The Algebraic K-Theory of Finite Fields

Callum Sutton

19 June 2020

In this report we define the *algebraic K-groups* of a ring and compute these groups in the case of a finite field $\mathbb{F}_q$, where $q = p^d$. In Section 1 we give Quillen's definition of algebraic K-theory in terms of the $(+)$-construction, commenting on how this definition relates to other notions of algebraic K-theory. The remainder of the report is dedicated to giving a survey of Quillen's proof in [Qui72] that the K-theory groups of $\mathbb{F}_q$ are given by:

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 2i, n > 0 \\ \mathbb{Z}/(q^i - 1) & n = 2i - 1, n > 0. \end{cases}$$

In Section 2 we compute the homotopy groups of $F\Psi^q$, where $F\Psi^q$ is the homotopy fibre of the map $(1 - \psi^q) : BU \to BU$. Here ψ^q represents the qth Adams operation on reduced (topological) K-theory. In Section 3 the *Brauer lift* of a modular representation is used to define a map $\alpha : \mathrm{BGL}(\mathbb{F}_q) \to F\Psi^q$ and hence a map $\alpha^+ : \mathrm{BGL}(\mathbb{F}_q)^+ \to F\Psi^q$. In Section 4 we show that this map is a homotopy equivalence by showing that it induces an isomorphism on rational and mod l homology for all primes l. In Sections 4.1-4.3 we show that both spaces have trivial rational and mod p homology. In Sections 4.4-4.5 we consider mod l cohomology for l a prime distinct from p, finding generators for $H^*(F\Psi^q; \mathbb{F}_l)$ and determining its ring structure. Cohomology classes in $H^*(F\Psi^q; \mathbb{F}_l)$ give rise to cohomology classes for $\mathrm{BGL}_n(\mathbb{F}_q)$ and in Section 4.5 we describe $H^*(\mathrm{BGL}_n(\mathbb{F}_q); \mathbb{F}_l)$ in terms of these classes. An argument involving limits in Section 4.6 allows us to conclude that α^* is an isomorphism on all cohomology groups.

Contents

1 Introduction to algebraic K-theory

In this section we will define the algebraic K-groups of an associative, unital ring R. Before we move on to Quillen's definition of the higher algebraic K-groups, we will discuss the definitions of the groups K_0 and K_1 which already existed prior to his work (note that the group K_2 had also been defined, but we will not discuss this original definition). The beginnings of algebraic K-theory were in algebraic geometry: In the course of his work on the Grothendieck-Riemann-Roch theorem Grothendieck defined the ring $K_0(X)$ of a smooth, quasi-projective scheme X in terms of the coherent sheaves on X (see [BS58]). In the case of a ring R we define $K_0(R)$ to be the Grothendieck group of the monoid of finitely generated projective R-modules. If $R = F$ is a field, then the finitely generated projective F-modules are just F-vector spaces, which are characterised by their dimension. Therefore, we have $K_0(F) \cong \mathrm{Gr}(\mathbb{N}, +) \cong \mathbb{Z}$ for every field F.

Next we turn our attention to $K_1(R)$, the so-called *Whitehead group* of R. To define this group we begin with the general linear group $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices with coefficients in R. Given $g \in \mathrm{GL}_n(R)$, the matrix $g \oplus (1) = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ belongs to $\mathrm{GL}_{n+1}(R)$, giving us a map $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$. Putting these maps together we obtain a diagram:

$$\mathrm{GL}_1(R) \hookrightarrow \mathrm{GL}_2(R) \hookrightarrow \mathrm{GL}_3(R) \hookrightarrow \cdots .$$

We define $\mathrm{GL}(R)$ to be the direct limit of this diagram. This can be thought of as the increasing union $\bigcup_{n\geq 1} \mathrm{GL}_n(R)$. That is, we identify two matrices if they can be made equal by adding 1s to the diagonal. We define $K_1(R)$ to be the abelianisation of this group: $K_1(R) := \mathrm{GL}(R)^{ab}$. To make this more explicit we will determine the commutator subgroup $[\mathrm{GL}(R), \mathrm{GL}(R)]$ of this group. Let δ_{ij} be the matrix with (i, j)th entry 1 and the remaining entries 0. We say that the matrices $e_{ij}(r) = I + r\delta_{ij}$ for $i \neq j$ are *elementary* and we define $E_n(R)$ to be the subgroup generated by the elementary matrices. Our map $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$ sends $E_n(R)$ into $E_{n+1}(R)$, so in the limit we obtain a subgroup $E(R)$ of $\mathrm{GL}(R)$. We will show that $[\mathrm{GL}(R), \mathrm{GL}(R)] = E(R)$.

Lemma 1.1 (Lemma 1.3.2 of Chapter 3 of [Wei13])**.** *The subgroup $E(R)$ of $\mathrm{GL}(R)$ is perfect: we have $[E(R), E(R)] = E(R)$.*

Proof. Let $e_{ij}(r) \in E(R)$. We can choose a representative of this matrix with dimension at least three and so we can choose $k \neq i, j$. The result follows from the matrix identity,

$$e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$$

□

Lemma 1.2 (Whitehead [Whi50])**.** *The commutator subgroup of $\mathrm{GL}(R)$ is $E(R)$.*

Proof. We will follow the proof given in Lemma 3.1 of [Mil71]. Since $E(R)$ is perfect we know that $E(R) \subset [\mathrm{GL}(R), \mathrm{GL}(R)]$. To prove the reverse inclusion we observe that for $g \in \mathrm{GL}_n(R)$ we have the identity,

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ g^{-1} & I \end{pmatrix} \begin{pmatrix} I & I-g \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} I & I-g^{-1} \\ 0 & I \end{pmatrix}.$$

It can be shown directly that each of these factors is a product of elementary matrices. Let $g, h \in \mathrm{GL}(R)$, then (after adding 1s along the diagonal so that it makes sense to multiply these matrices) we have,

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & 0 \\ 0 & hg \end{pmatrix} = \begin{pmatrix} ghg^{-1}h^{-1} & 0 \\ 0 & I \end{pmatrix} = [g, h].$$

Now we note that the matrices on the left all belong to $E(R)$, so we've shown that $[g, h] \in E(R)$. Therefore, $[\mathrm{GL}(R), \mathrm{GL}(R)] = E(R)$. □

This result allows us to write $K_1(R) = \mathrm{GL}(R)/E(R)$. In particular, if $R = F$ is a field then the subgroup $E(F)$ generated by the elementary matrices is all of $SL(F)$ (because we can row reduce a determinant 1 matrix to the identity using these elementary matrices), so $K_1(F) = \mathrm{GL}(F)/E(F) = \mathrm{GL}(F)/SL(F) \cong F^\times$. This means that $K_1(\mathbb{F}_q) \cong \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)$.

We observe that $H_1(\mathrm{BGL}(R); \mathbb{Z}) \cong \pi_1(\mathrm{BGL}(R))^{ab} = \mathrm{GL}(R)^{ab} = K_1(R)$. This indicates that we might be able to define higher K-groups using the higher homotopy or homology groups. However, the higher homotopy groups of $\mathrm{BGL}(R)$ all vanish, meaning that it would not be useful to define $K_n(R) = \pi_n(\mathrm{BGL}(R))$. Therefore, we would like to define a topological space which approximates the homology of $BGL(R)$ but whose higher homotopy groups do not vanish. To define such a space, we first need to understand *acyclic maps*. These are maps $f : X \to Y$ such that the homotopy fibre $F(f)$ of f has the homology of a point. We will define the homotopy fibre in Section 2 but for now note that $f : X \to X^+$ being acyclic means that f_* is an isomorphism on all homology groups (see Lemma 1.6 of Chapter 4 of [Wei13]) and that $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective (by the long exact sequence of a fibration).

Proposition 1.3 (Theorem 2.1 of [Ger73]). *Let X be a based CW complex with E a perfect subgroup of $\pi_1 X$. Then there is an acyclic map $f : X \to X^+$ such that E is the kernel of the surjective map $f_* : \pi_1 X \to \pi_1 X^+$.*

Proof. The idea of this is to attach two and three-cells to the complex to kill the perfect subgroup, for details see Proposition 4.40 of [Hat02]. $\square$

If we apply this construction to the perfect subgroup $E(R) \subset \mathrm{GL}(R) = \pi_1(\mathrm{BGL}(R))$ we obtain a space $\mathrm{BGL}(R)^+$ and a map $f : \mathrm{BGL}(R) \to \mathrm{BGL}(R)^+$ such that

$$\pi_1(\mathrm{BGL}(R)^+) = \pi_1(\mathrm{BGL}(R))/E(R) = GL(R)/E(R) = K_1(R).$$

Thus, we've found a topological space which computes $K_1(R)$. Generalising this to higher values of n leads to the following definition.

Definition 1.1. Let R be a ring, then we define $K_n(R) = \pi_n(\mathrm{BGL}(R)^+)$ for $n \geq 1$.

From this definition it's not obvious that K_n is a functor but in fact this is true. Since $\mathrm{BGL}(R)^+$ is path-connected, if we define $K_0(R)$ by $\pi_0(\mathrm{BGL}(R)^+)$, then we just get $K_0(R) = 0$ for any ring R. This conflicts with the established definition of $K_0(R)$, so the definition only works for $n \geq 1$. This can be fixed by defining $K_n(R) = \pi_n(\mathrm{BGL}(R)^+ \times K_0(R))$ but because we have already computed $K_0(\mathbb{F}_q)$ we will just work with the definition $K_i(R) = \pi_i \, \mathrm{BGL}(R)^+$ which works for $i > 0$. Note that this definition also agrees with the original definition of $K_2(R)$ using the Steinberg group, for details see Corollary 1.7.1 of [Wei13].

Before continuing, we will make a brief historical remark on this definition of the higher algebraic K-groups, since it seems somewhat ad-hoc to define invariants of a ring using the homotopy groups of a topological space. In his work on the Adams conjecture in [Qui71] Quillen constructed maps $\mathrm{BGL}_n(\mathbb{F}_q) \to F\Psi^q$ (the space $F\Psi^q$ is the homotopy fibre of the map $(1 - \psi^q)$, its construction and the construction of this map will be discussed in greater detail below) and saw that in the limit this map became an isomorphism on all homology groups. However, the map $\mathrm{BGL}(\mathbb{F}_q) \to F\Psi^q$ is not a homotopy equivalence because the spaces have different fundamental groups. The (+)-construction means that we can replace $\mathrm{BGL}(\mathbb{F}_q)$ with a space $\mathrm{BGL}(\mathbb{F}_q)^+$ so that the induced map becomes a homotopy equivalence. Quillen saw that if algebraic K-theory was defined in this way, then this would allow him to infer the K-groups of $\mathbb{F}_q$ from the homotopy groups of $F\Psi^q$. Therefore, in some sense this definition was made specifically so that we could determine the K-groups of $\mathbb{F}_q$. Quillen later gave a more general definition of K-groups in terms of the *Q-construction* introduced in [Qui73] and in [Gra76] it was shown by Grayson that the K-groups computed using either method agree (whenever both make sense). This means that the calculation of $K_n(\mathbb{F}_q)$ detailed in this report holds for more modern definitions of K-theory.

