

Columbia University
Department of Physics
QUALIFYING EXAMINATION

Monday, January 14, 2013

3:10PM to 5:10PM

Classical Physics

Section 2. Electricity, Magnetism & Electrodynamics

Two hours are permitted for the completion of this section of the examination. Choose 4 problems out of the 5 included in this section. (You will not earn extra credit by doing an additional problem). Apportion your time carefully.

Use separate answer booklet(s) for each question. Clearly mark on the answer booklet(s) which question you are answering (e.g., Section 2 (Electricity etc.), Question 2, etc.).

Do **NOT** write your name on your answer booklets. Instead, clearly indicate your **Exam Letter Code**.

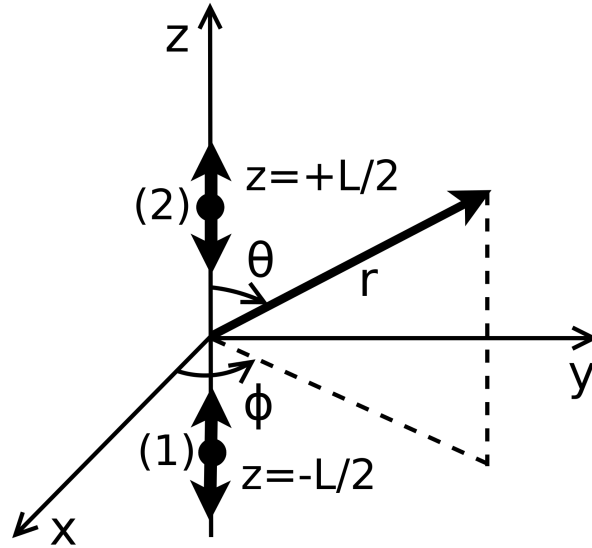
You may refer to the single handwritten note sheet on $8\frac{1}{2}$ " \times 11" paper (double-sided) you have prepared on Classical Physics. The note sheet cannot leave the exam room once the exam has begun. This note sheet must be handed in at the end of today's exam. Please include your Exam Letter Code on your note sheet. No other extraneous papers or books are permitted.

Simple calculators are permitted. However, the use of calculators for storing and/or recovering formulae or constants is NOT permitted.

Questions should be directed to the proctor.

Good Luck!

1. Consider two very tiny oscillating electric dipoles which are located on the z -axis as shown in the diagram.



The first dipole is located at $z = -L/2$ and has a time-varying electric dipole moment given by

$$\vec{D}_1 = D_0 \hat{z} \cos \left(\omega t + \frac{\alpha}{2} \right) = \text{Re} \left\{ D_0 \hat{z} e^{-i(\omega t + \frac{\alpha}{2})} \right\} \quad (1)$$

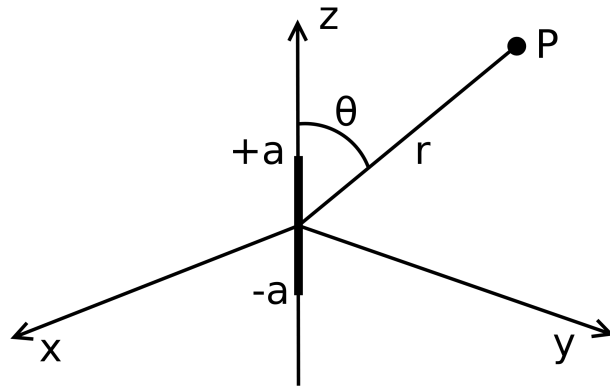
The second dipole is located at $z = +L/2$ and has a time-varying electric dipole moment given by

$$\vec{D}_2 = D_0 \hat{z} \cos \left(\omega t - \frac{\alpha}{2} \right) = \text{Re} \left\{ D_0 \hat{z} e^{-i(\omega t - \frac{\alpha}{2})} \right\} \quad (2)$$

The two dipoles therefore oscillate in the same direction (in the z -direction where \hat{z} is a unit vector in the z -direction) with the same amplitude D_0 and the same angular frequency ω but are out of phase by phase angle α . D_0 , ω , L and α are given positive real constants. You should express your answers in this problem using these constants. Consider a field at point (r, θ, ϕ) in the radiation zone ($r \rightarrow \infty$) and keep only the leading $1/r$ terms.

