Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 17, 2014
1:00PM to 3:00PM
General Physics (Part I)
Section 5.

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on General Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. The sun has mass $M \approx 2 \times 10^{33}$ g, radius $R \approx 7 \times 10^{10}$ cm and luminosity $L \approx 4 \times 10^{33}$ erg s^{-1}. Radiation escapes from the surface "photospheric" layer, and you are asked to estimate the parameters of this layer. You can assume that radiation at the photosphere has approximately Planck spectrum.

   (a) Estimate surface temperature.

   (b) Estimate the thickness of the photospheric layer H. You can assume that it is made of hydrogen.

   (c) Assuming opacity $\kappa \sim 0.5$ cm^2 g^{-1} in the photospheric layer, estimate its mass density ρ.

   (d) Estimate the ratio of radiation pressure to gas pressure in the photospheric layer.

   Gravitational constant $G = 6.7 \times 10^{-8}$ cm^3 g^{-1} s^{-2}

   Radiation constant: $a = 7.6 \times 10^{-15}$ erg K^{-4} cm^{-3}. (Energy density of Planck radiation is aT^4.)

   Stefan-Boltzmann constant $\sigma = ac/4 = 5.7 \times 10^{-5}$ erg cm^{-2} s^{-1} K^{-4}

   Boltzmann constant: $k = 1.4 \times 10^{-16}$ erg K^{-1}

   Proton mass: $m_p = 1.7 \times 10^{-24}$ g

2. For this problem a chain is defined as a classical one dimensional extended object which lives on the links of a two dimensional square lattice of lattice constant a and has energy per unit length (tension) $\mathcal{T}$. We will ignore any interactions of the chain with itself (i.e. there is no extra energy if the chain crosses itself or if two segments of the chain lie on the same link of the lattice).

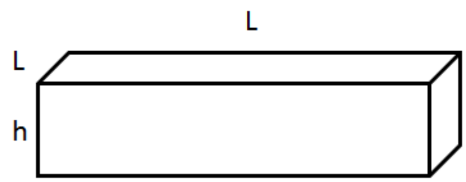
   Suppose that one end of the chain is fixed at the origin but that the length of the chain (number of links) and the position of the other end can fluctuate.

   (a) Assuming that the chain is in thermal equilibrium at temperature T please find
      i. The partition function of the chain
      ii. The free energy of the chain
      iii. The expectation value of the length of the chain (number of segments)

   (b) The free energy found in (a) only makes sense for temperatures less than a temperature T^*. Please
      i. find an expression for T^*.
      ii. show that T^* is the temperature of a phase transition separating two phases. Characterize the two phases.
      iii. give an expression for the behavior of the length of the chain as T approaches T^* from below.

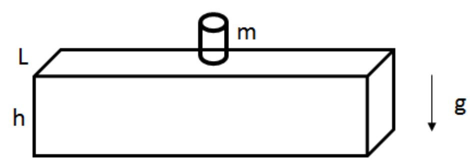
3. Consider a point mass moving freely along x in the interval $[a, b]$. At the two extrema $x = a, b$ there are two walls. When our particle hits either wall, it bounces back elastically.
   (a) Draw the trajectory of the particle in *phase space*.
   (b) Suppose now that the two walls start moving, very slowly, as a result of some external forces. Compute the adiabatic invariant. [*Recall that the adiabatic invariant is the area enclosed by the orbit in phase space.*]
   (c) Given your answer to item (b), under what conditions is energy approximately conserved?
   (d) Explain in physical terms why your answer to item (c) makes sense. Is energy conserved in each collision?
   (e) Show that on time scales much longer than the particle 'round trip' time, this system obeys an ideal gas-like law,
   $$P \cdot V = nk_B T.$$
   What is the thermodynamic interpretation of the adiabatic invariant you computed in item (b)?

