2013 Comprehensive Exam
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Note: I didn't go as in depth in this year's solutions as in others...

1. Order of magnitude estimation:

Small meteors striking the earth will burn up in the 10-km thickness of the atmosphere, while larger ones will survive to reach the surface. Consider a solid iron meteor (density = 8000 kg/m^3, heat of fusion = 250 J/g, heat of vaporization = 6000 J/g). Do an order of magnitude estimate of the minimum size for such meteors that will survive to strike the Earth's surface. Hint: the drag force on a very fast object moving through a fluid is $F_D = \frac{1}{2}\rho_{fluid}v^2 C_D A$, where A is the cross-sectional area of the object, v is the speed of the object, ρ_{fluid} is the density of the fluid, and the drag coefficient C_D is a constant of order 1. You may assume that the initial velocity of the meteor is of the same order as the Earth's orbital velocity around the Sun.

1. To get initial velocity, two methods:

a.
$F_{cent} + F_{grav} = 0$, isolate v
b. $\omega = \frac{v}{r} = \frac{2\pi}{365d}$

2. When the meteor is travelling through the atmosphere, it is compressing the air in front of it, causing it to heat up. This heats up the meteor which result in it burning up. The kinetic energy turns to heat, which causes it to slow down as it travels through the atmosphere. The change in kinetic energy and potential energy must be less than the energy lost due to vaporization/fusion for it to survive.

It will burn up if the energy is greater than (6000 + 250)*mass of meteor.

$$\Delta KE + \Delta U > (6000 + 250) * M_{meteor}$$

We have ΔKE (KE_1 from initial velocity, KE_2 from drag force – solve for v and plug in (note at terminal velocity, $F_D = Mg$), we have ΔU ($mgh_1 = mg(10km)$,1 and $mgh_2 = 0$) we can get the mass from m = volume*density. The drag force has an area term, and we have a volume term from the mass, so just plug in A and V assuming it's a sphere and solve for r, the radius of the meteor.

2.

A. A sphere of uniform density with mass 0.1 kg and radius 5 cm is set spinning with a rotational speed of 10 revolutions per second. What is the quantum mechanical limit to the angular accuracy with which the direction of the spin can be aligned?

B. An atomic transition produces a spectral line with wavelength 400 nm when the atom decays from some state A to the ground state. The spectral line is measured with a diffraction grating containing 10,000 lines. What is the longest lifetime for state A that it is possible to measure using this spectrometer?

A. Heisenberg's uncertainty principle:

$$\Delta x \Delta p > \frac{\mathrm{h}}{2}$$

Extrapolating this to the angular version,

$$\Delta \theta \Delta L > \frac{\mathrm{h}}{2}$$

Where $\Delta L = I\omega$, $I = 2/5mr^2$ for a sphere. This gives ~ 1.05×10^{-31} rev. (should this be in rad? Is it already in radians?) Multiple by 2pi to convert to radians.

B. Using diffraction gratings is common and useful to separate spectral lines associated with atomic transitions. It separates the different colours of light more than dispersion in a prism. **Used to separate diff wavelengths of light.** Conditions are same as for double slits.

When you have the diffraction gratings, $N = \frac{\lambda}{\Delta\lambda}$. $\Delta\lambda$ is the minimum resolvable wavelength. This is also called the resolvance. We can also invoke our time energy uncertainty:

$$\Delta E \Delta t > \frac{\mathrm{h}}{2}$$

So, we can find the longest lifetime that is possible to measure if we find the ΔE of this transition (*Note: wouldn't this be the shortest lifetime? Because this is Δt?)*

$E = \frac{hc}{\lambda}$. Take derivative with respect to λ:

$\Delta E = \frac{-hc\Delta\lambda}{\lambda^2}$ (should there be a negative here now? From the derivative?)

Now, just plug in Δλ and then plug into Δt equation. Result:

$$\Delta t < \frac{\lambda N}{4\pi c} = 1.06 \times 10^{-12} s$$

(I used the negative to switch around the sign, hehe. Don't know if that is a thing)

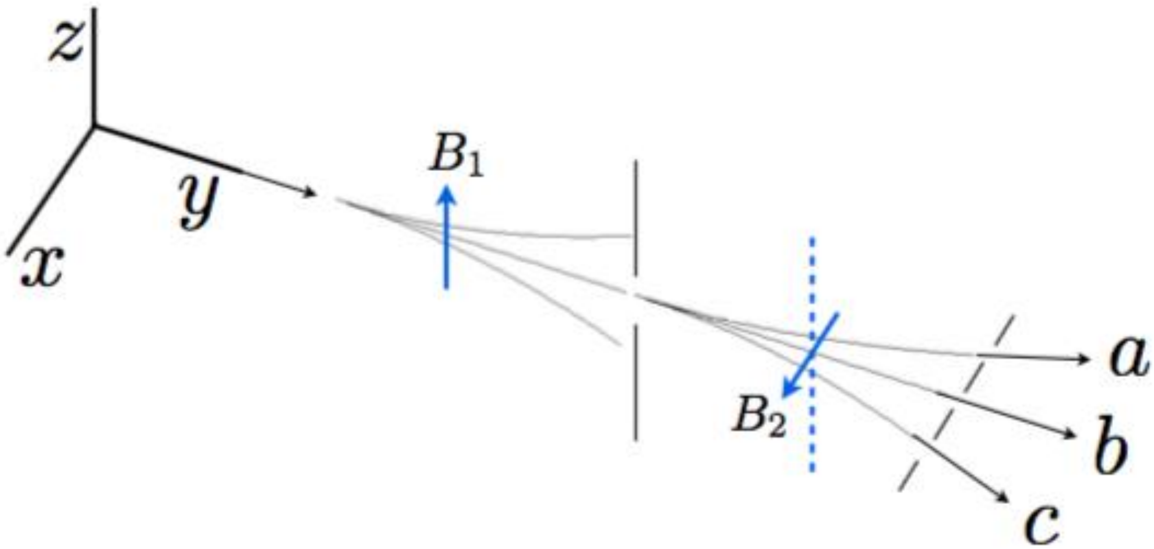


Figure 1: Positronium beam passing through a double Stern-Gerlach apparatus

3. A beam of ground state positronium particles (an "atom" composed of an electron and its anti-particle a positron) enters the double Stern-Gerlach apparatus shown below. Assume the particles are in an equal mixture of the two spin configurations — that is, there is an equal number of para-positronium with a total spin $S = 0$ and ortho-positronium with a spin $S = 1$. Assume also that the beam is completely unpolarized. The positronium first encounters a magnetic field along the z axis with a gradient increasing in the z direction (in the direction of the arrow) and the beam splits into three. The un-deflected beam then flies through a hole in the beam-stop and encounters a magnetic field directed *perpendicular* to the first one and along the x axis. The beam splits again.

3A) What fraction of the total beam flux will pass through the first beam-stop? What fraction of the total beam flux will pass through to each of the outputs labeled a, b, and c? Write down the spin state for the positronium passing through each of these outputs in the z basis.

3B) Suppose the positronium is disassociated at the outputs a, b, and c so that the spin of the electron and positron can be measured independently and the electron and positron spin projections along the z axis are measured. What is the chance of finding both the electron and the positron in the *same* spin state for each of the outputs.

4. An electron is constrained to move in a ring with radius R (in the XY plane).

A) Find the eigenstates and eigenvalues for the electron. (Ignore the electron's spin.)

B) Now an electric field of strength E is applied along the X-direction. Find the leading (non-zero) correction to the ground state energy.

C) Calculate the polarizability of the system (the ratio of the electric dipole moment to the electric field strength).

A. An electron is constrained to move around a ring. To find the eigenstates/eigenvalues, we need to solve the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\boldsymbol{x})\psi = E\psi$$

I am assuming potential is 0 (it's constrained so… no where for it to go.)
We can use polar coordinates. The Laplacian operator is:

In plane polar coordinates Laplace's equation takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Because we are stuck in this ring, the derivative with respect to *r* is zero, so only the last term survives. This gives us (I'm not using real equations because it slows down word):

$$-\frac{h^2}{2m} * (\frac{1}{r^2}\frac{d^2}{d\theta^2})\,\Psi = E\,\Psi$$

...

$$\frac{d^2\psi}{d\theta^2} = -\frac{2mr^2}{h^2}\,E\,\Psi = -\omega^2\Psi$$

where $\omega = \sqrt{\frac{2mEr^2}{h^2}}$.

