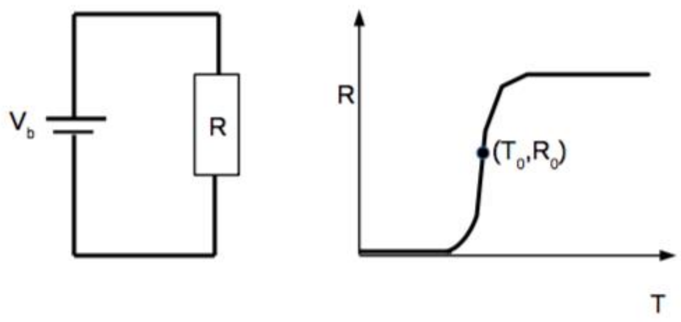
2018 Solution Explanations
By Cassandra M ☺



1. A transition edge sensor is a thin superconducting film that is used to detect tiny temperature changes. It is hooked up to a constant voltage source V_b (see drawing on left) and is in thermal contact with a cold bath of temperature T_{bath}. The sensor has a heat capacity C, and let T denote its temperature. Heat flows from the sensor to the bath at a rate given by $P(T) = K(T^2 - T_{bath}^2)$. V_b is chosen so that the current flowing through the film is large enough to balance it on the transition between its normal and superconducting state (see drawing on right). At some temperature T_0 the sensor therefore has a finite resistance R_0 and is in a state of equilibrium. If the temperature of the sensor increases from T_0 to $T_0 + \delta T$, where δT is very small, the resistance of the sensor will change quickly, due to the steep (but finite) slope α of the superconducting transition curve.

A. Write down an expression for T_0.

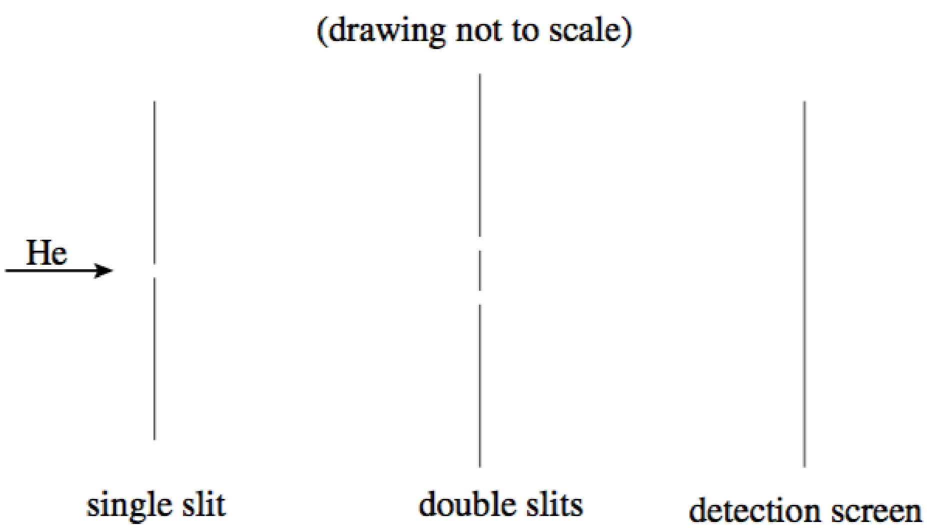
B. Supppose that at $t = 0$ the sensor quickly absorbs some energy, increasing its temperature from equilibrium by a small δT. Calculate the temperature of the sensor as a function of time for $t > 0$. You should linearize the relevant differential equation for T by using a Taylor series expansion.

1. Here we just want an expression for the equilibrium temperature of the sensor, before the temp increases. I'm assuming P is power, but what units are K?

$$P(T) = K(T^2 - T_{bath}^2) = \frac{V^2}{R} \rightarrow T_0^2 = \frac{V_b^2}{R_0} + T_{bath}^2$$

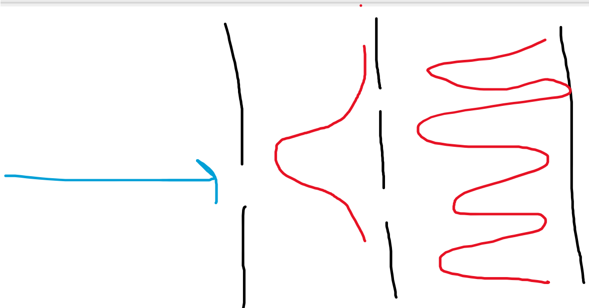
$$\boxed{T_0 = \sqrt{\frac{V_b^2}{R_0} + T_{bath}^2}}$$

2. Now the temperature increases by a small amount. $P = \frac{E}{t}$, so maybe we can get a t in here somehow.



2. A beam of He atoms moving at 1350 m/s is normally incident onto a screen with a narrow slit of width d. A second screen with two slits in it sits 64 cm downstream of the first. The two slits are 1 μm wide and their centres are 8 μm apart. A "detection screen" sits a further 64 cm downstream, and records the hit locations of each helium atom. The atoms form an interference pattern on this screen.

Calculate the spacing of the interference bands on the detection screen. How small must d be in order for the interference pattern to be seen?



In reality, on the detection screen the amplitude of the peaks would be decreasing, but it's just my bad drawing skills.

The spacing of the interference bands is independent of the first slit, and only depends on the double slits.

$$y = \frac{m\lambda D}{d_2}$$

Use De Broglie's relation to find lambda.

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{6.626e-34}{(4 * 1.66e-27)(1350)} = 7.35e-11\ m$$

The distance to the first maxima is

$$y = \frac{7.35e - 11 \ * (0.64m)}{8e - 6m} = 5.9 \ \mu m$$

So the spacing between them is this times 2.