4. Two otherwise empty (rectangular) cavities (A and B) have walls with negligible heat capacity. One (cavity A) has an adjustable roof height h_A and lengths L_A for the other dimensions. It is filled with blackbody radiation initially at temperature T_A. Cavity B and its contents is similarly described by h_B, L_B and T_B. $T_A > T_B$



   (a) Suppose h_A and h_B are fixed. An 'ideal' cyclic engine works to extract heat from cavity A, convert some of it to work, and put some into cavity B. After many such cycles, the engine is left in its initial state. What is the minimum radiation temperature in cavity A ($T_A^{'}$) and the maximum in cavity B ($T_B^{'}$)?

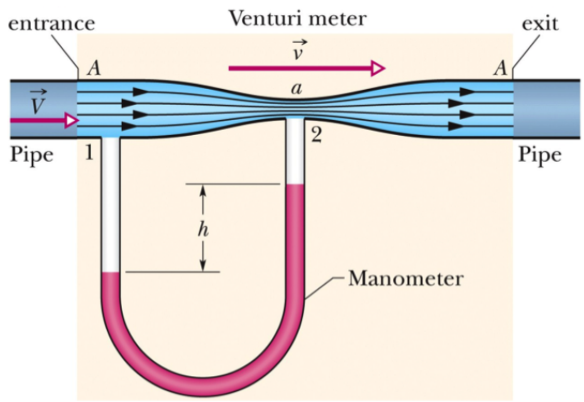
   (b) Suppose h_A and h_B are not fixed but a mass m_A on the roof of A is sufficient to give the initial altitude h_A and similarly m_B gives the initial h_B. A cyclic engine operates to transfer heat between the two cavities and extract work. When the maximum work has been extracted, what is the final $h_B^{'}$?



   (c) The cavity A is isolated and then its height is expanded slowly from h to $h^{'}$. What change takes place in its temperature ($T^{'}$)?

   (d) Suppose we lived in n-dimensions instead of $n = 3$ dimensions. How would the result of (c) above be changed?

5. A venturi meter is inserted is used to measure the flow speed of water (with density $\rho = 10^3$ kg/m^3) through a pipe, as pictured below. The meter is inline with the pipe, connecting two sections of pipe. The cross-sectional area A of the entrance and exit of the meter matches the pipe's cross-sectional area. At the entrance and exit, the water has a speed V; in between, it flows through a narrower throat of cross-sectional area a with a speed v. A manometer connects the entrance with the throat, and shows a height difference h between the fluid levels in the two arms of the manometer. If $A = 20$ cm^2, $a = 8$ cm^2, and $h = 1$ cm, what is the flow speed in the pipe?



6. A classical gas in three dimensions is constrained by a wall to move in the $x \geq 0$ region of space. The atoms of the gas do not interact. A potential

$$V(x) = \frac{1}{2}\alpha x^2 \qquad x \geq 0$$

attracts the atoms to the wall and the atoms are free to move in an area A in the y and z directions. If the gas is at uniform temperature T show that the number of particles per unit area varies as

$$N(x) = 2N\sqrt{\frac{\alpha\beta}{2\pi}}e^{-\alpha\beta x^2/2}$$

where N is the total number of particles per unit area. You may need to recall that

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

In the usual derivation of the ideal gas law, one uses Newton's laws and the virial theorem to show that, for each particle, -$\langle \vec{q}_i \cdot \vec{F}_i \rangle = 3K_\text{B}T$, where the angle brackets denote the time average. Then $-\langle \sum_{k=1}^{N} \vec{q}_k \cdot \vec{F}_k \rangle$ is evaluated and shown to be PV. Repeat this derivation in a slab of thickness dx and area A but in the presence of the potential given above. Show that the ideal gas law is still locally obeyed and that the pressure is given by

$$P(x) = Ce^{-\alpha\beta x^2/2}$$

Find C.

General:

The sun has mass $M \approx 2 \times 10^{33}$ g, radius $R \approx 7 \times 10^{10}$ cm and luminosity $L \approx 4 \times 10^{33}$ erg s^{-1}. Radiation escapes from the surface "photospheric" layer, and you are asked to estimate the parameters of this layer. You can assume that radiation at the photosphere has approximately Planck spectrum.