This is simple harmonic motion. Let's use complex exponentials instead of sine and cosines.

There are two solutions (eigenvectors) to this, in the form
$\psi_{\pm} = Ae^{\pm i\omega\theta}$

To find the constant A, normalize:
$\int |\psi|^2 d\theta = 1 \rightarrow A = \frac{1}{\sqrt{2\pi}}$

The eigenvalues are $E = \frac{h^2\omega^2}{2mr^2}$.

What is ω? We need a boundary condition. Because the particle is confined to this ring,

$$\psi(\theta) = \psi(\theta + 2\pi)$$

$e^{\pm i\omega\theta} = e^{\pm i\omega\theta}e^{\pm 2\pi i\omega}$

$$1 = e^{\pm 2\pi i\omega}$$

EULER!! This equation is only true if ω is an integer, because $1 = \cos(2\pi\omega) + isin(2\pi\omega)$. So therefore $\omega = 0, \pm 1, \pm 2$ .... etc. So the ground state is non-degenerate and all the other states have degeneracy 2, because w is squared in the E equation and therefore the energy is the same for $\pm\omega$.

B. Now an electric field is applied in the x-direction, and we want to find the leading order correction to the ground state, where $\omega = 0$. This is called the Stark Effect – when the electron ring is placed in a uniform magnetic field, the energy levels are shifted, and degeneracy is broken (it's like the Zeeman effect for electric fields). Now there is a non-zero potential, which causes a perturbation. It's for the ground state, so we can use non-degenerate perturbation theory.

Our new potential is the same as any non-quantum potential, which is
$$V = -qE_0x = -qE_0rcos\theta$$

First order:

$$\boxed{E_n' = \langle n^o | H' | n^o \rangle}$$

Where
$$H' = -qE_0rcos\theta$$

Doing this gives an integral from 0 to 2pi over $cos\theta$, which is just 0. So, we need to use second order perturbation theory.

$$E_n^{(2)} = \sum_{k\neq n} \frac{|\langle\phi_k|H_1|\phi_n\rangle|^2}{E_n^{(0)} - E_k^{(0)}}$$

I'm obviously not going to write out all the math on word. But basically, we want the leading correction to the ground state energy. Because it's the correction to the *ground state energy*, we can let n = 0 here and instead vary k. We want to do the minimum amount of work, so let's just sum over the k = -1 and k = 1 states (should I be doing more? Or just one of them and not sum at all? Not sure….)

$$\frac{|<\psi_{-1}| - qE_0 rcos\theta|\psi_0>|^2}{-E_{-1}} + \frac{|<\psi_{+1}| - qE_0 rcos\theta|\psi_0>|^2}{-E_{+1}}$$

You can either do integration by parts twice, or expand cos in terms of complex exponentials. $\psi_0 = \frac{1}{\sqrt{2\pi}}, E^0 = 0$.

In the end, I got something like (note: Initially I wrote this without using word equations, so I probably made some errors when rewriting):

$$E_0^2 = (1 + cos2\theta)\left(\frac{qE_0 r^2}{h}\right)^2 \frac{m}{\pi}$$

C.

I believe it is true we can say $P = \frac{p}{E} = \frac{q<rcos\theta>}{\_E}$ (where E is the applied field).However, we must find the expectation value of the dipole moment using the new wavefunction, so we must find the correction to the wavefunction using perturbation theory, and then $\psi_{new} = \psi_{old} + \psi'$.

5. Two spin-1/2 particles located at sites 1 & 2 respectively interact with an exchange interaction $H = J\vec{S}_1 \cdot \vec{S}_2$, where J is the coupling constant.

A) For two spins in an initial state of $|\text{up}\rangle\ |\text{down}\rangle$, what are the probabilities of finding two spins in $|\text{up}\rangle|\text{up}\rangle$, $|\text{down}\rangle|\text{down}\rangle$ and $|\text{down}\rangle\ |\text{up}\rangle$ states at time t later.

B) Now assume the exchange interaction is modulated periodically, $J = J(t) = J_0 \cos\omega t$. Find out the corresponding probabilities in Part A.

We know our Hamiltonian, it's $H = JS_1 \cdot S_2$. We know the initial spin state, so we want to time evolve our initial state and then find the probabilities of finding two spins in the other states at later times.

Initial state = | ↑↓⟩. This isn't a standard state, but we can write it as a linear combination of the singlet state and the m = 0 triplet state.

$$\psi(0) = |\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{2}(\uparrow\downarrow + \downarrow\uparrow) + (\uparrow\downarrow - \downarrow\uparrow)\right) = \frac{1}{2}(|1\ 0> \ +|0\ 0>)$$

To time evolve, we need to find the time evolution operator and tack it on.

$$|\psi(t)\rangle = e^{-iHt/\hbar}\,|\psi(0)\rangle\,.$$

We need to calculate how the Hamiltonian acts on our states, which are $|1\ 0> \ +|0\ 0>$.

$$JS_1 \cdot S_2 \Big|1\ 0 \geq \frac{J}{2}(S^2 - S_1^2 - S_2^2)\Big|1\ 0> = \frac{Jh^2}{2}(2 - \frac{3}{4} - \frac{3}{4})|1\ 0> = \frac{J}{4}h^2|1\ 0>$$

(Remember $S^2|s\ m> \ -h^2 s(s+1)|s\ m>$)

$$JS_1 \cdot S_2 \Big|0\ 0 \geq \frac{J}{2}(S^2 - S_1^2 - S_2^2)\Big|0\ 0> = \frac{Jh^2}{2}(0 - \frac{3}{4} - \frac{3}{4})|1\ 0> = \frac{-3J}{4}h^2|1\ 0>$$

So, then our state evolved with time looks like

$$\psi(t) = \frac{1}{2}(|1\ 0> e^{\frac{-iJht}{4}} + |0\ 0> e^{\frac{3iJht}{4}})$$

And then we can just use the probability density formula to find the probably of being in diff states at time t.

For example:

$|\langle\uparrow\uparrow|\psi(t)\rangle|^2$

For the states up up and down down, the probability is 0 because they are orthogonal to states |1 0> and |0 0 >. It's non-zero for the state down up.

| < ↓↑ | psi(t)>|2
= | ½ (1/root2 exp(-iJht/4) <↓↑|↓↑>) - ½ (1/root2 exp(+3iJht/4) <↓↑|↓↑>)|^2

Just do the math…. I got $\frac{1}{8}(1-\cos(Jht))$.

B. Now our Hamiltonian is not time dependent. We will have to solve the Schrödinger equation if the Hamiltonian does not commute with itself at different times (I think), i.e.

$$[H(t_1), H(t_2)] \neq 0$$

BUT, it does. So, we can just do what we did before, except our time evolution operator needs to account for this:

$$U(t, t_0) = \exp\left[\frac{-i}{\hbar}\int_{t_0}^{t} H(t')\,dt'\right]$$

So, I think we basically get the same thing except our exponential has a $J_0\sin(wt)$ in it. So, our new probability of being in the state down down is

$$\frac{1}{8}(1-\cos(J_0\sin(\omega t)t)).$$

SEEMS A LITTLE WEIRD BUT OK!

6. Consider N independent spin-1/2 particles at temperature T, each having a magnetic moment μ. An external magnetic field B is further applied.

A) How many microscopic configurations or microstates are there for a given total energy E of the N-spin system?

B) Use Boltzmann statistics to calculate the probability of finding the N-spin system with energy E.

C) Find the most probable E for the N-spin system.

D) Calculate the distribution function of E when N is very, very large.

A. There are N spin ½ particles, which means each particle can either have spin up (+1/2) or spin down (-1/2). We know the Hamiltonian for spin ½ particles in a magnetic field:

$H = -\mu \cdot B$, where μ is the magnetic moment and B the mag field.

