Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 13, 2020

10:00AM to 12:00PM

Classical Physics

Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

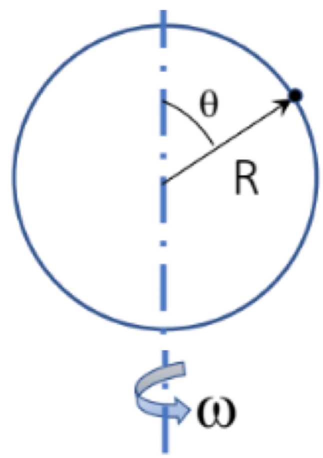
Questions should be directed to the proctor.

Good Luck!

1. A vertical circular wire loop with its center at the origin rotates about the z-axis with a constant angular velocity ω. A bead can move freely without friction along the loop.

   a) At what positions can the bead remain at a constant angle along the loop? The answers depend on the value of ω; you must give the answers for all cases.

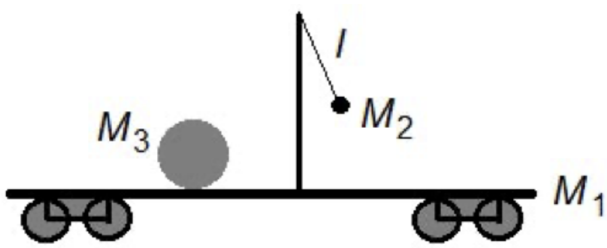
   b) Which of these positions are stable equilibria and which are unstable one?



2. Consider a train car of mass M_1 able to move without friction in one dimension to which is mounted a pendulum composed of point mass M_2 suspended by a massless rod of length ℓ. A drum of mass M_3 is placed on the car for part (b). The drum is symmetric about an axis directed out of the plane of the diagram, has moment of inertia I about this axis and radius R. It is free to roll without slipping on the top surface of the car. Solve both parts (a) and (b) in the approximation that the motion of the pendulum is in the small angle approximation.

   (a) With only the car and pendulum present describe in words the two independent modes of the system. What is the frequency of oscillation if the system starts with the car and pendulum at rest but the pendulum makes a non-zero angle θ_0 with the vertical direction.

   (b) Now include the rolling drum in the system and describe in words the three independent modes of the system. What is the frequency of oscillation if the system starts with the car, drum and pendulum at rest but the pendulum makes a non-zero angle θ_0 with the vertical direction.



3. A ball is bouncing vertically and perfectly elastically in a standing elevator. The maximum height of the bouncing ball is h_0. The upward acceleration of the elevator then changes very slowly from 0 to $g/8$. Using adiabatic invariants, find the new maximum height of the bouncing ball.

4. There is a toy called a celt or rattleback (see figure for a representation of the spinning rattleback) which has the strange property that when placed on a table (with friction) and spun in one specific direction it slows down, then rattles and starts to spin in the opposite direction. When confronted with this behavior you might be tempted to say that it appears as if the law of conservation of angular momentum is violated. The full analysis of this motion is complex, however it can at least be shown simply that the vertical component of the angular momentum is not conserved under a special condition for $\mathbf{r}$, which is the vector from the center of mass to the point of contact with the table. What is that condition? Use the following notation:

   $\mathbf{v}_{\mathrm{CM}}$ is the velocity of the center of mass;

   $\mathbf{r}$ is the vector from the center of mass to the point of contact;

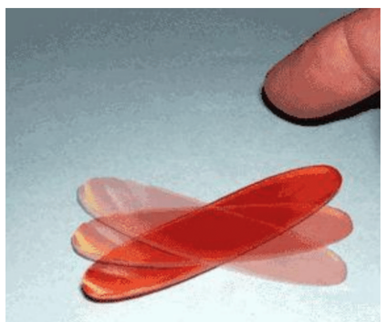
   $\mathbf{F}$ is the net force exerted by the table on the rattleback at the point of contact of the rattleback with the table;

   $\mathbf{L}$ is the angular momentum of the rattleback;

   M is the mass of the rattleback;

   $\hat{\mathbf{y}}$ is a unit vector in the upward vertical direction.

   You may also find the following vector identity useful: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.



5. Consider the three following Lagrangians:

   1. $L = e^{\lambda t}\,\frac{m}{2}\dot{x}^2$
   2. $L = \frac{m}{2}\dot{x}^2 e^{2\gamma x}$
   3. $L = \frac{\dot{x}}{\omega x}\tan^{-1}\left(\frac{\dot{x}}{\omega x}\right) - \frac{1}{2}\ln\left(\dot{x}^2 + \omega^2 x^2\right)$

   (a) For one of the Lagrangians write out the equation of motion for the corresponding system. For a second, *different* Lagrangian, obtain the Hamiltonian and write out the Hamiltonian equations of motion.

   (b) For one of the Lagrangians analyzed in part a, provide the general solution to the Lagrange or Hamiltonian equation of motion for $t > 0$ in terms of relevant initial conditions at $t = 0$. Clearly indicate the initial values that you have assumed.

   (c) For the third Lagrangian not analyzed in part a, identify as many conserved quantities for the physical system as you can find and indicate what continuous transformations of position and/or time the conserved quantities can be associated with. Ideally, show how the transformation affects the Lagrangian and how the conserved quantity follows from the form of the transformation.

Question 1 Solution

The generic Lagrangian is

$$L = T - V = \frac{1}{2}M\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - MgR\cos\theta.$$

Here we have $r = R$, $\dot{r} = 0$, and $\dot{\phi} = \omega$, so this becomes

$$\begin{aligned} L &= \frac{1}{2}MR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) - MgR\cos\theta \\ &= \frac{1}{2}MR^2\dot{\theta}^2 - V_{\text{eff}}, \end{aligned}$$

where $V_{\text{eff}} = MgR\cos\theta - \frac{1}{2}MR^2\omega^2\sin^2\theta$. The stationary points are the zeros of $V'_{\text{eff}}(\theta)$. We compute

$$\begin{aligned} V'_{\text{eff}}(\theta) &= -MgR\sin\theta - MR^2\omega^2\sin\theta\cos\theta \\ &= -MR^2\omega^2\sin\theta\left(\frac{g}{R\omega^2} + \cos\theta\right). \end{aligned}$$

Thus the stationary points are at $\sin\theta = 0 \implies \theta = 0, \pi$ and $\cos\theta = -\dfrac{g}{R\omega^2}$ (for $\dfrac{g}{R\omega^2} \leq 1$).

If $\dfrac{g}{R\omega^2} > 1$, then there is a minimum of the effective potential at $\theta = \pi$ and a maximum at $\theta = 0$. Thus $\theta = \pi$ is a stable equilibrium point, and $\theta = 0$ is unstable, as expected.

If $\dfrac{g}{R\omega^2} < 1$, there is a minimum at $\theta = \cos^{-1}\left(-\frac{g}{R\omega^2}\right)$ and maxima at $\theta = 0, \pi$. Thus $\theta = \cos^{-1}\left(-\frac{g}{R\omega^2}\right)$ is a stable point, and $\theta = 0, \pi$ are unstable.

Question 2 Solution

(a) Describe the state of the system by two coordinates: θ, the angle that the pendulum makes with vertical direction, and x, the location of the train car. One mode is simple translation with $x(t) = x_0 + v_0 t$ and $\theta = 0$. The second mode is oscillation with the location of the center of mass fixed. The frequency can be found by solving Newton's equations for the motion of the pendulum and the car:

$$M_1 \ddot{x} = M_2 g \theta, \quad M_2 \left(\ell \ddot{\theta} + \ddot{x} \right) = -M_2 g \theta.$$

Eliminating $\ddot{x}$ from the second equation, one finds

$$M_2 \ell \ddot{\theta} = -M_2 \left(1 + \frac{M_2}{M_1} \right) g\theta,$$

giving an oscillation frequency

$$\omega = \sqrt{\frac{g}{\ell} \left(1 + \frac{M_2}{M_1} \right)}.$$

(b) Two modes are simple to describe: i) Simple translation of the entire system at constant velocity with $\theta = 0$ and ii) The drum rolls at constant velocity while the car and pendulum are at rest and $\theta = 0$. The third, oscillatory mode can be obtained from the four equations for the acceleration of each of the three masses and the angular acceleration of the drum. Introducing the force F that the car exerts on the drum and the angle ϕ through which the drum has rotated, we can write

$$\begin{aligned} M_1 \ddot{x} &= M_2 \theta - F \qquad & M_2 \left(\ell \ddot{\theta} + \ddot{x} \right) &= -M_2 g \theta \\ M_3 \left(R\ddot{\phi} + \ddot{x} \right) &= F & I\ddot{\phi} &= -FR. \end{aligned}$$

Eliminating F, $\ddot{\phi}$, and $\ddot{x}$ from these equations leaves

$$M_2 \ddot{\theta} = -M_2 g \left(1 + \frac{M_2}{M_1 + \frac{M_3}{1 + M_3 R^2/I}} \right),$$

which yields the frequency

$$\omega = \sqrt{\frac{g}{\ell} \left(1 + \frac{M_2}{M_1 + \frac{M_3}{1 + M_3 R^2/I}} \right)}.$$

Question 3 Solution

In the frame of the elevator, the energy of the ball is

$$E = \frac{p^2}{2m} + mg'y,$$

where $g' = g + a$ is the effective gravitational acceleration, with a the acceleration of the elevator.

The adiabatic invariant is

$$J = \oint p\,dy = 2\int_0^h p\,dy.$$

Conservation of energy implies

$$E = mg'h \implies p = m\sqrt{2g'(h-y)},$$

so

$$J = 2m\int_0^h dy\,\sqrt{2g'(h-y)} = \frac{4\sqrt{2}}{3}m\sqrt{g'}h^{3/2}.$$

Thus

$$h_0^{3/2}\sqrt{g_0'} = h_f^{3/2}\sqrt{g_f'}.$$

Since $g_0' = 0$ and $g_f' = \frac{9}{8}g$, we find

$$h_f = \left(\frac{8}{9}\right)^{1/3} h_0 = \frac{2}{3^{2/3}}h_0.$$

Question 4 Solution

The following basic equation pertains to the motion:

$$\mathbf{F} - Mg\hat{\mathbf{y}} = M\frac{d\mathbf{v}_{\mathrm{CM}}}{dt}.$$

So,

$$\mathbf{F} = M\left(\frac{d\mathbf{v}_{\mathrm{CM}}}{dt} + g\hat{\mathbf{y}}\right),$$

and from the torque about the center of mass

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F}.$$

Substituting for $\mathbf{F}$,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times M\left(\frac{d\mathbf{v}_{\mathrm{CM}}}{dt} + g\hat{\mathbf{y}}\right) = M\left(\mathbf{r} \times \frac{d\mathbf{v}_{\mathrm{CM}}}{dt} + g\mathbf{r} \times \hat{\mathbf{y}}\right).$$

Now we find the vertical component of the angular momentum, which we want to show is not conserved:

$$\frac{d}{dt}\left(\hat{\mathbf{y}} \cdot \mathbf{L}\right) = M\hat{\mathbf{y}} \cdot \left(\mathbf{r} \times \frac{d\mathbf{v}_{\mathrm{CM}}}{dt} + g\mathbf{r} \times \hat{\mathbf{y}}\right) = M\hat{\mathbf{y}} \cdot \left(\mathbf{r} \times \frac{d\mathbf{v}_{\mathrm{CM}}}{dt}\right) + M\hat{\mathbf{y}} \cdot \left(g\mathbf{r} \times \hat{\mathbf{y}}\right).$$

The second term vanishes since the vectors are perpendicular to each other, and the first term can be rewritten using the given vector identity

$$\frac{d}{dt}\left(\hat{\mathbf{y}} \cdot \mathbf{L}\right) = M\left(\hat{\mathbf{y}} \times \mathbf{r}\right) \cdot \frac{d\mathbf{v}_{\mathrm{CM}}}{dt}.$$

This then tells us the condition we need for the vertical component of the angular momentum to not be conserved, namely when

$$\hat{\mathbf{y}} \times \mathbf{r} \neq 0,$$

in other words, when $\mathbf{r}$ does not lie along the vertical direction.

If you want more detail on this complicated motion (e.g. why does the rattleback change rotation when spun in one direction and not the other, i.e. exhibit chirality? It is not evident from what we showed above), see the full analysis by Bondi in Proc. R. Soc. Lond. A 405, 265-274 (1986).