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Solution:

(a) Flux of blackbody radiation from the surface is σT^4, hence the solar luminosity may be written as

$$L = 4\pi R^2 \sigma T^4 \qquad \Rightarrow \qquad T = \left(\frac{L}{4\pi R^2 \sigma}\right)^{1/4} \sim 6 \times 10^3 \text{ K}.$$

(b) Density in the solar atmosphere changes with altitude on the hydrostatic scale-height H such that the change in potential energy, Hgm_p, is comparable to kT. Here $g = GM/R^2$ is the gravitational acceleration at the solar surface. This gives the characteristic thickness of the photospheric layer,

$$H = \frac{kT}{gm_p} = \frac{R^2\, kT}{GM\, m_p} \sim 2 \times 10^7 \text{ cm}.$$

(c) Radiation escapes from the layer of optical depth ~ 1. Thus, the photospheric layer satisfies the condition $H\kappa\rho \sim 1$, which gives

$$\rho \sim \frac{1}{\kappa H} \sim 10^{-7} \text{ g cm}^{-3}.$$

(d) Radiation pressure is $P_{\rm rad} \sim aT^4/3$ and gas pressure is $P_{\rm gas} \sim (\rho/m_p)kT$, so

$$\frac{P_{\rm rad}}{P_{\rm gas}} \sim \frac{m_p\, aT^3}{3k\, \rho} \sim 10^{-4}.$$

QUALS 2014

AJM

1. Stat Mech Problem

Two Dimensional Tethered Chain

For this problem a chain is defined as a classical one dimensional extended object which lives on the links of a two dimensional square lattice of lattice constant a and has energy per unit length (tension) $\mathcal{T}$. We will ignore any interactions of the string with itself (i.e. there is no extra energy if the chain crosses itself or if two segments of the chain lie on the same link of the lattice).

Suppose that one end of the chain is fixed at the origin but that the length of the chain (number of links) and the position of the other end can fluctuate.

(a) Assuming that the chain is in thermal equilibrium at temperature T please find

(i) The partition function of the chain

(ii) The free energy of the chain

(iii) The expectation value of the length of the chain (number of segments)

(b) The free energy found in (a) only makes sense for temperatures less than a temperature T^*. Please

(i) find an expression for T^*.

(ii) show that T^* is the temperature of a phase transition separating two phases. Characterize the two phases

(iii) give an expression for the behavior of the length of the chain as T approaches T^* from below.

Solution: Two Dimensional Tethered Chain

(a)

(i)

$$Z = \sum_{lengths\ n} \left(\sum_{configurations\ of\ length\ n} e^{-n\frac{\mathcal{T}a}{k_B T}} \right) \tag{1}$$

Because each link can go in one of 4 directions, the number of configurations is 4^n so

$$Z = \sum_{lengths\ n} e^{n\left(ln4 - \frac{\mathcal{T}a}{k_B T}\right)} = \frac{1}{1 - e^{ln4 - \frac{\mathcal{T}a}{k_B T}}} \tag{2}$$

(ii) $F = -k_B T ln Z = k_B T ln \left(1 - e^{ln4 - \frac{\mathcal{T}a}{k_B T}}\right)$

(iii)

$$< n >= \frac{1}{Z} \sum_{lengths\ n} n e^{n\left(ln4 - \frac{\mathcal{T}a}{k_B T}\right)} = \frac{e^{ln4 - \frac{\mathcal{T}a}{k_B T}}}{1 - e^{ln4 - \frac{\mathcal{T}a}{k_B T}}} \tag{3}$$

(b)

(i) Clearly, only for for $T < \mathcal{T}a/\left(k_B ln4\right)$ the sum converges and the expression makes sense.