To finish this section, we will note some properties of the space $\mathrm{BGL}(R)^+$. First we make note of the universal property of $\mathrm{BGL}(R)^+$.

Proposition 1.4 (Universal property of $\mathrm{BGL}(R)^+$). *The map $f : \mathrm{BGL}(R) \to \mathrm{BGL}(R)^+$ induces a bijection,*

$$f^* : [\mathrm{BGL}(R)^+, Z] \xrightarrow{\sim} [\mathrm{BGL}(R), Z]$$

for any space Z such that the fundamental group of each component of Z has no non-trivial perfect subgroups.

Proof. This result is stated on page 583 of [Qui72]. $\square$

Proposition 1.5. *The space $\mathrm{BGL}(R)^+$ is an H-space.*

Proof. The direct sum of matrices induces a map $\mathrm{GL}_n(R) \times \mathrm{GL}_m(R) \to \mathrm{GL}_{m+n}(R) \subset \mathrm{GL}(R)$, giving us a map $\mathrm{BGL}_m(R) \times \mathrm{BGL}_m(R) \to \mathrm{BGL}(R) \to \mathrm{BGL}(R)^+$ (note that $B(G \times H) \simeq BG \times BH$ because $\pi_1(X \times Y) \cong \pi_1 X \times \pi_1 Y$). Note that we can also think of $\mathrm{BGL}(R)$ as the space $\mathrm{BGL}(R) = \varinjlim \mathrm{BGL}_n(R)$, so this gives us a map $\mathrm{BGL}(R) \times \mathrm{BGL}(R) \to \mathrm{BGL}(R)^+$. But the fundamental group of $\mathrm{BGL}(R)^+$ is $K_1(R)$ which is abelian and therefore has no non-trivial perfect subgroup. Therefore, this map factors through the map $\mathrm{BGL}(R) \times \mathrm{BGL}(R) \to (\mathrm{BGL}(R) \times \mathrm{BGL}(R))^+ \simeq \mathrm{BGL}(R)^+ \times \mathrm{BGL}(R)^+$. Here we've used that $(X \times Y)^+ \cong X^+ \times Y^+$ because if $X \to X^+$ and $Y \to Y^+$ are acyclic, then $X \times Y \to X^+ \times Y^+$ is as well. Thus, we've produced a map $\mu : \mathrm{BGL}(R)^+ \times \mathrm{BGL}(R)^+ \to \mathrm{BGL}(R)^+$, which gives $\mathrm{BGL}(R)^+$ the structure of an H-space. These maps are shown in the following diagram.

$$\begin{array}{ccc}
\mathrm{BGL}_m(R) \times \mathrm{BGL}_n(R) & \xrightarrow{\oplus} & \mathrm{BGL}_{m+n}(R) \\
\downarrow & & \downarrow \\
\mathrm{BGL}(R) \times \mathrm{BGL}(R) & \longrightarrow & \mathrm{BGL}(R) \\
\downarrow & & \downarrow \\
\mathrm{BGL}(R)^+ \times \mathrm{BGL}(R)^+ & \xrightarrow{\mu} & \mathrm{BGL}(R)^+
\end{array}$$

□

2 The Space $F\Psi^q$

Next we will introduce a certain topological space $F\Psi^q$ whose homotopy groups are easy to compute, with the ultimate goal of proving that $F\Psi^q \simeq \mathrm{BGL}(\mathbb{F}_q)^+$. Recall that we have

$$K(X) = [X, BU \times \mathbb{Z}]$$

for BU the classifying space for complex vector bundles. If X is also assumed to be path-connected then we have

$$\widetilde{K}(X) = [X, BU].$$

In [Ada62] Adams showed that the Adams operations on $\widetilde{K}(-)$ are represented by maps $BU \to BU$ which, by an abuse of notation, we also call ψ^q. Now we define $F\Psi^q$ to be the space of homotopy fixed points of the map $\psi^q : BU \to BU$. This is the pullback of the maps $(\mathrm{id}, \psi^q) : BU \to BU \times BU$ and the map $\Delta : BU^I \to BU \times BU$ which sends $\gamma \mapsto (\gamma(0), \gamma(1))$.

$$\begin{array}{ccc}
F\Psi^q & \xrightarrow{p} & BU^I \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \Delta} \\
BU & \xrightarrow{(\mathrm{id}\,, \psi^q)} & BU \times BU
\end{array} \qquad (*)$$

We know that $F\Psi^q$ is the set of $(x, \gamma) \in BU \times BU^I$ such that γ is a path from x to $\psi^q(x)$. Before proceeding we will need to discuss the notion of a fibration.

Definition 2.1 (Page 375 of [Hat02])**.** A map $p : E \to B$ is a *fibration* if for any space X, any homotopy $g_t : X \to B$ and any $\widetilde{g}_0 : X \to E$ lifting g_0, we can lift g_t to a homotopy $\widetilde{g}_t : X \to E$. That is, we can always construct the dotted map in the following diagram,

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\widetilde{g}_0"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "g"] \arrow[ur, dashed, "\widetilde{g}"] & B
\end{tikzcd}$$

If $F = p^{-1}(*)$ then we will denote this fibration by $F \to E \to B$.

The important result we will need concerning fibrations is the following:

Theorem 2.1 (Theorem 4.41 of [Hat02])**.** *Let $F \to E \to B$ be a fibration, then we have a long exact sequence of homotopy groups,*

$$\cdots \to \pi_{n+1}B \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1}F \to \cdots .$$

Now given a based map $f : X \to Y$, we can construct an associated fibration so that we can apply the previous lemma. We define the path space P_f of f to be the pullback of f and the map $p : Y^I \to Y$ which sends $\gamma \mapsto \gamma(0)$. Note that we did not require f to be based to construct the path space, although we will need this assumption to construct the homotopy fibre.

$$\begin{array}{ccc}
P(f) & \longrightarrow & Y^I \\
\downarrow & & \downarrow{\scriptstyle p} \\
X & \xrightarrow{f} & Y
\end{array}$$

This is the space $Pf = \{(x,\gamma) \in X \times Y^I \mid f(x) = \gamma(0)\}$. The map $Pf \to Y$ is a fibration by Proposition 4.64 of [Hat02] and the fibre of the basepoint $y_0 \in Y$ is $F(f) = \{(x,\gamma) \in X \times Y^I \mid f(x) = \gamma(0), \gamma(1) = y_0\}$. This space is called the *homotopy fibre* of f. Now we note that $P(f)$ has a subspace $\{(x, c_{f(x)})\} \simeq X$, and $P(f)$ deformation retracts onto this subspace because we can shrink paths back to the constant path at the initial point. Therefore we have a long exact sequence

$$\cdots \to \pi_{n+1}Y \to \pi_n F(f) \to \pi_n X \to \pi_n Y \to \cdots .$$

If we define the *path space* PY of Y to be the space of based maps $I \to Y$, where the basepoint of I is 1 then we also could have defined Ff to be the pullback $X \times_Y PY$ of the maps $f : X \to Y$ and $p_1 : PY \to Y$, which sends $\gamma \mapsto \gamma(0)$. The fact that these definitions are equivalent is immediate from writing both spaces out as sets. Note that PX is the path space of the inclusion $\{x_0\} \hookrightarrow X$, so the projection $PX \to X$ is a fibration. The inverse image of the basepoint is

$$\{x_0\} \times_X PX \simeq \{\gamma \mid \gamma(0) = \gamma(1) = x_0\} = \Omega X,$$

the loop space of X, giving us the *path space fibration* $\Omega X \to PX \to X$. To sum up the discussion of fibrations, we observe that if we have a based map $f : X \to Y$ then we have the following cartesian square. Because pullbacks of fibrations are fibrations (see page 49 of [May99]), both vertical maps are fibrations.

$$\begin{tikzcd}
F(f) \arrow[r] \arrow[d] & PY \arrow[d, "p_1"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Now we can return to the space $F\Psi^q$. We want to show that $F\Psi^q$ is the homotopy fibre of the map $BU \xrightarrow{1-\psi^q} BU$. We can extend our original diagram to the right using the map $d : BU \times BU \to BU$ which represents the difference operation $(\alpha, \beta) \mapsto \alpha - \beta$ on K-theory.

$$\begin{tikzcd}
F\Psi^q \arrow[r, "p"] \arrow[d, "\phi"] & BU^I \arrow[r, "m"] \arrow[d, "\Delta"] & P(BU) \arrow[d, "p_1"] \\
BU \arrow[r, "(\mathrm{id}, \psi^q)"] & BU \times BU \arrow[r, "d"] & BU
\end{tikzcd} \qquad (**)$$

The map $BU^I \to P(BU)$ sends γ to a path from the basepoint to $d(\gamma(0), \gamma(1))$. Note that the vertical map on the right is the pathspace fibration, with fibre ΩBU. The vertical map in the middle is also a fibration, to prove this we need to show that given the following square we can lift the homotopy h to a homotopy $\widetilde{h}$.

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "i_0"] & BU^I \arrow[d] \\
Y \times I \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & BU \times BU
\end{tikzcd}$$

Let $(y,t) \in Y \times I$. We need to find a path γ from $h_1(y,t)$ to $h_2(y,t)$. We know that $f(y)$ is a path from $h_1(y,0)$ to $h_1(y,0)$. Now we also have a path $\left(h_1|_{y\times[0,t]}\right)^{-1}$ from $h_1(y,t)$ to $h_1(y,0)$. Similarly we have a path $h_1|_{y\times[0,t]}$ from $h_2(y,0)$ to $h_2(y,t)$. Concatenating these three paths gives the required path from $h_1(y,t)$ to $h_2(y,t)$. This defines a map $\widetilde{h} : Y \times I \to BU^I$ which lifts h.

