Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 15, 2014
1:00PM to 3:00PM
Modern Physics
Section 3. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 3 (QM), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider a one-dimensional, simple harmonic oscillator with an additional, anharmonic x^4 term:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x^4$$

   (a) What are the energy eigenvalues of H when $\lambda = 0$?

   (b) What is the shift in the lowest eigenvalue found in part (a) to first order in λ?

   (c) Find this energy shift to order λ^2.

   (d) Using a Gaussian trial function, make a variational estimate of the energy of the ground state. (You need only obtain an explicit equation whose solution would give this estimate). Solve this equation to first order in λ and compare your result with that in part (b).

2. Suppose we have a system with N non-interacting spins, with the state described by:

$$|\Psi\rangle = |\Psi\rangle^{(1)} |\Psi\rangle^{(2)} ... |\Psi\rangle^{(N)} ,$$

where each spin is in an identically prepared superposition of spin up (+) and down (-):

$$|\Psi\rangle^{(1)} = |\Psi\rangle^{(2)} = ... = |\Psi\rangle^{(N)} = a |+\rangle + b |-\rangle ,$$

where $|a|^2 + |b|^2 = 1$, and $|+\rangle , |-\rangle$ form an orthonormal set. The inner product between different spins (e.g. spin (1) and spin (2)) is zero. Let us define an operator that acts on $|\Psi\rangle$:

$$\hat{P} \equiv \frac{1}{2N} \sum_{i=1}^{N} \left(\sigma^{(i)} + 1\right) ,$$

where each $\sigma^{(i)}$ acts only on the corresponding spin $|\Psi\rangle^{(i)}$, with the understanding that acting on $|+\rangle$ returns 1, and acting on $|-\rangle$ returns -1.

Compute the norm of the state $\left(\hat{P} - |a|^2\right) |\Psi\rangle$ i.e. find

$$\langle\Psi| \left(\hat{P} - |a|^2\right)^2 |\Psi\rangle ,$$

for an arbitrary N. Does the $N \to \infty$ limit correspond to what you expect? Hint: if you think about what $\hat{P}$ means physically, you would realize you have *derived* the Born rule - without talking about the collapse of the wavefunction at all.

3. Use the trial wave function

$$\Psi(x) = \left(\frac{a}{\pi}\right)^{1/4} e^{-ax^2/2}$$

to estimate, and bound, the ground state energy of the Hamiltonian

$$H = \frac{p^2}{2m} - V_0\delta(x)$$

for one-dimensional motion in a δ-function potential. $V_0 > 0$. Compare your answer with the exact answer $E_0^{exact} = -\frac{mV_0^2}{2\hbar^2}$

4. For a quantum mechanical particle of mass m, consider the (active) Galilean boost transformation

$$\vec{x} \to \vec{x}' = \vec{x} - \vec{v_0}t,$$

where v_0 is a constant velocity vector. For simplicity set $\hbar = 1$.

(a) On physical grounds, argue that the probability density and current

$$\rho(\vec{x}, t) \equiv |\psi(\vec{x}, t)|^2, \qquad \vec{J}(\vec{x}, t) = -\frac{i}{2m}\left[\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right]$$

should transform as

$$\rho(\vec{x}, t) \to \rho(\vec{x}', t), \qquad \vec{J}(\vec{x}, t) \to \vec{J}(\vec{x}', t) + \rho(\vec{x}', t)\vec{v_0}$$

(b) Derive the corresponding transformation law for the wave function $\psi(\vec{x}, t)$. [*Hint: use a "polar" representation for the wave function,* $\psi(\vec{x}, t) = R(\vec{x}, t)e^{i\theta(\vec{x}, t)}$.]

(c) Check that, in the absence of a potential, the Schrödinger equation is invariant under such a transformation.

5. Consider a free particle in one dimension with mass m. At time $t = 0$ the expectation value of its position is $\langle x \rangle_0$, with a variance $(\Delta x)_0^2 = \langle x^2 \rangle_0 - \langle x \rangle_0^2$. Find the variance at some later time t. Express your answer in terms of expectation values of operators at $t = 0$.

