Coxeter Groups and the Davis Complex

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1 Introduction

A key insight of geometric group theory is that we can understand the algebraic structure of groups by studying the geometric properties of spaces which they act on. In particular, if a group acts on a space with non-positive curvature in a way which respects this geometric structure, then we can infer a great deal about the algebraic structure of the group.

We will consider *Coxeter groups* from the perspective of geometric group theory, associating to any Coxeter group a *CAT(0) space* known as the Davis complex which has an action of our original group. In the case of right-angled Coxeter groups, this construction will be simplified: every cell in the Davis complex will in fact be a cube. This will allow us to prove that these spaces are CAT(0) using a theorem of Gromov. We will also show that a Coxeter group acts on its associated Davis complex *geometrically*, so that Coxeter groups are an example of what we refer to as CAT(0) groups. Finally, we will briefly discuss some of the properties of CAT(0) groups.

2 Artin and Coxeter Groups

We begin by defining Artin and Coxeter groups and surveying some useful results about these groups found in [1] and [4].

An Artin group is a group in which the only relations between generators are braid relations of the form, $\underbrace{abab...}_{m \text{ letters}} = \underbrace{babab...}_{m \text{ letters}}$ where m is an integer satisfying $2 \leq m \leq \infty$. We exclude the possibility that $m = 1$ since the generators are assumed to be distinct, and we also adopt the convention that if $m_{ij} = \infty$ then x_i and x_j satisfy no relation. Therefore, an Artin group has a presentation of the form [4],

$$\mathcal{A} = \langle S \mid \mathcal{R} \rangle = \left\langle x_1, ..., x_n \;\middle|\; \underbrace{x_i x_k ...}_{m_{i,k}} = \underbrace{x_k x_i ...}_{m_{i,k} = m_{k,i}}, i \neq k \right\rangle$$

We call the pair (A, S) an Artin system, where the relations between the generators S are defined by the *Coxeter matrix*, the symmetric matrix whose ijth entry is m_{ij}. For each Artin system (A, S) we have an associated Coxeter system (W, S) where we add additional relations which dictate that every generator has order two. Therefore, the Coxeter system (W, S) related to the previous Artin system has the presentation:

$$W(S) = \langle S \mid \mathcal{R} \rangle = \left\langle x_1, ..., x_n \;\middle|\; \underbrace{x_i x_k ...}_{m_{i,k}} = \underbrace{x_k x_i ...}_{m_{i,k} = m_{k,i}}, i \neq k, x_i^2 = e, \forall i = 1, ...n \right\rangle.$$

Henceforth we will only be concerned with Coxeter groups. Coxeter groups were originally studied by H.S.M. Coxeter as reflection groups of finite dimensional Euclidean space. Indeed, if W has n generators, then there is a canonical linear representation $W \hookrightarrow GL_n(\mathbb{R})$ [5, p.117]. We will not define this representation because the only information we will need is that it gives an action of any Coxeter group on some finite dimensional Euclidean space.

We call a Coxeter group for which all entries m_{ij} in the Coxeter matrix are either 2 or ∞ a *right-angled Coxeter group.* In such groups every generator has order 2 and the only relations between the generators are commuting relations of the form $ab = ba$. We will be particularly interested in right-angled Coxeter groups because the constructions detailed in Section 5 for arbitrary Coxeter groups are greatly simplified in the right-angled case. These groups are referred to as 'graph groups' because all the information of their presentation can be encoded in terms of a graph whose vertex set is the generating set and in which there is an edge between any two commuting generators. Such a graph is useful because we are able to deduce information about the associated group from its generating graph.

We observe that the right-angled Coxeter group defined by the complete graph K_n is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$, since all generators commute and every generator has order 2. The right-angled Coxeter group associated to n isolated points is the free product of n copies of $(\mathbb{Z}/2\mathbb{Z})$, because the only relations in the group are those which say that every generator has order 2.

Given a graph Γ, the Coxeter group generated by any induced subgraph Δ of Γ will be a subgroup of $W(\Gamma)$. This is because every edge in Γ between vertices of $V(\Delta)$ belongs to Δ, so we are considering the subgroup generated by a subset of the generating set in which the relations between the generators in the subgroup are the same as the relations between these elements in the larger group [1, p.294]. We can think of these as being subgroups which are themselves Coxeter groups. For an arbitrary Coxeter system with generating set S, we call the subgroup generated by $T \subset S$ a *special subgroup*.

We call a subset T of the generating set *spherical* if the special Coxeter subgroup it generates is finite. A right-angled Coxeter group W is finite if and only if all the vertices in the defining graph are adjacent: if we have two vertices a and b which are not adjacent then the subgroup they generate $\langle a, b\rangle$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ because there is no relation in the larger group which involves both a and b. Therefore, for a right-angled Coxeter group the spherical subsets of S are the sets of vertices which belong to a complete subgraph of Γ.

Suppose there is a directed edge labelled with the generator $a \in W$ from x to xa in the Cayley graph of a Coxeter group W. Then there is also a directed edge from xa to x labelled a because $(xa)a = xa^2 = xe = x$. Therefore, when considering a Coxeter group it will be more convenient to consider the undirected Cayley graph in which there is an undirected edge labelled a between the vertices x and xa. We will see that such a graph is *simplicial*, which will be useful when we want to define the Davis complex of a Coxeter group in Section 5.

3 CAT(κ) Spaces

Now that we have defined Coxeter groups, we consider the metric spaces which they will act on. To define a CAT(κ) space we begin by defining *model spaces*, M_κ^n, for n-dimensional space with curvature κ. We will define M_κ^n to be a n-dimensional Riemannian manifold with constant sectional curvature κ.

We begin by defining M_0^n to be $\mathbb{R}^n$ with the usual Euclidean metric. This space is a Riemannian manifold with the Riemannian metric,

$$g_0 = \sum_{i=1}^{n} dx^i \otimes dx^i.$$

If $\kappa > 0$ then we define M_κ^n to be the n-dimensional sphere of radius $\frac{1}{\sqrt{\kappa}}$. As a submanifold of $\mathbb{R}^{n+1}$ the metric g_0 induces a metric on M_κ^n, where we think of the tangent space at a point $x \in S^n$ as a vector subspace of $T_x\mathbb{R}^{n+1}$. This manifold has constant sectional curvature κ [2, p.15].