Since the left square is cartesian we also know that $F\Psi^q \to BU$ is a fibration (with fibre ΩBU). The commutative diagram $(**)$ means that we have a map $F\Psi^q \to F(1-\psi^q)$, since $F(1-\psi^q)$ is defined to be the pullback of the two maps in this square. We can use the fact that the maps $F\Psi^q \to BU$ and $F(1-\psi^q) \to BU$ are both fibrations with the same fibre to show that this map $F\Psi^q \to F(1-\psi^q)$ is in fact a homotopy equivalence. That is, $F\Psi^q \simeq \mathrm{hofib}(1-\psi^q)$. Therefore, we can apply the long exact sequence of a fibration:

$$\begin{tikzcd}
\pi_{n+1}BU \arrow[r, "1-\psi^q"] \arrow[d, equal] & \pi_{n+1}BU \arrow[r] \arrow[d, equal] & \pi_n F\Psi^q \arrow[r] & \pi_n BU \arrow[r, "1-\psi^q"] \arrow[d, equal] & \pi_n BU \arrow[d, equal] \\
\widetilde{K}(S^{n+1}) \arrow[r, "1-\psi^q"] & \widetilde{K}(S^{n+1}) & & \widetilde{K}(S^n) \arrow[r, "1-\psi^q"] & \widetilde{K}(S^n)
\end{tikzcd}$$

Now we know that ψ^q is multiplication by q^i on $\widetilde{K}(S^{2i}) \cong \mathbb{Z}$ so when $n = 2i$ this sequence becomes

$$0 \to \pi_{2i} F\Psi^q \to \mathbb{Z} \xrightarrow{q^i} \mathbb{Z} \to ...$$

and so $\pi_{2i} F\Psi^q \cong 0$. Similarly, when $n = 2i + 1$ we have,

$$\mathbb{Z} \xrightarrow{1-q^{i+1}} \mathbb{Z} \to \pi_{2i+1} F\Psi^q \to 0$$

and so $\pi_{2i+1} F\Psi^q \cong \mathbb{Z}/(q^{i+1} - 1)$. To summarise, we have shown for $n \geq 1$ that

$$\pi_n(F\Psi^q) = \begin{cases} 0 & n = 2i \\ \mathbb{Z}/(q^{i+1} - 1) & n = 2i + 1. \end{cases}$$

Our goal is now to show that $F\Psi^q$ is homotopy equivalent to the space $\mathrm{BGL}(\mathbb{F}_q)^+$, which would allow us to conclude that $K_n(\mathbb{F}_q) = \pi_n(F\Psi^q)$. From the previous calculation we see that $\pi_1(F\Psi^q) = K_1(\mathbb{F}_q)$, so it's at least reasonable that such a construction might be possible. Before progressing to the construction of this map we will detail some additional useful properties of the space $F\Psi^q$.

Definition 2.2. A topological space X is *simple* (or *abelian*), if $\pi_1(X)$ is abelian and acts trivially on $\pi_n(X)$ for all n.

In particular, we know that all H-spaces are simple. The map $BU \times BU \to BU$ which represents addition on K-theory lifts to give a map $\nu : F\Psi^q \times F\Psi^q \to F\Psi^q$ (for details see page 555 of [Qui72]). This map gives $F\Psi^q$ the structure of an H-space which gives the space $[X, F\Psi^q]$ a multiplicative structure. The fact that $F\Psi^q$ is simple also follows from the fact that for a fibration $F \to E \to B$, the action $\pi_1(F, x_0) \to \mathrm{Aut}(\pi_n(F, x_0))$ factors through $i_* : \pi_1(F, x_0) \to \pi_1(E, x_0)$ (see Exercise 4.3.10 of Hatcher). Since $\pi_1 BU = 0$ we get that $\pi_1(F\Psi^q)$ acts trivially on $\pi_n(F\Psi^q)$.

Proposition 2.2. *[Lemma 1(ii) of [Qui72]] If X is a space with $[X, U] = 0$, where U is the direct limit of the unitary groups $U(n)$, then*

$$\phi_* : [X, F\Psi^q] \to [X, BU]^{\psi^q}$$

is a bijection which respects the multiplicative structure on $[X, F\Psi^q]$. Here $[X, BU]^{\psi^q} = \{[f] \in [X, BU] \mid \psi^q \circ f \simeq f\}$. That is, $[X, F\Psi^q]$ can be thought of as the subgroup of $\widetilde{K}(X)$ which is invariant under the Adams operation ψ^q.

Proof. By the universal property of the pullback, a map $X \to F\Psi^q$ is specified by a map $f : X \to BU$ and a map $g : X \to BU^I$ such that $(f(x), \psi^q(f(x))) = \Delta(g(x)) = (g(x)(0), g(x)(1))$. If we write $g_t(x) = g(x)(t)$ then g is a homotopy from f to $\psi^q \circ f$. That is elements of $[X, F\Psi^q]$ can be thought of as pairs (f, g_t) where $f : X \to BU$ and g_t is a homotopy from f to $(\psi^q \circ f)$. The map ϕ_* is just projection onto the first element of this pair. Now let $h \in [X, BU]^{\psi^q}$. That is, h is a map with $\psi^q \circ h \simeq h$, so we can choose a homotopy h_t from h to $\psi^q \circ h$, and then $\phi_*(h, h_t) = h$, so ϕ_* is surjective.

Now we want to show that ϕ_* is injective. Suppose $\phi_*(f) = \phi_*(g)$, then $\phi \circ f \simeq \phi \circ g$ as maps $Y \to BU$. Therefore, we have a homotopy $h : X \times I \to BU$ from $\phi \circ f$ to $\phi \circ g$. This information is shown in the following diagram:

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, "i"'] & F\Psi^q \arrow[d, "\phi"] \\ X \times I \arrow[r, "h"] \arrow[ur, dashed, "\widetilde{h}"] & BU \end{tikzcd}$$

Because $F\Psi^q \to BU$ is a fibration we can lift h to $\widetilde{h} : X \times I \to F\Psi^q$. We have $\phi \circ \widetilde{h}(x, 0) = h(x, 0) = \phi(f(x))$ and $\phi \circ \widetilde{h}(x, 1) = \phi(g(x))$. From this we see that h is almost a homotopy from f to g. Note that $U \simeq \Omega BU$ by Bott periodicity, so because $\phi : F\Psi^q \to BU$ is a fibration with fibre ΩBU, we can use the fact that $[X, \Omega BU] = [X, U] = 0$ to obtain a homotopy from f to g. Therefore, ϕ_* is injective. This map respects the multiplicative structure on both spaces, because the H-space map ν came from lifting the addition map on $BU \times BU \to BU$ in the first place. $\square$

3 Constructing a map $BGL(\mathbb{F}_q) \to F\Psi^q$

Our goal now is to find a map $\mathrm{BGL}(\mathbb{F}_q) \to F\Psi^q$. The main ingredient in constructing this map will be the Brauer lift of a modular representation. Since $\pi_1 F\Psi^q = \mathbb{Z}/(q-1)$ is abelian, the universal property of $\mathrm{BGL}(R)^+$ says that such a map will factor through a map $\mathrm{BGL}(R)^+ \to F\Psi^q$.

3.1 λ-rings and Adams operations

In this section we will discuss the definition of Adams operations in the more general context of a λ-ring. These are rings in which we have some notion of exterior powers. This will allow us to define Adams operations on the complex representation ring $R(G)$. More details about λ-rings can be found in Atiyah's paper [AT69] or in Chapter 2 §4 of [Wei13].

Definition 3.1 (Page 257 of [AT69])**.** A λ-ring is a commutative unital ring R with a family of operations $\left(\lambda^k : R \to R\right)_{k\geq 0}$ such that $\lambda^0(x) = 1$, $\lambda^1 = \mathrm{id}$ and

$$\lambda^k(x+y) = \sum_{i=0}^{k} \lambda^i(x)\lambda^{k-i}(y)$$

for all $x, y \in R$.

If we define $W(R) = 1 + tR[\![t]\!]$ to be the group of *big Witt vectors*, with group operation given by multiplication of power series, then the last requirement says exactly that we have a group homorphism $\lambda_t : R^+ \to W(R)$ defined by $\lambda_t(r) = \sum_{i=0}^{\infty} \lambda^i(r)t^i$. The two examples of λ-rings we will be interested in are topological K-theory $K(X)$ of a space X and $R(G)$, the ring of complex representations of a finite group G. The λ-ring structure on $R(G)$ is induced from the exterior power of representations: If V is a G-module, then the exterior power $\Lambda^k(V)$ inherits the structure of a G-module from the action $g \cdot (v_1 \wedge \cdots \wedge v_k) = (g \cdot v_1) \wedge \cdots \wedge (g \cdot v_k)$.

Now $\mathbb{Z}$ has the structure of a λ-ring with λ-structure defined by the map $\lambda_t : \mathbb{Z} \to W(\mathbb{Z})$, $\lambda_t(m) = (1+t)^m$. We say that a λ-ring R is *augmented* if we have a λ-ring homomorphism $\varepsilon : R \to \mathbb{Z}$. In the case of $K(X)$ or $R(G)$ the augmentation is given by taking the dimension of a virtual bundle or a virtual representation. For any R we can give $W(R)$ the structure of a λ-ring by requiring multiplication to satisfy $(1+at) * (1+bt) = 1 + abt$ and requiring the λ-structure to satisfy $\lambda^n(1+at) = 1$ for $n > 1$ (see page 259 of [AT69]). We say that R is a *special λ-ring* if the map $\lambda_t : R \to W(R)$ is a homomorphism of λ-rings. Both $R(G)$ and $K(X)$ are special λ-rings (see Theorem 1.5(i) and 1.5(ii) of [AT69]). In any augmented special λ-ring we are able to define the Adams operation $\psi^k(r)$ to be the coefficient of t^k in the power series,

$$\psi_t(r) = \varepsilon(r) - t\frac{d}{dt}\log \lambda_{-t}(r).$$

These operations are ring homomorphisms which satisfy the properties of Theorem 2.20 of [Hat17], interpreted in the more general setting of a λ-ring. That is, the operations ψ^k are natural, we have $\psi^i \circ \psi^j = \psi^{ij}$ and $\psi^k(l) = l^k$ for 'line elements' l. In the case of $R(G)$ the line elements are one dimensional representations, so the Adams operations on $R(G)$ satisfy $\psi^k(\chi) = \chi^k$ for one-dimensional representations of G. Now for a sum of one-dimensional representations $\chi_1 \oplus \chi_2 \oplus \cdots \oplus \chi_n$ we get,

$$\psi^k(\chi_1 + \cdots + \chi_n) = \psi^k(\chi_1) + \cdots + \psi^k(\chi_n) = \chi_1^k + \cdots \chi_n^k.$$

By Theorem 4.2.3 of Chapter 2 of [Wei13] a splitting principle holds in $R(G)$ because $R(G)$ is a special λ-ring, so to prove statements concerning the Adams operations we can assume that representations split as sums of one-dimensional representations. Therefore, the previous equality shows that for any character χ of G we have

$$(\psi^k\chi)(g) = \chi(g^k)$$

for all $g \in G$. In fact, many authors take this to be the definition of the Adams operations on $R(G)$. Whilst this definition is undoubtedly simpler, we have not used it because it isn't clear how one would show that the natural map $R(G) \to K(BG)$ is a λ-ring homomorphism using this definition.