(ii) T^* separates a phase with finite length chains from a phase where the chains have infinite length (lengths proliferate)

(iii) As $T \to T^*_-$

$$< n > \to \frac{T^*}{T^* - T} \tag{4}$$

(**If you want a more complicated problem one can make it one dimensional and ask for the mean square value of the position**)

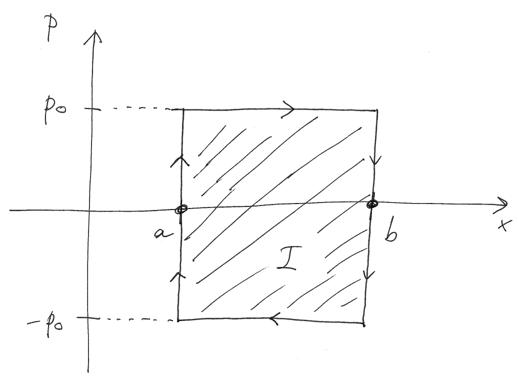
3 General (Stat Mech): Single particle thermodynamics

Consider a point mass moving freely along x in the interval $[a, b]$. At the two extrema $x = a, b$ there are two walls. When our particle hits either wall, it bounces back elastically.

1. Draw the trajectory of the particle in *phase space*.
2. Suppose now that the two walls start moving, very slowly, as a result of some external forces. Compute the adiabatic invariant. [*Recall that the adiabatic invariant is the area enclosed by the orbit in phase space.*]
3. Given your answer to item 2, under what conditions is energy approximately conserved?
4. Explain in physical terms why your answer to item 3 makes sense. Is energy conserved in each collision?
5. Show that on time scales much longer than the particle 'round trip' time, this system obeys an ideal gas-like law,
$$P \cdot V = nk_BT \; . \tag{24}$$
What is the thermodynamic interpretation of the adiabatic invariant you computed in item 2?

Solution

1.



2. $I = \text{area} = 2(b-a)p_0$.

3. $E = H = p_0^2/(2m)$. Energy is approximately constant if p_0 is. Given the adiabatic invariant above, to have energy conservation we need $(b-a) = \text{const}$, i.e. the two walls have to move in the same direction at the same speed—not necessarily at constant speed though.

4. Energy is not conserved in each collision: if you bounce elastically off a receding wall, you end up with *less* energy than you started with. However, what you lose by this collision, you get back when you hit the other wall, if the two walls are moving a the same speed. (Recall that the conservation of the adiabatic invariant involves a time-average over a full cycle.) So, under the condition of item 2, the *average* energy is conserved.

5. The natural definitions of pressure, temperature and volume for our system are,

$$P = \left\langle \left| \frac{dp}{dt} \right|_{\text{one wall}} \right\rangle = \frac{2p_0}{\tau} = \frac{p_0^2}{m(b-a)} \tag{25}$$

$$\tfrac{1}{2}k_B T = E = \frac{p_0^2}{2m} \tag{26}$$

$$V = (b-a) \ , \tag{27}$$

where τ denotes the round-trip time, $\langle \dots \rangle$ denotes time averaging over a few cycles, and of course the number of particles is one. We get precisely eq. (24).

Moving the walls adiabatically slowly, without injecting "heat" into the system, does not change entropy. Therefore, our adiabatic invariant should be associated with entropy in this thermodynamical interpretation of the system. We can work out the actual relation between the two quantities from the first law of thermodynamics,

$$dE = TdS - PdV \ , \tag{28}$$

which, according to the "dictionary" (25)–(27), gives

$$dS = k_B\Big(\frac{dp_0}{p_0} + \frac{dV}{V}\Big)\ . \tag{29}$$

On the other hand, just from $I = 2p_0V$, we have

$$dI = 2\Big(V dp_0 + p_0 dV\Big) = I\frac{1}{k_B}dS \tag{30}$$

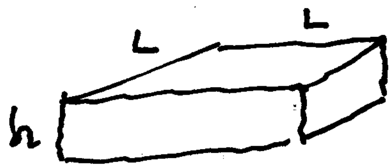
Integrating this ODE we get

$$S = k_B \log I + \text{const}\ . \tag{31}$$

Mal Ruderman

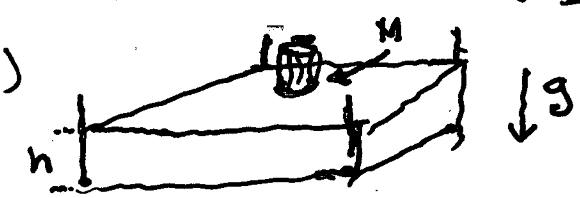
General (Thermo)