If $\kappa < 0$ then M_κ^n is defined by scaling the metric on the hyperboloid model of n-dimensional hyperbolic space. We define the Minkowski metric on $\mathbb{R}^{n+1}$ by

$$\langle x, y\rangle_{-1} := -x_0y_0 + \sum_{i=1}^{n} x_iy_i$$

and define $\mathbb{R}^{n,1}$ to be the set $\mathbb{R}^{n+1}$ with this metric. Similarly to how we defined M_κ^n as a submanifold of $\mathbb{R}^{n+1}$ when $\kappa > 0$, when $\kappa < 0$ we define M^n_κ to be the submanifold of $\mathbb{R}^{n,1}$ of length $\frac{1}{\kappa}$ vectors,

$$M^n_\kappa = \left\{ x \in \mathbb{R}^{n,1} \mid \langle x, y \rangle_{-1} = \frac{1}{\kappa} \right\}$$

There is a pseudo-Riemannian metric on $\mathbb{R}^{n,1}$ defined by [2, p.93]

$$\sum_{i=1}^{n} dx^i \otimes dx^i - dx^0 \otimes dx^0.$$

and the restriction of this metric to M^n_κ is a Riemannian metric [2]. Thus, for any n and any $\kappa \in \mathbb{R}$ we have exhibited M^n_κ as a manifold. As a Riemannian manifold, M^n_κ has an induced distance function with respect to which it is a geodesic metric space.

We would like to understand the curvature of an arbitrary metric space (X, d) but if X is not a Riemannian manifold then we cannot apply the usual tools of Riemannian geometry to calculate sectional curvature. Therefore, we define curvature by comparing triangles in X to triangles in some Riemannian manifold whose curvature we do understand. Informally, a geodesic metric space is CAT(κ) if geodesic triangles in X are no fatter than geodesic triangles in M^2_κ. An arbitrary metric space is then said to have curvature at most κ if it is locally CAT(κ). To make this statement precise we will need to define the *comparison triangle* for an arbitrary geodesic triangle Δ in X.

If we have three points $x, y, z \in X$ which are pairwise connected by fixed geodesic segments, then we call this a *geodesic triangle* $\Delta(x, y, z)$. If we can choose three points $x', y', z' \in M^2_\kappa$ which are pairwise the same distance apart as x, y and z then we join these points by geodesic segments and call the resulting figure a *comparison triangle* for x, y and z [2, I.2.13]. The only obstruction to constructing such a triangle in M^n_κ is when $\kappa > 0$. In this case the distance between points cannot be arbitrarily far apart: a comparison triangle exists in M^n_κ if and only if $d(x, y) + d(y, z) + d(z, x) \leq \frac{2\pi}{\sqrt{\kappa}}$, the circumference of a great circle [2, p.159]. Given a point s lying on a geodesic segment in the triangle, we say that s' lying on the corresponding geodesic segment in M^2_κ is a *comparison point* if both points are the same distance from the endpoints of the geodesic segment on which they lie. We then define a metric space to be CAT(κ) if for any geodesic triangle Δ for which we can define a comparison triangle Δ', all points in Δ satisfy the CAT(κ) inequality. This says that given two points $s, t \in \Delta$ and comparison points $s', t' \in \Delta'$ we have,

$$d_X(s, t) \leq d_\kappa(s', t'),$$

[2, II.1.1]. If the geodesic segments in X are linearly parametrised as $\gamma_i : [0, 1] \to X$ and the corresponding geodesic segments in $\mathbb{R}^2$ are parametrised as $\gamma'_i : [0, 1] \to \mathbb{R}^2$ for $i, j \in \{1, 2, 3\}$ then this says that

$$d_X(\gamma_i(s), \gamma_j(t)) \leq d_\kappa(\gamma'_i(s), \gamma'_j(t)), \quad \forall s, t \in [0, 1].$$

Henceforth, we will only be concerned with CAT(0) spaces, so we will discuss several properties of these spaces which will be used throughout.

Lemma 3.1. [2, II.1.4 (3)] *Metric balls in CAT(0) spaces are convex: the geodesic segment between any two points lying within a ball does not leave the ball*

Proof. Let X be CAT(0), let $x_0 \in X$ and let $R > 0$. If we choose $x, y \in B_R(x_0)$ then we can form a geodesic triangle $\Delta(x, y, x_0)$ and a comparison triangle $\Delta'(x', y', x'_0)$ in M^2_0. Then, by the definition of a comparison triangle, $d(x', x'_0) = d(x, x_0) \leq R$ and $d(y', y'_0) = d(y, y_0) \leq R$, so x' and y' belong to the ball $B_R(x'_0)$ in M^2_0. Pick any point z lying along the geodesic between x and y in X and a comparison point z' in M^2_0, then by the CAT(0) inequality we know that,

$$d(z, x_0) \leq d(z', x'_0) \leq R$$

since balls in M^2_0 are convex, so any point z' lying along the geodesic between x' and y' lies within $B_R(x'_0)$. Therefore, the geodesic segment between x and y is contained within $B_R(x_0)$. □

We say that a topological space X is *contractible* there is a homotopy from the identity map to a constant map. For CAT(0) spaces we have the following result:

Proposition 3.2. [2, II.1.4 (4)]*: A CAT(0) space X is contractible.*

To prove this statement we will require two lemmas concerning the geodesic segments in a CAT(0) space.

Lemma 3.3. [2, II.1.4 (3)]*: Geodesic segments in CAT(0) spaces are unique.*

Proof. If there are two geodesics γ and γ' between $x, y \in X$ then we can choose a point z lying along γ and form a geodesic triangle $\Delta(x, y, z)$. In the comparison triangle $\Delta'(x', y', z')$ in $\mathbb{R}^2$ we know that $d(x', z') + d(z', y') = d(x', y')$, which implies that z' lies on a line between x' and y'. Therefore, since geodesic triangles in X are no thicker than geodesic triangles in $\mathbb{R}^2$, and the triangle in $\mathbb{R}^2$ is degenerate we conclude that both geodesic segments are the same [12]. $\square$

Fix a point $x_0 \in X$, then we can choose a unique (linearly parametrised) geodesic $\gamma_x : [0, 1] \to X$ from x to x_0.

