Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 9, 2017
11:00AM to 1:00PM
Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

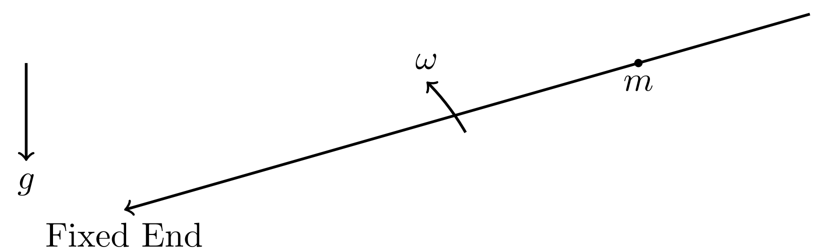
Good Luck!

1. A particle of mass m is constrained to slide without friction along a straight rod, as in the figure below. The rod is rotating in a vertical plane about one end with a constant angular velocity ω, in the presence of a uniform gravitational field g. The rod is horizontal at $t = 0$.

   (a) Find the Lagrangian and the differential equations of motion for the mass m.

   (b) Solve the differential equations assuming that at $t = 0$ the mass is at rest a distance d from the fixed end of the rod.

   (c) Show that there is a certain critical angular frequency ω_c such that, if $\omega < \omega_c$, the mass will eventually hit the fixed end of the rod and, if $\omega > \omega_c$, the mass will eventually fly off the rotating end of the rod. Find ω_c.



2. In Newtonian physics, the circular orbit of two gravitationally bound mass M objects is a stable configuration. General relativity teaches us this is an approximation, as the system must lose energy due to gravitational radiation.

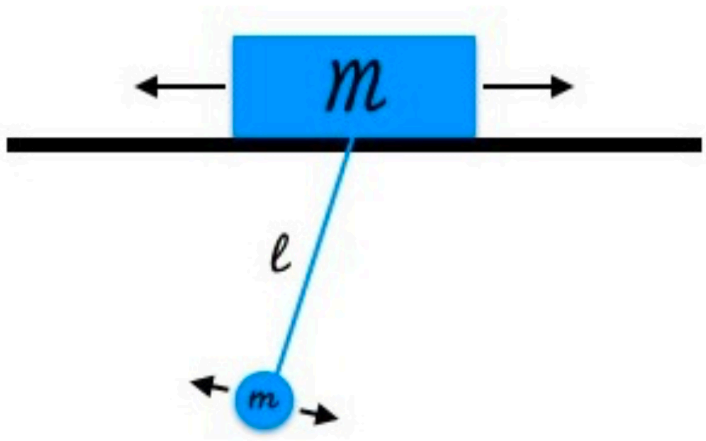
   Consider the case of two mass M black holes in a circular orbit of very large radius R. Using only Newtonian physics, estimate the total energy lost due to gravitational radiation before the black holes merge. You may assume the black holes merge when they first touch, and that the radius of the black hole is given by the Newtonian condition for the escape velocity to be the speed of light.

3. A simple pendulum of mass m and length l is connected at its point of support to a block of mass M. The block is free to slide along a frictionless horizontal track.

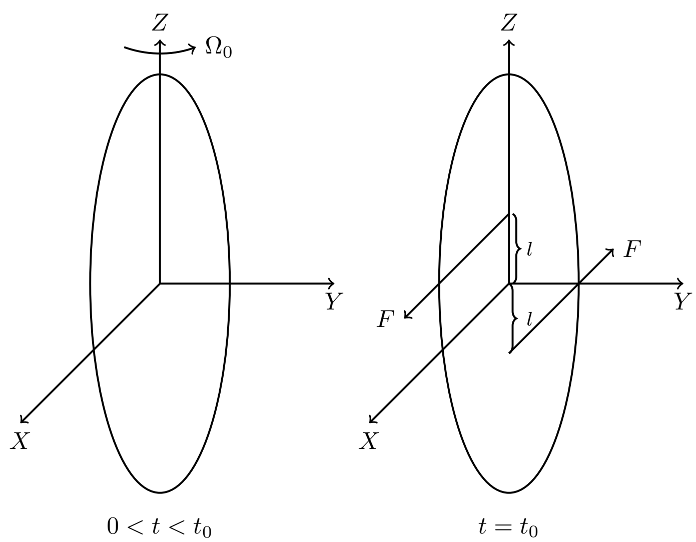
   (a) Find the Lagrangian for this system (assuming it is placed in a uniform gravitational field). Find Lagrange's equations.