Now for the second question. The interference won't be seen if the location of the second slits are in the minima of the pattern from the first slits.

For a single slit, the minima occur for integer numbers of m:

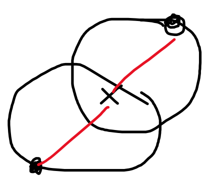
$$y = \frac{m\lambda D}{d} \rightarrow d = \frac{m\lambda D}{y}$$

Y is 4 micrometers.

$$d = \frac{7.35e - 11 \ * (0.64m)}{4e - 6m} = 12\mu m$$

So d should be smaller than this.

3. A neutron star is in a binary system with another star. It has an orbital period of 42 minutes around the centre of mass and its orbit is circular. The orbital velocity of the neutron star around the centre of mass is 11 km/sec while the companion has an orbital velocity of 770 km/sec. Find the masses of the companion star and the neutron star.



Let's start with determining some facts about the system, for example the radius of each planet's orbit. The neutron star travels $2\pi r$ every 42 minutes, so therefore

$$v = \frac{x}{t} \rightarrow 2\pi r = x = vt \rightarrow r_n = \frac{v_n T}{2\pi} = 4.4e6m$$

What about the companion star? In a binary system, each has the same orbital period. So we can use the exact same relation, with a different v:

$$r_n = \frac{v_c T}{2\pi} = 3.1e8m$$

As it's a binary system, both stars will be orbiting around a common centre of mass. By definition, the center of mass is:

$$x_{com} = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}$$

If we take the center of mass as x = 0, we get the relation

$$0 = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2} \rightarrow 0 = x_1 m_1 + x_2 m_2 \rightarrow x_1 m_1 = -\, x_2 m_2$$

Note however, if the COM is the 0 point then one of the x positions must be negative. So, taking that into account, $r_1 m_1 = r_2 m_2 \rightarrow m_1 = \frac{r_2 m_2}{r1} = 70.4 m_2$.

Ok. So we only need one more thing - Kepler's third law is:

$$T^2 = \frac{4\pi r^3}{m}$$

For a binary system, this is

$$T^2 = \frac{4\pi (r_1 + r_2)^3}{m_1 + m_2} = \frac{4\pi (r_1 + r_2)^3}{71.4 m_2}$$

Note: this is something I didn't know.. that for a binary system, we could just add the masses and the radii. Hm.

Therefore $m_1 = \frac{4\pi (r_1+r_2)^3}{71.4 T^2} = \frac{4\pi (4.4e6+3.1e8)^3}{71.4(42*60)^2} = 8.61e17\ kg$

And $m_1 = \frac{1}{70.4} * 8.61e17 = 1.22e16\ kg$

4. A neutron star is composed primarily of neutrons packed to nuclear densities, and the mass of a neutron star is two times the mass of the Sun.

A. Roughly calculate the radius of a neutron star.

A neutron star will often accrete matter onto its hard surface from a companion star. Assume this accretion flow is spherically symmetric and is falling onto the neutron star surface at a rate of approximately $dm/dt = 10^{-9}$ solar masses per year.

B. Estimate the total accretion luminosity (power) emitted by a neutron star at the given accretion rate.

C. Assuming the neutron star radiates like a black body, what is the typical energy of the emitted photons? In what wavelength band (e.g. radio? visible?) does this radiation predominantly occur?

A. The weird thing about this problem is you can either do it the (seemingly?) incorrect way, or the legitimate way, and they almost give the same answer. I wonder if the non-legit way is actually a good approximation?

The non-legit way is determining the neutron density, ($\rho_n = m_n/V_n$) which we probably actually couldn't do on an exam because we maybe don't know the neutron radius (0.8fm). But then, we say this is the density of the star as well, and use the standard mass volume relation to find the radius:

$$V = \frac{2M_{sun}}{\rho} = \frac{4}{3}\pi R^3$$

Because neutron stars are degenerate, they don't follow this mass radius relation. But I got the radius as 17 km, which is kind of correct. An ok approximation?

The "legit" way is to find the real mass radius relation, by setting the degeneracy pressure equal to the gravitational pressure:

$$P_{degen} = P_{grav} = \frac{F_{Grav}}{A} = \frac{GM^2}{4\pi R^4} = \frac{\pi^3 h^2}{15m_n}\left(\frac{3N}{\pi V}\right)^{5/3} = \frac{\pi^3 h^2}{15m_n}\left(\frac{3M}{\pi V m_n}\right)^{5/3}$$

By rearranging for M and R, we get something like:

$$R = \frac{\pi^3 h^2}{15m_n}\left(\frac{9}{\pi^2 m_n 4}\right)^{\frac{5}{3}} M^{-1/3} = 12.4km$$

(Note: I have noticed a slight math error but I don't think it changes the answer that much, so note my numerical values are probably not correct. I'm over it!)

B. The power is energy/time. What is the energy of the emitted accretion by the neutron star? I believe it would be kinetic energy.

$$P = \frac{\frac{1}{2}mv^2}{t} \rightarrow \frac{dP}{dt} = \frac{\frac{1}{2}dmv^2}{dt} = \frac{1}{2}\frac{dm}{dt}v^2$$

For velocity, I'd assume we can use conservation of force to find it, considering it's spherically symmetric...

$$F_G + F_{cent} = 0 = \frac{GMm}{r^2} + \frac{mv^2}{r} \rightarrow v^2 = \frac{GM}{r}$$

$$\frac{dP}{dt} = \frac{1}{2}\frac{dm}{dt}\frac{GM}{r} = 6.9e30\ W$$

NOTE from future: should I have actually used c for the velocity? Because the luminosity is from the emitted photons, so it doesn't necessarily need to be the speed of the accreting material...

