2014 Solution Explanations
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1. Consider a spherical unicellular organism. Oxygen is transported through the cell by diffusion, as governed by the diffusion equation:

$$D\nabla^2 N(\vec{x},t) = \frac{\partial N}{\partial t}$$

where $N(\vec{x},t)$ is the concentration of oxygen (molecules/m$^3$) at any point in space or time and $D$ is a constant.

Assume that oxygen is consumed at a constant uniform rate $\alpha$ [units: molecules/(m$^3$·s)] throughout all parts of the cell's interior. Suppose that the cell is bathed in a fluid with constant oxygen concentration $N_0$.

Calculate the largest possible size of the cell if the steady state oxygen concentration in all parts of the cell must be greater than some level $N_{min}$. You may assume that oxygen transport across the cell's membrane occurs through diffusion at the same speed as diffusion through the interior of the cell.

I think my solution to this is probably way too simple. But that's ok.
First, we need to get our ODE in the correct form based on the question.

Oxygen is consumed at a constant, uniform rate alpha. Note that the units of this are the same as N/t. We're also told this is **steady state**, which means dN/dt = 0.

$$D\nabla^2\left(\frac{d^2N}{dt^2}+\frac{d^2N}{dx^2}\right) = \frac{dN}{dt} = 0 = \nabla^2\left(\frac{d^2N}{dx^2}\right)$$

Our organism is spherical, so we should use spherical coordinates: $D\nabla^2 N(x) = D\nabla^2 N(r)$
We also must account for alpha: $D\nabla^2 N(r) = \alpha$.

One thing I'm unsure about is if alpha Is positive or negative on RHS. This equation is telling us how oxygen is transported THROUGH the cell (from outside?) and alpha is the rate the cell consumes oxygen (also from outside to inside). So, I think they're both positive.

$$\nabla^2 N(r) = \frac{\alpha}{D}$$

This looks like the Poisson equation: $\nabla^2\phi(r) = -\frac{\rho}{\epsilon_0}$ the general solution to which is the Green's function:

$$N(r) = -\frac{1}{4\pi}\int \frac{\alpha(r')d^3r'}{D|r-r'|}$$

I added a negative because the solution to $\nabla^2\phi(r) = -\frac{\rho}{\epsilon_0}$ doesn't have one, and the signs are switched. But the number of molecules/m$^3$ shouldn't be negative so I don't see how this could have a negative. But we're integrating in terms of r' so can we just say r is 0 (as in r at the origin equals 0, which is true) and cancel out the negatives?

$$N(r) = \frac{1}{4\pi}\int \frac{\alpha dV'}{Dr'} = \frac{1}{4\pi}\frac{\alpha}{D}\frac{4\pi r^2}{2}$$

Everything must be greater than $N_{min}$, so can we say:

$$N(r) = N_{min} = \frac{\alpha}{D}\frac{R^2}{2} \rightarrow R = \sqrt{\frac{2DN_{min}}{\alpha}}$$

This doesn't include $N_0$ (I feel like it should… how easily it can maintain Nmin surely depends on the surrounding fluid $N_o$? Or is that irrelevant and it just depends on how the cell can efficiently absorbed oxygen?).

Note from the future: I think maybe No was supposed to come in as an integration constant, which would leave the final answer being $R = \sqrt{\frac{2D(N_{min}-N_0)}{\alpha}}$

2. Order of magnitude estimation: Could a snowflake cooled to 10 microkelvins be lifted with an ordinary permanent magnet, 10 cm across, with a strength of 0.2 T measured at a distance of 10 cm from either pole, acting on the induced nuclear polarization? The proton's magnetic moment is $2.8\mu_N$. Oxygen is a spin-0 nucleus. Ignore any contributions from electrons. Explain fully your reasoning.

What will cause the snowflake to lift? If the magnetic force pointing upwards is greater than the gravitational force pulling the snowflake downwards.

Assume the snowflake is 0.1 g. $F_{grav} = mg = \frac{0.1}{1000}kg * 9.8\frac{m}{s^2} = 9.8x10^{-4}N$

What about $F_{magnet}$? The dipole force on the snowflake $F_{dip} = \nabla(m \cdot B)$, where B is the external field, which is 0.2 T at 10 cm. Let's find m first, the magnetic moment of the snowflake.

The snowflake has a magnetic moment from the protons within it. The magnetic moment of ONE proton is 2.8 uN. How many protons is in our total snowflake?

$$N = 6.022 \times 10^{23}\frac{mol}{g} \cdot \frac{1}{\frac{18g}{mol}} * 0.1g = 3.34 \times 10^{21}$$

But we're told that oxygen is spin 0, which means it doesn't contribute to the magnetic moment (I wouldn't have known this even though it seems obvious now). So only 2/18ths of these contribute:

$$N = \frac{2}{18} * \ 3.34 \times 10^{21} \ = 3.7 \times 10^{20} \rightarrow m_{tot} = 2.8\mu N * \ N = 5.2 \ \times 10^{-6}\, J/T$$

Now we have m, so we just need B. We have B, which is 0.2T at 10cm. However, the force is a gradient… so we need to find how B varies with location, aka we need a formula for B. Do this by modelling the magnet as a dipole.

$$B_{dipole} = \frac{\mu_0}{4\pi r^3}(3(m \cdot \hat{r})\,\hat{r} - m) = \frac{\mu_0}{4\pi r^3}(2M)$$

Assume r is in the same direction as m at 10 cm, because magnetic moment is from the south pole to north pole of a bar magnet (so along y axis).

We can find M by plugging in B at r = 0.1 cm

$$M = \frac{B4\pi r^3}{2\mu_0} = \frac{0.2\,T \cdot 4\pi(0.1m)^3}{2\mu_0} = 1000\, J/T$$

And $F_{dip} = \nabla\left(m \cdot \frac{2M\mu_0}{4\pi r^3}\right) = \frac{2Mm\mu_0}{4\pi}\ (-\frac{3}{r^4})$

I'm assuming r is 10 cm again? That is how far the magnetic is from the dipole?

Here, $F_{dip} = 3.12\text{x}10^{-5}$ N. This is LESS than $F_{grav}$. Which means the snowflake cannot be picked up! However, this very much so depends on what we assume for the mass of the snowflake. Also, I just ignored its temperature… should that be important? I don't think so.

3. A spin is initially prepared in the $|s = 1/2, s_z = +1/2\rangle$ state. A periodically reversing magnetic field with period $T$ is applied along the $x$ direction. For each period, $B = B_0$ when $T/2 > t > 0$ and $B = -B_0$ when $T > t > T/2$. Find $\langle S_y \rangle$ and $\langle S_x \rangle$ at $t = (n + 1/2)T$ and at $t = nT$, where $n$ is an integer.

