C^*-Algebras and the Gelfand-Naimark Theorem

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1 Introduction

The space of bounded linear operators on a Banach space X naturally inherits a Banach space structure. However, since for any $S, T \in \mathcal{B}(X)$ we have $S \circ T \in \mathcal{B}(X)$ and $\|S \circ T\| \leq \|S\|\|T\|$, the space $\mathcal{B}(X)$ also has the structure of a non-commutative unital ring. This leads us to define a Banach algebra to be a Banach space over $\mathbb{C}$ which is also an associative algebra where the norm satisfies $\|xy\| \leq \|x\|\|y\|$. In the study of operator algebras we apply the theory of rings to these algebras so as to gain further insight into functional analysis. Moreover, for a Hilbert space H, the space $\mathcal{B}(H)$ has additional structure coming from the fact that we can take the adjoint of an operator and stay in $\mathcal{B}(H)$. We take $\mathcal{B}(H)$ as a model for the class of *C^*-algebras.* We define a C^*-algebra to be a Banach algebra A with an involution $(*) : A \to A$ satisfying $x^{**} = (x^*)^* = x$, $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$ and $(\lambda x)^* = \overline{\lambda} x^*$, for all $x, y \in A$ and $\lambda \in \mathbb{C}$. We also require that A satisfies the 'C^*-identity', $\|x^*x\| = \|x\|^2$.
In this essay we will show that we do not lose too much information through this abstraction, since, by the Gelfand-Naimark Theorem, any C^*-algebra is isomorphic (in a suitable sense) to an algebra of operators on a Hilbert space. We will follow the approach taken by Arveson in [1], but we will also refer to [3] and [4]. We will restrict our attention to the case of unital C^*-algebras. However, this is not a serious restriction, since we can extend any C^*-algebra to a unital C^*-algebra by adjoining a multiplicative identity and many of the results outlined in this essay go through for non-unital C^*-algebras with only minor modifications.

We will begin by making some definitions and sketching some basic results concerning C^*-algebras. Throughout, A will denote a C^*-algebra unless otherwise noted. We define the *spectrum* of an element $x \in A$ to be the set $\sigma(x) = \{\lambda \in \mathbb{C} \mid \lambda e - x \notin A^\times\}$, where $A^\times$ denotes the invertible elements in the ring A. The *spectral radius* $\nu(x)$ of x is then defined as $\sup\{|\lambda| \mid \lambda \in \sigma(x)\}$. Using Cauchy's integral formula one can show[1] that $\nu(x) = \lim_{k\to\infty} \|x^k\|^{1/k}$. Since $\|x^k\|^{1/k} \leq \|x\|^{k/k} = \|x\|$, this implies that $\nu(x) \leq \|x\|$ in general. However, if $x \in A$ is self-adjoint ($x = x^*$) then the C^* identity gives, $\|x^2\| = \|x^*x\| = \|x\|^2$, so by induction we have that $\|x^{2^k}\| = \|x\|^{2^k}$. Therefore,

$$\nu(x) = \lim_{k\to\infty} \|x^{2^k}\|^{2^{-k}} = \lim_{k\to\infty} \|x\| = \|x\|.$$

Therefore, for any $x \in A$, because x^*x is self adjoint we have that $\|x\|^2 = \nu(x^*x)$. That is, the C^*-identity implies that the norm is uniquely determined by the $*$-algebra structure on A.

Now we turn our attention to defining the morphisms in the category of C^*-algebras. We would like morphisms between C^*-algebras to respect both the algebraic structure and the involution, so we consider only those algebra homomorphisms $\pi : A \to B$ which satisfy $\pi(x^*) = \pi(x)^*$, the so-called *$*$-homomorphisms.* We would also like our morphisms to respect the analytic structure on A, but we see that because the norm on a C^*-algebra is uniquely determined by the algebra structure, any $*$-homomorphism is automatically bounded:

Proposition 1.1. *Let $\pi : A \to B$ be a $*$-homomorphism of C^*-algebras. Then $\|\pi\| \leq 1$.*

Proof. First we see that for any $y \in A$ we have $\sigma_B(\pi(y)) \subset \sigma_A(y)$. This is because,

$$\lambda \in \sigma(\pi(y)) \iff \lambda e - \pi(y) \notin B^\times \iff \pi(\lambda e - y) \in B^\times \iff \lambda e - y \in \pi^{-1}(B^\times).$$

[1]See Proposition 2.1.3 (v) of [3].

