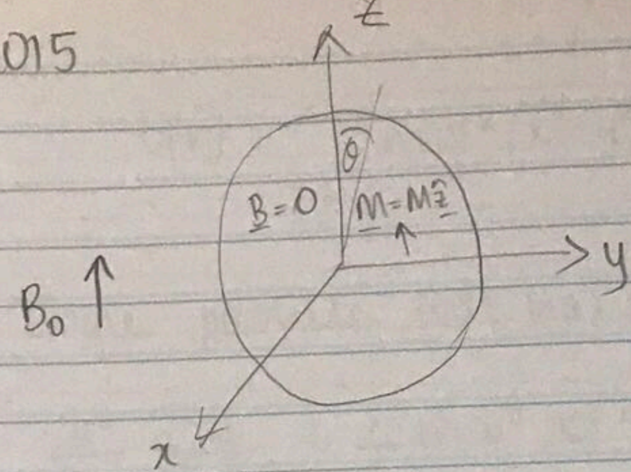
2015

Q2



Field inside uniformly magnetised sphere of total moment $\underline{M}$: $\underline{B} = \frac{2\mu_0}{3}\underline{M}$

$$\Rightarrow \quad B_{in} = B_0 = \frac{2\mu_0}{3} M$$

$$\Rightarrow M = \frac{3}{2}\mu_0 B_0$$

Or At surface need $B_0 = B_{in}$

B_{in} = dipole field. Equate to find M.
Plug back into dipole field & add B_0 to find field outside. Should go to B_0 at infinity.

$K_b = \underline{M} \times \hat{n}$

4. $V(r) = \frac{1}{2} m\omega^2 (x^2+y^2+z^2)$ Two electrons, a & b

A. Single particle will have:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2} m\omega^2 x^2 \psi = E\psi$$ and same in y and z

$$\psi_0(x,y,z) = A \exp\left(-\frac{m\omega}{2\hbar}(x^2+y^2+z^2)\right)$$

$N = n_x + n_y + n_z$

$$E_n = \left(n_x + n_y + n_z + \frac{3}{2}\right)\hbar\omega$$

<u>Ground</u> $\psi_0 = \psi_a(N=0)\,\psi_b(N=0)\,\frac{|\uparrow\downarrow - \downarrow\uparrow\rangle}{\sqrt{2}}$ $S=0$

Energy $= 3\hbar\omega$ Degeneracy $= 1$

Require spin antisymm as spatial state is symm

<u>First excited</u> $\psi_1 = \left(\psi_a(N=0)\psi_b(N=1) - \psi_a(N=1)\psi_b(N=0)\right)$

$* \; |\uparrow\uparrow\rangle$ $S=1$

or $* \; |\downarrow\downarrow\rangle$ $S=1$

$* \; \frac{|\uparrow\downarrow + \downarrow\uparrow\rangle}{\sqrt{2}}$ $S=1$

OR $\psi_1 = \left(\psi_a(N=0)\psi_b(N=1) + \psi_a(N=1)\psi_b(N=0)\right)$

$* \; \frac{|\uparrow\downarrow - \downarrow\uparrow\rangle}{\sqrt{2}}$

Energy $= 4\hbar\omega$ Degeneracy $= 4$

$= \underline{12}$

(maybe x3 for 3D options?) ✓

$$S^2|\psi\rangle = \hbar^2 s(s+1)|\psi\rangle$$

B. $$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\psi + \alpha\vec{S}_1\cdot\vec{S}_2\psi = E\psi$$

$$(S_1+S_2)^2 = S_1^2 + S_2^2 + 2S_1\cdot S_2$$

$$\rightarrow \quad S_1\cdot S_2 = \frac{1}{2}\left(S^2 - S_1^2 - S_2^2\right)$$

$$S=0$$

$$\vec{S}_1\cdot\vec{S}_2\frac{|\uparrow\downarrow-\downarrow\uparrow\rangle}{\sqrt{2}} = \frac{\hbar^2}{2}\left(-\frac{3}{4}-\frac{3}{4}\right)$$

$$= \frac{3}{4}\hbar^2$$

→ Interaction lowers energy of ground state by $\frac{3\alpha\hbar^2}{4}$ but degeneracy still 1.

$$\vec{S}_1\cdot\vec{S}_2|\uparrow\uparrow\rangle = \frac{\hbar^2}{2}\left(2-\frac{3}{4}-\frac{3}{4}\right)$$

$$= +\frac{\hbar^2}{4}$$

And same for $|\downarrow\downarrow\rangle$ and $\frac{|\uparrow\downarrow+\downarrow\uparrow\rangle}{\sqrt{2}}$

→ Interaction lowers energy of singlet state

$$\psi_1 = \left(\psi_a(N=0)\psi_b(N=1) + \psi_a(N=1)\psi_b(N=0)\right) * \frac{|\uparrow\downarrow-\downarrow\uparrow\rangle}{\sqrt{2}}$$

by $\frac{\alpha\hbar^2}{4}$

And raises the triplet states

$$\psi_1 = (\psi_a(N=0)\psi_b(N=1) - \psi_a(N=1)\psi_b(N=0))$$

$* \quad |\uparrow\uparrow\rangle$

$* \quad |\downarrow\downarrow\rangle$

$* \quad \frac{|\uparrow\downarrow + \downarrow\uparrow\rangle}{\sqrt{2}}$

by $\frac{3\alpha\hbar^2}{4}$

So splits degeneracy of 1st excited state into 9 and 3

singlet & triplet $\times 3$ due to spatial degeneracy x, y & z.