The total particles are composed of the spin up and spin down particles, so

$$N = n_\uparrow + n_\downarrow$$

The energy of the spin up particles are $E = -\mu B$ and the energy of the spin down particle is $E = +\mu B$. So we can write the total energy as

$E_{tot} = \mu B(n_\downarrow - n_\uparrow)$

It's not just $E_{tot} = \mu BN$ – it would be if every particle had $E = \mu B$, but that's not the case because some particles have $E = -\mu B$

We want to determine the number of microstates, aka the multiplicity. The formula for this is

$$Q = \frac{N!}{N_\uparrow! N_\downarrow!}.$$

We want to write everything in terms of the variables we have, so use $E_{tot} = \mu B(n_\downarrow - n_\uparrow)$ and $N = n_\uparrow + n_\downarrow$ to write N↑ in terms of N and E, and the same for N down.

N↑ = ½ (N - E/μB)
N↓ = ½ (N + E/μB) I think

Then plug in!

Q = N! / (½ (N - E/μB))! (½ (N + E/μB))!

Or something to that effect.

B. We want the probability using Boltzmann stats. There is a formula for this:

$$P(s) = \frac{1}{Z} e^{-E(s)/kT}$$

The numerator can stay the same, with energy E. What about the partition function, Z?

$$Z = \sum_s e^{-E(s)/kT}$$

Z is the sum over all possible states. Well, there are only 2 states, spin up and spin down (assuming one particle). So Z for one particle is:

$$Z = e^{\beta\mu B} + e^{-\beta\mu B}$$

To generalize this to N indistinguishable particles:

$$Z = \frac{\left(e^{\beta\mu B} + e^{-\beta\mu B}\right)^N}{N!}$$

Anyway, this leaves us

$$P = \frac{1}{Z} e^{-\frac{E}{kT}} = \frac{N!\, e^{-\frac{E}{kT}}}{(e^{\beta\mu B} + e^{-\beta\mu B})\hat{}N} = \frac{N!\, e^{-\frac{E}{kT}}}{\cosh^N \beta\mu B}$$

C. There is also a formula for this, it's

$$E_{avg} = \frac{-1}{Z}\frac{dZ}{d\beta}$$

I think ended up getting something like

$$\boxed{E_{avg} = -\frac{N\mu B tanh\left(\frac{\mu B}{kT}\right)}{kT}}$$

D.

$$P = \frac{N!\, e^{-\frac{E}{kT}}}{\cosh^N \beta\mu B}$$

When N is large, $N! \approx N^N E^{-N}\sqrt{2\pi N}$?

7. Consider a uniformly charged, solid sphere with mass M of radius R, carrying a total charge Q and spinning with angular velocity ω about the z axis. An external magnetic field $\vec{B} = B_0\hat{x}$ is applied. Calculate the frequency of precession of the spin axis.

Okay, I'm kind of missing the last step on this one…. But I'll go through what I know. The sphere is spinning. This induces a magnetic field and a magnetic moment, because the charges are moving. The frequency of precession is $\frac{d\theta}{dt} = \omega_p$.

The external magnetic field is going to induce a torque on the sphere:

$$\tau = m \times B_0$$

The torque is also $\tau = \frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt}$.

Let's find the torque… so we need to find the magnetic moment.

To find magnetic moment of the sphere, we can find the magnetic moment for a slice and then integrate.

$m = IA \rightarrow dm = dIA + IdA$. For each segment area stays the same, so $dm = dIA$.

Think about cutting a sphere into hundreds of flat disc slices. What is the area of these faces? $A = \pi r^2$, where r is the radius of that slice. And $r = Rsin\theta$, where R is the radius of the total sphere (easy to see with trig)

$$A = \pi R^2 \sin^2 \theta$$

What about dI? Current is the rate of change of charge over time. We can say that the change in current is like the change of charge times the frequency of the rotation.

$$dI = dq \cdot f = \frac{dq\omega}{2\pi}$$

What about dq? This is a sphere, so $dq = \rho dV$.

$$dI = \frac{\omega}{2\pi}\rho dV$$

therefore

$$dm = \frac{\omega}{2\pi}\rho\pi r^2 \sin^2 \theta \, dV$$

I'm a bit confused as to why we can make R go down to little r…

$$m = \int \frac{\omega}{2\pi}\rho\pi r^2 \sin^2 \theta \, r^2 sin\theta dr d\theta d\phi$$

Integrate phi from 0 to 2pi, theta from 0 to pi, and r from 0 to R.

After integrating and writing subbing $\rho = \frac{q}{\frac{4}{3}\pi R^3}$, I got

$$m = \frac{QR^2\omega}{5}\hat{z}$$

(In exam conditions, could maybe just use dimensional analysis… units of magnetic moment are $\frac{C}{s}m^2$, which is of course $q\omega R^2$.

(It’s in the z direction because of the RHR – fingers curled around spinning sphere; thumb points up)

So, the torque is

$$\tau = m \times B_0 = \frac{QR^2\omega B_0}{5} = \frac{Idw_p}{dt} = \frac{\frac{2}{5}MR^2dw_p}{dt}$$

$$\frac{dw_p}{dt} = \frac{QR^2\omega B_0}{2MR^2} = \frac{Q\omega B_0}{2M}$$

The problem is I want $\frac{d\theta_p}{dt}$ I think. So, I think there is another step or I did this the wrong way…. ☹

8. A long, solid, and well insulated rod has one end attached to a cold temperature reservoir so that the entire rod is initially at the reservoir temperature. The rod is characterized by a length d, a known specific heat C, density ρ, and thermal conductivity κ. If the other end of the rod is suddenly connected to a high temperature reservoir at a temperature that is ΔT higher, then

A. At $t = 0$, find the heat intensity (power per unit area) that flows into the low temperature reservoir.

B. At thermal equilibrium, find the heat intensity that flows into the low temperature reservoir.

C. Do an order of magnitude estimate of the time it takes for this equilibrium to be established.

Initially, when the rod is only connected to the low temperature reservoir, it is in thermal equilibrium; the temperature remains constant with time and with position – everything is temperature Tc. Suddenly, one end of the rod is connected to a high temperature reservoir at temp Th = Tc + ΔT. Now the rod is no longer in equilibrium, and heat will begin to flow from the high temp reservoir to the low temp one. So the temperature will begin increasing with distance from Th to the Tc reservoir.

A. At t = 0, no heat would have reached the low temp reservoir yet… so this is 0?

B. Let’s use the equation for the rate of heat conduction (Fourier’s law).

$$\frac{dQ}{dt} = -kA\frac{dT}{dx}$$

Note that Q is in units of J, so $\frac{Q}{At}$ is J/s A, or W/m2 (this is units of heat intensity). So our heat intensity is $\frac{dQ}{Adt}$.

$$\frac{dQ}{Adt} = -kA\frac{dT}{dx} = -kA\frac{(T_h - T_c)}{d} = -kA\frac{(T_c + \Delta T - T_c)}{d} = -kA\frac{(\Delta T)}{d}$$

So, the heat intensity flowing into the low temperature reservoir

$$\boxed{q = -kA\frac{\Delta T}{d}}$$

C. Should we actually be solving the heat equation? Using boundary conditions we have? Or can we use dimensional analysis? Looking at the heat equation

$$\frac{dT}{dt} = -\frac{k}{c\rho}\frac{dT^2}{dx^2}$$

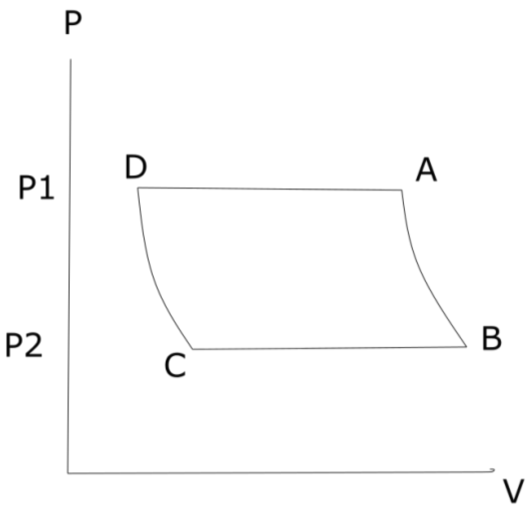
we have the variables t, k, c, p, and d, and T. Using dimensional analysis (aka just isolating t in the heat equation), we get

$$\boxed{t \approx \frac{c\rho d^2}{k}}$$

This makes sense because the time for equilibrium should increase with larger distance, larger density, and larger specific heat capacity (because this is the amount of heat you need to add to raise the temperature), and it should decrease with higher conductivity, k. It should also be faster the smaller the change in temperature is!

