

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 13, 2017
3:00PM to 5:00PM
General Physics (Part II)
Section 6.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

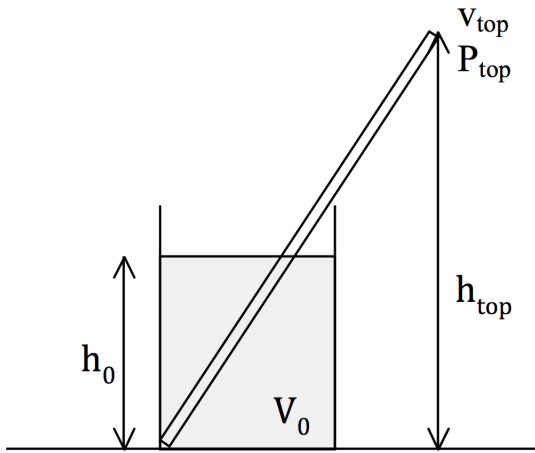
You may refer to the single handwritten note sheet on $8\frac{1}{2}'' \times 11''$ paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. A glass is filled to height h_0 with a volume of water V_0 with density ρ . A straw with uniform cross sectional area A is used to drink the water by creating a pressure at the top of the straw (P_{top}) that is less than atmospheric pressure (P_{Atm}). The top of the straw is a distance h_{top} above the bottom of the glass.
- (a) Determine the pressure (P_{top}) at the top of the straw as a function of the height (h) of the water in the glass so that the velocity of water coming out of the top of the straw remains constant at a value of v_{top} .
 - (b) How much work is done by the person in order to drink all the water in the glass with the constant velocity v_{top} ?
 - (c) How much time does it take to drink all the water in the glass with the constant velocity v_{top} ?



2.

- (a) Imagine one day the sun ceases to shine. The temperature on the earth will be cold such that the atmosphere is liquefied, and the earth surface will be covered by an ocean of liquid air. Estimate the average depth of this liquid air ocean. Compare it to the average depth of the true ocean, which is on the order of 5 km.
- (b) Estimate the largest height of a mountain on earth. Are the typical heights of mountains on the moon higher or lower than those on earth?

Hint: Assume that rock melts at 1000 Kelvin and the average mass number of rock is 30. The rest mass of the proton and neutron is about 1 GeV, and 1 eV corresponds to about 10,000 K of temperature. As a crude approximation, you can assume that the latent heat of fusion is of the same order as kT_m , where T_m is the melting temperature.

3. Uniform plasma with electron number density n is in thermal equilibrium with Planck radiation of temperature T and energy density $U = aT^4$, where $a = \pi^2 k^4 / 15c^3 \hbar^3$ is the radiation constant and k is the Boltzmann constant. The electrons (mass m_e) and photons interact through Thomson scattering with cross section $\sigma_T = (8\pi/3)r_e^2$ where $r_e = e^2/m_e c^2$ is the classical electron radius.
- (a) Find the photon mean free path λ .
 - (b) Estimate the electron mean free path to photon scattering, l .
Hint: consider the time between successive scatterings from the electron point of view.
 - (c) Estimate the average number of scatterings experienced by a photon as it diffuses through a distance $L \gg \lambda$.

4. A two-dimensional electron gas (2DEG) is separated from a conducting metal plane by a dielectric of known thickness. The 2DEG can be patterned into any shape and with any configuration of electrical contacts, and is cooled in a cryostat to 2K with a variable magnetic field.
- (a) Describe a transport measurement that could be made to deduce the electron density in a channel formed by the 2DEG.
 - (b) The capacitance formed by the conducting metal plane and the 2DEG is typically very small, and therefore difficult to measure directly. Describe a transport measurement that could allow determining the relative permittivity of the dielectric between the conducting metal plane and the 2DEG.

Note: Be sure to indicate all quantities to be measured in the experiments.

