Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 12, 2018
10:00AM to 12:00PM
General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" × 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider a star of mass M and radius R. Assume that the star has a uniform density and is composed of Hydrogen gas which is 100% ionized and well virialized. Assume the gas obeys the ideal gas law. The star radiates energy and therefore gradually contracts.
   (a) Estimate the star temperature in terms of mass M and radius R. Is the temperature increasing or decreasing with contracting radius R? How do you explain this behavior?
   (b) What is the electron de Broglie wavelength of the star's gas in terms of temperature?
   (c) Define the critical density of the star ρ_0 as the density at which quantum effects change the pressure. Express ρ_0 in terms of the electron de Broglie wavelength. Will the star continue to contract to $\rho > \rho_0$?
   (d) Using the above results, find an expression for the maximum temperature a star can reach as a function of its mass. Estimate this temperature for a star of mass $M = 2 \times 10^{33}$ g (mass of the sun).

   For your numerical estimates you may find useful:

   Gravitational constant $G = 6.67 \times 10^{-8}$ cm^3 g^{-1} s^{-2}
   Planck constant $\hbar = 1.05 \times 10^{-27}$ erg s
   Electron mass $m_e = 0.91 \times 10^{-27}$ g
   Proton mass $m_p = 1.67 \times 10^{-24}$ g
   Classical electron radius $r_e = e^2/m_e c^2 = 2.82 \times 10^{-13}$ cm
   Proton radius $r_p \sim 0.9$ fm$= 0.9 \times 10^{-13}$ cm

2. A system with two nondegenerate energy levels, E_0 and E_1 ($E_1 > E_0 > 0$) is populated by N distinguishable particles at temperature T.

   (a) What is the average energy per particle? Express answer in terms of E_0, E_1 and $\Delta E = E_1 - E_0$.

   (b) What is the average energy per particle as $T \to 0$? Express answer in terms of E_1 and ΔE.

   (c) What is the average energy per particle as $T \to \infty$? Express answer in terms of $E_0 + E_1$ and ΔE.

   (d) What is the specific heat at constant volume, c_V, of this system? Express answer in terms of ΔE.

   (e) Compute c_V in the limits $T \to 0$ and $T \to \infty$ and make a sketch of c_V versus $\Delta E/kT$.

3. This is a problem on the one-dimensional Ising model. Consider a system with $N+1$ spins with the Hamiltonian:

$$H = -J \sum_{i=0}^{N-1} s_{i+1} s_i \tag{1}$$

where each s_i can take on only two possible values: $+1$ or -1. Let us define $G \equiv \beta J$, where β is the usual inverse temperature. Thus, the partition function Z takes the form

$$Z = \sum_{\{s\}} \exp \left[G \sum_{i=0}^{N-1} s_{i+1} s_i \right] \tag{2}$$

where the symbol $\{s\}$ denotes all possible combinations of the values of the spins.

(a) Compute Z. You should be able to express it as an analytic function of G (and N). Hint: it is useful to define a variable $t_i \equiv s_{i+1} s_i$ which also takes the values $+1$ or -1. You might also want to assume first that s_0 is fixed and consider a sum over all possible combinations of $t_0, t_1, ..., t_{N-1}$; then sum over $s_0 = \pm 1$.

(b) What is the free energy F? What is its low temperature ($\beta \to \infty$) limit? Be careful to distinguish between the $J > 0$ (ferromagnetic) and $J < 0$ (anti-ferromagnetic) possibilities.

4. In a system of electrons confined to a two-dimensional plane, the energy of the states of electrons is represented as:

$$E[j, p_x, p_y] = \frac{1}{2m_e}(p_x^2 + p_y^2) + jE_z, \qquad E_z = 0.01 \text{ eV},$$

where jE_z, $j = 0, 1, 2, \ldots$, represents the quantized energy of electron oscillations along the z-direction normal to the (x, y)-plane, and (p_x, p_y) are the components of electron momentum in the (x, y)-plane. The (x, y)-plane has a (large) surface area A.

(a) Evaluate the density of states for each value of the quantum number j.

(b) Assume that the electron density is $n = 10^{12}$ cm^{-2}. Find the chemical potential at temperature $T = 0$ K. Describe qualitatively what happens when the temperature is raised to $T \sim 100\,\text{K}$.

