Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 8, 2018
2:00PM to 4:00PM
Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. An otherwise free non-relativistic charged particle having mass m and charge e moves in a uniform magnetic field $\vec{B}$ pointing in the $\hat{z}$ direction.

   (a) Assume that at $t = 0$ the particle is located at the origin and moving with velocity $\vec{v}_0$ in the x-direction: $\vec{v}_0 = v_0\hat{x}$. Determine the particle's subsequent position $\vec{r}(t)$ and velocity $\vec{v}(t)$ as a function of time and describe the resulting motion (ignoring radiation damping).

   (b) If the initial velocity $\vec{v}_0$ has both an x- and a z-component, $\vec{v}_0 = v_{0x}\hat{x} + v_{0z}\hat{z}$ find the subsequent postion $\vec{r}(t)$ and describe the resulting motion.

2. A crystal is composed of a collection of identical atoms. The atoms are modeled as point particles with charge q and mass m coupled with spring constant k to fixed atomic sites of opposite charge $-q$. The number density of atoms in this crystal is N. Damping effects are accounted for by the damping constant Γ. A plane polarized electromagnetic wave of frequency ω propagates in the solid. Use the following notation: $\omega_0^2 = k/m$.

   (a) Derive an expression for the complex dielectric function $\epsilon(\omega)$.

   (b) Sketch the real and imaginary parts of $\epsilon(\omega)$.

3. You have a very long ideal solenoid with radius R, N turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical shells of length l. One is inside the solenoid at radius $a < R$ and carries a charge $+Q$ uniformly distributed over its surface. The other is outside the solenoid at radius $b > R$ and carries charge $-Q$ uniformly distributed over its surface. Note that $l >> R$ and ignore fringe fields.

   (a) What is the angular momentum of this system?

   (b) As the current in the solenoid is gradually reduced to zero, the cylinders begin to rotate. Calculate the final angular momentum of each and show that their sum is equal to the initial angular momentum of the system.

4. Consider in electrostatics an infinitely extended uniform charge density, $\rho(\vec{x}) = \rho_0$.

   (a) Prove that, despite $\rho(\vec{x})$ being invariant under translations and rotations, there is no unique solution to the Poisson equation that is preferred in terms of symmetries.

   (b) Suppose now that the charge distribution is smoothly cut off in a spherical fashion at very large distances, i.e.

$$\rho(\vec{x}) = \rho_0 f(|\vec{x}|/R) ,$$

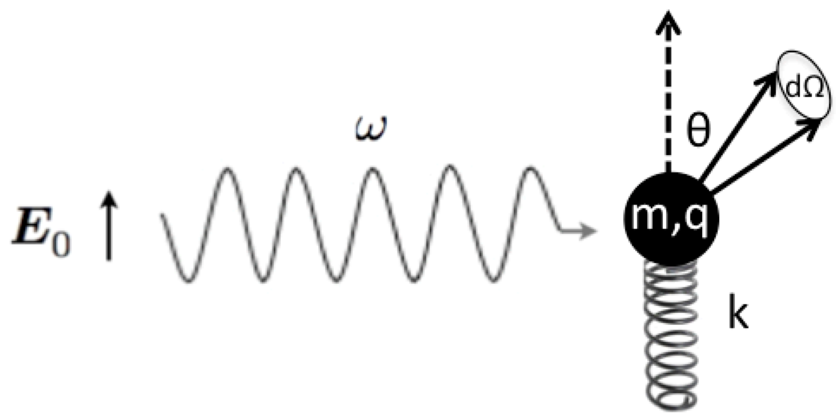
   where $f = 1$ for $|\vec{x}|/R \ll 1$, and $f = 0$ for $|\vec{x}|/R \gg 1$. Argue that this removes the ambiguity in the Poisson problem. Compute the potential for $|\vec{x}|/R \ll 1$ and for $|\vec{x}|/R \gg 1$ (up to additive constants).

5. A linearly polarized electromagnetic wave of amplitude $\vec{E}_0$ and frequency ω is incident on a classical particle of mass m and charge q attached to a spring of spring constant k, as shown in the figure below. Assume that the particle is free to move in three dimensional space, i.e. it is mechanically an isotropic 3-D classical harmonic oscillator.