Now for a finite group G we have a ring homomorphism $R(G) \to K(BG)$ which sends a representation (V, ρ) of G to the vector bundle $EG \times_\rho V = EG \times V/(xg, v) \sim (x, \rho(g)v)$. Since the λ-ring structure on both $K(BG)$ and $R(G)$ is defined in terms of the exterior power on **Vect**, we see that this map is in fact a λ-homomorphism. Therefore, we can take ψ^q-invariants on either side of our map, giving us a homomorphism

$$R(G)^{\psi^q} \to K(BG)^{\psi^q} = [BG, BU \times \mathbb{Z}]^{\psi^q} \to [BG, BU]^{\psi^q}$$

where we have projected onto reduced K-theory. Now, by the loop-suspension adjunction we have,

$$[BG, \Omega BU] = [\Sigma BG, BU] = \widetilde{K}^0(\Sigma BG) = \widetilde{K}^1(BG).$$

Now the Atiyah-Segal completion theorem implies that $K^1(BG) = 0$ (see Corollary 7.3 of [Ati61]). In fact this only requires the Atiyah-Segal completion theorem for finite groups, which was proved by Atiyah in [Ati61]. For details on the more general result for compact Lie groups, see [AS69]. Hence $\widetilde{K}^1(BG) = 0$ also and so Proposition 2.2 tells us that $[BG, BU]^{\psi^q} \cong [BG, F\Psi^q]$. Thus, we've constructed a group homomorphism

$$R(G)^{\psi^q} \to [BG, F\Psi^q].$$

Recall that $[BG, F\Psi^q]$ inherits a group structure from $[BG, BU] = \widetilde{K}(BG)$.

3.2 Brauer lifts

In this section we will use the Brauer lift to define a map $R_{\mathbb{F}_q}(G) \to R(G)$, where $R_{\mathbb{F}_q}(G)$ is the ring of finite dimensional $\mathbb{F}_q$ representations of G. To begin with we fix an embedding $\iota : \overline{\mathbb{F}_q}^\times \to \mathbb{C}^\times$. Given an $\mathbb{F}_q$ representation V of a group G, we get an $\overline{\mathbb{F}_q}$ representation $V \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ of G simply by extending the scalars to the algebraic closure. We will denote this representation by $\overline{V}$. This defines a homomorphism $R_{\mathbb{F}_q}(G) \to R_{\overline{\mathbb{F}_q}}(G)$. Now let V be an $\overline{\mathbb{F}_q}$ representation of G. We define the *Brauer character* of V to be the function

$$\chi_V(g) = \sum_i \iota(\lambda_i)$$

where $\{\lambda_i\}$ is the set of eigenvalues of g acting on V, counted with multiplicity. In [Gre55] Green showed that χ_V is a virtual character of G (for more details see Chapter 18 §1 of [Ser77]).

If E and E' are $\overline{\mathbb{F}_q}$ representations of G, then the set of eigenvalues of g acting on $E \oplus E'$ is the union of the eigenvalues for g acting on E and g acting on E'. Therefore, $\chi_{E\oplus E'} = \chi_E + \chi_{E'}$, so the Brauer lift defines a (group) homomorphism $\varphi : R_{\overline{\mathbb{F}_q}}(G) \to R(G)$. Combining this with the extension of scalars map we obtain a homomorphism $R_{\mathbb{F}_q}(G) \to R(G)$.

Let V be an $\mathbb{F}_q$ representation of G, then we claim that $\varphi(\overline{V})$ is fixed by ψ^q. Fix $g \in G$ and let $\{\lambda_i\} \subset \overline{\mathbb{F}_q}$ be the set eigenvalues of g acting on $\overline{V}$. Because $\overline{V}$ comes from the $\mathbb{F}_q$-representation V we know that all the λ_i belong to the subfield of $\overline{\mathbb{F}_q}$ fixed by the Frobenius endomorphism $x \mapsto x^q$. That is $\lambda_i^q = \lambda_i$ for all i. The eigenvalues of g^q are $\{\lambda_i^q\}$ (when the order of g is prime to p then g is diagonalisable, so this follows immediately) and therefore,

$$\chi_{\overline{V}}(g^q) = \sum_i \iota(\lambda_i^q) = \sum_i \iota(\lambda_i) = \chi_{\overline{V}}(g).$$

Therefore, we've established that if we start with an $\mathbb{F}_q$-representation V of G and lift to an element $\varphi(\overline{V})$ of $R(G)$, then $\psi^q(\varphi(\overline{V})) = \varphi(\overline{V})$. Thus, our map $R_{\mathbb{F}_q}(G) \to R(G)$ from earlier is in fact a map,

$$R_{\mathbb{F}_q}(G) \to R(G)^{\psi^q}.$$

Composing with the previous map $R(G)^{\psi^q} \to [BG, F\Psi^q]$ we have produced a homomorphism

$$R_{\mathbb{F}_q}(G) \to [BG, F\Psi^q].$$

Given a representation E (this definition holds if E is either an element of $R(G)^{\psi^q}$ or $R_{\mathbb{F}_q}(G)$) we write $E^\sharp : BG \to F\Psi^q$ for the associated homotopy class of maps. Given a complex representation E of G we write E^+ for the associated homotopy class of maps $BG \to BU$.

Now consider $G = \mathrm{GL}_n(\mathbb{F}_q)$. Associated to the defining representation of $\mathrm{GL}_n(\mathbb{F}_q)$ on $\mathbb{F}_q^n$ is a homotopy class of maps $\alpha_n : \mathrm{BGL}_n(\mathbb{F}_q) \to F\Psi^q$. We can pull back representations along the inclusion

$\mathrm{GL}_n(\mathbb{F}_q) \hookrightarrow \mathrm{GL}_{n+1}(\mathbb{F}_q)$ giving us a map $R_F(\mathrm{GL}_{n+1}(\mathbb{F}_q)) \to R_F(\mathrm{GL}_n(\mathbb{F}_q))$ for any field F. All the maps we've defined involving extending scalars, taking Brauer lifts and forming vector bundles from representations commute with the map $R_F(\mathrm{GL}_{n+1}(\mathbb{F}_q)) \to R_F(\mathrm{GL}_n(\mathbb{F}_q))$. Therefore, the following diagram commutes for all n.

$$\begin{tikzcd}
\mathrm{BGL}_n(\mathbb{F}_q) \arrow[rr] \arrow[dr, "\alpha_n"] & & \mathrm{BGL}_{n+1}(\mathbb{F}_q) \arrow[dl, "\alpha_{n+1}"] \\
& F\Psi^q &
\end{tikzcd}$$

By the universal property of the colimit we get a homotopy class of maps $\alpha : \mathrm{BGL}(\mathbb{F}_q) = \varinjlim \mathrm{BGL}_n(\mathbb{F}_q) \to F\Psi^q$.

Theorem 3.1 (Quillen, [Qui72]). *The map $\alpha : \mathrm{BGL}(\mathbb{F}_q) \to F\Psi^q$ is an isomorphism on all homology groups.*

Before we consider the proof of this result, we will see how it tells us the K-theory of $\mathbb{F}_q$. For this we will need a generalised version of Whitehead's theorem.

Lemma 3.2 (Prop 4.74 of [Hat02]). *Let $f : X \to Y$ be a map of simple spaces. Then f is a homotopy equivalence if and only if f induces isomorphisms on all integral homology groups.*

Proof. This is proved in [Hat02] using obstruction theory and the relative Hurewicz theorem. Alternatively, it follows from a more general result of Dror, see [Dro71]. □

Corollary 3.2.1. *The K-theory of a finite field $\mathbb{F}_q$ is given by*

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 2i, i > 0 \\ \mathbb{Z}/(q^{i+1} - 1) & n = 2i + 1. \end{cases}$$

Proof. Since $\mathrm{BGL}(\mathbb{F}_q)^+$ and $F\Psi^q$ are both simple spaces the previous lemma says that it's enough to prove that the map $\mathrm{BGL}(\mathbb{F}_q)^+ \to F\Psi^q$ is an isomorphism on all homology groups. Since α factors through $\mathrm{BGL}(\mathbb{F}_q)^+ \to F\Psi^q$ we have the following diagram for all $i \geq 0$.

$$\begin{tikzcd}
H_i(\mathrm{BGL}(\mathbb{F}_q)) \arrow[r, "\alpha_*"] \arrow[d, "f_*"] & H_i(F\Psi^q) \\
H_i(\mathrm{BGL}(\mathbb{F}_q)^+) \arrow[ur] &
\end{tikzcd}$$

The map f_* is an isomorphism on all integral homology groups because f is acyclic and α_* is an isomorphism on all homology groups by Theorem 3.1. Therefore, $H_i(\mathrm{BGL}(\mathbb{F}_q)^+) \to H_i(F\Psi^q)$ is an isomorphism for all i. This means that $\mathrm{BGL}(\mathbb{F}_q)^+$ and $F\Psi^q$ are homotopy equivalent, so because we already computed the homotopy groups of $F\Psi^q$ in Section 2, we're done. □

4 A sketch of the proof of Theorem 3.1

First we will use the universal coefficient theorem to reduce our problem. Henceforth, k will denote the finite field $\mathbb{F}_q$ with $q = p^d$.

Theorem 4.1 (Theorem 3A.3 of [Hat02]). *If C is a chain complex of free abelian groups, then there are natural short exact sequences,*

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to \mathrm{Tor}(H_{n-1}(C), G)$$

Corollary 4.1.1 (Corollary 3A.7(b) of [Hat02]). *A map $f : X \to Y$ induces isomorphisms on homology with $\mathbb{Z}$ coefficients if and only if it induces isomorphisms on homology with $\mathbb{Q}$ and $\mathbb{F}_l$ coefficients for all primes l.*

Proof. This is a corollary of the universal coefficient theorem in homology, for details see [Hat02]. □

By this result it will suffice to show that α_* is an isomorphism on $H_*(-;\mathbb{Q})$ and $H_*(-;\mathbb{F}_l)$. Since these homology groups are in fact vector spaces over a field, the universal coefficient theorem for cohomology tells us that homology and cohomology are dual to one another, so in each case we can show either that α_* is an isomorphism on homology or that α^* is an isomorphism on cohomology. The remainder of the report will be dedicated to giving a survey of Quillen's proof of this result in [Qui72]. We will aim to give a general idea of the constructions and ideas which go into this proof, rather than trying to cover every detail.

4.1 Rational and mod p homology of $F\Psi^q$

In this section we will show that $\widetilde{H}_*(F\Psi^q;\mathbb{Q}) = \widetilde{H}_*(F\Psi^q;\mathbb{F}_p) = 0$. To prove this we will need to understand the notion of a *Serre class*. A *Serre class* is a non-empty collection of abelian groups $\mathcal{C}$ such that for any short exact sequence $0 \to A \to B \to C \to 0$, B is in $\mathcal{C}$ if and only if A and C are both in $\mathcal{C}$. We will be interested in Serre classes satisfying the following two properties:

1. If $A, B \in \mathcal{C}$ then $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}(A, B) \in \mathcal{C}$.
2. If $A \in \mathcal{C}$ then $H_n(K(A, 1);\mathbb{Z}) \in \mathcal{C}$ for all $n > 0$.