Question 5 Solution

(a) The Lagrange equations of motion and general solutions for the three different Lagrangians are:

1. $L = e^{\lambda t}\,\frac{m}{2}\dot{x}^2$

$$\frac{d}{dt}\left(e^{\lambda t} m\dot{x}\right) = 0 \to me^{\lambda t}\left(\lambda\dot{x} + \ddot{x}\right) = 0 \to \ddot{x} = -\lambda\dot{x}.$$

The velocity, $\dot{x}$ has general solution $\dot{x} = Ce^{-\lambda t}$ and the displacement has general solution,

$$x = C_1 + C_2 e^{-\lambda t}.$$

2. $L = \frac{m}{2}\dot{x}^2 e^{2\gamma x}$

$$\frac{d}{dt}\left(m\dot{x}e^{2\gamma x}\right) - \gamma m\dot{x}^2 e^{2\gamma x} = 0 \to me^{2\gamma x}\left(\ddot{x} + 2\gamma\dot{x}^2 - \gamma\dot{x}^2\right) = 0 \to \ddot{x} = -\gamma\dot{x}^2$$

The velocity satisfies

$$1/\dot{x} = C + \gamma t \to \dot{x} = \frac{1}{C + \gamma t}.$$

Then, the position satisfies

$$x = \int dt\,\frac{1}{C + \gamma t} = \frac{1}{\gamma}\ln\left(C + \gamma t\right) + C'.$$

3. $L = \frac{\dot{x}}{\omega x}\tan^{-1}\left(\frac{\dot{x}}{\omega x}\right) - \frac{1}{2}\ln\left(\dot{x}^2 + \omega^2 x^2\right)$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}} &= \frac{1}{\omega x}\tan^{-1}\left(\frac{\dot{x}}{\omega x}\right) + \frac{\dot{x}}{\omega^2 x^2}\left(1 + \frac{\dot{x}^2}{\omega^2 x^2}\right)^{-1} - \frac{\dot{x}}{\dot{x}^2 + \omega^2 x^2} \\
&= \frac{1}{\omega x}\tan^{-1}\left(\frac{\dot{x}}{\omega x}\right) + \frac{\dot{x}}{\omega^2 x^2}\left(\frac{\omega^2 x^2}{\omega^2 x^2 + \dot{x}^2}\right) - \frac{\dot{x}}{\dot{x}^2 + \omega^2 x^2} \\
&= \frac{1}{\omega x}\tan^{-1}\left(\frac{\dot{x}}{\omega x}\right) \\
\frac{\partial L}{\partial x} &= -\frac{\dot{x}}{\omega x^2}\tan^{-1}\left(\frac{\dot{x}}{\omega x}\right) - \frac{1}{1 + (\dot{x}/\omega x)^2}\frac{\dot{x}}{\omega x}\frac{\dot{x}}{\omega x^2} - \frac{\omega^2 x}{\dot{x}^2 + \omega^2 x^2} \\
&= -\frac{\dot{x}}{\omega x^2}\tan^{-1}\left(\frac{\dot{x}}{\omega x}\right) - \frac{1}{x}.
\end{aligned}$$

The equation of motion is therefore

$$-\frac{\dot{x}}{\omega x^2}\tan^{-1}\left(\frac{\dot{x}}{\omega x}\right) + \frac{1}{\omega x}\frac{\omega^2 x^2}{\dot{x}^2 + \omega^2 x^2}\frac{\omega x\ddot{x} - \omega\dot{x}^2}{\omega^2 x^2} + \frac{\dot{x}}{\omega x^2}\tan^{-1}\left(\frac{\dot{x}}{\omega x}\right) + \frac{1}{x} = 0.$$

This simplifies to

$$\frac{x\ddot{x}-\dot{x}^2}{\dot{x}^2+\omega^2 x^2}+1=0 \implies \ddot{x}+\omega^2 x=0,$$

which has general solution of either form:

$$x=C\cos(\omega t-\phi),\text{ or } x=A\cos(\omega t)+B\sin(\omega t)\,.$$

(b) The Hamiltonian, $H(x,p)$ can be obtained using the Legendre transformation, $H=p\dot{x}-L$. Evaluating the Hamiltonian and the corresponding equations of motion for the three Lagrangians

1. $L=e^{\lambda t}\,\frac{m}{2}\dot{x}^2$ The generalized momentum is

$$p=\frac{\partial L}{\partial \dot{x}}=e^{\lambda t}\,m\dot{x}\to\dot{x}=\frac{p}{m}e^{-\lambda t}.$$

Then,

$$H=\frac{p^2}{m}e^{-\lambda t}-\frac{m}{2}e^{\lambda t}\left[\frac{p}{m}e^{-\lambda t}\right]^2=\frac{p^2}{m}e^{-\lambda t}\left(1-\frac{1}{2}\right)=\frac{p^2}{2m}e^{-\lambda t}.$$

The Hamiltonian equations of motion are, then,

$$\dot{x}=\frac{\partial H}{\partial p}=\frac{p}{m}e^{-\lambda t}, \tag{1}$$

$$\dot{p}=-\frac{\partial H}{\partial x}=0. \tag{2}$$

Then, since p is a constant, the general solution can be written,

$$x=C-\frac{p}{m\lambda}e^{-\lambda t}.$$

2. $L=\frac{m}{2}\dot{x}^2e^{2\gamma x}$ The generalized momentum is

$$p=\frac{\partial L}{\partial \dot{x}}=m\dot{x}e^{2\gamma x}\to\dot{x}=\frac{p}{m}e^{-2\gamma x},$$

so

$$H=\frac{p^2}{m}e^{-2\gamma x}-\frac{m}{2}e^{2\gamma x}\left[\frac{p}{m}e^{-2\gamma x}\right]^2=\frac{p^2}{m}e^{-2\gamma x}\left(1-\frac{1}{2}\right)=\frac{p^2}{2m}e^{-2\gamma x}.$$

Then, the Hamiltonian equations of motion are:

$$\dot{x}=\frac{\partial H}{\partial p}=\frac{p}{m}e^{-2\gamma x} \tag{3}$$

$$\dot{p}=-\frac{\partial H}{\partial x}=\frac{p^2}{2m}\left(-2\gamma\right)e^{-2\gamma x}=-2\gamma H. \tag{4}$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 13, 2020
2:00PM to 4:00PM
Classical Physics
Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

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Questions should be directed to the proctor.

Good Luck!

1. A plane wave is normally incident on a perfectly reflecting mirror. A glass photographic plate is placed on the mirror and forms a small angle α to the mirror. The photographic emulsion is nearly transparent. But when later developed, a striped pattern is found due to the action of the wave. Predict the spacing of the stripes. Ignore reflections or attenuation due to the photographic plate.

2. Consider a spherical capacitor with a fixed radius a for the outer spherical shell and vacuum between the shells. If the electric field at the surface of the inner spherical shell cannot exceed a value E_0, for what radius b of the inner shell is the stored energy maximized, and how much energy can be stored?

3. Imagine that an ideal magnetic dipole $\mathbf{m}$ is located at the origin of an inertial system S' that moves with speed v in the x direction with respect to inertial system S. In S', the vector potential can be expressed as

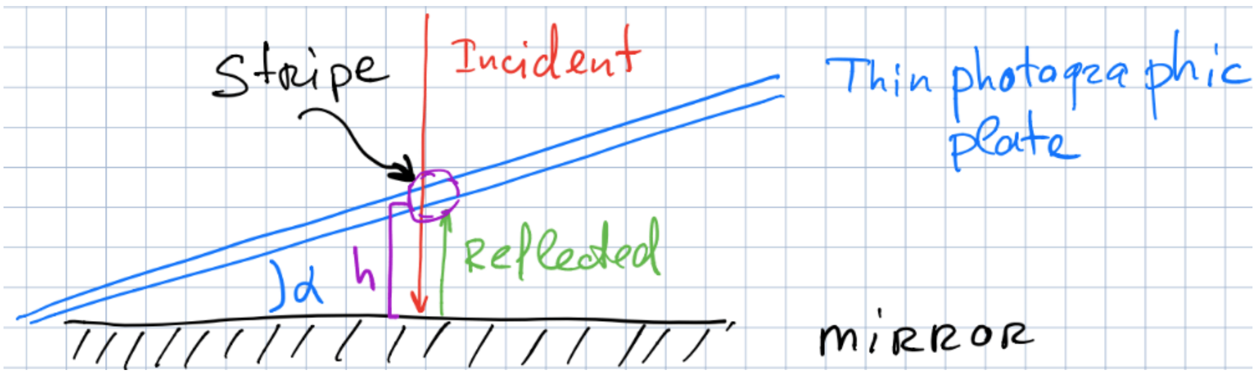
$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}'}{|\mathbf{r}'|^2},$$

and the scalar potential is zero. Find the scalar potential V in S (make certain all elements of the answer are expressed for an observer in S).

4. A spherical shell with radius R and uniform surface charge density σ spins with angular frequency ω around a diameter. Find the magnetic field at its center.

5. Consider a magnetic field $\mathbf{B}$ with energy density $U_B = 10^{-16}\, m_e c^2/r_e^3$ where m_e is electron mass and r_e is the classical electron radius. An electron is injected with velocity $\mathbf{v}_0 \perp \mathbf{B}$; $v_0 << c$. How many circular orbits will it make around $\mathbf{B}$ before its kinetic energy $E_0 = m_e v_0^2/2$ is reduced to $E_1 = 10^{-2}\, E_0$?

Question 1 Solution



For constructive interference, we have

$$\frac{2h}{\lambda}(2\pi) + \pi = 2\pi n \implies h = \frac{2n-1}{4}\lambda$$

Since $h = d \sin\alpha$,

$$d = \frac{(2n-1)}{4}\frac{\lambda}{\sin\alpha}$$

Thus the stripes are regularly spaced a distance $\lambda/\sin\alpha$ apart. The first one is at distance $\lambda/4\sin\alpha$ from the point of contact.

Question 2 Solution

For convenience, let $b = ka$, with $0 < k < 1$. If E_0 is the field at radius ka, then for $r > ka$, $E = E_0 \frac{(ka)^2}{r^2}$. Energy stored in the field is

$$\begin{aligned}
U &= \frac{\epsilon_0}{2} \int E^2 \, dV \\
&= \frac{\epsilon_0}{2} \int_{ka}^{a} \left(E_0 \frac{(ka)^2}{r^2} \right)^2 4\pi r^2 \, dr \\
&= 2\pi\epsilon_0 k^4 a^4 E_0^2 \int_{ka}^{a} \frac{1}{r^2} \, dr \\
&= 2\pi\epsilon_0 k^4 a^4 E_0^2 \left(\frac{1}{ka} - \frac{1}{a} \right) \\
&= 2\pi\epsilon_0 a^3 E_0^2 \left(k^3 - k^4 \right).
\end{aligned}$$

We take the derivative to find the maximum:

$$3k^2 - 4k^3 = 0 \implies k = \frac{3}{4}.$$

Thus the maximum stored energy is achieved at $b = \frac{3}{4}a$. The stored energy is

$$U = 2\pi\epsilon_0 a^3 E_0^2 \left(\left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^4 \right) = \frac{27}{128} \pi\epsilon_0 a^3 E_0^2.$$

Alternatively, we can solve this problem using the capacitance of the sphere,

$$C = 4\pi\epsilon_0 \frac{ab}{a-b} = 4\pi\epsilon_0 a \frac{k}{1-k}.$$

If we let Q be the charge on the inner sphere, the electric field at the inner surface is

$$E = \frac{Q}{4\pi\epsilon_0 (ka)^2} \implies Q = 4\pi\epsilon_0 k^2 a^2 E_0.$$

Thus the energy is

$$U = \frac{Q^2}{2C} = \frac{\left(4\pi\epsilon_0 k^2 a^2 E_0\right)^2}{8\pi\epsilon_0 a \frac{k}{1-k}} = 2\pi\epsilon_0 a^3 E_0^2 \left(k^3 - k^4 \right),$$

agreeing with our expression above.