   (b) What are the constants of motion?

   (c) What are the eigenfrequencies associated with small oscillations about the equilibrium position of this system? Describe the motion of the system associated with each frequency.



4. A symmetrical top with $I_1 = I_2 \neq I_3$ is initially placed with its center of mass at the origin of a fixed inertial coordinate system XYZ and with the z body axis (axis 3) along the Z axis as indicated in the figure. At $t = 0$ the top spins about its z body axis with angular frequency Ω_0. Between $t = 0$ and $t = t_0$, the top simply spins about the z axis with its center of mass fixed. At t_0, two equal and opposite forces of magnitude F are exerted in the $+X$ and $-X$ direction and at $Z = +l$ and $-l$ respectively. The forces are very large but act for a short time Δt with $F\Delta t$ of finite size. Give the motion of the top subsequent to the application of the forces F and $-F$ in the absence of gravity.



5. A cylinder with mass m and radius r is suspended from the ceiling as shown in Fig. 1. On one side, a spring of stiffness k is inserted. There is sufficient friction between the rope and the cylinder to prevent sliding of the cylinder. Find the oscillation frequency of the suspended cylinder.

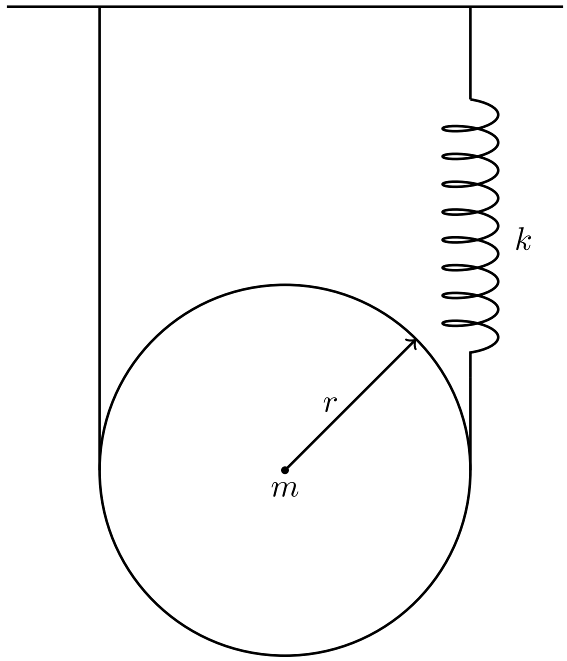


Figure 1: A suspended cylinder.

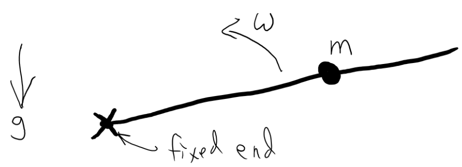
Humensky
Section 1-1

1. Classical Mechanics: A particle of mass m is constrained to slide without friction along a straight rod, as in the figure below. The rod is rotating in a vertical plane about one end with a constant angular velocity ω, in the presence of a uniform gravitational field g. The rod is horizontal at $t = 0$.

a. (6 pts) Find the Lagrangian and the differential equations of motion for this system.

b. (5 pts) Solve the differential equations assuming that at $t = 0$ the mass is at rest a distance d from the fixed end of the rod.

c. (4 pts) Show that there is a certain critical angular frequency ω_c such that, if $\omega < \omega_c$, the mass will eventually hit the fixed end of the rod and, if $\omega > \omega_c$, the mass will eventually fly off the rotating end of the rod. Find ω_c.



2. Relativity: In Newtonian mechanics, power can be found from $P = \frac{dE}{dt} = \vec{F} \cdot \vec{v}$, where E is the total energy of a particle moving with velocity $\vec{v}$ and acted on by a net force $\vec{F}$. Show that this relation is also valid in relativistic mechanics (assume that Newton's second law is valid under special relativity).