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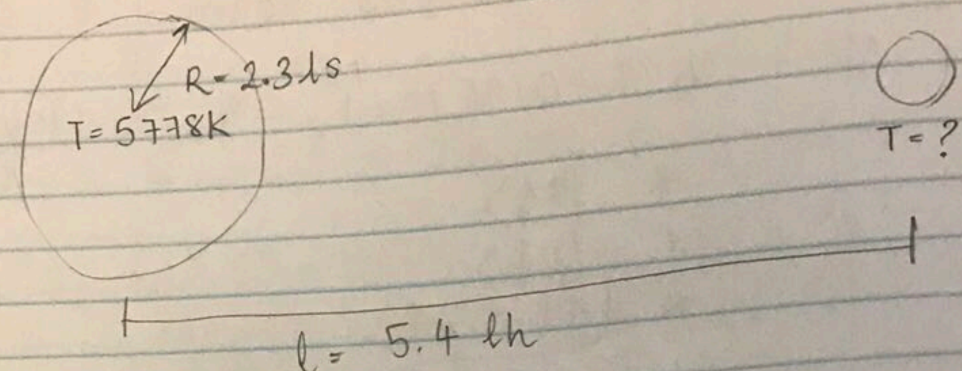


Stefan Boltzmann $j = \sigma T^4$

$\frac{\text{total E}}{\text{area} * \text{time}}$

Black body spectrum $\rho(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3 (e^{\hbar\omega/kT} - 1)} = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$

$\frac{dE}{d\omega}$

Assume Pluto small & far away enough that point absorber.

$$P_{sun} = \sigma T^4 \times 4\pi R_{sun}^2$$

$$= 5.67 \times 10^{-8} \times 5778^4 \times 4\pi \times (6.9 \times 10^8)^2$$

$R_{sun} = 2.3 \times 3 \times 10^8 = 6.9 \times 10^8$ m.

$l = 5.4 \times 60 \times 60 \times 3 \times 10^8$
$= 5.8 \times 10^{12}$ m.

$\Longrightarrow P_{sun} = 3.78 \times 10^{26}$ W

At Pluto ~~Flux~~ $= \frac{P_{sun}}{4\pi l^2} = \frac{3.78 \times 10^{26}}{4\pi \times (5.8 \times 10^{12})^2}$

$= 0.894 \frac{W}{m^2}$ woah so little!

$c = f\lambda$
$= \frac{\omega \lambda}{2\pi}$

$\omega = 2\pi f$

Wien's Law
$\lambda_{max} = b/T$

Now, how much of that is reflected

Approximate Reflectivity as 0.6 (60% reflected) $\lambda = 0 \rightarrow 10\,\mu m$
0 $\lambda = 10 \rightarrow \infty\,\mu m$

0.6 $\omega = \frac{2\pi c}{\lambda}$ $\infty \rightarrow 1.9\times10^{14}\ \text{rads}^{-1}$

$$P = \int_{1.9\times10^{14}}^{\infty} \frac{\hbar\omega^3}{\pi^2c^3(e^{\hbar\omega/kT}-1)}\,d\omega$$

ω = large
$e^{\hbar\omega/kT} \gg 1$

$$\approx \frac{\hbar}{\pi^2c^3}\int_{1.9\times10^{14}}^{\infty} \omega^3 e^{-\frac{\hbar\omega}{kT}}\,d\omega$$

$x = \frac{\hbar\omega}{kT}$

$$= \frac{\hbar}{\pi^2c^3}\left(\frac{kT}{\hbar}\right)^4\int_{0.25117}^{\infty} x^3 e^{-x}\,dx$$

$$= \frac{(kT)^4}{\pi^2c^3\hbar^3}\Big[-e^{-x}(x^3+3x^2+6x+6)\Big]_{0.2517}^{\infty}$$

$$= \frac{4.05\times10^{-77}}{8.88\times10^{17} \times 1.11\times10^{-68} \times 1\times10^{-34}}\left(0+6\right)$$

$P = 2.46\times10^{8}\ W$ ← basically negligable percentage of total power is in this freq range

So maybe just say $P_{Pluto} = 0.894\ W$
$= \sigma T^4$

Do need to worry about Pluto's radius
But then it cancels out!

$$T = \sqrt[4]{\frac{0.894}{\sigma}} = 63\,K.$$

This is not far off from correct answer = 50K.

Find λ_{max} is in 0.6 reflectivity region.

$N_e = \frac{N}{V}$ Δx

6. Heisenberg uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$

$N_e = \frac{N}{V}$ $\Delta x \simeq N_e^{-1/3}$ ← would not have known

$\Delta p = m \Delta v$ → $\Delta p \geq \frac{\hbar}{2} \frac{1}{\Delta x}$

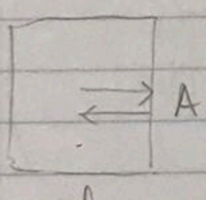
non relativistic

$\Delta v \geq \frac{\hbar}{2m_e} N_e^{1/3}$

Imagine a particle in a box, colliding elastically w walls:

☆ not sure

$P = FA \simeq p \frac{N}{V} v$



Pressure = flux of momentum

why is it okay to change to Δv from v?

$\Rightarrow$ $P_{degen} \sim N_e m_e (\Delta v)^2$

Center of White dwarf

$P_{grav} \sim \frac{GM^2}{R^4}$

Stable: $P_{degen} \sim P_{grav}$

$N_e \cancel{m_e} \frac{\hbar^2}{4m_e^2} N_e^{2/3} \sim \frac{GM^2}{R^4}$

ignoring constants.