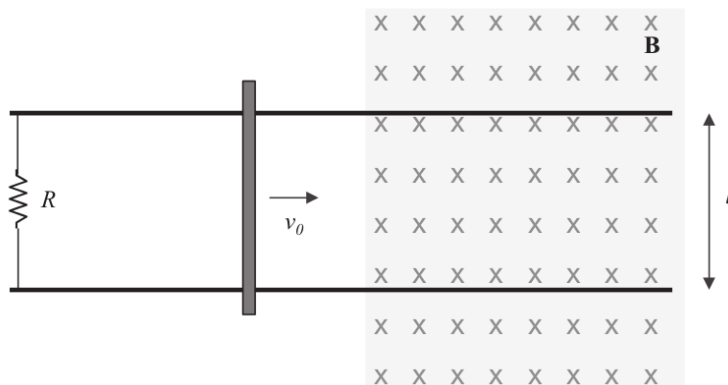
- (a) Find the electric field \vec{E}_{RAD} at (r, θ, ϕ) in the radiation zone.
- (b) Find the time-averaged power detected per unit solid angle ($dP/d\Omega$) in the (θ, ϕ) direction in the radiation zone.

2. A linear uniform charge distribution of λ coulomb/meter extends along the z -axis from $z = -a$ to $z = +a$. Find a series expansion for $V(r, \theta)$ in terms of Legendre polynomials valid to all orders, and for $r > a$.

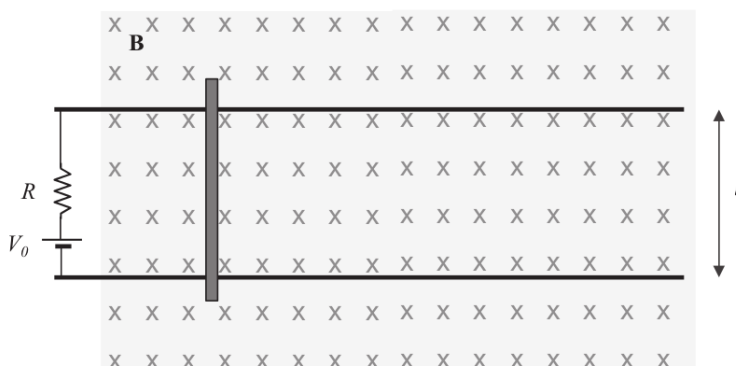


3. Suppose we have a point charge q in uniform motion with relativistic velocity v in the x direction. What is the electric field configuration sourced by this charge? To be concrete, suppose the charge position as a function of time t is $\vec{x}_q(t) = vt\hat{x}$, and we are interested in the electric field at some arbitrary location and time.

4. A conducting rod of mass m slides on frictionless, conducting rails whose separation is l in a region of constant magnetic field \mathbf{B} (direction into the page). The rails are connected to a resistor R . Assume that the conducting rod and rails have negligible resistances.
- (a) At time $t = 0$, the rod just enters into the region of magnetic field with a constant velocity v_0 . Find the velocity of the rod at $t > 0$.

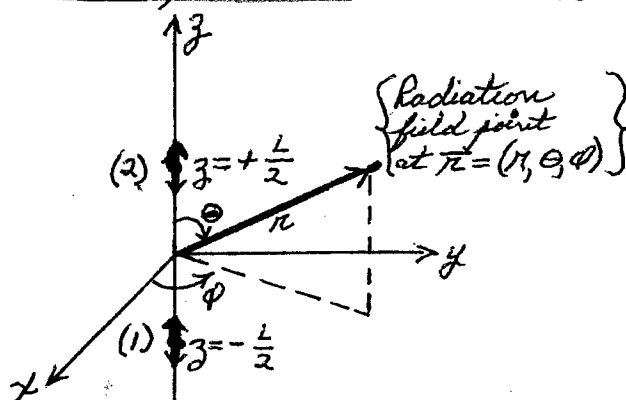


- (b) Now a battery with voltage V_0 is connected to the rail in series with the resistor as you find in the following figure. Assume that the rod is at rest at $t = 0$. When the rod is at rest, there is no induced EMF, and the current is purely driven by the battery. This current in turn pushes the rod due to the Lorentz force. Once the rod slides, there is an induced EMF. Find the velocity of the rod $v(t)$ for $t > 0$. In this problem we assume that the applied magnetic field \mathbf{B} is much larger than the magnetic field generated by the rails.



