Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 12, 2015
1:00PM to 3:00PM
Classical Physics
Section 1. Classical Mechanics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 1 (Classical Mechanics), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

You may refer to the single handwritten note sheet on $8\frac{1}{2}$" × 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Two identical billiard balls of radius R and mass M rolling with velocities $\pm\vec{v}$ collide elastically, head-on. Assume that after the collision they have both reversed motion and are still rolling.

   (a) Find the impulse which the surface of the table must exert on each ball during its reversal of motion.

   (b) What impulse is exerted by one ball on the other?

   Recall that a time varying force $F(t)$ exerts an impulse $I = \int F(t)dt$.
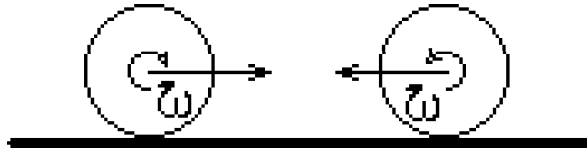
2. A uniform ball of radius R rolls without slipping between two rails such that the horizontal distance is d between the two contact points of the rails to the ball.

   (a) Find the relationship between the speed of the center-of-mass of the ball, v_{cm} and the ball's angular speed ω. Discuss your result in the limits $d = 0$ and $d = 2R$.

   (b) Consider a simpler problem in which the ball rolls without slipping on an inclined plane. If the ball begins from rest, find its center-of-mass speed, v_{cm}, after it has descended a vertical distance h.

   (c) Replace the ramp with the two rails and again find v_{cm} as a function of the vertical distance h that the ball has descended after beginning at rest. In each case, the work done by friction should be ignored.

   (d) Which speed in parts (b) and (c) is smaller? Why? Answer in terms of how the loss of potential energy is shared between the gain in translational and rotational kinetic energies. Show that your result has the correct behavior in the limits $d = 0$ and $d = 2R$.

Note that for a uniform sphere $I = \frac{2}{5}MR^2$.

3. Consider a particle of mass m constrained to move on a frictionless cylinder of radius R, given by the equation $\rho = R$ in cylindrical polar coordinates (ρ, ϕ, z), as shown in the left-hand figure below. In addition to the force of constraint (the normal force of the cylinder), two additional forces act on the mass. The first is the downward force of gravity, $\vec{F}_{\text{grav}} = -mg\hat{z}$. The second is $\vec{F}_{\text{Hooke}} = -k\vec{r}$ directed toward the origin. This is the three-dimensional version of the Hooke's-law force.

(a) Using z and ϕ as generalized coordinates, find the Lagrangian L.

(b) Write down and solve Lagrange's equations and describe the motion.

(c) Slightly distort the cylinder considered above into a right-cylindrical cone which has radius R at $z = 0$ and whose wall makes a small angle θ with the wall of the original cylinder, as shown in the right-hand figure below. Find the new Lagrangian L and then simplify it by keeping only those terms of zeroth or first order in θ.

(d) Again write down and solve Lagrange's equations in this small-θ approximation.
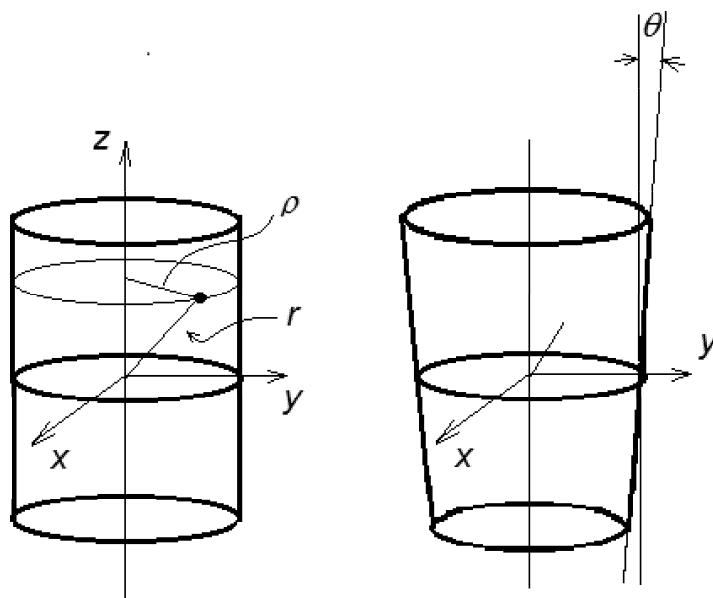
4. A very long plank of mass m and height h rests on two cylinders of radius R that rotate towards each other at a constant angular velocity, ω. The plank is subject to a constant vertical gravitational acceleration, $-g$. The cylinders are at equal height and are separated by a horizontal distance $2d$. There is a coefficient of friction, μ, between the plank and the rollers that is independent of the velocity of the plank relative to the rollers. In this problem you will analyze the motion of the plank in terms of the displacement, x, of its center position relative to the midpoint of the two rollers.
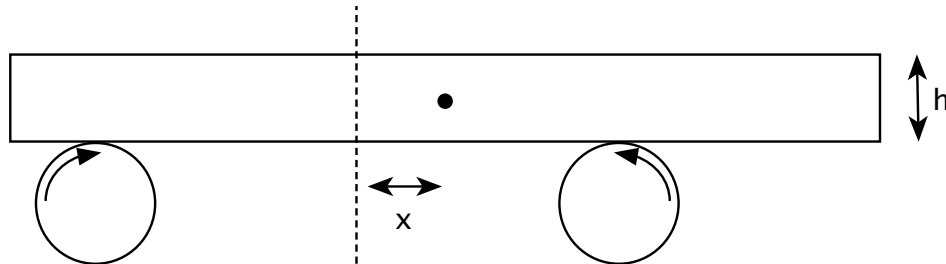
(a) Suppose that for $t < 0$, the plank is held fixed with $x = x_0 < d$ and that at $t = 0$ it is released. Find $x(t)$ for $t > 0$ in the limit that h is negligibly small.