Some numerical values:

1 eV $= 1.6 \times 10^{-12}$ erg $= 1.6 \times 10^{-19}$ Joule, and equivalent to about 10000 K
$m_e = 9.1 \times 10^{-28}$ g
$e = 4.8 \times 10^{-10}$ esu$=1.6 \times 10^{-19}$ C
$\hbar = 1.05 \times 10^{-27}$ erg s $= 1.05 \times 10^{-34}$ Joule $\cdot$ s

5. Earth's atmosphere is composed of a wide range of gases including water vapor. Estimate the ratio of the number of water molecules to the total number of gas particles (atoms and molecules) in a cubic meter of air in typical ambient conditions using the following information:

   - Typical ambient temperature $T_A = 293$ K and relative humidity $\rho = 50\%$.
   - Saturation vapor pressure of water approximately doubles with every 10 K rise in temperature.
   - Boiling temperature T_B of water at typical ambient pressure is 373 K.

6. Consider a system of N independent identical harmonic oscillators whose energy is given by

$$E = \sum_{i=1}^{N} \left(m_i + \tfrac{1}{2}\right)\hbar\omega,$$

where $m_i = 0, 1, 2, \ldots$ are the oscillator excitation quantum numbers. The harmonic oscillators describe the motion of atoms in a solid. This model was introduced by Einstein in 1905 to understand thermodynamic properties (e.g. the heat capacity) of solids.

   (a) Determine the number of ways, $\Omega(E)$, that this energy can be obtained. Hint: $\Omega(E)$ can be understood as the number of ways of putting $M \equiv \sum_{i=1}^{N} m_i$ indistinguishable balls in N labeled boxes.

   (b) Find the entropy S in terms of N and M, assuming N and M are both large.

   (c) Using the relation between entropy S, energy E, and temperature T, find T in terms of N and M.

   (d) Express E in terms of N and T. Which form does E take in the limiting cases $k_{\mathrm{B}}T \ll \hbar\omega$ and $k_{\mathrm{B}}T \gg \hbar\omega$?

A star raises its temperature by gravitational contraction. Assume a star has mass M, radius R, a uniform density, and is composed of Hydrogen gas which is 100% ionized. The gas remains well virialized as the star contracts. Assume the gas obeys the ideal gas law.

a) Estimate the temperature of a star in terms of mass M and radius R.

The star will contract until quantum effects (electron degeneracy pressure) become important, at which point further contraction will not raise its temperature, since the star behaves as an electron gas.

b) What is the electron de Broglie wavelength of the star's gas in terms of temperature?

c) Define the critical density of the star as the density at which quantum effects become important. Express the critical density in terms of the electron de Broglie wavelength.

d) Using the above results, find an expression for the maximum temperature a star can reach as a function of its mass. Estimate this temperature for a star of order one solar mass, $M \sim 2\times10^{33}$ g.

e) Estimate the temperature required for two protons to surmount the Coulomb barrier and initiate fusion. If your answer in d) is much less than that in e), explain qualitatively how it is that the fusion reaction can start?

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Soln. General Hailey

a.) $\frac{3}{2}(N_p+N_e)KT \simeq f\frac{GM^2}{2R}$ ie $2\langle K\rangle+\langle U\rangle=0$
$f=$ numerical correction to pot. Energy

$N_p=N_e \quad M\simeq N_p m_p$

$\Rightarrow KT \simeq GMm_p f/6R$ $\underline{\text{Ans}}$

b.) $\lambda_e = h/p \quad p^2/2m_e = \frac{3}{2}KT \quad \lambda_e \simeq \frac{h}{\sqrt{3m_e KT}}$ $\underline{\text{Ans}}$

c.) critical density for quantum effects is where interparticle e^- spacing $\simeq \lambda_e$ or

$$\lambda_e \simeq (V/N_e)^{1/3} \simeq \left(\frac{V}{N_p}\right)^{1/3} \simeq \left(\frac{m_p V}{m_p N_p}\right)^{1/3} \simeq \left(m_p/\rho_c\right)^{1/3}$$

$\rho_c = m_p/\lambda_e^3$ $\underline{\text{Ans}}$

d.) $KT_{max} = \frac{f\,GMm_p}{6\left(\frac{3M}{4\pi\rho_c}\right)^{1/3}}$. sub in for ρ_c from b, c

$$KT_{max} = f\frac{GMm_p}{6}\left(\frac{4\pi}{3M}\right)^{1/3}\left\{\frac{m_p}{\left(h/\sqrt{3m_e KT_{max}}\right)^3}\right\}^{1/3}$$

