Differential algebraic geometry and the Geometric Mordell Conjecture

Callum Sutton

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National
University

Declaration

The work in this thesis is my own except where otherwise stated.

Callum Sutton

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Abstract

Let F be a function field of one variable over a field k. In this thesis, we use methods of differential algebraic geometry to prove Mordell's Conjecture for non-hyperelliptic curves over function fields. We develop the foundations of algebraic geometry from the functor of points perspective. In particular, we study the structure of group schemes, algebraic groups and abelian varieties. Our construction of jet and arc schemes leads into an account of the theory of differential algebraic geometry, in which we discuss δ-rings, D-schemes and $\mathcal{D}_X$-modules. We construct differential characters of abelian varieties and use these maps, in conjunction witth the Mordell–Weil Theorem, to study the F-rational points on a curve.

Contents

Notation and terminology

The definition of other terms used in the text may be found using the index.

Notation

$\mathbb{A}^n$	Affine n-space
$\mathcal{O}_X$	The structure sheaf of a scheme X
$\mathcal{O}(X)$	The coordinate ring of a k-functor X
$\mathbb{P}^n$	Projective n-space
$J(-)$	The jet scheme functor
k	An associative, commutative, unital ring
$\mathrm{Hom}_k(-,-)$	Morphisms in the category of k-algebras
$\mathbf{Alg}_k$	The category of k-algebras
$\mathbf{Mod}_k$	The category of k-modules
$\mathbf{Sch}_S$	The category of schemes relative to a base scheme S
$\mathbf{GrSch}_G$	The category of group schemes relative to a base group scheme G
$\mathbf{DSch}_S$	The category of D-schemes relative to a base D-scheme S

Chapter 1

Introduction

The problem of finding non-trivial integer solutions to the equation

$$X^n + Y^n = Z^n \tag{1.1}$$

when $n \geq 3$ is a notoriously difficult problem in arithmetic geometry, introduced by Fermat in 1637 and finally resolved by Wiles in 1995. In 1922, Mordell asked instead whether homogeneous equations in three variables of genus greater than one* have at most finitely many solutions (see page 192 of [50]). The *Fermat curve* described by Equation 1.1 has genus $(n-1)(n-2)/2$ which is greater than one when $n \geq 4$, which means that Mordell's conjecture would imply a weak form of Fermat's Last Theorem. Mordell's conjecture was resolved by Faltings in 1983 who proved that a non-singular projective curve C of genus $g \geq 2$ defined over a number field k has finitely many k-rational points [30]. Here a k-rational point may be thought of as a solution to an equation with coordinates in k, although we will later reinterpret this concept as a morphism of schemes $\operatorname{Spec} k \to C$.

For a curve E of genus one defined over a number field, a related result known as the Mordell–Weil Theorem (proved by Weil in [68]) implies that the set of rational points of E is either empty or a finitely-generated abelian group. In the latter case we call E an *elliptic curve*.

In a 1940 letter to Simone Weil, André Weil articulated the importance of analogy in studying the ostensibly unrelated fields of number theory and geometry (see [69]). His insight was that because of the similarities between the rings $\mathbb{C}[t]$ and $\mathbb{Z}$, many problems in geometry could fruitfully be 'translated' into number theory and vice versa, often with the ring $\mathbb{F}_q[t]$ acting as an intermediary. Indeed, the proof that there are no solutions to Equation 1.1 over $\mathbb{C}[t]$ is elementary in comparison with the proof of Fermat's Last Theorem. Therefore, we might hope to simplify Mordell's Conjecture further by seeking $\mathbb{C}(t)$-rational points on curves of genus greater than one, rather than

*By this we mean that the curve defined by the equation has genus greater than one.

$\mathbb{Q}$-rational points. Indeed, such a translation is possible and Mordell's Conjecture for function fields, or the *Geometric Mordell Conjecture*,[†] was established prior to Faltings' proof of the analagous result for number fields. The Geometric Mordell Conjecture will be one of our primary interests in this thesis, so here we will give an overview of the history of the result.

The first published proof of the Geometric Mordell Conjecture was given by Manin in 1963 [44]. Some of the ideas of Manin's approach, which made systematic use of the *Gauss–Manin connection*, will be discussed in Appendix A.[‡] Many other proofs of this result have now been published: In [57] Paršin deduces the Geometric Mordell Conjecture from *Shafarevich's Conjecture* by analysing the branched covers of a curve of genus $g \geq 1$. In [46] McMullen gives a proof of the Geometric Mordell Conjecture based on Paršin's approach, but using topological techniques involving the mapping class groups of Riemann surfaces. In this thesis we will present a proof of the Geometric Mordell Conjecture based on the proof given by Alexandru Buium in the book [13], which built upon the ideas developed in his papers [12, 14, 15]. Here we state the version of Mordell's Theorem which will be proved in Chapter 6, although we defer the task of precisely defining all the terms involved until the body of the text.

Theorem 1.1 (The Geometric Mordell-Weil Theorem, Theorem 2.1 of [13])**.** *Let F/k be a function field of one variable and let X be a smooth, projective, (non-hyperelliptic) curve over F of genus at least 2. Assume further that $\overline{X} = X \times_{\operatorname{Spec} F} \operatorname{Spec} \overline{F}$ does not descend to k. Then the set $X(F)$ of F-rational points of X is finite.*

From our perspective, a significant difference between $\mathbb{C}(t)$ and a number field is the existence of the differential operator $\frac{d}{dt}$ on $\mathbb{C}(t)$. In [13], this observation allows Buium to approach the Geometric Mordell Conjecture through the paradigm of *differential algebraic geometry*, in which the basic geometric objects are those cut out by differential rather than polynomial equations. Our goal in this thesis is to use *jet schemes* to situate the foundations of differential algebraic geometry within standard algebraic geometry, rather than as a separate field. This approach was suggested by Buium in Sections 3.3.14-3.3.17 of [13] and the appendix to [12]. Work on this translation was initiated by the 'Buium seminar' in 2011 [6]. We emphasise that all the main ideas of the proof presented in this thesis are due to Buium: our role has mainly been to give an account of the foundations of algebraic geometry in terms of jet functors and to 'translate' Buium's proof into this language.

[†]Note that despite the name, this result is no longer conjectural. We follow Buium [13] in this naming convention.

[‡]Manin's paper [44] was later found by Coleman to contain a gap. Coleman partially resolved this issue by proving a weaker form of the theorem of the kernel which was sufficient to establish Mordell's conjecture, and Chai later proved that Manin's original statement followed from results of Deligne in Hodge theory [21, 17].

In order to study differential algebraic geometry, we must first understand the language of algebraic geometry. To this end, Chapter 2 is dedicated to constructing the category of *schemes* using the functorial viewpoint. Moreover, when studying rational points on curves, the ability to embed a curve in an *abelian variety* known as the *Jacobian* is important, because such an embedding allows us to carry out our analysis in the category of *algebraic groups*. These objects will be defined in Chapter 3 as instances of a more general object known as a *group scheme*. Furthermore, we will give an overview of the important structure results which simplify the study of the category of algebraic groups. In particular, we will see in Theorem 3.4 that the category of *commutative* algebraic groups is abelian.

In Chapter 4 we turn our attention to the foundations of differential algebraic geometry. The basic algebraic object we will study is a *δ-ring*, which is a ring in which we have a notion of differentiation. We then define a class of schemes which we will call *D-schemes*, so that the category of *affine D-schemes* is dual to the category of δ-rings. In Section 4.4 we introduce *$\mathcal{D}_X$-modules*, which do not appear explicitly in Buium's work, but will be useful in our analysis.

The remainder of the thesis is dedicated to proving the Geometric Mordell Conjecture using the theory developed in the earlier chapters. In Chapter 5 we will construct *δ-characters* of abelian varieties, which are used in conjunction with the Geometric Mordell–Weil Theorem 3.20 to establish the Geometric Mordell conjecture for non-hyperelliptic curves in Chapter 6.