Question 3 Solution

Start by applying the Lorentz transformation $V = \gamma\left(V' + vA'_x\right)$, realizing that $V' = 0$. So

$$V = \gamma v \frac{\mu_0}{4\pi} \frac{m_y z - m_z y}{\left(r'\right)^3}.$$

We still need to transform r', which is

$$(r')^2 = \gamma^2 \left(R^2 - \frac{v^2}{c^2} R^2 \sin^2\theta\right),$$

where $\mathbf{R}$ is the vector in S from the (instantaneous) location of the dipole to the point of observation and θ is the angle between $\mathbf{R}$ and $\hat{\mathbf{x}}$. And thus we have

$$\begin{aligned} V &= \gamma v \frac{\mu_0}{4\pi} \frac{m_y z - m_z y}{\gamma^3 R^3 \left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}} \\ &= \frac{\mu_0}{4\pi} \frac{\mathbf{v} \cdot (\mathbf{m} \times \mathbf{R}) \left(1 - \frac{v^2}{c^2}\right)}{R^3 \left(1 - \frac{v^2}{c^2}\sin^2\theta\right)^{3/2}}. \end{aligned}$$

Question 4 Solution

We divide the sphere into rings centered on the z axis. These rings have width $dw = R\,d\theta$. The speed of each ring is $v = \omega R \sin\theta$. Focusing on one ring at angle θ, the charge passing a given point in time dt is

$$dq = \sigma(dw)(vdt) = \sigma\omega R^2 \sin\theta\, d\theta\, dt.$$

Thus the ring current is

$$I = \frac{dq}{dt} = \sigma\omega R^2 \sin\theta\, d\theta.$$

The Biot-Savart field $d\mathbf{B}$ produced by a small piece of the ring with length $d\ell$ has magnitude

$$dB = \frac{\mu_0}{4\pi}\frac{I\,d\ell}{R^2}.$$

Integrating over the whole ring, the horizontal components cancel, with the vertical components ($dB\,\sin\theta$) add. The length of the ring is $\ell = 2\pi R\sin\theta$. Thus the contribution to the net field due to a ring of width $d\theta$ at angle θ is

$$\begin{aligned}
\mathbf{B}_{\text{ring}} &= \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{I\ell}{R^2}\sin\theta \\
&= \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\frac{(\sigma\omega R^2\sin\theta\,d\theta)\,(2\pi R\sin\theta)}{R^2}\sin\theta \\
&= \hat{\mathbf{z}}\frac{1}{2}\mu_0\sigma\omega R\sin^3\theta\,d\theta.
\end{aligned}$$

Integrating over θ from 0 to π, and doing so by rewriting $\sin^3\theta = \sin\theta\,(1-\cos^2\theta)$, yields

$$\begin{aligned}
\mathbf{B} &= \hat{\mathbf{z}}\frac{1}{2}\mu_0\sigma\omega R\int_0^\pi \sin\theta\left(1-\cos^2\theta\right)\,d\theta \\
&= \hat{\mathbf{z}}\frac{1}{2}\mu_0\sigma\omega R\left(\frac{1}{3}\cos^3\theta - \cos\theta\right)\Bigg|_0^\pi \\
&= \hat{\mathbf{z}}\frac{2}{3}\mu_0\sigma\omega R.
\end{aligned}$$

Question 5 Solution

The electron with velocity $\mathbf{v} \perp \mathbf{B}$ experiences acceleration due to the Lorentz force:

$$\mathbf{a} = \frac{-e\,\mathbf{v} \times \mathbf{B}}{m_e c}.$$

It rotates around $\mathbf{B}$ with Larmor radius r found from $a = v^2/r$. The rotation period is

$$T = \frac{2\pi r}{v} = \frac{2\pi v}{a} = 2\pi\,\frac{m_e c}{eB}.$$

The acceleration a gives the second time derivative of the dipole moment $|\ddot{\mathbf{d}}| = ea$, and the electron loses energy due to the dipole radiation with rate given by the Larmor formula:

$$\frac{dE}{dt} = -\frac{2\ddot{\mathbf{d}}^2}{3c^3} = -\frac{2e^4v^2B^2}{3m_e^2c^5} = -\frac{4e^4B^2}{3m_e^3c^5}\,E \implies \ln\frac{E_0}{E(t)} = \frac{4e^4B^2t}{3m_e^3c^5}.$$

The energy is reduced to E_1 after time

$$t_1 = \frac{3m_e^3c^5}{4e^4B^2}\ln\frac{E_0}{E_1}.$$

Using $U_B = B^2/8\pi$ and $r_e = e^2/m_ec^2$ one finds the number of Larmor orbits

$$N_1 = \frac{t_1}{T} = \frac{3m_e^2c^4}{8\pi e^3 B}\ln\frac{E_0}{E_1} = \frac{3\,\ln(E_0/E_1)}{(8\pi)^{3/2}\,(r_e^3U_B/m_ec^2)^{1/2}} = \frac{3\,\ln 100}{10^{-8}(8\pi)^{3/2}} \approx 1.1\times 10^7.$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 15, 2020
10:00AM to 12:00PM
Modern Physics
Section 1. Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider the two-dimensional simple harmonic oscillator with Hamiltonian

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}k(x^2 + y^2).$$

   (a) Write down the general equation for the energy eigenvalues of this Hamiltonian.

   (b) Write down the energy eigenvalues for the ground state and the first excited state. What are the degeneracies of these states, if any?

   (c) Write down the energy eigenfunctions for the ground state and the first excited state.

   A perturbation is added to the above Hamiltonian of the form $H' = bxy$, where $b << 1$.

   (d) Find the energy eigenvalues of the perturbed Hamiltonian $H_0 + H'$ for the ground and the first two excited states.

   (e) Sketch the energy level diagram for the ground and first two excited states of the perturbed Hamiltonian.

2. Consider a particle constrained to move in two dimensions (xy-plane) on a circular ring. The only variable of the system is the azimuthal coordinate angle ϕ. The state of the system is described by a wave function $\Psi(\phi)$, which must satisfy the symmetry

$$\Psi(\phi + 2\pi) = \Psi(\phi).$$

The wave function must also be normalized, i.e.,

$$\int_0^{2\pi} |\Psi(\phi)|^2 \, d\phi = 1.$$

Suppose initially that there is no potential applied to the particle. The Hamiltonian for the system thus consists only of the kinetic energy of the particle, which can be written

$$H_0 = \frac{L_z^2}{2K},$$

where the operator $L_z = -i\hbar \frac{d}{d\phi}$ and K is a constant.

(a) Calculate the eigenvalues and normalized eigenfunctions of the Hamiltonian H_0. Which of the energy levels are degenerate?

Now suppose that the particle also experiences a potential, which can be treated as a small perturbation to the overall Hamiltonian:

$$H' = -\lambda \cos(2\phi),$$

where λ is a constant.

(b) Considering the perturbed Hamiltonian, calculate the lowest *non-vanishing* correction to the ground state energy due to H'. The following integral may be useful:

$$\frac{1}{2\pi} \int_0^{2\pi} e^{in\phi} \cos(m\phi) \, d\phi = \frac{1}{2} \left(\delta_{m,n} + \delta_{m,-n} \right),$$

where $\delta_{i,j}$ is the Kronecker δ-function.

(c) Calculate the ground state wave function to first order in λ.

3. Consider a hydrogen atom. The spin-orbit interaction is written as

$$H_{\text{so}} = A\mathbf{S} \cdot \mathbf{L},$$

where $\mathbf{S}$ is the spin of the electron, $\mathbf{L}$ is the orbital angular momentum and $A = e^2/2m^2c^2r^3$.

(a) Describe in words the origin of the spin-orbit interaction.

(b) Construct the basis of wavefunctions/eigenstates that diagonalizes H_{so}.

(c) Obtain the spin-orbit interaction energies for hydrogen in the state with radial quantum number $n = 2$. You may express your results in terms of the matrix elements of A without explicitly evaluating those matrix elements.

4. A 2×2 matrix is parameterized as $\hat{\rho} = \frac{1}{2}(A\,\mathbb{1} + \boldsymbol{\sigma} \cdot \mathbf{B})$, where $\mathbb{1}$ is the identity matrix, A is a real number, $\mathbf{B}$ is 3-dimensional vector of real numbers, and $\boldsymbol{\sigma}$ represents the 3 Pauli spin matrices:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Find the conditions on A and $\mathbf{B}$ if this matrix is a valid density matrix for a pure state. You may find this identity (written using Einstein summation convention) useful:

$$\sigma_i \sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k.$$

(b) Assuming the 2-state system is spin-1/2, and the matrix is in the z-representation, find the values of A and $\mathbf{B}$ that maximize the expectation value of $\langle \hat{S}_y \rangle$, the y-component of the spin operator.

(c) What conditions on A are $\mathbf{B}$ are needed to represent the density matrix for a mixed state?

5. Consider a particle of mass m moving in the three-dimensional potential

$$V(r, \theta, \phi) = V_0 \delta(r - a),$$

where V_0 is a constant. Suppose the angular momentum $\ell = 0$. Then the Schrödinger equation for the radial wavefunction is

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{2m}{\hbar^2}\left(E - V_0(r - a)\right) \right) R(r) = 0.$$

In terms of $k = \sqrt{\frac{2mE}{\hbar^2}}$, give

(a) the solution to the Schrödinger equation when $0 < r < a$,

(b) the solution to the Schrödinger equation when $r > a$.

(c) Give the matching condition for parts (a) and (b) at $r = a$.

(d) If one writes $R(r) = h_0^{(2)}(kr) + e^{2i\delta} h_0^{(1)}(kr)$ for $r > a$, can you evaluate $e^{2i\delta}$? What is δ when V_0 is very large? (*Note the difference between the Dirac delta function and the phase shift, δ.*)

The solutions to the differential equation

$$\left(\frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} + 1 - \frac{\ell(\ell+1)}{z^2} \right) f(z) = 0,$$

are the spherical Bessel functions $j_\ell(z)$ and $n_\ell(z)$, or equivalently, the spherical Hankel functions $h_\ell^{(1)}(z)$ and $h_\ell^{(2)}(z)$. For $\ell = 0$, we have

$$j_0(z) = \frac{\sin(z)}{z}, \quad n_0(z) = -\frac{\cos(z)}{z}, \quad h_0^{(1)}(z) = \frac{e^{iz}}{iz}, \quad h_0^{(2)}(z) = -\frac{e^{-iz}}{iz}.$$

Question 1 Solution

(a) This is just two independent one-dimensional simple harmonic oscillator, so the general solution is just $E_{n_x,n_y} = E_{n_x} + E_{n_y} = (n_x + n_y + 1)\hbar\omega$, with $\omega^2 = k/m$.

(b) The energies are $E_{\text{ground}} = E_{0,0} = \hbar\omega$ and $E_1 = E_{01} = E_{10} = 2\hbar\omega$. The first excited state is doubly degenerate.

(c) The wavefunction for the ground state is $\psi_0 = \psi_0(x)\psi_0(y)$. The wavefunction for the first excited state is $\psi_1 = \psi_0(x)\psi_1(y)$ or $\psi_1(x)\psi_0(y)$. The $\psi_{n_x}(x)$ and $\psi_{n_y}(y)$ are just the solutions to the one-dimensional simple harmonic oscillator. Recalling that $\psi_0 \propto e^{-\beta^2 x^2/2}$ and $\psi_1 \propto xe^{-\beta^2 x^2/2}$, where $\beta^2 = m\omega/\hbar$. Normalizing, we obtain

$$\psi_0(x) = \left(\frac{\beta}{\pi^{1/2}}\right)^{1/2} e^{-\beta^2 x^2/2}$$

$$\psi_1(x) = \left(\frac{2\beta^3}{\pi^{1/2}}\right)^{1/2} xe^{-\beta^2 x^2/2},$$

and the same for the y direction.

(d) The ground state energy is unchanged, $E_{\text{ground}} = \hbar\omega$, since

$$\langle 00|xy|00\rangle = \int_{-\infty}^{\infty} \psi_0(x)x\psi_0(x)\,dx \int_{-\infty}^{\infty} \psi_0(y)y\psi_0(y)\,dy = 0$$

by parity.

Computing the energy of the first excited state involves finding the eigenvalues of the perturbing Hamiltonian, which in this basis is

$$H' = b\begin{pmatrix} \langle 10|xy|10\rangle & \langle 10|xy|01\rangle \\ \langle 01|xy|10\rangle & \langle 01|xy|01\rangle \end{pmatrix}.$$

The diagonal elements are (by parity)

$$\langle 1|x|1\rangle \langle 0|y|0\rangle = 0, \quad \langle 0|x|0\rangle \langle 1|y|1\rangle = 0.$$

The off-diagonal elements obey

$$\langle 01|xy|10\rangle = \langle 10|xy|01\rangle = \left(\langle 0|x|1\rangle\right)^2.$$

Thus we only need to calculate

$$\int_{-\infty}^{\infty} \psi_0(x)x\psi_1(x)\,dx = \frac{1}{\sqrt{2}\beta}.$$

Thus

$$H' = b\begin{pmatrix} 0 & 1/2\beta^2 \\ 1/2\beta^2 & 0 \end{pmatrix}.$$

Aside: This is way easier using

$$\hat{x} = \frac{1}{\sqrt{2}\beta}\left(\hat{a}_x + \hat{a}_x^\dagger\right) \qquad \hat{y} = \frac{1}{\sqrt{2}\beta}\left(\hat{a}_y + \hat{a}_y^\dagger\right),$$

where

$$\hat{a}_x^\dagger \left|00\right\rangle = \left|10\right\rangle, \quad \hat{a}_y^\dagger \left|00\right\rangle = \left|01\right\rangle, \quad \hat{a}_x \left|10\right\rangle = \left|00\right\rangle, \quad \hat{a}_y \left|01\right\rangle = \left|00\right\rangle, \quad \text{etc.}$$

Then right away $\langle 01|xy|10\rangle = 1/2\beta^2$ by inspection.

To find the eigenvalues of H', we solve

$$\det\begin{pmatrix} -\lambda & b/2\beta^2 \\ b/2\beta^2 & -\lambda \end{pmatrix} = 0 \implies \lambda = \pm\frac{b}{2\beta^2}.$$

Thus the perturbed energies are $\hbar\omega$ and $2\hbar\omega \pm b/2\beta^2$.

(e) We find

$2\hbar\omega + b/2\beta^2$

$2\hbar\omega - b/2\beta^2$

$\hbar\omega$

Question 2 Solution

(a) The eigenvalues of $H_0 = -\dfrac{\hbar^2}{2K}\dfrac{d^2}{d\phi^2}$ are $E_n = \dfrac{m^2\hbar^2}{2K}$, where m is an integer. The normalized eigenfunctions are

$$\Psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}.$$

All the energy levels except $m = 0$ are doubly degenerate, since $m^2 = (-m)^2$.