But $A^\times \subset \pi^{-1}(B^\times)$ since a non-zero ring homomorphism sends units to units, so this implies that $\lambda e - y \notin A^\times$, or $\lambda \in \sigma(y)$. This means that $\nu(\pi(y)) \leq \nu(y)$. Let $x \in A$, then since x^*x and $\pi(x^*x)$ are self-adjoint we have,

$$\|\pi(x)\|^2 = \|\pi(x)^*\pi(x)\| = \|\pi(x^*x)\| = \nu(\pi(x^*x)) \leq \nu(x^*x) = \|x^*x\| = \|x\|^2,$$

This proves that $\|\pi(x)\| \leq \|x\|$, so $\|\pi\| \leq 1$. $\square$

2 Commutative C^*-algebras

Given a compact Hausdorff space X, the space $C(X)$ of continuous functions $X \to \mathbb{C}$ is a C^*-algebra with multiplication defined by $(fg)(x) = f(x)g(x)$ and involution defined by $f^*(x) = \overline{f(x)}$. This is a commutative C^*-algebra and the function $1 : X \to \mathbb{C}$ given by $x \mapsto 1$ is a multiplicative identity. In this section we will prove that any commutative C^*-algebra is isomorphic to a space of this form. To construct this space we consider the non-zero homomorphisms $\chi : A \to \mathbb{C}$, which we call *characters* and define $\hat{A}$ to be the set of all characters on A. In the proof of Theorem 1.1.1 in [1] the convergent series $\sum_{n=0}^{\infty} z^n/n! \in A$ is used to prove that if A is a commutative C^*-algebra, then any character is also a $*$-homomorphism and therefore has norm at most 1 in A^*. But since $\chi(e) = \chi(e \cdot e) = \chi(e)\chi(e)$ amd χ is not the zero map, we conclude that $\chi(e) = 1$, so every character has norm exactly one. Therefore, $\hat{A}$ is a subspace of the unit ball in A^* and thereby inherits the weak* topology. We would like to show that $\hat{A}$ is a compact Hausdorff space in this topology.

First we see that $\hat{A}$ is Hausdorff, for if we have distinct points $\chi_1, \chi_2 \in \hat{A}$, we can choose $x \in A$ such that $\chi_1(x) \neq \chi_2(x)$. If we let $2\delta = |\chi_1(x) - \chi_2(x)|$, then the weak* open sets $V_{\delta,\chi_1,x}$ and $V_{\delta,\chi_2,x}$ are disjoint. Moreover, $\hat{A}$ is weak* closed: If $\varphi \in (\hat{A})^c$ then either φ is not multiplicative or it is the zero map. Suppose φ is the zero homomorphism, then the weak* open set $V_{1/2,\varphi,e} = \{\psi \in A^* \mid |\psi(e)| < 1/2\}$ is contained in the complement, since $\chi(e) = 1$ for all $\chi \in \hat{A}$. Similarly, if φ is not multiplicative then we can choose a neighbourhood U of $\varphi(a)\varphi(b)$ which does not contain $\varphi(ab)$. By the continuity of multiplication in $\mathbb{C}$ we can choose balls $B_{\delta_1}(\varphi(a))$ and $B_{\delta_2}(\varphi(b))$ such that for any $x \in B_{\delta_1}(\varphi(a))$ and $y \in B_{\delta_2}(\varphi(b))$, xy lies in U. Taking $\delta = \min\{\delta_1, \delta_2\}$ we see that if $\psi \in V_{\delta,\varphi,a,b}$ then $\psi(a) \in B_{\delta_1}(\varphi(a))$ and $\psi(b) \in B_{\delta_2}(\varphi(b))$ so $\psi(a)\psi(b) \in U$. Therefore, $\psi(ab) \neq \psi(a)\psi(b)$ and $\hat{A}$ is closed. The unit ball in A^* is compact by the Banach-Alaoglu theorem, so $\hat{A}$ is compact as a closed subset of a compact set. Therefore, $\hat{A}$ is compact Hausdorff and so $C(\hat{A})$ is a C^*-algebra.

Given an element $x \in A$ we can define an element φ_x of $C(\hat{A})$ by $\varphi_x(\chi) = \chi(x)$. Pointwise evaluation maps are always continuous in the weak* topology, so we can define the 'Gelfand map' $\Gamma : A \to C(\hat{A})$ by $x \mapsto \varphi_x$. To prove that this is an isomorphism we will require the following results.