$\Rightarrow \quad KT_{max} \simeq \frac{1}{4}\frac{G^2}{h^2}m_p^{8/3}m_e M^{4/3}f^2$ $\underline{\text{Ans}}$

For $M\simeq$ mass sun $KT_{max} \simeq 3\text{ KeV } f^2$

$f=3/5$ gives $\simeq 1\text{ KeV} - 10^7\text{K}$ $\underline{\text{Ans}}$

e. The Coulomb barrier is $\frac{e^2}{r} = e^2/\hbar c\,\frac{\hbar c}{r}$

for $r\simeq 1\text{ fm}$ $U_c \simeq \alpha\,\frac{197\text{MeV-fm}}{1\text{fm}} \simeq 1\text{MeV}$

This is much larger than mean energy of a proton (or even p at the center of sun)

fusion happens by QM tunneling through barrier.

PROBLEM 5-2 SOLUTIONS

(a)

$$u = \frac{E_0 e^{-E_0/kT} + E_1 e^{-E_1/kT}}{e^{-E_0/kT} + e^{-E_1/kT}}$$

$$u = \frac{E_0 + E_1 e^{-\Delta E/kT}}{1 + e^{-\Delta E/kT}}$$

(b)

As $T \to 0, 1/kT \to \infty$

$u = E_0 - \Delta E$

(c)

As $T \to \infty, 1/kT \to 0$

$u = \frac{1}{2}(E_0 + E_1)$

(d)

$E_{total} = NE = N\frac{E_0 + E_1 e^{-\Delta E/kT}}{1 + e^{-\Delta E/kT}}$

$C_V = \frac{\partial E_{total}}{\partial T}$

$C_V = \frac{-1}{kT^2} N \cdot \frac{E_1 e^{-\Delta E/kT}(-\Delta E)E_0 + (E_0 + E_1 e^{-\Delta E/kT})e^{-\Delta E/kT}\Delta E}{(1+e^{-\Delta E/kT})^2}$

$C_V = \frac{-1}{kT^2} N \cdot \frac{-(\Delta E)^2 e^{-\Delta E/kT}}{(1+e^{-\Delta E/kT})^2}$

$C_V = \frac{N \cdot \Delta E^2 e^{-\Delta E/kT}}{kT^2(1+e^{-\Delta E/kT})^2}$

(e)

In the limit $T \to 0$, $kT \to 0$ and $e^{-\Delta E/kT} \to 0$

$C_V = Nke^{-\Delta E/kT}(\Delta E/kT)^2$

In the limit $T \to \infty$, $kT \to \infty$ and $e^{-\Delta E/kT} \to 1$

$C_V = \frac{Nk}{4}(\frac{\Delta E}{kT})^2$

Statistical mechanics problem. This is a problem on the one-dimensional Ising model. Consider a system with $N+1$ spins with the Hamiltonian:

$$H = -J \sum_{i=0}^{N-1} s_{i+1} s_i \tag{1}$$

where each s_i can take on only two possible values: $+1$ or -1. Let us define $G \equiv \beta J$, where β is the usual inverse temperature. Thus, the partition function Z takes the form

$$Z = \sum_{\{s\}} \exp \left[G \sum_{i=0}^{N-1} s_{i+1} s_i \right] \tag{2}$$

where the symbol $\{s\}$ denotes all possible combinations of the values of the spins.

a. Compute Z. You should be able to express it as an analytic function of G (and N). Hint: it is useful to define a variable $t_i \equiv s_{i+1} s_i$ which also takes the values $+1$ or -1. You might also want to assume first that s_0 is fixed and consider a sum over all possible combinations of $t_0, t_1, ..., t_{N-1}$; then sum over $s_0 = \pm 1$.

b. What is the free energy F? What is its low temperature ($\beta \to \infty$) limit? Be careful to distinguish between the $J > 0$ (ferromagnetic) and $J < 0$ (anti-ferromagnetic) possibilities.

c. Compute the two-point correlation $\langle s_j s_k \rangle$ defined by:

$$\langle s_j s_k \rangle = Z^{-1} \sum_{\{s\}} s_j s_k \exp \left[G \sum_{i=0}^{N-1} s_{i+1} s_i \right] \tag{3}$$

where you can assume $j > k$. What is its high temperature limit ($\beta \to 0$ or $G \to 0$)? How about the low temperature limit?