5. Consider a cylindrical capacitor of length L . The capacitor consists of an inner conducting wire of radius a and an outer conducting shell of radius b . The space between ($a < r < b$) is filled with a non-conducting material which has a dielectric constant ϵ . In all cases, neglect any end effects or fringing fields.
- (a) What is the value of the electric field as a function of the radial position, r , when the capacitor has charge Q on it?
 - (b) What is the capacitance?
 - (c) Now suppose the dielectric is pulled out partially while the capacitor is connected to a battery of constant potential V . What is the force required to hold the dielectric in this position?

Classical Physics
Electromagnetism - Allan Blaser.



Consider two very tiny oscillating electric dipoles which are located on the z-axis as shown in the diagram.

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$$\vec{D}_1 = D_0 \hat{z} \cos\left(\omega t + \frac{\alpha}{2}\right) = \text{Real Part} \left\{ D_0 \hat{z} e^{-i\left(\omega t + \frac{\alpha}{2}\right)} \right\}$$

The second dipole is located at $z = +L/2$ and has a time-varying electric dipole moment given by

$$\vec{D}_2 = D_0 \hat{z} \cos\left(\omega t - \frac{\alpha}{2}\right) = \text{Real Part} \left\{ D_0 \hat{z} e^{-i\left(\omega t - \frac{\alpha}{2}\right)} \right\}$$

The two dipoles therefore oscillate in the same direction (in the z-direction where \hat{z} is a unit vector in the z-direction) with the same amplitude D_0 and the same angular frequency ω but are out of phase by phase angle α . D_0 , ω , L , and α are given positive real constants. You should express your answers in this problem using these constants. Consider a field point at (r, θ, ϕ) in the radiation zone ($r \rightarrow \infty$) and keep only the leading $1/r$ terms.

- Find the electric field \vec{E}_{RAD} at (r, θ, ϕ) in the radiation zone.
- Find the time-averaged power detected per unit solid angle ($dP/d\Omega$) in the (θ, ϕ) direction in the radiation zone.

Electromagnetism Solution - Allan Blair

(a) For electric dipole radiation (in the radiation zone, with $r \rightarrow \infty$):

$$\vec{E}(\vec{r}, t) = \frac{1}{rc^2} \left\{ \hat{r} \times \left[\hat{r} \times \ddot{\vec{D}}(t') \right] \right\}$$

retarded $t' \approx t - \frac{r}{c} + \frac{\hat{r} \cdot \vec{x}'}{c}$
(where \vec{x}' = the location of the dipole)

Using complex notation and then taking the real part at the end,
 $\vec{D}_1 = D_0 \hat{z} e^{-i(\omega t + \frac{\alpha}{2})}$ at $z = -\frac{L}{2}$ and $\vec{D}_2 = D_0 \hat{z} e^{-i(\omega t - \frac{\alpha}{2})}$ at $z = +\frac{L}{2}$.

