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(5) We start w/ the schrödinger eqn for a steady-state soln:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi = \left(\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right)\left(-\frac{\hbar^2}{2m}\right)$$

Assume ψ is separable into $R(r)$ and $Y(\theta,\varphi)$ components.
Since $l=0$ we don't care about $Y(\theta,\varphi)$ as it has no contribution to the energy ($l(l+1)=0$) so for now I just consider $\psi=\psi(r)$.
Letting $u(r)=\psi(r)\cdot r$, the schrödinger eqn simplifies to:

$$\left(\frac{-\hbar^2}{2m}\right)\frac{1}{r}\frac{\partial^2}{\partial r^2}u = \frac{E-V}{r}u \Rightarrow \frac{\partial^2 u}{\partial r^2} = (E-V)u\left(\frac{-2m}{\hbar^2}\right)$$

I will solve for u in each region of r, then use boundary conditions to connect them. Let $a=0.5$ fm and $b=3$ fm.

Region 1: $r<a$
in this region $V=+\infty$, so $\psi(r<a)$ must be zero (and thus $u=0$)

Region 2: $a<r<b$
in this region $V=-V_0$, and for a bound state E must be greater than V, so:

$$\frac{\partial^2 u}{\partial r^2} = -\frac{2m}{\hbar^2}(E+V_0)u$$

the standard solution here is $u(r)=A\exp(ik_1 r)+B\exp(-ik_1 r)$ with $k_1=\sqrt{2m(E+V_0)}/\hbar$

Region 3: $r>b$
in this region $V=0$, and E must be <0 for a ground state, so:

$$\frac{\partial^2 u}{\partial r^2} = \frac{-2mE}{\hbar^2}u \quad \text{(remember } E<0)$$

for which the solution is $u(r)=C\exp(k_2 r)+D\exp(-k_2 r)$ with $k_2=\sqrt{-2mE}/\hbar$. Since $u(\infty)=0$, C must be 0 and $u(r)=D\exp(-k_2 r)$.

The wavefunc must be continuous at a and b, so:

$$0 = A\exp(ik_1 a)+B\exp(-ik_1 a)$$
$$A\exp(ik_1 b)+B\exp(-ik_1 b) = D\exp(-k_2 b)$$

$$\frac{k_1}{k_2} = \sqrt{\frac{2m(E+V_0)}{-2mE}} = i\sqrt{1+\frac{V_0}{E}}$$

The first gives:

$$u_i(r) = A\sin(k_1 r - k_1 a)$$

and the second becomes

$$A\sin(k_1 b - k_1 a) = D\exp(-k_2 b)$$

Additionally, the derivative at b must also be continuous:

$$k_1 A\cos(k_1(b-a)) = -k_2 D\exp(-k_2 b)$$
$$D = -\frac{k_1}{k_2}A\cos(k_1(b-a))\exp(k_2 b)$$
$$= -i\sqrt{1+\frac{V_0}{E}}A\cos(k_1(b-a))\exp(k_2 b)$$

(5) Continuing from the previous page:

$$A\sin(k_1(b-a)) = -\frac{k_1}{k_2}A\cos(k_1(b-a))$$

$$\tan(k_1(b-a)) = -\frac{k_1}{k_2} = -i\sqrt{1+\frac{V_0}{E}}$$ * why imaginary!? oops

$$\frac{\exp(2ik_1(b-a))-1}{i(\exp(2ik_1(b-a))+1)} = -i\sqrt{1+\frac{V_0}{E}}$$

$$\exp(2ik_1(b-a))-1 = \sqrt{1+\frac{V_0}{E}}\left(\exp(2ik_1(b-a))+1\right)$$

$$\exp(2ik_1(b-a))\left[1-\sqrt{1+\frac{V_0}{E}}\right] = \left[1+\sqrt{1+\frac{V_0}{E}}\right]$$

$$\exp(2ik_1(b-a)) = \left[\frac{1+\sqrt{1+V_0/E}}{1-\sqrt{1+V_0/E}}\right]$$

$$k_1 = \frac{-i}{2(b-a)}\log\left[\frac{1+\sqrt{1+V_0/E}}{1-\sqrt{1+V_0/E}}\right] = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

- I don't know how to solve this for the allowed values of E, probably because I messed up some algebra.
- There's also an "i" in this expression, which may also be a sign that I messed up some algebra.

The rest of the plan was to find E in terms of V_0, and then find V_0 such that the energy E was zero. This would provide the minimum V_0 for a bound state to exist, as bound states would just have $E<0$.

the radial wavefunc

If there was such a ground state, it looks like it would be symmetric. The $\ell=0$ angular wavefunctions are also symmetric.