$\frac{\hbar^2}{m_e} N_e^{5/3} \sim \frac{GM^2}{R^4}$

$M = \rho \frac{4}{3} \pi R^3$

$\rho = N_e \times m_e$

$N_e = \frac{3M}{4 m_e \pi R^3}$

$$\frac{\hbar^2}{m_e}\left(\frac{3}{4}\frac{M}{m_e \pi R^3}\right)^{5/3} \sim \frac{GM^2}{R^4}$$

$$\frac{\hbar^2}{m_e^{8/3}}\frac{M^{5/3}}{R^5} \sim \frac{GM^2}{R^4}$$

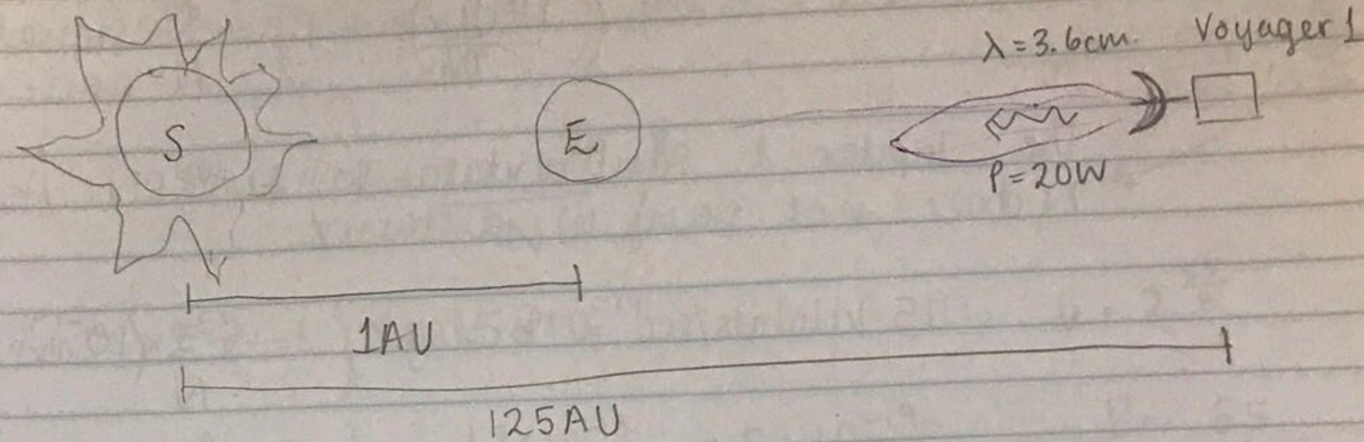
$$\frac{\hbar^6}{m_e^8}\frac{M^5}{R^{15}} \sim \frac{G^3 M^6}{R^{12}}$$

$$M \sim \frac{1}{R^3}\frac{\hbar^6}{G^3 m_e^8}$$

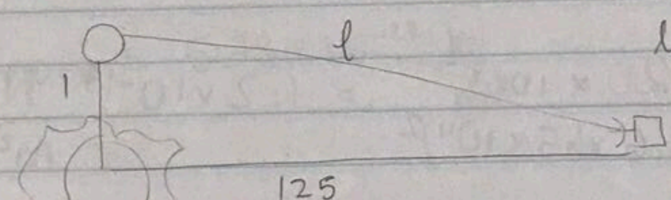
So as R increases M decreases → unexpected!

$$\boxed{R \propto M^{-1/3}} \quad \checkmark$$

7.

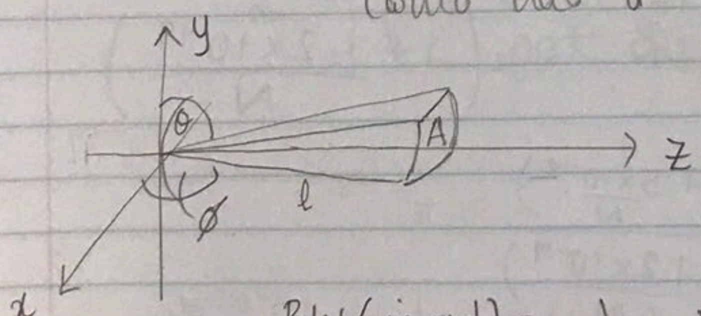


A. Assuming lines up like in picture → distance antenna to E = 124 AU.



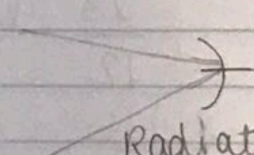
$l = \sqrt{1^2 + 125^2} = 125 \text{ AU}$

Could had d = 125AU if as shown.



~~Feel like I'm missing something here?~~

$$BW(\text{in rad}) = \frac{\lambda}{D} = \frac{3.6 \text{ cm}}{3.6 \text{ m}} = \frac{1}{100} \text{ rad.}$$



Radiation in lobe.

$A \approx l^2 BW^2$

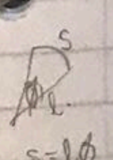
$l = 124 \text{ AU}$

$$\Rightarrow Flux_{Earth} = \frac{P}{l^2 \left(\frac{1}{100}\right)^2}$$

$$= \underline{\underline{5.78 \times 10^{-22} \frac{W}{m^2}}}$$

Beamwidth $\sim k \frac{\lambda}{D}$

From dim analysis & know narrower for ↑D.



Noise is Gaussian & doesn't change

B. Bit Rate = $B \log_2\left(1+\frac{S}{N}\right)$

S = signal
N = noise

Use Jupiter & Saturn data to find N. Assume N does not vary w distance

$$115 \text{ kilobits/sec} = B \log_2\left(1 + \frac{5\times10^{-19}}{N}\right)$$

$$\text{Flux}_{\text{Jupiter}} = \frac{20 \times 100^2}{(4.2 \times 1.5 \times 10^{11})^2} = 5.0\times10^{-19}\ \frac{W}{m^2}$$