The spin is initially in the state $\psi(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (spin up in z-direction). A B field is applied in the x-direction. The Hamiltonian is: $H = -\gamma S \cdot B$. Since B is in the x-direction, only $S_x$ survives.

$$H = -\gamma S_x \cdot B = \mp\gamma B_0 \frac{h}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Where the negative is for when $T/2 > t > 0$ $(B = B_0)$ and the positive is for when $T > t > \frac{T}{2}(B = -B_0)$

Find the eigenvalues and vectors: find det(Sx – λI) as usual. This gives $\mp\gamma B_0\frac{h}{2}$ as the eigenvalues, and $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ \pm 1\end{bmatrix}$ as the vectors for T/2 > t > 0. When T > t > T/2, the sign in front of the eigenvalues switch.

I'm going into more detail than normal just to remind myself. Solve Schrödinger:

$$ih\frac{d\chi}{dt} = H\chi \rightarrow ih\frac{d\chi}{\chi} = Hdt \rightarrow ih\int\frac{d\chi}{\chi} = \int Hdt \rightarrow ln\chi + \chi_0 = Ht + t_0$$

$\chi_{\pm}(t) = e^{t_0 - x_0}e^{H_{\pm}t} = a_{\pm}e^{H_{\pm}t}$

(should be a plus or minus everywhere). Switching to psi and putting it all together:
(Also I dropped a gamma, that should be in here)

$\psi(t) = \frac{a}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}e^{-\frac{iB_0}{2}t} + \frac{b}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}e^{\frac{iB_0}{2}t}$ T/2 > t > 0
$\psi(t) = \frac{a}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}e^{\frac{iB_0}{2}t} + \frac{b}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}e^{\frac{-iB_0}{2}t}$ T > t > T/2

Use initial conditions to find constants (also knowing that |a|^2 + |b|^2 = 1)

$$\psi(0) = \begin{bmatrix}1\\0\end{bmatrix} = \frac{a}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}e^0 + \frac{b}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}e^0$$

This gives $a = b = \frac{1}{\sqrt{2}}$. So now there is a ½ in front of everything, not a $\frac{1}{\sqrt{2}}$.

Can we write this as one equation? Can we use the fact that our values only change by a negative sign?

For 0 < t < T/2:

$\langle S_x\rangle = \langle\psi|S_x|\psi\rangle = (\frac{1}{\sqrt{2}}\chi_+e^{\frac{iB_0}{2}t} + \frac{1}{\sqrt{2}}\chi_-e^{\frac{-iB_0}{2}t})\frac{h}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}(\frac{1}{\sqrt{2}}\chi_+e^{\frac{-iB_0}{2}t} + \frac{1}{\sqrt{2}}\chi_-e^{\frac{+iB_0}{2}t})$
$= (\frac{1}{\sqrt{2}}\chi_+e^{\frac{iB_0}{2}t} + \frac{1}{\sqrt{2}}\chi_-e^{\frac{-iB_0}{2}t})\frac{h}{2}(\frac{1}{\sqrt{2}}\chi_+e^{\frac{-iB_0}{2}t} - \frac{1}{\sqrt{2}}\chi_-e^{\frac{+iB_0}{2}t})$
$= \frac{h}{4}(\chi_+e^{\frac{iB_0}{2}t} + \chi_-e^{\frac{-iB_0}{2}t})\left(\chi_+e^{\frac{-iB_0}{2}t} - \chi_-e^{\frac{+iB_0}{2}t}\right) = \frac{h}{4}\Big(\chi_+^2 - \chi_-^2 + \chi_+\chi_1 e^{\frac{+iB_0}{2}t}e^{\frac{+iB_0}{2}t} -$
$\chi_+\chi_1 e^{\frac{+iB_0}{2}t}e^{\frac{+iB_0}{2}t}\Big) = 0.$

Note from future: would have been way easier to first put the equations in terms of sins and cosines, like this:

$$\psi(t) = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{-\frac{iB_0}{2}t} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}e^{\frac{iB_0}{2}t} = \begin{bmatrix}\cos\left(\frac{\gamma B_0}{2}t\right)\\ isin\left(\frac{\gamma B_0}{2}t\right)\end{bmatrix}$$

This is the same as for when B = -$B_0$. Sy?

$$\langle S_y\rangle = \langle\psi|S_y|\psi\rangle = \left(\frac{1}{\sqrt{2}}\chi_+ e^{\frac{iB_0}{2}t} + \frac{1}{\sqrt{2}}\chi_- e^{\frac{-iB_0}{2}t}\right)\frac{h}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\left(\frac{1}{\sqrt{2}}\chi_+ e^{\frac{-iB_0}{2}t} + \frac{1}{\sqrt{2}}\chi_- e^{\frac{+iB_0}{2}t}\right)$$
$$= \left(\frac{1}{\sqrt{2}}\chi_+ e^{\frac{iB_0}{2}t} + \frac{1}{\sqrt{2}}\chi_- e^{\frac{-iB_0}{2}t}\right)\frac{h}{2}\left(\frac{1}{\sqrt{2}}(-i)\,\chi_- e^{\frac{-iB_0}{2}t} + \frac{i}{\sqrt{2}}\chi_+ e^{\frac{+iB_0}{2}t}\right)$$
$$= \frac{ih}{4}\left(\chi_+ e^{\frac{iB_0}{2}t} + \chi_- e^{\frac{-iB_0}{2}t}\right)\left(-\chi_- e^{\frac{-iB_0}{2}t} + \chi_+ e^{\frac{+iB_0}{2}t}\right) = \frac{ih}{4}\left(\chi_+^2 e^{\frac{iB_0}{2}t} - \chi_-^2 e^{\frac{-iB_0}{2}t} + \chi_+\chi_1 - \chi_+\chi_1\right)$$

$$= \frac{ih}{4}\left(e^{\frac{iB_0}{2}t} - e^{\frac{-iB_0}{2}t}\right) = \frac{2\mathrm{i}^2\mathrm{h}}{4}\sin\frac{B_0 t}{2} = -\frac{h}{4}\sin\frac{B_0 t}{2}$$

What about the differences in t = nT and t = (n+1/2)T?

$$\langle S_y\rangle = -\frac{h}{4}\sin\frac{\gamma B_0 nT}{2};\ -\frac{h}{4}\sin\frac{\gamma B_0(n+\frac{1}{2})T}{2}$$

It's periodic… so when n = 0, <Sy> = 0. Therefore <Sy> should equal 0 for all nT, because n is an integer. I'm not 100% sure about when t = n+1/2T, though... I think it would not equal 0 then, maybe?

Same for <Sx>, <Sx> = 0 always. (The true reason for this is because our spin is precessing around the x-axis, always in the yz plane).

4. A thin door of height $H$ and width $D$ is suspended on its hinges whose axis $\hat{z}'$ is inclined to the vertical $\hat{z}$ by an angle $\theta$. What is the period of (small) oscillations of this door as it flaps back and forth around axis $\hat{z}'$?