Solution (Lam). This is a standard textbook problem, proposed by Lenz and solved by Ising. Let's rewrite Z:

$$\begin{aligned} Z = \sum_{s_0} \sum_{\{t\}} \exp \left[G \sum_{i=0}^{N-1} t_i \right] &= \sum_{s_0} \sum_{\{t\}} \prod_{i=0}^{N-1} \exp \left[G t_i \right] \\ &= \sum_{s_0} (e^{-G} + e^{G})^N = 2(e^{-G} + e^{G})^N \, . \end{aligned} \tag{4}$$

The free energy is

$$F = -\beta^{-1} \ln Z = -\beta^{-1} \left[\ln 2 + N \ln \left(e^{-G} + e^{G} \right) \right] \, . \tag{5}$$

Note that F scales linearly with N in the large N limit, as expected. For $J > 0$, the low temperature limit corresponds to $G \to \infty$, which gives $F \to -NG/\beta = -NJ$. For $J < 0$, the low temperature limit corresponds to $G \to -\infty$, which gives $F \to NG/\beta = NJ$. These give the expected ground state energy in the ferromagnetic case and antiferromagnetic case respectively (i.e. it is $-N|J|$ in both cases). By the way, the $\ln 2$ is the entropy of the lowest energy level (i.e. there are 2 possible ground states). The two-point correlation is

$$\begin{aligned}
\langle s_j s_k \rangle &= Z^{-1} \sum_{\{s\}} s_j s_k \exp\left[G \sum_{i=0}^{N-1} s_{i+1} s_i \right] \\
&= Z^{-1} \sum_{\{s\}} (s_j s_{j-1})(s_{j-1} s_{j-2}) ... (s_{k+2} s_{k+1})(s_{k+1} s_k) \exp\left[G \sum_{i=0}^{N-1} s_{i+1} s_i \right] \\
&= Z^{-1} \sum_{s_0} \sum_{\{t\}} t_{j-1} t_{j-2} ... t_{k+1} t_k \exp\left[G \sum_{i=0}^{N-1} t_i \right] \\
&= Z^{-1} \sum_{s_0} \sum_{\{t\}} t_{j-1} t_{j-2} ... t_{k+1} t_k \prod_{i=0}^{N-1} \exp\left[G t_i \right] \\
&= Z^{-1} 2 (-e^{-G} + e^{G})^{j-k} (e^{-G} + e^{G})^{N-(j-k)} \\
&= \left(\frac{e^G - e^{-G}}{e^G + e^{-G}} \right)^{j-k} \\
&= (\tanh G)^{j-k} .
\end{aligned} \tag{6}$$

In the high temperature limit $G \to 0$, $\langle s_j s_k \rangle \to 0$ i.e. the spins become uncorrelated as expected. In the low temperature limit, for $J > 0$, we have $G \to \infty$, and so $\langle s_j s_k \rangle \to 1$; for $J < 0$, $G \to -\infty$ instead, in which case $\langle s_j s_k \rangle \to (-1)^{j-k}$. This is consistent with the expectation that, at zero temperature, all spins align in the ferromagnetic case, whereas the spins alternate in the antiferromagnetic case.

Some extra comments: it is useful to define (assuming $G > 0$ for simplicity)

$$e^{-2\lambda} \equiv \tanh \mathrm{G} \tag{7}$$

such that

$$\langle s_j s_k \rangle = e^{-2\lambda (j-k)} , \tag{8}$$

where $1/(2\lambda)$ has the interpretation of a correlation length. Note the curious duality, that the roles of λ and G can be reversed: $e^{-2G} = \tanh \lambda$. The mapping allows one to connect the 1D Ising model to a quantum mechanics problem of a single spin (see the corresponding problem in the quantum mechanics section).

Section 5: General Physics, part I

In a two-dimensional electron system the energy of the states of electrons can be represented as:

$$E[j, p_x, p_y] = (j + ½)\,E_z + (1/2m_o)\,(p_x^2 + p_y^2)$$

Where $(j+ ½)\,E_z$ represents the energy for motion along the z-direction that is normal to the (x,y) plane. p_x and p_y are the components of electron momentum in the (x,y) plane. $E_z = 0.01$eV. The allowed values of the quantum number j are j= 0, 1, 2 … . The allowed values of electron momentum are subject to quantization within a square of area $A=L^2$, where L is the length of the sides of the square.