(b) The first-order correction to the ground state is

$$E_0^{(1)} = \left\langle \Psi_0^{(0)}(\phi) \middle| H' \middle| \Psi_0^{(0)}(\phi) \right\rangle = -\frac{\lambda}{2\pi}\int_0^{2\pi} \cos(2\phi)\, d\phi = 0.$$

The second-order correction is

$$\begin{aligned}
E_0^{(2)} &= \sum_{m\neq 0} \frac{\left|\left\langle \Psi_0^{(0)} \middle| H' \middle| \Psi_m^{(0)} \right\rangle\right|^2}{E_0^{(0)} - E_m^{(0)}} \\
&= \sum_{m\neq 0} \frac{\left| -\frac{\lambda}{2\pi}\int_0^{2\pi} e^{im\phi}\cos(2\phi)\, d\phi \right|^2}{-m^2\hbar^2/2K} \\
&= -\frac{\lambda^2 K}{2\pi^2\hbar^2} \sum_{m\neq 0} \frac{1}{m^2} \left| \int_0^{2\pi} e^{im\phi}\cos(2\phi)\, d\phi \right|^2 \\
&= -\frac{\lambda^2 K}{2\hbar^2} \sum_{m\neq 0} \frac{1}{m^2} \left|\delta_{2,m} + \delta_{2,-m}\right|^2 \\
&= -\frac{\lambda^2 K}{4\hbar^2}.
\end{aligned}$$

(c) The first-order correction to the ground state wavefunction is

$$\begin{aligned}
\left|\Psi_0^{(1)}\right\rangle &= \sum_{m\neq 0} \frac{\left\langle \Psi_m^{(0)} \middle| H' \middle| \Psi_0^{(0)} \right\rangle}{E_0^{(0)} - E_m^{(0)}} \left|\Psi_m^{(0)}\right\rangle \\
&= \frac{\lambda/2}{2\hbar^2/K}\left|\Psi_2^{(0)}\right\rangle + \frac{\lambda/2}{2\hbar^2/K}\left|\Psi_{-2}^{(0)}\right\rangle \\
&= \frac{\lambda K}{4\hbar^2}\frac{1}{\sqrt{2\pi}}\left(e^{2i\phi} + e^{-2i\phi}\right) \qquad \text{(from part (b))} \\
&= \frac{\lambda K}{2\sqrt{2\pi}\hbar^2}\cos(2\phi).
\end{aligned}$$

Question 3 Solution

(a) In the rest frame of the electron the proton is moving with velocity $-\mathbf{v}$, which produces a magnetic field $\mathbf{B} \sim \mathbf{L}$. The coupling of this field to the intrinsic magnetic moment from $\mathbf{S}$ is the spin-orbit interaction H_{so}.

(b) We rewrite the spin-orbit interaction $H_{\text{so}} = A\mathbf{S} \cdot \mathbf{L}$ as

$$H_{\text{so}} = \frac{A}{2}\left(J^2 - L^2 - S^2\right),$$

where $\mathbf{J} = \mathbf{S} + \mathbf{L}$ is the total angular momentum. The states with quantum numbers J, L, and S make this Hamiltonian diagonal. The states can be written as $|j, m, l, s\rangle$, where m is the z-component of J. In this basis the matrix elements of H_{so} are

$$\langle H_{\text{so}}\rangle = \langle A(r)\rangle_{nl}\hbar^2\left(j(j+1) - l(l+1) - \frac{3}{4}\right).$$

(c) For $n = 2$, the allowed values of l are 0 or 1. The allowed values of j are $1 \mp 1/2$. The value $j = 1/2$ is obtained from $l = 0, 1$. There are two terms. The value $j = 3/2$ is obtained from $l = 1$ only. There is one term. There are also two matrix elements:

$$\langle A(r)\rangle_{2,0} = A_0, \quad \langle A(r)\rangle_{21} = A_1.$$

The spin-orbit interaction energies are then obtained with the above equations.

Question 4 Solution

(a) A valid density matrix requires $\mathrm{Tr}\,[\hat{\rho}] = 1 \implies A = 1$ (since Pauli matrices are traceless). For a pure state, it must also satisfy $\hat{\rho}^2 = \hat{\rho}$ so

$$\hat{\rho} = \frac{1}{2}\left(\mathbb{1} + \boldsymbol{\sigma}\cdot\mathbf{B}\right) \implies \hat{\rho}^2 = \left[\frac{1}{2}\left(A\,\mathbb{1} + \boldsymbol{\sigma}\cdot\mathbf{B}\right)\right]^2 = \frac{1}{2}\cdot\frac{1}{2}\left[\mathbb{1} + 2\boldsymbol{\sigma}\cdot\mathbf{B} + (\boldsymbol{\sigma}\cdot\mathbf{B})^2\right].$$

Using

$$(\boldsymbol{\sigma}\cdot\mathbf{B})^2 = (\sigma_i B^i)(\sigma_j B^j) = \delta_{ij}B^iB^j\,\mathbb{1} + i\epsilon_{ijk}B^iB^j\sigma_k = \mathbf{B}\cdot\mathbf{B}\mathbb{1} + i(\mathbf{B}\times\mathbf{B})\cdot\boldsymbol{\sigma} = \mathbf{B}\cdot\mathbf{B}\mathbb{1},$$

we get

$$\hat{\rho}^2 = \frac{1}{2}\cdot\frac{1}{2}\left(\mathbb{1} + 2\vec{\sigma}\cdot\mathbf{B} + (\mathbf{B}\cdot\mathbf{B})\,\mathbb{1}\right), \tag{1}$$

which requires $\mathbf{B}\cdot\mathbf{B} = 1$ to satisfy $\hat{\rho}^2 = \hat{\rho}$.

(b) We find

$$\langle \hat{S}_y \rangle = \mathrm{Tr}\left[\hat{\rho}\hat{S}_y\right] = \frac{1}{2}\mathrm{Tr}\left[\left(\mathbb{1} + \sigma\cdot\mathbf{B}\right)\frac{\hbar}{2}\sigma_y\right] = \frac{\hbar}{2}\cdot\frac{1}{2}\mathrm{Tr}\left[(\sigma_x B_x + \sigma_y B_y + \sigma_z B_z)\sigma_y\right],$$

but $\sigma_x\sigma_y = i\sigma_z$, which is traceless, and similarly for $\sigma_z\sigma_y$, while $\sigma_y\sigma_y = \mathbb{1}$, so $\mathbf{B} = (0,1,0)$ maximizes $\langle \hat{S}_y \rangle$ to $\frac{\hbar}{2}$, which makes physical sense.

(c) For a mixed state, we must still have $\mathrm{Tr}\,[\hat{\rho}] = 1$, so we still require $A = 1$ (of course). But a mixed state is signified by $\mathrm{Tr}\,[\hat{\rho}^2] < \mathrm{Tr}\,[\hat{\rho}]$; using results from part (a) we see that the necessary condition is $|\mathbf{B}|^2 < 1$.

Those seeking more information should look at the Wikipedia article on the Bloch sphere.

Question 5 Solution

(a) For $0 < r < a$, the differential equation is

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{2mE}{\hbar^2}\right) R(r) = 0,$$

which we can rewrite as

$$\left(\frac{d^2}{d(kr)^2} + \frac{2}{kr}\frac{d}{d(kr)} + 1\right) R(kr) = 0.$$

Thus the solution is $R(kr) = Cj_0(kr)$. We do not include the $n_0(kr)$ solution because it is not regular at $r = 0$.

(b) For $r > a$, we have the same differential equation, but we find

$$R(kr) = C_1 h_0^{(1)}(kr) + C_2 h_0^{(2)}(kr).$$

(c) The matching conditions are

$$Cj_0(ka) = C_1 h_0^{(1)}(ka) + C_2 h_0^{(2)}(ka)$$

and

$$\int_{a-\epsilon}^{a+\epsilon} dr \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2 - \frac{2mV_0}{\hbar^2}\delta(r-a)\right) R(r) = 0.$$

The second condition implies

$$C_1 h_0^{(1)\prime}(ka) + C_2 h_0^{(2)\prime}(ka) - Cj_0'(ka) = \frac{2mV_0 C}{\hbar^2 k} j_0(ka).$$

(d) Using part (c), we eliminate C, finding

$$C_1 h_0^{(1)\prime}(ka) + C_2 h_0^{(2)\prime}(ka) - \frac{j_0'(ka)}{j_0(ka)}\left(C_1 h_0^{(1)}(ka) + C_2 h_0^{(2)}(ka)\right) = \frac{2mV_0}{\hbar^2 k}\left(C_1 h_0^{(1)}(ka) + C_2 h_0^{(2)}(ka)\right).$$

Thus

$$e^{2i\delta} = \frac{C_2}{C_1} = -\frac{h_0^{(1)\prime}(ka) - \left(\frac{j_0'(ka)}{j_0(ka)} + \frac{2mV_0}{\hbar^2 k}\right) h_0^{(1)}(ka)}{h_0^{(2)\prime}(ka) - \left(\frac{j_0'(ka)}{j_0(ka)} + \frac{2mV_0}{\hbar^2 k}\right) h_0^{(2)}(ka)}.$$

For $V_0 \to \infty$,

$$e^{2i\delta} = -\frac{h_0^{(1)}(ka)}{h_0^{(2)}(ka)} = e^{2ika},$$

so $\delta = ka$.

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Wednesday, January 15, 2020
2:00PM to 4:00PM
Modern Physics
Section 4. Relativity and Applied Quantum Mechanics

Two hours are permitted for the completion of this section of the examination. Choose <u>4 problems</u> out of the 5 included in this section. (You will <u>not</u> earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 4 (Relativity and Applied Quantum Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. The deuteron is a bound state of a neutron and a proton. The deuteron ground state binding energy is $E_B = -2.22$ MeV. Model the proton-neutron potential as a square well of unknown depth V_0 and range $a = 2.0$ fm, and assume that the ground state is an s-wave.

   (a) Using the fact the the deuteron is a very weakly bound state, make a rough estimate for V_0. You may take the neutron and proton masses to be equal with $Mc^2 = 940$ MeV. You may find it useful to use $\hbar c = 197.3$ MeV-fm.

   (b) Improve your estimate by taking into account the boundary conditions for the radial wave-function at $r = a$ to find an approximate solution to the resulting transcendental equation, keeping in mind that the deuteron is loosely bound.

2. A laser beam has photon number density n. Find the photon density n' in a frame moving along the beam with velocity v.

3. Consider the following simplified model of neutrino oscillation, involving only a two-level system:

   We begin by assuming that there are two unique neutrino mass states, represented by $|\nu_1\rangle$ and $|\nu_2\rangle$, each with mass m_1 and m_2, respectively, and that each is an eigenstate of the Hamiltonian in the two-dimensional Hilbert Space spanned by these two basis vectors, i.e.:

$$\mathbf{H}\,|\nu_1\rangle = E_1\,|\nu_1\rangle\,,$$
$$\mathbf{H}\,|\nu_2\rangle = E_2\,|\nu_2\rangle\,,$$

   and $\langle\nu_i|\nu_j\rangle = \delta_{ij}$.

   We also assume that when neutrinos are produced in a weak interaction they are always produced as so-called "flavor eigenstates," defined in terms of the mass states as follows:

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle\,,$$

   referred to as an electron neutrino, and

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle\,,$$

   referred to as a muon neutrino.

   Given a neutrino is produced at time $t = 0$ as a pure muon neutrino eigenstate, calculate the probability that, after some time $t > 0$ measured in the lab frame, this neutrino would be detectable (through a weak interaction) as an electron neutrino.

   When is this probability maximal?

4. A particle of charge q and rest mass m_0 is accelerated from rest by a uniform electric field $\mathbf{E} = \frac{V_0}{L}\hat{\mathbf{x}}$, which is generated by applying a potential difference V_0 between two parallel metal plates separated by a distance L.

   (a) What is the kinetic energy of the particle when it reaches the second plate? Recall that $dE = \mathbf{F} \cdot d\mathbf{r}$.

   (b) What is the particle's Lorentz factor γ?

   Suppose that the particle passes through a hole in the second plate into a region where the electric field $\mathbf{E} = 0$, but there is a uniform magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$. The particle is observed to move in a circular arc of radius r.

   (c) Derive an expression for r in terms of B_0, V_0, c, and the mass-to-charge ratio m_0/q.

   (d) Compute $\mathbf{E}$ and $\mathbf{B}$ in the rest frame of the particle immediately after it emerges from the hole, expressed in terms of B_0 and the dimensionless velocity β.

5. A stream of protons traveling at velocity $v = 0.9900c$ are directed into a two slit experiment where the slit separation is $d = 4.00 \times 10^{-9}$m. An interference pattern results on the viewing screen. What is the angle between the center of the pattern and the second minimum? You can assume the distance between the screen with the slits and the viewing screen is much larger than any other relevant distance.