9. The reversible Brayton engine cycle, $A \to B \to C \to D \to A$, consists of two isentropic changes from $A \to B$ and $C \to D$ as well as two isobaric state changes from $B \to C$ and $D \to A$. During the step $D \to A$, the gas absorbs heat Q_{DA}. During the step $B \to C$, the gas expels heat Q_{BC} which can be considered 'wasted'. Let the efficiency of the engine be W_{ABCD}/Q_{DA}, and let the working substance be an ideal gas. Calculate the efficiency of the engine in terms of the gas's heat capacities C_P and C_V and the ratio of pressure at the isobaric steps.

This is a “Carnot engine” type question, where we need to draw a PV diagram. The PV diagram kind of looks like this:



Where A-B and C-D are adiabatic, and B-C and D-A are isobaric (constant pressure). The goal is to find the efficiency of the engine in terms of Cp, CV, and p2/p1. We're given the efficiency is W_{ABCD}/Q_{DA}, so all we need to do to find W_{ABCD} and Q_{DA} in terms of these variables.

First, let's find the W for each step, and then sum these to find W_{ABCD}.

A-B
Adiabatic so Q = 0. $W_{AB} = \Delta U = \frac{Nfk}{2}(T_B - T_A)$.
We know that $C_v = \frac{Nfk}{2}$ = NfK/2 because $C_v = \left(\frac{dU}{dT}\right)_V$.

$$W_{AB} = \mathrm{C_v}(T_B - T_c)$$

C-D
This is the same thing because it's adiabatic.

$$W_{CD} = \mathrm{C_v}(T_A - T_D)$$

B-C
Now this is isobaric, so pressure doesn't change. We can use our handy work formula.

$$W_{BC} = -PdV = -P_2(V_c - V_B)$$

D-A
Again,

$$W_{DA} = -PdV = -P_1(V_D - V_A)$$

Summing all of these,

$$W_{ABCD} = \mathrm{C_v}(T_B - T_c) + \mathrm{C_v}(T_A - T_D) - P_2(V_c - V_B) - P_1(V_D - V_A)$$

This isn't very helpful, but we know the substance is an ideal gas. So, we can use $P_1V_A = NkT_A$ to change our volumes into temperatures…

$$V_c - V_B = Nk(T_c - T_B)/P_2$$
$$V_D - V_A = \frac{Nk(T_D - T_A)}{P_1}$$

$$W_{ABCD} = (\mathrm{C_v} + Nk)(T_B - T_c + T_A - T_C)$$

NICE!!!

Ok, now let's find Q_{DA}.

$$\Delta U = Q + W_{DA} \rightarrow Q = \Delta U - W_{DA} = \frac{Nkf}{2}(T_A - T_D) + P(V_A - V_D) = (C_V + Nk)(T_A - T_D)$$

Efficiency = W_{ABCD}/Q_{DA} = $(C_V + Nk)(T_D-T_C+T_B-T_A)/(C_V + Nk)(T_A-T_D)$
= $(T_D-T_C+T_B-T_A)/(T_A-T_D)$ = $-(T_A-T_D)/(T_A-T_D) + (T_B-T_C)/(T_A-T_D)$
= $-1 + (T_B-T_C)/(T_A-T_D)$

Now we need to write this in terms of P1, P2, Cv, and Cp. I guess we didn't really need to write it in terms of Cv earlier, but whatever. I think the key to getting this is to recall that the adiabatic constant, γ, is equal to Cp/Cv. (I think if you didn't realize that, this would be hopeless). So, if we can get our adiabatic constant in here, we can at least get a Cp and a Cv. Our cycle is adiabatic from A->B and C-D. So

$P_1V_A^{\gamma} = P_2V_B^{\gamma}$
$P_2V_C^{\gamma} = P_1V_D^{\gamma}$

This isn't helpful to us; we want it in terms of P and T. Using the ideal gas law (but really looking at my formula sheet), we can also write

$P_1^{(1-\gamma)}T_A^{\gamma} = P_2^{(1-\gamma)}T_B^{\gamma}$
$P_2^{(1-\gamma)}T_C^{\gamma} = P_1^{(1-\gamma)}T_D^{\gamma}$

Referring to our E = $-1 + (T_B-T_C)/(T_A-T_D)$

If we plug in Tb and Tc, we can get it in terms of Ta and Tc… do some cancelling…

Use $T_B = (P_1/P_2)^{(1-\gamma)/\gamma}\, T_A$ and same for T_C

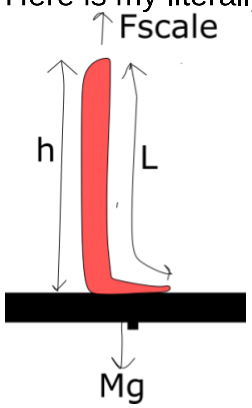
$E = -1 + (P_1/P_2)^{(1-\gamma)/\gamma}(T_A - T_D)/(T_A-T_D)$

$$\boxed{\mathbf{E = -1 + (P_1/P_2)^{(1-\gamma)/\gamma}}}$$

Where gamma = Cp/Cv!!!!!!!!!!!!!!
Honestly, I kind of hate this question because it isn't challenging conceptually, but it depends on if you can see the relations.

10. A long uniform rope with mass M and length L is suspended vertically above a scale, with the lower end of the rope almost touching the surface of the scale. When the upper end of the rope is released, the rope falls onto the scale's surface. Each segment of the rope comes to rest immediately (ideal non-elastic) as soon as it hits the scale. Sketch the reading on the scale as a function of time, and calculate the maximum scale reading during this process.

Here is my literally horrible drawing:



I'm ashamed. Looking at our FB diagram, there are only two forces on the rope: the weight of gravity and the normal force upward (the force from the scale). The normal force, F_{scale} is the basically reading that the scale will say.

On the rope:

$F_{tot} = F_{scale} - Mg$ (positive is in the +y direction)

Where to go from here? We need to account for the momentum from the rope falling on the scale. Thanks to Newton, we know

$$F_{tot} = \frac{dp}{dt}$$

Something we need to account for is the fact that, momentum is changing with time because the mass is changing with time, or at least the mass that's still in the air. Let h be the height of the rope that's still in the air. Define $\lambda = M/L = m/h$

$$p = mv = \lambda hv$$

What is velocity? It's the change of this height with time.

$$p = \lambda h \frac{dh}{dt}$$

Let's take the derivative of this with respect to time because we need dp/dt. This is simple chain rule.

$$\frac{dp}{dt} = \lambda(\dot{h})^2 + \lambda h\ddot{h} = F_{scale} - Mg$$
$$F_{scale} = \lambda(\dot{h})^2 + \lambda h\ddot{h} + Mg$$

So we will get our scale reading if we find h' and h'' (velocity and acceleration). Note the only downward force is from gravity, so we can say h'' = -g. What about velocity? Use conservation of energy.

$$KE_2 - KE_1 = -(U_2 - U_1)$$
$$½\, mv^2 - 0 = mgL - mgh$$
$$v^2 = \dot{h}^2 = 2g(L - h)$$

$$F_{scale} = \lambda 2g(L - h) - \lambda hg + Mg$$
$$F_{scale} = \lambda 2g(L - h) - \lambda hg + \lambda Lg$$
$$F_{scale} = 3\lambda g(L - h)$$

WOW!!!! So pristine!! The only thing changing in here is h, which is decreasing as the rope falls, so the max value will be when h = 0 (when the last piece of rope hits the scale) when

$$\boxed{F_{max} = 3\lambda gL = 3Mg.}$$

What about the time? Can we find the force in terms of time and plot that?