$$\therefore \vec{E}_1 = \frac{-\omega^2 D_0}{rc^2} [\hat{r} \times (\hat{r} \times \hat{z})] e^{-i[\omega(t - \frac{r}{c} - \frac{\hat{r} \cdot \hat{z} L}{2}) + \frac{\alpha}{2}]}$$

$$\text{and } \vec{E}_2 = \frac{-\omega^2 D_0}{rc^2} [\hat{r} \times (\hat{r} \times \hat{z})] e^{-i[\omega(t - \frac{r}{c} + \frac{\hat{r} \cdot \hat{z} L}{2}) - \frac{\alpha}{2}]}.$$

$$\hat{r} \times (\hat{r} \times \hat{z}) = \hat{r} \times (-\hat{\phi} \sin \theta) = +\hat{\theta} \sin \theta \quad \text{and} \quad \hat{r} \cdot \hat{z} = \cos \theta.$$

$$\therefore \vec{E} = \vec{E}_1 + \vec{E}_2 = -\frac{\omega^2 D_0}{c^2} \frac{e^{i(br - \omega t)}}{r} \left[e^{i(\frac{L}{2} \cos \theta + \frac{\alpha}{2})} + e^{-i(\frac{L}{2} \cos \theta + \frac{\alpha}{2})} \right]$$

$$= -\frac{2\omega^2 D_0}{c^2} \frac{e^{i(br - \omega t)}}{r} \cos \left[\frac{L}{2} \cos \theta + \frac{\alpha}{2} \right] [\hat{\theta} \sin \theta]$$

$$\text{where } k \equiv \frac{\omega}{c}.$$

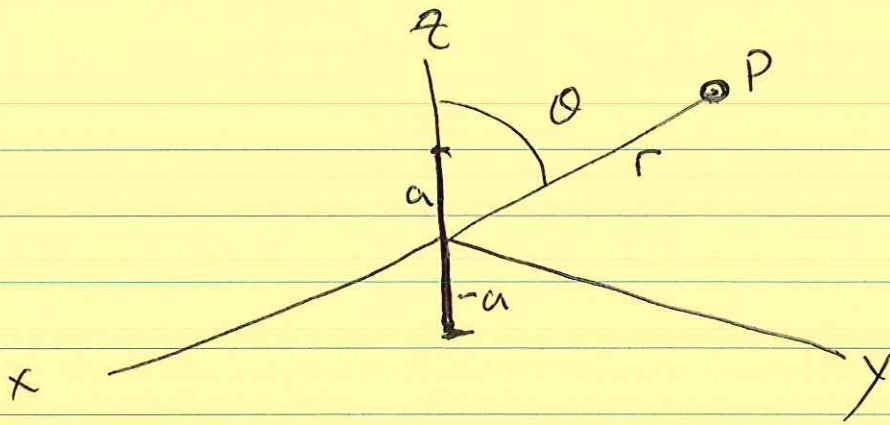
Taking the real part, $\vec{E}_{\text{physical}} = \frac{-2\omega^2 D_0}{c^2} \frac{\cos(br - \omega t)}{r} \cos \left[\frac{L}{2} \cos \theta + \frac{\alpha}{2} \right] [\hat{\theta} \sin \theta]$

(b) $\vec{S} = \frac{c}{4\pi} (\vec{E} \times \vec{B}) = \frac{c}{4\pi} |\vec{E}|^2 \hat{r}$, using $\vec{E}_{\text{physical}}$ for \vec{E} .

$$\therefore \vec{S} = \frac{c}{4\pi} \frac{4\omega^4 D_0^2}{c^4} \frac{\cos^2(br - \omega t)}{r^2} \cos^2 \left(\frac{L}{2} \cos \theta + \frac{\alpha}{2} \right) [\sin^2 \theta] \hat{r}$$

$$\left(\frac{dP}{d\Omega} \right)_{\text{time avg}} = \left[\left(\vec{S} \right)_{\text{time avg}} \cdot \hat{r} \right] r^2 \quad \text{with} \quad \left[\cos^2(br - \omega t) \right]_{\text{time avg}} = \frac{1}{2}.$$

$$\therefore \left(\frac{dP}{d\Omega} \right)_{\text{time avg}} = \frac{\omega^4 D_0^2}{2\pi c^3} \left[\cos^2 \left(\frac{L}{2} \cos \theta + \frac{\alpha}{2} \right) \right] [\sin^2 \theta]$$



Griffiths
E-M
Quas

A linear uniform charge distribution of λ coulomb/meter extends along the z -axis from $z = -a$ to $z = +a$. Find a series expansion for $V(r, \theta)$ in terms of Legendre polynomials valid to all orders, and for $r > a$.