N will be in $\frac{W}{m^2}$

$$\text{Flux}_{\text{Saturn}} = \frac{20 \times 100^2}{(8.6 \times 1.5 \times 10^{11})^2} = 1.2\times10^{-19}\ \frac{W}{m^2}$$

$$60 \text{ kilobits/sec} = B \log_2\left(1 + \frac{1.2\times10^{-19}}{N}\right)$$

$$\frac{115}{60} = \frac{\log_2\left(1 + \frac{5\times10^{-19}}{N}\right)}{\log_2\left(1 + \frac{1.2\times10^{-19}}{N}\right)}$$

$$\frac{23}{12} \log_2\left(1 + \frac{1.2\times10^{-19}}{N}\right) = \log_2\left(1 + \frac{5\times10^{-19}}{N}\right)$$

$$\left(1 + \frac{1.2\times10^{-19}}{N}\right)^{\frac{23}{12}} = 1 + \frac{5\times10^{-19}}{N}$$

$$2^{115} = \left(1 + \frac{5\times10^{-19}}{N}\right)^{B} \qquad \log_2 k = \frac{55}{B}$$

$$2^{60} = \left(1 + \frac{1.2\times10^{-19}}{N}\right)^{B}$$

$$(N + 1.2\times10^{-19})\, 2^{\frac{55}{B}} = N + 5\times10^{-19} \qquad k = 2^{\frac{55}{B}}$$

$$N = \frac{k \times 1.2\times10^{-19} - 5\times10^{-19}}{(1-k)} \qquad B = \frac{55}{\log_2 k}$$

$$2^{115} = \left(1 + \frac{5\times\cancel{10^{-19}}\,(1-k)}{1.2\times\cancel{10^{-19}}\,k - 5\times\cancel{10^{-19}}}\right)^{\frac{55}{\log_2 k}}$$

$$\text{Flux}_{\text{Pluto}} = 5.78\times10^{-22}\ \frac{W}{m^2}$$

A lot of algebra!

$$2^{r} = \left(1 + \frac{5.78\times10^{-22}}{N}\right)^{B}$$

$$2^{\frac{115}{55}\log_2 k} = 1 + \frac{5(1-k)}{1.2k - 5}$$

$$k^{\frac{115}{55}} = \frac{1.2k - 5 + 5 - 5k}{1.2k - 5}$$

$$(1.2k - 5)\, k^{\frac{23}{11}} = -3.8k$$

$$1.2k^{\frac{23}{11}} - 5k^{\frac{12}{11}} + 3.8 = 0$$

$$\log_2 3 = A$$

$$A \ln 2 = \ln 3$$

$$A = 1.58$$

$k = 1$ solves this

$k \approx 3$ solves this (Wolfram)

$$\Rightarrow B = 34.7$$

$$N = 7\times10^{-20}$$

So then put in (dubious) numbers found:

$$r = B \log_2 \left(1 + \frac{5.78 \times 10^{-22}}{N} \right)$$

$$2^{\frac{r}{34.7}} = \left(1 + \frac{5.78 \times 10^{-22}}{7 \times 10^{-20}} \right)$$

$$\frac{r}{34.7} \ln 2 = \ln (1.008)$$

$$r = 34.7 \ln$$

$$r = 0.41 \text{ kilobits/sec.}$$

Maybe?

Taylor expand expression.

8. A. $v = K \rho^a g^b \lambda^c$

$g = \frac{F}{m} = \frac{ma}{m} = a.$

Dimensional analysis

$\lambda = l.$

$\rho = \frac{m}{l^3}$ $\quad g = \frac{l}{t^2}$

$$\frac{l}{t} = \left(\frac{m}{l^3}\right)^a \left(\frac{l}{t^2}\right)^b l^c$$

$$= \left(\frac{l}{t^2}\right)^{1/2} l^{1/2}$$

google

$a = 0,\ b = \frac{1}{2},\ c = \frac{1}{2}$

$v \propto g^{1/2} \lambda^{1/2}$

B. $v_p = \frac{\omega}{k}$ $\quad v_g = \frac{\partial \omega}{\partial k}$

$\lambda = \frac{2\pi}{k}$

$\psi_1(x,t) = \sin(\omega t - k_1 x)$

$\psi_2(x,t) = \sin((\omega + \Delta\omega)t - k_2 x)$

$\omega_{beat} = \Delta\omega$

$v_p = K\sqrt{\frac{g}{k}}$

$\frac{\omega}{k} = K\sqrt{\frac{g}{k}}$

$\omega = K\sqrt{gk}$

$v_g = \frac{\partial \omega}{\partial k} = \frac{K}{2}\sqrt{\frac{g}{k}}.$

$$\boxed{2v_g = v_p}$$

$k = \frac{1}{g}\left(\frac{\omega}{K}\right)^2$

$$k_2 = k_1 + \frac{1}{g}$$

$$k_1 = \frac{1}{g}\left(\frac{\omega}{K}\right)^2 \qquad k_2 = \frac{1}{g}\left(\frac{\omega + \Delta\omega}{K}\right)^2$$

$$v_{p_1} = v_{p_2} \qquad v_p = \frac{\omega}{k_1} = \frac{\omega + \Delta\omega}{k_2}$$

$$\Delta\omega = \frac{k_2}{k_1}\omega - \omega$$

$$= \omega\left(\frac{k_2}{k_1} - 1\right)$$

$$= v_p\left(k_2 - k_1\right)$$

$$= \frac{v_p}{gK^2}\left(\cancel{\omega^2} + 2\omega\Delta\omega + \Delta\omega^2 - \cancel{\omega^2}\right)$$

$$-\frac{\cancel{\Delta\omega}\, gK^2}{v_p} + 2v_p k_1 \cancel{\Delta\omega} + \Delta\omega^{\cancel{2}} = 0.$$

$$\Delta\omega = \left(\frac{gK^2}{v_p} - 2v_p k_1\right)$$

10. At what atomic no. Z do atomic electrons move at relativistic speeds?

$$pc > \frac{mc^2}{2} \qquad E^2 = p^2c^2 + m^2c^4$$

$$\sqrt{E^2 - m^2c^4} > \frac{mc^2}{2} \qquad E = \gamma mc^2$$

$$KE = (\gamma - 1)mc^2$$

$$p = \gamma mv$$

$$E^2 > m^2c^4 + \frac{m^2c^4}{4}$$

$$E^2 \geqslant \frac{5}{4} m^2c^4$$

$$\rightarrow \gamma^2 \geqslant \frac{5}{4}$$

$$\gamma > \sqrt{\frac{5}{4}}$$

$p = \hbar k$ Or maybe more useful to keep as condition on p

$$\frac{-\hbar^2}{2\mu} \nabla^2 \psi_n \quad - \frac{Ze^2}{4\pi\varepsilon_0 r} \psi_n = E_n \psi_n \qquad \mu = \frac{m_e M_{nucleus}}{m_e + M_{nucleus}}$$