Theorem 4.2 (Theorem 5.7 of [Hat08])**.** *Let $\mathcal{C}$ be a Serre class satisfying conditions 1 and 2 and let X be path-connected and simple. Then $\pi_n(X) \in \mathcal{C}$ for all n if and only if $H_n(X;\mathbb{Z}) \in \mathcal{C}$ for all $n > 0$.*

Proof. This is an application of the Serre spectral sequence, for details see [Hat08] or Chapter 10 of [DK01]. ☐

Let P be a subset of the set of all primes. Then the class $\mathcal{F}_P$ of all finite abelian groups A such that no element of A has order divisible by elements of P is a Serre class satisfying properties 1 and 2 (by Lemma 10.2 of [DK01]). We write $\mathcal{F}_p$ for the Serre class of finite groups whose elements do not have order divisible by p. We've shown that $\pi_{2i}(F\Psi^q) = 0$ and $\pi_{2i+1}(F\Psi^q) = \mathbb{Z}/(q^{i+1} - 1)$, which is cyclic of order prime to p, so the groups $\pi_n(F\Psi^q)$ belong to $\mathcal{F}_p$ for all n. Therefore, the previous result implies that the groups $H_n(X;\mathbb{Z})$ are finite and have elements with order coprime to p. Note that $\mathrm{Tor}(H_{n-1}(F\Psi^q), \mathbb{Q}) = 0$ because $\mathbb{Q}$ is torsion free, so for all $n > 0$ the universal coefficient theorem says that

$$H_n(F\Psi^q;\mathbb{Q}) \cong (H_n(F\Psi^q) \otimes \mathbb{Q}) \oplus \mathrm{Tor}(H_{n-1}(F\Psi^q), \mathbb{Q}) = H_n(F\Psi^q) \otimes \mathbb{Q} = 0$$

because $H_n(F\Psi^q)$ is torsion. Because all elements of $H_i(F\Psi^q)$ have order prime to p we know that $\mathrm{Tor}(H_{n-1}(F\Psi^q), \mathbb{F}_p) = 0$ and $H_n(F\Psi^q) \otimes \mathbb{F}_p = 0$ (see the discussion on page 266 of [Hat02]). Therefore, for all $n > 0$ we have,

$$H_n(F\Psi^q;\mathbb{F}_p) \cong (H_n(F\Psi^q) \otimes \mathbb{F}_p) \oplus \mathrm{Tor}(H_{n-1}(F\Psi^q), \mathbb{F}_p) = 0.$$

4.2 Group cohomology and rational cohomology of $\mathrm{GL}(k)$

We will begin with a discussion of group cohomology. Let G be a group and let A be a $\mathbb{Z}$-module, then we define the group cohomology of G by

$$H^i(G; A) = H^i(BG; A).$$

This formula makes sense because Theorem 1B.8 of [Hat02] says that the homotopy type of a $K(G, 1)$ is uniquely determined by G. There is a more algebraic definition of group cohomology with coefficients in any G-module, but we will not need this notion here. For more details about this concept, see [Ser79]. Our definition of group cohomology is something of a notational convenience rather than a new concept. Note that by the universal coefficient theorem we have,

$$H^1(G; A) \cong \mathrm{Hom}(H_1(BG), A) \cong \mathrm{Hom}(\pi_1(BG)^{ab}, A) \cong \mathrm{Hom}(G^{ab}, A) \cong \mathrm{Hom}(G, A),$$

where the last equality is by the universal property of the abelianisation. That is, elements of $H^1(G; A)$ can be thought of as group homomorphisms $G \to A$. We claim that if G is a finite group then $H^i(G;\mathbb{Z})$ is always torsion. This also follows from Theorem 5.1 above: the groups $\pi_n(BG)$ belong to the Serre class $\mathcal{F}$ of finite groups for all n and so $H_n(BG;\mathbb{Z})$ is torsion for all $n > 0$ and so

$$H_n(BG;\mathbb{Q}) = H_n(BG) \otimes \mathbb{Q} = 0.$$

Applying this to the finite groups $\mathrm{GL}_n(k)$ we have that

$$H^i(\mathrm{GL}_n k; \mathbb{Q}) = 0$$

for all $i > 0$ and all n. In order to pass to $\mathrm{GL}(k)$ we will need to apply the following result.

Proposition 4.3 (Prop 3.33 of [Hat02]). *Suppose X is the union of a directed set of subspaces X_α with the property that each compact set in X is contained in some X_α, then the natural map*

$$\varinjlim H_i(X_\alpha; G) \to H_i(X; G)$$

is an isomorphism.

Note that for each inclusion $\mathrm{GL}_n(k) \to \mathrm{GL}_{n+1}(k)$ we have an associated map $\mathrm{BGL}_n(k) \to \mathrm{BGL}_{n+1}(k)$. Using this we can show that $\mathrm{BGL}(k)$ is the increasing union of the spaces $\mathrm{BGL}_n(k)$. Moreover, because we are working with CW complexes, the compactness property also holds. Therefore, the theorem tells us that

$$H^i(\mathrm{BGL}(k); \mathbb{Q}) = \varinjlim H^i(\mathrm{BGL}_n(k); \mathbb{Q}) = \varinjlim 0 = 0.$$

4.3 Mod p cohomology of $\mathrm{GL}(\mathbb{F}_q)$

Let k denote the field $\mathbb{F}_q$. Now we turn our attention to the groups $H^i(\mathrm{GL}(k); \mathbb{F}_p)$. We will not present a complete proof of this result.

Theorem 4.4 (Theorem 6 of Quillen). *$H^i(\mathrm{GL}_n(k); \mathbb{F}_p) = 0$ for $0 < i < d(p-1)$ and all n.*

Proof. The proof of this given in Section 11 of [Qui72] is quite technical. The fact that the subgroup U of upper-triangular matrices with all eigenvalues 1 is a p-Sylow subgroup of $\mathrm{GL}_n(k)$ allows us to reduce the problem to showing that $H^i(U)^T = 0$ for $0 < i < d(p-1)$, where T is the subgroup of diagonal matrices. The proof uses the algebraic group structure of $\mathrm{GL}_n(\overline{\mathbb{F}_q})$. □

Corollary 4.4.1 (Corollary 2 of [Qui72]). *The groups $H^i(\mathrm{GL}(k); \mathbb{F}_p)$ are trivial for all $i > 0$.*

Proof. Suppose that the groups $H^i(\mathrm{GL}(k); \mathbb{F}_p)$ are not all trivial, and let i be the least positive integer such that $H^i(\mathrm{GL}(k); \mathbb{F}_p) \neq 0$. Let $x \in H^i(\mathrm{GL}(k); \mathbb{F}_p)$ be a non-zero element. Let E be an n-dimensional $\mathbb{F}_q$-representation of a finite group G, then we have an associated homomorphism $E^\sharp : G \to \mathrm{GL}_n k \to \mathrm{GL}(k)$. This gives us a map $(E^\sharp)^* : H^i(\mathrm{GL}(k); \mathbb{F}_p) \to H^i(G; \mathbb{F}_p)$. We define

$$x(E) = (E^\sharp)^*(x) \in H^i(G; \mathbb{F}_p).$$

We claim $x(E) = 0$ for all representations E. The idea of the proof is that given a representation of G we can extend scalars to another finite field k' whose order has been chosen so that $H^i(\mathrm{GL}_n(k'); \mathbb{F}_p) = 0$ for all n. If we choose d' such that $dd'(p-1) > i$ and choose a field extension k'/k of degree d' then the previous theorem applied to k' will say that $H^i(\mathrm{GL}_n(k'); \mathbb{F}_p) = 0$ for all n. Starting with E, if we extend scalars to k' we get a representation $E \otimes_k k'$. Since $H^i(\mathrm{GL}_n k'; \mathbb{F}_p) = 0$ we know that if we view $E \otimes_k k'$ as a k-representation then we have $x(E \otimes_k k') = 0$. However, we know that $E \otimes_k k' \cong E^{\oplus d'}$ as a representation over k, and the fact that x lives in the smallest non-zero cohomology group forces it to be additive over direct sums, so we have

$$0 = x(E \otimes_k k') = x(E^{\oplus d'}) = d' x(E).$$

Therefore, if we also assume that $(d', p) = 1$ then this implies that $x(E) = 0$ (such a choice of d' is possible), whence the claim.

Now because $H_i(\mathrm{GL}(k); \mathbb{F}_p) = \varinjlim H_i(\mathrm{GL}_n(k); \mathbb{F}_p)$ taking duals means that $H^i(\mathrm{GL}(k); \mathbb{F}_p) = \varprojlim H^i(\mathrm{GL}_n(k); \mathbb{F}_p)$. Consider the defining representation $E_n : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$. This induces a map $(E_n^\sharp)^* : H^i(\mathrm{GL}(k); \mathbb{F}_p) \to H^i(\mathrm{GL}_n(k); \mathbb{F}_p)$. By the argument above, $(E_n^\sharp)^*(x) = 0$ for all n, so because $H^i(\mathrm{GL}(k); \mathbb{F}_p)$ is the limit of these groups, we conclude that $x = 0$. □

4.4 Calculation of the mod l cohomology of $F\Psi^q$ for $l \neq p$

Now we turn our attention to the case where the coefficient field is $\mathbb{F}_l$, for $l \neq p$. This is the only case where our spaces have non-trivial cohomology, so we will need to be more careful here. We will have to find explicit generators for the cohomology of both spaces, and show that the map $\alpha^* : H^i(\mathrm{BGL}(\mathbb{F}_q);\mathbb{F}_l) \to H^i(F\Psi^q)$ sends these generators to one another. For ease of notation we will write $H^i(X)$ for $H^i(X;\mathbb{F}_l)$. If we refer to integral cohomology in this section then we will use the notation $H^i(-;\mathbb{Z})$ to avoid confusion. Let r be the order of q in $\mathbb{F}_l$. That is, let r be the least positive integer such that $q^r \equiv 1 \mod l$. This means that the generator of $\mathbb{Z}/(q^{jr}-1)$ has order divisible by l, so we can define a ring homomorphism $\mathbb{Z}/(q^{jr}-1) \to \mathbb{F}_l$ for any j by sending a generator of one group to the other. This map will be important for constructing cohomology classes in $H^*(F\Psi^q;\mathbb{F}_l)$.

A key idea in this section is that we can use the structure of the ring $H^*(C_n;\mathbb{F}_l)$ for a cyclic group C_n to understand the rings $H^*(F\Psi^q)$ and $H^*(\mathrm{BGL}_n(k))$. In Example 3E.2 of [Hat02], the Bockstein homomorphism is used to show that for odd l,

$$H^*(BC;\mathbb{F}_l) \cong \mathbb{F}_l[u] \otimes \Lambda_{\mathbb{F}_l}[v],$$

with $|u| = 2$ and $|v| = 1$. Recall from Example 3.13 of [Hat02] that $\Lambda_{\mathbb{F}_l}$ denotes the exterior algebra over $\mathbb{F}_l$. For a ring R the exterior product $\Lambda_R[y_1, \cdots, y_n]$ is a free R-module with the basis $\{y_{i_1} \cdots y_{i_k} \mid i_1 < \cdots < i_k\}$ with multiplication defined by $y_i y_j = -y_j y_i$ if $i \neq j$ and $y_i^2 = 0$. In the case $l = 2$ the cohomology ring $H^*(BC;\mathbb{F}_2)$ depends on q, making matters more complicated. Therefore, to simplify proceedings we will assume from here on that $l \neq 2$. For details on how to make the argument work for $l = 2$, see [Qui72].