We assume that the reader of this thesis is familiar with basic commutative algebra, category theory and sheaf theory. We hope to give a fairly self-contained introduction to algebraic geometry based on the functor of points, but at times we will necessarily have to view schemes as certain types of locally-ringed spaces. Therefore, we assume that the reader is familiar with the definition of a scheme in terms of locally-ringed spaces and has an understanding of the basic theory of $\mathcal{O}_X$-modules. We refer the reader to Sections II.2 and II.5 of [34] for an introduction to this material. Furthermore, we will sometimes adopt the complex-analytic viewpoint, so an understanding of basic algebraic topology (a familiarity with the first three chapters of [35] will be sufficient) and differential geometry is assumed.

Chapter 2

Scheme theory from the functorial viewpoint

2.1 The functor of points of a scheme

In this thesis we will in general emphasise the *functor of points* approach to scheme theory, thinking of schemes as certain kinds of functors rather than as locally ringed spaces. Therefore, we will begin with a discussion of how the functor of points perspective arises within the 'standard' approach to algebraic geometry. We refer the reader unacquainted with the definition of schemes in terms of locally ringed spaces to the textbooks [34] and [64] for an introduction. In this section we will refer to a scheme defined in terms of a locally ringed space as a 'scheme'. Given a 'scheme' X we can define a contravariant functor $h_X : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ from the category of schemes to the category of sets by $Y \mapsto \mathrm{Mor}(Y, X)$. This is called the *functor of points of* X. A morphism $X_1 \to X_2$ defines a natural transformation $\mathrm{Mor}(-, X_1) \to \mathrm{Mor}(-, X_2)$ and so we obtain a functor $\mathbf{Sch} \to [\mathbf{Sch}^{\mathrm{op}}, \mathbf{Set}]$. Indeed, this construction makes sense for any locally small category and defines a functor $\mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$, which we call the *Yoneda embedding*. The Yoneda Lemma 2.1 tells us that the functor $\mathrm{Mor}(-, X)$ captures all the information of the object X of $\mathscr{C}$. This is a standard result in category theory, so we will not give a proof here, for details see page 61 of [42].

Theorem 2.1 (The Yoneda lemma)**.** *Let $\mathscr{C}$ be a locally small category, let $F : \mathscr{C}^{op} \to$ $\mathbf{Set}$ be a contravariant functor and let X be an object of $\mathscr{C}$. Then there is a natural isomorphism between the set of natural transformations* $\mathrm{Mor}(-, X) \to F$ *and* $F(X)$*. Moreover, the Yoneda embedding* $\mathscr{C} \to [\mathscr{C}^{op}, \mathbf{Set}]$ *is an equivalence of $\mathscr{C}$ with a full subcategory of the functor category* $[\mathscr{C}^{op}, \mathbf{Set}]$.

In fact, when $\mathscr{C}$ is the category of schemes we can go even further: given a scheme X we can restrict the functor $h_X : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ to the subcategory of affine schemes.

Since the category of affine schemes is dual to the category of rings we obtain a functor $\mathbf{Alg}_{\mathbb{Z}} \to \mathbf{Set}$ defined by $A \mapsto \mathrm{Mor}(\mathrm{Spec}\, A, X)$. (Note that for a commutative unital ring k, $\mathbf{Alg}_k$ will denote the category of commutative k-algebras). Because schemes are covered by affine schemes, this functor captures all the information of the scheme X.

Proposition 2.2 (Proposition VI-2 of [29]). *The functor $h_X : \mathbf{Sch} \to [\mathbf{Alg}_{\mathbb{Z}}, \mathbf{Set}]$ is fully faithful.*

Sketch. Let X and Y be schemes and suppose we have a morphism $\varphi : h_X \to h_Y$. We need to prove that φ comes from a unique morphism of schemes. Let $\{U_\alpha = \mathrm{Spec}\, B_\alpha\}$ be an open affine cover of X. Then we have open immersions $i_\alpha : U_\alpha \hookrightarrow X$ which gives us morphisms $\varphi_{B_\alpha}(i_\alpha) : U_\alpha \to Y$. Because φ is a natural transformation, these morphisms glue to define a morphism $X \to Y$. The uniqueness of this morphisms derives from the fact that morphisms of schemes which agree on an open cover coincide. □

For a commutative unital ring k, we call an object of the functor category $[\mathbf{Alg}_k, \mathbf{Set}]$ a *k-functor*. When we work over algebraically closed fields, much of our intuition from classical algebraic geometry carries over to the locally ringed space $(X, \mathcal{O}_X)$. However, when working over other fields, or even rings, the connections are not always so clear. A key insight, due to Grothendieck, is that we may instead develop the foundations of algebraic geometry beginning with k-functors. Grothendieck highlighted that an object does not need to have an underlying 'honest' topological space to be considered a 'geometric' object: In the next section we will define schemes as certain kinds of k-functors to which we may apply geometric reasoning. Indeed, the axiom of choice is required to show that when A is a non-zero ring the topological space $\mathrm{Spec}\, A$ contains a point. It makes sense that we avoid invoking this result within the foundations of our subject. That being said, the locally ringed space $(X, \mathcal{O}_X)$ remains an important tool in understanding a scheme so we will certainly not dispose of it completely, but we will not think of this as the primary object of study.

As a first example, let k be a commutative ring and consider a polynomial $f \in k[x_1, ..., x_n]$. The vanishing locus of f is defined to be the set,

$$V(f) = \{(a_1, ..., a_n) \in k^n \mid f(a_1, ..., a_n) = 0\}.$$

The classical viewpoint is to study this solution set in the case where k is an algebraically closed field. However, it makes sense to evaluate f at a point of A^n for any k-algebra A, giving us a new solution set for each k-algebra. Moreover, if we have a homomorphism $\varphi : A \to B$ of k-algebras, then given a solution $(a_1, ..., a_n) \in A^n$ to $f(x_1, ..., x_n) = 0$, we get another solution $(\varphi(a_1), ..., \varphi(a_n)) \in B^n$. That is, f defines a functor $X : \mathbf{Alg}_k \to \mathbf{Set}$, where $X(A) = \{(a_1, ..., a_n) \in A^n \mid f(a_1, ..., a_n) = 0\}$. Note that the set of points $(a_1, ..., a_n) \in A^n$ such that $f(a_1, ..., a_n) = 0$ identifies with the set of k-algebra homomorphisms $k[x_1, ..., x_n] \to A$ which send f to zero. That is, $X(A) =$

$\mathrm{Hom}_k(k[x_1,...,x_n]/(f), A)$. Note that this is precisely the functor of points of the 'scheme' $\mathrm{Spec}\, k[x_1,...,x_n]/(f)$. Thus, we may think about affine schemes as a way to keep track of a set of solutions to systems of polynomial equations in different k-algebras.

Of course, when we consider the functor $\mathbf{Alg}_k \to \mathbf{Set}$, we are ignoring any topology which the set $X(A)$ might have (for instance $X(\mathbb{C})$ will often inherit a natural topology as a subspace of $\mathbb{C}^n$), but the Yoneda lemma tells us that this information may be recovered by knowing $X(A)$ for every k-algebra A.*

Note that in the sequel, if it is useful to make a distinction, we will use the notation $\mathbf{Sch}'_k$ for the category of schemes in the sense of EGA (so that an object in $\mathbf{Sch}'_k$ is a locally ringed space which is locally isomorphic to the prime spectrum of a k-algebra).

2.2 Foundations of functorial geometry

In this section we will develop the foundations of functorial algebraic geometry following Chapter 1 of Jantzen's book [37] and the book of Demazure–Gabriel [26]. In particular, none of the material of this section is original

2.2.1 Affine Schemes and k-functors

Recall that a k-functor is an object of the functor category $[\mathbf{Alg}_k, \mathbf{Set}]$. A morphism of k-functors is a natural transformation. A *sub-k-functor* Y of a k-functor X is a k-functor such that $X(A) \subset Y(A)$ for every k-algebra A and $Y(\varphi) = X(\varphi)|_{Y(A)}$ for all $A, B \in \mathbf{Alg}_k$ and $\varphi \in \mathrm{Hom}_k(A, B)$.