5. An astronaut wants to have some coffee with milk. At her disposal are some coffee (200 ml) at 97°C and some milk (200 ml) at 7°C . She would like to use all the coffee and milk for the drink. Unfortunately she has to finish some work that will take her 15 minutes before she can begin to consume her drink. She would like her drink to be as hot as possible when she drinks it. She is trying to find out if she should mix the coffee and milk before or after finishing her work.

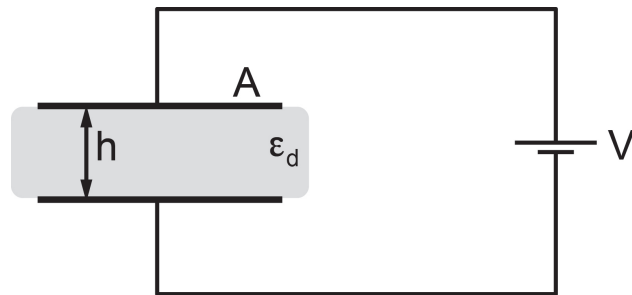
The astronaut starts her analysis by assuming that her coffee and milk are exchanging heat with their surroundings primarily through black body radiation. She assumes her coffee and milk are suspended in air as liquid spheres due to zero gravity. The mixture would also be a sphere suspended in air. The environment is at the room temperature (300 K). She also assumes that the specific heat capacity and density of coffee and milk are the same as water, and that all liquids are in thermal equilibrium throughout their volume as they cool down or warm up.

- (a) Based on her assumptions, should the astronaut mix the coffee and milk before or after her work? You can use the following approximation valid for small ΔT

$$\int_{T_1}^{T_1+\Delta T} \frac{1}{(T^4 - T_0^4)} dT \approx \frac{\Delta T}{T_1^4 - T_0^4} \quad (1)$$

- (b) What is the maximum temperature of the coffee when she begins to drink it?

6. In a parallel plate capacitor, electrical forces squeeze the dielectric layer that separates the two plates. If the dielectric material is compliant, the reduction in the distance between parallel plates is no longer negligible. To explore potential uses of this phenomenon, we would like to determine the changes in electrical and mechanical properties of this system in response to an applied voltage.



- Write an expression for the electrical force pulling the capacitor plates in terms of plate area A , plate separation h , dielectric permittivity ϵ_d , and the applied voltage V .
- Assume that the dielectric layer behaves as a linear spring with a spring constant k , and it has an initial thickness of h_0 . Determine the voltage V necessary to reduce the plate separation from h_0 to h .
- The spring constant depends on the Young's modulus E_d , area A , and thickness h_0 by $k = E_d A / h_0$. Using this relationship, express your result in (b) in terms of E_d , ϵ_d , h , and h_0 .
- Once the plates in (b) reach their equilibrium separation h , a small external force, ΔF , is applied to the parallel plates to compress the dielectric layer by Δh . Determine the effective spring constant $k_{eff}(h) = -\Delta F / \Delta h$.
- Evaluate $k_{eff}(h)$ at $h = 2h_0/3$. Interpret your result.

Quals Problem Fluids

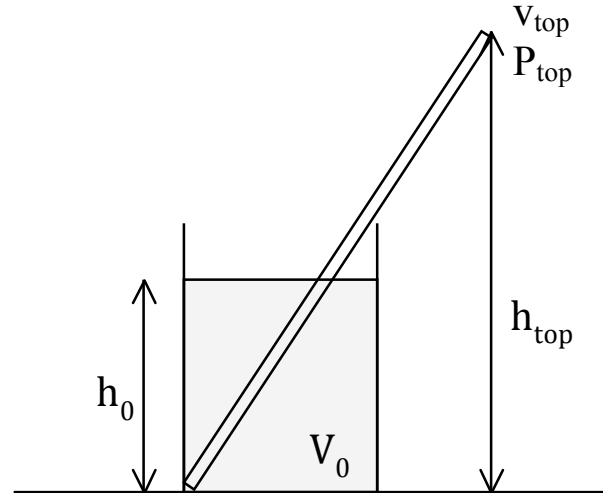
M. Shaevitz
Dec, 2016

A glass is filled to height h_0 with a volume of water V_0 with density ρ . A straw with uniform cross sectional area A is used to drink the water by creating a pressure at the top of the straw (P_{top}) that is less than atmospheric pressure (P_{Atm}). The top of the straw is a distance h_{top} above the bottom of the glass.