NOTE: see better method below
The time to fall is $t = x/v = L/(2gL)^{1/2}$. Isolating L, $L = 2t^2 g$, and plugging into F,

$$F = 3\lambda g(2t^2 g - h)$$

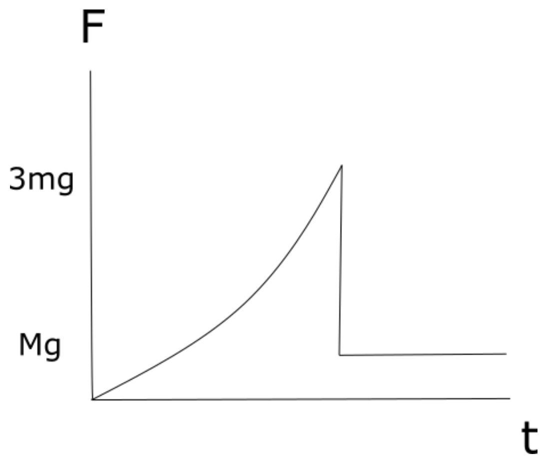
Don't know if this is correct, but it at least tells us that the force is going to rise like a quadratic until h = 0, and then the scale will settle on the true weight, which is Mg.

----------- BETTER METHOD--------
Actually can use kinematics formulas:

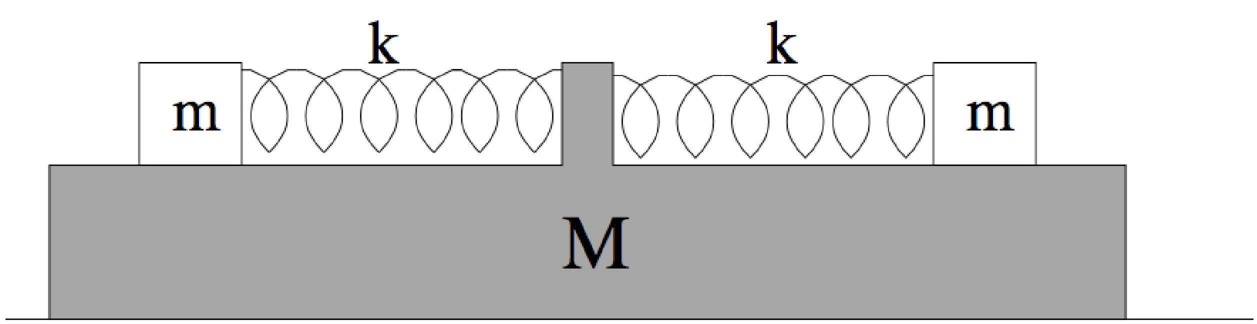
$$y = y_0 + ½\, gt^2$$

Here, y = h and y_0 = L. so
$h = L - ½\, gt^2 \rightarrow$

$$\boxed{F = 3\lambda g(L - L + ½\, gt^2) = \frac{3}{2}\lambda g^2 t^2}$$



11. Consider, as depicted in the figure below, a platform of mass M sitting on a frictionless surface and free to move in the horizontal (right-left) direction. Two identical blocks, both of mass m, are connnected to a post which is fixed to the platform by two identical springs with spring constants k. The blocks are free to move on the platform, also in the horizontal direction and friction at all of the interfaces can be neglected. Find the normal mode frequencies of the system in terms of the constants k, M and m. Describe the nature of the motion of each of the normal modes.



We want to find the normal mode frequencies. To do that, we need to find the equations of motion for the system them, and solve them (the normal frequencies are omega (w)).

There are two options: Just write the equations of motion by looking, or write the Lagrangian and use the Euler Grange equations to find the it. I personally find using the Lagrangian easier.

Let x = motion of M, x_1 = motion of m (left), x_2 = motion of m (right). When $x_1$and x_2 are zero, the springs are not displaced. Positive x is in the right direction.

$$L = T - V$$

$$T \;=\; ½\, M\, \dot{x}^2 \;+\; ½\, m(\dot{x} + \dot{x}_1)^2 \;+ ½\, m(\dot{x} + \dot{x}_2)^2$$

The T of the large mass is obvious, but for the smaller masses it's slightly more tricky: the mass is moving both because of the spring action, but also because M is moving. So the velocity of that block is equal to the sum of both of those movements.

$$V \;=\; ½kx_1^2 \;+\; ½kx_2^2$$

The potential energy is just equal to the potential from the springs. The large mass doesn't have any potential energy associated with it.

$$L = \; ½\, M\, \dot{x}^2 \;+\; ½\, m(\dot{x} + \dot{x}_1)^2 \;+ ½\, m(\dot{x} + \dot{x}_2)^2 - ½kx_1^2 - \; ½kx_2^2$$

To find the equations of motion, use the E-L equations for x, x1, and x2:

$$\frac{\partial L}{\partial f} - \frac{d}{dt}\frac{\partial L}{\partial f'} = 0$$

I won't write the math here (it's trivial really) but I got:

$$m(\ddot{x} \;+\; \ddot{x}_1) \;=\; -kx_1$$
$$m(\ddot{x} \;+\; \ddot{x}_2) \;=\; -kx_2$$
$$\ddot{x}(M + 2m) + m(\ddot{x}_2 \;+\; \ddot{x}_1) \;=\; 0$$

There are two ways to solve this: play around with the equations (ie letting x1+x2 = y, and solving) or using the matrix determinant method. I'm going to use the matrix method, as it's a lot more reliable than trying to solve it by force.

Make a matrix in the form **Mx'' = -Kx**

this looks like:

$$\begin{bmatrix} m & m & 0 \\ m & 0 & m \\ M + 2m & m & m \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = - \begin{bmatrix} 0 & 0 & k \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_1 \\ x_2 \end{bmatrix}$$

Note the negative in **Mx'' = -Kx!!** I ignored it initially and got negative roots. K is the negative of what you think it is.

Find the eigenvalues of this matrix:
det(**K –** w_2**M**)**a** = 0 (w_2 is in place of the stereotypical lambda variable).

Finding the determinant of this matrix gives something to the effect of:
$-w_2m(w_2m(k- w_2m)) + w_2m(-w_2m(k- w_2m))- w_2(M+2m)(k- w_2m)_2 = 0$

Set this equal to 0 and solve to find the eigenvalues. Obviously $w^2 = 0$ is an eigenvalue. And setting $k - w^2 m = 0$ (as this term can be divided out) gives $w^2 = k/m$, which is a classic. Since there are 3 equations, we need 3 normal modes. To find the last, solve the above equation for w. This gives

$w^2 = \frac{k(M+2m)}{mM}$

Therefore, the normal modes are the square roots of:

$\boxed{w^2 = \frac{k(M+2m)}{mM}, w^2 = \frac{k}{m}, w^2 = 0}$

What about describing the nature of the normal modes? I think this is just finding the eigenvectors and seeing what it's like. So plug back in **K –** w^2**M** and see what we get:

We can also guess it's of the form $Ae^{i\omega t}$, and by solving we can find the constants out front.

$w = 0$:

$$x_1 = x_2 = 0$$

As a vector:

$$y(t) = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

So the springs aren't really moving. Not enough info to find X, it's just doing whatever.

$$w = \sqrt{k/m}$$
$$x_! = -x_2$$
$$x = 0$$
$$y(t) = \begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix} e^{i\sqrt{\frac{k}{m}}t}$$

The base is stationary, and the springs are moving opposite to each other.

$$w^2 = \frac{k(M+2m)}{mM}:$$

$$x1 = x2$$
$$x = -\frac{2m}{M+2m}$$

$$y(t) = \begin{vmatrix} -\frac{2m}{M+2m} \\ 1 \\ 1 \end{vmatrix} e^{i\sqrt{\frac{k(M+2m)}{mM}}t}$$

Masses are moving in same direction; base is moving opposite direction.