Hydrogen-like atom energies:

Use Virial Theorem $\quad E = \frac{V}{2} = -T$ why?

$$V = \frac{-Ze^2}{4\pi\varepsilon_0 r} \qquad T = \frac{1}{2}\mu v^2 = \frac{p^2}{2\mu} \qquad p = \mu v = \frac{n\hbar}{r}$$

$$\frac{Ze^2}{8\pi\varepsilon_0 r} = \frac{n^2\hbar^2}{2\mu r^2} \rightarrow r = \frac{n^2\hbar^2}{\cancel{2}\mu} \frac{\cancel{8}^{4}\pi\varepsilon_0}{Ze^2}$$

So then $E = \frac{-Ze^2}{8\pi\varepsilon_0} \frac{Z\mu e^2}{4n^2\hbar^2\pi\varepsilon_0}$

$$= \frac{-Z^2e^4\mu}{32\pi^2\hbar^2\varepsilon_0^2} \frac{1}{n^2}$$

So as n increases, E decrease. Relativistic motion will occur in ground state

$m_{nucleus} = Z(m_p + m_n)$ assume same number of protons & neutrons (true for lower Z elements)

$$\rightarrow \mu = \frac{Zm_e(m_p + m_n)}{Z(m_p + m_n) + m_e}$$

For $n = 1$ $E = \sqrt{p^2c^2 + m_e^2c^4}$

$$= \frac{Z^2e^4\mu}{32\pi^2\hbar^2\varepsilon_0^2}$$

At limit of relativistic $p^2c^2 = \frac{m^2c^4}{4}$

$$\Rightarrow E = \sqrt{\frac{5}{4} m_e^2c^4}$$

Slot numbers in to find Z

$$\sqrt{\frac{5}{4}}\, c^2 = \frac{Z^3e^4}{32\pi^2\hbar^2\varepsilon_0^2} \frac{(m_p + m_n)}{(Z(m_p + m_n) + m_e)}$$

$$3.1 \times 10^{-71}(3.34 \times 10^{-27} Z + 9.1 \times 10^{-31}) = Z^3\, 6.6 \times 10^{-76}$$

$\times 3.34 \times 10^{-27}$

$$3.1 \times 3.34 \times 10^5\, Z + 3.1 \times 9.1 \times 10^1 = Z^3\, 6.6 \times 3.34$$

$1.04\times10^6 Z + 282 = 22.0 Z^3$

$22Z^3 - 1\times10^6 Z - 282 = 0.$

$Z \approx 213$ apparently solves this.

Maybe could have made my life easier by approximating

$$\mu \approx m_e$$

$$\Rightarrow \sqrt{\frac{5}{4}}\,c^2 = \frac{Z^2 e^4}{32\pi^2\hbar^2\varepsilon_0^2}$$

$Z \approx 41990$?? uh oh.

Try again:

$$V = \frac{-Ze^2}{4\pi\varepsilon_0 r} \qquad T = pc$$

Virial theorem

$$E = \frac{V}{2} = -T$$

$$\frac{Ze^2}{8\pi\varepsilon_0 r} = pc$$

say $r \approx a_0$

$$\rightarrow \quad Z > \frac{4\pi\varepsilon_0 a_0\, mc^2}{e^2}$$

$$> 19000$$

$$\boxed{\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{\hbar}{mca_0}}$$

11. $|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle \pm |\downarrow\downarrow\rangle \right.$

$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle \right.$

$H_z = -\vec{m}.\vec{B_z}$ $\quad\quad \vec{m} = \mu \vec{s}$

$\boxed{U_z(t,0) = e^{-\frac{iE_z t}{\hbar}}}$ $\quad\quad H_z \psi = E_z \psi$

Check $\quad i\hbar \frac{\partial}{\partial t} U_z = H U_z$

$i\hbar \frac{-iE}{\hbar} U_z = H U_z$

$E U_z = H U_z$

And similarly $\quad \boxed{U_x(t,0) = e^{-\frac{iE_x t}{\hbar}}}$ $\quad\quad H_x \psi = E_x \psi$

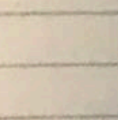
$\vec{B_z} = B_z \hat{z}$ $\quad\quad \vec{B_x} = B_x \hat{x}$

A. $U_z(t_A, 0)|z_+\rangle = -i|z_+\rangle$ $\quad\quad S_z = \frac{\hbar}{2}\sigma_z$

$H_z|z_+\rangle = -\mu S_z B_z |z_+\rangle$

$= -\mu B_z \frac{\hbar}{2} |z_+\rangle$ $\quad\quad E_z = -\mu B_z \frac{\hbar}{2}$

$U_z(t,0) = -i = e^{-\frac{i\pi}{2}} = e^{-\frac{iE_z t}{\hbar}} = e^{i\frac{3}{2}\pi}$



$\frac{E_z t}{\hbar} = -3\frac{\pi}{2}$

$$t_A = -\frac{3\hbar}{2}\pi \frac{1}{E_z}$$

$$= \frac{3\cancel{\hbar}}{\cancel{2}}\pi \frac{\cancel{2}}{\mu B_z \cancel{\hbar}}$$

$$= \frac{3\pi}{\mu B_z}$$

B. $U_z(t_A, 0)|z_-\rangle = e^{-\frac{i\mu B_z t_A}{2}}|z_-\rangle$

$E_z = \mu B_z \frac{\hbar}{2}$

+ve.

$$= e^{-i\frac{3\pi}{2}}|z_-\rangle$$

$$= i|z_-\rangle$$

+ve.