a) Determine the pressure (P_{top}) at the top of the straw as a function of the height (h) of the water in the glass so that the velocity of water coming out of the top of the straw remains constant at a value of v_{top} .

b) How much work is done by the person in order to drink all the water in the glass with the constant velocity v_{top} ?

c) How much time does it take to drink all the water in the glass with the constant velocity v_{top} ?



Solution:

a) Use coordinate system with +y up and zero at the bottom of the glass.

Assume velocity of the water at the top of the fill is zero.

Compare point at the top of liquid with height h to point at the top of the straw with height h_{top} .

$$P_{Atm} + \rho gh = P_{top} + \frac{1}{2} \rho v_{top}^2 + \rho gh_{top}$$

$$P_{top} = P_{Atm} - \frac{1}{2} \rho v_{top}^2 - \rho g(h_{top} - h)$$

$$b) Area_{glass} = A_{glass} = V_0 / h_0$$

Calculate the work against gravity to raise the water to the top of the straw.

$$W_{gravity} = \int_0^{h_0} \rho g A_{glass} (h_{top} - h) dh = \rho g A_{glass} h_0 \left(h_{top} - \frac{h_0}{2} \right) = \rho g V_0 \left(h_{top} - \frac{h_0}{2} \right)$$

Calculate the work to increase the kinetic energy all the water to a velocity v_{top}

$$W_{KE} = \frac{1}{2} \rho V_0 v_{top}^2$$

$$W_{total} = \rho V_0 \left(g \left(h_{top} - \frac{h_0}{2} \right) + \frac{1}{2} v_{top}^2 \right)$$

$$\text{Could also use: } W = \int F dx = \int (P_{Atm} - P_{top}) A dx = \int (P_{Atm} - P_{top}) A_{glass} dh$$

$$= \int_0^{h_0} \left(\frac{1}{2} \rho v_{top}^2 + \rho g(h_{top} - h) \right) A_{glass} dh \Rightarrow \text{Which gives the same result.}$$

$$c) Flow_Rate = Av_{top} \Rightarrow t_{total} = \frac{V_0}{Av_{top}}$$

① 10 m water = 1 atmosphere

air is N_2 and O_2 . Liquid air density is the same as water.

Ocean of liquid air is ~ 10 meters.

or

height of air ~ 10 km

When air liquifies it loses factor of 1000

$\Rightarrow 10$ meters

② assume mountain shift down by 1 layer of atoms.

$$mgh = k_B T_m \quad \leftarrow \text{melting temperature}$$

$$h = \frac{k_B T_m}{mg}$$

$$10,000 \text{ K} = 1 \text{ eV}$$

$$T_m = 1000 \text{ K} = 0.1 \text{ eV}/k_B$$

$$= \frac{(k_B T_m)}{mc^2} \cdot \frac{c^2}{g} = \frac{(0.1 \text{ eV})}{30 \times 10^9 \text{ eV}} \cdot \frac{(3 \times 10^8 \text{ m/s})^2}{10 \text{ m/s}^2}$$

$$= 3 \times 10^4 \text{ m} \approx 30 \text{ km}$$

Problem:

Uniform plasma with electron number density n is in thermal equilibrium with Planck radiation of temperature T and energy density $U = aT^4$, where $a = \pi^2 k^4 / 15c^3 \hbar^3$ is the radiation constant and k is the Boltzmann constant. The electrons (mass m_e) and photons interact through Thomson scattering with cross section $\sigma_T = (8\pi/3)r_e^2$ where $r_e = e^2/m_e c^2$ is the classical electron radius.

(a) Find the photon mean free path λ .