(b) Next solve the problem posed in part (a) assuming that h is small but not negligible. Over what range of x_0 is your solution valid?

(c) Explain why your solution to part (b) fails when h and x_0 are not properly chosen and describe qualitatively the true motion of the plank that results.

5. A pendulum of length L and mass m is connected to a block also of mass m that is free to move horizontally on a frictionless surface. The block is connected to a wall with a spring of spring constant k. For the special case where

$$\sqrt{\frac{k}{m}} = \sqrt{\frac{g}{L}} = \omega_0 \tag{1}$$

determine:

(a) The frequencies of the normal modes of this system for small oscillations around the equilibrium positions.

(b) The motion of each of the normal modes.

Classical Mechanics
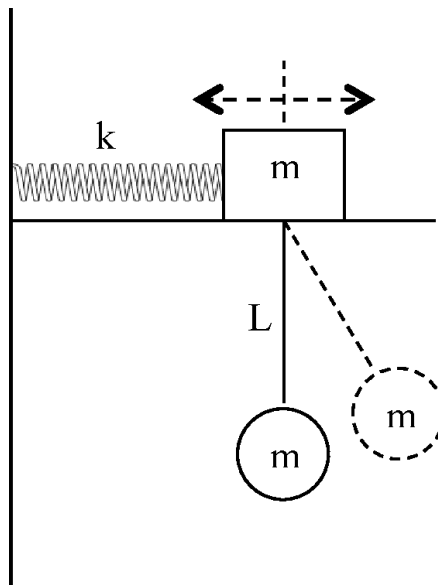
1.   a)   Only the table exerts a torque
about the cm of each ball

$\therefore \int F_t(t) R \, dt = \Delta L = 2(\frac{2}{5} MR^2) \omega$

$+ \; I_{table} = \frac{4}{5} MR^2 \omega / R = \frac{4}{5} MR\omega$

for each ball this is directed
toward the center of the fissure

b)   Newton's Law require that the
change in momentum of the
ball equal the total impulse

For the left-hand ball, with +
to the right

$I_{table} + I_{other} = -2M\omega R$

$+ \; I_{other} = -2M\omega R - \frac{4}{5} MR\omega =$

$= - \frac{14}{5} M\omega R$

2. (a)

$$\ell = \sqrt{R^2 - d^2}$$

$$v_{cm} = \omega \ell = \omega \sqrt{R^2 - d^2}$$

(b)
$$\frac{1}{2} M v_{cm}^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2 = Mgh$$

$$\left(\frac{1}{2} + \frac{1}{5}\right) v_{cm}^2 = gh$$

$$v_{cm} = \sqrt{\frac{10}{7}gh}$$

(c)
$$\frac{1}{2} M v_{cm}^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\left(\frac{v_{cm}}{\ell}\right)^2 = Mgh$$

$$\frac{1}{2} + \frac{1}{5}\left(\frac{R^2}{R^2 - d^2}\right) v_{cm}^2 = gh$$

$$v_{cm} = \left[\frac{gh}{\frac{1}{2} + \frac{1}{5}\frac{R^2}{R^2 - d^2}}\right]^{1/2}$$

d) speed decreases as d increases and a larger fraction of K.E. is rotational

(a) $\quad T = \frac{1}{2} m \left(R^2 \dot{\phi}^2 \right) + \frac{1}{2} m \dot{z}^2 = \frac{1}{2} m \left(R^2 \dot{\phi}^2 + \dot{z}^2 \right)$