Two otherwise empty (rectangular) cavities (A and B) have walls with negligible heat capacity. One (cavity A) has an adjustable roof height h_A and lengths L_A for the other dimensions. It is filled with blackbody radiation initially at temperature T_A



Cavity B and its contents is similarly described by h_B, L_B, and T_B. $T_A > T_B$.

a) Suppose h_A and h_B are fixed. An "ideal" cyclic engine works to extract heat from cavity A, convert some of it to work, and put some into cavity B. After many such cycles the engine is left in its initial state. What is the minimum radiation temperature in cavity A (T_A') and the maximum in cavity B (T_B')?

b) 

Suppose h_A and h_B are not fixed but a mass m_A on the roof of A is sufficient to give the initial altitude h_A and, similarly m_B gives the initial h_B. A cyclic engine operates to transfer heat between the two cavities and extract work. When the maximum work has been extracted what is the final h_B' ?

c) The cavity A is isolated and then its height is expanded slowly from h to h'. What change takes place in its temperature (T')?

d) Suppose we lived in n-dimensions instead of $n = 3$ dimensions, How would the result of c) above be changed?

Suggested points/10 Answer to General (Thermo) Mal Ruderman

Answers are based on entropy of blackbody radiation $S \propto T^3 V$. Students should know this, or use $S \propto \frac{U}{T} \propto \frac{T^4 V}{T} = T^3 V$

$$\begin{pmatrix} V_A = h_A L_A^2 \\ V_B = h_B L_B^2 \end{pmatrix}$$

(4) a) $V_A T_A^3 + V_B T_B^3 =$ initial S

Final Temperatures $T_A' = T_B' = T_F$

& $S' = (V_A + V_B) T_F^3$

$$S = S' \Rightarrow \boxed{T_F^3 = \frac{V_A T_A^3 + V_B T_B^3}{V_A + V_B}.}$$

(3) b) $\overset{\text{At end}}{h_A'} = 0$ and $T_B' = T_B$ since pressure of radiation is constant

$(T_A' = T_A)$

$$\therefore \boxed{(h_B' - h_B) L_B^2 T_B^3 = h_A L_A^2 T_A^3}$$

(2) c)

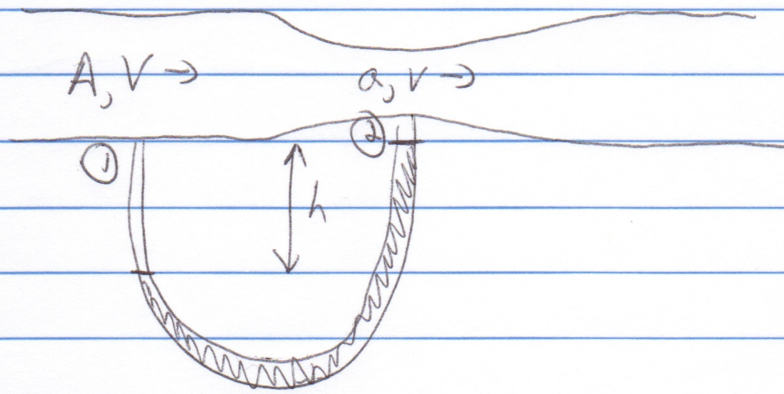
$S \propto V T^3$ $\quad V = L^2 h$

$$\therefore T \propto \left(\frac{1}{h}\right)^{1/3} \qquad \boxed{\frac{T'}{T} = \left(\frac{h}{h'}\right)^{1/3}}$$

(1) d) $S \propto \frac{(k_B T)^n}{T} k_B T$ $\quad \boxed{T'/T = (h/h')^{1/n}}$

Gen Phys I

(5) (Humensky)