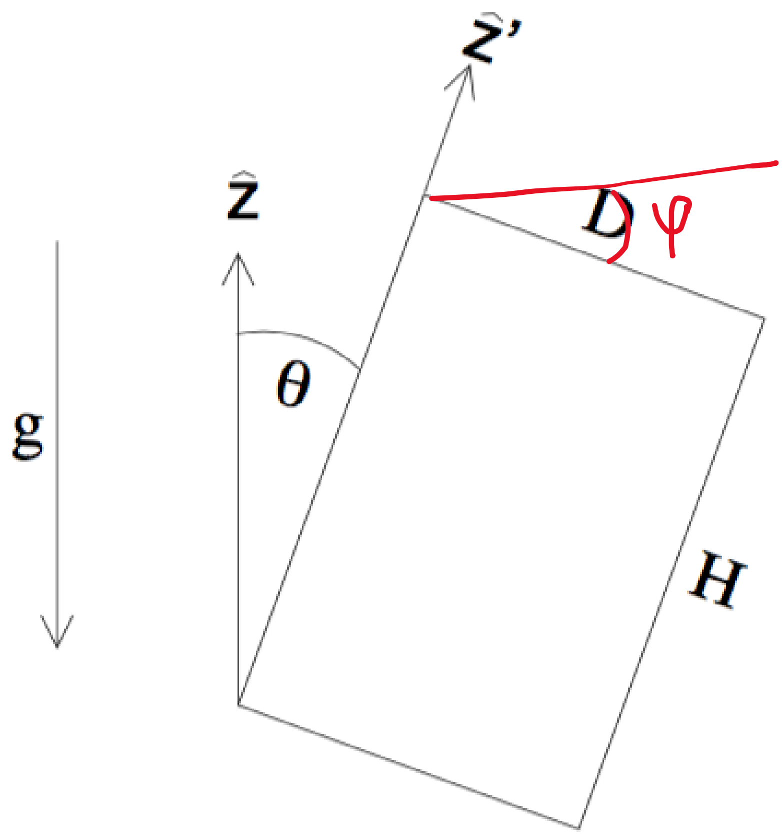



Figure 1: Swinging door tilted at an angle.

The hardest part about this question is the set up. Whenever it asks for the "period of small oscillations" you know you need to find the equation of motion. Let's use the Lagrangian method.

$$\mathcal{L} = T - V; T = \frac{1}{2}mv^2; V = mgh$$

We need to change the form of these though. Note that theta is a CONSTANT, and we care about the motion of the door flapping about z'. Define a new angle phi like I drew on the image. We want to know the period of oscillations of phi.

Kinetic Energy

Let's use the form of kinetic energy T= $\frac{1}{2}I\dot{\phi}^2$. Now we need to find the moment of inertia of the door. Let's find the moment of inertia along the bottom of the door (D).

$$I = \int r^2 dm = \int \rho r^2 dV = \rho \int_0^t dt \int_0^H dH \int_0^D r^2 dr = \frac{\rho t H D^3}{3} = \frac{\frac{M}{tHD} tHD^3}{3} = \frac{MD^2}{3}$$

Not sure if it's cool that I randomly introduced thickness. Anyway T= $\frac{1}{6}MD^2\dot{\phi}^2$.

Potential Energy

$V = mgh$, where h is along the z xaxis. So, we need to write h in terms of variables we care about. This part really confuses me. One way is:

$$z = x' sin\theta; x' = xcos\phi \rightarrow z = xcos\phi sin\theta \ldots z = Dcos\phi sin\theta?$$

And then

$$V = -mgDcos\phi sin\theta?$$

(NOTE FROM FUTURE: This should be H, not D, no? Since potential energy depends on the height?)

This feels very wrong. But I proceed.

$$\mathcal{L} = T - V = \frac{1}{6}MD^2\dot{\phi}^2 + mgDcos\phi sin\theta$$

Euler Lagrange equations for phi:

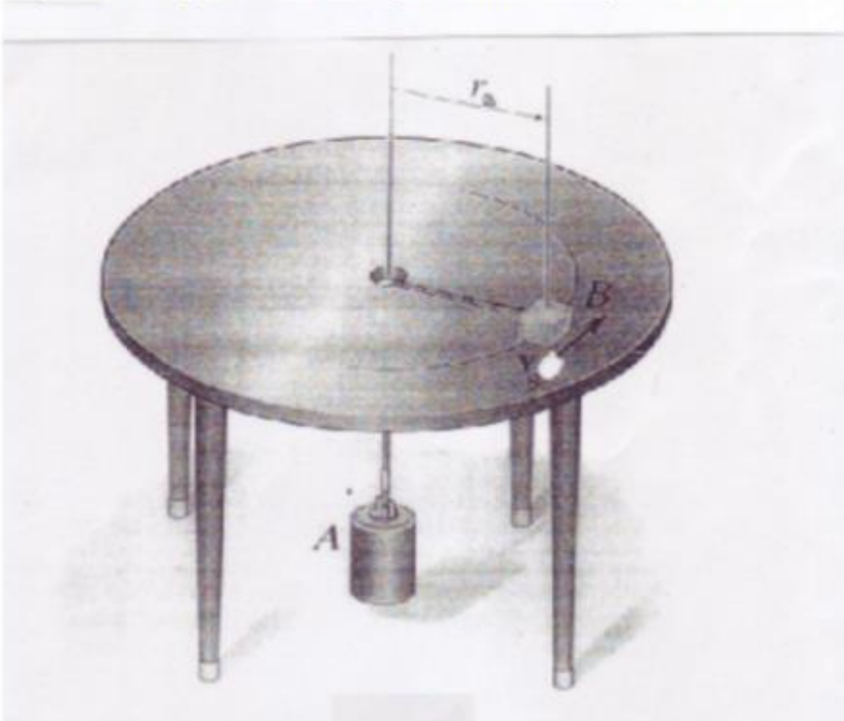
$$\frac{d}{dt}\left(\frac{d\mathcal{L}}{d\dot{\phi}}\right) = \frac{d\mathcal{L}}{d\phi} = \frac{d}{dt}\left(\frac{1}{3}MD^2\dot{\phi}\right) = -mgDsin\theta sin\phi = \frac{1}{3}MD^2\ddot{\phi}$$

"small oscillations" let sinphi = phi.

$$-\frac{mgDsin\theta\phi}{\frac{1}{3}MD^2} = \dddot{\phi}$$

This is a sinusoidal equation with a frequency $\omega = \sqrt{\frac{3gsin\theta}{D}}$ and therefore a period $T = 2\pi\sqrt{\frac{D}{3gsin\theta}}$. Units work out! But I feel weird about how I set up the potential energy equation.

5. A weight $B$ with mass $m_B$ lies on a frictionless table as depicted in the diagram below. It is connected to a cord which is threaded through a hole in the centre of the table. The weight $B$ is initially set in circular motion with an initial velocity $v_B$ and at radius $r_B$. The other end of the cord is attached to a weight $A$ of mass $m_A$ which is initially at rest a height $h$ above the floor. Then the weight is released from rest, and it will exert a downward force on the cord. Describe the subsequent motion of the weight $A$. Find an equation for the parameter regime where the weight $A$ will eventually hit the floor.




We want to describe the motion of the weight A, so this is a typical equation of motion question. The hardest part is the setup. Let's find the Lagrangian

The total length of the string is $L = r_A + r_B$.