Quals - Mechanics Solution

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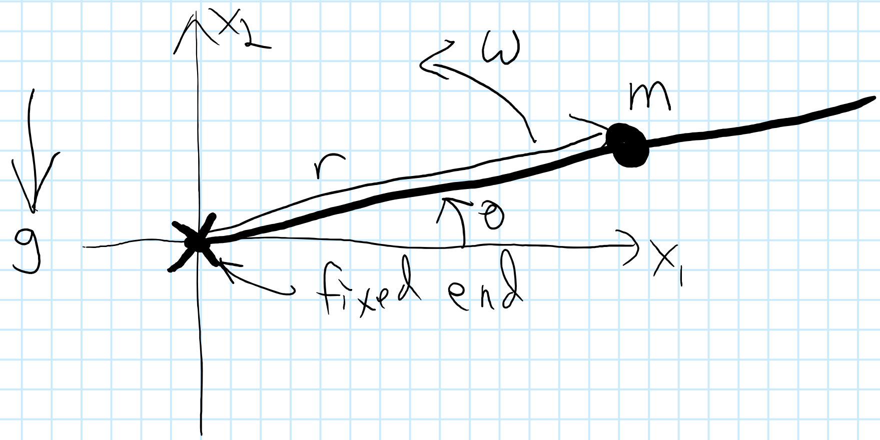
a) Use cylindrical coords r, θ - but $\theta = \omega t$ due to constant ω, so only r is a degree of freedom.



$x_1 = r\cos\omega t$, $x_2 = r\sin\omega t$

$\Rightarrow \dot{x}_1 = \dot{r}\cos\omega t - r\omega\sin\omega t$, $\dot{x}_2 = \dot{r}\sin\omega t + r\omega\cos\omega t$

$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) = \frac{1}{2}m[\dot{r}^2 + r^2\omega^2]$ (cross terms cancel)

$U = mgx_2 = mgr\sin\omega t$

$\Rightarrow L = T - U = \boxed{\frac{1}{2}m(\dot{r}^2 + r^2\omega^2) - mgr\sin\omega t = L}$

$\frac{\partial L}{\partial r} = mr\omega^2 - mg\sin\omega t$, $\frac{\partial L}{\partial \dot{r}} = m\dot{r}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$

$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 = \cancel{m}r\omega^2 - \cancel{m}g\sin\omega t - \cancel{m}\ddot{r}$

$\Rightarrow \boxed{\ddot{r} = r\omega^2 - g\sin\omega t}$

✓

b) Test solution: $r(t) = Ae^{\omega t} + Be^{-\omega t} + C\sin\omega t + D\cos\omega t$

$\Rightarrow \ddot{r} = A\omega^2 e^{\omega t} + B\omega^2 e^{-\omega t} - C\omega^2\sin\omega t - D\omega^2\cos\omega t$

Plugging into eqn of motion →

$$A\omega^2 e^{\omega t} + B\omega^2 e^{-\omega t} - C\omega^2 \sin\omega t - D\omega^2 \cos\omega t = \omega^2 A e^{\omega t} + \omega^2 B e^{-\omega t} + \omega^2 C \sin\omega t + \omega^2 D \cos\omega t - g\sin\omega t$$

$\Rightarrow D = 0$, $C = \frac{g}{2\omega^2}$, use initial conditions to get A and B:

$r(0) = A + B = d$, $\dot{r}(0) = \omega(A-B) + \frac{g}{2\omega} = 0$

$\Rightarrow \omega(A+B) = \omega d \rightarrow$ add

$\Rightarrow 2\omega A + \frac{g}{2\omega} = \omega d \Rightarrow A = \frac{1}{2}d - \frac{1}{4}\frac{g}{\omega^2}$

$\Rightarrow B = d - A = d - \frac{1}{2}d + \frac{1}{4}\frac{g}{\omega^2} = \frac{1}{2}d + \frac{1}{4}\frac{g}{\omega^2} = B$

$$\Rightarrow \boxed{r(t) = \left(\frac{d}{2} - \frac{g}{4\omega^2}\right)e^{\omega t} + \left(\frac{d}{2} + \frac{g}{4\omega^2}\right)e^{-\omega t} + \frac{g}{2\omega^2}\sin\omega t}$$

c) At late times, the $e^{\omega t}$ term dominates

$\Rightarrow$ if $\left(\frac{d}{2} - \frac{g}{4\omega^2}\right) > 0$ then $r(t) \rightarrow \infty$ and mass flies off rod, but if $\left(\frac{d}{2} - \frac{g}{4\omega^2}\right) < 0$ then $r(t) \rightarrow 0$ and mass will hit pivot.