We know that $H^*(BU;\mathbb{Z}) \cong \mathbb{Z}[\widetilde{c_1}, \widetilde{c_2}, ...]$ and $H^*(BU;\mathbb{F}_l) = \mathbb{F}_l[c_1, c_2, \cdots]$ for the universal Chern classes $\widetilde{c_i}$ and c_i in integral and mod l cohomology. We have a map $\phi : F\Psi^q \to BU$ which allows us to pull these classes back to cohomology classes in $H^*(F\Psi^q;\mathbb{Z})$ (resp. $H^*(F\Psi^q)$). It turns out these classes do not generate the ring $H^*(F\Psi^q)$, so we will have to find more cohomology classes by diagram chasing.

Let $f : X \to Y$ be a map, then we can form the mapping cylinder $M_f = (X \times I) \cup_f Y$. We will write $H^i(f;A)$ for the relative cohomology group $H^i(M_f, X; A)$. Note that if (M_f, X) is a good pair then this is just $\widetilde{H}^i(C_f;A)$, for C_f the mapping cone. Then we have the long exact sequence of the pair (M_f, X)

$$\cdots \to H^{i-1}(X;A) \xrightarrow{\delta} H^i(f;A) \xrightarrow{j} H^i(Y;A) \xrightarrow{f^*} H^i(X;A) \to \cdots$$

where we've used the fact that $M_f \simeq Y$. Here j is induced by the inclusion of Y into M_f. We form these long exact sequences for the maps $\phi : F\Psi^q \to BU$ and $\Delta : BU^I \to BU \times BU$ and then the commuting square $(*)$ means that we have a map $\Gamma^* : H^{2i}(\Delta;A) \to H^{2i}(\phi;A)$ which makes the following diagram involving both long exact sequences commute.

$$\begin{tikzcd}
 & H^{2i-1}(BU^I;A) \arrow[r] \arrow[d] & H^{2i}(\Delta;A) \arrow[r, "j'"] \arrow[d, "\Gamma^*"] & H^{2i}(BU \times BU) \arrow[r, "\Delta^*"] \arrow[d, "(\mathrm{id},\psi^q)^*"] & H^{2i}(BU^I) \arrow[d] \\
H^{2i-1}(BU;A) \arrow[r] & H^{2i-1}(F\Psi^q;A) \arrow[r, "\delta"] & H^{2i}(\phi;A) \arrow[r, "j"] & H^{2i}(BU;A) \arrow[r, "\phi^*"] & H^{2i}(F\Psi^q;A)
\end{tikzcd}$$

Note that the map $BU^I \to BU$ given by $\gamma \mapsto \gamma(1)$ is a homotopy equivalence with homotopy inverse $BU \to BU^I$ given by $x \mapsto c_x$. This is because we can shrink paths back to constant paths at their endpoints. Therefore, we have that $H^{2i-1}(BU^I;A) = H^{2i-1}(BU;A) = 0$, since the generators of $H^*(BU;A)$ live in even degrees. Therefore, our diagram becomes:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & H^{2i}(\Delta;A) \arrow[r, "j'"] \arrow[d, "\Gamma^*"] & H^{2i}(BU \times BU) \arrow[r, "\Delta^*"] \arrow[d, "(\mathrm{id},\psi^q)^*"] & H^{2i}(BU^I) \arrow[d] \\
0 \arrow[r] & H^{2i-1}(F\Psi^q;A) \arrow[r, "\delta"] & H^{2i}(\phi;A) \arrow[r, "j"] & H^{2i}(BU;A) \arrow[r, "\phi^*"] & H^{2i}(F\Psi^q;A)
\end{tikzcd}$$

Recall that we have defined the universal integral Chern classes $\widetilde{c_i} \in H^{2i}(BU;\mathbb{Z})$ and the mod l Chern classes $c_i \in H^{2i}(BU;\mathbb{F}_l)$. Note that the composition $BU \hookrightarrow BU^I \xrightarrow{\Delta} BU \times BU$ is the diagonal map so because the map $BU \hookrightarrow BU^I$ is a homotopy equivalence, we can identify Δ^* with the map on cohomology induced by the diagonal map. Let $\widetilde{c}_i \otimes 1 - 1 \otimes \widetilde{c}_i \in H^*(BU \times BU) \cong H^*(BU) \otimes H^*(BU)$. Then we have,

$$\Delta^*(\widetilde{c}_i \otimes 1 - 1 \otimes \widetilde{c}_i) = \widetilde{c}_i - \widetilde{c}_i = 0$$

because $\Delta^* : H^*(BU \times BU) \to H^*(BU)$ is multiplication on cohomology. By the exactness of the top row we can choose $z \in H^{2i}(\Delta; A)$ such that $j'(z) = \widetilde{c}_i \otimes 1 - 1 \otimes \widetilde{c}_i$. Now by the commutativity of the diagram we get that

$$j'(\Gamma^* z) = (\mathrm{id}, \psi^q)^*(j'(z)) = (\mathrm{id}, \psi^q)^*(\widetilde{c}_i \otimes 1 - 1 \otimes \widetilde{c}_i) = \widetilde{c}_i \cdot 1 - 1 \cdot \psi^q(\widetilde{c}_i) = (1 - q^i)\widetilde{c}_i.$$

Where we've use that $\psi^q(\widetilde{c}_i) = q^i \widetilde{c}_i$. Therefore, if we reduce coefficients mod $(q^i - 1)$ we get that $j'(\Gamma^*(z)) = 0$, and so by exactness there is a unique element $\widetilde{e}_i \in H^{2i-1}(F\Psi^q; \mathbb{Z}/(q^i - 1))$ such that $\delta(\widetilde{e}_i) = \Gamma^*(z)$ in $H^{2i}(\phi; \mathbb{Z}/(q^i - 1))$. A diagram chase shows that $\beta_{q^i - 1}(\widetilde{e}_i) = \phi^*(\widetilde{c}_i)$, where β_{q^i-1} is the Bockstein homomorphism associated to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/(q^i - 1) \to 0$. Finally, we are able to define e_{jr} as the image of $\widetilde{e_{jr}}$ under the map $H^{2jr-1}(F\Psi^q; \mathbb{Z}/(q^{jr} - 1)) \to H^{2jr-1}(F\Psi^q)$ induced by the ring homomorphism $\mathbb{Z}/(q^{jr} - 1) \to \mathbb{F}_l$ discussed above. Eventually we will be able to prove that these classes along with the classes $\phi^*(c_{jr})$ generate $H^*(F\Psi^q)$.

4.5 The algebra structure of $H^*(F\Psi^q; \mathbb{F}_l)$

Let E be a complex representation of a finite group G, then E defines an element $E^+ \in [BG, BU] = \widetilde{K}(BG)$. That is, a representation E defines an element E^+ in K-theory and so it makes sense to take the Chern classes of E^+. We write $c_i(E)$ for this Chern class associated to a representation. Explicitly, these are defined by:

$$\widetilde{c}_i(E) = (E^+)^*(\widetilde{c}_i) \in H^{2i}(BG; \mathbb{Z}), \quad c_i(E) = (E^+)^*(c_i) \in H^{2i}(BG; \mathbb{F}_l).$$

Note that if E is a representation of dimension n then it defines a vector bundle of the same dimension and so $\widetilde{c}_i(E) = 0$ for $i > n$. If $\psi^q E = E$ then we can use the associated map $E^\sharp : BG \to F\Psi^q$ to define the classes,

$$\widetilde{e}_i(E) = (E^\sharp)^*(\widetilde{e}_i) \in H^{2i-1}(BG; \mathbb{Z}), \quad e_{jr}(E) = (E^\sharp)^*(e_{jr}) \in H^{2jr-1}(BG; \mathbb{F}_l).$$

Note that by the way we've defined them, these characteristic classes of representations satisfy the usual properties of Chern classes. If $\psi^q E = E$ then we know that ψ^q is multiplication by q^i on $H^{2i}(BU)$ and so,

$$c_i(E) = c_i(\psi^q E) = \psi^q(c_i(E)) = q^i c_i(E)$$

where we've used the naturality of the Adams operations and Chern classes. This equation means that $c_i(E) = 0$ whenever q^i is not 1. Since we know that r is the multiplicative order of q in $\mathbb{F}_l^\times$ we conclude that $c_i(E) = 0$ unless $i \equiv 0 \bmod r$. We will first investigate these classes in the case of the finite group $C = \mathbb{Z}/(q^r - 1)$. Let ζ be a primitive q^rth root of unity, then the map $C \to \mathbb{C}^\times$ given by $1 \mapsto \zeta$ is a complex character which we will also call ζ. If we define

$$W = \zeta \oplus \zeta^q \oplus \cdots \oplus \zeta^{q^{r-1}}.$$

then we compute the character of $\psi^q(W)$ as,

$$\begin{aligned} \psi^q\left(\zeta \oplus \zeta^q \oplus \cdots \oplus \zeta^{q^{r-1}}\right) &= \sum_{a=0}^{r-1} \psi^q\left(\zeta^{q^a}\right) \\ &= \sum_{a=0}^{r-1} \zeta^{q^{a+1}} = \zeta^q + ... + \zeta^{q^{r-1}} + \zeta. \end{aligned}$$

This is the character of W, so $\psi^q(W) = W$. Therefore, the argument above allows us to define the classes $\widetilde{e}_i(W)$ and $e_{jr}(W)$.