C. Here we can use our handy blackbody equations.

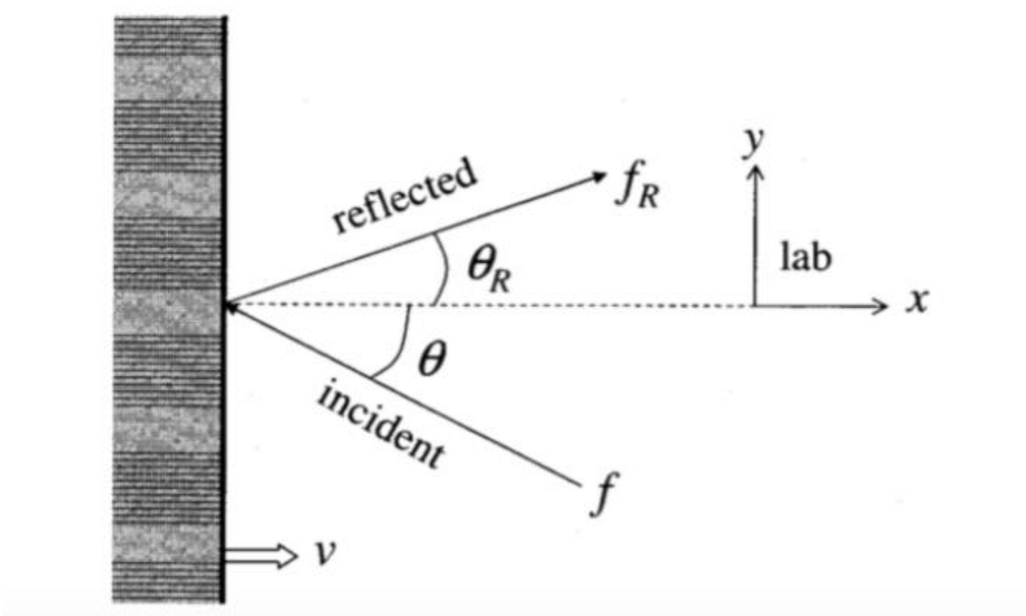
$$\lambda = \frac{b}{T}, P = A\sigma T^4$$

$$T = \left(\frac{P}{4\pi R^2 \sigma}\right)^{1/4} = \left(\frac{6.9e30}{4\pi * 12400^2 * 1.38e-23}\right)^{1/4} = 1.26e11\ K$$

Skeptical… $\lambda = \frac{b}{T} = \ \lambda = \frac{2.897\times10^{-3} Km}{1.26e11\ K} = 2.29e-14\ m = 23fm.$
This seems wrong. This is smaller than gamma rays.

$$E = hf = \frac{hc}{\lambda} = 8.6e-12\ J$$

5. A monochromatic beam of light is incident on a flat mirror. With respect to the laboratory, the mirror is traveling at relativistic speed v in the $+x$ direction. (The plane of the mirror is perpendicular to the x-axis.) In the lab frame, the incident light beam has frequency f and travels at angle θ with respect to the x-axis. Find the frequency f_R and the angle θ_R of the reflected light beam as measured in the lab frame.



Note: I'm really bad at relativity, so I expect this solution is incorrect.
Our mirror is moving with velocity v, so this could potentially just be a simple boost question. Can we boost the four-momentum vector? Our reflected photon:

The frequency of reflected light is always the same as the incident light, - I think this is true for the angle of reflection as well. So… can we find f' in the mirror frame by boosting, say that this is equal to f'_R in the mirror frame, and then find that in the lab frame by boosting again? And same for the angle?

The incident light:

$$\begin{bmatrix} E' \\ E'cos\theta' \\ E'sin\theta' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ Ecos\theta \\ Esin\theta \end{bmatrix}$$

The primes indicate the mirror frame (which we're given in the question) and we want to determine those on the right, in the lab frame.

$$E' = \gamma E - \beta\gamma Ecos\theta = hf' = \gamma hf - \beta\gamma hfcos\theta \rightarrow f' = f\gamma(1 - \beta cos\theta)$$

$$E'sin\theta' = Esin\theta \rightarrow sin\theta' = \frac{f}{f'}sin\theta$$

So this is the frequency of light in the lab frame, and $f' = f_R{}',\ \theta' = \theta_R'$. Now let's boost this back to the lab frame (? Am I not going to get the same thing?)

$$\begin{bmatrix} E_R{}' \\ E_R'cos\theta_R{}' \\ E_R'sin\theta_R{}' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_R \\ E_R cos\theta_R \\ E_R sin\theta_R \end{bmatrix}$$

$$f_R' = f_R(\gamma - \beta\gamma cos\theta_R) \rightarrow f_R = \frac{f_R'}{\gamma(1 - \beta cos\theta_R)}$$

$$f_R' sin\theta'_R = f_R\ sin\theta_R \rightarrow sin\theta_R = \frac{f_R'}{f_R}sin\theta'_R = \frac{f_R'}{f_R}\frac{f}{f_R}sin\theta = \frac{f}{f_R}sin\theta$$

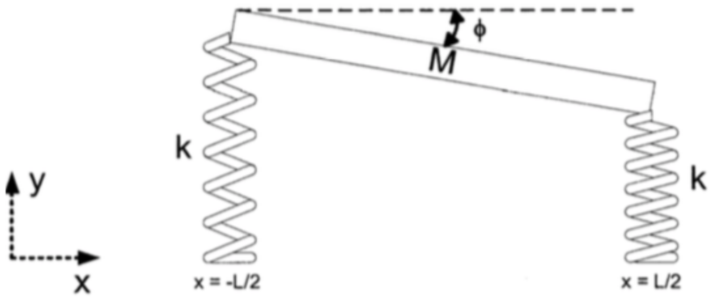
Plugging in f'…

$$f_R = f\frac{(1 - \beta cos\theta)}{(1 - \beta cos\theta_R)} = f\frac{(1 - \frac{v}{c}cos\theta)}{\left(1 - \frac{v}{c}cos\theta_R\right)}$$

$$sin\theta_R = \frac{f}{f_R}sin\theta$$

Hm. Could be reasonable? Everything needs to be plugged in and solved for, though...