N. Christ November 26, 2013

Quals Quantum Mechanics Problem

1. Consider a one-dimensional, simple harmonic oscillator with an additional, anharmonic x^4 term:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x^4$$

   (a) What are the energy eigenvalues of H when $\lambda = 0$? [2 points]

   (b) What is the shift in the lowest eigenvalue found in part (a) to first order in λ? [6 points]

   (c) Find this energy shift to order λ^2. [8 points]

   (d) Using a Gaussian trial function, make a variational estimate of the energy of the ground state. (You need only obtain an explicit equation whose solution would give this estimate.) Solve this equation to first order in λ and compare your result with that in part (b). [4 points]

Suggested Solution

1. (a) $E^{(0)} = \hbar\omega(n + \frac{1}{2})$.

   (b) Express x in terms of raising and lowering operators: $x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)$ and evaluate

$$\begin{aligned} E_0^{(1)} &= \lambda \langle 0|x^4|0\rangle \\ &= \lambda \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0|(a+a^\dagger)^4|0\rangle \\ &= \lambda \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0|a(a+a^\dagger)^2 a^\dagger|0\rangle \\ &= \lambda \left(\frac{\hbar}{2m\omega}\right)^2 \langle 0|\Big(a^2+1\Big)\Big((a^\dagger)^2+1\Big)|0\rangle \\ &= 3\lambda \left(\frac{\hbar}{2m\omega}\right)^2 \end{aligned}$$

   (c) Sum over two intermediate states $|n\rangle$, $n = 2$ and 4:

$$\begin{aligned} E_0^{(2)} &= \sum_{n=2,4} \frac{\Big|\langle n|\lambda x^4|0\rangle\Big|^2}{E_0^{(0)} - E_n^{(0)}} \\ &= \lambda^2 \left(\frac{\hbar}{2m\omega}\right)^4 \left\{\frac{72}{-2\hbar\omega} + \frac{24}{-4\hbar\omega}\right\} \\ &= -\frac{13\hbar^3\lambda^2}{4m^4\omega^5} \end{aligned}$$

   (d) Use the variational wave function $\psi(x) = Ne^{-\frac{m\omega}{2\hbar}\alpha x^2}$. Scaling the three terms in H by the appropriate factor of α and using the result from (a):

$$\begin{aligned} E(\alpha) &= \frac{1}{4}\hbar\omega(\alpha + \frac{1}{\alpha}) + 3\lambda\left(\frac{\hbar}{2m\omega}\right)^2 \frac{1}{\alpha^2} \\ &= \frac{1}{4}\hbar\omega\left\{\alpha + \frac{1}{\alpha} + \frac{\epsilon}{\alpha^2}\right\} \end{aligned}$$

   where $\epsilon = 3\lambda\hbar/(m^2\omega^3)$. Thus, we must solve $1 - 1/\alpha^2 - 2\epsilon/\alpha^3 = 0$. Which to first order in ϵ requires $\alpha = 1 - \epsilon$, leading an energy estimate which agrees with that found in (b).

QM problem. Suppose we have a system with N non-interacting spins, with the state described by:

$$|\Psi\rangle = |\Psi\rangle^{(1)}|\Psi\rangle^{(2)}...|\Psi\rangle^{(N)}\,, \tag{1}$$

where each spin is in an identically prepared superposition of spin up (+) and down (-):

$$|\Psi\rangle^{(1)} = |\Psi\rangle^{(2)} = ... = |\Psi\rangle^{(N)} = a|+\rangle + b|-\rangle\,, \tag{2}$$

where $|a|^2 + |b|^2 = 1$, and $|+\rangle, |-\rangle$ form an orthonormal set. The dot product between different spins (e.g. spin (1) and spin (2)) is zero. Let us define an operator that acts on $|\Psi\rangle$:

$$\hat{P} \equiv \frac{1}{2N}\sum_{i=1}^{N}(\sigma^{(i)} + 1)\,, \tag{3}$$

where each $\sigma^{(i)}$ acts only on the corresponding spin $|\Psi\rangle^{(i)}$, with the understanding that acting on $|+\rangle$ returns 1, and acting on $|-\rangle$ returns -1.