Question 1 Solution

(a) For s-wave states, and a two-body system with reduced mass μ, the Schrödinger equation becomes

$$-\frac{\hbar^2}{2\mu}\frac{1}{r}\partial_r^2(r\psi) + V(r)\psi = E\psi$$

which implies $u(r) = r\psi(r)$ satisfies a simple one-dimensional Schrödinger equation. The boundary condition at $r = 0$ requires $u(r) \propto \sin k_0 r$ for $r < a$, with $k_0^2 = \frac{2\mu}{\hbar^2}(V_0 + E)$. (In this expression the well-depth is $-V_0$, that is, V_0 is a positive number while E is negative for a bound state.)

If the state is very weakly bound, the wave function barely "turns over" at $r = a$, that is, $k_0 a \approx \frac{\pi}{2}$, which implies

$$V_0 \approx \frac{\hbar^2}{2\mu}\left(\frac{\pi}{2a}\right)^2 - E = 27.8 \text{ MeV}.$$

(b) To refine this estimate, we must take into account the wave function for $r > a$, which goes as e^{-qr} with $q^2 = -\frac{2\mu}{\hbar^2}E$. Matching $u(r$ and $u'(r)$ at $r = a$ leads to

$$\tan k_0 a = -\frac{k_0}{q},$$

which is the transcendental expression mentioned in the statement of the problem.

To obtain an approximate solution, use the fact that $k_0 a$ is only slightly greater than $\frac{\pi}{2}$ (again, the "loosely bound" condition) to express $k_0 a = \frac{\pi}{2} + \delta$ to rewrite the above equation as

$$\cot\delta = \frac{k_0}{q}.$$

We expect $\delta << 1$, leading to the approximate solution $\delta \approx \frac{q}{k_0}$, so

$$k_0 a = \frac{\pi}{2} + \delta = \frac{\pi}{2} + \frac{q}{k_0},$$

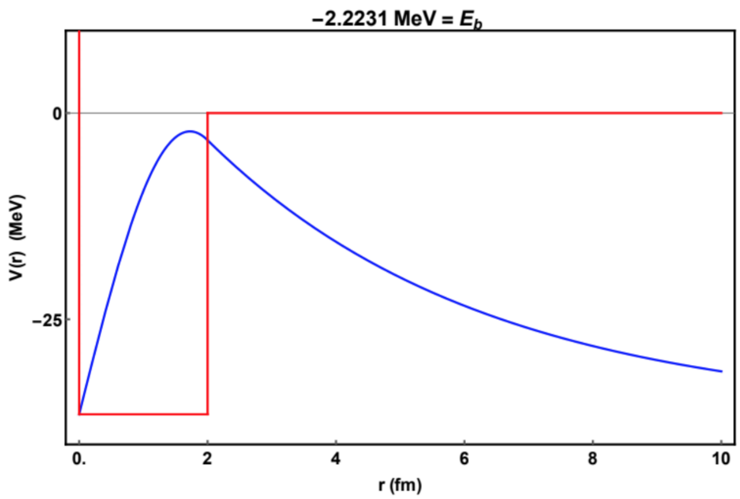
which is a quadratic equation for k_0 with solutions

$$k_0 = \frac{\frac{\pi}{2} \pm \sqrt{(\frac{\pi}{2})^2 + 4aq}}{2a}.$$

We take the + solution since that gives us $\frac{\pi}{2}$ + a small correction. Numerically, this results in $k_0 = \frac{1.82}{a}$, which then gives

$$V_0 \approx \frac{\hbar^2}{2\mu}\left(\frac{1.82}{a}\right)^2 - E = 36.7\,\text{MeV},$$

very close to the value of 36.5 MeV obtained by exact (numerical) solution to the transcendental equation. This differs significantly from the zeroth order estimate because the $\frac{\pi}{2} \approx 1.57 \to 1.82$ enters as the square in the expression for the potential depth, even though the wave-function does indeed "barely turn over", as shown in the below figure:



−2.2231 MeV = E_b
0
−25
V(r) (MeV)
0.
2
4
6
8
10
r (fm)

Question 2 Solution

Solution using 4-flux

Choose the x-axis along the beam. The four-flux of photon number is $F^\mu = cn(1,1,0,0)$. The corresponding flux in the moving frame $F^{\mu'} = cn'(1,1,0,0)$ is related to F^μ by Lorentz transformation:

$$F^{0'} = \gamma F^0 - \gamma\beta F^1 = \gamma(1-\beta)cn \implies n' = \gamma(1-\beta)n,$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{-1/2}$.

Solution using worldlines

Let S be the beam cross section. Consider a piece of the beam of length $l = x_A - x_B$. It contains $N = nlS$ photons. As the beam propagates with the speed of light c, the boundaries of the piece, x_A and x_B, also move with c. Their worldlines in the (ct, x) plane satisfy

$$x_A = ct_A, \qquad x_B = ct_B - l, \tag{1}$$

(the reference time $t = 0$ was chosen when $x_A = 0$). When the same piece is viewed in the moving frame, its cross section is $A' = A$ (the Lorentz boost along x does not touch y,z), and its length is changed to l'. Then

$$n'l'A' = n\,lA = N = \text{const.} \implies \frac{n'}{n} = \frac{l}{l'}.$$

It remains to find l'. For a pair of events (ct_A, x_A) and (ct_B, x_B), Lorentz transformation of the 4-vector connecting the events gives

$$c(t_A - t_B) = \gamma c(t'_A - t'_B) + \gamma\beta(x'_A - x'_B) \tag{2}$$
$$x_A - x_B = \gamma(x'_A - x'_B) + \gamma\beta c(t'_A - t'_B). \tag{3}$$

The meaning of length l' is $l' = x'_A - x'_B$ at $t'_A = t'_B$, so Eqs. (2) and (3) give

$$c(t_A - t_B) = \gamma\beta l'$$
$$x_A - x_B = \gamma l'.$$

Using $(x_A - x_B) - c(t_A - t_B) = l$ (see Eq. 1), one finds

$$\gamma\, l' - \gamma\beta\, l' = l \implies l = \gamma(1-\beta)l' \implies n' = \gamma(1-\beta)n.$$

Question 3 Solution

At $t = 0$, the initial state is given as $|\nu_\mu\rangle = \cos\theta\,|\nu_1\rangle - \sin\theta\,|\nu_2\rangle$. After some time t, we know each mass state will evolve according to:

$$\psi(t) = e^{-i\mathbf{H}t/\hbar}\psi(0) = e^{-iEt/\hbar}\psi(0).$$

Therefore, the time-evolved neutrino state is given by:

$$|\nu_\mu(t)\rangle = -\sin\theta e^{-iE_1t/\hbar}\,|\nu_1\rangle + \cos\theta e^{-iE_2t/\hbar}\,|\nu_2\rangle\,.$$

The probability that this state will be detectable as an electron neutrino is represented by the quantum-mechanical overlap of the time-evolved state and the definite electron neutrino state:

$$\begin{aligned}
P_{\mu\to e} &= |\langle\nu_e|\nu_\mu(t)\rangle|^2 \\
&= \left|(\cos\theta\,\langle\nu_1| + \sin\theta\,\langle\nu_2|)\left(-\sin\theta\,|\nu_1\rangle\,e^{-iE_1t/\hbar} + \cos\theta\,|\nu_2\rangle\,e^{-iE_2t/\hbar}\right)\right|^2 \\
&= \left|-\sin\theta\cos\theta e^{-iE_1t/\hbar} + \cos\theta\sin\theta e^{-iE_2t/\hbar}\right|^2 \\
&= \sin^2\theta\cos^2\theta\left|e^{-iE_2t/\hbar} - e^{-iE_1t/\hbar}\right|^2 \\
&= \sin^2\theta\cos^2\theta\left(1 + 1 - e^{iE_2t/\hbar}e^{-iE_1t/\hbar} - e^{-iE_2t/\hbar}e^{iE_1t/\hbar}\right) \\
&= \sin^2\theta\cos^2\theta\left(2 - 2\cos\left(E_2t/\hbar - E_1t/\hbar\right)\right) \\
&= 4\sin^2\theta\cos^2\theta\sin^2\left(\frac{E_2 - E_1}{\hbar}t\right) \\
&= \sin^2(2\theta)\sin^2\left(\frac{E_2 - E_1}{\hbar}t\right).
\end{aligned}$$

The probability oscillates with a frequency of $(E_2 - E_1)/(2\pi\hbar)$, and reaches its maximum whenever $t = (2n+1)\frac{\pi}{2}\hbar/(E_2 - E_1)$.

Question 4 Solution

(a) We first note that

$$dE = \mathbf{F} \cdot d\mathbf{r} = q\mathbf{E} \cdot d\mathbf{r} = \frac{qV_0}{L}\, dx.$$

Thus the kinetic energy of the particle when it reaches the second plate is

$$K = \int_0^E dE' = \int_0^L \frac{qV_0}{L}\, dx = qV_0.$$

(b) We find

$$K = (\gamma - 1)m_0c^2 = qV_0 \implies \gamma = 1 + \frac{qV_0}{m_0c^2}.$$

(c) Since the acceleration is perpendicular to the velocity, the relativistic version of Newton's second law yields

$$q\mathbf{v} \times \mathbf{B} = \gamma m_0 \mathbf{a}$$
$$qvB_0\,(\hat{\mathbf{x}} \times \hat{\mathbf{z}}) = \gamma m_0 \omega^2 r \hat{\mathbf{r}}.$$

Here $\hat{\mathbf{r}} = -\hat{\mathbf{y}}$, so we have

$$q\omega r B_0 = \gamma m_0 \omega^2 r \implies \omega = \frac{qB_0}{\gamma m_0}.$$

We then find that r is

$$\begin{aligned} r &= \frac{v}{\omega} = \frac{\beta c}{\omega} \\ &= \frac{\beta c \gamma m_0}{qB_0} \\ &= \frac{m_0 c}{qB_0}\gamma\sqrt{1-\gamma^{-2}} \\ &= \frac{m_0 c}{qB_0}\left(1 + \frac{qV_0}{m_0c^2}\right)\sqrt{1 - \frac{1}{(1+\frac{qV_0}{m_0c^2})^2}} && \text{(from part (b))} \\ &= \frac{1}{B_0}\sqrt{2V_0\frac{m_0}{q} + \frac{V_0^2}{c^2}} && \text{(No need to simplify this far.)} \end{aligned}$$

(d) The fields in the lab frame are

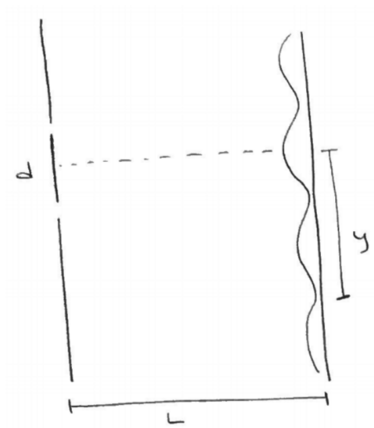
$$\mathbf{E} = 0, \quad \mathbf{B} = B_0\hat{\mathbf{z}}.$$

Thus the fields in the rest frame of the particle are

$$\mathbf{B}' = \gamma B_0 \hat{\mathbf{z}} = \frac{B_0}{\sqrt{1-\beta^2}}\hat{\mathbf{z}}, \quad \mathbf{E}' = \gamma(-\beta B_0)\hat{\mathbf{y}} = -\frac{\beta B_0}{\sqrt{1-\beta^2}}\hat{\mathbf{y}}.$$

Question 5 Solution

We let $L >> d$ be the distance from the slits to the viewing screen and let y be the distance up the screen, as in the figure.



The distances the protons travel from slits 1 and 2 to the screen are

$$\ell_1 = \sqrt{L^2 + \left(y + \frac{d}{2}\right)^2} \approx L + \frac{1}{2L}\left(y + \frac{d}{2}\right)^2$$

$$\ell_2 = \sqrt{L^2 + \left(y - \frac{d}{2}\right)^2} \approx L + \frac{1}{2L}\left(y - \frac{d}{2}\right)^2$$

The path length difference is thus

$$\ell_1 - \ell_2 \approx \frac{dy}{L}.$$

For the second minimum, we set this equal to $3\lambda/2$, yielding

$$y = \frac{3}{2}\frac{\lambda L}{d}.$$

The deBroglie wavelength is

$$\lambda = \frac{h}{p} = \frac{h}{\gamma m v} = \frac{4.1356 \times 10^{-15} \text{ eV} \cdot \text{s}}{\frac{0.99c}{\sqrt{1-0.99^2}}(938 \times 10^6 \text{ eV}/c^2)} = 1.88 \times 10^{-16} \text{ m}.$$

Thus the angle is

$$\theta \approx \tan\theta = \frac{y}{L} = \frac{3\lambda}{2d} = 7.06 \times 10^{-8} \text{ rad}.$$

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 17, 2020
10:00AM to 12:00PM
General Physics
Section 5. Thermodynamics and Statistical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 5 (Thermodynamics and Statistical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider a gas contained in volume V at temperature T. The gas is composed of N distinguishable particles of zero rest mass, so that the energy E and momentum p of the particle are related by $E = pc$. The number of single-particle energy states in the range p to $p + dp$ is $4\pi V p^2\, dp/h^3$. Find the equation of state and the internal energy of the gas, and compare with an ordinary gas.