6. A thin bar with uniform density of length L and mass M is supported by two massless springs. The springs have identical spring constants k and unloaded lengths l. The centre of mass of the bar is constrained to move only vertically with displacement $y(t)$, but the bar can rotate in the xy plane with angle $\phi(t)$.



A. Determine the Lagrangian of the system for small displacements, taking gravity into account.

B. Solve the Euler-Lagrange equations of motion to determine the frequencies and eigenvectors for the normal modes of the system.

C. Suppose at $t = 0$ the end of the bar at $x = L/2$ is depressed by a *small* amount d, while the other end is held at its equilibrium position. (In other words, $y_1(t = 0) = l$ and $y_2(t = 0) = l - d$). If all initial generalized velocities are zero, find expressions for $y(t)$ and $\phi(t)$.

As usual the Lagrangian is $\mathcal{L} = T - V$.

The kinetic energy is that of the bar moving in the y direction, and the rotational KE of the bar moving in the phi plane.

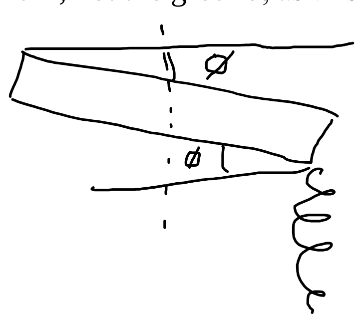
$$T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\phi}^2$$

I'll just leave moment of inertia as a variable for now. For a bar rotating about its center of mass, $I = \frac{1}{12}ML^2$.

For potential energy, we have the gravitational PE from the bar being suspended, and also the potential of the spring constants.

$$V = -Mgy + \frac{1}{2}k(y_1)^2 = \frac{1}{2}k(y_2)^2$$

Now, y_1 and y_2. Initially I got these wrong, because for some reason I kept thinking y was the distance from the ground to the top of the dotted line, but it's really from the ground to the spring (except taking into account the distance from equilibrium, l… this is also easy to forget… it's the distance from equilibrium, not the ground, as this isn't from gravity!)



This displacement distance can be described using $d = \frac{L}{2} sin\phi \sim \frac{L}{2}\phi$ because we have small displacements. Therefore the Lagrangian is

$$\mathcal{L} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\phi}^2 + Mgy - \frac{1}{2}k\left(y - l + \frac{L}{2}\phi\right)^2 - \frac{1}{2}k\left(y - l - \frac{L}{2}\phi\right)^2$$

B. Euler Lagrange.

$$\frac{d}{dt}\left(\frac{dL}{d\dot{y}}\right) = \frac{dL}{dy}$$

$$M\ddot{y} = Mg - k\left(y - l + \frac{L}{2}\phi\right)^1 - k\left(y - l - \frac{L}{2}\phi\right)^1 = Mg - k(2y - 2l)$$

$$\ddot{y} = g - \frac{2k}{m}(y - l) = -\frac{2k}{m}y + \left(g + \frac{2kl}{m}\right)$$

This is a sinusoidal ODE! The solution (note adding a constant just… adds a constant) is:

$$y(t) = c_1 \sin\left(\sqrt{\frac{2k}{m}}t\right) + c_2 \cos\left(\sqrt{\frac{2k}{m}}t\right) + \frac{m\left(g + \frac{2kl}{m}\right)}{2k}$$

For phi:

$$I\ddot{\phi} = \frac{-kL}{2}\left(y - l + \frac{L}{2}\phi\right)^1 + \frac{kL}{2}\left(y - l - \frac{L}{2}\phi\right)^1 = \frac{-kL}{2}\left(y - l + \frac{L}{2}\phi - y + l + \frac{L}{2}\phi\right) = \frac{-kL^2\phi}{2}$$

$$\ddot{\phi} = \frac{-kL^2\phi}{I2}$$

Which has the solution

$$\phi(t) = c_3 \sin\sqrt{\frac{kL^2}{2I}}t + c_4 \cos\sqrt{\frac{kL^2}{2I}}t$$

So those are the eigenvectors, and the normal modes are $\omega_1 = \sqrt{\frac{k}{2I}}L, \omega_2 = \sqrt{\frac{2k}{M}}$

Now putting in I, $\omega_1 = \sqrt{\frac{12kL^2}{2ML^2}} = \omega_1 = \sqrt{\frac{6k}{M}}$. Nice!