Lemma 3.4. [2, II.1.4 (1)]*: If $x_n \to x$ then $\gamma_{x_n} \to \gamma_x$ uniformly. That is, the geodesic segment γ_x depends continuously on $x \in X$.*

Proof. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ large enough that $n > N$ implies that $d(x_n, x) < \varepsilon$. There exist geodesics γ'_{x_n} and γ'_x in $\mathbb{R}^2$ which form two sides of a comparison triangle for $\{x_n, x, x_0\}$. Then for any $t \in [0, 1]$ the CAT(0) inequality gives that,

$$d(\gamma_{x_n}(t), \gamma_x(t)) \leq d(\gamma'_{x_n}(t), \gamma'_x(t))$$

But the linear segment from $\gamma'_{x_n}(t)$ to $\gamma'_x(t)$ is parallel to the base of the triangle since $\{\gamma'_{x_n}(t), \gamma'_x(t), x_0\}$ form the vertices of a triangle which is similar to the triangle on $\{x'_n, x', x'_0\}$. Therefore, $d(\gamma'_{x_n}(t), \gamma'_x(t)) \leq d(\gamma'_{x_n}(1), \gamma'_x(1))$ because the length of any line in a Euclidean triangle parallel to the base is shorter than the base itself.
Therefore, we have,

$$d(\gamma_{x_n}(t), \gamma_x(t)) \leq d(\gamma'_{x_n}(t), \gamma'_x(t)) \leq d(\gamma'_{x_n}(1), \gamma'_x(1)) = d(x_n, x) < \varepsilon$$

by the definition of a comparison triangle. Therefore, $\gamma_{x_n} \to \gamma_x$ uniformly on $[0, 1]$. $\square$

Proof of Proposition 3.2. Because we can move each point in X towards x_0 continuously along the unique geodesic γ_x we can define a homotopy $F : [0, 1] \times X \to X$ by $F(t, x) = \gamma_x(t)$. Note that F is continuous by Lemma 3.4 and we have that $F(0, x) = \gamma_x(0) = x$ whereas $F(1, x) = \gamma_x(1) = x_0$, so F is indeed a homotopy [5, p.501]. $\square$

Finally we note that if $\kappa \leq 0$ then any CAT(κ) space is also CAT(0) because triangles in M^2_κ are thinner than those in M^2_0, so these results go through for any negatively curved metric space. Note that it is also true that if $\kappa' \leq \kappa$ then a CAT(κ') space is also a CAT(κ) space [2, II.1.12 (1)]. However, we will not need this result because we will primarily be concerned with CAT(0) spaces.

4 Complexes

In order to construct the Davis complex we will need to understand some results concerning simplicial and cell complexes, so we discuss the construction of these objects here as well as some of their properties.

Firstly, a *convex polytope* is the convex hull of any set of points $\{p_0, p_1, ..., p_k\}$ in $\mathbb{R}^n$. This is the smallest convex set which contains these points. If we require that our set of $(k + 1)$ points do not lie in any plane of dimension less than or equal to k, then we obtain a k-simplex. Concretely, this is the convex hull of $k + 1$ points $\{p_0, ..., p_k\}$ in $\mathbb{R}^n$ which do not lie in a single n-plane. That is, the vectors $\{p_i - p_0\}_{i=1}^n$ are linearly independent [5]. Given such points, the simplex they span is the set

$$\left\{ \sum_{i=0}^{k} \lambda_i p_i \;\middle|\; \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0 \right\} \subset \mathbb{R}^n,$$

which is denoted $[v_0, ..., v_k]$ [7, p.9]. We denote the simplex spanned by the basis vectors in $\mathbb{R}^{n+1}$ by Δ^n.

The *faces* of a simplex are the simplices spanned by some subset of the vertex set. We then define a *simplicial complex* to be a collection of simplices in some ambient space which are glued together along faces so that any two simplices in the complex are either disjoint or they share a face [13, p.169]. The dimension n of this simplicial complex is the maximum dimension of its simplices. An isomorphism of simplicial complexes is a bijective map between two complexes such that the image of the vertex set of a simplex spans a simplex.

We define an *abstract simplicial complex* or *simplicial set* K to be a non-empty set V (the vertices of the complex) and a collection $\mathcal{S}$ of subsets of V such that the singleton sets $\{v\}_{v\in V}$ belong to $\mathcal{S}$ and if $S \in \mathcal{S}$, then any non-empty $T \subseteq S$ also belongs to the collection [2, p.123]. Given an n-simplex S in K (an element of $\mathcal{S}$ with cardinality n) we can define a (non-abstract) simplex in the real vector space V spanned by S, $\mathbb{R}^{|S|}$. The geometric realisation of the simplex is the convex hull of the basis S in $\mathbb{R}^{|S|}$, which we denote $\Delta^{|S|-1}$ as above [2, p.124]. The geometric realisation of the abstract simplicial complex K is defined to be the union of the geometric realisations of the simplices of K. If $T \subset S$ and $T \subset S'$ then we identify the simplices corresponding to S and S' along the face which represents T [2, p.124].

We note that we can also realise any poset as a simplicial complex in a similar manner, by converting a chain of length $k+1$, $a_0 \leq a_1 \leq .... \leq a_k$ in the poset to a k-simplex in the vector space generated by the elements of the chain [2, p.370].

Another generalisation of the notion of a simplicial complex is that of a *cell* or *CW complex*. We build a cell complex X inductively, beginning with a set of vertices or 0-cells which we refer to as the 0-skeleton X^0. We then attach 1-cells or arcs which pass between the points in X^0 to form the 1-skeleton X^1. In general we construct the n-skeleton X^n by attaching n-cells to the $(n-1)$-skeleton X^{n-1} [7, p.5]. A *cellulation* of a topological space is a homeomorphism from the space to a cell complex which allows us to think of the topological space as a cell complex [5].

The Davis complex will be a *convex cell complex*, this is a CW complex in which every cell is a convex polytope. We conceive of each polytope as belonging to an ambient Euclidean space, so it has an inherited metric and then the cell complex has the length metric, in which the distance between two points is the infimal length of all paths between the points [5, p.232]. If every cell in the CW complex is isometric to $[0,1]^n$, then we call the complex a *cube complex*. To form a cube complex K, we take a disjoint union of cubes $\bigsqcup_{\lambda\in L}[0,1]^{n_\lambda}$ and a family of isometries between faces of the cubes and then glue the cubes together along these isometric faces by identifying points x and x' such that $x' = f(x)$ for some isometry f [11, p.5]. A cube complex is itself a convex cell complex so it has the natural metric induced by the Euclidean metric on each cube [4].

The *link* of a vertex v in a convex cell complex is an object which conveys local information about the neighbourhood of the vertex. We form the link of a vertex by intersecting a small $(n-1)$-dimensional sphere centred on a vertex with the simplex, so that the vertices will be the points where the sphere intersects the edges incident to v [12] [11, p.5]. In general the k-skeleton of the link will be the intersection between the sphere and the $(k+1)$-skeleton of the complex so that a set of vertices in the link will span a simplex if the edges they correspond to all belong to a single face in the cell complex. For instance, the link of a vertex in the cube complex $[0,1]^2$ is the path graph P_2, as shown in Figure 1.