2. Consider a mole of helium gas in container of size one liter at room temperature. Assume the helium is an ideal, noninteracting gas in the classical limit. The single-particle partition function for an atom is given by $Z_1 = V\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}$.

   (a) Please compute the free energy of the gas (use Stirling's approximation).

   (b) Now compute the chemical potential.

   (c) Now assume that this container is connected to a large reservoir of helium gas, which has the same pressure and temperature. The container is free to exchange particles with the reservoir. Consider an ideal gas of particles of mass M in a volume V in thermal equilibrium with a reservoir with which it can exchange particles. Please compute the fluctuation in particle number $\frac{\sqrt{\langle N(t)-N\rangle^2}}{N}$ where the brackets indicate a long time average.

   (d) Assuming the number density is the same as in parts (a) and (b), how big should the container have been so that the fluctuation in particles defined in (c) is 10%? (Work this out numerically.)

3. For an elastic filament it is found that, at a finite range in temperature, a displacement x requires a force (tension)

$$\mathcal{T} = ax + bT + cTx$$

where a, b, and c are constants. Furthermore, the heat capacity C of the filament at constant displacement is proportional to temperature, i.e.

$$C_x = A(x)T.$$

(a) Use an appropriate Maxwell relation to calculate the partial derivative of the entropy, S, with respect to displacement at fixed temperature T, i.e. calculate $(\partial S/\partial x)_T$.

(b) Show that the coefficient A in the equation for specific heat, above, has to in fact be independent of x, i.e.

$$\frac{dA}{dx} = 0.$$

(c) Give the expression for $S(T, x)$ assuming $S(0, 0) = S_0$.

(d) Calculate the heat capacity at constant tension, i.e. $C_{\mathcal{T}}$ as a function of displacement and temperature.

4.

(a) The Bose Einstein Condensation temperature T_{BEC} is the temperature at which a macroscopic number of bosons in a gas enter the ground state. At this temperature, the thermal wave length (or spread of the wave function) of a boson becomes comparable to the average distance between adjacent bosons. Use this fact to estimate T_{BEC} for an ideal monoatomic, non-relativistic Bose gas system in three dimensions which has a particle density, n (bosons per volume) and the boson mass, m. It is not necessary to derive T_{BEC} rigorously using the partition function and Riemann zeta function. We will, however, accept such an answer if it is correctly done.

(b) For spin-1/2 fermions in 3 dimensions, one can calculate the Fermi temperature T_F from the Fermi energy $\epsilon_F = k_B T_F$ for non-interacting Fermi gas with density n and mass of each fermion m. Show that T_{BEC} and T_F has the same dependence on n and m. This can be done either by using the uncertainty principle or by calculating ϵ_F for free Fermions in a 3-dimensional box.

(c) When one resorts to a rigorous calculation, T_{BEC} and T_F are comparable within a factor of 2. Explain why this is so with qualitative arguments. We are not asking you to perform a rigorous calculation of these numbers, but rather remember what T_{BEC} and T_F stand for and explain why they are comparable.

(d) Describe one phenomenon in the real world which can be attributed to the Bose Einstein Condensation, and describe an experimental method to measure the transition temperature.

5. A mixed stated in quantum mechanics is described by a density matrix rather than by a vector in the Hilbert space. The density matrix, $\hat{\rho}$, is an operator that allows to compute expectation values of observables through

$$\langle \hat{O} \rangle = \mathrm{Tr}\big(\hat{\rho}\, \hat{O}\big) \ . \tag{1}$$

The trace 'Tr' is defined as

$$\mathrm{Tr}\big(\dots \big) = \sum_n \langle n | \cdots | n \rangle \ , \tag{2}$$

where the sum is taken over any complete orthonormal set of states $\{|n\rangle\}_n$.

Consider then a 1-D harmonic oscillator with proper frequency ω, at finite temperature $T = 1/\beta$. The corresponding density matrix (setting $k_{\mathrm{B}} = 1$) is

$$\hat{\rho} = \frac{e^{-\beta \hat{H}}}{\mathrm{Tr}\big(e^{-\beta \hat{H}}\big)} \ , \tag{3}$$

where $\hat{H}$ is the Hamiltonian. Notice that $\mathrm{Tr}(\hat{\rho}) = 1$.

(a) Compute the average energy:

$$E \equiv \langle \hat{H} \rangle \ . \tag{4}$$

(b) Compute the so-called von Neumann entropy:

$$S \equiv -\mathrm{Tr}\big(\hat{\rho}\, \log \hat{\rho}\big) \ . \tag{5}$$

(c) For E and S computed above, discuss the low-temperature ($T \ll E_0$) and high-temperature ($T \gg E_0$) limits.

(d) For the high-temperature limit of item 3 above, consider changing the temperature by an infinitesimal amount dT. Verify the thermodynamical identity

$$dE = T dS \ . \tag{6}$$

[Hint: *To compute traces, use the orthonormal basis made up of energy eigenstates. In this basis, all traces relevant for the problem reduce to geometric series or derivatives thereof (with respect to β). Also, by using directly the expression* (3), *you can relate the computation of S in part (b) to that of E in part (a).*]

6. *Fermions with peculiar dispersion*

One (unproven) theory says that matter in the core of a neutron star can be viewed as a gas of fermions with the excitation spectrum

$$\varepsilon_{\pm}(k) = \pm \frac{\hbar^2}{2M} (k - k_F)^2$$

where the $-$ pertains to $k < k_F$ and the $+$ to $k > k_F$.

Suppose that at $T = 0$ all of the states with energy less than 0 are filled and all of the states with energy greater than zero are empty.

(a) What is the chemical potential at $T = 0$?

(b) Assuming that the density of states in reciprocal space is constant around $|k| = k_F$, i.e. approximating $\int d^3k \to k_F^2 \int dq$ with $q = k - k_F$ and neglecting any temperature dependence of the chemical potential find the temperature dependence of the mean energy of the system $E(T) - E(T = 0)$.

(c) Using the same approximations as in (b), find the specific heat and give the leading temperature dependence as $T \to 0$.

Question 1 Solution

The partition function for N particles is

$$Z = \int dV_1 \cdots dV_n \frac{d^3p_1 \dots d^3p_N}{h^{3N}}\, e^{-E/k_BT}.$$

Substituting the given density of states gives

$$\begin{aligned} Z &= \left(\frac{4\pi V}{h^3} \int_0^\infty dp\, e^{-pc/k_BT} p^2 \right)^N \\ &= \left(\frac{8\pi V}{h^3} \left(\frac{k_BT}{c} \right)^3 \right)^N . \end{aligned}$$

The free energy is $F = -k_BT \ln Z$, and $dF = -p\, dV - SdT$, so

$$p = -\left(\frac{\partial F}{\partial V} \right)_T = k_BT \frac{\partial \ln Z}{\partial V} = \frac{Nk_BT}{V}.$$

The internal energy is

$$U = -\frac{\partial \ln Z}{\partial (1/k_BT)} = 3Nk_BT.$$

For an ordinary gas, we have $p = Nk_BT/V$ and $U = \frac{3}{2}Nk_BT$, so the pressure is the same, but the energy is off by a factor of 2.

Question 2 Solution

(a) The N-particle partition function is given by $Z = Z_1^N/N!$. The free energy of an ideal gas is $F = -k_B T \ln(Z)$. Plug and chug to get

$$F(N, V, T) = -Nk_B T - Nk_B T \ln\left(\frac{V}{N\lambda_T^3}\right).$$

with $\lambda_T = \sqrt{\dfrac{2\pi\hbar^2}{mk_B T}}$.

(b) The chemical potential is

$$\mu = \frac{\partial F}{\partial N} = -k_B T \ln \frac{V}{N\lambda_T^3}.$$

(c) Examination of the formula for the grand canonical partition function shows that

$$\begin{aligned}
\left\langle (N - \langle N \rangle)^2 \right\rangle &= T^2 \frac{\partial^2}{\partial \mu^2} \ln \Omega \\
&= T \frac{\partial N}{\partial \mu} \\
&= T \left(\frac{\partial \mu}{\partial N}\right)^{-1}.
\end{aligned}$$

From the result of (b) we evaluate this as

$$T\left(\frac{\partial \mu}{\partial N}\right)^{-1} = N,$$

so the relative number fluctuation is just $\frac{1}{\sqrt{N}}$ or $\frac{1}{\sqrt{nV}}$.

(d) From the answer to (c), $\frac{1}{\sqrt{nV}} = 0.1$ or $V = \frac{100}{n} = 166\ \mathrm{nm}^3$.

Question 3 Solution

(a) If F is the free energy then

$$\left(\frac{\partial S}{\partial x}\right)_T = -\frac{\partial^2 F}{\partial x \partial T} = -\frac{\partial}{\partial T}\frac{\partial F}{\partial x} = \frac{\partial \mathcal{T}}{\partial T} = b + cx$$

(b) From the relation $C = T\,(\partial S/\partial T)$ we find $A(x) = \partial S/\partial T$ so $\partial A(x)/\partial x = \partial^2 S/\partial T \partial x$ and taking the derivative with respect to T of the answer in (a) shows the result is 0

(c) Because the cross derivative $\partial^2 S/\partial x \partial T = 0$ we have

$$S(x,T) = S_x(x) + S_T(T),$$

and integrating the results of the previous problems

$$S(x,T) = S(0,0)AT + bx + \frac{1}{2}cx^2.$$

(d) The change in tension is

$$d\mathcal{T} = (a + cT)dx + (b + cx)dT$$

so if $d\mathcal{T} = 0$ then $\frac{dx}{dT} = -\frac{b+cx}{a+cT}$ and

$$C_{\mathcal{T}} = T\frac{\partial S}{\partial T}_{\mathcal{T}} = T\left(\frac{\partial S}{\partial T}\right)_x + T\left(\frac{\partial S}{\partial x}\right)_T \frac{dx}{dT} = AT - T\frac{(b+cx)^2}{a+cT}.$$

Question 4 Solution

Examples of possible solutions:

(a) At temperature T, $p^2/2m \sim k_BT \implies p = \sqrt{2mk_BT}$. Let $\lambda = h/p$ by the thermal wavelength. In a BEC, λ becomes comparable to the inter-boson distance $n^{-1/3}$. Therefore, $p^2 = h^2 \times n^{2/3} = 2mk_BT_{\text{BEC}}$, so

$$T_{\text{BEC}} \sim \frac{h^2}{2k_B}\frac{n^{2/3}}{m}.$$

An exact calculation gives

$$T_{\text{BEC}} = \frac{1}{2\pi\zeta(3/2)^{2/3}}\frac{h^2}{k_B}\frac{n^{2/3}}{m} \approx 0.084\frac{h^2}{k_B}\frac{n^{2/3}}{m}.$$

(b) Suppose you divide the volume V into a small sub-cube of the edge size Δx. In each sub-cube, we can put two fermions (spin up and down). Thus $N = 2V/(\Delta x)^3$ where N is the total number of fermions. Thus $\Delta x = 2^{1/3}(V/N)^{1/3}$. By the uncertainty principle, $\Delta p \Delta x \sim \hbar$. Then

$$\frac{(\Delta p)^2}{2m} \sim \epsilon_{\text{Fermi}} \sim \hbar^2 2^{-2/3} 2^{-1}\frac{n^{2/3}}{m}.$$

So

$$T_{\text{Fermi}} \sim \frac{1}{2^{5/3}}\frac{\hbar^2}{k_B}\frac{n^{2/3}}{m}.$$

An exact calculation for a free electron gas in three dimensions yields

$$T_{\text{Fermi}} = \frac{(3\pi^2)^{2/3}}{2}\frac{\hbar^2}{k_B}\frac{n^{2/3}}{m}.$$

(c) As you can see above, exact results for T_{BEC} and T_{Fermi} are comparable within a factor 2. This can be understood if you remember that both energy scales represent a boundary between quantum mechanical (boson or fermion) particles and higher energy classical particles. Below these energy scales, bosons and fermions form quantum degenerate liquid. In the classical gas at higher temperatures, distinction between bosons and fermions becomes unnecessary.

(d) Examples of Bose-Einstein condensates and methods to measure the condensation temperate:

  (a) <u>Superfluid Helium-4:</u> Superfluidity can be verified by vanishing viscosity measured by torsion oscillator, critical opalescence, lambda-shaped peak of the specific heat, and more.

(b) Superconductivity: BCS superconductors undergo a special case of BEC at the critical temperature T_c, where formation of bosons from fermions (pair formation) and condensation of newly formed bosons occur simultaneously. Critical temperature can be measured as zero resistivity, diamagnetism due to Meissner effect, specific heat anomaly, and many other phenomena.

(c) BEC of cold atoms: BEC of dilute cold boson gas can be observed from the velocity distribution peaking at zero momentum of the gas below T_{BEC}.