The kinetic energy depends on both A and B. It is simply for mass A: $T_A = \frac{1}{2} m_A \dot{x_A}^2$
We want this in terms of our x defined earlier:

$$T_A = \frac{1}{2} m_A \dot{x_A}^2 = \frac{1}{2} m_A \left( \frac{d}{dt}(L - x) \right)^2 = \frac{1}{2} m_A \dot{x}^2$$

As the string is moving, mass B has a linear and a rotational component.

$$T_A = \frac{1}{2} m_B \dot{x}^2 + \frac{1}{2} m_B v_{rot}^2 = \frac{1}{2} m_B \dot{x}^2 + \frac{1}{2} m_B x^2 \dot{\theta}^2$$

This is why we chose x in reference

The potential energy only depends on weight A, so $V = -m_a g(L - x)$

Our Lagrangian is then

$$\mathcal{L} = \frac{1}{2}m_B\dot{x}^2 + \frac{1}{2}m_Bx^2\dot{\theta}^2 + \frac{1}{2}m_A\dot{x}^2 + m_Ag(L-x)$$

Euler Lagrange:

$$\frac{d}{dt}\left(m_Bx^2\dot{\theta}\right) = 0$$

Angular momentum is conserved! Because torque is zero.

$$\left(m_Bx^2\dot{\theta}\right) = a_1 \rightarrow \dot{\theta} = \frac{a_1}{m_Bx^2}$$

$$(m_B + m_A)\ddot{x} = m_Bx\dot{\theta}^2 - m_Ag = m_Bx\left(\frac{a_1}{m_Bx^2}\right)^2 - m_Ag = \frac{a_1^2}{m_Bx^3} - m_Ag$$

We can't really solve this. I won't try.

But for the constant, because a is released from rest, x''(0) = 0, and x(0) = $r_B$. (Because of how we defined x)

$$0 = \frac{a_1^2}{m_Br_B^3} + m_Ag \rightarrow a_1^2 = m_Agm_Br_B^3$$

$$\boxed{(m_B + m_A)\ddot{x} = \frac{m_Agr_B^3}{x^3} - m_Ag}$$

When A will eventually hit the floor, x'(t)=x''(t)=0, and x(t) = $r_B$-h?

$$\boxed{(m_B + m_A)(0) = \frac{m_Agr_B^3}{(r_B-h)^3} - m_Ag}$$

**Pause:** Would it actually make more sense to define two variables for the length of the string, x and r? instead of having everything in terms of x? Then we could find the motion of A independent of x? That would look like:

$$\mathcal{L} = \frac{1}{2}m_B\dot{r}^2 + \frac{1}{2}m_Br^2\dot{\theta}^2 + \frac{1}{2}m_A\dot{x}^2 + m_Agx$$

Where now x is the height of the mass A. Euler Lagrange (x):

$$\frac{2}{2}m_A\ddot{x} = m_Ag \rightarrow \ddot{x} = g \rightarrow x(t) = \frac{g}{2}t^2 + a_1$$

$$m_B\ddot{r} = m_B r\dot{\theta}^2 \rightarrow \ddot{r} = r\dot{\theta}^2$$

$$\frac{d}{dt}\left(m_B r^2 \dot{\theta}\right) = 0 \rightarrow m_B r^2\dot{\theta} = a_2 \rightarrow \dot{\theta}^2 = \left(\frac{a_2}{m_B r^2}\right)^2$$

$\dot{\theta}$ is conserved. Plug theta dot into r equation.

$$\ddot{r} = r\dot{\theta}^2 = r\left(\frac{a_2}{m_B r^2}\right)^2 = \frac{a_2^2}{m_B^2 r^3}$$

We have the constraint $x + r = L \rightarrow \ddot{x} + \ddot{r} = 0$

Note: I don't like this method as much, ignore.

6. As a first approximation, the neutron-proton interaction can be described by the attractive square-well potential

$$V(r) = -V_0\theta(a - r)$$

where $\theta(x) = 1$ if $x \geq 0$ and $\theta(x) = 0$ if $x < 0$. This potential has two parameters, the depth of the well $V_0$ and the range of the interaction $a$. The range is of order the pion Compton wavelength $a \approx \lambda_\pi = \hbar/(m_\pi c)$ with $m_\pi c^2 =$ 139 MeV and $\hbar c = 197$ MeV·fm . Estimate $V_0$ using the fact that the neutron and proton have only one bound state, the deuteron, with binding energy of $-2.22MeV$. Use a proton mass equal to a neutron mass, $m_N = m_P = 940$ MeV.

We are looking at the neutron proton interaction inside an atom in their bound state, the deuteron. We are describing it by the square well potential:

$$\begin{aligned} V(r) &= -V_0 \quad a - r > 0 \rightarrow r < a \\ V(r) &= 0 \qquad a - r < 0 \rightarrow r > a \end{aligned}$$

(because $x = a - r$)

This is an atom (spherical) so use the radial wavefunction.

We know the energy of the bound state, so we simply need to find the energy in terms of V0 and then we can solve for V0. Solve the Schrödinger equation.

Bound state: E < 0.
First, for r <a:

$$\left(\frac{-h^2}{2m}\frac{d^2u}{dr^2} + \frac{h^2 l(l+1)}{2mr^2} - V_0\right)u = Eu$$

We already know the potential does not have the l term in it (that is part of the effective potential) so we can say l = 0.

$$\left(\frac{-h^2}{2m}\frac{d^2u}{dr^2} - V_0\right)u = Eu \rightarrow \frac{d^2u}{dr^2} = -\frac{2m}{h^2}(E + V_0)u = -l^2u$$

The solution is

$$u(r) = Asin(lr) + Bcos(lr)$$

(Note: a lot of this is unnecessary, but I'm writing it all out for my own benefit).
Should there be a condition for r = 0? Should u = 0 at r = 0? $u(0) = B = 0$. This would make sense for symmetry reasons. In that case only the sin term lives.
Now, r > a, V = 0:

$$\frac{d^2u}{dr^2} = -\frac{2m}{h^2}Eu = k^2u$$

We take the negative inside the k because this is a bound state and E < 0, otherwise there would be a negative in the square root. The solution is

$$u(r) = Ce^{kr} + De^{-kr}$$

When r is infinite, the +kr term blows up so C = 0.