Find eigenstates of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = E\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \pm\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$\beta = \alpha$
$\beta = -\alpha$ $\rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$|z_\pm\rangle = \frac{1}{\sqrt{2}}(|x_+\rangle \pm |x_-\rangle)$$

$$H_x |x_\pm\rangle = -\mu B_x \vec{S}_x |x_\pm\rangle$$
$$= \mp \mu B_x \frac{\hbar}{2} |x_\pm\rangle.$$

$$U_x(t_A, 0)|z_\pm\rangle = \frac{1}{\sqrt{2}}\left(U_x(t_A, 0)|x_+\rangle \pm U_x(t_A, 0)|x_-\rangle \right)$$
$$= \frac{1}{\sqrt{2}}\left(e^{+i\frac{\mu B_x t_A}{2}} |x_+\rangle \pm e^{-i\frac{\mu B_x t_A}{2}} |x_-\rangle \right)$$

Assume $B_x = B_z$

$$U_x(t_A, 0)|z_\pm\rangle = \frac{1}{\sqrt{2}}\left(-i|x_+\rangle \pm i|x_-\rangle \right)$$
$$= \frac{i}{\sqrt{2}}\left(-|x_+\rangle \pm |x_-\rangle \right)$$
$$= \frac{i}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \pm \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} \right)$$

$$U_x(t_A, 0)|z_+\rangle = i\begin{pmatrix}0\\-1\end{pmatrix} = -i|z_-\rangle$$

$$U_x(t_A, 0)|z_-\rangle = i\begin{pmatrix}-1\\0\end{pmatrix} = -i|z_+\rangle$$

c. Four Bell states: ① $\frac{1}{\sqrt{2}}\left(|++\rangle + |--\rangle\right)$

② $\frac{1}{\sqrt{2}}\left(|++\rangle - |--\rangle\right)$

$U_x = U_x(t_A, 0)$
$U_z = U_z(t_A, 0)$

③ $\frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle \right)$
④ $\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle \right)$

$U_z |z_+\rangle = -i\,|z_+\rangle$
$U_z |z_-\rangle = i\,|z_-\rangle$
$U_x |z_+\rangle = -i\,|z_-\rangle$
$U_x |z_-\rangle = -i\,|z_+\rangle$

① → ② Apply iU

② → ③ Apply U_x and then U_z

③ → ④

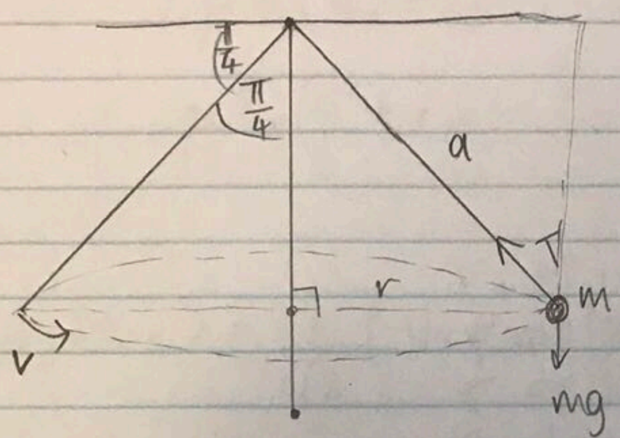
④ → ①

Run out of time
but I think I could
get this

Hilroy

12

A



$$T\cos\frac{\pi}{4} = mg$$

$$2r^2 = a^2$$

$$r = \frac{a}{\sqrt{2}}$$

$$T\sin\frac{\pi}{4} = \frac{mv^2}{r}$$

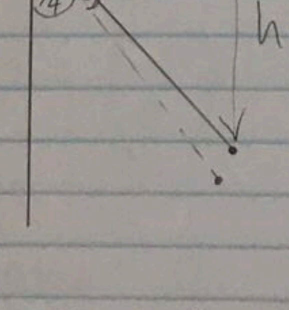
$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$v = \sqrt{\frac{a}{\sqrt{2}}\,\frac{1}{m}\,mg}$$

$$v^2 = \frac{ga}{\sqrt{2}}$$

(doesn't look good?)

B



$$F = T\sin\theta = T\sin\left(\frac{\pi}{4} + \varepsilon\right)$$

$$m \frac{\partial^2 x}{\partial t^2} = F$$

$$m \frac{\partial^2 \varepsilon}{\partial t^2} = \frac{\partial F}{\partial \varepsilon} \varepsilon$$

$$m \ddot{\varepsilon} = T \cos\left(\frac{\pi}{4} + \varepsilon\right) \varepsilon$$

Taylor expand about equilibrium

$$V(\theta) = V\left(\frac{\pi}{4}\right) + \frac{1}{2} V''\left(\frac{\pi}{4}\right) \varepsilon^2$$

$$F = -\frac{dV}{dx} = -V''\left(\frac{\pi}{4}\right) \varepsilon$$

$$V = \frac{1}{2} m v^2 - mgh$$

put in terms of θ

$$= \frac{1}{2} \frac{mga\sin^2\theta}{\cos\theta} - mga\cos\theta$$

$$= \frac{mga}{2}\left(\frac{\sin^2\theta - 2\cos^2\theta}{\cos\theta}\right)$$

$T\cos\theta = mg$

$h = a\cos\theta$

$T\sin\theta = \frac{mv^2}{r}$

$r = a\sin\theta$

$$\frac{dV}{d\theta} = \frac{mga}{2}\left(\frac{6\cos^2\theta\sin\theta + (\sin^2\theta - 2\cos^2\theta)\sin\theta}{\cos^2\theta}\right)$$