   (a) Calculate the differential cross section $d\sigma/d\Omega$ for the light to scatter into a solid angle $d\Omega$. Express your results in terms of the angle θ with respect to the polarization direction.

   (b) Calculate the total cross section.

   (c) Consider the above system as a semi-classical model for a valence electron of a nitrogen atom in the Earth's atmosphere, where the "spring" represents the restoring force holding the electron at its equilibrium position in the atom.
   Using your result from part 2, explain why the sunset is red. Make sure to justify any approximations made.



1. An otherwise free non-relativistic charged particle having mass m and charge e moves in a uniform magnetic field $\vec{B}$ pointing in the $\hat{z}$ direction.
   (a) Assume that at $t = 0$ the particle is located at the origin and moving with velocity $\vec{v}_0$ in the x-direction: $\vec{v}_0 = v_0\hat{x}$. Determine the particle's subsequent position $\vec{r}(t)$ and velocity $\vec{v}(t)$ as a function of time and describe the resulting motion (ignoring radiation damping).
   (b) If the initial velocity $\vec{v}_0$ has both an x- and a z-component, $\vec{v}_0 = v_{0x}\hat{x}+v_{0z}\hat{z}$ find the subsequent postion $\vec{r}(t)$ and describe the resulting motion.

Solution:

(a) Starting with the Lorentz force law:

$$m\frac{d\vec{v}}{dt} = \frac{e}{c}\vec{v} \times \vec{B}$$

we conclude that the velocity v rotates with a constant angular velocity $\vec{\omega} = -\frac{e}{mc}\vec{B}$. For the initial conditions in the problem this implies that

$$\vec{v}(t) = v_0\Big(\cos(\omega t), -\sin(\omega t), 0\Big)$$

where $\omega = eB/mc$ is the cyclotron frequency. This equation can be integrated to find:

$$\vec{r}(t) = \frac{v_0}{\omega}\Big(\sin(\omega t), \cos(\omega t) - 1, 0\Big).$$

Thus, the particle moves in a circle of radius $R = v_0/\omega = (v_0 mc)/(eB)$.

(b) Since no forces act in the z-direction the motion will be the same as that found in part (a) except for an additional constant velocity v_{0z} in the z-direction:

$$\vec{r}(t) = \Big(\frac{v_{0x}}{\omega}\sin(\omega t), \frac{v_{0x}}{\omega}(\cos(\omega t)) - 1), v_{0z}t\Big)$$

so the motion is now helical.

~~Problem. Section 2 (d.basov db3056@columbia.edu)~~ for lvg

2. A crystal is composed of a collection of identical atoms with charge q and mass m coupled with the spring constant k to a fixed atomic site. The number density of atoms in this crystal is N. Damping effects are accounted with the damping constant Γ. A plane polarized electromagnetic wave of frequency ω propagates in the solid. Use the following notation: $\omega_0^2 = \frac{k}{m}$.

A. Derive an expression for the complex dielectric function $\varepsilon(\omega)$.
B. Sketch the real and imaginary parts of $\varepsilon(\omega)$.

Solution.
A.

Harmonic osc: $\frac{d^2\vec{r}}{dt^2} + \Gamma \frac{d\vec{r}}{dt} + \omega_0^2 \vec{r} = q\vec{E}$

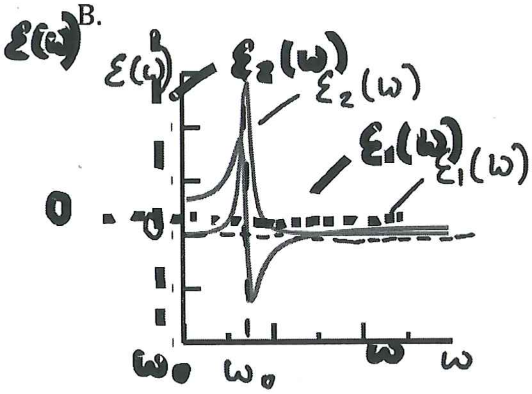
$$\vec{r}(\omega) = \frac{q}{m} \frac{E}{\omega_0^2 - \omega^2 - i\omega\Gamma}$$

dipole moment: $\vec{p}(\omega) = \frac{Nq^2}{m} \frac{E}{\omega_0^2 - \omega^2 - i\omega\Gamma}$

$$\vec{P} = \chi(\omega)\vec{E} \; ; \quad \varepsilon(\omega) = 1 + 4\pi\chi(\omega)$$

$$\rightarrow \varepsilon(\omega) = 1 + \frac{4\pi N e^2}{m} \frac{1}{\omega_0^2 - \omega^2 - i\omega\Gamma}.$$

B.