To find the energies we need to look at the BCs. The wavefunction and the derivative of the wavefunction should be continuous at r = a. Note, we can either do this to find the constants or to find k or l for the energies. In this case it's the latter.

$$u(a) = Asin(la) = De^{-ka}$$
$$\frac{du(a)}{dr} = Alcos(la) = -Dke^{-ka}$$

Divide the equations by each other.

$$\frac{Asin(la)}{Alcos(la)} = \frac{De^{-ka}}{-Dke^{-ka}}$$
$$\frac{\tan(la)}{l} = -\frac{1}{k} \rightarrow l = -ktan(la)$$

Energy looks like:

For mass – we should use the reduced mass. I totally would have thought it's just the mass of a proton + neutron, but we're considering this as one state.

$$m = \frac{m_N m_p}{m_N + m_p} = \frac{m_{n^2}}{2m_n} = 470\ MeV$$

$$E + V_0 = \frac{n^2\pi^2 h^2}{2m(2a)^2} \rightarrow V_0 = \frac{\pi^2 n^2 h^2}{2m(2a)^2} - E_0$$
$$= \frac{\pi^2(1)(6.58 \times 10^{-22} Mevs)^2}{2 * 470\frac{MeV}{c^2} * \left(2 * \frac{197}{139} \times 10^{-15} m\right)^2} + 2.22\ MeV$$

$$V_0 = 51.5\ MeV + 2.22\ MeV = 53.7\ MeV$$

7. A heat pipe is a tube containing a small amount of a liquid and the balance of the volume filled with the liquid's vapor, and is used to transfer heat efficiently in confined spaces. For example, heat pipes are often used to cool the processor in laptops. In this problem, a water-based heat pipe is 5 mm inner diameter and 30 cm long, and held in a vertical orientation so most of the liquid is at the bottom of the tube. Heat can be transferred very rapidly along the heat pipe if the liquid end is heated and the other end cooled, as liquid from the hot end evaporates, then recondenses when it reaches the other end. Provide an estimate of the maximum rate of heat transfer by such a heat pipe, including carefully spelling out assumptions you are making. Try to provide numerical estimates for as many parameters as possible that go into your calculations.

8. The outer 100 km of the earth is nearly solid rock (the lithosphere). The inner boundary of the lithosphere is defined by the depth where the temperature becomes so hot (around 2000°C) that rock begins to flow. Within this inner boundary, liquid rock can flow, transfering heat by convection so temperature gradients are much smaller. (The core temperature reaches 5000-7000K.) An order of magnitude estimate for thermal conductivity in rock is 1 W/m/K. The primary source of heat in the interior of the earth is believed to be radioactive decay of various elements, especially $^{232}$Th and $^{238}$U. Estimate:

A. by how much the outer surface temperature of the earth is raised due to this inner heat

B. roughly how much heat is generated per cubic meter within the earth.

A.
There are two ways:
1. Determine how much of this heat makes it to the surface (heat transfer from core to surface? Or heat transfer through rock from bottom of lithosphere at 2000 deg to surface?)

$$\frac{dQ}{dt} = -\frac{kA(T_2 - T_1)}{L}$$

We have k, T1, L, and A. But we don't have dQ/dt.

2. Determine how hot the earth is without the core, and then the difference must be from the core. This is probably a very crude estimate, because it ignores heat greenhouse gases and

atmospheric effects. But I'm not sure how to do it otherwise.

For this method, we need to determine the amount the temperature is raised by the Sun.

Using the Stefan-Boltzmann Law:

$$P_{sun} = 4\pi r_{sun}^2 \sigma T_{sun}^4$$

This amount obviously doesn't all reach the earth. We need to scale by the distance to the earth (this gives the flux, the P/area)

$$f_{earth} = \frac{4\pi r_{sun}^2 \sigma T_{sun}^4}{4\pi d^2}$$

Where d is 1 AU. What if we're not given this number? We can use the known fact that it takes ~ 8 minutes for light to reach the earth?

$$d = v * t = 3 \times \frac{10^8 m}{s} \times 8m \times \frac{60s}{m} = 1.44 \times 10^{11} m$$

Good enough.

This is the flux, but we want the power. We need to multiple by the area, but the earth looks like a disk from the suns perspective so we drop the factor of 4. Also note:

$$P_{abs} = (1-a)P_{recieved}$$

Where a is the albedo of earth, which is 0.3 (should I memorize this)? This means 30% of the sun's energy is reflected.

$$P_{earth} = \frac{(1-a)\pi r_{earth}^2 r_{sun}^2 \sigma T_{sun}^4}{4\pi} = 4\pi r_{earth}^2 \sigma T_{earth}^4$$

Isolate $T_{earth}$. This gives $T_{earth}$ = 254 K!

The average temp on earth is ~ 14 degrees, which is 287 K. 287 K – 254 K = 33 K. So, the core of the earth heats up the surface by 33 K, which is -260 degrees Celsius (lol).

B. Again, this makes me think we are supposed to calculate the decay energies of Th and U and determine how much heat this would generate. This would require a lot of assumptions, because we don't know how much of these elements are within the earth.

But we can do a different way: We know the heat transfer through the lithosphere is generated from the heat in the core. The total heat transfer through the lithosphere must equal the total heat transfer through the rest of the earth, or else the heat would not be conserved.

$$\frac{dQ}{dt} = q = -\frac{kA(T_2 - T_1)}{l} = -\frac{kA(33K - 2273K)}{100e3\ m}$$

The area of the lithosphere is $A_{lith} = A_{earth} - A_{belowlith} = 4\pi(6.4e6m)^2 - 4\pi(6.4e6m - 100e3m)^2 = 1.5e13\ m^2$

$$q_{lithosphere} = -1\frac{W}{mK} \times 1.5e13m^2 \times \left(-\frac{2240K}{100e3\ m}\right) = 3.36e11\ W$$

$$\frac{q_{earth}}{m^2} = \frac{q_{lithosphere}}{4\pi r_{earth}^2} = \frac{0.65mW}{m^2}$$

So, the heat transfer per cubic area is 0.65mW/m$^2$.

9. If you rub a balloon against your hair and then bring it close to a stream of water flowing from a faucet, you will observe the water to bend towards the balloon (as in the figure).

Figure 2: A stream of water bending towards a balloon.

**(a)** Why does this occur? Describe how the electrical charges are distributed on the balloon and in the water in order to produce the attractive force observed.

**(b)** In this situation, why does the water bend toward and not away from the balloon? Under what condition(s) could the water be made to bend away from the balloon?

Ah, classic question.

a) By rubbing the balloon against my hair, it causes the balloon to pick up some charge. Either the balloon collects some electrons from my hair, or my hair collects some electrons from the balloon. Let's assume the balloon picked up a negative charge it is now net negative, and hence has an electric field.

Water has no net charge. Objects that are either positively or negatively charged attract neutral objects because of polarization. In the case of water molecules, they are a bit more special because they are already polar: there is more negative charge near the oxygen atom and more positive charge near the hydrogen atoms (but net neutral).

When the negative balloon comes near the water molecules, they will orient themselves so negative oxygen atoms are farther from the balloon, and the hydrogens are closer. Because unlike charges attract and like charges repel, the water will be drawn to the balloon. There also may an induced dipole moment within the atoms: the electrons moving away from the balloon and the protons closer.

b). This is because the water is polar which allows it to be attracted to the balloon. It could bend away from the balloon if there were negative ions dissolved in it which caused it to be net negative, in which case it would be repelled from the also negative balloon.

10. A electric eel of length 2 m generates a voltage difference between its head and tail using 5000 electrocyte cells. These cells are connected in series, and each produces a potential difference of 140 mV during a shock, as well as having an internal resistance. When measured in air (out of water!), the open circuit potential difference between the head and tail is measured to be 600 V. Note that some "leakage current" flows back through the skin of the eel. If the tail is then "shorted" to the head through an ammeter, the measured current is 1 amp.