Let $f : X \to X'$ be a morphism of k-functors and let Y' be a subfunctor of X. Then for any k-algebra A, $f_A : X(A) \to X'(A)$ is a map of sets and so we may define a subfunctor $f^{-1}(Y')$ of X by

$$f^{-1}(Y')(A) = f_A^{-1}(Y'(A)).$$

We call $f^{-1}(Y')$ the inverse image of Y' under f. Moreover, given a family $(Y_i)_{i \in I}$ of subfunctors of a k-functor X, their *intersection* is defined in the obvious way:

$$\left(\bigcap_{i \in I} Y_i\right)(A) = \bigcap_{i \in I} Y_i(A).$$

Note that obvious definition of the union of subfunctors will not give us the union in the category of schemes. Finally, given morphisms $f : X \to S$ and $g : Y \to S$ of k-functors, we can define the *fibre product* $X \times_S Y$ using the fibre product in $\mathbf{Set}$:

$$(X \times_S Y)(A) = X(A) \times_{S(A)} Y(A) = \{(x, y) \in X(A) \times Y(A) \mid f_A(x) = g_A(y)\}.$$

The fact that this is the fibre product in the category of k-functors is immediate, but note that we will see in Section 2.3.4 that this functor is the fibre product in the category

*Indeed we can recover the topology on X by knowing $X(K)$ for every k-algebra K which is a field of schemes. The ease of defining fibre products in functorial geometry is one advantage of this approach.

Now fix a k-algebra R. In the previous section we saw that the functor of points of the affine 'scheme' $\operatorname{Spec} R$ is the functor $A \mapsto \operatorname{Hom}_k(R, A)$. Note that throughout this thesis we will use the notation $\operatorname{Hom}_k(-,-)$ to denote the morphisms in the category of k-algebras. This observation leads us to define a k-functor $\operatorname{Spec} R : \mathbf{Alg}_k \to \mathbf{Set}$ defined on objects by

$$(\operatorname{Spec} R)(A) = \operatorname{Hom}_k(R, A)$$

and on morphisms by $(\varphi : A \to B) \mapsto \varphi^*$, where $\varphi^* : \operatorname{Hom}_k(R, A) \to \operatorname{Hom}_k(R, B)$ is defined by, $\varphi^*(\psi) = \varphi \circ \psi$. We call this functor the *spectrum* of R. We say that a k-functor is an *affine scheme* if it is isomorphic to the spectrum of some k-algebra (that is, if it is representable). A morphism of affine schemes is simply a morphism of the underlying k-functors. Our first example of an affine scheme is affine n-space $\mathbb{A}^n$. This is the functor $\mathbf{Alg}_k \to \mathbf{Set}$ defined by $\mathbb{A}^n(B) = B^n$ and $\mathbb{A}^n(\varphi) : (b_1, ..., b_n) \mapsto (\varphi(b_1), ..., \varphi(b_n))$. The functor $\mathbb{A}^n$ is representable because we have

$$(\operatorname{Spec} k[x_1, ..., x_n])(B) = \operatorname{Hom}_k(k[x_1, ..., x_n], B) = B^n = \mathbb{A}^n(B).$$

That is, $\mathbb{A}^n = \operatorname{Spec} k[x_1, ..., x_n]$.

For a k-functor X, we define the *coordinate ring* of X to be the k-algebra

$$\mathcal{O}(X) = \operatorname{Mor}(X, \mathbb{A}^1).$$

The set $\mathcal{O}(X)$ has the structure of a ring with addition defined by $(f+g)_R(x) = f_R(x) + g_R(x)$ and multiplication defined by $(fg)_R(x) = f_R(x)g_R(x)$. Moreover, we have a ring homomorphism $k \to \mathcal{O}(X)$ defined by sending $\lambda \in k$ to the constant morphism $c_\lambda : X \to \mathbb{A}^1$ at λ. Thus, $\mathcal{O}(X)$ has the structure of a k-algebra. If $X = \operatorname{Spec} R$ is an affine scheme then by the Yoneda lemma 2.1 we have

$$\mathcal{O}(X) = \operatorname{Mor}(\operatorname{Spec} R, \mathbb{A}^1) = \mathbb{A}^1(R) = R,$$

which means that for any affine scheme X we have $X = \operatorname{Spec} \mathcal{O}(X)$. In particular, the functor $X \mapsto \mathcal{O}(X)$ is an equivalence between the category of affine k-schemes and $\mathbf{Alg}_k^{\mathrm{op}}$.

For an arbitrary k-functor X we call $\operatorname{Spec} \mathcal{O}(X)$ the *affinisation* of X and we have a morphism $\psi : X \to \operatorname{Spec} \mathcal{O}(X)$ defined on R-points to be the map $X(R) \to \operatorname{Hom}(\mathcal{O}(X), R)$ given by $x \mapsto [f \mapsto f_R(x)]$.

Note that by the Yoneda Lemma we can (and often will) conflate points in the set $X(R)$ with maps $\operatorname{Spec} R \to X$. That is, given $x \in X(R)$ we will write x for the map $\operatorname{Spec} R \to X$ defined on A-points to be the map $\operatorname{Hom}_k(R, A) \to X(A)$ given by $f \mapsto X(f)(x)$. The fact that the fibre product of k-functors also defines the fibre product in the category of affine k-schemes follows immediately from the fact that the tensor product is the coproduct in the category of k-algebras. If $X_1 \to S$ and $X_2 \to S$ are morphisms of affine k-schemes, then we've shown that $X_1 \times_S X_2$ is an affine scheme with

$$\mathcal{O}(X_1 \times_S X_2) = \mathcal{O}(X_1) \otimes_{\mathcal{O}(S)} \mathcal{O}(X_2)$$

and in particular $\mathcal{O}(X_1 \times X_2) = \mathcal{O}(X_1) \otimes_k \mathcal{O}(X_2)$. Note that if $S = \operatorname{Spec} A$ is affine, then we will sometimes write $X_1 \times_A X_2$ for $X_1 \times_S X_2$.

Now let X be an affine scheme and let T be a subset of $\mathcal{O}(X)$. Then we can define a subfunctor $V(T)$ of X by

$$\begin{aligned} V(T)(A) = \{x \in X(A) \mid f_A(x) = 0, \forall f \in T\} &\cong \{\alpha : \mathcal{O}(X) \to A \mid \alpha(T) = 0\} \\ &= \operatorname{Hom}(\mathcal{O}(X)/(T), A) \end{aligned}$$

where (T) is the ideal generated by T. Therefore, the previous calculation shows that $V(T) = \operatorname{Spec} \mathcal{O}(X)/(T)$, where (T) is the ideal generated by T. Subfunctors of this form are called *closed subfunctors* of X. Note that, like the closed subsets of a topology, arbitrary intersections of closed subfunctors are again closed:

$$\bigcap_{i \in I} V(T_i) = \left\{ x \in X(A) \,\middle|\, f_A(x) = 0, \forall f \in \bigcup_{i \in I} T_i \right\} = V\left(\bigcup_{i \in I} T_i \right).$$

On the other hand, an *open subfunctor* of X is defined to be a subfunctor of the form

$$D(I)(A) = \{x \in X(A) \mid (f_A(x) \mid f \in I) = A\}$$

where $(f_A(x) \mid f \in I)$ denotes the ideal generated by the elements $f_A(x) \in A$ for all $f \in I$. Note that for a field K we have

$$D(I)(K) = \bigcup_{f \in I} \{x \in X(A) \mid f_K(x) \neq 0\}$$

since $f_K(x)$ generates the unit ideal of K if and only if it is non-zero. One may show directly that for ideals I and J we have $D(I \cap J) = D(I) \cap D(J) = D(IJ)$, see Section I.1.5 of [37]. Moreover, if $I = (f)$ then we have

$$\begin{aligned} D(f)(A) &= \{x \in X(A) \mid (f_A(X) \mid f \in I) = (1)\} \\ &= \{\alpha \in \operatorname{Hom}(\mathcal{O}(X), A) \mid \alpha(f) \in A^\times\} = \operatorname{Hom}(\mathcal{O}(X)_f, A) \end{aligned}$$

by the universal property of localisation. Therefore, we have that $D(f) = \operatorname{Spec} \mathcal{O}(X)_f$, which we can think of as saying that the complement of a hyperplane in X is affine.