(b) Estimate the electron mean free path to photon scattering, l .

[Hint: consider the time between successive scatterings from the electron point of view.]

(c) Estimate the average number of scatterings experienced by a photon as it diffuses through a distance $L \gg \lambda$.

Solution:

(a)

$$\lambda = \frac{1}{n\sigma_T}.$$

(b) Photons bombard electrons with the speed of light. From the electron point of view, the characteristic time between successive scatterings is

$$t_{sc} \sim \frac{1}{n_\gamma \sigma_T c},$$

where n_γ is the number density of photons. The mean energy of a blackbody photon is $\bar{E} \approx 3kT$, and hence

$$n_\gamma \sim \frac{aT^4}{3kT} = \frac{aT^3}{3k}.$$

Electrons move with thermal speeds $v \sim (kT/m_e)^{1/2}$, and their mean free path to scattering is $l \sim vt_{sc}$,

$$l \sim \frac{(kT/m_e)^{1/2}}{n_\gamma \sigma_T c} \sim \frac{3k(kT/m_e c^2)^{1/2}}{\sigma_T a T^3} \sim \alpha^{-2} \frac{\hbar}{m_e c} \left(\frac{kT}{m_e c^2} \right)^{-5/2},$$

where $\alpha = e^2/\hbar c \approx 1/137$.

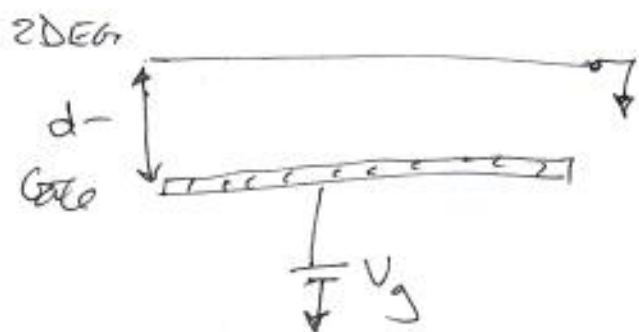
(c) Photon diffusion is a random walk with steps $\sim \lambda$ on the timescale $\tau \sim \lambda/c$. The diffusion length L is related to time t by

$$L^2 = Dt,$$

where $D \sim \lambda^2/\tau = c\lambda$ (more precisely $D = c\lambda/3$, where 3 is associated with the dimension of space). The number of scatterings is

$$N \sim \frac{ct}{\lambda} \sim \left(\frac{L}{\lambda} \right)^2.$$

To determine the relative permittivity, ϵ , consider the 2DEG / gate to be a simple parallel plate capacitor -



from a parallel plate cap model, the carrier density induced in the 2DEG will be

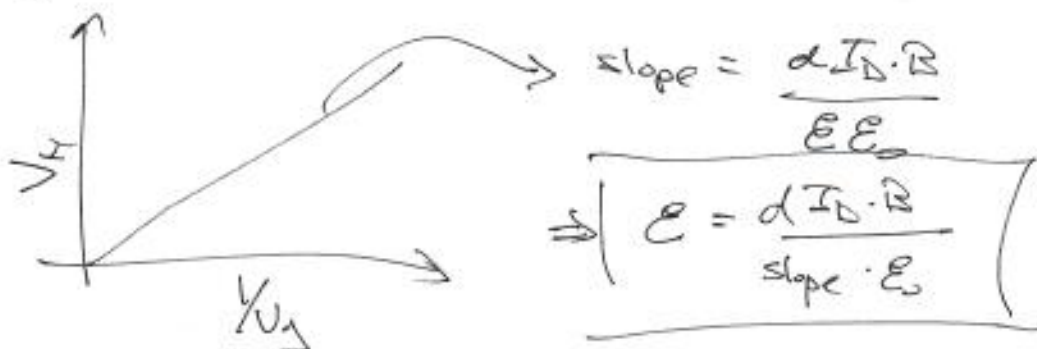
$$n = \frac{C_g V_g}{e} \quad (3)$$

where $C_g = \frac{\epsilon \epsilon_0}{d}$ is the capacitance
 ϵ is permittivity of the dielectric
 ϵ_0 is permittivity of free space.
 d is dielectric thickness