C. I think using the y1 and y2 values, I can get total y. Is it just $y = l - (l - d) = d$?

In this case,

$$y(0) = d = 0 + c_2(1) + \frac{m\left(g + \frac{2kl}{m}\right)}{2k} \rightarrow c_2 = d - \frac{m\left(g + \frac{2kl}{m}\right)}{2k} = d - (\frac{mg}{2k} + l)$$

Then taking the derivative and setting y'(0)=0

$$y'(t) = c_1\sqrt{\frac{2k}{m}}\cos\left(\sqrt{\frac{2k}{m}}t\right) - c_2\sqrt{\frac{2k}{m}}\sin\left(\sqrt{\frac{2k}{m}}t\right) + 0$$

$$y'(0) = 0 = c_1\sqrt{\frac{2k}{m}}(0) \to c_1 = 1$$

$$y(t) = (d - \left(\frac{mg}{2k} + l\right))\cos\left(\sqrt{\frac{2k}{m}}t\right) + \frac{m\left(g + \frac{2kl}{m}\right)}{2k}$$

We already said $\frac{L}{2}\phi = d$, so can we say the initially phi is $\phi_0 = \frac{2d}{L}$? (If anyone is reading this, I am totally not putting a lot of thought into this and not thinking twice so ignore my simplicity). If this is correct, just do the same thing for the phi equation.

7. A hydrogen atom in its ground state has a wavefunction given by $\Psi_0(r,\theta,\phi) \propto e^{-r/a_0}$, where $a_0 = 5.3\times10^{-11}$ m is the Bohr radius. When placed in an external electric field $\vec{E} = E_0\hat{z}$, the new wavefunction, to lowest order in perturbation theory, becomes:

$$\Psi(r,\theta,\phi) \propto \Psi_0\left[1 - \left(\frac{E_0}{A}\right)\left(\left(\frac{r}{a_0}\right) + \frac{1}{2}\left(\frac{r}{a_0}\right)^2\right)\cos\theta\right]$$

Here A is a parameter characterizing the unperturbed atom.

Use this wavefunction to calculate the electric dipole moment of the atom in the external field, to first order in E_0.

Hint:

$$\int_0^\infty dr\; e^{-r}r^n = n!$$

The dipole moment is generally $p = qd$. Can we say our quantum dipole moment is $p = q < d > = q < rcos\theta >$?

I will be using $\int_{-\infty}^{-\infty} x^n e^{-\frac{x}{a}}dx = n!\,a^{n+1}$

Therefore, normalizing the ground state gives $\int\int\int_{-\infty}^{-\infty} A^2 e^{-\frac{2r}{a}}r^2 sin\theta dr d\theta d\pi = 1 \to A = \frac{1}{\sqrt{a^3\pi}}$

$$p = e\int\int\int \frac{1}{a^3\pi}e^{-\frac{2r}{a_0}}\left(1 - \left(\frac{E}{A}\right)\left(\left(\frac{r}{a}\right) + \frac{1}{2}\left(\frac{r}{a}\right)^2\right)cos\theta\right)^2 r^2 sin\theta dr d\theta d\pi$$

I’m not going to do all of this math here. Note that squaring causes one of the terms to go away, because it includes an E_2. I believe the end answer actually is shockingly simple (I did a lil pluggy pluggy into Wolfram).

8. An electron spin is oriented upward along $+\hat{z}$, and subject to a magnetic field B_x along $+\hat{x}$. The precession period of the spin around the magnetic field direction is T. The operator S_z is measured repeatedly, N times within a time interval $T/2$ (assume the measurements are evenly spaced in time). What is the probability that all N measurements returned $+\hbar/2$, in the limit of large N? (Express your answer in terms of N.)

Our Hamiltonian is the same as always: $H = -\mu \cdot B = -\gamma S_x B_x = -\frac{\gamma h B}{2}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

Finding the eigenvectors and values of this matrix gives:

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalue $\lambda = -\frac{\gamma h B}{2}$ and $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda = \frac{\gamma h B}{2}$

Using the time evolution operator $U = e^{-\frac{i}{h}\int H dt}$, our wavefunction becomes

$$\psi(t) = \frac{a}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-\frac{\gamma i h B}{2}t} + \frac{b}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{\frac{\gamma i h B}{2}t}$$

To find a and b, use our initial condition:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{a}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{b}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This gives a = b = ½.

Reducing this further, we can get $\psi(t) = \begin{bmatrix} \cos\left(\frac{\gamma h B t}{2}\right) \\ i\sin\left(\frac{\gamma h B t}{2}\right) \end{bmatrix}$

Now. Let’s start with finding the probability in general of measuring h/2 in the Sz basis.

$$\mathrm{P} = | < [1\ 0]\begin{bmatrix} \cos\left(\frac{\gamma h B t}{2}\right) \\ i\sin\left(\frac{\gamma h B t}{2}\right) \end{bmatrix} > |^2 = \cos^2\left(\frac{\gamma h B t}{2}\right)$$

The kind of tricky part is realizing that since it started in +z, and it’s periodic, the probability must equal 1 at T.

$1 = \cos^2\left(\frac{\gamma hBT}{2}\right)$, which means $\frac{\gamma hBT}{2} = 2\pi$, or $T = \frac{4\pi}{\gamma hB}$

We make a measurement every $\mathrm{t} = \frac{\mathrm{T}}{2\mathrm{N}} \rightarrow \cos^2\left(\frac{\gamma hBT}{4N}\right) = \cos^2\left(\frac{\gamma hB}{4N}\frac{4\pi}{\gamma hB}\right) = \cos^2\left(\frac{\pi}{N}\right)$

Probability is multiplicative, so if we make N measurements, the total prob of getting this is $\cos^{2\mathrm{N}}\left(\frac{\pi}{N}\right)$. If N is large, it means what is inside the cos is very small, so we can Taylor expand cosine.

Taylor expanding:

$$\boxed{P = \left(1 + \frac{1}{2}\left(\frac{\pi}{\mathrm{N}}\right)^2 + \cdots\right)^{2N}}$$

I think this is enough for the large limit.