12. High-permeability materials are often used for magnetic shielding applications. The relative permeability of iron is 5000 and the saturation field is 2 T. Do an order of magnitude estimation of the following:

A) Estimate the magnetic field in the center of an iron tube with 2 cm diameter, 30 cm length, and 1 mm wall thickness subject to the Earth's magnetic field (0.05 mT), oriented transverse to the axis of the tube.

B) An MRI magnet creates a 7 T magnetic field, with a bore the size of a person. You wish to create a magnetic shield that will eliminate stray fields from this magnet beyond a radius of 3 m. Assuming you have available a material of density $\rho = 8$ g/cm^3 with infinitely high relative permeability, but saturation magnetization of 0.5 T, what is the minimum mass of the shield?

I'm not even going to attempt this right now, because I have no hope.

13. Liquid helium-3 (an isotope of helium with atomic weight 3) at very low temperature is a good approximation of a Fermi gas. Given that the density of liquid He-3 at 100mK is 0.08 g/cm^3:

A) Would you characterize He-3 at 100 mK as a highly degenerate, weakly degenerate, or classical Fermi gas?

B) Do an order of magnitude estimation of the specific heat of He-3 at this temperature.

A) We say a gas is degenerate when all the quantum states are filled below the fermi energy, ϵ_f, while all the states above this energy are unoccupied. In a normal gas, most of the electron energy levels are unfilled and the electrons are free to move to other states. When the particle density increases, electrons are forced to fill the higher energy states, even at low temps. Boltzmann stats no longer apply at these temperatures.

The condition for a fermion gas to be degenerate is when $kT << \epsilon_f$, so the average thermal energy is much smaller than the fermi energy (the fermi energy is the energy difference between the highest and lowest occupied states).

The formula for ϵ_f:

$$\epsilon_f = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3}$$

We know T is 100mK, so $kT = 1.38 \times 10^{-24} J$.

To find $\in_f$, we need the number density of particles, N/V:

$$\frac{N}{V} = \frac{\frac{m_{tot}}{m_{HE}}}{V} = \frac{\rho}{m_{He}} = 1.606 \times 10^{28}/m^3$$

Plugging this in and calculating $\in_f$, I got $\in_f$ = $3.69 \times 10^{-19} J$, which indicates this is a highly degenerate gas.

$$\boxed{3.69 \times 10^{-19} J >> 1.38 \times 10^{-24} J}$$

Note also, for m I used the mass of an electron, not the mass of the He-3 atom, because it's the energy difference between the electron's states (is this correct?)

B. Schroeder basically goes through this in his thermodynamics book, so I'm wondering if we are supposed to derive it and not just state it? But the total energy of a degenerate fermi gas is

$U = \frac{3}{5} N e_f + \frac{\pi^2}{4} \frac{N}{e_f} (kT)^2$

We know that $C_V = (dU/dT)_{N,V}$, so taking the derivative of U we get

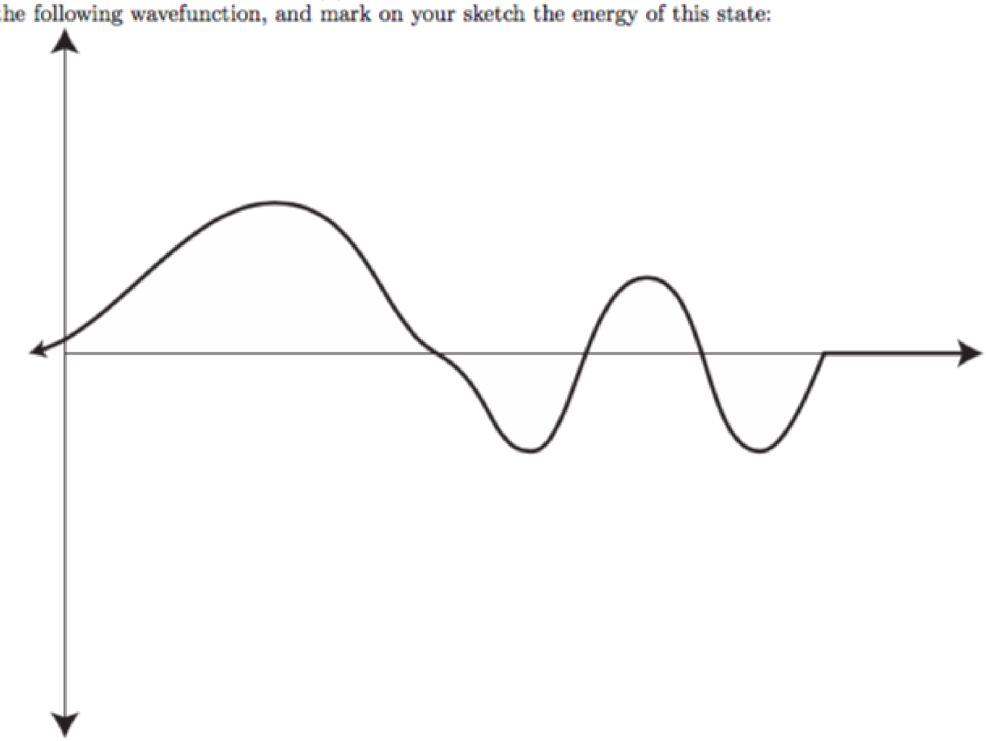
$C_v = \frac{\pi^2}{2} \frac{N}{e_f} k^2 T$

Although, what about N? We have N/V but we don't have the volume. Should we assume a box of $1m^3$? Just for the estimate? Then $N = 1.606 \times 10\text{^}28$, leading to

$$C_V = 4.09\, J/K$$

Seems… normal? I guess it depends on the mass. Assuming the volume of $1m^3$, the mass would be 80 kg, so the specific heat would be 0.05 J/kgK. Seems small.

14. Sketch a 1D potential V(x) that would have as one of its bound states the following wavefunction, and mark on your sketch the energy of this state:



Explain clearly in words the most important features of your potential and why they must be present.

This is the plot of a wave function. It seems like this is a finite/infinite square well type question (think about an infinite box where the wave functions are sinusoidal).

Think: What sort of potentials create wavefunctions that look like this? How do we get the wavefunction in the first place? Solving the Schrodinger equation, which looks like this:

$$\nabla^2\Psi = -\frac{2m}{h^2}(E-V)\Psi$$

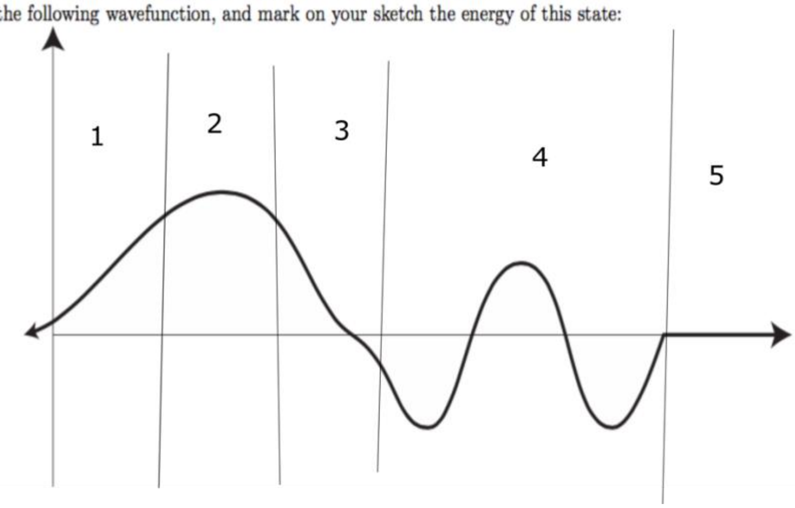
When $E > V$ (which is a classically allowed state) the solution to this is sinusoidal: $\Psi(x) = Asin(kx) + Bcos(kx)$ (where k is the square root of $\frac{2m}{h^2}(E-V)$)

When $E < V$ (which is a classically forbidden state), the solution is exponential:

$$\Psi(x) = Ce^{kx} + De^{-kx}$$

So, the wavefunction should look sinusoidal when $V < E$ and exponential when $V > E$. Oh well. I'll split this into 5 sections:

14. Sketch a 1D potential V(x) that would have as one of its bound states the following wavefunction, and mark on your sketch the energy of this state:



The reason I choose these 5 is because this is where it appears the wavefunction switches from curving away from (aka exponential) to towards (sinusoidal) the x-axis. V should be greater than E in sections 1 and 3, and less than E in sections 2 and 4.