2.3 Schemes

2.3.1 Sheaves in Grothendieck topologies

Before we can define a scheme we will need to understand what it means for a category to have a topology. In this section we will develop the theory of Grothendieck topologies only as far as we need to define a scheme. Once we have discussed morphisms of schemes, we will be able to give more examples of Grothendieck topologies. For a more comprehensive discussion of these concepts we refer the reader to the texts by Moerdijk–Mac Lane [43] and Vistoli [66].

Recall that a *presheaf* on a category $\mathscr{C}$ is a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$. If $\mathscr{C}$ is the category $\mathbf{Top}(X)$ of open subsets of a topological space X, then we can ask whether or not F is a sheaf. To say that F is a sheaf means that we can understand the sections of F on an open set U by considering sections on an open cover of U. However, in an arbitrary category there are no open sets: all we have to work with are maps into and out of an object $U \in \mathscr{C}$, so the problem of defining what it means for a collection of morphisms $\{U_i \to U\}$ to be a covering of U is more subtle. Thankfully, Grothendieck determined the correct way to generalise open coverings to a category.

Definition 2.3 (Definition 2.24 of [66])**.** A *Grothendieck topology* on a category $\mathscr{C}$ is a collection of families of maps $\{U_i \to U\}_{i \in I}$ in $\mathscr{C}$ called *coverings*, satisfying the following axioms:

(i). If $V \to U$ is an isomorphism, then $\{V \to U\}$ is a covering of U.

(ii). If $\{U_i \to U\}$ is a covering and we have a morphism $V \to U$, then the fibre products $U_i \times_U V$ exist and the collection $\{U_i \times_U V \to V\}$ is a covering.

(iii). If $\{U_i \to U\}$ is a covering and for each i we choose a covering $\{V_{ij} \to U_i\}$, then the collection $\{V_{ij} \to U_i \to U\}$ is a covering.

A category along with a Grothendieck topology is called a *site*.

Remark 2.4. Note that in [43] what we have called a Grothendieck topology in Definition 2.3 is called a 'base' of a Grothendieck topology.

Example 2.5. Our first example of a Grothendieck topology is a topology in the usual sense: In $\mathbf{Top}(X)$ we can take the coverings of U to be the collection $\{V \hookrightarrow U\}$ of open inclusions into U. In $\mathbf{Top}(X)$ the intersection of sets is a categorical fibre product, meaning that the three conditions of Definition 2.3 are satisfied.

Example 2.6 (The Zariski topology on $\mathbf{Aff}_k$)**.** In Section 2.2.1 we defined open and closed subfunctors of k-functors, so it should come as no surprise that these objects are related to Grothendieck topologies. The *Zariski topology* on the category $\mathbf{Aff}_k$ is the topology in which a covering of $\operatorname{Spec} A$ is a collection of morphisms $\{\operatorname{Spec} B_i \to \operatorname{Spec} A\}_{i \in I}$ where B_i is isomorphic to A_{f_i} for some $f_i \in A$ and $(f_i, i \in I) = A$. Note that we can always replace a cover $\{\operatorname{Spec} A_{f_i} \to \operatorname{Spec} A\}_{i \in I}$ by a finite cover, because if $(f_i) = A$, then we can choose r_i such that $\sum_{i=1}^{n} r_i f_i = 1$ and hence $(f_1, ..., f_n) = A$. Proving that this is indeed a topology on $\mathbf{Aff}_k$ follows from basic commutative algebra.

Now that we know the right definition of a topology on a category, we can define a sheaf on a site. If $\mathscr{C}$ is a site and $F : \mathscr{C}^{op} \to \mathbf{Set}$ is a presheaf, then F is a *sheaf* if the following diagram is an equaliser,

$$F(U) \xrightarrow{F(p_i)} \prod_i F(U_i) \underset{F(\mathrm{pr}_2)}{\overset{F(\mathrm{pr}_1)}{\rightrightarrows}} \prod_{(j,k)} F(U_i \times_U U_k)$$

for all objects $U \in \mathscr{C}$ and every covering $\{p_i : U_i \to U\}$ of U. Here the first map is induced by the projections $U_i \to U$, the upper map is induced by $\mathrm{pr}_1 : U_j \times_U U_k \to U_j$ and the lower map is induced by $\mathrm{pr}_2 : U_j \times_U U_k \to U_k$. Expanding what it means for this diagram to be an equaliser we see that if we have elements $a_i \in F(U_i)$ such that $F(\mathrm{pr}_1)a_i = F(\mathrm{pr}_2)a_j$ for all i and j, then there exists a unique $a \in F(U)$ such that $F(p_i)a = a_i$ for all i. Note that a presheaf is called *separated* if it satisfies only the identity axiom: for any $a, b \in F(U)$ such that $F(p_i)a = F(p_i)b$ for all i, we have $a = b$.

Definition 2.7. A category which is equivalent to the category $\mathbf{Shv}(\mathscr{C})$ of sheaves on a site $\mathscr{C}$ is called a *topos*.

2.3.2 The definition of a scheme

Recall that we have the Yoneda Embedding $\mathbf{Sch}'_k \to [\mathbf{Alg}_k, \mathbf{Set}]$. We would like to understand which functors are in the image of this embedding and therefore deserve to be called schemes. Clearly, for a functor $\mathbf{Sch}'^{\mathrm{op}}_k \to \mathbf{Set}$ to be considered a scheme it must be representable. However, since we made the restriction to the subcategory $[\mathbf{Alg}_k, \mathbf{Set}]$, determining which functors are schemes requires more work.

Firstly, for $X : \mathbf{Alg}_k \to \mathbf{Set}$ to represent a geometric object we need some notion of how functions glue on open sets. Therefore, we require that X be a sheaf in the Zariski topology on $\mathbf{Alg}_k$ (as defined in Example 2.6). That is for any k-algebra A and $f_i \in A$ generating the unit ideal, we require that Diagram 2.1 be an equaliser diagram.

$$X(A) \to \prod_i X(A_{f_i}) \rightrightarrows \prod_{i,j} X(A_{f_i f_j}) \tag{2.1}$$

We call a k-functor satisfying equation 2.1 a *local functor* or a *Zariski sheaf*. Note that it's enough to check this condition for all finite sets $\{f_1, ..., f_n\}$ of elements in A generating the unit ideal.

Now, for a functor to be considered an 'algebraic' object, we should also require that 'locally' it look like an affine scheme. Therefore, we need to understand what it means for a k-functor to have an open cover.

Definition 2.8. For a k-functor X, we say that a subfunctor $Y \subset X$ is *open* (resp. *closed*) if for any k-algebra A and morphism $f : \operatorname{Spec} A \to X$, $f^{-1}(Y)$ is an open (resp. closed) subscheme of $\operatorname{Spec} A$ in the sense of the previous section. More generally, we say that a morphism of functors $X \to Y$ is an *open immersion* (resp. a *closed immersion*) if it is an isomorphism of X onto an open (resp. closed) subfunctor of Y.

Let X_i be an open subfunctor of X. Then for any morphism $\operatorname{Spec} A \to X$, the pullback $\operatorname{Spec} A \times_X X_i$ is an open subfunctor of $\operatorname{Spec} A$ (by the definition of an open subfunctor). We say that a family of open sub-k-functors $\{X_i\}_{i\in I}$ of X is an open covering of X if for any morphism $\operatorname{Spec} A \to X$, the set $\{D(I_j) = \operatorname{Spec} A \times_X X_j \to \operatorname{Spec} A\}_{j\in I}$ satisfies $\sum_j I_j = A$. Alternatively, by Section I.1.7 of [37], a family $\{X_i\}_{i\in I}$ of open subfunctors of X is an open cover if and only if $X(K) = \bigcup_i X_i(K)$ for every k-algebra K which is a field.