A. Draw a Thevenin equivalent circuit for the eel, giving numerical values for all of its components. (Hint: A Thevenin equivalent circuit consists only of resistors and ideal voltage sources.)

B. The eel is released into freshwater with a resistivity of 200 $\Omega \cdot m$. Do an order of magnitude estimate of the electrical resistance to the current flowing through the water, and use this to calculate the current pushed by the eel through the water.

C. An electric ray lives in saltwater, which has a resistivity of 0.23 $\Omega \cdot m$. From the standpoint of delivering the maximum power to the water, explain why the electric ray will produce a lower voltage (around 50 V) but higher current than the eel. (Note: the ray has the same number of electrocytes as an eel, but places some of them in parallel rather than all in a series.)

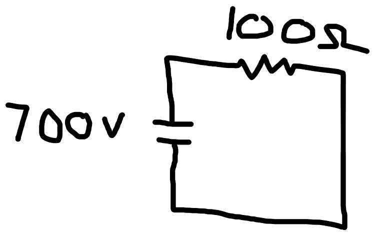
A. Our circuit consists only of a resistor and battery (no capacitor). So we need to find the battery voltage and the internal resistance. There are 5000 cells connected in series, each having voltage 140 mV. In series, the total voltage is the sum of all of the voltage components:

$$V_{tot} = \varepsilon = 0.14\text{V} \times 5000 = 700\text{V}$$

There is an internal resistance, r, which is easy to calculate – we know the measured (effective) voltage is 600V when the current is 1 A, so there is a voltage difference of 100V at 1A.

$$V_{eff} = V_{batt} - Ir \rightarrow r = (V_{batt} - V_{eff})/I = 100V/1A = 100\Omega$$

Circuit kind of looks like this:



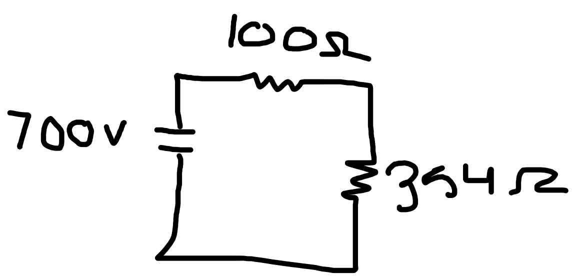

B. The resistivity is a property of a material that quantifies how strongly it resists electric current. It also depends on the size of the material.

$$\rho = \frac{AR}{l}\,[\Omega \cdot m] \rightarrow R = \frac{\rho l}{A}$$

We are determining the resistance R to the current flowing through the water. The eel is displacing a eel-sized chunk of water, so we can use the eel values for l and A. Assuming the eel is ~ 30cm. The area of water the eel is pushing on is basically a circle the size of the eel head, so A = 4pi($r_{head}$)$^2$.

$$R = \frac{200\Omega\mathrm{m} * 2m}{4\pi 0.3m^2} = 354\Omega$$

Now, this creates a circuit like




Is this correct? The eel and the internal resistance are like properties of the emf, and then the water resistance is like the real resistor in the circuit.

In series,

$$R_{equiv} = \sum R_i = 454\Omega \rightarrow \mathrm{I} = \frac{\mathrm{V}}{\mathrm{R}} = \frac{700}{454} = 1.54A$$

C. The power is $P = VI = I^2V = \frac{V^2}{R}$.

Just to encompass everything, note the Power can be broken up into the power dissipated to heat due to the internal resistor, and the power output.

$$P_{tot} = VI = P_{lost} + P_{output} = I^2 r + I^2 R$$

The resistivity of salt-water is lower, so assuming the same size electric ray, R will be lower. (R = 0.4 ohms) Power is proportional to 1/R, so a lower resistance means greater power. Because some parts of the circuit are in parallel, R will be lower as it is, because $1/R_{equiv} = \sum \frac{1}{R_i}$, so the equivalent resistance is now 0.4 ohms.

Therefore, the voltage could be lower anyway to have higher power output. However, now that some of the electrocytes are in parallel, the total voltage is equal to the voltage across the individual electrocytes (whereas before total voltage was the sum). This is the opposite for current – in series, the total current is equal to the current through each component, but in parallel the current is the sum through each component. So the voltage can be lower and current higher, and higher output.

11. Rotating liquid mirror

A. A rotating container of radius $R$ holds reflective liquid mercury ($\rho = 13.6\ \mathrm{g/cm^3}$) at its bottom. In its steady state, the container and the mercury all rotate at the same angular velocity $\omega$. and the surface of the mercury acquires a concave curvature. By considering how the centripetal acceleration of a small volume of fluid at the surface is provided by the radial component, calculate the height of the fluid's surface as a function of the distance from the axis of rotation.

B. This curved surface acts as a mirror. If the cylinder has a diameter of 2 m and rotates at 20 rotations per minute, determine the focal length of this mirror (which has its axis in the vertical direction).

A. The rotating mercury is in its steady state; none of the particles are moving with time in the radial direction or with height. They are moving with angular velocity omega perpendicular to the radial direction, though. The fact that they're not moving means the surface must be an equipotential (aka constant everywhere) – if it was not, the particles would attempt to move to a region of lower potential.

$$F_{eff} = F_g + F_{cent} = -mg + \frac{mv^2}{r} = -mg + m\omega^2 r; \omega = \frac{v}{r}$$

Therefore, the potential is

$$V_{eff} = -\frac{dF}{dx} = +mgz - \frac{m\omega^2 r^2}{2} = constant = A$$

$$\boxed{z = \frac{\omega^2 r^2}{2g} + A}$$

B.

$$\omega = 2\pi f = 2\pi\left(\frac{20}{60}\right) = \frac{2.1}{s}$$

I know the focal length is $f_l = \frac{g}{2\omega^2}$, but I'm not sure where the constant comes from. To get half way there, we can say the focal length depends on the height z and the radius r: $f_l = kr^a z^b$, and increasing the height causes the focal length to lower, while increasing the radius causes it to increase. So we can assume $f_l = kr^2 z^{-1} = \frac{kr^2}{z} = \frac{k2gz}{\omega^2 z} = \frac{2kg}{\omega^2}$, but I don't know where to get the k = ¼ from.

12. Find the distribution of the total momentum $\vec{P}$ of $N$ identical free particles at temperature $T$. Calculate the most probable speed of the centre of mass of the collection. Hint: The velocity distribution for each particle follows the Maxwell distribution.

13. On July 16, 1945, the United States detonated the first atomic bomb in a test code-named Trinity. British physicist G.I. Taylor estimated the yield of the nuclear explosion from a photograph of the fireball using simple dimensional analysis. He reasoned that the radius $R$ of the blast should initially depend only on the energy $E$ of the explosion, the time $t$ after the detonation, the density $\rho = 10^{-3}$ g/cm$^3$ of the air, and a dimensionless constant, $S$, related to the ratio of specific heats of air.