9. Do an order of magnitude estimate of the amount of energy stored in the magnetic field surrounding the Earth. (The field strength on the Earth's surface is $\sim 5 \times 10^{-5}$ T.) Compare this to an order of magnitude estimate of the amount of electrical energy used in the greater Vancouver area each year.

We want to find the total energy stored outside the earth (not a formula for it) which means we need to integrate. Energy density and B-field are related by:

$$E_B = \frac{1}{2}B^2/\mu_0$$

Work out the units – this is energy density (J/V), so we need to integrate over volume to find the total energy.

$$\int\int\int_R^\infty \frac{1}{2\mu_0}B^2r^2 sin\theta dr d\theta d\phi$$

We only know B at the surface. We know the earth has a dipole field, which decreases proportional to $\frac{1}{r^3}$ (Important! At first, I said $\frac{1}{r^2}$). So, in our integral, we want B_0, but we're given B_R, where R is the earth radius. But we know how it scales:

$$B_R \propto \frac{B_0}{R^3}$$

So we can say

$$B_0 \propto \frac{B_R R^3}{r^3}$$

$$4\pi \int_R^\infty \frac{1}{2\mu_0} (\frac{B_0 R^3}{r^3})^2 r^2 dr = \frac{2\pi}{\mu_0} \int \frac{B_R^2 R^{6^1}}{r^4} \, dr$$

Therefore

$$E_{tot} = \frac{-6\pi B_R^2 R_E^6}{\mu_0} \left(\frac{1}{\infty^3} - \frac{1}{R_E^3} \right) = \frac{6\pi B_R^2 R_E^3}{\mu_0} \sim 9.6 \times 10^{18} J$$

For the total electrical energy used in Vancouver in a year, this is a bit tougher because it requires knowledge of how much a general home uses, and then are buildings (e.g. Triumf) that use way more energy than this. So this is a very approximate guess.

$$E_E = \frac{1}{2} E^2 / \epsilon_0$$

A standard lightbulb is 120W. I'll assume an average house uses 100x this? Honestly probably more. I mean, ovens. Whatever though. The population of Vancouver is 675000, so:

$$E_E = 120W \times 100 \times 675000 \times (365 * 24 * 3600) \times 10 = 2.55 \times 10^{18} J$$

What the heck! So similar. The multiple of 10 at the end is to account for non-residential large uses of energy. The earths B field is weak AF.

10. The Sun is in mechanical equilibrium between gravitational and pressure forces. Make an order of magnitude estimate of the temperature T of the centre of the Sun.

Hydrostatic equilibrium tells us that:

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \rightarrow P = \int_0^R \frac{-GM\rho dr}{r^2} = \frac{GM\rho}{R}$$

Let's just ignore the 0 in the integral... lol.

We know all of the values on the right, so we need to find the pressure in terms of temperature. Note that using the luminosity won't work, because that depends on the *surface* temperature. We care about the temperature of the gas molecules inside the sun. Ideal gas law:

$$PV = NkT \rightarrow \frac{NkT}{\frac{4}{3}\pi r^3}$$

$$\frac{dP}{dr} = \frac{d}{dr}\left(\frac{NkT}{\frac{4}{3}\pi r^3} \right) = -\frac{NkT}{\frac{4}{3}\pi r^4}$$

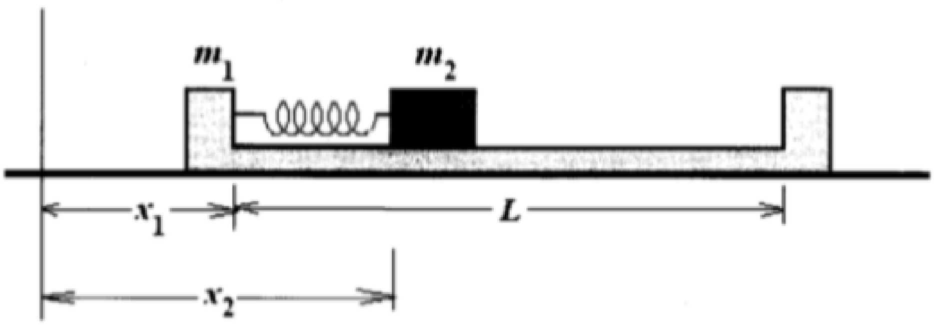
$$-\frac{NkT}{\frac{4}{3}\pi r^4} = -\frac{GM^2}{\frac{4}{3}\pi r^5} \rightarrow NkT = \frac{GM^2}{R} \rightarrow T = \frac{GM^2}{RNk}$$

How to find the number atoms inside the sun? Assume it's all hydrogen for simplicity.

$$T = \frac{GM^2}{RNk} = \frac{GMm_H}{Rk} = \frac{6.67e-11*2e30*1.67e-27}{7e8*1.38e-23} = 23e6\ K$$

Correct order of magnitude!

12. A block of mass m_2 slides inside a cavity of length L inside a second block with mass m_1, which rests on a horizontal table . The masses m_1 and m_2 are connected by a massless spring with spring constant k and equilibrium length $l \ll L$. Intially both blocks are at rest and located at $x_1 = 0$ and $x_2 = l - \Delta l$, where Δl is the initial compression of the spring.



A. If the mass m_1 slides without friction on the table and m_2 slides without friction on the second block, find the positions $x_1(t)$ and $x_2(t)$ as a function of time.

B. If mass m_1 exerts a frictional force on m_2 proportional to their relative velocity, $F_{1\text{ on }2} = -\sigma(\dot{x}_2 - \dot{x}_1)$, again determine the resulting motion of the two masses. (Continue to assume that m_1 slides without friction on the table, and also assume that σ is small.)