Figure 1: The link of a vertex in $[0,1]^2$

In the case of cube complexes, the link will always be a union of simplices, but it may fail to be a simplicial complex: if we join two copies of $[0,1]^2$ along two edges as shown in Figure 2, then the link of the vertex which is incident to the two edges we join has the link shown on the right, which is not a simplicial complex. Moreover, if the cell complex is not a cube complex then the link of a vertex will not necessarily be comprised of simplices: the link of a vertex in an octahedron is a square, since four edges meet at any vertex.

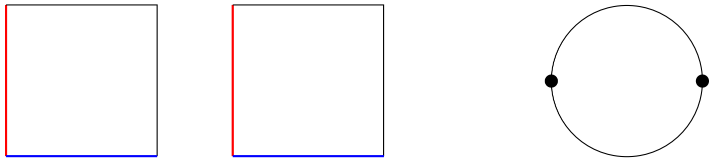

Figure 2: A cube complex such that the link of a vertex is not a simplicial complex

We say that a simplicial complex or set K is a *flag complex* if the vertices of every complete graph in the 1-skeleton of K span a simplex [11, 2]. This means that flag complexes are uniquely specified (up to isomorphism) by their 1-skeletons. We would like to consider cube complexes which are non-positively curved in the sense of section 3. Although this is not true in general, there is a necessary and sufficient condition due to Gromov [6], the proof of which may be found in Chapter II.5 of [2] and Appendix I of [5]:

Theorem 4.1 (Gromov)**.** *A finite dimensional cube complex has non-positive curvature (is locally CAT(0)) if and only if the link of each vertex is flag.*

This is in fact two conditions: we require that the link of each vertex is a simplicial complex and that this complex is flag. Intuitively, if the link of a vertex is flag, then if we choose three points lying on edges near the vertex, then either the shortest path between them in the ambient space stays within the complex, or the edges they lie on form a tree with the vertex, which is non-positively curved. As an example, the link of any vertex in the cube complex $[0,1]^3$ is shown in Figure 3.

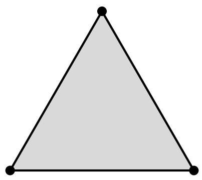

Figure 3: The link of a vertex in $[0,1]^3$.

The complete graph K_3 in the 1-skeleton spans a 2-simplex so the link of every vertex is flag which implies that $[0,1]^3$ has non-positive curvature. If instead we were to consider the 2-skeleton of $[0,1]^3$ then the link of any vertex would be K_3. The three vertices of K_3 are adjacent, but do not span a simplex, so the 2-skeleton of $[0,1]^3$ is not a metric space of non-positive curvature.

Theorem 4.1 tells us that a cube complex has non-positive curvature. To obtain the stronger result that the space is CAT(0) we need the complex to be simply connected:

Lemma 4.2. *A simply connected, complete metric space X of non-positive curvature is CAT(0).*

Proof. The Cartan-Hadamard theorem for metric spaces ([2, II.4.1 (2)]) implies that the universal cover of a complete, connected metric space of non-positive curvature is CAT(0). If X is simply connected, then it is its own universal cover, so X is CAT(0) by the Cartan-Hadamard theorem. □

A proof of the Cartan-Hadamard Theorem is detailed in Chapter II.4 of [2].

5 The Davis Complex

Given a Coxeter group W, we associate with it a CAT(0) space called the *Davis Complex* which we denote Σ. We will give a general overview of the construction of the Davis complex Σ before describing it in more detail.

The most natural way to obtain a space which a given group acts on is to find its Cayley graph. However, we would also like the space to be CAT(0), which is only true if the Cayley graph is a tree: any Cayley graph Δ is a cube complex and the link of any vertex in the Cayley graph consists of isolated points, so Theorem 4.1 implies that this graph has non-positive curvature. However, if there are relations in the group then the Cayley graph will contain a loop, and is therefore not simply-connected. Therefore, we will attach 2-dimensional convex polytopes to fill any loop in the Cayley graph so that the complex becomes simply-connected. This is the *Cayley 2-complex* of a Coxeter group [10]. The Cayley 2-complex is not necessarily a cube complex unless W is a right-angled Artin group and even if W is right-angled, the link of a vertex in this complex is not necessarily flag: the Cayley 2-complex of $(\mathbb{Z}/2\mathbb{Z})^3$ is the 2-skeleton of $[0,1]^3$ and the link of any vertex in this complex is the complete graph on 3-vertices, which is not flag. Therefore, we continue to attach convex polytopes of higher dimensions to the complex until we obtain a space which satisfies the properties we would like. We will show that the complex obtained by this procedure is CAT(0).

Throughout we will consider three running examples. The first will be the right-angled Coxeter group W_1 generated by the disjoint union of the path graph P_2 and an isolated point. This gives the presentation $\langle a, b, c \mid ab = ba, a^2 = b^2 = c^2 = e\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2 * (\mathbb{Z}/2\mathbb{Z})$. Part of the (undirected) Cayley graph of this group is shown in Figure 4.

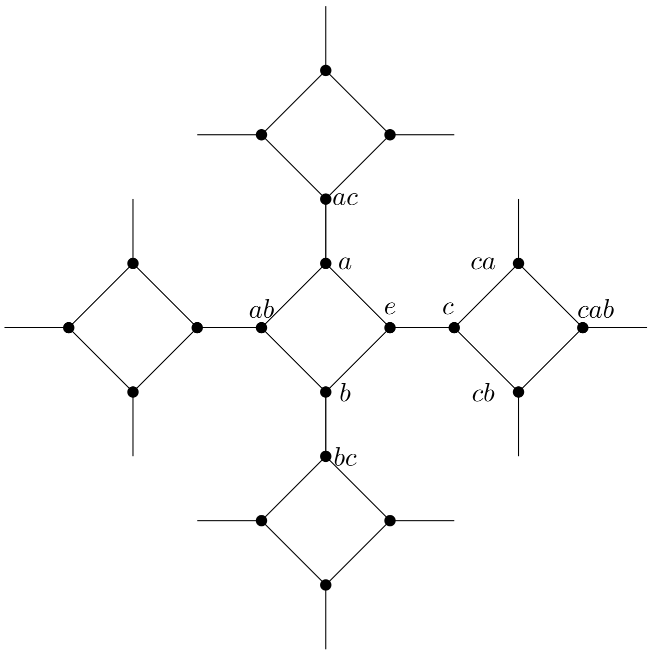

Figure 4: The Cayley graph of the example group W_1.

The second running example will be the (non right-angled) Coxeter group

$$W_2 = \langle a, b, c \mid aba = bab, bcb = cbc, a^2 = b^2 = c^2 = e\rangle.$$

Part of the Cayley graph of this group is shown in Figure 5.