Find an expression for $R(t) = Sf(E, \rho, t)$. Given that $S$ is of order unity, find the approximate energy, $E$, produced in the explosion. Express your result in tons of TNT where 1 ton TNT $= 4.2 \times 10^9$J.

This is simple dimensional analysis.

$$R = Sf(E, \rho, t) = SE^a \rho^b t^c \approx [kg\ m^2 s^{-2}]^a [kgm^{-3}]^b [s]^c$$

Intuitively, we know the energy and time should increase the radius; the density of surrounding air should decrease it. So the density of air is in the denominator. Let a=1, b = -1, and then c=2 to cancel out the seconds.

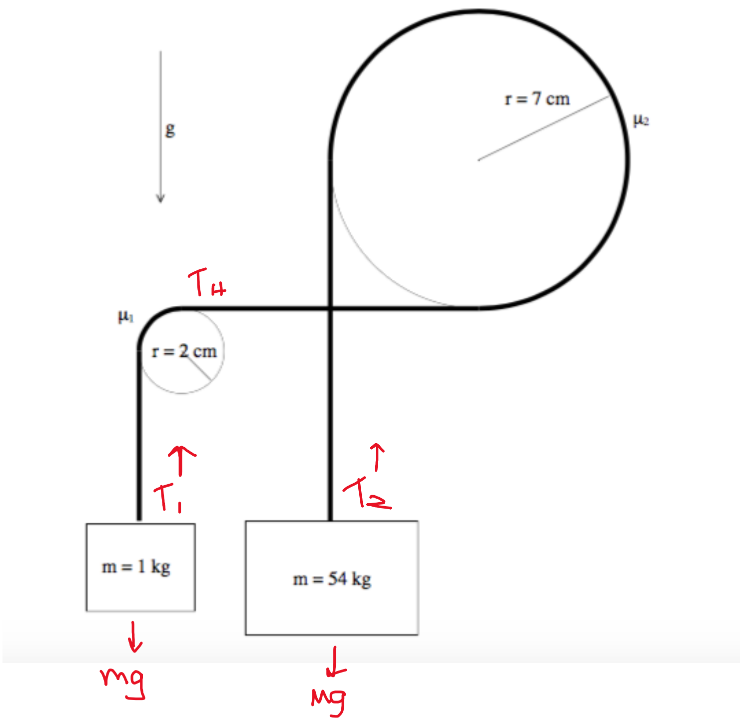
$$R = \frac{[kg\ m^2 s^{-2}][s]^2}{[kgm^{-3}]} = [m^5]$$

Therefore $R = \left(\frac{Et^2}{\rho}\right)^{\frac{1}{5}}$

$$E = \frac{R^5 \rho}{t^2} = \frac{(150m)^5 \left(\frac{1kg}{m^3}\right)}{(0.025s)^2} = 1.2 \times 10^{14} J = 29\ tons\ TNT$$

Where R = 150m was an estimate.
I'm skipping Q 14 because it seems too hard.

15. A rope wraps around two fixed cylinders as shown in the figure. The cylinders are made from two different materials and their coefficients of static friction are $\mu_1$ and $\mu_2$. Cylinder one and two have a radius of 2 cm and 7 cm, respectively. A mass of 54 kg hangs from the end of the rope attached to cylinder 2 and a mass of 1 kg is attached to the other end. Find the conditions on the coefficients of friction $\mu_1$ and $\mu_2$ such that the masses do not slip.

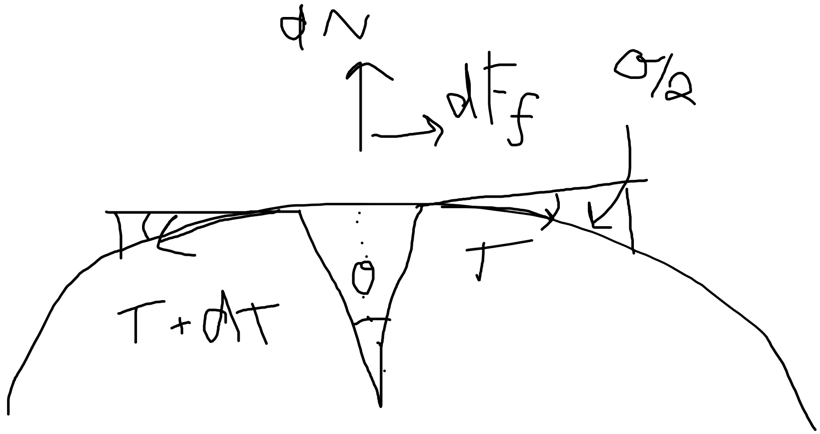



We can use the Capstan equation for this. Unfortunately, I never would have thought to derive it on the spot. Because of the interaction of frictional forces and tension, the tension on a line wrapped around a cylinder may be different on either side.

$$T_L = T_H e^{\mu\theta}$$

Where $T_L$ is the applied tension from the load, $T_H$ is the tension on the other side of the pulley, and $\theta$ is the angle covered by the rope (because this is where there is tension). For cylinder one, $\theta \sim \frac{\pi}{4}$, for cylinder two, $\theta \sim \frac{3\pi}{2}$.

Let's derive first.




See drawing above where we are calculating the forces on a small segment of the cylinder. T is our normal tension in the rope ($T_H$) while on the other side the tension must be higher (this is the side that is holding the load, m or M). Find the forces in the x and y direction as usual. No slipping, so acceleration = 0 .

$$\sum F_X = dF_f + T\cos\left(\frac{d\theta}{2}\right) - (T + dT)\cos\left(d\frac{\theta}{2}\right)$$
$$= \mu dN + T\cos\left(\frac{d\theta}{2}\right) - (T + dT)\cos\left(\frac{d\theta}{2}\right) = 0$$
$$\rightarrow \mu dN = dT\cos\left(\frac{d\theta}{2}\right)$$

$$\sum F_y = dN - T\sin\left(\frac{d\theta}{2}\right) - (T + dT)\sin\left(\frac{d\theta}{2}\right) = 0$$
$$\rightarrow dN = (2T + dT)\sin\left(\frac{d\theta}{2}\right)$$

To get the math worth out in the end, dT needs to go away here. It disappears because we can neglect two differentials that are multiplied (too small), which will happen with the small angle approximation. Now let's set the dN equations equal.

$$dN = \frac{dT}{\mu}\cos\left(\frac{d\theta}{2}\right) = (2T)\sin\left(\frac{d\theta}{2}\right)$$

Small angle approximation

$$\frac{dT}{\mu} = (2T)\frac{d\theta}{2} \rightarrow \int_{T_H}^{T_L}\frac{dT}{T} = \int_0^{\theta}\mu d\theta \rightarrow \ln\left(\frac{T_L}{T_H}\right) = \mu\theta \rightarrow T_L = T_H e^{\mu\theta}$$

Now, back to the problem.
$T_H$ is the same for both pulleys, so let's set them equal in this equation. T1 and T2 are the TL's for each pulley.