$\Rightarrow \frac{d}{2} - \frac{g}{4\omega_c^2} = 0 \Rightarrow \boxed{\omega_c = \sqrt{\frac{g}{2d}}}$

1 Problem: Black Hole Merger

In Newtonian physics, the circular orbit of two gravitationally bound mass M objects is a stable configuration. General relativity teaches us this is an approximation, as the system must lose energy due to gravitational radiation.

Consider the case of two mass M black holes in a circular orbit of very large radius R. Using only Newtonian physics, estimate the total energy lost due to gravitational radiation before the black holes merge. You may assume the black holes merge when they first touch, and that the radius of the black hole is given by the Newtonian condition for the escape velocity to be the speed of light.

2 Solution: Black Hole Merger

Re-derive virial theorem averages to get factors of 2 right:

$$\frac{GM^2}{(2r)^2} = \frac{Mv^2}{r} \quad \Rightarrow \quad KE_{Tot} = Mv^2 = \frac{GM^2}{4r} = -\frac{1}{2}\,V_{Pot}$$

$$\Rightarrow \quad E_{Tot} = KE_{Tot} + V_{Pot} = \frac{1}{2}V_{Pot} = -\frac{1}{2}\frac{GM^2}{2r} \quad \checkmark$$

Schwarzschild radius given by

$$\frac{GMs}{R_s} = \frac{1}{2}mc^2 \quad R_s = \frac{2GM}{c^2} \quad ,$$

which happens to be exactly correct for static black hole.

"very large radius R" $\quad \Rightarrow \quad E_{Tot} \simeq 0$ initially.

The final total energy is for $r = R_s \Rightarrow$

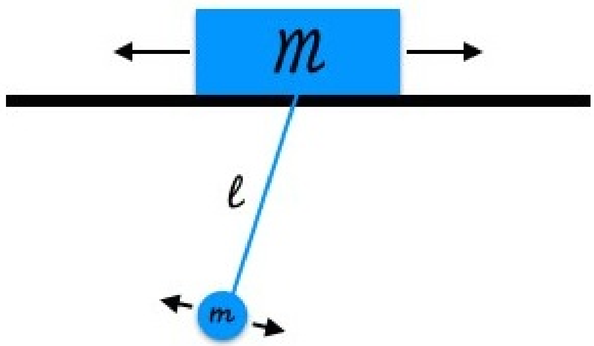
$$E_{Tot}^{final} = -\frac{GM^2}{4R_s} = -\frac{GM^2}{4 \cdot 2GM/c^2} = -\frac{1}{8}Mc^2 \quad ,$$

which is not too far off the LIGO observed value of $(3.0 \pm 0.5)M_\odot$ for the merger of a $29M_\odot$ and $36M_\odot$ black holes.

This problem could be extended to calculate the power radiated, but this would require the understanding that nearly all the radiation takes place in the last couple of orbits.

2017 Quals Problem: Classical Mechanics

A simple pendulum of mass m and length l is connected at its point of support to a block of mass M. The block is free to slide along a frictionless horizontal track.



a. Find the Lagrangian for this system (assuming it is placed in a uniform gravitational field). Find Lagrange's equations.
b. What are the constants of motion?
c. What are the eigenfrequencies associated with small oscillations about the equilibrium position of this system? Describe the motion of the system associated with each frequency.

Rosen
Section 1-3

2017 Quals Problem: Classical Mechanics
Solution

a.
Kinetic energy: $T = \frac{1}{2}M(\dot{x}_M^2 + \dot{y}_M^2) + \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2)$,
Potential energy: $V = mgy_m$,
where $x_M = x$, $y_M = 0$, $x_m = x + l\sin\theta$ and $y_m = -l\cos\theta$.