Solution:

$$V(r, \theta) = \sum_{n=0}^{\infty} A_n r^{-(n+1)} P_n(\cos \theta) + B_n r^n P_n(\cos \theta)$$

 $B_n = 0$ for all n

$$\Rightarrow V(r, \theta) = \sum_{n=0}^{\infty} A_n r^{-(n+1)} P_n(\cos \theta)$$

Need A_n : At $\theta=0$ $P_n = 1$ and

$$E_z(\theta=0) = \frac{\lambda}{4\pi\epsilon_0} \int_{-a}^a \frac{\lambda dz}{(r-z)^2} = \frac{\lambda}{4\pi\epsilon_0} \left(\frac{1}{r-a} - \frac{1}{r+a} \right)$$

$$E_z = \frac{\lambda}{4\pi\epsilon_0} \frac{2a}{(r^2 - a^2)} = \frac{1}{4\pi\epsilon_0} \frac{2a\lambda}{r^2} \left(1 + \frac{a^2}{r^2} + \frac{a^4}{r^4} + \dots \right)$$

$$E_z = -\frac{\partial V(r, \theta)}{\partial r} = \sum_{n=0}^{\infty} (n+1) A_n r^{-(n+2)} P_n$$

Matching coefficients:

$$A_0 = 2a\lambda; \quad 3A_2 = 2a^3\lambda; \quad 5A_4 = 2a^5\lambda \dots$$

$$\Rightarrow V(r, \theta) = \frac{2\lambda}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \left(\frac{a}{r} \right)^{(2n+1)} P_{2n}(\cos \theta)$$

EM Problem 3. Suppose we have a point charge q in uniform motion with relativistic velocity v in the x direction. What is the electric field configuration sourced by this charge? To be concrete, suppose the charge position as a function of time t is $\vec{x}_q(t) = vt\hat{x}$, and we are interested in the electric field at some arbitrary location and time.

Solution. The easiest way to solve this problem is by Lorentz transformation. Let us call the rest frame of the charge the $'$ frame. The electric field in this frame is simple:

$$\vec{E}' = \frac{q\vec{x}'}{(x'^2 + y'^2 + z'^2)^{3/2}} \quad (1)$$

where we have put the charge at $x' = y' = z' = 0$. Consider boosting to the lab frame with $x = \gamma(vt' + x')$, $t = \gamma(t' + vx')$, $y = y'$, $z = z'$, or in other words $\mathbf{x} = \mathbf{\Lambda}\mathbf{x}'$ where $\mathbf{\Lambda}$ is the Lorentz matrix. ($\gamma = 1/\sqrt{1-v^2}$). To figure out how the electric field transforms, recall that it is part of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and thus $F_{01} = E_1$, $F_{02} = E_2$, $F_{03} = E_3$, $F_{12} = B_3$, $F_{23} = B_1$, $F_{31} = B_2$. The field strength transforms as $\mathbf{F} = (\mathbf{\Lambda}^{-1})^T \mathbf{F}' \mathbf{\Lambda}^{-1}$. There is no magnetic field in the $'$ frame. Thus we find

$$E_1 = E'_1, \quad E_2 = \gamma E'_2, \quad E_3 = \gamma E'_3 \quad (2)$$

Rewriting $x' = \gamma(-vt + x)$, we thus have

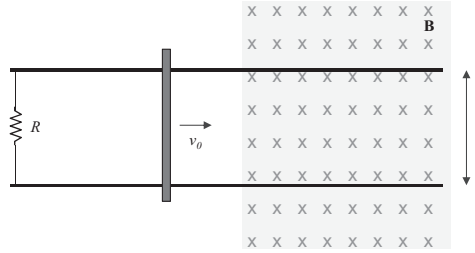
$$\vec{E} = \frac{q\gamma\vec{r}(t)}{(\gamma^2 r_x(t)^2 + r_y^2 + r_z^2)^{3/2}} \quad (3)$$

where $\vec{r}(t)$ is the separation vector from the instantaneous position of the charge $(vt, 0, 0)$ to the point of interest (x, y, z) . It is remarkable that the electric field points from the instantaneous charge position (at time t , the time of measurement of the field), rather than the charge position at the retarded time.