Proposition 4.5 (Prop 1 of [Qui72]). *Let $v \in H^1(BC) \cong Hom(C, \mathbb{F}_l)$ be the class of the homomorphism $C \to \mathbb{F}_l$ sending $1 \mapsto 1$ and let $u = c_1(\zeta) \in H^2(BC)$. Then*

$$c_i(W) = \begin{cases} 1 & i = 0 \\ (-1)^{r-1} u^r & i = r \\ 0 & \textit{otherwise.} \end{cases}$$

and

$$e_{jr}(W) = \begin{cases} (-1)^{r-1}u^{r-1}v & j = 1 \\ 0 & j \neq 1. \end{cases}$$

Proof. By the correspondence between Chern classes of representations and vector bundles, we have that $\widetilde{c}_i(W) = 0$ for $i > r$ because W is an r-dimensional representation. Therefore $c_i(W) = 0$ for $i > r$. We know $c_0(W) = 1$ by definition, and we've shown that $c_i(W) = 0$ unless $i \equiv 0 \mod r$. Therefore, it remains to compute $c_r(W)$. Let $\widetilde{v} \in H^1(C; \mathbb{Z}/(q^r-1)) = \mathrm{Hom}(C, \mathbb{Z}/(q^r-1)) = \mathrm{Hom}(\mathbb{Z}/(q^r-1), \mathbb{Z}/(q^r-1))$ be the class of the identity map and let $\widetilde{u} = \widetilde{c}_1(\zeta) \in H^2(BC; \mathbb{Z})$. Note that these classes are sent to u and v respectively when we reduce modulo l. Applying the Whitney sum formula we have,

$$\widetilde{c}_r(W) = \sum_{a_{i_1}+\ldots+a_{i_k}=r} \widetilde{c}_{i_1}\left(\zeta^{q^{a_{i_1}}}\right) \cdots \widetilde{c}_{i_k}\left(\zeta^{q^{a_{i_k}}}\right)$$

But because ζ^{q^a} is a one-dimensional representation $\widetilde{c}_i(\zeta^{q^a}) = 0$ for $i > 1$ and so this sum becomes,

$$\prod_{a=0}^{r-1} \widetilde{c}_1(\zeta^{q^a}) = \prod_{a=0}^{r-1} q^a \widetilde{c}_1(\zeta) = \prod_{a=0}^{r-1} q^a \widetilde{u}$$

where we've used the fact that the first Chern class turns tensor products of line bundles into sums (for vector bundles and hence for representations). Now we note that $0 + r + \cdots + r - 1 = \frac{r(r-1)}{2}$ and so we have shown that

$$\widetilde{c}_r(W) = q^{r(r-1)/2}\widetilde{u}^r.$$

We know that $q^{r/2} \equiv -1 \mod l$ (as $\left(q^{r/2}\right)^2 = q^r \equiv 1 \mod l$, but $q^{r/2}$ is not congruent to 1 mod l) and hence $q^{r(r-1)/2} \equiv (-1)^{r-1} \mod l$. Therefore, when we reduce the previous equation mod l we get that,

$$c_r(W) = (-1)^{r-1}u^r.$$

Now we consider the Bockstein long exact sequence,

$$H^{2i-1}(BC; \mathbb{Z}) \to H^{2i-1}(BC; \mathbb{Z}/n) \xrightarrow{\beta_n} H^{2i}(BC; \mathbb{Z}).$$

We know that BC is an infinite dimensional Lens space, so Example 2.43 of [Hat02] says that $H^{2i-1}(BC; \mathbb{Z}) = 0$ for all i. Therefore, the map $\beta_n : H^{2i-1}(BC; \mathbb{Z}/n) \to H^{2i}(BC; \mathbb{Z})$ is always injective and so it will suffice to compute $\beta_{q^i-1}(\widetilde{e}_i(W)) = \widetilde{c}_i(W)$. We can show by a diagram chase that $\widetilde{u} = \beta_{q^r-1}(\widetilde{v})$, so interpreting elements of $H^*(BC; \mathbb{Z})$ as coefficients we obtain,

$$\beta_{q^r-1}(q^{r(r-1)/2}\widetilde{u}^{r-1}\widetilde{v}) = q^{r(r-1)/2}\widetilde{u}^{r-1}\beta_{q^r-1}(\widetilde{v}) = q^{r(r-1)/2}\widetilde{u}^r = \widetilde{c}_r(W).$$

Therefore, we have,

$$\beta_{q^r-1}\widetilde{e}_i(W) = \widetilde{c}_i(W) = \begin{cases} 0 & i > r \\ \beta_{q^r-1}(q^{r(r-1)/2}\widetilde{u}^{r-1}\widetilde{v}) & i = r \end{cases}$$

By the injectivity of β_{q^r-1} we have a formula for $\widetilde{e}_i(W)$ which reduces mod l to the correct formula. $\square$

Henceforth we will write c_i for the image of $c_i \in H^{2i}(BU)$ in $H^{2i}(F\Psi^q)$ under ϕ^*. We introduce symbols t and s with $s^2 = 0$ and then we define the power series,

$$c_{ts} = 1 + \sum_{j \geq 1} \left(c_{jr}t^j + e_{jr}t^{j-1}s\right).$$

If we have $\xi \in R(G)$ such that $\psi^q\xi = \xi$ then it makes sense to define,

$$c_{ts}(\xi) = 1 + \sum_{j \geq 1} \left(c_{jr}(\xi)t^j + e_{jr}(\xi)t^{j-1}s\right).$$

In Section 4 of [Qui72] Quillen shows that these power series satisfy the formula $c_{ts}(\xi + \eta) = c_{ts}(\xi)c_{ts}(\eta)$. The previous calculation shows that if we let $x = (-1)^{r-1}u^r$ and $y = (-1)^{r-1}u^{r-1}v$ then

$$c_{ts}(W) = 1 + xt + ys.$$

Theorem 4.6 (Theorem 1 of [Qui72]). *The algebra structure of $H^*(F\Psi^q)$ is given by,*

$$H^*(F\Psi^q) \cong \mathbb{F}_l[c_r, c_{2r}, ...] \otimes \Lambda_{\mathbb{F}_l}[e_r, e_{2r}, ...].$$

Proof. To prove this we need to show that the monomials $c_r^{\alpha_1} c_{2r}^{\alpha_2} \cdots e_r^{\beta_1} e_{2r}^{\beta_2} \cdots$ for $\alpha_j \geq 0$ and $\beta_j \in \{0, 1\}$ form a basis for $H^*(F\Psi^q)$ and that $e_{jr}^2 = 0$ for all j. Our strategy will be to infer this from the algebra structure on $H^*(C^m)$.

Consider the $m$fold product of cyclic groups C^m (recall that $C \cong \mathbb{Z}/(q^r - 1)$). We have surjective group homomorphisms $p_i : C^m \to C$ for $1 \leq i \leq m$. We can pull back the representation W of C along these maps to obtain representations W_i of C^m. Then the representation $\bigoplus_{i=1}^m W_i$ is ψ^q invariant as a direct sum of ψ^q-invariant representations, so we have an associated map $BC^m \to F\Psi^q$ which induces a ring homomorphism $H^*(F\Psi^q; \mathbb{F}_l) \to H^*(BC^m; \mathbb{F}_l)$. We then define $\overline{c_{jr}} = c_{jr}(\bigoplus W_i)$ and $\overline{e_{jr}} = e_{jr}(\bigoplus W_i)$.

An argument involving the Eilberg-Moore spectral sequence associated to the pullback square (*) can be used to show that these monomials span all of $H^*(F\Psi^q)$. For details see Lemma 2 and the proof of Theorem 1 of [Qui72]. Next we will show that these monomials are linearly independent. To do this, it will suffice to show that for any m, the monomials $\overline{c_r}^{\alpha_1} \overline{c_{2r}}^{\alpha_2} \cdots \overline{c_{m_r}}^{\alpha_m} \overline{e_r}^{\beta_1} \overline{e_{2r}}^{\beta_2} \cdots \overline{e_{m_r}}^{\beta_m}$ are linearly independent in $H^*(BC^m)$. Then by increasing m we can conclude that the monomials we started with must be linearly independent. Let u_i and v_i be the pullbacks of $u, v \in H^*(C)$ along the projection $p_i : C^m \to C$ then if we define $x_i = (-1)^{r-1} u_i^r$ and $y_i = (-1)^{r-1} u_i^{r-1} v_i$, so the formula $c_{ts}(\eta + \xi) = c_{ts}(\eta) c_{ts}(\xi)$ means that,

$$c_{ts}\left(\bigoplus W_i\right) = \prod_{i=1}^m c_{ts}(W_i) = \prod_{i=1}^m (1 + x_i t + y_i s).$$

By comparing the coefficients of t^j and $t^{j-1}s$ on either side we obtain,

$$\overline{c_{jr}} = \sum_{i_1 < \cdots < i_j} x_{i_1} \cdots x_{i_j}$$

$$\overline{e_{jr}} = \sum_{\substack{i_1 < \cdots < i_j \\ 1 \leq k \leq j}} x_{i_1} \cdots \widehat{x}_{i_k} \cdots x_{i_j} y_{i_k}.$$

It can be shown (see Lemma 9 of [Qui72]) that the inclusion

$$\mathbb{F}_l[\overline{c_r}, \cdots, \overline{c_{mr}}] \otimes \Lambda[\overline{e_r}, \cdots, \overline{e_{mr}}] \to \mathbb{F}_l[x_1, \cdots, x_m] \otimes \Lambda[y_1, \cdots, y_m]$$

is injective (this is a variation of the result that the symmetric polynomials are algebraically independent over the coefficient ring, see Theorem 6.1 of [Lan02]). From this it follows that these monomials are linearly independent in $H^*(BC^m)$.

Finally we need to show that $e_{jr}^2 = 0$. Because $y_i = (-1)^{r-1} u_i^{r-1} v_i$ we know that $y_i^2 = 0$ (it involves a $v_i^2 = 0$ term). Consider the polynomial $\overline{e_{jr}}^2$. All the summands with a y_i^2 term vanish and for each $y_i y_j$ term with $i \neq j$ we get a $y_j y_i$ term which is otherwise the same, so because $y_i y_j = -y_j y_i$ all these terms cancel too. Therefore, $\overline{e_{jr}}^2 = 0$. Because we have found a basis of monomials for $H^*(F\Psi^q)$ we can write $e_{jr}^2 = \lambda_1 f_1 + \cdots + \lambda_n f_n$ for monomials f_i from the spanning set. Since $\overline{e_{jr}}^2 = 0$ we get that

$$0 = \lambda_1 \overline{f_1} + \cdots + \lambda_n \overline{f_n}$$

so because the monomials $\overline{f_1}, ..., \overline{f_n}$ are independent in $H^*(BC^m)$ we conclude that $\lambda_1 = \cdots = \lambda_n = 0$. That is, $e_{jr}^2 = 0$.

□

4.6 The mod l cohomology of $\mathrm{BGL}(\mathbb{F}_q)$

Now we turn our attention to the cohomology groups of the finite general linear groups $\mathrm{GL}_n(\mathbb{F}_q)$. Let μ_l be the group of lth roots of unity in $\overline{k}$, then the extension $k(\mu_l)/k$ has degree r. If we let $C = k(\mu_l)^\times \cong \mathbb{Z}/(q^r - 1)$ then C acts on $k(\mu_l)$ by multiplication. Thinking of $k(\mu_l)$ as an r-dimensional

k-vector space L, this action gives L the structure of a C-representation. We would like to show that the Brauer lift of L is the complex representation W. To see this we observe that the map $k(\mu_l) \otimes_k \overline{k} \to \overline{k}^r$ given by $z \otimes w \mapsto \left(z^{q^a} w\right)_{a=0}^{r-1}$ is an isomorphism of k-algebras by the normal basis theorem of Galois theory. This tells us that for $x \in C$, the eigenvalues of x acting on $L \otimes_k \overline{k}$, are $x, x^q, \cdots, x^{q^{r-1}}$ and so if χ is the character of $L \otimes_k \overline{k}$ then we have

$$\chi(x) = \operatorname{Tr}\left(x \mid L \otimes_k \overline{k}\right) = \sum_{a=0}^{r-1} x^{q^a}.$$

Now suppose z generates $k(\mu_l)^\times$, then $\iota(z) = \zeta$ for ζ a q^rth root of unity in $\mathbb{C}$. Therefore, the Brauer lift of L has character χ_L defined by,

$$\chi_L(z) = \sum_{a=0}^{r-1} \iota\left(z^{q^a}\right) = \sum_{a=0}^{r-1} \zeta^{q^a}.$$

This is the same as the character of W, so we conclude that L lifts to the complex representation W.