$\underbrace{\qquad}$

$V_\phi = R \frac{d\phi}{dt}$

$U = \frac{1}{2} k \left(\sqrt{R^2 + z^2} \right)^2 + mgz = \frac{1}{2} k \left(R^2 + z^2 \right) + mgz$

$$\boxed{\mathcal{L} = T - U = \left[\frac{1}{2} m \left(R^2 \dot{\phi}^2 + \dot{z}^2 \right) \right] - \left[\frac{1}{2} k \left(R^2 + z^2 \right) + mgz \right]}$$

(b) $\quad \dfrac{\partial \mathcal{L}}{\partial z} = \dfrac{d}{dt} \dfrac{\partial \mathcal{L}}{\partial \dot{z}}, \quad \dfrac{\partial \mathcal{L}}{\partial \phi} = \dfrac{d}{dt} \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad \leftarrow \text{lagrange's equations}$

$\dfrac{\partial \mathcal{L}}{\partial z} = -kz - mg$

$\dfrac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z}$

$\dfrac{d}{dt} \dfrac{\partial \mathcal{L}}{\partial \dot{z}} = m\ddot{z}$

$\left. \right\} \quad m\ddot{z} = -kz - mg$

$z(t) = A \cos(\omega t + \delta) + z_0$

$\dot{z} = A\omega(-1) \sin(\omega t + \delta)$

$\ddot{z} = A\omega^2(-1) \cos(\omega t + \delta)$

$m \left[A\omega^2(-1) \cos(\omega t + \delta) \right] = -k \left[A \cos(\omega t + \delta) + z_0 \right] - mg$

$\qquad\qquad = -k \left[A \cos(\omega t + \delta) \right] - kz_0 - mg$

$\qquad\qquad\qquad\qquad\qquad\qquad \underbrace{\qquad\qquad}$

$m(-1)\omega^2 = -k \quad \rightarrow \quad \omega^2 = \dfrac{k}{m} = 2\pi f$

$\qquad\qquad\qquad\qquad\qquad\qquad -kz_0 - mg = 0$

$\omega = \sqrt{\dfrac{k}{m}} = 2\pi f$

$\qquad\qquad\qquad\qquad\qquad\qquad z_0 = -\dfrac{mg}{k}$

A must be small

$f = \dfrac{1}{2\pi} \sqrt{\dfrac{k}{m}} \quad \leftarrow \text{resonant frequency}$

$$\boxed{z(t) = A \cos(\omega t + \delta) - \dfrac{mg}{k}}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mR^2 \dot{\phi}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mR^2 \ddot{\phi}$$

$$0 = mR^2 \ddot{\phi}$$

$$\ddot{\phi} = 0 \qquad \text{no acceleration}$$

$$\dot{\phi} = \omega_\phi \qquad \text{constant speed}$$

$$\boxed{\phi(t) = \omega_\phi t + \phi_0}$$

(c)

$$p = R + z \tan \theta$$

$$z \tan \theta$$

$$T = \frac{1}{2} m (R + z \tan\theta)^2 \dot{\phi}^2 + \frac{1}{2} m \dot{p}^2 + \frac{1}{2} m \dot{z}^2$$

$$\dot{p} = \dot{z} \tan\theta$$

$$T = \frac{1}{2} m (R + z \tan\theta)^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 \left[1 + \tan^2\theta \right]$$

$$U = \frac{1}{2} k \left(\sqrt{(R + z\tan\theta)^2 + z^2} \right)^2 + mgz$$

$$U = \frac{1}{2} k \left((R + z\tan\theta)^2 + z^2 \right) + mgz$$

$$\boxed{\mathcal{L} = T - U = \left[\frac{1}{2} m (R + z\tan\theta)^2 \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 \left[1 + \tan^2\theta \right] \right] - \left[\frac{1}{2} k \left((R + z\tan\theta)^2 + z^2 \right) + mgz \right]}$$

if $\theta = $ small, then $\tan\theta \to \theta$, $\tan^2\theta \to 0$, $\cos\theta \to 1$

$$\boxed{\mathcal{L} = \left[\frac{1}{2} m (R^2 + 2Rz\theta) \dot{\phi}^2 + \frac{1}{2} m \dot{z}^2 \right] - \left[\frac{1}{2} k \left((R^2 + 2Rz\theta) + z^2 \right) + mgz \right]}$$