Neutrons are fermions, though, meaning their overall wavefunction must be antisymmetric. Thus, the spinor must be antisymmetric! For two spin-1/2 particles this means the singlet state $\chi \sim |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ which has a total spin of zero.

Thus, with $\ell=0$ and $s=0$ the total angular momentum $j=\ell+s$ must also be zero!

⑥ a) just an infinite square well with width D, so $\boxed{E_n = \frac{n^2\hbar^2\pi^2}{2mD^2}}$ 6a)

c) $K = \frac{1}{2}mv^2 = \frac{n^2\pi^2\hbar^2}{2mD^2} \Rightarrow V = \frac{\pi\hbar}{mD}n$ is the classical speed of the particle.

Force is just $\frac{dp}{dt}$ so, for an elastic collision with momentum change $\Delta p = 2mv = 2\frac{\pi\hbar n}{D}$ in a time $\Delta t = \frac{D}{v} = \frac{mD^2}{\pi\hbar n}$ the force is

$$\boxed{F = \frac{\Delta p}{\Delta t} = 2\frac{\pi^2\hbar^2 n^2}{mD^3}}\quad 6c)$$

b) in this case the force is just the rate of change in energy as the walls move: (since $\Delta E = \int \vec{F}\cdot d\vec{x}$)

$$F = \frac{\partial E}{\partial D} \Rightarrow \boxed{F = -\frac{\pi^2\hbar^2 n^2}{mD^3}}\quad 6b)$$

which is half as strong as the classical force but, notably, is in the opposite direction.

⑧ a) The distance (along x) between x(t) and the center of mass of the door is (assuming x(t) is the position of the hinge):

$$X_{cm}(t) = x(t) - \frac{L}{2}\cos\phi$$

of the CoM

so the total velocity can be described as

$$\dot{x}_{cm} = \dot{x} + \frac{L}{2}\sin\phi\dot{\phi}$$

The total kinetic energy (with I = moment of inertia) is then:

$$T = \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\left(\dot{x} + \frac{L}{2}\sin\phi\dot{\phi}\right)^2$$

There is no potential energy. Taking derivatives for the lagrangian:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = I\dot{\phi} + m\left(\dot{x} + \frac{L}{2}\sin\phi\dot{\phi}\right)\frac{L}{2}\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = I\ddot{\phi} + m\ddot{x} + \frac{mL^2}{4}\left[\ddot{\phi}\sin\phi\cos\phi + \sin\phi\cos^2\phi\dot{\phi}\dot{\phi}^2 - \sin^2\phi\cos\phi\dot{\phi}^2\right] - \frac{mL}{2}\dot{x}\sin\phi\dot{\phi}$$

$$= I\ddot{\phi} + m\ddot{x} + \frac{mL^2}{4}\sin\phi\cos\phi\left[\ddot{\phi} + \dot{\phi}^2(\cos\phi - \sin\phi)\right] - \frac{mL}{2}\dot{x}\sin\phi\dot{\phi}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{1}{2}m\left(\dot{x} + \frac{L}{2}\sin\phi\dot{\phi}\right)\frac{L}{2}\cos\phi\dot{\phi}$$

$$I = \frac{mL^2}{12}$$

Now $\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right)$ so:

8a) $$\boxed{\frac{mL}{4}\cos\phi\dot{x}\dot{\phi} + \frac{mL^2}{8}\sin\phi\cos\phi\dot{\phi}^2 = I\ddot{\phi} + m\ddot{x} + \frac{mL^2}{4}[\ldots] - \frac{mL}{2}\dot{x}\sin\phi\dot{\phi}}$$

this looks a bit too complicated but let's continue anyways

b) constant acceleration → $\ddot{x} = 0$, $\dot{x} = at$

small angles → $\cos = 1$ $\sin = \phi$

$$\frac{mL}{4}at\phi\dot{\phi} + \frac{mL^2}{8}\phi\dot{\phi}^2 = I\ddot{\phi} + \frac{mL^2}{4}\phi\left[\ddot{\phi} + \dot{\phi}^2(1-\phi)\right] - \frac{mL}{2}at\phi\dot{\phi}$$

aaand we're stuck... oops

It's clear my expression for kinetic energy is incorrect. I need some way to include the KE from the motion of the cart and not just the rotation of the door. My $\dot{x}_{cm}$ is probably wrong, but I can't just do $\frac{1}{2}m\dot{x}^2$ (or similar) as that just "decouples" ϕ and x in the lagrangian.