(d) Other possible case includes BEC of neutron star, BEC of excitons in semiconductors, and possible BEC of pre-formed pairs in underdoped regions of high-T_c cuprate and other unconventional superconductors. One can refer to literatures on how to measure T_{BEC} in these systems

Question 5 Solution

For the one-dimensional Harmonic oscillator, the energy eigenstates $\{|n\rangle\}_{n=0,1,2,\ldots}$ are such that

$$\hat{H}\,|n\rangle = E_0\left(n+\frac{1}{2}\right)|n\rangle\,, \qquad n=0,1,2,\ldots,$$

where $E_0 = \hbar\omega$ is the energy gap between two adjacent states. Since $\hat{\rho}$ is a function of $\hat{H}$ alone, $\hat{\rho}$ is also diagonal in this basis:

$$\hat{\rho}\,|n\rangle = \rho_n\,|n\rangle\,,$$

with eigenvalues

$$\rho_n = \frac{e^{-\beta E_0(n+1/2)}}{\sum_{m=0}^{\infty} e^{-\beta E_0(m+1/2)}}$$

The sum at the denominator is a geometric series,

$$e^{-\beta E_0/2}\sum_{m=0}^{\infty}\left(e^{-\beta E_0}\right)^m = \frac{e^{-\beta E_0/2}}{1-e^{-\beta E_0}}$$

and so the eigenvalues of $\hat{\rho}$ are

$$\rho_n = \left(1-e^{-\beta E_0}\right)e^{-\beta E_0 n}$$

(a) We have

$$\begin{aligned}
E = \mathrm{Tr}\left(\hat{\rho}\hat{H}\right) &= \left(1-e^{-\beta E_0}\right)E_0\sum_{n=0}^{\infty}\left(e^{-\beta E_0 n}\right)\left(n+\frac{1}{2}\right) &(1)\\
&= \left(1-e^{-\beta E_0}\right)E_0\left(-\frac{1}{E_0}\frac{\partial}{\partial\beta}+\frac{1}{2}\right)\sum_{n=0}^{\infty}\left(e^{-\beta E_0}\right)^n &(2)\\
&= \frac{1}{2}E_0 - \left(1-e^{-\beta E_0}\right)\frac{\partial}{\partial\beta}\frac{1}{1-e^{-\beta E_0}} &(3)\\
&= E_0\left(\frac{1}{2}+\frac{e^{-\beta E_0}}{1-e^{-\beta E_0}}\right) &(4)
\end{aligned}$$

(b) Given the definition of the thermal density matrix in the problem, we have

$$\begin{aligned}
S &= -\mathrm{Tr}\left(\hat{\rho}\,\log\hat{\rho}\right)\\
&= \beta\,\mathrm{Tr}\left(\hat{\rho}\hat{H}\right) + \mathrm{Tr}\left(\hat{\rho}\right)\log\left(\mathrm{Tr}\left(e^{-\beta\hat{H}}\right)\right)\\
&= \beta E + \log\left(\frac{e^{-\beta E_0/2}}{1-e^{-\beta E_0}}\right)\\
&= \beta\big(E-\frac{1}{2}E_0\big) - \log\left(1-e^{-\beta E_0}\right)\\
&= \beta E_0\frac{e^{-\beta E_0}}{1-e^{-\beta E_0}} - \log\left(1-e^{-\beta E_0}\right).
\end{aligned}$$

(c) At low temperatures, we have $e^{-\beta E_0} \ll 1$, and so

$$E = \frac{1}{2} E_0 \left(1 + \mathcal{O}\left(e^{-\beta E_0}\right)\right), \qquad S = \mathcal{O}\left(\beta E_0 \, e^{-\beta E_0}\right) << 1.$$

This makes sense: at very low temperatures compared to the gap, the probability of being in an excited state is exponentially small. So, up to exponentially small corrections, the system is in the ground state, with energy $E_0/2$ and zero entropy.

At high temperatures, we have $e^{-\beta E_0} \simeq 1 - \beta E_0$, and so:

$$E \approx \frac{1}{\beta} = T, \qquad S \approx -\log(\beta E_0) = \log \frac{T}{E_0} >> 1$$

The energy is consistent with equipartition, while the entropy is very high, and of the right order of magnitude: all states up to $E \sim T$ are equally populated; there are $N \sim E/E_0 \sim T/E_0$ of them, and the entropy is $S \sim \log N$.

(d) For an infinitesimal temperature variation dT, from above we get

$$dE \simeq dT \ , \qquad dS \simeq \frac{dT}{T},$$

which is consistent with $dE = TdS$.

Question 6 Solution

(a) The chemical potential at $T = 0$ is zero.

(b) The internal energy is (making the indicated approximation and noting that $\varepsilon_- = -\varepsilon_+$)

$$E = \int dk \, \frac{\varepsilon(k)}{e^{\frac{\varepsilon(k)}{T}} + 1} \approx \frac{k_F^2}{2\pi^2} \int_0^\infty dq \, \frac{\varepsilon_+(q)}{e^{\frac{\varepsilon_+(q)}{T}} + 1} - \frac{\varepsilon_+(q)}{e^{\frac{-\varepsilon_+(q)}{T}} + 1}$$

or

$$E(T) - E(0) = \frac{k_F^2}{2\pi^2} \int_0^\infty dq \, \varepsilon_+(q) \left(1 - \tanh \frac{\varepsilon_+(q)}{2T}\right).$$

(c) Define $u = \frac{\hbar^2 q^2}{4MT}$ so $q = \sqrt{\frac{4MTu}{\hbar^2}}$ and $dq = \sqrt{\frac{MT}{\hbar^2}} \frac{du}{\sqrt{u}}$ so

$$E(T) - E(0) = \frac{k_F^2}{\pi^2} \left(\frac{M}{\hbar^2}\right)^{\frac{3}{2}} T^{\frac{3}{2}} \int_0^\infty du \, \sqrt{u} \, (1 - \tanh u) \, ,$$

so the specific heat is $C = \dfrac{\partial E}{\partial T} \propto T^{\frac{1}{2}}$.

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Friday, January 17, 2020
2:00PM to 4:00PM
General Physics
Section 6. Various Topics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 6 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 6 (General Physics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" $\times$ 11" paper (double-sided) you have prepared on Modern Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. In answering the questions below, make reasonable estimates for any needed parameters/quantities not already provided.

   Consider an idealized Sun and Earth as blackbodies in otherwise empty space. The Sun has a surface temperature $T_S = 6000$ K, and heat transfer processes on the Earth are effective enough to keep the Earths surface temperature uniform. The radius of the Earth is $R_E = 6.4 \times 10^6$ m, the radius of the Sun is $R_S = 7.0 \times 10^8$ m, and the Earth-Sun distance is $d = 1.5 \times 10^{11}$ m. The mass of the Sun is $M_S = 2.0 \times 10^{30}$ kg.

   (a) Find the temperature of the Earth.

   (b) Find the radiation force on the Earth.

   (c) Compare these results with those for an interplanetary granule in the form of a spherical, perfectly conducting black body with a radius $R = 0.1$ cm, moving in a circular orbit around the Sun at a radius equal to the Earth-Sun distance d.

   (d) At what distance from the Sun would a metallic particle melt if its melting temperature $T_m = 1550$ K?

   (e) For what size particle would the radiation force calculated in part (c) be equal to the gravitational force from the Sun at a distance d?

   Note: You will need the values of the Stefan-Boltzmann constant σ and the gravitational constant G.

2. A small bit of solid matter is released at the top of out atmosphere. It is pulled down by gravity through three atmospheric gas layers, described below, before reaching the earth's surface. What is its characteristic fall velocity while passing through each layer?

   Each of the three layers has its own characteristic gas mass density ρ of mass m molecules moving about with a randomized thermal velocity v. These molecules scatter elastically on neighbors with a cross-section σ.

   The bit of matter has a fixed mass M and radius R. In a vacuum it would have an earth directed acceleration g. Because of the drag force on the bit from its passage through the local atmosphere with downward speed V, its downward acceleration is essentially canceled.

   In layer I, the first it passes through, the air molecules' mean free path for scattering $\lambda >> R$ and V is supersonic $V > v$. In the second, layer II, with large ρ, $\lambda >> R$ but $V < v$. In the third, layer III, $\lambda < R$ ansd $V << v$. The drag force on the falling bit becomes proportional to RV. (Stokes' Law is appropriate.)

   What is the characteristic V, as a function of R, g, M, ρ, v, σ, and m in each layer?

3. When it rains, is it better to run fast or to walk slowly (assuming you don't want to get too wet)? Motivate your answer with an actual computation. In particular, you should compute the total amount of water hitting you as you move from A to B, as a function, among other things, of your speed. You can adopt several simplifications:

    1. You are a parallelepiped, gliding smoothly at constant (horizontal) speed v.
    2. The density of rain drops is so high that you can treat rain as a continuum, with constant mass density ρ_{rain}, moving vertically downwards at constant speed v_{rain}.
    3. The angle at which rain hits you does not matter, nor does its speed relative to you: you are perfectly absorbent.
    4. Ignore, of course, relativistic effects.

4. Consider normal modes of vibration of a linear chain of identical atoms of mass M in the harmonic approximation. Assume that coupling between atoms exist only for nearest neighbor atoms. These couplings are represented by single equal springs with force constant K. Normal modes have wave vector $\mathbf{k}$ and frequency ω. Also assume that the chain has N atoms and length $L = Na$.

   (a) Write the equation of motion for displacements $\mathbf{u}$ of normal modes and obtain the normal mode frequencies $\omega(k)$ as function of the magnitude of the wavevector $|\mathbf{k}| = k$.

   (b) Longitudinal normal modes have $\mathbf{u} \parallel \mathbf{k}$ and transverse modes have $\mathbf{u} \perp \mathbf{k}$. How many longitudinal modes and transverse modes exist for each value of wave vector. Why are modes with wave vectors $\mathbf{k}$ and $-\mathbf{k}$ degenerate

   (c) Show that for long wavelengths the equation of motion reduces to a continuum elastic wave equation. What is the speed of sound?

   (d) Use the cyclic boundary conditions to obtain the density of states of modes. How many modes are in the first Brillouin zone of the linear chain?

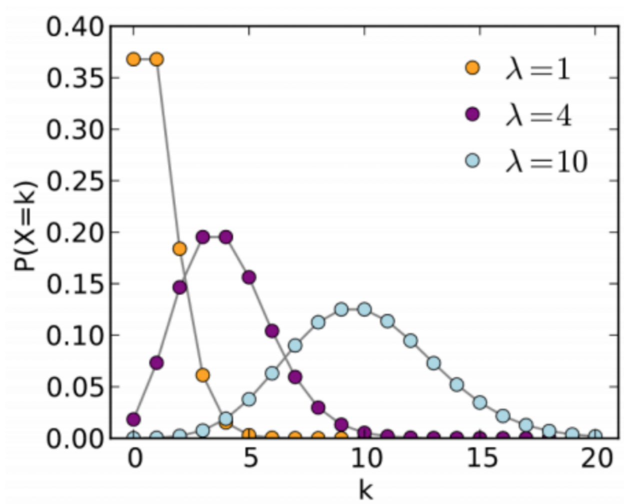
   (e) In a Debye model $\omega(k)$ is approximated by a linear dispersion $\omega(k) = c_s k$, where c_s is the speed of sound in the chain. Obtain the specific heat due to longitudinal vibrations of the chain of identical atoms in the Debye approximation and obtain the expression for the heat capacity (specific heat at constant volume) in the limit of $T \to 0$.

5. Please read the entire question before beginning and note that there are equations at the end of the problem. The hints are there to benefit you but you can still do the problem in any way you choose.

   In this part of the problem, we will explore both the Poisson distribution and the normal distribution and their relation. All given equations can be used without proof - in all other cases you must show your work!

   The Poisson distribution (form is given at the bottom of the page) is a discrete probability distribution defined for integers $k \geq 0$ and mean $\lambda > 0$ that expresses the probability of a given number of events occurring in a fixed interval of time and/or space. In physics, we often have an expectation for a given rate of some type of event so we use the Poisson distribution to describe the probability of a given number of events occurring in a given time interval assuming that our expectation is the rate multiplied by our given time interval. The Poisson distribution is a limit of the Binomial distribution with $p \to 0$ and $N \to \infty$.

   Below is an image of the Poisson distribution for different mean values.



   (a) Show that the mean (expectation value) and the variance of the Poisson distribution are both equal to λ.

   Hint 1: Use may use the exponential summation given below.

   Hint 2: Remember that $\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$.

   (b) Show that in the limit of $\lambda >> 1$ and $k \sim \lambda$ (k on the order of λ) that the Poisson distribution can be approximated by the normal distribution.

   Hint 1: Let $x = k = \lambda(1 + \delta)$ where we know $\delta << 1$ since $k \sim \lambda$.

   Hint 2: You will need to use Stirling's Approximation (see equations below).

   Hint 3: To approximate $(1 + x)^{ax}$ where $x << 1$, we can first use the natural logarithm to get it in a more convenient form and then exponentiate the result.

   Dark matter experiments, in their simplest form, are counting experiments. You have an expectation for your background over some period of time T. These experiments then take data over this time period T and look at the total of number of events to determine whether they see a statistically significant excess of events or not (this excess of course being from dark matter interactions!).