As a representation, we know that L defines a map $C \to \mathrm{GL}_r(k)$. This representation is faithful, because we are considering the action of $k(\mu_l)^\times$ on $k(\mu_l)$ by multiplication: if $r \cdot x = x$ for all $x \in k(\mu_l)$, then $r = 1$. Now the Galois group $\Gamma = \mathrm{Gal}(k(\mu_l)/k) \cong \mathbb{Z}/r$ is generated by the Frobenius endomorphism which sends $z \mapsto z^q$. This group acts on $C = k(\mu_l)^\times$ so we have a homomorphism $\Gamma \to \mathrm{Aut}(C)$ which allows us to write the semi-direct product $\Gamma \ltimes C$. This semi-direct product acts on L by $(\sigma, z) \cdot x = \sigma(z) \cdot x$. Therefore, we have a Γ-equivariant embedding $C \hookrightarrow \mathrm{GL}_r(k)$, where the action of Γ on C corresponds to an inner-automorphism of $\mathrm{GL}_r(k)$ (because it comes from a representation). That is we have a map $H^*(\mathrm{GL}_r\, k)^\Gamma \to H^*(C)^\Gamma$. By Proposition 3 of Chapter 7 of [Ser79] we know that inner automorphisms act trivially on group cohomology, so we have in fact constructed a map,

$$H^*(\mathrm{GL}_r(k)) \to H^*(C)^\Gamma.$$

Now we can extend this result. Let $n = mr+e$ for some $0 \leq e < r$. Then we obtain a representation L^m of $S_m \ltimes (\Gamma \ltimes C)^m$ where S_m permutes the factors of L^m. Taking the direct sum of this representation with an e-dimensional trivial representation we obtain an $(S_m \ltimes \Gamma^m)$-equivariant embedding $C^m \hookrightarrow \mathrm{GL}_n(k)$, where $S_m \ltimes \Gamma^m$ acts by inner-automorphisms on $\mathrm{GL}_n(k)$ (by the same argument as before). Therefore, we have constructed a map,

$$H^*(\mathrm{GL}_n\, k) \to H^*(C^m)^{S_m \ltimes \Gamma^m} \cong \left(\bigotimes_m H^*(C)^\Gamma\right)^{S_m}$$

where we have applied the Künneth formula with $B(C^m) \simeq BC^m$ and the fact that the action of Γ on $\bigotimes_m H^*(C)$ commutes with the tensor product (since it comes from an action on $H^*(C)$ in the first place). Recall that the Brauer lifting gives us a map $\mathrm{BGL}_n(k) \to F\Psi^q$ and hence a map $H^*(F\Psi^q) \to H^*(\mathrm{GL}_n(\mathbb{F}_q))$. Recall that in Section 4.5 we used the representation $\bigoplus W_i$ to define a map $H^*(F\Psi^q) \to H^*(C^m)$. Putting these maps together we obtain the following diagram.

$$\begin{tikzcd} H^*(F\Psi^q) \arrow[rr, "\bigoplus W_i"] \arrow[dr] & & H^*(C^m) \\ & H^*(\mathrm{GL}_n(k)) \arrow[ur, "L^m \oplus \mathbf{1}^e"] & \end{tikzcd}$$

This diagram commutes precisely because L lifts to W. It turns out that taking the $(S_m \ltimes \Gamma^m)$-invariant elements of $H^*(C^m)$ gives exactly the image of this map:

Theorem 4.7 (Corollary to Theorem 3 of [Qui72]). *The homomorphism*

$$H^*(\mathrm{GL}_n\, k) \to H^*(C)^{S_m \ltimes \Gamma^m}$$

constructed above is an algebra isomorphism. (If $l = 2$ then it is merely injective, but we are ignoring this case.)

Proof. The proof of this result is quite technical so we will just give a quick overview. Let $q^r - 1 = l^a h$ for $(h, l) = 1$. We say that a subgroup H of $\mathrm{GL}_n\, k$ has exponent l^a if $g^{l^a} = 1$ for all $g \in H$. Then we can tell whether an element $x \in H^i(\mathrm{GL}_n\, k; \mathbb{F}_l)$ is zero by restricting the element to all subgroups with exponent l^a: If x restricts to 0 in all these cohomology groups, then it had to be zero to begin with (see Lemma 13 of [Qui72]). Then we can show that all such subgroups are conjugate to subgroups of C^m (recall that we have embedded C^m inside $\mathrm{GL}_n\, k$), see Lemma 13 of [Qui72]. From this it follows that the map $H^*(\mathrm{GL}_n\, k) \to H^*(C^m)$ is injective. $\square$

Now we have everything we need to deduce the algebra structure of $H^*(\mathrm{GL}_n\, k)$. Consider the map $\alpha_n^* : H^*(F\Psi^q) \to H^*(\mathrm{GL}_n\, k)$. Since this comes from lifting the defining representation k^n of $\mathrm{GL}_n\, k$ to a map $\mathrm{BGL}_n\, k \to F\Psi^q$ we know that it sends $c_{jr} \mapsto c_{jr}(k^n)$ and $e_{jr} \mapsto e_{jr}(k^n)$ for $jr \leq n$ (recall that we defined characteristic classes of ψ^q-invariant representations and hence of any $\mathbb{F}_q$-representation). From the proof of Theorem 4.7 we know that the kernel of the map $H^*(F\Psi^q) \to H^*(C^m)$ is the ideal generated by the c_{jr} and the e_{jr} for $jr > n$, so we can conclude that

$$H^*(\mathrm{GL}_n\, k) \cong \mathbb{F}_l[c_r(k^n), c_{2r}(k^n), ..., c_{mr}(k^n)] \otimes \Lambda[e_r(k^n), e_{2r}(k^n), ..., c_{mr}(k^n)].$$

Recall that $m = \lfloor \frac{n}{r} \rfloor$. From this it is intuitive that when n goes to infinity α^* becomes an isomorphism on all cohomology groups. We will comment briefly on how to make this argument more rigorous. We've shown that $H_i(\mathrm{GL}(k)) = \varinjlim H_i(\mathrm{GL}_n(k))$, so taking the dual of either side gives,

$$H^i(\mathrm{GL}(k)) = \varprojlim H^i(\mathrm{GL}_n(k))$$

where we note that because we're working with vector spaces over a field, homology and cohomology are dual to one another. Comparing the algebra structure of $H^*(\mathrm{GL}_n\, k)$ and $H^*(F\Psi^q)$ we see that for any i we can choose large enough n so that the extra generators of $H^*(F\Psi^q)$ all live in a degree greater than i. This means that $\alpha_n : \mathrm{BGL}_n(k) \to F\Psi^q$ is an isomorphism on $H^i(-)$ for large enough n. Since $\alpha : \mathrm{BGL}(k) \to F\Psi^q$ is the colimit of the maps $\alpha_n : \mathrm{BGL}_n(k) \to F\Psi^q$ we conclude that $\alpha : \mathrm{BGL}(k) \to F\Psi^q$ is an isomorphism on cohomology for all i. This concludes the proof of Theorem 3.1 because we have now shown that α induces an isomorphism on homology/cohomology groups with coefficients in $\mathbb{Q}$ or $\mathbb{F}_l$ for any prime l.

References

[Ada62] J.F. Adams, *Vector Fields on Spheres*, Ann. Math. **75** (1962), 603–632.

[AS69] M F Atiyah and G B Segal, *Equivariant K-Theory and Completion*, J. Differ. Geom. **3** (1969), 1–18.

[AT69] M. F. Atiyah and D. O. Tall, *Group representations, λ-rings and the J-homomorphism*, Topology **8** (1969), no. 3, 253–297.

[Ati61] M.F. Atiyah, *Characters and cohomology of finite groups*, Inst. des Hautes Études Sci. Publ. Mathématiques **9** (1961), 23–64.

[BS58] Armand Borel and Jean-Pierre Serre, *Le théorème de Riemann-Roch*, Bull. Soc. Math. Fr. **86** (1958), 97–136.

[DK01] James F. Davis and Paul Kirk, *Lecture Notes in Algebraic Topology*, American Mathematical Society, Providence, RI, 2001.

[Dro71] Emmanuel Dror, *A generalization of the Whitehead theorem*, Symp. Algebr. Topol. (Berlin-Heidelberg) (Peter J. Hilton, ed.), Springer, 1971, pp. 13–22.

[Ger73] S.M. Gersten, *Higher K-Theory of rings*, High. K-Theories (Berlin-Heidelberg) (Hyman Bass, ed.), Springer Berlin Heidelberg, 1973, pp. 3–42.

[Gra76] Daniel Grayson, *Higher algebraic K-theory: II*, Springer Berlin Heidelberg, Berlin-Heidelberg, 1976.

[Gre55] J.A. Green, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc. **80** (1955), 402–447.

[Hat02] Allen Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

[Hat08] ———, *Spectral sequences*, available online at: `https://pi.math.cornell.edu/~hatcher/SSAT/SSATpage.html`, 2008.

[Hat17] ———, *Vector Bundles & K-Theory*, 2.2 ed., available online at: `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`, 2017.

[Lan02] Serge Lang, *Algebra*, third ed., Springer-Verlag, New York, 2002.

[May99] J.P. May, *A concise course in algebraic topology*, University of Chicago Press, Chicago, IL, 1999.

[Mil71] John Milnor, *Introduction to algebraic K-theory*, Princeton University Press, Princteon, NJ, 1971.

[Qui71] Daniel Quillen, *The Adams Conjecture*, Topology **10** (1971), no. 1, 67–80.

[Qui72] ———, *On the cohomology and K-Theory of the general linear Groups over a finite field*, Ann. Math. **96** (1972), 552–586.

[Qui73] ———, *Higher algebraic K-theory: I*, Springer Berlin Heidelberg, Berlin-Heidelberg, 1973.

[Ser77] Jean-Pierre Serre, *Linear representations of finite groups*, Springer-Verlag, New York-Heidelberg, 1977.

[Ser79] ———, *Local fields*, Springer-Verlag, New York-Berlin, 1979.

[Wei13] Charles A. Weibel, *The K-Book*, American Mathematical Society, Providence, RI, 2013.

[Whi50] J.H.C. Whitehead, *Simple homotopy types*, Am. J. Math. **72** (1950), 1–57.