Lagrangian:
$L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\cos\theta\,\dot{\theta}\dot{x} + mgl\cos\theta$

Lagrange's equations:
For x: $(M+m)\ddot{x} + ml\cos\theta\,\ddot{\theta} - ml\sin\theta\,\dot{\theta}^2 = 0$
For θ: $ml^2\ddot{\theta} + ml\cos\theta\,\ddot{x} - ml\sin\theta\,\dot{x}\dot{\theta} + mgl\sin\theta = 0$

b.
Constants of motion: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$ so $\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + ml\cos\theta\,\dot{\theta} = \text{const}$
Also: $E = T + V = \text{const}$

c.
Small angle approx:

$$\hat{T} = \begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}, \quad \hat{V} = \begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}$$

So:

$$\det\begin{pmatrix} -\omega^2(M+m) & -\omega^2 ml \\ -\omega^2 ml & +mgl - \omega^2 ml^2 \end{pmatrix} = 0$$

Eigenfrequencies:
$\omega^2 = 0$: corresponds to the block and the pendulum moving together. I.e., x changes but not θ.
$\omega^2 = \frac{g}{l}\frac{M+m}{M}$: corresponds to oscillations of the pendulum. In the limit $M \gg m$, the block doesn't move.

Mechanics

Solution to problem 4

(4) Force acts for short time so top does not change its position between t_0 and $t_0+\Delta t$, however there is a change in angular momentum.

$$t<t_0 \quad \vec{L}_{before} = (0, 0, I_3\Omega_0)$$

$$t>t_0 \quad \vec{L} = (0, 2F\ell\Delta t, I_3\Omega_0)$$

Since there are no forces on the top after $t_0+\Delta t$, $\vec{L}$ maintains its value and the problem is now that of a free symmetrical top which precesses about $\vec{L}$ which lies in the YZ plane at an angle $\theta = \tan^{-1}\left[\frac{2F\Delta t\ell}{I_3\Omega_0}\right]$.

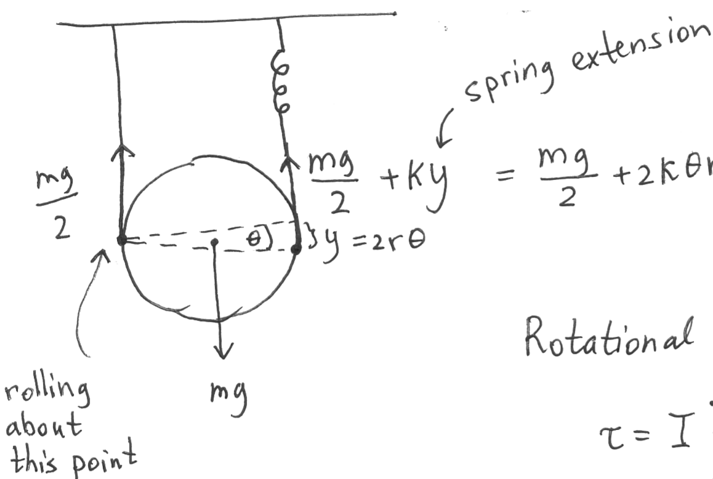
The rate of precession is

$$\Omega_{pr} = \frac{|\vec{L}|}{I_1} = \frac{\sqrt{(I_3\Omega_0)^2 + 4F^2(\Delta t)^2\ell^2}}{I_1}$$

while the rate of rotation about its own z-axis is

$$\Omega_z = \frac{\vec{L}\cos\theta}{I_3} = \Omega_0$$

Suspended cylinder and spring. SOLUTION.



$$= \frac{mg}{2} + 2k\theta r$$

Rotational 2nd law:

$$\tau = I\ddot{\theta}$$

$$\cancel{mgr} - \cancel{\frac{mg}{2}}\times 2r - 2k\theta r \times 2r = \underbrace{\left(\frac{1}{2}mr^2 + mr^2\right)}_{=I=\frac{3}{2}mr^2}\ddot{\theta}$$

$$\ddot{\theta} = -\frac{4kr^2\times 2}{3mr^2}\theta = -\frac{8}{3}\frac{k}{m}\theta$$

$$\omega = \sqrt{\frac{8}{3}\frac{k}{m}}, \quad \boxed{f = \frac{1}{2\pi}\sqrt{\frac{8}{3}\frac{k}{m}}}$$