In section 5, the wavefunction is 0. This must mean that the potential is infinite at this point, and therefore the probability of a particle being found here is 0.

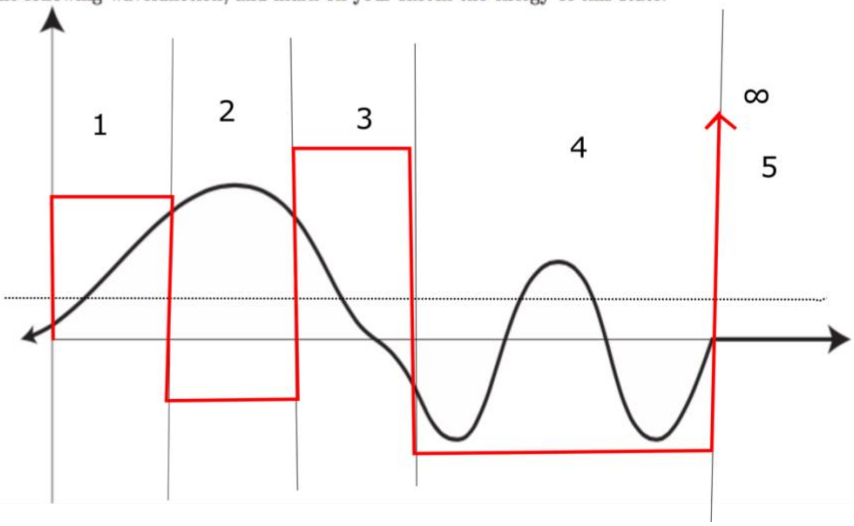
Because $\Psi(x) \sim Asin(kx)$, where V is contained inside k, when V is larger, the frequency will be larger – therefore, $|V_4| > |V_2|$.

This is similar for exponentials: when $e^\wedge kx$ is larger, the exponential should be steeper - therefore, $|V_3| > |V_1|$.

Based on what I just said, I recover something like this:

Note: I drew this overtop the initial graph, but of course the y-axis is different in both graphs, so the where I drew E (the dotted line) is arbitrary in relation to the graph of the wavefunction. V is the red line. I also drew it as squares which is maybe unnecessary…

14. Sketch a 1D potential V(x) that would have as one of its bound states the following wavefunction, and mark on your sketch the energy of this state:



In section 3, we switch from a classically forbidden to a classically allowed region. I believe during this region the particle is tunneling, which is why the amplitude of the sinusoidal wave decreases after penetrating the barrier. Also note the curvy part in 3: I think this is just the fact that the wavefunction is a combination of exponentials.

15. A free electric dipole $\vec{p}$ sits in the middle of a dielectric sphere of radius a, and points in the $+z$ direction. The sphere is a linear isotropic dielectric with dielectric constant $\epsilon > \epsilon_0$. Calculate the potential for $r > a$.

A reminder that a dielectric is a fancy way of saying an insulator. All the charges in this material are attached to molecules/atoms, and cannot just move around freely. They can move around within the atom/molecule, by either stretching or rotating.

If a dielectric is placed in an electric field, E pulls the electrons and the nucleus apart, and this is balanced with the force pulling the electrons and nucleus together. This creates a tiny dipole moment, p, pointing the same direction as E (aka the material is polarized now)

To determine the potential, we need to solve Laplace's equation:
$\nabla^2 V = 1/e_0\ \rho = 0$

This is a boundary condition type problem, with a common workflow, which is:
1. Determine boundary conditions
2. Use Laplace equation to determine potential inside and outside
3. Combine Laplace equation with BCs and use the series of equations to solve for the unknown constants

4. Plug in unknown constants -> have potential!

1. Boundary conditions
First two are same as usual:

(i) $V_{out} = V_{in}$
(ii) $\epsilon_{out} E_{out} = \epsilon_{in} E_{in}$
This is the same as
$\epsilon_{out} \frac{V_{out}}{dr} = \epsilon_{in} \frac{V_{in}}{dr}$ (because there is no charge on the surface…)

The next two depend on our specific problem.
Because there is no external electric field,
(iii) $V \to 0 \; at \; \infty$

Finally, what about when $r \to 0$? Because there is a dipole in the middle of our sphere,
(iv) $V \to V_{dipole}$ at r = 0, $V_{dipole} = \frac{pcos\theta}{4\pi\epsilon_{in} r^2}$

The general solution to Laplace's equation is:

$$V(r,\theta) = \sum_{l=0} (A_l r^l + \frac{B_l}{r^{l+1}}) P_l(cos\theta)$$

So let's put together our boundary conditions. First (i). What is V_{out}? We know it must be 0 at infinity, so the A term must disappear.

$$V(r,\theta)_{out} = \sum_{l=0} (\frac{B_l}{r^{l+1}}) P_l(cos\theta)$$

V_{in}? Now the B term must disappear, because we want V to equal V_{dipole} when r = 0.

$$V(r,\theta)_{in} = \frac{pcos\theta}{4\pi\epsilon r^2} + \sum_{l=0} (A_l r^l) P_l(cos\theta)$$

Note I am using ϵ, and not ϵ_0, where $\epsilon = \epsilon_0 \epsilon_r$, where ϵ_r is the dielectric constant. This is to reflect that the potential is reduced by a factor of the dielectric constant.

$$\sum_{l=0} (\frac{B_l}{R^{l+1}}) P_l(cos\theta) = \frac{pcos\theta}{4\pi\epsilon R^2} + \sum_{l=0} (A_l R^l) P_l(cos\theta)$$

Can we find any of our constants? Firstly, notice (or somehow remember for the exam? Ugh) that the Legendre polynomial when $l = 1$ is $cos\theta$, so when $l = 1$, the $cos\theta$ term cancels out. So, we have a different solution for both $l = 1$ and $l \neq 1$.

When $l = 1$,

$$B_1 = \frac{p}{4\pi\epsilon} + AR^3$$

When $l \neq 1$, we only equate the terms with the Legendre polynomial (Why? This confuses me... think about it…)

$$B_l = A_l R^{2l+1}$$

So now why have our B constant. We just need to find the A constant and we are finished. Let's use our second boundary condition, $\epsilon_{out} \frac{V_{out}}{dr} = \epsilon_{in} \frac{V_{in}}{dr}$.

$$\frac{dV(r,\theta)_{out}}{dr} = \sum_{l=0} (\frac{-(l+1)B_l}{r^{l+2}}) P_l(cos\theta)$$

$$\frac{dV(r,\theta)_{in}}{dr} = \frac{-2pcos\theta}{4\pi\epsilon r^3} + \sum_{l=0} (A_l l r^{l-1}) P_l(cos\theta)$$

$$\epsilon_0 \sum_{l=0} \left(\frac{-(l+1)B_l}{r^{l+2}}\right) P_l(cos\theta) = \epsilon(\frac{-2pcos\theta}{4\pi\epsilon r^3} + \sum_{l=0} (A_l l r^{l-1}) P_l(cos\theta)\ )$$

Note there is an epsilon nought in front of the V_{out} term, because it's just air. But the epsilon in front of the V_{in} term reflects our dielectric constant. Now we can solve for A. First, when l = 1:

$$\epsilon_0 \left(\frac{-(2)B_1}{R^3}\right) cos\theta = \epsilon\left(\frac{-2pcos\theta}{4\pi\epsilon R^3} + \ A_1\right) cos\theta$$

Plug in B_1….

$$\epsilon_0 \left(\frac{-2(\frac{p}{4\pi\epsilon} + AR^3)}{R^3}\right) = \epsilon\left(\frac{-2p}{4\pi\epsilon R^3} + \ A_1\right)$$

$$(\frac{-p\epsilon_0}{2\pi\epsilon R^3} - 2A_1\epsilon_0) = \left(\frac{-2p}{4\pi R^3} + \ A_1\epsilon\right)$$

$$A_1 = \ -\frac{p}{2\pi\epsilon R^3} \frac{(\epsilon_0 + \ \epsilon)}{(\epsilon + 2\epsilon_0)}$$

I think I have a negative sign wrong somewhere…. Whatever… This is at least a ¾, ok.