Section 2- Problem 3
Humensky

~~Electromagnetism Problem~~:

3

You have a very long ideal solenoid with radius R, N turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical shells of length l. One is inside the solenoid at radius $a < R$ and carries a charge $+Q$ uniformly distributed over its surface. The other is outside the solenoid at radius $b > R$ and carries charge $-Q$ uniformly distributed over its surface. Note that $l >> R$ and ignore fringe fields.

A. What is the angular momentum of this system?

B. As the current in the solenoid is gradually reduced to zero, the cylinders begin to rotate. Calculate the final angular momentum of each and show that their sum is equal to the initial angular momentum of the system.

C. Now suppose that instead of decreasing the current, we leave it constant but connect a weakly conducting radial spoke between the cylinders. Calculate the final total angular momentum of the system when the charge on the cylinders has dropped to zero; note that they will rotate as a single rigid body.

Solution:

A. Initially, there is an electric field $\vec{E} = \frac{Q}{2\pi\epsilon_0 l}\frac{1}{r}\hat{r}$ in the region between the cylinders, and a magnetic field $\vec{B} = \mu_0 NI\hat{z}$ inside the solenoid. The momentum density in the region $a < r < R$ is $\vec{p}/V = \epsilon_0\left(\vec{E}\times\vec{B}\right) = -\frac{\mu_0 NIQ}{2\pi lr}\hat{\phi}$. The angular momentum density is $\vec{L}/V = \vec{r}\times\vec{p}/V = -\frac{\mu_0 NIQ}{2\pi l}\hat{z}$. Multiply by the volume $V = \pi\left(R^2 - a^2\right)l$ to get the total angular momentum: $\vec{L}_{EB} = -\frac{1}{2}\mu_0 NIQ\left(R^2 - a^2\right)\hat{z}$.

B. When the current is turned off, the changing magnetic field induces an azimuthal electric field, given by Faraday's law:

$$\begin{aligned}\vec{E} &= -\frac{1}{2}\mu_0 N\dot{I}\frac{R^2}{r}\hat{\phi}, \quad (r > R)\\ \vec{E} &= -\frac{1}{2}\mu_0 N\dot{I}r\hat{\phi}, \quad (r < R)\end{aligned}$$

Thus the torque on the outer cylinder is $\vec{\tau}_b = \vec{r}\times(-Q\vec{E}) = \frac{1}{2}\mu_0 NQR^2\dot{I}\hat{z}$, and it picks up an angular momentum $\vec{L}_b = \frac{1}{2}\mu_0 NQR^2\hat{z}\int_t^0 \frac{dI}{dt}dt = -\frac{1}{2}\mu_0 NIQR^2\hat{z}$. Similarly, the torque on the inner cylinder is $\vec{\tau}_a = \frac{1}{2}\mu_0 NQa^2\dot{I}\hat{z}$ and its angular momentum increase is $\vec{L}_a = \frac{1}{2}\mu_0 NIQa^2\hat{z}$. Thus, the total angular momentum change for the cylinders is $\vec{L}_a + \vec{L}_b = \vec{L}_{EB}$.