Theorem 2.1. *(Gelfand-Mazur Theorem) Let A be a unital Banach algebra which is also a division algebra (every non-zero element is invertible). Then $A = \mathbb{C}e$.*

Proof. Firstly we note that $\mathbb{C}e$ always embeds in A. However, by Proposition 2.1.3 (iii) of [3] the spectrum of an element in a Banach algebra is non-empty. Therefore, given $x \in X$ we can choose $\lambda \in \sigma(x)$, so $\lambda e - x$ is not invertible. But A is a division algebra, so $\lambda e - x = 0$, or $x = \lambda e$. That is, every element of A belongs to $\mathbb{C}e$. $\square$

A *left ideal* in A is a linear subspace I of A such that whenever $x \in I$ we have that $rx \in I$. An *ideal* is a two-sided ideal. We say an ideal is maximal if it is not the whole algebra and is not strictly contained in any other ideal. Following [3], by considering the ideal structure in A we can prove the following result:

Lemma 2.2. *Let A be commutative and let $x \in A$, then $\sigma(x) = \{\chi(x) \mid \chi \in \hat{A}\}$.*

Proof. First we will prove that the maximal ideals in A are of the form $M_\psi = \ker \psi$ for $\psi \in \hat{A}$. Let I be a maximal ideal, then by Proposition 2.2.1 of [3] I is also closed. Therefore, A/I is a Banach algebra, but since I is maximal, a result for commutative rings implies that A/I is also field and therefore a division algebra. The Gelfand-Mazur theorem implies that $A/I \cong \mathbb{C}$, so the quotient map $\pi : A \to A/I \cong \mathbb{C}$ is a character, so $I = M_\pi$. We note that $(A^\times)^c$ is a two-sided ideal, because if x is not invertible then neither is rx or xr for any $r \in A$. Therefore, if $\lambda e - x$ is not invertible it must be contained in some maximal ideal. The inverse is also true because if $\lambda e - x$ is invertible then there exists y with $(\lambda e - x)y = e$. Then for any $\psi \in \hat{A}$ we have $(\lambda - \psi(x))\psi(y) = 1$, so $\psi(x) \neq \lambda$ for all $\psi \in \hat{A}$ and so $\lambda e - x$ is not contained in any maximal ideal. Thus,

$$\lambda \in \sigma(x) \iff \lambda e - x \notin A^{\times} \iff \exists \chi \in \hat{A},\ \lambda e - x \in M_{\chi} \iff \exists \chi \in \hat{A},\ \chi(\lambda e - x) = 0$$

$$\iff \exists \chi,\ \varphi_x(\chi) = \chi(x) = \lambda \iff \lambda \in \operatorname{im} \varphi_x.$$

Hence $\sigma(x) = \operatorname{im} \varphi_x = \{\chi(x) \mid \chi \in \hat{A}\}$. This implies that $\|\Gamma(x)\| = \sup_{\chi \in \hat{A}} |\chi(x)| = \sup\{|\lambda| \mid \lambda \in \sigma(x)\} = \nu(x)$. □

We say that a subset F of $C(X)$ *separates points* if for distinct points $x, y \in X$ we can choose $f \in F$ such that $f(x) \neq f(y)$. We then have the following standard result which is stated in [5].

Theorem 2.3. *(Stone-Weierstrass Theorem) Let X be compact Hausdorff and let $A \subset C(X)$ be a subalgebra. If A is a closed under the $(*)$ operation defined above, contains $\mathbb{C}$ and separates points, then A is dense in $C(X)$.*

Now we are in a position to prove the main result of this section.

Theorem 2.4. *The Gelfand map $\Gamma : A \to C(\hat{A})$ given by $x \mapsto \varphi_x$ is an isomorphism of C^*-algebras.*