The third running example will be $(\mathbb{Z}/2\mathbb{Z})^3 \cong \langle a, b, c \mid ab = ba, bc = cb, ca = ac, a^2 = b^2 = c^2 = e\rangle$, whose Cayley graph is the 1-skeleton of $[0,1]^3$.

To define the Davis complex formally we will first need a number of definitions. We define $\mathcal{S}$ to be the poset of spherical subsets of the generating set S, ordered by inclusion. The poset obtained by removing the empty set $\mathcal{S}_{>\emptyset}$ is a simplicial set with vertex set S (the set of generators), since if $T \in \mathcal{S}_{>\emptyset}$ and $A \subset T$ is non-empty, then A must also generate a finite subgroup in W. We call this simplicial set the *nerve* of (W, S), denoted $N(W, S)$ or N_Γ if W is right-angled [5, p.123]. Explicitly, the k-simplices of this simplicial set are the spherical subsets of cardinality $k + 1$.

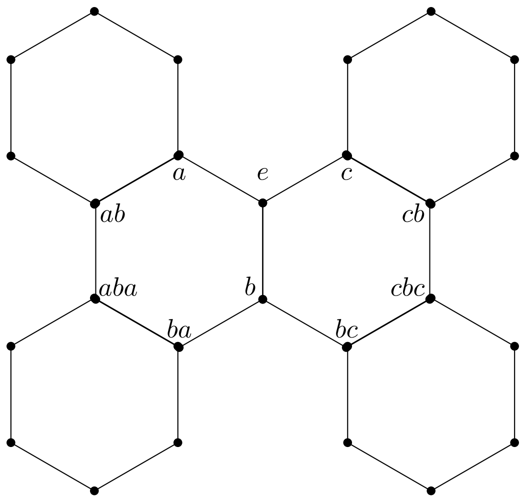

Figure 5: The Cayley graph of the second running example W_2.

The nerve of our first example W_1 has vertices labelled a, b and c. The non-empty spherical subsets are $\{a\}$, $\{b\}$, $\{c\}$ and $\{a, b\}$. Therefore the vertices a and b span a 1-simplex, so the nerve is isomorphic to the defining graph for W_1.

For our second example W_2 the non-empty spherical subsets are $\{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}$ and $\{b, c\}$. Therefore, the nerve is isomorphic to the path graph on three vertices. However, the nerve of a Coxeter system is not always a graph: in the case of $(\mathbb{Z}/2\mathbb{Z})^3$, the spherical sets are $\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}$ and $\{a, b, c\}$. The three vertices a, b and c span a 2-simplex, so the nerve is a two-dimensional simplicial complex isomorphic to the simplicial complex shown in Figure 3.

We observed that in the first example the nerve of the group W is isomorphic to the defining graph Γ. In fact the nerve of a right-angled Coxeter group is always isomorphic to the flag complex generated by its defining graph.

Proposition 5.1. ([5, 7.1.8])*: If W_Γ is a right-angled Coxeter group, then N_Γ is a flag complex.*

Proof. Suppose we have a non-empty subset T of W whose elements are pairwise adjacent in the 1-skeleton of N_Γ. If two vertices are adjacent in N_Γ then they generate a finite subgroup, so they must commute. Therefore, all the elements of T must commute in W, so they generate $(\mathbb{Z}/2\mathbb{Z})^{|T|}$. This is a finite group, so T is spherical and therefore spans a simplex in N_Γ. $\square$

For a right-angled Coxeter group the simplices in the 1-skeleton of N_Γ are those formed by the size two spherical subsets. But these are the sets of commuting generators, so the 1-skeleton of the nerve is isomorphic to the defining graph Γ. Therefore, we can think of the nerve of W_Γ as the flag complex generated by the graph Γ.

We define a *spherical subgroup* to be a finite subgroup W_T of W generated by a spherical subset T. Then $W\mathcal{S}$ is defined to be the set of all cosets of spherical subgroups viewed as subsets of the generating set S [5, p.125]. This can be expressed as the disjoint union

$$W\mathcal{S} = \bigsqcup_{T \in \mathcal{S}} W/W_T.$$

We allow $T = \emptyset$, in which case $W_\emptyset = \{e\}$ and $W/W_\emptyset = W$. $W\mathcal{S}$ is partially ordered by inclusion. The following result gives a more concrete formulation of this partial order.

Proposition 5.2. [Section 1.2 of [3]] *With the notation as above, we have $wW_T \leq w'W_{T'}$ if and only if $T \subset T'$ and $w^{-1}w' \in W_{T'}$*

Proof. If $wW_T \subset w'W_{T'}$, then $W_T \subset w^{-1}w'W_{T'}$ so because $e \in W_T$ we also know that $e \in w^{-1}w'W_{T'}$. This means that $w^{-1}w'W_{T'} = W_{T'}$, since this is the only coset which contains the identity, and hence

$w^{-1}w' \in W_{T'}$. We know that if $W_T \subset W_{T'}$ then $T \subset T'$ since if we have $x \in T$ but $x \notin T'$ then $x \notin W_{T'}$ but $x \in W_T$. The converse is clearly also true. $\square$

We define the Davis complex Σ to be the simplicial complex which is the geometric realisation of this poset [5, p.126]. The vertex set of Σ is the set $W\mathcal{S}$ of cosets and each n-simplex is a chain of length $(n+1)$ in $W\mathcal{S}$. We observe that W acts on a simplex of Σ by left multiplication,

$$g \cdot (w_0 W_{T_0} \subset ... \subset w_n T_n) = g w_0 W_{T_0} \subset ... \subset g w_n T_n.$$

The 1-skeleton of the Davis complex for $\mathbb{Z}/2\mathbb{Z}$ is shown in Figure 6. Each of the small triangles should span a 2-simplex as suggested by the shading in the top left corner. In this case there are no chains of length 4 in $W\mathcal{S}$ so the dimension of the simplicial complex is 2.

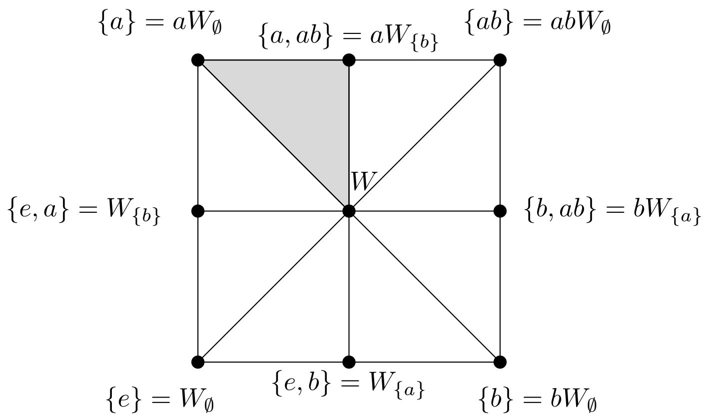

Figure 6: The Davis complex Σ for $\mathbb{Z}/2\mathbb{Z}$.