When $l \neq 1$, (Note I plugged in Bl from the beginning)

$$A_l l R^{l-1} = \left(\frac{-(l+1)(A_l R^{2l+1})}{R^{l+2}}\right)$$

Did A_l just cancel out on me? It looks like the R term cancels out and I get something like, l = ½. That is impossible though because l is integers... so I say that $A_l = 0$ or else this doesn't make sense (is this legit? Idk).
Let's just roll with it. Plugging A1 and Al into B1 and Bl,

$$B_l = 0$$
$$B_1 = \frac{p}{4\pi\epsilon} + \left(\frac{-p}{2\pi\epsilon R^3}\frac{(\epsilon_0 + \epsilon)}{(\epsilon + 2\epsilon_0)}\right)R^3$$

Now, we can plug this into our original equation for V_{out}!!

$$V(r,\theta)_{out} = \frac{\frac{p}{4\pi\epsilon} + \left(\frac{-p}{2\pi\epsilon R^3}\frac{(\epsilon_0 + \epsilon)}{(\epsilon + 2\epsilon_0)}\right)R^3}{r^2} cos\theta$$

Where the cos is from the Legendre polynomial, and only the l = 1 term has survived. Thinking about it, it makes sense – it would not make sense if there were multiple potentials depending on the value of l (I think). Also, I obviously did not simplify. Note that $\epsilon = \epsilon_0\epsilon_r$, it isn't the same e defined in the question (ϵ_r is). I'm sure there are math mistakes here but I think this is good from a physics perspective.

16.

Active galactic nuclei (AGN), presumably powered by supermassive black holes, often eject back-to-back relativistic jets of material. The motion of material in these jets can be measured by observing the passage of lumps of material along the jet using radio telescopes.

Suppose that an AGN is at some large, known distance d from us, with the jet pointing (generally) towards Earth at some angle θ. (You may safely assume that $d \gg$ the size of the AGN+jet.) Let β be the speed of material in the jet, in units of c. Demonstrate that for a range of θ the apparent velocity of motion of the material in the transverse direction can appear to exceed c. Determine the range of θ for which the illusion of such superluminal motion can occur, and calculate the largest possible apparent transverse velocity for a given β.

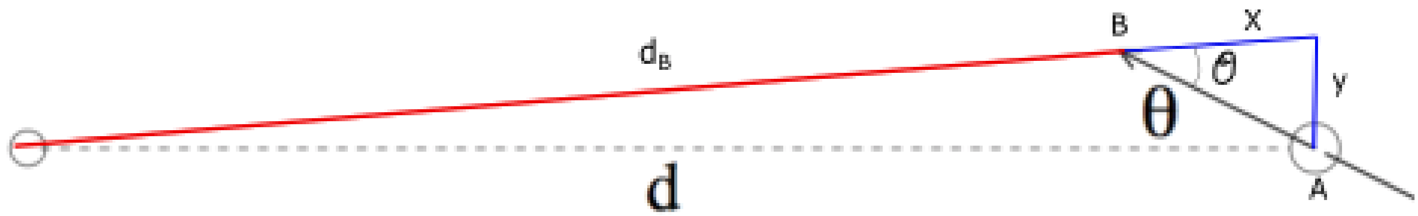
(Remarkably many such systems have actually been observed. Even more remarkably this effect was actually predicted in advance!)

Okay. To be honest, I don't think I would even be able to figure this one out on an exam. The question is somewhat simple but I feel like I wouldn't think of the solution. Whatever. See here: https://en.wikipedia.org/wiki/Superluminal_motion

So, the light from the jet is at an angle θ. We know the speed of material in the jet, which is β. We want to show that the apparent velocity (which is the velocity we see the jet having from earth) in the transverse direction (the y-direction) can appear to exceed c.

So, we need to find an expression for the velocity in the transverse direction. And then we can see if it's greater than c. All we're going to do is define all of our distances and angles in terms of beta, and theta, and solve for the transverse velocity.

See my reference image:



I have defined points A and B, A is where the jets originate and B is the "end" of the jet (the jets are measured by observing lumps of material along the jet, so pretend there is a lump here). By the way, I ignored the gamma term and just assumed it was ~ 1. Is this ok? I don't know. It ends up cancelling out anyway.

We're told θ is very small from Earth.

Let's focus on our smaller triangle. From the jet:

$$x' = l' \cdot cos\theta = \ v \cdot dt' \cdot cos\theta = \beta c dt' cos\theta; y' = \beta c \ dt' \ sin\theta$$

Where l is the length of the hypotenuse of that blue triangle, which we can write in terms of velocities (should I have a gamma in front of this?).

The primes represent that this is from the jet's perspective. What is time here. dt is the time it takes the light emitted from the jet emitted at A to reach B.

The apparent velocity, v_y, is $v_{app} = \frac{y}{dt}$, where y and dt are no longer primes. We don't know these variables yet, but once we find them, we have the answer.

What about dt? We can find t_A and t_B and from earth and subtract them. Use Lorentz transformation for time:

$t = \gamma(t' + \frac{d\prime}{c})$, since it's light travelling to us, the time it takes to get to us is the time it's emitted plus the time it takes to get to us (obviously lmao).

Therefore

$t_A = \gamma(t_A' + \frac{d}{c});\ \ t_B = t_B' + d_B/c$

Because theta is so small, can we assume $d_B + x = d$? Therefore $t_B = \gamma(t_B' + (d - x')/c)$. Ignoring gamma:

$$dt = t_B - t_A = \ t_B' + \frac{d - x'}{c} - \ t_A' - \frac{d}{c} = dt' - \frac{x'}{c} = dt' - \ v \cdot dt' \cdot \frac{cos\theta}{c} =$$
$$= dt' - \beta \cdot dt' \cdot cos\theta = dt'(1 - \beta cos\theta)$$

Plugging this into v_{app},

$$v_{app} = \frac{y}{dt} = \frac{\beta cdtsin\theta}{dt' - \beta \cdot dt' \cdot cos\theta}$$

Hm. I want to be able to cancel out the dts. One of them is not prime though. Should the y also be prime? Then we get

$$v_{trans} = \frac{y}{dt} = \frac{\beta cdt'sin\theta}{dt' - \beta \cdot dt' \cdot cos\theta} = \frac{\beta csin\theta}{1 - \beta cos\theta}$$

$$\beta_{trans} = \frac{\beta sin\theta}{1 - \beta cos\theta}$$

This checks out but I'm confused about the primes. When should something be prime and when should it not be prime? If the velocity was in the x-direction, then y' = y and it would make sense… but y should not equal y' here? I think y would only be equal to y' if it's perpendicular to the velocity direction. Well… I guess thinking about it, it's not really moving is it? Like, the light along l is moving, but the length y is not changing. So in that case it would make sense that y = y'…

Ignoring that… This is just math from here. To find the largest possible velocity, find $\frac{d\beta_{app}}{d\theta}$, set this equal to 0, find the value of beta (I found $\beta = cos\theta \rightarrow \theta_{max} = \arccos\beta$) then plug this back in to β_{app} which gives

$$\beta_{maxT} = \frac{cos\theta sin\theta}{1 - \cos^2\theta} = \frac{cos\theta}{sin\theta} = \arctan(arccos\beta)$$

or

$$\beta_{\mathrm{maxT}} = \frac{\beta sin \arccos\beta}{1 - \beta cos \arccos\beta} = \frac{\beta\sqrt{(1-\beta^2)}}{1-\beta^2} = \frac{\beta}{\sqrt{(1-\beta^2)}}$$

For the range of theta,

$$v_{app} = \frac{\beta csin\theta}{1 - \beta cos\theta} > c \rightarrow \frac{\beta sin\theta}{1 - \beta cos\theta} > 1 \rightarrow \beta(sin\theta + cos\theta) > 1$$

Is this enough? Just saying I hate relativity.