C. Now the magnetic field remains constant but the electric field is changing with time as a current flows through the connecting spoke. The magnetic field in the region $a < r < R$ is $\vec{B} = \mu_0 NI\hat{z}$. For a current $I'(t)$ in the radial spoke, the Lorentz force on a segment dr of the spoke is $d\vec{F} = I'dr\hat{r}\times\vec{B} = -\mu_0 NII'dr\hat{\phi}$. Thus, the torque on the spoke is $\vec{\tau} = \int_{spoke}\vec{r}\times d\vec{F} = \mu_0 NII'\int_a^R rdr(-\hat{z}) = -\mu_0 NII'(R^2 - a^2)\hat{z}/2$. The final angular momentum, after the cylinders are completely discharged, is $\vec{L} = \int\vec{\tau}dt = -\frac{1}{2}\mu_0 NI(R^2 - a^2)Q\hat{z} = \vec{L}_{EB}$ since $\int I'dt = Q$.

Nicolis

4. Consider in electrostatics an infinitely extended uniform charge density, $\rho(\vec{x}) = \rho_0$.

1. Prove that, despite $\rho(\vec{x})$'s being invariant under translations and rotations, there is no unique solution to the Poisson equation that is preferred in terms of symmetries.
2. Suppose now that the charge distribution is smoothly cut off in a spherical fashion at very large distances, i.e.

$$\rho(\vec{x}) = \rho_0 f(|\vec{x}|/R) \; , \tag{1}$$

where $f = 1$ for $|\vec{x}|/R \ll 1$, and $f = 0$ for $|\vec{x}|/R \gg 1$. Argue that this removes the ambiguity in the Poisson problem. Compute the potential for $|\vec{x}|/R \ll 1$ and for $|\vec{x}|/R \gg 1$ (up to additive constants).

Solution

1. There is no solution for the potential $V(\vec{x})$ that has the same symmetries as the charge distribution:

   For $V(\vec{x})$ to be invariant under rotations around a given origin, it has to be a function of $|\vec{x}|$ only. However, any such function won't be invariant under translations (i.e., change of origin), unless it is a constant. But a constant has vanishing laplacian, thus violating Poisson's equation.

   One can have a rotationally invariant $V(|\vec{x}|)$ that solves Poisson's equation, but for any constant vector $\vec{a}$, $V(|\vec{x} - \vec{a}|)$ is an equally good solution. There are infinitely many solutions of this form, all related by translations.

2. The cutoff function f picks an origin—the center of the sphere—thus violating the original charge distribution's invariance under translations. Now one can look for solutions for V that are isotropic, $V = V(|\vec{x}|)$, but not homogeneous.

   For $r \equiv |\vec{x}| \ll R$, Poisson's equation reads

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{d}{dr}(r^2 V'(r)) = -\frac{\rho_0}{\epsilon_0} \; , \tag{2}$$

   and the solution (regular at the origin) is

$$V(r) = -\frac{\rho_0}{\epsilon_0}\frac{r^2}{6} + \text{const} \; . \tag{3}$$

   For $r \gg R$, we have a

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + \text{const} \tag{4}$$

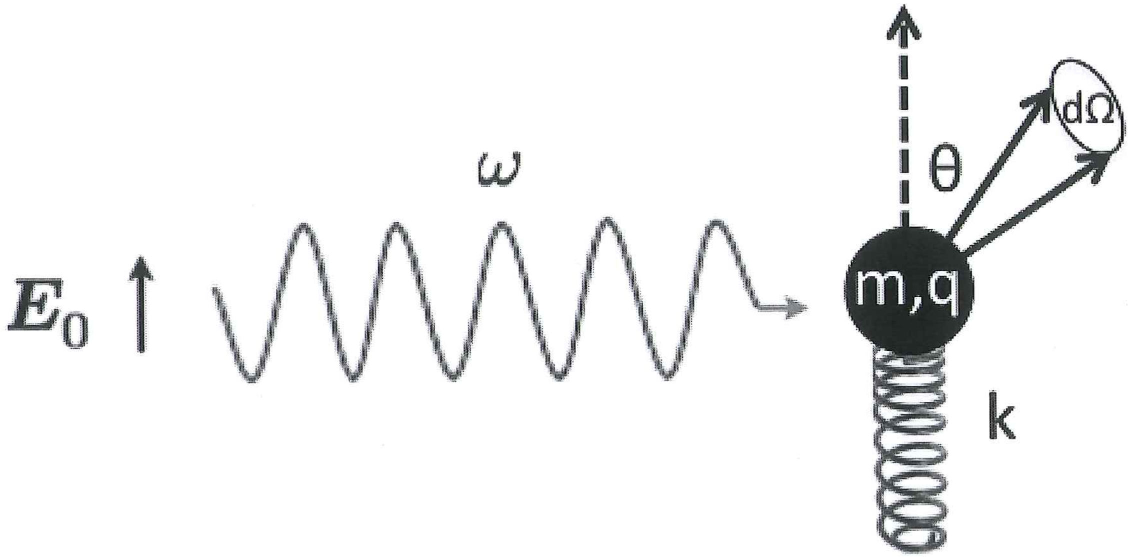
   vacuum solution, with total charge given by