From (1) we also have

$$V_H = \frac{I_D}{C_g V_g} \cdot R = \frac{\alpha I_D \epsilon \cdot R}{\epsilon \epsilon_0 \cdot V_g}$$

to determine ϵ , we can measure V_H which varies with V_g . Plotting V_H vs $1/V_g$ now gives a linear response



~~in both cases~~

Two primary sources of error are uncertainty

i) uncertainty in the experimental quantities, such as the applied current, I_D , or magnetic field, B .

One way to minimize this, eg when determining the value of E , would be to measure the slope of V_H vs V_g for various values of I_D and B .

ii) Since in both cases the quantity measured is a voltage, voltage noise sets a limit on the data resolution.

This can be improved by using lock-in techniques, increasing I_D to increase signal-to-noise etc.

b) For infinite square well potential, the energy level spacing is

$$\Delta E = \frac{\hbar^2 \pi^2}{2mL^2} \quad \text{where } m, \text{ electron mass} \\ L, \text{ well width}$$

In order to resolve energy levels it requires

$$\Delta E > k_B T$$

$$\Rightarrow \frac{\hbar^2 \pi^2}{2mL^2} > k_B T$$

$$\Rightarrow L^2 < \frac{\hbar^2 \pi^2}{2m k_B T}$$

$$\boxed{L < \frac{\hbar \pi}{\sqrt{2m k_B T}}}$$

Take $\hbar \sim 1 \times 10^{-34}$

$\pi \sim 3$

$m \sim 9 \times 10^{-31}$

$k_B \sim 1.4 \times 10^{-23}$

$T \sim 2 \text{ K}$ (this is the lowest temp achieved by the cryostat)

$$\Rightarrow L < \frac{(10^{-34})(3)}{\sqrt{2(9 \times 10^{-31})(1.4 \times 10^{-23})(2)}} \\ \sim 5 \times 10^{-8} \text{ m.}$$

Construction should be less than $\sim 50 \text{ nm}$ wide

c) In order to resolve spin splitting, requires that the Zeeman energy be larger than $k_B T$

1.e.

$$\Delta E_{Zeeman} > k_B T$$

$$g \mu_B B \cdot S > k_B T$$

$$B > \frac{k_B T}{g \mu_B \cdot S}$$

g - g factor
 μ_B - Bohr magneton
S - electron spin

take $k_B \sim 1 \times 10^{-23}$

$$g \approx 2$$

$$\mu_B \sim 9 \times 10^{-24}$$

$$S = 1/2$$

$$T = 2K$$

$$B > \frac{(1 \times 10^{-23}) / 2}{2 \cdot 9 \times 10^{-24} \cdot 1/2}$$

$$\sim 2 \text{ Tesla}$$

∴ Magnetic field should be larger than $\sim 2T$.
to resolve spin splitting @ this temperature.

An astronaut wants to have some coffee with milk. At her disposal are some coffee (200 ml) at 97 °C and some milk (200 ml) at 7 °C. She would like to use all the coffee and milk for the drink. Unfortunately she has to finish some work that will take her 15 minutes before she can begin to consume her drink. She would like her drink to be as hot as possible when she drinks it. She is trying to find out if she should mix the coffee and milk before or after finishing her work.

The astronaut starts her analysis by assuming that her coffee and milk are exchanging heat with their surroundings primarily through black body radiation. She assumes her coffee and milk are suspended in air as liquid spheres due to zero gravity. The mixture would also be a sphere suspended in air. The environment is at the room temperature (300 K). She also assumes that the specific heat capacity and density of coffee and milk are the same as water, and that all liquids are in thermal equilibrium throughout their volume as they cool down or warm up.