$A = 20cm^2$
$V = ?$
$a = 8cm^2$
$h = 1cm$

Bernoulli eqn: $\frac{1}{2}\rho V^2 = \Delta p + \frac{1}{2}\rho v^2$

Continuity eqn: $AV = av \rightarrow v = \left(\frac{A}{a}\right)V$

Pressure $\Delta p = p_2 - p_1 = -\rho g h$

$\Rightarrow \frac{1}{2}\cancel{\rho} V^2 = -\cancel{\rho} g h + \frac{1}{2}\cancel{\rho}\left(\frac{A}{a}\right)^2 V^2$

$V^2 = \left(\frac{A}{a}\right)^2 V^2 - 2gh$

$2gh = V^2\left[\left(\frac{A}{a}\right)^2 - 1\right]$

$V = \sqrt{\frac{2gh}{\left(\frac{A^2}{a^2}\right)-1}} = \sqrt{\frac{2(9.8\frac{m}{s^2})(0.01m)}{\left(\frac{20}{8}\right)^2 - 1}} = \boxed{0.037\frac{m}{s}}$

Statistical Mechanics Quals Problem
Fall 2013

Robert Mawhinney

A classical gas in three dimensions is constrained by a wall to move in the $x \geq 0$ region of space. The atoms of the gas do not interact. A potential

$$V(x) = \frac{1}{2}\alpha x^2 \qquad x \geq 0$$

attracts the atoms to the wall and the atoms are free to move in an area A in the y and z directions. If the gas is at uniform temperature T show that the number of particles per unit area varies as

$$N(x) = 2N\sqrt{\frac{\alpha\beta}{2\pi}}e^{-\alpha\beta x^2/2}$$

where N is the total number of particles per unit area. You may need to recall that

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

In the usual derivation of the ideal gas law, one uses Newton's laws and the virial theorem to show that, for each particle, -$\langle \vec{q}_i \cdot \vec{F}_i \rangle = 3K_\mathrm{B}T$, where the angle brackets denote the time average. Then $-\langle \sum_{k=1}^N \vec{q}_k \cdot \vec{F}_k \rangle$ is evaluated and shown to be PV. Repeat this derivation in a slab of thickness dx and area A but in the presence of the potential given above. Show that the ideal gas law is still locally obeyed and that the pressure is given by

$$P(x) = Ce^{-\alpha\beta x^2/2}$$

Find C.

Solution

The partition function for the canonical ensemble imediately gives the spatial distribution as

$$N(x) \propto e^{-\alpha\beta x^2/2} \tag{1}$$

To get the normalization, we just evaluate $\int_0^\infty dx$. This gives

$$\frac{1}{2}\sqrt{\frac{2\pi}{\alpha\beta}} \tag{2}$$

The integral must evalute to N for N particles, so we have

$$N(x) = 2N\sqrt{\frac{\alpha\beta}{2\pi}}e^{-\alpha\beta x^2/2} \tag{3}$$

We now need

$$-\langle\sum_{k=1}^{N} \vec{q}_k \cdot \vec{F}_k\rangle \tag{4}$$

for a slab of area A and thickness dx. There is a contribution in the interior of the volume due to the external force $F(x) = -\alpha x$. There are also contributions at the boundaries, due to the pressure, which we denote as $P(x)$.

$$-\langle\sum_{k=1}^{N} \vec{q}_k \cdot \vec{F}_k\rangle = -\int dV x F(x) + \int d\vec{A} \cdot \vec{q}\, P(x) \tag{5}$$

$$= -A\int_x^{x+dx} dx(-\alpha x^2)N(x) + 2P(x)Adx + A(x+dx)P(x+dx) - AxP(x) \tag{6}$$

$$= AN(x)\alpha x^2 dx + 3P(x)Adx + Ax\frac{\partial P(x)}{\partial x} \tag{7}$$

The first and third terms in the previous equation cancel if the pressure is given by $N(x)$ times $1/\beta$. Assuming a Gaussian dependence for $P(x)$ we find

$$3k_{\mathrm{B}}TN(x)Adx = 3AdxP(x) \tag{8}$$

$$C = 2Nk_{\mathrm{B}}T\sqrt{\frac{\alpha\beta}{2\pi}} \tag{9}$$