$$Q = \rho_0 \int d^3x \, f(|\vec{x}|/R) \; . \tag{5}$$

~~E&M (Metzger)~~

5. A linearly polarized electromagnetic wave of amplitude $\vec{E}_0$ and frequency ω is incident on a classical particle of mass m and charge q attached to a spring of spring constant k, as shown in the figure below. Assume that the particle is free to move in three dimensional space, i.e. it is mechanically an isotropic 3-D classical harmonic oscillator.

1. Calculate the differential cross section $d\sigma/d\Omega$ for the light to scatter into a solid angle $d\Omega$. Express your results in terms of the angle θ with respect to the polarization direction.
2. Calculate the total cross section.
3. Consider the above system as a semi-classical model for a valence electron of a nitrogen atom in the Earth's atmosphere, where the "spring" represents the restoring force holding the electron at its equilibrium position in the atom.
   Using your result from part 2, explain why the sunset is red. Make sure to justify any approximations made.



Solution:

1. The equation of motion for the position $\vec{r}$ of the particle is

$$\ddot{\vec{r}} + \omega_0^2 \vec{r} = q\frac{\vec{E}_0}{m}\cos(\omega t), \tag{1}$$

where $\omega_0 = \sqrt{k/m}$. Assuming a harmonic form to the solution $\vec{r} = \vec{r}_0 \cos(\omega t)$ gives

$$\vec{r}_0 = \frac{q^2 \vec{E}_0}{m(\omega_0^2 - \omega^2)} \tag{2}$$

The classical dipole moment of the charge is then

$$\vec{d} = q\vec{r} = \frac{q^2 \vec{E}_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t) \tag{3}$$

The differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{S_0} \cdot \frac{dP}{d\Omega}, \tag{4}$$

where

$$S_0 = \frac{c}{4\pi}|\vec{E}_0|^2 \tag{5}$$

is the incident intensity and

$$\frac{dP}{d\Omega} = \frac{\ddot{\vec{d}} \cdot \ddot{\vec{d}}}{4\pi c^3}\sin^2\theta \tag{6}$$

is the differential power emission of the dipole radiation, where θ is the angle between $\vec{E}_0$ and the wave vector of the emitted radiation. Thus,

$$\frac{d\sigma}{d\Omega} = \left(\frac{q^2}{mc^2}\right)^2 \frac{\omega^4 \sin^2\theta}{(\omega_0^2 - \omega^2)^2} \tag{7}$$

2. Integrating over solid angle

$$\sigma_{\text{tot}} = \int_0^{2\pi} d\phi \int_0^{\pi} \frac{d\sigma}{d\Omega}\sin\theta d\theta = \frac{8\pi}{3}\left(\frac{q^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2} \tag{8}$$

3. The frequency of the Sun's visual wavelength radiation is much less than the natural frequency of the atomic "spring", which corresponds to the frequency corresponding to the ionization (typical ionization energies of valence electrons are several eV, i.e. in the ultra-violet portion of the spectrum). For $\omega \ll \omega_0$, equation (7) reduces to the Rayleigh scattering limit

$$\sigma = \frac{8\pi}{3}\left(\frac{q^2}{mc^2}\right)^2 \frac{\omega^4}{\omega_0^4} \tag{9}$$

The fact that $\sigma \propto \omega^4$ shows that high frequency ("blue") radiation from the sun passing through the atmosphere will be preferentially scattered over lower frequency ("red") emission. This results in the observed sunlight appearing redder when it passes through a large column of atmospheric gas near the sunet or sunrise.