(a) Based on her assumptions, should the astronaut mix the coffee and milk before or after her work? You can use the following approximation valid for small ΔT

$$\int_{T_1}^{T_1+\Delta T} \frac{1}{(T^4 - T_0^4)} dT \approx \frac{\Delta T}{T_1^4 - T_0^4}$$

(b) What is the maximum temperature of the coffee when she begins to drink it?

Solution:

(a) The general idea is that heat is lost/gained from the environment by radiation. Let us consider the post-mixing and pre-mixing scenarios separately.

Post-mixing:

Initially, the temperatures of the coffee and milk are T_c^i and T_m^i respectively. The coffee loses heat and the milk gains heat from the environment. After time Δt , the coffee and milk reach final temperature T_c^f and T_m^f respectively. Given that the quantity and characteristics of the coffee and milk are identical, the final

temperature reached will be $T_d^f = \frac{T_m^f + T_c^f}{2}$

To calculate the final temperatures of coffee and milk, we calculate the amount of heat lost in time dt , and convert that number to a change in temperature dT as follows:

$$\text{Heat lost by coffee in time } dt: \quad dQ = \sigma A (T_c^4 - T_0^4) dt$$

$$\text{Heat gained by milk in time } dt: \quad dQ = \sigma A (T_0^4 - T_m^4) dt$$

$$\text{Change in temperature of coffee in time } dt: \quad dQ = mC_v dT_c$$

$$\text{Change in temperature of milk in time } dt: \quad dQ = mC_v dT_m$$

Differential equation for temperature of coffee and milk as a function of time:

$$mC_v dT_c = -\sigma A (T_c^4 - T_0^4) dt$$

$$mC_v dT_m = \sigma A (T_0^4 - T_m^4) dt$$

These can be integrated to get the final temperatures of milk and coffee before mixing:

$$\int_{T_c^i}^{T_c^f} \frac{1}{(T_c^4 - T_0^4)} dT_c = -\frac{\sigma A}{mC_v} \Delta t$$

$$\Delta T_c = -\frac{\sigma A}{mC_v} \Delta t ((T_c^i)^4 - T_0^4)$$

Put in numbers:

mass $m = 0.2$ kg

Radius of coffee sphere = 3.63 cm

Area of coffee sphere = 165 cm²

Heat capacity = 4.2 J/g

Initial temperature of coffee = 370 K

Time = 15 min = 900 seconds

Gives:

$$\Delta T_c = -10.6 \text{ K}$$

So final temperature of coffee is 86.4 C

Similarly for milk:

$$\Delta T_m = 2.0 \text{ K}$$

So final temperature of milk is 9 C

After mixing the two together, final drink temperature = 47.7 C

Pre-mixing:

Initially, the temperatures of the coffee and milk are T_c^i and T_m^i respectively. On mixing, the drink has a temperature $T_d^i = \frac{T_m^i + T_c^i}{2} = 52$ C. Subsequently, the drink cools down by radiation as above. Additional points to note are that the total mass of the drink is now $2m$, and the surface area is now $2^{2/3} A$. Following the steps above, one obtains the differential equation for the cooling of the drink with temperature:

$$2mC_v dT_d = -2^{2/3} \sigma A (T_d^4 - T_0^4) dt$$

Which can also be integrated to get the final temperature of the drink.

$$\Delta T_d = -\frac{\sigma A}{2^{1/3} mC_v} \Delta t ((T_d^i)^4 - T_0^4)$$

Plug in numbers to get:

$$\Delta T_d = -2.4 \text{ K}$$

So final temperature of the drink is 49.6 C

Part (a) can also be answered without explicit calculation:

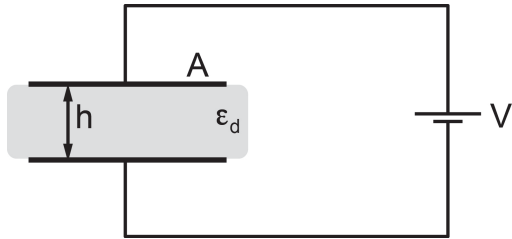
The key observation in this problem is that the rate of change of temperature is proportional to the fourth power of temperatures (as one expects for radiation). With this in mind, we can use the following logic to evaluate the post-mixing and pre-mixing scenarios without doing explicit calculations:

(a) post-mixing – the initial temperature of milk is close to room temperature already, so the temperature of milk does not increase much. The initial temperature of coffee is high, so it will lose heat quickly to the environment.