Proof. The map is clearly a homomorphism because any character is a homomorphism. It is a $*$-homomorphism because any character is a $*$-homomorphism. Next we see that Γ is an isometry. By Lemma 2.2 we know that $\|\Gamma(x^*x)\| = \nu(x^*x)$ and since x^*x is self-adjoint we have, $\|x^*x\| = \nu(x^*x)$. Therefore,

$$\|x\|^2 = \|x^*x\| = \nu(x^*x) = \|\Gamma(x^*x)\| = \|\Gamma(x)\|^2 .$$

Hence, Γ is an isometry and therefore a closed map, sending A isometrically onto a closed subspace of $C(\hat{A})$. To prove that Γ is surjective we note that $\Gamma(A)$ is closed under conjugation since $\Gamma(x^*) = \overline{\Gamma(x)}$. We also know that $\Gamma(A)$ contains the constant functions because $\Gamma(\lambda e)$ is the map $\varphi_{\lambda e}(\chi) = \lambda\chi(e) = \lambda$. Finally, $\Gamma(A)$ separates points, because given $\chi_1 \neq \chi_2$ there is some $x \in A$ such that $\chi_1(x) \neq \chi_2(x)$, so $\Gamma(x)\chi_1 \neq \Gamma(x)\chi_2$. Therefore, the Stone-Weierstrass Theorem implies that $\Gamma(A)$ is dense in $C(X)$. Because $\Gamma(A)$ is closed this means that Γ is surjective. □

Here we remark that if A was not assumed to be unital, then the same result holds, except that $\hat{A}$ is locally compact rather than compact, so we obtain an isomorphism to the space $C_0(\hat{A})$ of continuous functions which vanish at ∞. Moreover, we can also construct the space of characters for any Banach algebra, but we do not have this isomorphism unless we are in the C^*-algebra case.

This isomorphism means that the contravariant functor $X \mapsto C(X)$ is a duality between the category of compact Hausdorff spaces with continuous maps and the space of commutative, unital C^*-algebras with non-zero $*$-homomorphisms, whereby X is homeomorphic to Y if and only if $C(X) \cong C(Y)$ [2]. Thus, a compact Hausdorff space is determined up to isomorphism by its algebra of functions. This observation marks the starting point for the field of noncommutative geometry, in which we seek to apply the techniques used for studying compact spaces to arbitrary algebras. For more details see [2].

Often we will leverage the fact that the C^*-algebra generated by a normal element x (x satisfying $x^*x = xx^*$) and the identity is always commutative and unital, so we can apply Theorem 2.4. For instance, this implies that if x is normal then its norm is equal to its spectral radius. Moreover, by Proposition 1.4.3 of [4], the map $\hat{B} \to \sigma_A(x)$ given by $\chi \mapsto \chi(x)$ is a homeomorphism, so the Gelfand map gives an isomorphism $C(\sigma(x)) \to C(\hat{B}) \cong B$. Therefore, given a continuous function $f \in C(\sigma(x))$ we obtain an element in A which we denote $f(x)$. This is called the *continuous functional calculus* for normal elements in a C^*-algebra.

In particular this allows us to prove that any injective $*$-homomorphism is isometric. By the proof of Proposition 1.1 we note that it is enough to show that $\sigma(\pi(x)) \supset \sigma(x)$ for any x self-adjoint. Suppose to the contrary that $\sigma(\pi(x)) \subsetneq \sigma(x)$ then we can choose a non-zero $f : \sigma(x) \to \mathbb{R}$ which is supported on $\sigma(x) \setminus \sigma(\pi(x))$. On page 12 of [1] it is shown that $f(\pi(x)) = \pi(f(x))$. Therefore, because f vanishes on $\sigma(\pi(x))$ we conclude that $\pi(f(x)) = f(\pi(x)) = 0$. Since π is injective this means that $f(x) = 0$. But by Theorem 2.4, $\|f(x)\| = \|f\| > 0$ since f is not the zero map. This is a contradiction, so $\sigma(\pi(x)) = \sigma(x)$.

3 The GNS Construction

Next we turn out attention to noncommutative C^*-algebras. We will begin by describing a procedure for producing representations of a C^*-algebra. We define a representation of a C^*-algebra A on a Hilbert space H to be a $*$-homomorphism $\pi : A \to \mathcal{B}(H)$. We say that a representation is *faithful* if it is injective. We call a linear functional $f \in A^*$ *positive* if $f(z^*z) \geq 0$ for all $z \in A$. Given a positive linear functional f, the form $(\cdot,\cdot) : A \times A \to \mathbb{C}$ defined by $(x, y) = f(y^*x)$ is sesquilinear and satisfies $(x, x) \geq 0$ for all $x \in A$. Therefore, by the Cauchy-Schwarz inequality[2] for any $x, y \in A$ we have,

$$|f(y^*x)|^2 = |(x, y)|^2 \leq (x, x)(y, y) = f(x^*x)f(y^*y). \tag{1}$$

Moreover, for any $x, y, z \in A$ we have,

$$(xy, z) = f(z^*xy) = f((x^*z)^*y) = (y, x^*z). \tag{2}$$

In Proposition 1.6.2 of [1] the Cauchy-Schwarz inequality and the functional calculus are used to prove that every positive linear functional satisfies $f(e) = \|f\|$. We say that a positive linear functional f is a *state* if $f(e) = \|f\| = 1$.