Theorem 2.9 (Theorem VI-14 of [29])**.** *Any k-functor $X : \mathbf{Alg}_k \to \mathbf{Set}$ which is a local functor and has an open covering by affine schemes is in isomorphic to a functor in the image of the Yoneda embedding $\mathbf{Sch}'_k \to [\mathbf{Alg}_k, \mathbf{Set}]$.*

Theorem 2.9 allows us to make the following definition:

Definition 2.10. A k-functor X is a *k-scheme* if it is a local functor with an open cover by affine schemes.

Remark 2.11. Theorem 2.9 means that for any scheme $X : \mathbf{Alg}_k \to \mathbf{Set}$ there is a locally ringed space $(Y, \mathcal{O}_Y)$ such that $h_Y \cong X$ as functor $\mathbf{Alg}_k \to \mathbf{Set}$. We will denote this locally ringed space by $(|X|, \mathcal{O}_X)$. We can define a number of topological properties of a scheme in terms of $|X|$. For instance, we say that X is *Noetherian* if $|X|$ is a Noetherian as a topological space and we say that X is *irreducible* if $|X|$ is irreducible as a topological space.

Recall that we defined the coordinate ring of a k-functor X by $\mathcal{O}(X) = \operatorname{Mor}(X, \mathbb{A}^1)$ and we saw that if X and Y are affine k-schemes then we have $\mathcal{O}(X \otimes_k Y) \cong \mathcal{O}(X) \otimes_k \mathcal{O}(Y)$. In general this need not be true. However, if k is a field then we have the following result, although we omit the proof.

Lemma 2.12 (Proposition I, §2, 2.6 of [26])**.** *Let X and Y be quasicompact and quasiseparated schemes over a field k. Then the canonical morphism $\mathcal{O}(X) \otimes_k \mathcal{O}(Y) \to \mathcal{O}(X \times_k Y)$ is an isomorphism of k-algebras.*

2.3.3 Sheafification

Often k-functors will arise which are not schemes, but which we would like to be. The *sheafification* operation allows us to associate a sheaf to our functor, which might be a scheme. This operation will be particularly important when we construct quotients of algebraic groups. Let $\mathscr{C}$ be a site. We define a morphism of sheaves to be a natural transformation of functors, so that the category $\mathbf{Shv}_{\mathscr{C}}$ includes as a full subcategory of $\mathbf{PrShv}_{\mathscr{C}}$. We denote this inclusion by $i : \mathbf{Shv}_{\mathscr{C}} \hookrightarrow \mathbf{PrShv}_{\mathscr{C}}$. Proposition 2.13 tells us that i has a right adjoint which we call *sheafification.* We refer the reader to [66] for a proof.

Proposition 2.13 (Theorem 2.64 of [66]). *Let $\mathscr{C}$ be a site and let $F : \mathscr{C}^{op} \to$ **Set** be a presheaf on X. Then there exists a sheaf F^{sh} and a morphism $\phi : F \to F^{sh}$ which is universal among maps from F to sheaves. That is, if $f : F \to G$ is a morphism of sheaves then there exists a unique morphism $\theta : F^{sh} \to G$ such that $f = g \circ \phi$.*

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "\theta"] & G \\ F^{sh} \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$$

2.3.4 Examples of schemes

Affine schemes

The fact that an affine scheme $\operatorname{Spec} A$ is a scheme is not immediate from the definition. To prove that this is indeed the case we need to show that $\operatorname{Spec} A$ is a local functor. Note that this is equivalent to proving that the structure sheaf of an affine scheme is in fact a sheaf. For a proof of this result, see Theorem 4.1.2 of [64] or Proposition 2.2 of [34].

Fibre products

Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes. Then we will prove that the fibre product of $\mathbb{Z}$-functors $X \times_S Y$ is a scheme. If we can verify that $X \times_S Y$ is a scheme, then it is immediate that $X \times_S Y$ is the fibre product in the category of schemes.

To see that $X \times_S Y$ is a sheaf, suppose we have $(x_i, y_i) \in X(A_{f_i}) \times_{S(A_{f_i})} Y(A_{f_i})$ such that $(X \times_S Y)(\mathrm{pr}_1)(x_i, y_i) = (X \times_S Y)(\mathrm{pr}_2)(x_j, y_j)$ for all i and j. This means that $X(\mathrm{pr}_1)x_i = X(\mathrm{pr}_2)x_j$ and $Y(\mathrm{pr}_1)y_i = Y(\mathrm{pr}_2)y_j$ for all i and j. Therefore, the x_i (resp. the y_i) glue to a unique $x \in X(A)$ (resp. $y \in Y(A)$) such that $X(p_i)x = x_i$ (resp. $X(p_i)y = y_i$) for all i. To see that (x, y) is an element of $(X \times_S Y)(A)$ we observe that because f and g are morphisms of functors, they commute with restriction, and so we have

$$S(p_i)(f_A(x)) = f_{A_{f_i}}(x_i) = g_{A_{f_i}}(y_i) = S(p_i)(g_A(y))$$

Therefore, unique gluing in the sheaf S implies that $f_A(x) = g_A(y)$.

To obtain an open affine covering for $X \times_S Y$, choose open affine coverings $\{X_i\}$, $\{Y_j\}$ and $\{S_l\}$ for X, Y and S respectively. Then for any i, j, l the fibre product $X_i \times_{S_l} Y_j$ is affine and for any k-algebra K which is a field we have

$$(X \times_S Y)(K) = \bigcup_{i,j,k} X_i(K) \times_{S_l(K)} Y_j(K)$$

because if $(x, y) \in (X \times_S Y)(K)$, then $x \in X_i(K)$ for some i and $y \in Y_j(K)$ for some j and $(f|_{X_i})_K(x) = f_K(x) = g_K(y) = (g|_{Y_j})_K(x)$ belongs to some $S_l(K)$.

Sheaves of modules

We will not discuss any of the basic constructions concerning $\mathcal{O}_X$-modules in this thesis. For an introduction to this topic we refer the reader to Chapter II.5 of [34]. However, here we will comment on our conventions: We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* if for any morphism $f : \operatorname{Spec} A \to \mathcal{F}$, the pullback $f^*\mathcal{F}$ is isomorphic to $\widetilde{M}$ for some A-module M. Here $\widetilde{M}$ is the sheaf on $\operatorname{Spec} A$ define by $\widetilde{M}(D(f)) = M_f$ for a distinguished open $D(f)$. We say that a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if M can always be taken to be a finitely-generated A-module.† Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $x \in |X|$, then the *fibre* of $\mathcal{F}$ at x is the $\kappa(x)$-vector space $\mathcal{F}|_x \coloneqq \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$.

Grassmannians

As a first (concrete) example of a scheme which is not affine we will now define the *Grassmannian* scheme $\mathrm{Gr}_{n,r}$ in terms of a certain moduli problem. Let $r \le n$ be non-negative integers and consider the *Grassmannian functor* $\mathrm{Gr}_{n,r} : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ which sends a scheme X to the set of locally free rank r quotients of $\mathcal{O}_X^{\oplus n}$ up to isomorphism.‡ An object in this set can be thought of as an exact sequence $\mathcal{O}_X^{\oplus n} \to \mathcal{Q} \to 0$ and two such sequences are isomorphic if we have an isomorphism $\mathcal{Q}_1 \to \mathcal{Q}_2$ commuting with the maps from $\mathcal{O}_X^{\oplus n}$. Given a morphism $f : X \to Y$ of schemes we obtain an induced morphism $\mathrm{Gr}_{n,r}(Y) \to \mathrm{Gr}_{n,r}(X)$ by sending $\mathcal{O}_Y^{\bigoplus n} \to \mathcal{Q}$ to $\mathcal{O}_X^{\bigoplus n} \to f^*\mathcal{Q}$. This is well defined because f^* is right exact (by [63, Tag 01AJ]) and additive, and $f^*\mathcal{O}_Y = \mathcal{O}_X$.