$mv^2 = Ta\sin^2\theta$

$= \frac{mga\sin^2\theta}{\cos\theta}$

$u = \sin^2\theta - 2\cos^2\theta$

$du = 2\sin\theta\cos\theta + 4\cos\theta\sin\theta$

$v = \cos\theta$

$dv = -\sin\theta$

$$\frac{dV}{d\theta} = \frac{mga}{2}\left(\frac{4\cos^2\theta\sin\theta + \sin^3\theta}{\cos^2\theta}\right)$$

$$u = 4\cos^2\theta\sin\theta + \sin^3\theta$$
$$du = \cancel{4}(\cancel{8}\cos\theta\sin^2\theta + 4\cos^3\theta) + 3\sin^2\theta\cos\theta$$

$$v = \cos^2\theta$$
$$dv = 2\cos\theta\sin\theta$$

$$\frac{d^2V}{d\theta^2} = \frac{mga}{2}\left(\frac{(11\sin^2\theta\cos\theta + 4\cos^3\theta)\cos^2\theta - 2\cos\theta\sin\theta(4\cos^2\theta\sin\theta + \sin^3\theta)}{\cos^4\theta}\right)$$

Maybe there's a quicker way to do this?

$$= \frac{mga}{2}\left(\frac{\overset{3}{\cancel{11}}\sin^2\cos^3 + 4\cos^5 - \cancel{8\cos^3\sin^2} - 2\cos\sin^4}{\cos^4}\right)$$

$$= \frac{mga}{2}\left(\frac{3\sin^2\theta\cos^3\theta - 2\sin^4\theta\cos\theta + 4\cos^5\theta}{\cos^4\theta}\right)$$

$$V''\left(\frac{\pi}{4}\right) = \frac{mga}{2}\left(\frac{3\left(\frac{1}{\sqrt{2}}\right)^5 - 2\left(\frac{1}{\sqrt{2}}\right)^5 + 4\left(\frac{1}{\sqrt{2}}\right)^5}{\left(\frac{1}{\sqrt{2}}\right)^4}\right)$$

$$= \frac{5mga}{2\sqrt{2}}$$

$$\omega = \sqrt{\frac{V''(x_0)}{m}} = \sqrt{\frac{5ga}{2\sqrt{2}}}$$

$$= \sqrt{\frac{5}{2}}\ v$$

Another method

12 $L = \frac{1}{2} m\left(a^2 \sin^2\theta \, \dot{\phi}^2 + a^2 \dot{\theta}^2 \right) + mga\cos\theta$

$\ddot{\theta} = \sin\theta \left(\dot{\phi}^2 \cos\theta - \frac{g}{a} \right)$ ← get equation of motion and then Taylor expand in ε

$\ddot{\phi} = 0 = \frac{d}{dt}\left(ma^2 \sin^2\theta \, \dot{\phi} \right) = \frac{dL}{d\phi}$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

Put in $V(\theta) = V\left(\frac{\pi}{4}\right) + \frac{\partial V}{\partial \theta}\left(\frac{\pi}{4}\right)\left(\theta - \frac{\pi}{4}\right)$

$$+ \frac{1}{2}\frac{\partial^2 V}{\partial \theta^2}\left(\theta - \frac{\pi}{4}\right)^2$$

$$F = -\frac{\partial V}{\partial \theta} = -\left(\frac{\partial^2 V}{\partial \theta^2}\left(\frac{\pi}{4}\right)\left(\theta - \frac{\pi}{4}\right) + \frac{\partial V}{\partial \theta}\left(\frac{\pi}{4}\right) \right)$$

$$= -V''\left(\frac{\pi}{4}\right)\varepsilon$$

$$m\ddot{\varepsilon} = -V''\left(\frac{\pi}{4}\right)\varepsilon.$$

Have to be careful, used $\frac{1}{2}mv^2$ but only works

when $\frac{(mv \times r)^2}{2mr^2} = \frac{1}{2}mv^2$

Solve this by saying adiabatic $p^{1-\gamma} T^{\gamma} = \text{const}$

13 $dp = \rho g \, dh$ $\qquad pV = nRT$

$\rho V = Nm$ $\qquad \rho \, dV = -V d\rho \Rightarrow dV = -\frac{V d\rho}{\rho}$

$V dp + p dV = N k_B dT$

$V dp - \frac{Vp}{\rho} d\rho = N k_B T$

$V dp - \frac{N k_B T}{\rho} d\rho = N k_B T$

$p^{1-\gamma} T^{\gamma} = \text{cost}$

$\frac{dp}{p} = \frac{\gamma}{\gamma - 1} \frac{dT}{T}$ $\qquad \frac{\gamma}{\gamma-1} \frac{dT}{\cancel{T}} = \frac{mg}{R\cancel{T}} dz$

$dp = \rho g \, dh \rightarrow dp = \frac{mgP}{RT} dz$ $\qquad \frac{dT}{dz} = \frac{mg(\gamma-1)}{\gamma R}$

$p = \rho R T$

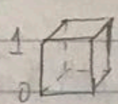
$T - T_0 = \frac{(\gamma - 1)}{\gamma} \frac{mg}{R} z$

put in numbers

$r_{Earth} = 6.4 \times 10^6$ m.

13. ________ 3000m T p

________ 0m 293K 1atm

 $\frac{dp}{dz} = -\rho g$ ← Deriving this formula

$p = FA = mgA$

$\Delta p = \int_1^\infty m(z)\, g(z)\, dz\, A - \int_0^\infty m(z)\, g(z)\, dz\, A$

$= \frac{A}{V} \int_1^0 \rho(z)\, g(z)\, dz$

not sure about this

$\frac{dp}{dz} \approxeq -\rho(z)\, g(z)$

Assuming ideal gas (will not work if humid) but says dry.

| (one mole | $pV = RT$ | $dp = \frac{R}{V} dT$ |

$p \frac{m}{\rho} = RT$

$\frac{dp}{dz} = -\frac{pm}{RT} g$

$\rho = \frac{m}{V}$

$r = r + z$

$g = \frac{GMm}{(r(1+\frac{z}{R}))^2} \approx \frac{GMm}{r^2}\left(1 - \frac{2z}{R}\right)$

$\frac{R}{\cancel{V}} \frac{dT}{dz} = -\frac{\cancel{RT}\, m}{\cancel{V}\, \cancel{RT}} g$

$= -\frac{mg}{R}$

Assume $g \approx$ constant over range.

$$T - T_0 = -\frac{mg}{R} z$$ ← dimensional analysis works.