Theorem 3.1. *(The GNS Construction) For any state $f \in A^*$, there exists a Hilbert space H, a representation $\pi : A \to \mathcal{B}(H)$ and a vector $\xi \in H$ such that for all $x \in A$, $f(x) = \langle \pi(x)\xi, \xi\rangle$.*

Proof. We know that given a positive linear functional, we can construct a sesquilinear form such that $(x, x) \geq 0$ for all $x \in A$. The idea of this proof is to construct a space on which this form induces an inner-product, by quotienting out by the vector subspace $I = \{z \in A \mid (z, z) = 0\}$. To do so we require that I is a left-ideal in A. Let $z \in I$ and $x \in A$, then by equations 2 and 1 we have,

$$|(xz, xz)|^2 = |(x^*xz, z)|^2 \leq (x^*xz, x^*xz)(z, z) = 0$$

Hence $(xz, xz) = 0$ and so $xz \in I$ and I is a left ideal in A. This allows us to define the quotient algebra A/I on which the form $(\cdot,\cdot)$ descends to an inner-product $\langle\cdot,\cdot\rangle$. We define $H = \overline{A/I}$ to be the completion of A/I under this inner-product.

Given $x \in A$ we can define a map $\pi(x) : A/I \to A/I$ by $(y + I) \mapsto (xy + I)$. This map is well-defined because I is a left ideal of A and it is linear because of the way addition is defined in a quotient ring. To prove that this is bounded we let $\eta = y + I \in A/I$ and consider the functional $g \in A^*$ defined by $g(z) = f(y^*zy)$. Then g is positive, because $g(z^*z) = f((zy)^*zy) \geq 0$, so $\|g\| = g(e) = f(y^*y)$. Therefore,

$$\|\pi(x)\eta\|^2 = \|xy + I\|^2 = f(y^*x^*xy) = g(x^*x) \leq \|g\|\|x^*x\| = f(y^*y)\|x\|^2 = \|x\|^2\|\eta\|^2.$$

Therefore, $\pi(x)$ is bounded on A/I, so it extends to an element of $\mathcal{B}(H)$. The fact that π is an algebra homomorphism is clear, and it is a $*$-homomorphism because for any $y+I, z+I \in A/I$, equation 2 gives,

$$\langle\pi(x)(y + I), z + I\rangle = \langle xy + I, z + I\rangle = (xy, z) = (y, x^*z) = \langle y + I, \pi(x^*)(z + I)\rangle,$$

so $\pi(x)^* = \pi(x^*)$. Finally we observe that if we set $\xi = e + N$ then for any $x \in A$ we have,

$$\langle\pi(x)\xi, \xi\rangle = f(e^*(xe)) = f(x).$$

□

4 The Gelfand-Naimark Theorem

Now we would like to use this construction to prove that every unital C^*-algebra is isomorphic to a sub C^*-algebra of $\mathcal{B}(H)$ for some Hilbert space H. We have already shown how to construct a representation given a state, so we now need to show that there are enough states for us to construct a faithful representation of A on some Hilbert space. We know that the spectrum of any self-adjoint element is real. If the spectrum is also non-negative then we have the following result.

[2] Note that to prove the Cauchy-Schwarz inequality we only require the fact that $\langle x, x\rangle \geq 0$, so the result still holds for positive semi-definite forms.

Proposition 4.1. *For $x \in A$ self-adjoint, the following are equivalent,*

1. $\sigma(x) \geq 0$.
2. $x = y^*y$ *for some* $y \in A$.
3. $x = y^2$ *for some* $y \in A$ *self-adjoint.*

Proof. (1) $\implies$ (3): If $\sigma(x) \geq 0$, then we have a continuous function $f : \sigma(x) \to \mathbb{R}$ given by $f(t) = \sqrt{t}$. Therefore, the functional calculus allows us to define an element $y = f(x) \in A$ which satisfies $x = y^2$. The implication (3) $\implies$ (1) comes from the fact that $\sigma(x) \subset \mathbb{R}$, so $\sigma(x) = \sigma(y^2) = (\sigma(y))^2 \geq 0$. Moreover, (3) $\implies$ (2) is true because $y^2 = y^*y$ for y self-adjoint.