(b) pre-mixing – the initial temperature after mixing is midway between coffee and milk – ie, much closer to room temperature than the initial temperature of coffee. The mixture will cool much more slowly than the case of the coffee alone. A small additional advantage in this case is that the ratio of surface area to mass is smaller than in the previous case.

Therefore, it is clear that pre-mixing the drink leads to a higher final temperature.

General Physics II capacitor problem and solution



In a parallel plate capacitor, electrical forces squeeze the dielectric layer that separates the two plates. If the dielectric material is compliant, the reduction in the distance between parallel plates is no longer negligible. To explore potential uses of this phenomenon, we would like to determine the changes in electrical and mechanical properties of this system in response to an applied voltage.

- Write an expression for the electrical force pulling the capacitor plates in terms of plate area A , plate separation h , dielectric permittivity ϵ_d , and the applied voltage V .
- Assume that the dielectric layer behaves as a linear spring with a spring constant k , and it has an initial thickness of h_0 . Determine the voltage V necessary to reduce the plate separation from h_0 to h .
- The spring constant k depends on the Young's modulus E_d , area A , and thickness h_0 by $k = E_d A / h_0$. Using this relationship, express your result in (b) in terms of E_d , ϵ_d , h , and h_0 .
- Once the plates in (b) reach their equilibrium separation h , a small external force, ΔF , is applied to the parallel plates to compress the dielectric layer by Δh . Determine the effective spring constant $k_{eff}(h) = -\Delta F / \Delta h$.
- Evaluate $k_{eff}(h)$ at $h = 2h_0/3$. Interpret your result.

Solution

- a) The force can be calculated from the energy density of the electric field as

$$F = -\frac{1}{2} \epsilon_d E^2 \times A$$

the negative sign indicates that the force is in the pulling direction. Substituting $E = V/h$, we obtain the following result:

$$F = -\frac{1}{2} \epsilon_d \frac{V^2}{h^2} A$$

there are various other derivations of this final formula (e.g. Differentiating total energy, including the battery, with respect to h).

- b) Spring force is balanced by the electrical force:

$$F_{total} = -k(h - h_0) - \frac{1}{2} \epsilon_d \frac{V^2}{h^2} A = 0$$

Rearranging the terms give:

$$V^2 = 2k \frac{h_0 - h}{\epsilon_d A} h^2$$

- c) Substituting $k = E_d A / h_0$ gives:

$$V^2 = 2 \frac{E_d A}{h_0} \frac{h_0 - h}{\epsilon_d A} h^2 = \frac{2E_d}{\epsilon_d} h^2 \left(1 - \frac{h}{h_0}\right)$$

Note that the result is independent of the surface area, A .

- d) Using linear approximation, we can write

$$\Delta F / \Delta h \approx dF / dh$$

Using the expression for F_{total} in (a), we can obtain the derivative as follows

$$\frac{dF}{dh} = -k + \epsilon_d \frac{V^2}{h^3} A$$

substituting V^2 gives:

$$\frac{dF}{dh} = -k \left(3 - 2 \frac{h_0}{h}\right)$$

note that $k_{eff}(h) = -\Delta F/\Delta h$, therefore

$$k_{eff} = k \left(3 - 2 \frac{h_0}{h} \right), \quad \text{alternatively } k_{eff} = \frac{E_d A}{h_0} \left(3 - 2 \frac{h_0}{h} \right)$$

e) For $h = 2h_0/3$

$$k_{eff} = 0$$

Effective spring constant vanishes. This means the parallel plate system is unstable. A small displacement of one plate towards the other will cause it to collapse onto the other plate.