By abuse of notation, we will also write $\mathrm{Gr}_{n,r}$ for the functor $\mathbf{Alg}_k \to \mathbf{Set}$ obtained by restriction. The Grassmannian functor is a Zariski sheaf because we can glue together locally free sheaves and we may also construct an open affine cover, for details of this proof see [63, Tag 089T]. Therefore, $\mathrm{Gr}_{n,r}$ is a scheme which we call the *Grassmannian scheme*. Note that, by definition, morphisms $S \to \mathrm{Gr}_{n,r}$ parametrise rank r locally-free quotients of $\mathcal{O}_S^{\oplus n}$.

We define *n-dimensional projective space over k* to be the scheme $\mathbb{P}^n \coloneqq \mathrm{Gr}(n+1, 1)$. Furthermore, if V is a finite-dimensional k-module, then we define the projectivisation of V to be the functor $\mathbb{P}(V) : \mathbf{Alg}_k \to \mathbf{Set}$ which sends A to the set of rank one projective A-module quotients of $A \otimes_k V^\vee$. Note that $\mathbb{P}(V)$ is isomorphic as a k-functor to $\mathbb{P}^{\dim V}$ and is therefore a scheme.

2.4 Morphisms of schemes

A morphism of schemes $f : X \to Y$ is simply a natural transformation of the underlying k-functors, which in general might not be particularly well-behaved. Therefore, there

†Note that some authors use a slightly different definition of a coherent sheaf. Here we follow [34].
‡We refer the reader unfamiliar with the concept of an $\mathcal{O}_X$-module to Section II.5 of [34]

are a number of additional conditions which we might impose on f so that it will interact with the scheme structure on X and Y in a way which is easier to understand. Describing some of these conditions will be the subject of this section. Moreover, we will see that many properties which we think of as being properties of morphims of schemes are in fact properties of morphisms of k-functors, and often it will be helpful to view them as such.

2.4.1 Geometric properties of morphisms

Here we will introduce three conditions which any property $\mathcal{P}$ of morphisms should satisfy to be thought of as a 'geometric' property.

1. The composition of two morphisms with property $\mathcal{P}$ should have property $\mathcal{P}$.

2. The next condition is *stability under base-change.* To introduce this concept, let $X : \mathbf{Alg}_k \to \mathbf{Set}$ be a k-functor and let B be a k-algebra. Then every B-algebra is a k-algebra and so we have an inclusion functor $\mathbf{Alg}_B \to \mathbf{Alg}_k$. We define the *base-change* of X to B to be the functor $X_B : \mathbf{Alg}_B \to \mathbf{Alg}_k \to \mathbf{Set}$ obtained by precomposing with the inclusion. Note that if $X = \operatorname{Spec} A$ is affine, then $(\operatorname{Spec} A)_B = \operatorname{Spec}(B \otimes_k A)$ because if R is a B-algebra, then a homomorphism from A to R over k is the same as a morphism from $A \otimes_k B$ to R over B. More generally, if we have a morphisms of schemes $f : X \to S$ and $g : Y \to S$, then the *base-change* of f along g is the pullback morphism $f : X \times_S Y \to Y$. We say that property $\mathcal{P}$ is stable under base-change if for any $f : X \to Y$ with property $\mathcal{P}$ and any $g : Z \to Y$, the morphism $X \times_Y Z \to Z$ has property $\mathcal{P}$.

   On the level of functors the base-change operation seems fairly innocuous, but the concept of changing base is often important. For instance, in many cases when working over a field, one may base change to the algebraic closure, where there are additional results such as the Nullstellensatz available.

3. Let τ be a Grothendieck topology on the category of schemes. We say that a property $\mathcal{P}$ is *τ-local on the target* if for any morphism $f : X \to Y$ and covering $\{U_i \to Y\}$ belonging to τ, f has property $\mathcal{P}$ if and only if each morphism $X \times_Y U_i \to U_i$ has property $\mathcal{P}$. If it suffices to check $\mathcal{P}$ on any Zariski open cover of the target by *affines*, then we say that $\mathcal{P}$ is *affine-local on the target.*

2.4.2 Finiteness conditions on morphisms

We say that f is *quasi-compact* if the morphism $f : |X| \to |Y|$ of topological spaces is quasi-compact in the sense that whenever V is a quasicompact open subset of $|Y|$,

$f^{-1}(V)$ is quasi-compact in $|X|$.[§] By an argument involving the *Affine Communication Lemma* 5.3.2 of [64], the property of being quasi-compact is affine-local on the target. Using the fact that quasi-compactness is affine-local on the target, one may check directly that quasi-compactness is stable under base-change. If the diagonal morphism $X \to X \times_Y X$ is quasi-compact, then we say that f is *quasi-separated.*

If we have a k-algebra $k \to A$, then A is *finitely generated* as a k-algebra if $A \cong k[x_1, ..., x_n]/I$ for some $x_i \in A$. We say that A is *finitely presented* if I can be taken to be a finitely generated ideal (if k is Noetherian then the finitely generated and finitely presented k-algebras coincide by the Hilbert basis theorem). Moreover, we can ask whether the morphism $k \to A$ makes A into a finitely-generated k-module. All of these properties of morphisms of rings have analogues for morphisms of schemes: if $f : X \to Y$ is a morphism of schemes then we say that f is *locally of finite type* if for any open affine subset $U = \operatorname{Spec} A$ of Y, and any open affine $\operatorname{Spec} B$ of $f^{-1}(U)$, B is a finitely-generated A-algebra. By the Affine Communication Lemma 5.3.2 of [64] it is enough to check that $f^{-1}(U)$ has an open affine cover by the spectra of finitely generated A-algebras. If each B in the previous definition is in fact a finitely-presented A-algebra, then we say that f is *locally of finite presentation.* Note that being locally of finite presentation is in fact a property of k-functors, see [63, Tag 049I]. If f is quasi-compact and locally of finite type then we say f is of *finite type.* Similarly, if f is quasi-compact, quasi-separated and locally of finite presentation, then we say that f is of *finite presentation.* These properties can easily be seen to be closed under composition. The property of being locally of finite type is affine-local on the target by Exercise 7.3.O of [64], from which we can prove directly that it is stable under base change.

Let k be a field. Then we say that X is *locally algebraic* (resp. *algebraic*) if the structure morphism $X \to \operatorname{Spec} k$ is locally of finite type (resp. is of finite type).[¶]. Note that if X is algebraic over k, then it is automatically quasi-separated by [63, Tag 01T7]. We observe that the base change of an algebraic scheme X to a field extension is algebraic.

2.4.3 Affine morphisms and closed immersions

We say that a morphism $f : X \to Y$ is affine if for any morphism $\operatorname{Spec} A \to Y$, the pullback $\operatorname{Spec} A \times_Y X$ is affine. In particular, if Y is affine, then X is also affine. Moreover, any morphism of affine schemes is affine because if we have $\operatorname{Spec} B \to \operatorname{Spec} A$ and $\operatorname{Spec} C \to \operatorname{Spec} A$, then $\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} C = \operatorname{Spec}(B \otimes_A C)$. The property

[§]Note that here, and in algebraic geometry in general, a quasi-compact set is what would ordinarily be called a compact set in other areas of mathematics (at least in English). Namely, a subset of a topological space is quasi-compact if every open cover has a finite subcover.

[¶]In Definition (I,§3, 2.2.1) Demazure & Gabriel define a scheme over a ring k to be algebraic if the structure morphism is of *finite presentation*, a concept which we have not defined. Instead we choose to only define the concept of an algebraic scheme over a field, where the two concepts are equivalent of being affine is clearly closed under composition. By Proposition 7.3.4 of [64], the property of being affine is affine-local on the target. Using these two results one may prove directly that affine morphisms are stable under base-change.

We say that a morphism $f : X \to Y$ is *finite* if it is affine and if, for any morphism $\operatorname{Spec} B \to Y$, the pullback $X \times_Y \operatorname{Spec} B \to \operatorname{Spec} B$ makes $\mathcal{O}(X \times_Y \operatorname{Spec} B)$ a finitely generated B-module. Again this property is clearly closed under composition. The property of being finite is affine-local on the target and closed under base-changes by Exercises 7.3.G and 9.4.B of [64].