Compute the norm of the state $(\hat{P} - |a|^2)|\Psi\rangle$ i.e. find

$$\langle\Psi|\left(\hat{P} - |a|^2\right)^2|\Psi\rangle\,, \tag{4}$$

for an arbitrary N. Does the $N \to \infty$ limit correspond to what you expect? Hint: if you think about what $\hat{P}$ means physically, you would realize you have *derived* the Born rule – without talking about the collapse of the wavefunction at all.

QM solution (Lam Hui). This is a *derivation* of the Born rule – that probability equals amplitude squared. Let me define $\tilde{P}$ as

$$\tilde{P} = \frac{1}{2N}\sum_{i=1}^{N}\sigma^{(i)}\,. \tag{5}$$

Therefore, $\hat{P} = \tilde{P} + \frac{1}{2}$. We find:

$$\begin{aligned}\langle\Psi|\left(\hat{P} - |a|^2\right)^2|\Psi\rangle &= \langle\Psi|\left(\tilde{P} - \frac{1}{2}(2|a|^2-1)\right)^2|\Psi\rangle \\ &= \langle\Psi|\tilde{P}^2|\Psi\rangle - (2|a|^2-1)\langle\Psi|\tilde{P}|\Psi\rangle + \frac{1}{4}(2|a|^2-1)^2\,. \end{aligned} \tag{6}$$

The first term gives

$$\langle\Psi|\tilde{P}^2|\Psi\rangle = \frac{N}{4N^2} + \frac{(N^2-N)}{4N^2}\left(|a|^2-|b|^2\right)^2\,, \tag{7}$$

where we have used $\sigma^{(i)}\left(a|+\rangle + b|-\rangle\right) = a|+\rangle - b|-\rangle$, and so the $i=j$ terms in $\tilde{P}^2$ return something proportional to $(a^*\langle+| - b^*\langle-|)\cdot(a|+\rangle - b|-\rangle) = |a|^2+|b|^2 = 1$, while the $i \neq j$ terms return something proportional to

$$\left[\,(a^*\langle+| + b^*\langle-|)\,\cdot\,(a|+\rangle - b|-\rangle)\,\right]^2 = \left(|a|^2-|b|^2\right)^2\,. \tag{8}$$

Similarly, the second term in Eq. (6) contains

$$\langle\Psi|\tilde{P}|\Psi\rangle = \frac{N}{2N}(|a|^2-|b|^2)\,. \tag{9}$$

Noting that $|a|^2-|b|^2 = 2|a|^2-1$, and putting everything together into Eq. (6) gives

$$\langle\Psi|\left(\hat{P}-|a|^2\right)^2|\Psi\rangle = \frac{1}{N}|a|^2|b|^2\,. \tag{10}$$

The miracle here is that all the $O(1)$ terms cancel out. We have therefore shown that in the $N\to\infty$ limit, $\left(\hat{P}-|a|^2\right)|\Psi\rangle$ is a state of zero norm i.e.

$$N\to\infty: \quad \hat{P}|\Psi\rangle = |a|^2|\Psi\rangle \tag{11}$$

By the usual rule of quantum mechanics, a state with a definite eigenvalue $|a|^2$ under a hermitian operator $\hat{P}$ would be said to have that precise quantum number. For us, $\hat{P}$ can be thought of as the probability operator whose eigenvalue is therefore the probability. Notice how we manage to discuss probability – as the amplitude squared – without talking about the collapse of the wavefunction at all. Of course this hinges on the acceptance of $\hat{P}$ as the probability operator, which seems reasonable from its definition.