$T_0 = 293K$

$$T = T_0 - \frac{29 \times 10^{-3} \times 9.8 \times 3000}{R}$$ ← 8.31

$m =$ molar mass

$\sim 80\%$ N_2 14×2 g/mol
$\sim 20\%$ O_2 16×2 g/mol

$$m \approx 0.8 \times 28 + 0.2 \times 32$$
$$= 29 g/mol$$

$T = 190K = -83°C$ seems much too cold!

Maybe can't make gravity assumption

$$dT = -\frac{mg}{R} dz$$

$$\approx -\frac{m}{R}\frac{GMm}{r^2}\left(1 - \frac{2z}{R}\right) dz$$

at surface

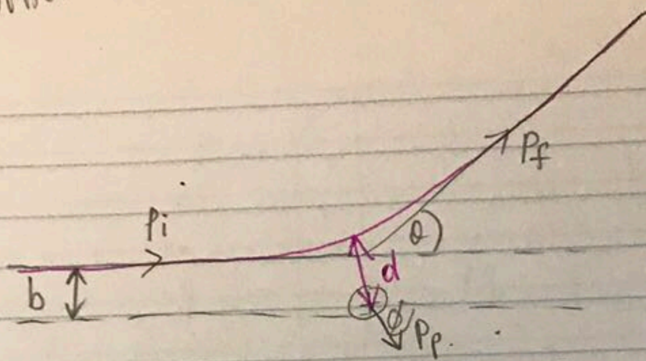
$$T - T_0 \simeq -\frac{mg}{R}\left(z - \frac{z^2}{R}\right)$$

$$T = T_0 - \frac{mgz}{R}\left(1 - \frac{z}{R}\right)$$

$= -83°C$ again

Orbit equations

14.



$$F = \frac{e^2}{4\pi \varepsilon_0 r^2} \qquad V = \frac{e^2}{4\pi \varepsilon_0 r}$$

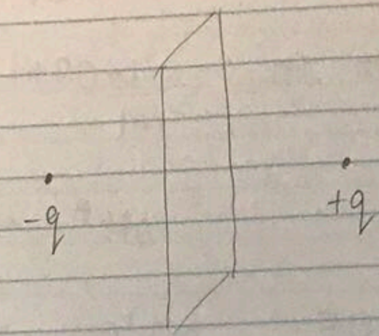
Cons of mom: $\vec{p_i} = \vec{p_p} + \vec{p_f}$

$$p_i = p_p \cos\phi + p_f \cos\theta$$

Cons of E: $\frac{p_i^2}{2m} = \frac{p_f^2}{2m} + \frac{p_p^2}{2m}$

2013

15.



Method of images
→ metal conductor,
want $E = 0$ on surface

$$F = \frac{q^2}{4\pi \varepsilon_0 \, (2d^2)}$$

$$t = q^{\alpha} \; \varepsilon_0^{\beta} \; d^{\gamma}$$

dimensional analysis from quantities in Force

$$t \propto \sqrt{d}$$

$$\rightarrow \quad \frac{t_1}{t_2} = \frac{\sqrt{d_1}}{\sqrt{d_2}}$$

2013

t_0 = now

16. Flux = $1400 \frac{W}{m^2}$ at top of atmosphere 1.5×10^{11} m

$4p \rightarrow {}^4He$

$t_\ell \rightarrow$ Lifetime when 70% of p $\rightarrow$ He. in core.

20% of mass.

At start 75% p 25% He

Energy in one PP reaction

$$\Delta E = (m_{He} - 4m_p)c^2$$
$$= 4.98 \times 10^{-12} \text{ J} \checkmark$$

Luminosity of sun = $P_{sun}(t=t_0) = 1400 \times 4\pi r^2$

$$= 4.0 \times 10^{26} \text{ W} \checkmark$$

So then we know

Reactions/sec $(t=t_0) = 7.9 \times 10^{37} \checkmark$

$M_{sun} = 2 \times 10^{30}$ kg.

Know this but can calculate using:

$$\frac{G M \cancel{m}}{r^2} = \frac{\cancel{m} v^2}{r}$$

$$M = \frac{v^2 r}{G} \qquad v = \frac{2\pi r}{T} \text{ (one year)}$$

$$= \frac{(2\pi)^2 r^3}{G T^2} = 2 \times 10^{30} \text{ kg}$$

yay!

$M_{core} = 4 \times 10^{29}$ kg

Assume $M_{core} \approx$ const and composition Sun = composition core.

$$\frac{dN_H}{dt} = \underbrace{-4 \times 7.9 \times 10^{37}}_{R} = -3.16 \times 10^{38}$$

4 required per reaction

assume const reaction rate since primordial times

$$N_H(t=0) = \frac{0.7 \times 2 \times 10^{30}}{1.6473 \times 10^{-27}} \times 0.2 = 1.70 \times 10^{56}$$

$$N_H = N_0 - Rt$$

Want to find t when $N_H(\text{core}) = 0$.

$$t = \frac{N_0}{R}$$

$$= \frac{1.70 \times 10^{56}}{3.16 \times 10^{38}} = 5.38 \times 10^{17} \text{ s}$$

$$= 1.71 \times 10^{10} \text{ yrs}$$

= 17.1 billion years.

Should be more like 10 billion years....

But right order of magnitude!

Hilroy