Lemma 2.14. *Let k be a field and let $f : X \to \operatorname{Spec} k$ be a finite morphism. Then $X(k)$ is a finite set.*

Proof. If $f : X \to \operatorname{Spec} k$ is finite, then X is affine, isomorphic to $\operatorname{Spec} B$ for a k-algebra B which is also a finitely generated k-vector space (say of dimension n). Let $x_1, \cdots, x_n$ be n elements of B which are linearly independent over k. Every other element is a linear combination of these elements, so we can write B as $k[x_1, ..., x_n]/I$ for some ideal I. Now since B is a finite k-algebra we can choose large enough n such that the set $\{1, x_i, x_i^2, \cdots, x_i^n\}$ is linearly dependent, so we can choose coefficients $a_j^{(i)} \in k$ such that $0 = a_0^{(i)} + \cdots + a_n^{(i)} x_i^n$. Therefore, the image of x_i in k must be a root of the polynomial $a_0^{(i)} + a_1^{(i)} t + \cdots + a_n^{(i)} t^n \in k[t]$, for some symbol t. That is, there are finitely many choices for the image of x_i in k. But this is true for all $i = 1, ..., n$ and so there are only finitely many morphisms $B \to k$. $\square$

Recall that we defined a closed immersion to be a morphism $X \to Y$ of k-functors such that for any morphism $\operatorname{Spec} A \to Y$ the pullback $\operatorname{Spec} A \times_Y X$ is affine and the induced morphism of k-algebras $A \to \mathcal{O}(\operatorname{Spec} A \times_Y X)$ is surjective. Being a closed immersion is clearly closed under composition (because the composition of two surjective morphisms is surjective). It is affine-local on the target by Exercise 8.1.D of [64] and it is closed under base-change by Section 9.2.(3) of [64]. If we have a morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ then to prove that it is a closed immersion it suffices to prove that $A \to B$ is surjective. This is true because if $A \to B$ is surjective and we have another morphism $A \to C$, then the induced morphism $C \to B \otimes_A C$ is surjective. Furthermore, we say that $f : X \to Y$ is *separated* if the diagonal morphism $X \to X \times_Y X$ is a closed immersion.

If $i : Z \to X$ is a closed immersion, then for each open affine $\operatorname{Spec} A$ of X, we know that the scheme-theoretic intersection of Z and $\operatorname{Spec} A$ is defined by an ideal of A. This defines a sheaf $\mathcal{J}$ of ideals on X, which is the kernel of the morphism $\mathcal{O}_X \to i_* \mathcal{O}_Z$. In fact in many cases we can go the other way: By the discussion of Section 13.5.4 of [64], any quasi-coherent sheaf of ideals $\mathcal{J}$ on X defines a corresponding closed subscheme of X.

Example 2.15. Let X be a scheme. The sheaf $\mathcal{J}$ of ideals on X defined on affine opens by $\mathcal{J}(\operatorname{Spec} A) = \sqrt{(0)}$ is quasi-coherent by Exercise 13.3.G of [64] and it therefore defines a closed subscheme Z of X. We observe that Z is *reduced* in the sense that $\mathcal{O}_Z(U)$ is nilpotent free for any open subset U of Z. Moreover, the underlying topological space of Z is homeomorphic to X, so we call this closed subscheme the *reduction* X_{red} of X.

The scheme-theoretic image

At this stage we are now able to consider the problem of defining the image of a morphism of schemes $f : X \to Y$. The k-functor image, $\operatorname{im} f(A) = \operatorname{im} f_A$ will not be a sheaf in general, and so we must adapt our approach somewhat. We define the *scheme-theoretic image* of f to be the smallest closed subscheme of Y through which f factors. As a closed subscheme of Y, the image of f is defined by a quasi-coherent sheaf of ideals $\mathcal{J}$. Based on the affine case we might expect that $\mathcal{J}$ is the kernel of the morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$. However, this kernel is not quasi-coherent in general. Instead, by [63, Tag 01R6], $\mathcal{J}$ is the largest quasi-coherent sheaf of ideals contained in $\ker[\mathcal{O}_Y \to f_*\mathcal{O}_X]$.

Note that if $f : X \to Y$ is quasi-compact and quasi-separated, then $f_*\mathcal{O}_X$ is quasi-coherent by Theorem 16.2.1 of [64]. Therefore, the kernel of the morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is also quasi-coherent and so the scheme-theoretic image of f is the closed subscheme of Y defined by $\ker[\mathcal{O}_Y \to f_*\mathcal{O}_X]$.

2.4.4 Flat morphisms

Let M be a k-module and consider the functor $-\otimes_k M : \mathbf{Mod}_k \to \mathbf{Mod}_k$. In general this is a right exact functor (see Section 2.2 of [28]), but if it is also left exact then we say that M is a *flat* k-module. Moreover, we say that a ring homomorphism $A \to B$ is *flat* if the morphism $-\otimes_A B$ is an exact functor. Extending this definition to arbitrary schemes, we say that a morphism $f : X \to Y$ of schemes is *flat at* $x \in |X|$ if $f_x : \mathcal{O}_{X,f(x)} \to \mathcal{O}_{X,x}$ is a flat ring homomorphism. The morphism f is *flat* if it is flat at each $x \in |X|$. Furthermore, f is called *faithfully flat* if it is also surjective as a morphism of topological spaces. By part (7) of [63, Tag 00HT], a morphism of affine schemes $\operatorname{Spec} A \to \operatorname{Spec} B$ is flat if and only if the morphism of rings $B \to A$ is flat.

Furthermore, a morphism is called *fppf* (or 'fidèlement plat et de présentation fini') if it is both faithfully flat and locally of finite presentation. In Section 3.2.1 we will see that *fppf* morphisms can be thought of as scheme-theoretic quotient maps.

2.4.5 The sheaf of differentials and unramified morphisms

Let $\phi : A \to B$ be an A-algebra and let M be a B-module. An A-linear derivation of B into M is an A-module homomorphism $D : B \to M$ such that $D(bc) = b\,D(c) + c\,D(b)$ for $b, c \in B$. Note that we have $D(1) = D(1 \cdot 1) = D(1) + D(1)$ and so $D(1) = 0$. By the A-linearity of D we then have $D(a) = D(a \cdot 1) = a \cdot D(1) = 0$. We define the *module of Kähler differentials* $\Omega_{B/A}$ to be the A-module generated by the symbols db for $b \in B$, subject to the relations $d(b+b') = db + db'$ for $b, b' \in B$, $d(bb') = b\, db' + b'\, db$ for $b, b' \in B$ and $d(a) = 0$ for $a \in \phi(A)$. We then have an A-linear derivation $B \to \Omega_{B/A}$ defined by $b \mapsto db$. Note that any A-linear derivation $D : B \to M$ factors uniquely through $d : B \to \Omega_{B/A}$.

As an example, if we write $B = A[x_1, x_2, ...]/I$, then we have

$$\Omega_{B/A} = \bigoplus_i B\, dx_i/(dr, r \in I).$$

In particular, if B is a finitely generated A-algebra, then $\Omega_{B/A}$ is a finitely generated module.

Furthermore, if we have morphisms $A \to B \to C$ then (by Proposition 8.2A of [34]) we obtain an exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0. \tag{2.2}$$

where first map sends $c \otimes db \mapsto c\, db$ and the second morphism sends $da \mapsto da$.

We would like to 'sheafify' this definition to define the *sheaf of differentials* $\Omega_{X/Y}$ of an arbitrary morphism of schemes $f : X \to Y$. Specifically, we choose an open affine cover $\{\operatorname{Spec} A_i\}$ of Y, then pull this back to an open cover $\{f^{-1}(\operatorname{Spec} A_i)\}$ of X. We can then choose an open affine cover $\{\operatorname{Spec} B^i_j\}$ of X subordinate to the original cover. On $\operatorname{Spec} B^i_j$ we have the sheaf $\widetilde{\Omega}_{B^i_j/A_i}$ and we can glue these to obtain a sheaf $\Omega_{X/Y}$ on X. By construction $\Omega_{X/Y}$ is a quasi-coherent sheaf. If X is a k-scheme then we will sometimes write Ω_X instead of $\Omega_{X/k}$. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes, then the exact sequence 2.2 sheafifies to give an exact sequence,

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0. \tag{2.3}$$

Furthermore, we define the *relative tangent sheaf of* $f : X \to Y$ to be the dual sheaf

$$\mathcal{T}_{X/Y} = \mathcal{H}om(\Omega_{X/Y}, \mathcal{O}_X).$$

Let X be a k-scheme and let $x \in |X|$. We define the *tangent space* to X at x to be $\operatorname{Hom}_{\kappa(x)}(\Omega_{X/k}|_x, \kappa(x))$.