D. Mueller

Sec 3 - 3

Quals Problem

Quantum Mechanics

Use the trial wave function

$$\psi(x) = \left(\frac{a}{\pi}\right)^{1/4} e^{-ax^2/2}$$

to estimate, and bound, the ground state energy of the Hamiltonian

$$H = \frac{p^2}{2m} - V_0\,\delta(x)$$

for one-dimensional motion in a δ-function potential. $V_0 > 0$. Compare your answer with the exact answer $E_0^{exact} = -\frac{mV_0^2}{2\hbar^2}$

Solution:

$$\langle\psi|H|\psi\rangle = \left(\frac{a}{\pi}\right)^{1/2} \int dx\, e^{-\frac{ax^2}{2}} \underbrace{\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \delta(x)\,V_0\right] e^{-\frac{ax^2}{2}}}_{-V_0 - \frac{\hbar^2}{2m}\int dx\,(-a + a^2x^2)\,e^{-ax^2}}$$

$$= \sqrt{\frac{a}{\pi}}\left[-V_0 - \frac{\hbar^2}{2m}\sqrt{a}\,\underbrace{\int_0^\infty dy\,(-1+y)\frac{1}{\sqrt{y}}\,e^{-y}}_{-\Gamma(\frac{1}{2}) + \Gamma(\frac{3}{2}) = -\sqrt{\pi}/2}\right]$$

$$E_{trial} = \sqrt{\frac{a}{\pi}}\left[-V_0 + \frac{\hbar^2}{4m}\sqrt{\pi a}\right]$$

$$E_{trial} = \frac{1}{\sqrt{\pi}}\left[-V_0\sqrt{a} + \frac{\hbar^2\sqrt{\pi}}{4m}a\right].$$ Set $\frac{dE_{trial}}{da} = 0$ to determine a

$$0 = -\frac{V_0}{2\sqrt{a}} + \frac{\hbar^2\sqrt{\pi}}{4m} \Rightarrow \sqrt{a} = \frac{2mV_0}{\sqrt{\pi}\,\hbar^2}$$

$$E_{variational} = -\frac{mV_0^2}{\pi\hbar^2}$$ exactly twice the exact answer

1 Quantum Mechanics: Galilean invariance

For a quantum mechanical particle of mass m, consider the (active) Galilean boost transformation

$$\vec{x} \to \vec{x}\,' = \vec{x} - \vec{v}_0\, t \,, \tag{1}$$

where $\vec{v}_0$ is a constant velocity vector. For simplicity set $\hbar = 1$.

1. On physical grounds, argue that the probability density and current

$$\rho(\vec{x},t) \equiv |\psi(\vec{x},t)|^2 \,, \qquad \vec{J}(\vec{x},t) = -\frac{i}{2m}\left[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right] \tag{2}$$

   should transform as

$$\rho(\vec{x},t) \to \rho(\vec{x}\,',t) \,, \qquad \vec{J}(\vec{x},t) \to \vec{J}(\vec{x}\,',t) + \rho(\vec{x}\,',t)\,\vec{v}_0 \tag{3}$$

2. Derive the corresponding transformation law for the wave function $\psi(\vec{x},t)$. [*Hint: use a "polar" representation for the wave function,* $\psi(\vec{x},t) = R(\vec{x},t)e^{i\theta(\vec{x},t)}$.]

3. Check that, in the absence of a potential, the Schröndinger equation is invariant under such a transformation.

Solution

1. At any given time, the probability of finding the particle in an infinitesimal volume d^3x does not change under a Galilean boost, because the infinitesimal volume element is unchanged:

$$d^3x' = d^3x \,. \tag{4}$$

   However, to have the same probability, one still has to track the *original* volume element, which is now moving at velocity $\vec{v}_0$. The only effect of eq. (1) on ρ is thus the change of argument

$$\rho(\vec{x},t) \to \rho(\vec{x}\,',t) \,. \tag{5}$$

   On the other hand, for the probability current density—like for all current densities—one also has to take into account that, in the $\vec{x}'$ coordinate system, all velocities get shifted by $\vec{v}_0$, so that the current density gets shifted by $\rho\vec{v}_0$:

$$\vec{J}(\vec{x},t) \to \vec{J}(\vec{x}\,',t) + \rho(\vec{x}\,',t)\,\vec{v}_0 \tag{6}$$