$$T_H = \frac{T_1}{e^{\frac{\mu 1 \pi}{4}}} = \frac{T_2}{e^{\frac{3\mu 2\pi}{2}}} \rightarrow \frac{T_1}{T_2} = e^{\pi(\frac{\mu 1}{4} - \frac{3\mu 2}{2})}$$

We can get a condition if we find T1 and T2. From the free body diagrams, and from the fact the masses aren't slipping (so acceleration = 0),

$$F_{net}^1 = T_1 - mg = 0, F_{net}^2 = T_2 - Mg = 0$$

$$\frac{m}{M} = e^{\pi\left(\frac{\mu 1}{4} - \frac{3\mu 2}{2}\right)} \rightarrow \ln\left(\frac{m}{M}\right) = \pi\left(\frac{\mu 1}{4} - \frac{3\mu 2}{2}\right) = \ln\left(\frac{1}{54}\right) = \frac{1}{4}\pi(\mu 1 - 6\mu 2)$$

$$\boxed{\mu_1 = \frac{4}{\pi}\ln\left(\frac{1}{54}\right) + 6\mu_2}$$

16. Electron and $\tau$-neutrino mixing is described by the Hamiltonian

$$H_{\text{mixing}} = -\omega\left[\sin(2\theta)\sigma_3 + \cos(2\theta)\sigma_2\right] + \frac{GN}{\sqrt{2}}\sigma_3$$

where $\sigma_i$ are Pauli matrices, $\omega = (m_\tau^2 - m_e^2)/2E$ with $E$ the relativistic neutrino energy, $\theta$ is the vacuum mixing angle, $G$ is the Fermi decay constant, $N$ is the density of electrons in matter and the electron neutrino is the state $\begin{bmatrix}1\\0\end{bmatrix}$ and the $\tau$ neutrino is the state $\begin{bmatrix}0\\1\end{bmatrix}$.

If the electron density in matter N is constant, at what distance L = ct would all electron neutrinos entering matter be converted to tau neutrinos?

Hint: the Pauli spin matrices are:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Somewhat standard question. We have the Hamiltonian for neutrino mixing, and we want to find the time that all the electron neutrinos are converted to taus. We need to evolve the Hamiltonian with time to see how the state evolves. Then we can find the probability of this state to be [0 1], and set that equal to one.

$$H = -\omega\left[\sin(2\theta)\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + \cos(2\theta)\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\right] + \frac{GN}{\sqrt{2}}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

Let $\sin(2\theta) = \alpha, \cos(2\theta) = \beta, \frac{GN}{\sqrt{2}} = \gamma$

Let $-\omega\alpha + \gamma = d$

$$H = \begin{bmatrix} d & \omega i\beta \\ -i\omega\beta & -d \end{bmatrix}$$

The time evolution operator is $U(t) = e^{-\frac{i}{h}\int H(t)dt}$

Our Hamiltonian is time independent which simplifies things. But we need to find the eigenvectors and values to time evolve our state.

$$det\begin{bmatrix} d-\lambda & \omega i\beta \\ -i\omega\beta & -d-\lambda \end{bmatrix} = (d-\lambda)(-d-\lambda) - \omega^2\beta^2 = 0 = -d^2 - d\lambda + \lambda d + \lambda^2 - \omega^2\beta^2$$
$$= \lambda^2 - \omega^2\beta^2 - d^2 \rightarrow \lambda = \pm\sqrt{\omega^2\beta^2 + d^2}$$
$$\lambda = \pm\sqrt{\omega^2\cos^2 2\theta + (-\omega\sin \quad 2\theta + \frac{GN}{\sqrt{2}})\text{^}2} = \pm\sqrt{\omega^2 + (\frac{GN}{\sqrt{2}})^2 - 2\frac{GN}{\sqrt{2}}\omega\sin(2\theta)}$$

These are the mixing eigenvalues (the energies). Now vectors.

First, positive lambda. I'm not going to plug in lambda.

$$\begin{bmatrix} d-\lambda & \omega i\beta \\ -i\omega\beta & -d-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Ok, this is borked. Should it be simplifying more?
$x = -\frac{\omega i\beta}{(d-\lambda)}y$, let y = 1

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-i\omega\beta}{(d-\lambda)} \\ 1 \end{bmatrix}$with pos eigenvalue
And
$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-i\omega\beta}{(d+\lambda)} \\ 1 \end{bmatrix}$ with neg eigenvalue. I'm skipping normalizing.

Could have also done $x = -\frac{d+\lambda}{\omega i\beta}y$

$\psi(t) =$ a$\begin{bmatrix} \frac{i\omega\beta}{(d-\lambda)} \\ 1 \end{bmatrix} e^{-\frac{i}{h}\lambda t}$ + b$\begin{bmatrix} \frac{i\omega\beta}{(d+\lambda)} \\ 1 \end{bmatrix} e^{+\frac{i}{h}\lambda t}$ = a $\begin{bmatrix} x_+ \\ 1 \end{bmatrix} e^{-\frac{i}{h}\lambda t}$ + b $\begin{bmatrix} x \\ 1 \end{bmatrix} e^{+\frac{i}{h}\lambda t}$

$$\chi_+ = \begin{bmatrix} x_+ \\ 1 \end{bmatrix}; \chi_- = \begin{bmatrix} x_- \\ 1 \end{bmatrix};$$

Find a and b constants. At time 0, $\psi(0) =$ a$\begin{bmatrix} x_+ \\ 1 \end{bmatrix}$+ b$\begin{bmatrix} x_- \\ 1 \end{bmatrix}$= $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$a = -b = \frac{1}{x_+ - x_-}$$

Then find $|< \psi_{tau}|\psi(t) > |^2$ = $|\begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(\text{a}\begin{bmatrix} x_+ \\ 1 \end{bmatrix} e^{-\frac{i}{h}\lambda t} - \text{a}\begin{bmatrix} x \\ 1 \end{bmatrix} e^{+\frac{i}{h}\lambda t}\right)|^2$

Set this equal to one and isolate t?

$|\begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(\text{a}\begin{bmatrix} x_+ \\ 1 \end{bmatrix} e^{-\frac{i}{h}\lambda t} - \text{a}\begin{bmatrix} x \\ 1 \end{bmatrix} e^{+\frac{i}{h}\lambda t}\right)|^2 = |\begin{bmatrix} 0 \\ 1 \end{bmatrix}\left(\text{a}\begin{bmatrix} x_+ e^{-\frac{i}{h}\lambda t} - x_- e^{+\frac{i}{h}\lambda t} \\ e^{-\frac{i}{h}\lambda t} - e^{+\frac{i}{h}\lambda t} \end{bmatrix}\right)|^2 = \left|a\left(e^{-\frac{i}{h}\lambda t} - e^{+\frac{i}{h}\lambda t}\right)\right|^2 = (a\left(-isin\left(\frac{\gamma t}{h}\right)\right))^2 = a^2 \sin^2 \frac{\gamma t}{h} = 1 \rightarrow L = \frac{\text{ch}}{\gamma} \arcsin \frac{1}{a^2}$