{DM} \ll M_{visible}$. This can modify the rotation curve

d.) baryon dark matter would lead to incorrect primordial elemental abundances eg $^4He/H$ ratio would be wrong. neutrinos can only be a sub-dominant component because hot neutrinos (low mass) could not produce the structures we see today. They would wash out galaxy density fluctuations, requiring top-down formation (from large scale objects down to galaxies). But observations clearly show galaxies formed first, then larger structures.

Many other answers...