2. Plugging the polar decomposition

$$\psi = Re^{i\theta} \tag{7}$$

   into eq. (2), one finds that in order to obey (3) one needs

$$R(\vec{x},t) \to R(\vec{x}\,',t) \,, \qquad \vec{\nabla}\theta(\vec{x},t) \to \vec{\nabla}\theta(\vec{x}\,',t) + m\vec{v}_0 \,, \tag{8}$$

   that is,

$$\psi(\vec{x},t) \to \psi(\vec{x}\,',t)\, e^{im\vec{v}_0\cdot\vec{x}} \tag{9}$$

   (times an irrelevant constant phase, related to the integration constant in (8)).

3. In the absence of a potential the Schröndinger equation reads

$$i\partial_t\psi + \frac{1}{2m}\nabla^2\psi = 0\ . \tag{10}$$

Under the replacement (9), we get extra contributions to $\partial_t\psi$ from the t-dependence of $\vec{x}\,'$,

$$\partial_t\big[\psi(\vec{x}\,', t)\, e^{im\vec{v}_0\cdot\vec{x}}\big]_{x=\text{const}} = e^{\cdots} \times \big[\partial_t\psi - \vec{v}_0\cdot\vec{\nabla}'\psi\big]\ , \tag{11}$$

and extra contributions to $\nabla^2\psi$, from the exponential factor,

$$\nabla^2\big[\psi(\vec{x}\,', t)\, e^{im\vec{v}_0\cdot\vec{x}}\big]_{t=\text{const}} = e^{\cdots} \times \big[\nabla'^{\,2}\psi + 2i\, m\vec{\nabla}'\psi\cdot\vec{v}_0\big]\ . \tag{12}$$

Plugging these expressions into (10), we get simply

$$e^{\cdots} \times \big[i\partial_t\psi + \frac{1}{2m}\nabla'^{\,2}\psi\big] = 0\ , \tag{13}$$

which is, apart from an irrelevant overall factor, the Schrödinger equation in the $\vec{x}\,'$ coordinate system.

Weinberg #1
Solution

Use $\frac{d\langle A\rangle}{dt} = \frac{1}{i\hbar}\langle [A,H]\rangle + \langle \frac{\partial A}{\partial t}\rangle$

$H = \frac{p^2}{2m}$

$\Rightarrow \langle p\rangle$ and $\langle p^2\rangle$ are t-independent

$$\frac{d\langle x\rangle}{dt} = \frac{1}{i\hbar}\langle [x,H]\rangle = \cancel{\frac{1}{i\hbar}}\langle \frac{p}{m}\rangle$$

$$\frac{d\langle x^2\rangle}{dt} = \frac{1}{i\hbar}\langle [x^2,H]\rangle = \frac{1}{m}\langle (xp+px)\rangle$$

$$\frac{d}{dt}(\Delta x)^2 = \frac{d}{dt}\langle x^2\rangle - 2\langle x\rangle\frac{d\langle x\rangle}{dt}$$

$$= \frac{1}{m}\langle (xp+px)\rangle - \frac{2}{m}\langle x\rangle\langle p\rangle$$

$$\frac{d^2}{dt^2}(\Delta x)^2 = \frac{2}{m^2}\langle p^2\rangle - \frac{2}{m^2}\langle p\rangle^2$$

$$\frac{d^3}{dt^3}(\Delta x)^2 = 0$$

$$\Rightarrow (\Delta x)^2_t = \frac{t^2}{m^2}(\Delta p)^2 + \frac{t}{m}\left(\langle xp+px\rangle_0 - \langle x\rangle_0\langle p\rangle\right) + (\Delta x)^2_0$$