Consider two detectors. In detector A we expect 1 background event in a single year and in detector B we expect 1000 background events in a year. We turn on our detectors and after one year we find that detector A counted a total of 11 events and detector B saw a total of 1010 events. Assume that we know our background perfectly.

(c) What is the probability that each detector's number of events is from background?

(d) If we can safely reject the background hypothesis, then it is possible that we can claim a direct detection of dark matter. In which detector is it less likely that the signal can be explained by background alone? Explain your reasoning.

$$\text{Discrete Expectation: } \langle k^a \rangle = \sum_k k^a p(k)$$

$$\text{Exponential Sum: } \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$$

$$\text{Expanding the Natural Log: } \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\text{Poisson Distribution: } p(k) = \frac{\lambda^k e^{-\lambda}}{k!},\ \lambda > 0,\ k = 0, 1, 2, \ldots$$

$$\text{Normal Distribution: } p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

$$\text{Stirling Approximation: } n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

6. Estimate the rms magnitude of the electric field associated with blackbody radiation in thermal equilibrium at room temperature. Note: $\sigma = 5.7 \times 10^{-8}$ W/(m^2·K^4).

Question 1 Solution

(a) The total power radiated from the Sun is

$$P_S = 4\pi R_S^2 \sigma \varepsilon T_S^4,$$

where $\epsilon = 1$ for a blackbody. Of this, the fraction that hits the Earth is $\frac{\pi R_E^2}{4\pi d^2}$, and in equilibrium, with the Earth also radiating as a blackbody, then:

$$\left(4\pi R_S^2\right) \sigma T_S^4 \left(\frac{\pi R_E^2}{4\pi d^2}\right) = \left(4\pi R_E^2\right) \sigma T_E^4,$$

so:

$$T_E = \sqrt{\frac{R_S}{2d}} T_S = 290 \text{ K}.$$

(b) The radiation pressure is:

$$P_{\text{rad}} = \frac{4\sigma}{3c} T_S^4 \left(\frac{4\pi R_S^2}{4\pi d^2}\right) = 7.1 \times 10^{-6} \text{ N/m}^2.$$

with the corresponding force on the Earth being:

$$F_E = \pi R_E^2 P_{\text{rad}} = 9.2 \times 10^8 \text{ N}.$$

(c) For the small granule, the temperature will be the same since it the same distance from the Sun as the Earth is. The radiation pressure on the granule is:

$$F_{\text{gr}} = \pi R^2 P_{\text{rad}} = 2.2 \times 10^{-11} \text{ N}.$$

(d) From the result of part (a), we can write the distance d_m at which the granule will melt:

$$d_m = \frac{1}{2} R_S \left(\frac{T_S}{T_m}\right)^2 = 5.2 \times 10^9 \text{ m}.$$

(e) Assume a spherical particle of mass m and radius r, then when the radiation and gravitational forces balance we have:

$$\frac{GM_S m}{d^2} = \pi r^2 P_{\text{rad}},$$

where $m = \frac{4}{3}\pi r^3 \rho$, and we can assume a density of say $\rho = 5.0 \times 10^3$ kg/m^3, then:

$$r = \frac{3}{4} \frac{P_{\text{rad}} d^2}{GM_S \rho} = 1.8 \times 10^{-7} \text{ m}.$$

Question 2 Solution

In layer I, we have

$$Mg \sim R^2V^2\rho \implies V \sim \left(\frac{Mg}{\rho R^2}\right)^{1/2}.$$

In layer II, we have

$$Mg \sim R^2Vv\rho \implies V \sim \frac{Mg}{\rho R^2 v}.$$

In layer III, we have

$$Mg \sim R\lambda Vv\rho \implies V \sim \frac{Mg}{\rho\lambda vR}.$$

Since $\lambda \sim \frac{m}{\rho\sigma}$, we have

$$V \sim \frac{Mg\sigma}{vmR}.$$

Question 3 Solution

Under the assumptions spelled out in the problem, it is better to run as fast as possible, $v \to \infty$.

Calling L the distance between A and B, the time it takes to cover that distance is $T = L/v$. Now call S_{top} the surface area of the top face of the parallelepiped (one's head), and S_{front} the surface area of the front face of the parallelepiped (one's body). During T, the mass of water that falls on one's head is $\rho_{\text{rain}} v_{\text{rain}} S_{\text{top}} T$, which is proportional to T, and thus inversely proportional to v. On the other hand, the mass intercepted by one's body is independent of T and thus of v: it is simply $\rho_{\text{rain}} S_{\text{front}} L$, the total mass of water contained in the 'tube' of cross section S_{front} connecting A to B.

So, the total mass of rain intercepted (and thus absorbed) in moving from A to B at constant speed v is

$$M_{\text{tot}} = \rho_{\text{rain}} L \cdot \left[S_{\text{top}} \frac{v_{\text{rain}}}{v} + S_{\text{front}} \right].$$

As a function of v, this is schematically

$$M_{\text{tot}} \sim \text{const} + 1/v,$$

which is minimized for $v \to \infty$.

Question 4 Solution

(a) The equation of motion is

$$M\ddot{u}(n) = -k\left(2u(n) - u(n-1) - u(n+1)\right),$$

where $u(n)$ is the displacement of the atom at position n in the chain. The normal mode frequencies are

$$\omega(k) = 2\sqrt{\frac{k}{M}}\left|\sin\left(\frac{1}{2}ka\right)\right| = \sqrt{\frac{2K}{M}(1-\cos ka)}.$$

(b) We have one longitudinal mode and two transverse modes. The modes with k and $-k$ are degenerate because there is no dependence on the direction of propagation.

(c) Taking $u(n) \propto e^{ikx(n)}$, where $x(n) = na$. In the long wavelength limit, we have

$$u(n) - u(n-1) \approx \frac{\partial u(n-1)}{\partial x(n-1)}ka$$

$$u(n) - u(n+1) \approx -\frac{\partial u(n+1)}{\partial x(n+1)}ka.$$

The equation of motion is

$$M\frac{\partial^2 u(n)}{\partial t^2} \approx -K\left(\frac{\partial u(n-1)}{\partial x(n-1)} - \frac{\partial u(n+1)}{\partial x(n+1)}\right)ka.$$

We also have

$$\frac{\partial u(n-1)}{\partial x(n-1)} - \frac{\partial u(n+1)}{\partial x(n+1)} \approx \frac{\partial^2 u(n)}{\partial x(n)^2}ka.$$

Thus the equation of motion is

$$\frac{\partial^2 u(n)}{\partial t^2} \approx \frac{K}{M}(ka)^2\frac{\partial^2 u(n)}{\partial x(n)^2}.$$

It follows that the speed of sound is $c_s = \sqrt{\frac{K}{M}}ka$

(d) The chain has N atoms and length $L = Na$. The density of states is $G(k) = \frac{L}{2\pi}$. We also have $dN = G(k)\,dk$, and in the first Brillouin zone:

$$\int_{-\pi/a}^{\pi/a} G(k)\,dk = N.$$

(e) The heat capacity at constant volume is

$$C_V = \frac{L}{\pi}(k_BT)\int_0^{x_D}\frac{x}{e^x-1}\,dx,$$

where $x_D = \hbar c_s k_D/k_BT$, where k_D is the Debye wavevector. For $T \to 0$,

$$C_V \approx \frac{L}{\pi}(k_BT)\int_0^{\infty}\frac{x}{e^x-1}\,dx = \frac{\pi}{6}L(k_BT).$$

Question 5 Solution

(a) We begin the problem by calculating the required expectation values: $\langle k \rangle$ and $\langle k^2 \rangle$

$$\langle k \rangle = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!}, \quad \langle k^2 \rangle = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k e^{-\lambda}}{k!}.$$

We start with the expansion of exponential: $e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$. Taking the derivative with respect to λ on both sides yields

$$e^\lambda = \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{k!} \implies \lambda e^\lambda = \sum_{k=1}^{\infty} \frac{k\lambda^k}{k!}.$$

Thus

$$\langle \lambda \rangle = e^{-\lambda} \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!} = e^{-\lambda} \lambda e^\lambda = \lambda.$$

Now we repeat the derivative trick for $\lambda e^\lambda = \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!}$:

$$e^\lambda + \lambda e^\lambda = e^\lambda (1+\lambda) = \sum_{k=0}^{\infty} \frac{k^2 \lambda^{k-1}}{k!} \implies e^\lambda \left(\lambda + \lambda^2\right) = \sum_{k=0}^{\infty} \frac{k^2 \lambda^k}{k!}$$

Thus we find

$$\left\langle k^2 \right\rangle = e^{-\lambda} \sum_{k=0}^{\infty} \frac{k^2 \lambda^k}{k!} = e^{-\lambda} e^\lambda \left(\lambda + \lambda^2\right) = \lambda + \lambda^2.$$

So the variance is

$$\sigma^2 = \left\langle k^2 \right\rangle - \langle k \rangle^2 = \lambda.$$

(b) We start with $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$ and want to get $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$ in the limit of $\lambda \to \infty$ and $k \sim \lambda$. For $k \sim \lambda$ only care about points relatively close to the mean (not tails). Following the hints, we define $x = k = \lambda(1+\delta)$ where δ is small. Thus

$$\begin{aligned}
p(x) &= \frac{\lambda^{\lambda(1+\delta)} e^{-\lambda}}{(\lambda(1+\delta))!} \\
&= \frac{\lambda^{\lambda(1+\delta)} e^{-\lambda}}{\sqrt{2\pi\lambda(1+\delta)}} e^{\lambda(1+\delta)} \lambda^{-\lambda(1+\delta)} (1+\delta)^{-\lambda(1+\delta)} && \text{(Stirling's approximation)} \\
&= \frac{e^{\lambda\delta}}{\sqrt{2\pi\lambda}} (1+\delta)^{-\lambda-\lambda\delta-\frac{1}{2}}.
\end{aligned}$$

Following the hint about the logarithm, we find

$$\begin{aligned}
\ln\left((1+\delta)^{-\lambda-\lambda\delta-\frac{1}{2}}\right) &= \left(-\lambda-\lambda\delta-\frac{1}{2}\right)\ln(1+\delta) \\
&\approx \left(-\lambda-\lambda\delta-\frac{1}{2}\right)\left(\delta-\frac{1}{2}\delta^2\right) \\
&= -\lambda\delta-\lambda\delta^2-\frac{1}{2}\delta+\frac{1}{2}\lambda\delta^2+\frac{1}{2}\lambda\delta^3+\frac{1}{4}\delta^2 \\
&\approx -\lambda\delta-\frac{1}{2}\lambda\delta^2.
\end{aligned}$$

Thus

$$\begin{aligned}
p(x) &\approx \frac{e^{\lambda\delta}}{\sqrt{2\pi\lambda}}e^{-\lambda\delta}e^{-\lambda\delta^2/2} \\
&= \frac{1}{\sqrt{2\pi\lambda}}e^{-\lambda\delta^2/2} \\
&= \frac{1}{\sqrt{2\pi\lambda}}e^{-(x-\lambda)^2/2\lambda}. && (\delta = \tfrac{x}{\lambda}-1=\tfrac{x-\lambda}{\lambda})
\end{aligned}$$

From part (a), we know $\mu = \langle x\rangle = \lambda$ and $\sigma^2 = \lambda$, so we simply have a Gaussian in the limit.

(c) The solution for this part comes from recognizing that we have a Poisson distribution for the background. For detector A we find

$$p(k=11;\lambda=1) = \frac{1\cdot e^{-1}}{11!} \approx 0,$$

so there is almost zero chance this is from the background. For detector B we can use the Gaussian approximation to find

$$p(k=1010;\lambda=1000) = \frac{1000^{1010}e^{-1000}}{1010!} \approx \frac{1}{\sqrt{2\pi\cdot 1000}}e^{-(1010-1000)^2/2\cdot 1000} = 0.12$$

(The Poisson calculation is numerically unstable.)

(d) The signal in detector A has only an extremely small chance of being from background so it is very possible that is has seen dark matter!

Question 6 Solution

Assuming unity emissivity, the power radiated by a blackbody surface is

$$P_{\mathrm{BBR}} = \sigma A T^4,$$

where $T \approx 290$ K is the temperature and A is the surface area.

On the other hand, the power transported by an electric-field wave is given by the Poynting vector $\vec{S}$ as

$$P_{\mathrm{P}} = \vec{S} \cdot \vec{A} = \frac{1}{c\mu_0} A \langle E^2 \rangle \cdot \frac{1}{2} \cdot \frac{1}{2},$$

where c is the speed of light and μ_0 is the permeability of free space. (The first factor of $\frac{1}{2}$ accounts for the fact that half of the BBR photons are directed toward the surface; the second factor of $\frac{1}{2}$ estimates the projection of the outward-going photons' average direction onto $\hat{A}$: to be precise, $\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = \frac{1}{2}$.)

Setting $P_{\mathrm{BBR}} = P_{\mathrm{P}}$, we find

$$\sqrt{\langle E^2 \rangle} = 2T^2 \sqrt{c\mu_0 \sigma} \approx 8\,\mathrm{V/cm}.$$