(a) Evaluate the density of states.

(b) Assume that the ~~areal~~ electron density is $n = 10^{12}$cm^{-2}. Find the chemical potential at the temperature T=0K. Describe qualitatively the changes that occur when the temperature is raised to 80K.

<u>Some numerical values:</u>

1eV = 1.6×10^{-12} ergs = 1.6×10^{-19} Joules, and equivalent to about 10000K degrees
1nm = 10^{-7} cm
$m_o = 9.1 \times 10^{-28}$ g = 9.1×10^{-31} kg
e = 4.8×10^{-10} esu = 1.6×10^{-19} C
$\hbar = 1.05 \times 10^{-27}$ ergs sec = 1.05×10^{-34} Joules sec

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Section 5 Problem # 4

Solution

(a) Assume that electron spin is $S = 1/2$. Then for each value of the quantum number j the DOS is given by

$$\frac{dn}{d\varepsilon} = \frac{m}{\pi \hbar^2}$$

(b) At $T = 0K$ the chemical potential is the Fermi energy. If all the electrons populate the $j = 0$ state the Fermi energy would be:

$$\varepsilon_F = 3.8 \text{ meV}$$

Because this value of ε_F is less than $\varepsilon_z = 10$ meV the state with $j = 0$ is the only are populated by electrons at $T = 0K$.

When the temperature is raised to $80K \simeq 8 meV$ electrons will populate states with $J = 1$.

Earth's atmosphere is composed of a wide range of gases including water vapor. Estimate the ratio of the number of water molecules to the total number of gas particles (atoms and molecules) in a cubic meter of air in typical ambient conditions using the following information:

i. Typical ambient temperature T_A and relative humidity ρ are $293°\mathrm{K}$ and 50%, respectively.
ii. Saturation vapor pressure of water approximately doubles with every $10°\mathrm{K}$ rise in water temperature.
iii. Boiling temperature T_B of water at typical ambient pressure is $373°\mathrm{K}$.

Solution:

At the boiling point, saturation vapor pressure of water becomes equal to atmospheric pressure. Since $T_B - T_A = 80°K$, we can express the ratio of the saturation vapor pressure of water vapor P_w at the ambient temperature to the atmospheric pressure P_A as:

$$\frac{P_w}{P_A} = 2^{-8}.$$

The number of particles in a given volume is proportional to partial pressure of those particles. Therefore the above ratio is equal to the ratio of the number of water molecules to the total number of gas particles in a saturated atmosphere.

Relative humidity is defined as the ratio partial pressure of water to the saturation vapor pressure of water. Therefore we can write:

$$\frac{n_\omega}{n_A} = \rho \frac{P_w}{P_A} = 2^{-9}.$$

The numerical value is $\sim 0.2\%$

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Problem Suggestion 2

Statistical Mechanics

Sebastian Will
Columbia University

(Dated: November 26, 2017)

QUANTUM HARMONIC OSCILLATORS IN AN EINSTEIN SOLID

Consider a system of N independent identical harmonic oscillators whose energy in a microcanonical ensemble is given by

$$E = \frac{1}{2}\hbar\omega N + \hbar\omega M.$$

The harmonic oscillators describe the motion of atoms in a solid. This model was introduced by Einstein in 1905 to understand thermodynamic properties (e.g. the heat capacity) of solids.

(a) Determine the number of ways, $\Omega(E)$, that this energy can be obtained. Note that

$$M = \sum_{i=1}^{N} m_i,$$

where m_i is the occuation number $(0, 1, 2, \dots)$ of a given harmonic oscillator. Hint: $\Omega(E)$ can be understood as the number of ways of putting M indistinguishable balls in N labelled boxes. It is also the number of ways of arranging $N-1$ partitions and M indistinguishable balls along a line. **6 mins**

(b) Find the entropy S in terms of N and M. Use Stirling's approximation $\ln(N!) \approx N \cdot \ln(N) - N$ to simplify the expression for S. **4 mins**

(c) Find the temperature T. **8 mins** Optional information: Note that

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N.$$

(d) Express E in terms of N and T. In order to simplify the expressions, assume that $M + N - 1 \approx M + N$. Which form does E take in the limiting cases $k_{\mathrm{B}}T \ll \hbar\omega$ and $k_{\mathrm{B}}T \gg \hbar\omega$? **6 mins**