(d) $\quad \dfrac{\partial \mathcal{L}}{\partial z} = mR\theta\dot{\phi}^2 - kR\theta - kz - mg$

$\dfrac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z}$

$\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{z}} = m\ddot{z}$

$$\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \quad m\ddot{z} = mR\theta\dot{\phi}^2 - kR\theta - kz - mg$$

$\dfrac{\partial \mathcal{L}}{\partial \phi} = 0$

$\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = m(R^2 + 2Rz\theta)\dot{\phi} = L_\phi \quad \longrightarrow \quad \dot{\phi} = \dfrac{L_\phi}{m(R^2+2Rz\theta)} \approx \dfrac{L_\phi}{mR^2}$

$\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0 = \dfrac{d}{dt}L_\phi \qquad \nearrow \quad L_\phi \text{ is constant}$

$\uparrow$ zeroth order in θ

$m\ddot{z} = mR\theta\left(\dfrac{L_\phi}{mR^2}\right)^2 - k(R\theta + z) - mg = \left[\theta\dfrac{L_\phi^2}{mR^3} - kR\theta - mg\right] - kz$

$\underbrace{\qquad\qquad}_{\theta\frac{L_\phi^2}{mR^3}} \qquad\qquad \underbrace{\qquad\qquad\qquad\qquad}_{\beta}$

$z(t) = A\cos(\omega t + \delta) + z_0$

$\dot{z} = A\omega(-1)\sin(\omega t + \delta)$

$\ddot{z} = A\omega^2(-1)\cos(\omega t + \delta)$

$m\left[A\omega^2(-1)\cos(\omega t + \delta)\right] = \beta - k\left[A\cos(\omega t + \delta) + z_0\right]$

$\beta - kz_0 = 0 \rightarrow z_0 = \dfrac{\beta}{k}$

$m\omega^2 = k$

$z_0 = \theta\dfrac{L_\phi^2}{kmR^3} - R\theta - \dfrac{mg}{k}$

$\omega = \sqrt{\dfrac{k}{m}}$

$$Z(t) = A \cos(\omega t + \delta) + \Theta \frac{L_\phi^2}{k m R^3} - R\Theta - \frac{mg}{k}$$

$\uparrow$ Small A

$\omega = \sqrt{\frac{k}{m}}$

$$\frac{d\dot\phi}{dz} = \frac{d}{dz}\left[L_\phi \left[m\left(R^2 + 2Rz\Theta \right) \right]^{-1} \right] = L_\phi(-1)\left[m\left(R^2 + 2Rz\Theta \right) \right]^{-2}\left[2mR\Theta \right]$$

$$\dot\phi \approx \frac{L_\phi}{mR^2} + \frac{1}{2}\left[L_\phi(-1)\left[2mR\Theta \right]\left[mR^2 \right]^{-2} \right](z)$$

$$\dot\phi \approx \frac{L_\phi}{mR^2} - L_\phi \frac{mR\Theta}{m^2 R^4} z = \frac{L_\phi}{mR^2}\left[1 - \frac{\Theta}{R} z \right]$$

$$\underbrace{\qquad\qquad}_{\frac{L_\phi}{mR^2} \frac{\Theta}{R} z}$$

$$\phi(t) = \frac{L_\phi}{mR^2} t - \frac{\Theta}{R} \frac{L_\phi}{mR^2} A \int \cos(\omega t + \delta)\, dt + \phi_0$$

$$\phi(t) = \frac{L_\phi}{mR^2} t - \frac{\Theta}{R} \frac{L_\phi}{mR^2} A \frac{1}{\omega} \sin(\omega t + \delta) + \phi_0$$

$\uparrow$ Small A

$\omega = \sqrt{\frac{k}{m}}$

(corrected)

4. (a) since the torque about cm is zero, if F_L^v & F_R^v are the vertical forces exerted by the rollers

$$F_R^v (d-x) = F_L^v (d+x) \qquad F_R^v + F_v^v = mg$$

$$F_L^v - F_R^v = -\frac{x}{d}(F_R^v + F_L^v) = -\frac{x}{d}mg$$

$$m\ddot{x} = \mu(F_L^v - F_R^v) = -x\frac{\mu}{d}mg$$

or SHM $\quad \ddot{x} = -\frac{\mu}{d}g x \qquad \omega = \sqrt{\frac{\mu}{d}g}$

$$x(t) = x_0 \cos(\omega t)$$

(b) Now horizontal forces exert a torque too:

$$F_R^v(d-x) + \mu F_L^v \frac{h}{2} = F_L^v(d+x) + \mu F_R \frac{h}{2}$$

or $\quad F_L^v - F_R^v = -\frac{x}{d-\mu\frac{h}{2}}(F_R^v + F_L^v)$

$$= -\frac{x}{d-\mu\frac{h}{2}}mg \qquad \frac{1}{2}\mu h < d$$

$$\downarrow \quad x(t) = x_0 \cos\omega t \quad \text{for} \quad \omega = \sqrt{\frac{g}{d-\mu h/2}}$$

note, e.g $\quad F_L^v = \frac{1}{2}(F_L^v - F_R^v) + \frac{1}{2}(F_L^v + F_R^v)$

$$= \frac{1}{2}\left[-\frac{x}{d-\mu h/2} + 1\right]mg$$

$$= \frac{1}{2}\frac{d-\mu h/2 - x}{d-\mu h/2}mg$$

since $F_L^v > 0 \qquad x < d-\mu h/2 \Rightarrow \qquad x_0 < d-\mu h/2$

c) Plank lifts off one of the rollers but falls back as cm moves toward that roller.

A pendulum of length L and mass m is connected to a block also of mass m that is free to move horizontally on a frictionless surface. The block is connected to a wall with a spring of spring constant k.

For the special case where $\sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{g}{L}} = \omega_0$, determine:

a) The frequencies of the normal modes of this system for small oscillations around the equilibrium positions.
b) The motion of each of the normal modes.
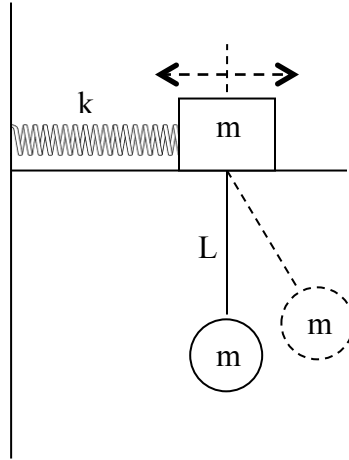
Solution:
Let x be the distance of the block from the wall and θ be the angle of the pendulum from the vertical.

Solution:

$$T = \frac{1}{2}m\left(L\dot{\theta} + \dot{x}\right)^2 + \frac{1}{2}m\dot{x}^2 \quad U = mgL(1 - \cos\theta) + \frac{1}{2}kx^2 = \frac{mgL}{2}\theta^2 + \frac{1}{2}kx^2$$

$$L = T - U = \frac{1}{2}m\left(L\dot{\theta} + \dot{x}\right)^2 + \frac{1}{2}m\dot{x}^2 - \frac{mgL}{2}\theta^2 - \frac{1}{2}kx^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(mL\left(L\dot{\theta} + \dot{x}\right)\right) + mgL\theta = mL^2\ddot{\theta} + mL\ddot{x} + mgL\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{d}{dt}\left(m\left(L\dot{\theta} + \dot{x}\right) + m\dot{x}\right) + kx = mL\ddot{\theta} + 2m\ddot{x} + m\omega_0^2 x = 0$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{L}}$$

Let $\theta = Ae^{i\omega t}$ and $x = Be^{i\omega t}$ then Lagrange's equation become

$$-AL^2\omega^2 - BL\omega^2 + AgL = -AL^2\omega^2 - BL\omega^2 + AL^2\omega_0^2 = 0$$

$$-AL\omega^2 - 2B\omega^2 + B\frac{k}{m} = -AL\omega^2 - 2B\omega^2 + B\omega_0^2 = 0$$

$$\begin{pmatrix} -L^2\omega^2 + L^2\omega_0^2 & -L\omega^2 \\ -L\omega^2 & -2\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$L^2\omega^4 - L^2\omega^2\omega_0^2 - 2L^2\omega_0^2\omega^2 + L^2\omega_0^4 = 0$$

$$\omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$$

a) $\omega = \dfrac{1}{2}\left(3 \pm \sqrt{5}\right)\omega_0$

b) $-AL^2\omega^2 - BL\omega^2 + AL^2\omega_0^2 = 0 \Rightarrow AL = B\left(\dfrac{\omega^2}{\omega_0^2 - \omega^2}\right)$

For $\omega = \dfrac{1}{2}\left(3 + \sqrt{5}\right)\omega_0 \Rightarrow AL = -\dfrac{\sqrt{5}+1}{2}B$

For $\omega = \dfrac{1}{2}\left(3 - \sqrt{5}\right)\omega_0 \Rightarrow AL = \dfrac{\sqrt{5}-1}{2}B$