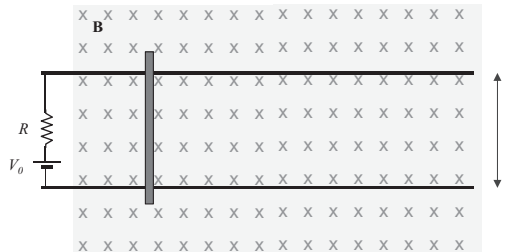
Electromagnetism Question

A conducting rod of mass m slides on frictionless, conducting rails whose separation is ℓ in a region of constant magnetic field \mathbf{B} (direction into the page). The rails are connected to a resistor R . Assume that the conducting rod and rails have negligible resistances.

(a) At time $t = 0$, the rod just enters into the region of magnetic field with a constant velocity v_0 . Find the velocity of the rod at $t > 0$.



(b) Now a battery with the voltage V_0 is connected to the rail in series with the resistor as you find in the following figure. Assume that the rod is at rest at $t = 0$. When the rod is at rest, there is no induced EMF, and the current is purely driven by the battery. This current in turn pushes the rod due to the Lorentz force. Once the rod slides, there is an induced EMF. Find the velocity of the rod $v(t)$ for $t > 0$. In this problem we assume that the applied magnetic field \mathbf{B} is much larger than the magnetic field generated by the rails.



Answers

(a) While the general solution can be obtained in the similar way as presented in (b) below, here we just use a simple energy consideration first. As the rod entered in the region with a finite magnetic field, current starts flow in the loop formed by the rod and two rails connected by the resistor. The EMF is given by

$$\varepsilon = -\frac{d\Phi}{dt} = -B\ell v(t) \quad (1)$$

where Φ is the magnetic flux within the current loop. Then the power dissipated in the resistor will be the same as the rate of kinetic energy decrease:

$$\frac{\varepsilon^2}{R} = -\frac{d(mv^2/2)}{dt} = -mv\frac{dv}{dt} \quad (2)$$

From Eq.(1) and (2), we obtain

$$-\gamma v = \frac{dv}{dt} \quad (3)$$

where

$$\gamma = \frac{(B\ell)^2}{mR} \quad (4)$$

Solving Eq(3) with the initial condition $v(t=0) = v_0$, we obtain

$$v(t) = v_0 e^{-\gamma t} \quad (5)$$

(b) When there is a current I presented in the loop, the Lorentz force on the rod is given by $B I \ell$, and thus,

$$m \frac{dv}{dt} = B I \ell \quad (6)$$

Along the loop,

$$V_0 = IR + \frac{d\Phi}{dt} \quad (7)$$

Plugging Eq.(1) and Eq.(7) into (8), we obtain

$$v_t = \frac{1}{\gamma} \frac{dv}{dt} + v(t) \quad (8)$$

where the terminal velocity is given by

$$v_t = \frac{V_0}{B\ell} \quad (9)$$

Note that Eq.(8) is the general form of Eq(3) in (a) for non zero V_0 . Solving this equation with the initial condition $v(t=0) = 0$, we have

$$v(t) = v_t(1 - e^{-\gamma t}) \quad (10)$$

2012 Qualifying Exam Problem – E&M – Tuts 12/1/12

Question:

Consider a cylindrical capacitor of length L . The capacitor consists of an inner conducting wire of radius a and an outer conducting shell of radius b . The space between ($a < r < b$) is filled with a non-conducting material which has a dielectric constant ϵ . In all cases, neglect any end effects or fringing fields.

- (a) What is the value of the electric field as a function of the radial position, r , when the capacitor has a charge Q on it?
- (b) What is the capacitance?
- (c) Now suppose the dielectric is pulled out partially while the capacitor is connected to a battery of constant potential, V . What is the force required to hold the dielectric in this position?