The description of the Davis complex as a simplicial complex is due to Davis. We also have a cellulation of Σ by convex polytopes due to Moussong [9], which aligns more closely with the geometry of the Cayley graph of W. The way we have depicted Σ in Figure 6 is suggestive of how we will cellulate Σ in general. We have a vertex $xW_\emptyset$ for each $x \in W$ so we can forget about the other vertices and attach convex polytopes whose vertices are of the form $xW_\emptyset$. We will clearly obtain a cube complex by attaching a copy of $[0,1]^2$ to the outer loop in Figure 6.

In general, to cellulate Σ, we will need objects known as Coxeter polytopes which are defined using the canonical action of a Coxeter group on $\mathbb{R}^n$ [5, p.128]. Given any finite Coxeter group W the canonical representation of W on $\mathbb{R}^n$ has a *fundamental chamber* whose translates cover the space and we can choose a point x belonging to the fundamental chamber [5, p.129]. We then define the Coxeter polytope to be the convex hull of $W \cdot x$, the orbit of the point x in the ambient space [8] [5, p.128]. In general the Coxeter polytope of a given group might be quite complicated, but in the case of right-angled Coxeter groups we have the following result:

Proposition 5.3. *Given a finite right-angled Coxeter group with n generators, the associated Coxeter polytope is* $[0,1]^n$.

Proof. Consider the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}$ by rotation about the point $\frac{1}{2}$. The point $1 \in \mathbb{R}$ belongs to the fundamental domain and its orbit is $\{0,1\}$, which has the convex hull $\{\lambda_0 \cdot 0 + \lambda_1 \cdot 1 \mid \lambda_0 + \lambda_1 = 1, \lambda_i \geq 0\} = [0,1]$. If $W \cong W_1 \times W_2$ then the orbit of a point $(x,y) \in W_1 \times W_2$ is $(W_1 \cdot x) \times (W_2 \cdot y)$, so the Coxeter polytope C_W is isomorphic to $C_{W_1} \times C_{W_2}$ [5, p.129].
Therefore, because any finite right-angled Coxeter group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ its Coxeter polytope is the n-cube $[0,1]^n$. $\square$

We have already observed that Σ contains a vertex $xW_\emptyset$ for each element of the group, and we define the 0-skeleton of Σ to be these vertices, which we identify with the set of elements of W. To define the n-skeleton of Σ, we attach the Coxeter polytope of the finite Coxeter group W_T along the vertices of each coset wW_T of W_T for each size n spherical subset T. In the example of $\mathbb{Z}/2\mathbb{Z}$ we begin with the vertices $\{W_\emptyset, aW_\emptyset, bW_\emptyset, abW_\emptyset\}$. Then for the spherical subsets of size 1, $\{a\}$ and $\{b\}$ we have the spherical cosets

$W_{\{a\}} = \{e, a\}$, $W_{\{b\}} = \{e, b\}$, $bW_{\{a\}} = \{b, ab\}$ and $aW_{\{b\}} = \{a, ab\}$. For each of these we attach a copy of $[0, 1]$ with one endpoint at each element of the coset. For instance $W_{\{a\}}$ gives us an edge between the vertices $W_\emptyset$ and $aW_\emptyset$. The 2-skeleton is then formed by attaching a copy of $[0, 1]^2$ along the coset $\{e, a, b, ab\}$. The final result is shown in Figure 7. We label the coset which generated each cell.

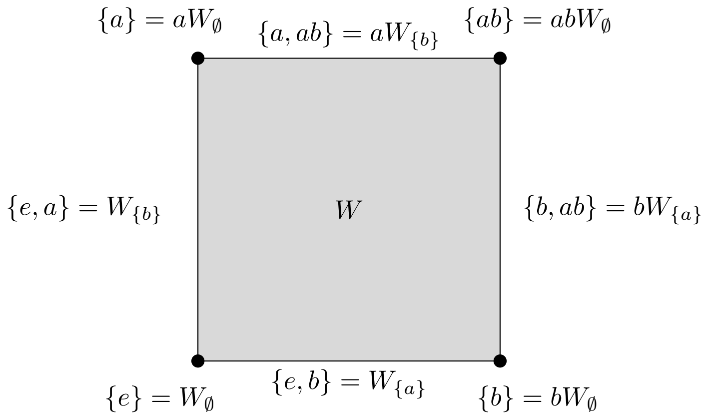

Figure 7: The cellulated Davis complex for $\mathbb{Z}/2\mathbb{Z}$.

We can given concrete descriptions of the 0, 1 and 2-skeletons of this complex in terms of other structures. We have already stated that the vertex set of Σ can be identified with W. An arbitrary cardinality 1 spherical subset is of the form $\{a\}$ for some generator $a \in S$. Then given $x \in W$, $xW_{\{a\}} = x\{e, a\} = \{x, xa\}$, which means we have an edge in the 1-skeleton between two vertices if they differ by right multiplication by a generator. That is, the 1-skeleton is the unoriented Cayley graph of W with respect to S. A cardinality two spherical subset is generated by any two generators which satisfy some relation, say a_1 and a_2, then the elements of $W_{\{a_1,a_2\}}$ form a loop in the Cayley graph and we attach the Coxeter polytope (a 2-cell) of $W_{\{a_1,a_2\}}$ along this loop. Since these are the only relations in a Coxeter group, the 2-skeleton is the Cayley 2-complex, in which every relation bounds a disk. We repeat this process for all spherical sets. In the case of a right-angled Coxeter group we attach a copy of $[0, 1]^n$ for every set of n commuting generators, so the Davis complex is a cube complex.

This description will allow us to construct the Davis complex for our running examples by extending their Cayley graphs. In W_1 the only spherical set of cardinality 2 is $\{a, b\}$, so we attach a copy of $[0, 1]^2$ for each coset $xW_{\{a,b\}}$. The result is shown in Figure 8 where the cosets which form the vertices of 2-cells have been marked.

For our second running example we have two cardinality 2 spherical subsets $\{a, b\}$ and $\{b, c\}$. These generate the subgroups $W_{\{a,b\}} = \{e, a, b, ab, ba, aba\}$ and $W_{\{b,c\}} = \{e, b, c, bc, cb, bcb\}$. Each coset of these two subgroups forms the vertices of a 2-cell which we attach to the graph. The Davis complex is an infinite series of hexagonal cells joined together along edges, as shown in Figure 9.