The implication (2) $\implies$ (3) is more technical so we will only sketch the proof, the details of which can be found in Section 1.7 of [1]. We begin by using the functional calculus to construct self-adjoint elements u and v such that $z^*z = u^2 - v^2$ and $uv = 0$. Then

$$(zv)^*zv = vz^*zv = v(u^2 - v^2)v = vu^2v - v^4 = -v^4.$$

Therefore, $\sigma((zv)^*zv) \leq 0$, which it can be shown implies that $zv = 0$. Therefore, $-v^4 = 0$, so because v is self-adjoint, $\|v\|^4 = \|v^4\| = 0$ and so $v = 0$. Thus, $z^*z = u^2$. ☐

Proposition 4.2. *Let $f \in A^*$ satisfy $\|f\| = f(e) = 1$. Then f is a state.*

Proof. First we show that for all $y \in A$ normal, $f(y)$ belongs to the closed convex hull K of $\sigma(y)$. Suppose not, then since K is the intersection of all convex sets containing $\sigma(y)$, we can choose a disk $B_R(z)$ which contains $\sigma(y)$ but not $f(y)$. This implies that $\sigma(y - ze)$ is contained in a disk of radius R. A simple calculation verifies that $y - ze$ is also normal so its norm agrees with its spectral radius and $\|y - ze\| = \nu(y - ze) \leq R$. Therefore, $|f(y - ze)| \leq \|f\|\|y - ze\| \leq R$. However, $|f(y - ze)| = |f(y) - z| > R$, a contradiction. By Proposition 4.1 we know that $\sigma(z^*z) \geq 0$, so the closed convex hull of this set is contained in the non-negative reals. Therefore, $f(z^*z) \geq 0$, so f is positive and is therefore a state. ☐

This allows us to show that given any self-adjoint element we can always find a state which does not vanish at that element.

Proposition 4.3. *Let $x \in A$ be self-adjoint. Then there exists a state f with $|f(x)| = \|x\|$.*

Proof. Theorem 2.1 applied to the commutative C^*-algebra generated by x and e gives that,

$$\|x\| = \|\Gamma(x)\| = \sup_{\chi \in \hat{B}} \chi(x)$$

But $\hat{B}$ is compact, so this supremum is attained, yielding a character $\chi \in \hat{B} \subset B^*$ with $\|\chi(x)\| = \|x\|$. We know that B is a linear subspace of A, so by Hahn-Banach, we can extend χ to a norm 1 functional $f \in A^*$ with $|f(x)| = \|x\|$. Since χ is a character we have $f(e) = \chi(e) = 1$, so because $\|f\| = f(e)$ for positive linear functionals, f is a state by Proposition 4.2. ☐

Given any non-zero $x \in A$, x^*x is self-adjoint so the previous result gives us a state f such that $|f(x^*x)| = \|x^*x\| = \|x\|^2$. Applying the GNS construction to f yields a representation $\pi_x : A \to \mathcal{B}(H_x)$ for some Hilbert space H_x. This allows us to prove our main result:

Theorem 4.4. *Let A be a C^*-algebra. Then there is a faithful representation of A on H for some Hilbert space H.*

Proof. Let $H = \bigoplus_{x \neq 0} H_x$ and consider the map $\pi = \bigoplus_{x \neq 0} \pi_x : A \to \mathcal{B}(H)$ defined by $\pi(y)(\bigoplus v_x) = \bigoplus \pi_x(y)v_x$. Given any non-zero $x \in A$, by Proposition 4.3 we have,

$$\|\pi_x(x)\xi\|^2 = \langle \pi_x(x)\xi, \pi_x(x)\xi \rangle = \langle \pi(x^*x)\xi, \xi \rangle = f(x^*x) = \|x\|^2 > 0.$$

If $\pi(y) = 0$, then $\pi_x(y) = 0$ for all $x \neq 0$. But by the previous inequality, if $y \neq 0$ then $\pi_y(y) \neq 0$, so we must have $y = 0$, which implies that π is injective. As shown above, any injective $*$-homomorphism is an isometry, so we have an isometric $*$-isomorphism of A onto a sub-algebra of $\mathcal{B}(H)$. ☐

We remark that the GNS construction is valid for any Banach algebra, but in the context of C^*-algebras we are able to guarantee that there are enough states for us to produce a faithful representation.

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