We can either use Lagrangian or just find the equations of motions directly. I choose Lagrangian.

$$\mathcal{L} = T - V$$

$$T = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_1\dot{x_2}^2$$

For potential energy, note that our x_1 and $x_2$is given. So we need to play around it to ensure that the potential energy describes the behaviour around equilibrium. Only the second mass has a potential energy term, the first mass is not relevant.

$$V = \frac{1}{2}k(x_2 - x_1 + l)^2$$

This, $x_2 - x_1 + l$, puts us at the equilibrium point of the spring.

$$\mathcal{L} = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_1\dot{x_2}^2 - \frac{1}{2}k(x_2 - x_1 + l)^2$$

Now solve E-L.

$$\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = \frac{dL}{dx}$$

$$m_1\ddot{x_1} = k(x_2 - x_1 + l)$$
$$m_2\ddot{x_2} = -k(x_2 - x_1 + l)$$

How to solve? We're getting a hint with the $x_2 - x_1$ that we should try and make a substitution like

$$y = x_2 - x_1 + l$$

$$m_1m_2\ddot{x_1} - m_1m_2\ddot{x_2} = km_1(x_2 - x_1 + l) + km_2(x_2 - x_1 + l) = yk(m_1 + m_2)$$

$$\ddot{y} = \frac{-yk(m_1 + m_2)}{m_1m_2} = \frac{-k}{\mu}y$$

Where μ is the reduced mass. The solution is

$$y(t) = Asin\left(\sqrt{\frac{k}{\mu}}t\right) + Bcos\left(\sqrt{\frac{k}{\mu}}t\right) = x_2(t) - x_1(t) + l$$

Initial conditions:

$$l - \Delta l - 0 + l = A(0) + B$$

So I think I may have made a sign error, I think the l's are supposed to cancel out. If not, we get $B = 2l - \Delta l$, and A = 0 from the velocity condition.

B. Our equations of motion change to:

$$m_1\ddot{x_1} = k(x_2 - x_1 + l) + \sigma(\ddot{x_2} - \ddot{x_1})$$
$$m_2\ddot{x_2} = -k(x_2 - x_1 + l) - \sigma(\ddot{x_2} - \ddot{x_1})$$

And so it is just a matter of solving the ODEs again. I believe it becomes something like:

$$\ddot{y} = -\frac{\sigma}{\mu}\dot{y} - \frac{k}{\mu}y$$

Which can be solved using the characteristic equation

$$r^2 + \frac{\sigma}{\mu}r + \frac{k}{\mu} = 0$$

Find the roots and the solution is of the form $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

The roots are:

$$r = \frac{\frac{-\sigma}{\mu} \pm \sqrt{\frac{\sigma^2}{\mu^2} - \frac{4\sigma k}{\mu^2}}}{2}$$

$$\mathrm{y}(t) = c_1 e^{(\frac{-\sigma}{\mu} + \sqrt{\frac{\sigma^2}{\mu^2} - \frac{4\sigma k}{\mu^2}})t} + c_2 e^{(-\frac{\frac{\sigma}{\mu} - \sqrt{\frac{\sigma^2}{\mu^2} - \frac{4\sigma k}{\mu^2}}}{2})t} = e^{-\frac{\sigma}{\mu}t}(c_1 e^{\left(\sqrt{\frac{\sigma^2}{\mu^2} - \frac{4\sigma k}{\mu^2}}\right)t} + c_2 e^{\left(-\sqrt{\frac{\sigma^2}{\mu^2} - \frac{4\sigma k}{\mu^2}}\right)t})$$

Whether or not the system is underdamped or overdamped depends on the value of the variables.

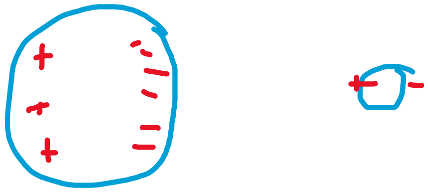
This makes sense, because we expect that now that there is a friction force, our equation of motion will look exponential as opposed to sinusoidal.

13. Describe what happens to an atom when it approaches a conducting sphere with a charge of +1 coulomb. How does the force on the atom depend on the distance from centre of the sphere? If the force is F at distance y from the centre of the sphere, what is the force when the distance is $2y$?

NOTE: Following solution almost certainly wrong lol.

Ok, what is happening here: We have our atom, consisting of protons and electrons approaching our conducting sphere. Charge of 1C, so pretty big charge.
If these were two point charges, we could just say $F = \frac{kq_1q_2}{r^2}$ and leave, but it's a bit more complicated.

When our atom approaches, the conducting sphere is going to become polarized. However… the atom will also become polarized. So creating maybe something like this:



Where the electrons in the sphere are attracted to the protons. Now, both the sphere and atom have dipoles with fields like

$$E_{dip} \propto \frac{p}{r^3} = \frac{qd}{r^3}$$

Where d is the distance between the dipoles, so ~ the sphere or atom diameter.

The sphere will become polarized according to $p_{sp} = \alpha E_{atom}$ where alpha is the polarizability. It will create a field $E_{sp} \propto \frac{p_{sp}}{r^3} = \frac{\alpha E_{atom}}{r^3}$.