In the first example, we observe that the link of every vertex is isomorphic to the defining graph. This is an example of a more general phenomenon:

Proposition 5.4. *The link of any vertex in the Davis (cell) complex of any Coxeter group is isomorphic to the nerve of the group, $N(W, S)$.*

Proof. We first prove this for the vertex $W_\emptyset \in \Sigma$. Let T be a non-empty spherical subset of cardinality n, then T spans an $(n-1)$-simplex in $N(W, S)$. In cellulating Σ we attached an n-cell with a vertex at $W_\emptyset$ for each spherical subgroup W_T and so the link of $W_\emptyset$ contains an $(n-1)$-simplex for each of these cells. This gives a correspondence between simplices in $N(W, S)$ and the link of $W_\emptyset$, so they are isomorphic as simplicial complexes.
To apply this result to an arbitrary vertex we note that the action of W on the vertices of Σ is simply the action of W on its Cayley graph, which is transitive. Therefore, the neighbourhood of a vertex $gW_\emptyset \in \Sigma$ is mapped isometrically onto a neighbourhood of $W_\emptyset$ by left-multiplication by g^{-1}. This means that the neighbourhood of any vertex looks the same, and their links are isomorphic as simplicial complexes. $\square$

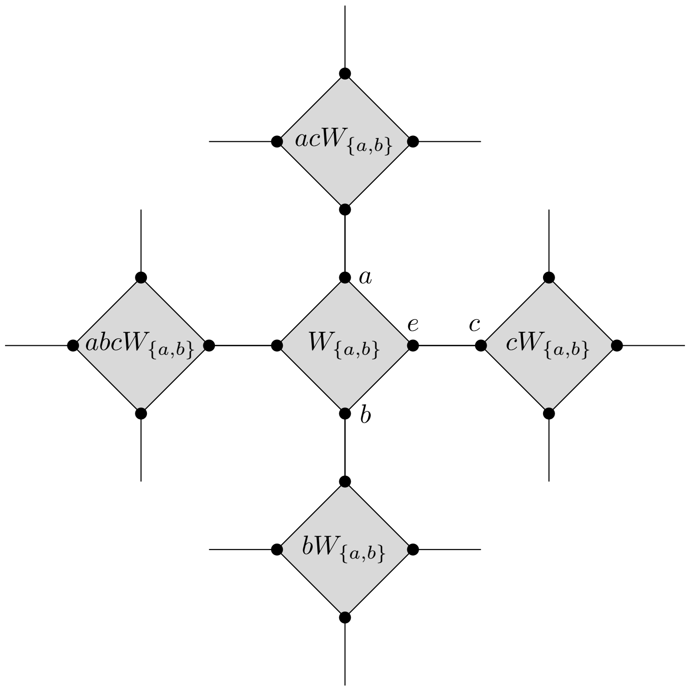

Figure 8: The cellulated Davis complex for our first running example.

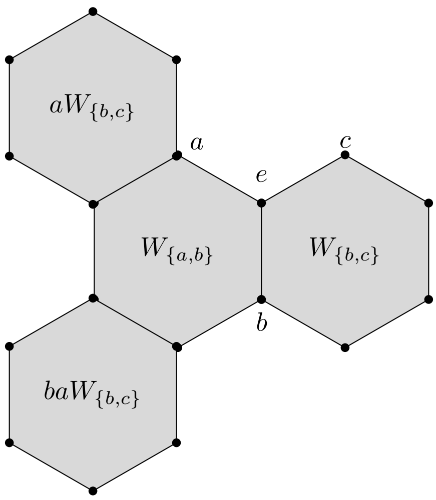

Figure 9: Part of the Davis complex of our second running example

In the case of right-angled Coxeter groups we proved that the nerve is the flag complex generated by the defining graph Γ, since the flag complex generated by a dimension one simplicial complex is unique. This means that the link of every vertex is isomorphic to the flag complex on Γ. Therefore, the theorem of Gromov above implies that the Davis complex Σ of a right-angled Coxeter group has non-positive curvature. To prove that this cube complex is in fact CAT(0) we can apply Lemma 4.2 once we have the following result.

Lemma 5.5. ([5, 7.3.5])*: The Davis complex of a Coxeter system* (W, S) *is simply connected.*

Proof. By Corollary 4.12 of [7] the inclusion of the 2-skeleton Σ^2 into Σ induces an isomorphism of fundamental groups. But we see that Σ^2 is simply-connected because we can push loops in Σ^2 homotopically into Σ^1 and any loop in Σ^1 bounds a disk in Σ^2 by construction, so it is null-homotopic [12]. $\square$

Since the Davis complex is complete, these results imply the following theorem, first proved by Gromov [6].

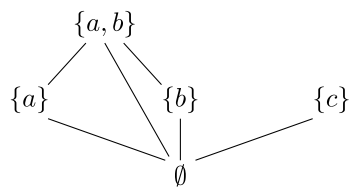

Figure 10: The partially ordered set $\mathcal{S}$ for our first running example W_1

Theorem 5.6. *The Davis complex of a right-angled Coxeter group is CAT(0).*

We cannot use 4.1 to prove that the Davis complex of any Coxeter group is CAT(0), since the Davis complex will not necessarily be a cube complex. However, in his PhD thesis [9], Moussong showed that the Davis complex of any Coxeter group is CAT(0) by showing that the link of any vertex is CAT(1). This proof is outlined in Appendix I of [5]. Thus, we know that for any Coxeter group we can construct a CAT(0) complex which the group acts on by isometries. However, we can deduce more information from this action if we can show that this action respects the geometric structure of Σ. We define a *geometric* action of a group G on a metric space X to be an action by isometries which is *cocompact* and *proper*. An action by isometries is cocompact if there exists a compact set K whose orbit is all of X, $G \cdot K = X$. An action by isometries is proper if for any $x_0 \in X$ there exists an $R > 0$ such that the set

$$\{g \mid gB_R(x_0) \cap B_R(x_0) \neq \emptyset\}$$

is finite. We will work towards the following result:

Theorem 5.7. *The action of a Coxeter group W on its associated Davis complex is geometric.*

To prove that the action is cocompact we will need to consider the *chamber* K for the action of W on Σ. We define K to be the geometric realisation of the poset $\mathcal{S}$. We can obtain this from the nerve of the graph by attaching a new vertex which belongs to every simplex. We can identify this with the join of the barycentric subdivision of the nerve with an additional vertex, since there is an edge from the vertex $\emptyset$ to every simplex of $N(W, S)$ [5, p.127]. The poset $\mathcal{S}$ for our first running example is depicted in Figure 10.