Solution:

- (a) For the case where there is no dielectric, we can use Gauss' law to find the electric field at a radius r

$$\begin{aligned}\int E dA &= \frac{Q_{\text{enclosed}}}{\epsilon} \\ E 2\pi r \ell &= \frac{\ell}{L} \left(\frac{Q}{\epsilon} \right) \\ E(r) &= \frac{Q}{2\pi\epsilon L} \times \frac{1}{r}\end{aligned}$$

- (b) To find the capacitance, we use

$$C = \frac{Q}{V}$$

We can use the solution to (a) to find V from

$V = -\int E dr$ (where r goes from a to b)

$$\begin{aligned}V &= -\frac{Q}{2\pi\epsilon L} \int \left(\frac{dr}{r} \right) \\ V_{ab} &= -\frac{Q}{2\pi\epsilon L} \ln\left(\frac{b}{a}\right)\end{aligned}$$

Therefore

$$C = \frac{Q}{V} = \frac{2\pi\epsilon L}{\ln\left(\frac{b}{a}\right)}$$

- (c) When we pull the dielectric out, then we will have a length x which has no dielectric in it, and a length $L - x$ which has dielectric in it. So the net capacitance will be that of two capacitors in parallel

$$C = C_{x-\text{no dielectric}} + C_{L-x-\text{with dielectric}}$$

Where the capacitance with no dielectric has a similar form to that found in part (b) except that we use ϵ_0 instead of ϵ . So

$$C = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)} \left[x + \frac{\epsilon}{\epsilon_0} (L - x) \right]$$

Where the change in capacitance is therefore given by

$$dC = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)} \left(1 - \frac{\epsilon}{\epsilon_0} \right) dx$$

The change in the work will be equal to the change in the potential energy. Recall that V remains constant since it is attached to a battery. So

$$Fdx + VdQ = \frac{1}{2} V^2 dC$$

Where

$$dQ = VdC$$

So if we substitute that in to the work equation, we get

$$Fdx + V(VdC) = \frac{1}{2}V^2dC$$

$$Fdx = -\frac{1}{2}V^2dC$$

So we can substitute in for dC and then substitute in the work equation to get

$$Fdx = -\frac{1}{2}V^2 \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)} \left(1 - \frac{\epsilon}{\epsilon_0}\right)dx$$

$$F = V^2 \frac{\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)} \left(\frac{\epsilon}{\epsilon_0} - 1\right)$$

Solution:

- (a) During the time the ball is slipping, it is accelerating with v decreasing, and ω increasing, the accelerations are caused by the frictional force:

$$a = -\frac{F}{m} = \frac{-\mu mg}{m} = -\mu g$$

$$\alpha = \frac{\tau}{I} = \frac{R\mu mg}{\left(\frac{2}{5}\right)mR^2} = \frac{5\mu g}{2R}$$

At time t when it first stops slipping and starts to roll without slipping, then the final velocity and final angular velocity is given by the usual kinematic equations using the above accelerations:

$$v = v_0 - \mu g t \quad (i)$$

$$\omega = \omega_0 + \frac{5\mu g}{2R} t = \frac{5\mu g}{2R} t \quad (ii)$$

And since it is not slipping, then v and ω are now related by:

$$v = \omega R \rightarrow \omega = \frac{v}{R} \quad (iii)$$

So substituting this in the earlier equation

$$\frac{v}{R} = \frac{5\mu g}{2R} t \rightarrow v = \frac{5\mu g}{2} t \quad (iv)$$

Now substitute (iv) in equation (i) for v above

$$\frac{5\mu g}{2} t = v_0 - \mu g t$$

Solving for t

$$t = \frac{2v_0}{7\mu g} \quad (v)$$

So we can use that value of t in the usual kinematic equation for constant acceleration

$$\Delta x = v_0 t - \frac{1}{2} \mu g t^2$$

$$\Delta x = \frac{12v_0^2}{49\mu g}$$

- (b) Substituting (v) in (i)

$$v = v_0 - \mu g \left(\frac{2v_0}{7\mu g} \right) = \frac{5}{7} v_0$$