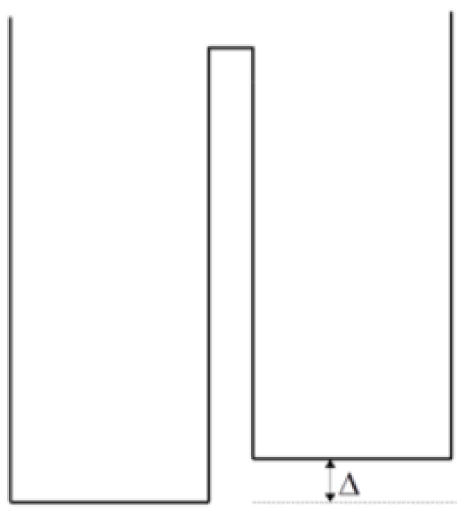
It will also feel an energy $U = -p_{sp} \cdot E_{atom} = \alpha E_{atom}^2 \propto \frac{\alpha p_{atom}^2}{r^6}$

So, the energy scales like 1/r6. Now, can I just say $F = -\frac{dU}{dr} \propto \frac{1}{r^7}$?

This would cause the force at 2y to be $F_2 = \frac{F}{128}$.

Does this make sense? I don't know. This seems weird. Now, if there were two independent dipoles that were already there, I would say $F = -\frac{dU}{dr} = -\frac{d}{dr}(p_1 \cdot E_2) \propto \frac{p_1 p_2}{r^4}$.

But… in our initial scenario, this is different, because the induced dipole is directly related to the E-field of the other dipole.



14. Consider a symmetric double well potential with a barrier much higher than the oscillation frequency at the bottom. (See above figure and initially assume $\Delta = 0$). Initially, a particle is in the left (L) side of the well. An experimentalist observes that the probability of finding the particle in the right hand side increases until it reaches a maximum value X at time T.

A. What is the value of X? With this information, can you predict what happens to the probability after that?

B. Describe qualitatively what happens to your prediction if there is a tilt in the double well potential so that L is Δ lower in energy than R? (The tilt is much smaller than the barrier height.) Does X depend on the tilt Δ?

A. The value of X is 1. The particle will go back and forth between the left and right well.
B. Now, the right well is higher energy – the particle will not like to be there as much, and will prefer the lower energy left well. So the value of X will be lower, the higher Δ is, the lower X will be.

15. A rubber band is stretched between two posts 10 cm apart; the tension in the rubber band is 10 N and the temperature is 25°C. Then, the temperature rises from 25°C up to 30°C, and as a result the tension in the band increases to 11 N. Next, the rubber band is stretched by an additional 1 cm. The rubber band stays at 30°C during this process by exchanging heat with the surrounding air. How much heat is exchanged, and is it released or absorbed by the rubber band? Assume the thermal expansion between the posts during the temperature rise is negligible. Hint: start by writing down a differential of internal energy U in terms of tension, length, temperature, and entropy, then work out a Maxwell relation to solve the problem.

$$dU = TdS - PdV = TdS - \frac{F}{A}dV = TdS - Fdl$$

The force here is tension! So $dU = TdS - \tau dl$, where tau is the tension force. We need to work out a Maxwell relation.

It is true that $\frac{dU}{dS}$ at constant l is T (similar for tau at constant S) so we can make the substitutions

$$dU = \left(\frac{dU}{dS}\right)_l dS + \left(\frac{dU}{dl}\right)_S dl$$

Because this is an exact differential, it's true that

$$\left(\frac{d}{dl}\left(\frac{dU}{dS}\right)_l\right)_S = \left(\frac{d}{dS}\left(\frac{dU}{dl}\right)_S\right)_l$$

$$\left(\frac{d}{dl}(T)_l\right)_S = \left(\frac{d}{dS}(-\tau)_S\right)_l$$

And so we get $\left(\frac{dT}{dl}\right)_S = \left(\frac{-d\tau}{dS}\right)_l$

Because $\Delta S = \frac{Q}{T}$, if we want to find the heat exchanged, we just need to find the change in entropy.

$$dS = -\frac{d\tau\, dl}{dT} = \frac{-(1N)(0.01m)}{5K} = -\frac{0.002J}{k}$$

$$\boxed{Q = -0.002\frac{J}{K} * 303K = -0.606J}$$

Because stretching the band increases the tension, the heat is released (because the temperature would want to rise). We can see this also by the fact that Q is negative.

16. In this problem we will consider the rotational kinetic energy of a hydrogen (H_2) molecule.

A. If the interatomic distance is R, what is the energy of the lowest possible excited rotational state of H_2?

B. Suppose that the protons' spins are parallel. What are the allowed values of the orbital angular momentum quantum number j of the molecule? (You should neglect the electrons' degrees of freedom here.)

C. Write down formulas for the partition function and the average rotational kinetic energy of the molecule if the protons' spins are parallel and the molecule is in thermal equilibrium with a thermal bath of temperature T.

A. Here we have an H_2 molecule. The energy associated with rotation is

$$E_{rot} = \frac{L^2}{2I} = \frac{L^2}{2\mu r^2}$$

My is the reduced mass. Because the hydrogens obviously have the same mass, the reduced mass is $\mu = \frac{m_H}{2}$.

Now let's quant-ify this (as in… make it quantum) using the fact that the angular momentum is quantized.

$$E_{rot} = \frac{l(l+1)h}{2\mu r^2}$$

In the lowest energy state… E=0? In the next one, $E_{rot} = \frac{1}{\mu r^2}$. Can 0 be an answer? It could just not be rotating I suppose…. AH: it says lowest EXCITED STATE, which means greater than the ground state.

B. Allowed values go from:

$$J_{max} \dots\dots J_{min} = |J_1 + J_2| \dots |J_1 - J_2|$$

In integer units of 1. The orbital angular momentum quantum number depends on the quantum orbital.