An n-simplex in K is defined by a chain of spherical subsets,

$$T_0 \subset T_1 \subset ... \subset T_n.$$

An n-simplex in Σ is defined by a chain of length $n+1$ in $W\mathcal{S}$,

$$w_0 W_{T_0} \subset w_1 W_{T_1} \subset .... \subset w_n W_{T_n}$$

This gives us a natural way to embed K in Σ by sending the simplex $T_0 \subset .... \subset T_n$ to $W_{T_0} \subset ... \subset W_{T_n}$. We identify K with its image under this inclusion. In the case of our first running example the chamber is shown in Figure 11, it is then shown embedded in the Davis complex using this map in Figure 12.

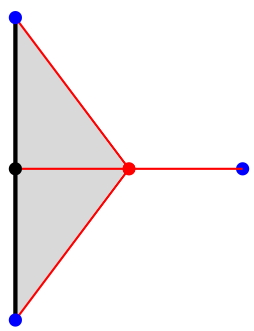

Figure 11: The chamber K of the first example W_1.

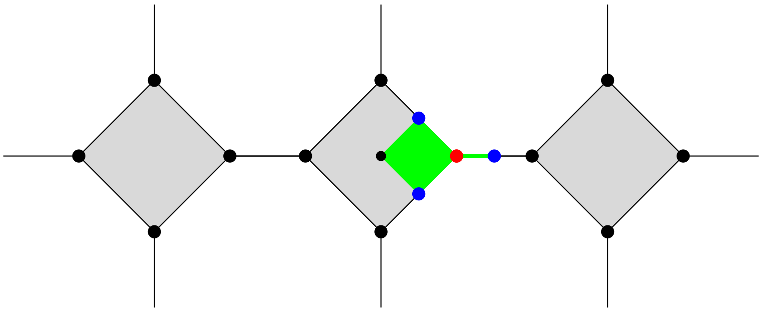

Figure 12: The chamber K embedded in the Davis complex of W_1

In the case of our second running example W_2 the link of any vertex is isomorphic to P_3, which is shown on the left in Figure 13. The chamber K is then shown on the right. This is then shown superimposed on the Davis complex in Figure 14.

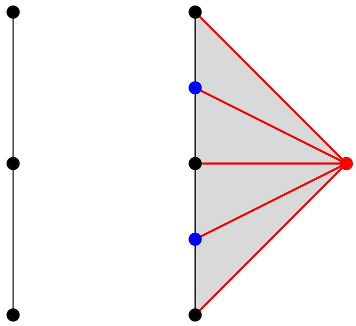

Figure 13: The nerve and the chamber K for the second example W_2.

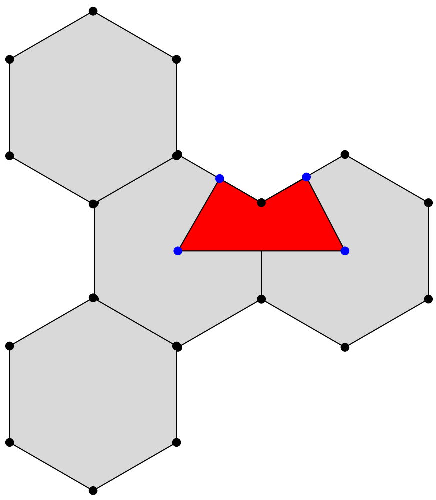

Figure 14: The chamber K embedded in the Davis complex of W_2

Considering Σ as a simplicial complex, we can show that the orbit of the chamber K is the entire Davis complex Σ.

Lemma 5.8. ([5, 7.2.3]) *The action of W on Σ is cocompact. In particular, W translates of the chamber K (the geometric realisation of $\mathcal{S}$) cover Σ.*

Proof. We know that the chamber K is compact as a finite dimensional simplicial complex on finitely many vertices. We will show that its orbit covers Σ. Let $w_0 W_{T_0} \subset ... \subset w_n W_{T_n}$ define an n-simplex in Σ. We proved above that if $wW_T \subset w' W_{T'}$ then $T \subset T'$ and $wW_{T'} = w'W_{T'}$, so we can rewrite the chain as,

$$w_0 W_{T_0} \subset w_0 W_{T_1} \subset ... \subset w_0 W_{T_n} = w_0 \left(W_{T_0} \subset W_{T_1} \subset ... \subset W_{T_n} \right).$$

Thus, w_0 sends a simplex belonging to K to the given chain, so W-translates of K cover Σ. □

Lemma 5.9. *The action of W on Σ is proper.*

Proof. To prove this we treat Σ as a cell-complex rather than a simplicial complex. First we prove that the stabiliser of any cell C in Σ is finite. The vertex set of a cell C corresponds to the elements of a spherical coset wW_T. Because the action of W on Σ sends vertices to vertices, for $x \in W$ to stabilise C it must send wW_T to itself. Therefore, $x(wW_T) = wW_T$, so $x \in wW_T w^{-1}$. This is a finite set because T is spherical and therefore at most finitely many elements stabilise the cell. In fact, because of the way we defined this action $wW_T w^{-1}$ is exactly the stabiliser of C.

Now we can prove that the action is proper: let $x_0 \in \Sigma$ and let D be the least dimension cell containing x_0, say D has dimension d. We can choose R to be less than half the distance from x_0 to any other cell of dimension d. Then $B_R(x_0)$ does not intersect any other cell of dimension d and it does not intersect any ball of radius R centred on a point in some other cell of dimension d. But we know that the action of W on Σ sends cells to cells of the same dimension, so for $g \cdot B_R(x_0)$ to have non-trivial intersection with $B_R(x_0)$, g must fix D. But we proved that the stabiliser of any cell is finite, so there are at most finitely many such g. □

Together these results imply that a Coxeter group W acts geometrically on its associated Davis complex Σ, so the Milnor-Schwarz Lemma implies that W with its word metric is quasi-isometric to the Davis complex. For this reason, such groups are referred to as *CAT(0) groups*. Many algebraic properties of CAT(0) groups are known, for instance CAT(0) groups have solvable word and conjugacy problems (see [2, p.441]). More details on the structure of CAT(0) groups can be found in [2]. There is an analogous CAT(0) cube complex known as the Salvetti complex which we can use to prove that right-angled Artin groups are CAT(0) groups [4]. However, for most Artin groups, despite their apparent similarity to Coxeter groups, it is not known whether or not such a construction is possible, and indeed for many classes of Artin groups it is not known whether the word problem is solvable [8]. Thus, the interaction between a Coxeter group and its Davis complex illustrates one of the basic ideas of geometric group theory: by constructing a space which a group will act on geometrically, we are able to determine algebraic information about a group.

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