We say that a morphism of schemes $f : X \to Y$ is *unramified* if it is locally of finite type and $\Omega_{X/Y} = 0$. Informally, this means that there are no tangent vectors of X which are 'vertical' with respect to the morphism to Y. The definition of an unramified morphism may also be recast in a different way which will also be useful for us: Let $f : X \to S$ be a morphism of schemes and suppose we have a diagram

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & S \end{tikzcd} \tag{2.4}$$

where I is a nilpotent ideal in A. If there is at most one dashed arrow $\operatorname{Spec} A \to X$ making the diagram commute, then we say that f is *formally unramified*. The following result tells us that to be unramified is the same as to be formally unramified and of finite presentation. A proof of this result may be found in [63, Tag 02H9].

Proposition 2.16. *A morphism $f : X \to Y$ is formally unramified if and only if $\Omega_{X/Y} = 0$.*

We observe that the definition of formally unramified makes sense for morphisms of arbitrary k-functors, because we never used the fact that X and Y were schemes. Furthermore, we can use Diagram 2.4 to make two more useful definitions about a morphism $f : X \to Y$ of k-functors. For any diagram of the form of Diagram 2.4, if there is always at least one dashed arrow making the diagram commute, then we say that f is *formally smooth*. If there is always exactly one dashed morphism making Diagram 2.4 commute, then we say that f is *formally étale*.

We then say that a morphism of schemes is *smooth* if it is both locally of finite presentation and formally smooth. Note that if $f : X \to Y$ is smooth, then the sheaf $\Omega_{X/Y}$ is locally free (see [63, Tag 02G1]). Similarly, we say that a morphism is *étale* if is both formally étale and locally of finite presentation.

Proposition 2.17. *Let $f : X \to S$ be a formally étale (resp. formally smooth, formally unramified) morphism of k-functors and let $Y \to S$ be a morphism of k-functors. Then the base change $X' = X \times_S Y \to Y$ is formally étale (resp. formally smooth, formally unramified).*

Proof. We will prove this for formally étale morphisms: The other two cases are analogous. Let A be a k-algebra and let I be a nilpotent ideal in A such that we have morphisms $\operatorname{Spec} A/I \to X'$ and $\operatorname{Spec} A \to Y$ making the square on the left in the following diagram commute.

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d] & X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[urr, dashed] & Y \arrow[r] & S \end{tikzcd} \tag{2.5}$$

Since the morphism $X \to S$ is formally étale we get a unique dashed arrow in Diagram 2.5. Now because the square on the right is cartesian we get a unique map $\operatorname{Spec} A \to X \times_S Y$ making the whole diagram commute. Therefore, $X \times_S Y \to Y$ is formally étale. □

The following result is a consequence of [63, Tag 00U3].

Proposition 2.18. *Let K/F be a finite separable field extension. Then* $\operatorname{Spec} K \to \operatorname{Spec} F$ *is a finite and étale morphism.*

Proof. Since K is a finite separable extension of F we can write K as $F[x]/(f)$ for some polynomial $f \in F[x]$. The morphism $F \to F[x]/(f)$ is evidently finite and it is flat because $F[x]/(f)$ is a field, so by [63, Tag 04FF] it will suffice to prove that the morphism $F \to F[x]/(f)$ is formally unramified. To prove this we observe that

$$\Omega_{K/F} \cong (F[x]/(f)\, dx)\,/(f'(x)\, dx) = 0.$$

Here we have used that f' is invertible in $F[x]/(f) = K$ because K/F is separable. □

Proposition 2.19. *Let k be a field and let $X \to \operatorname{Spec} k$ be an unramifed morphism of finite-type. Then X is finite over k.*

Proof. By [63, Tag 02G7] we have that X is a disjoint union $\coprod_i \operatorname{Spec} K_i$ where each K_i is a finite separable field extension of k. As X is of finite-type over k we conclude that this is a finite disjoint union. Therefore, the set of k-points of X is finite. □

2.4.6 Proper morphisms

We say that a morphism $f : X \to Y$ is *universally closed* if, for any morphism $Z \to Y$, the induced map of topological spaces $|X \times_Y Z| \to |Z|$ is a closed map. If, in addition, f is separated and of finite type, then we say that f is *proper.* If X is a k-scheme, then we say that X is *proper over k* if the structure morphism $X \to \operatorname{Spec} k$ is a proper map. We think of being proper over a field as a scheme-theoretic analogue of compactness, in a way which will be made more precise in Section 2.6.

2.4.7 Examples of Grothendieck topologies

Now that we have discussed morphisms of schemes, we will give two examples of Grothendieck topologies on the category of schemes.

Example 2.20 (The Zariski topology)**.** The *Zariski topology* on the category of schemes is the topology in which a covering of a scheme X is a jointly surjective collection of open immersions $\{U_i \to X\}$. By Example 2.5 this does indeed define a Grothendieck topology. Note that if X is an affine scheme, then we defined a covering of X to be a collection of inclusions of distinguished open sets. These two definitions are consistent because we can always refine any open cover $\{U_i \to X\}$ to an open cover $\{V_j \to X\}$ such that each V_j is a distinguished open of X.

Example 2.21 (The fppf site)**.** We define the *fppf topology* on the category of schemes by declaring the coverings of a scheme U to be collections of flat, finitely-presented maps to U which are jointly surjective as maps of topological spaces. The fppf topology will be useful later when we wish to define a quotient of algebraic groups. The fact that the fppf topology is a topology follows from the fact that an isomorphism is faithfully-flat and that both the class of flat morphisms and the class of finitely-presented morphisms are closed under composition and base change, as discussed in Sections 2.4.4 and 2.4.3.

2.5 Varieties and curves

Definition 2.22. A *variety* is an reduced and irreducible scheme which is separated and of finite type over a field k. We define a *curve* to be a variety of dimension one. The *genus* of a curve C is defined to be the dimension of $\Gamma(C, \Omega_{C/k})$ as a k-vector space.

Let C be a curve of genus $g \geq 1$, then $\Omega_{C/k}$ is generated by global sections (by Lemma IV.5.1 of [34]) and so we have an exact sequence,

$$\mathcal{O}_C \otimes_k \Gamma(C, \Omega_{C/k}) \to \Omega_{C/k} \to 0.$$

By the definition of projective space this quotient gives us a morphism

$$\varphi : C \to \mathbb{P}(\Gamma(C, \Omega_{C/k})^\vee)$$

called the *canonical map*.

Definition 2.23. We say that a curve C is *non-hyperelliptic* if the canonical map $\varphi : C \to \mathbb{P}(\Gamma(C, \Omega_{C/k})^\vee)$ is a closed embedding, in which case we call φ the *canonical embedding.*$^\|$ Note that this can only occur if $g \geq 3$.

Many constructions only make sense for smooth varieties, so at times it is convenient to approximate a projective variety by a smooth variety. In characteristic zero, Hironaka's theorem guarantees the existence of a smooth variety which is birational to the original variety, see [36]. In our case it will be sufficient to have an *alteration* of our variety, which is defined to be a surjective morphism from a smooth variety to the original variety. Such a construction is possible in any characteristic by de Jong's Theorem [25]. Note that although de Jong's Theorem was proved later than Hironaka's Theorem, its proof is in fact simpler.

Theorem 2.24 (de Jong [25], 1996)**.** *Let X be a projective variety over a field k. Then there exists a smooth projective variety $\overline{X}$ and a proper, surjective morphism $f : \overline{X} \to X$.*

2.6 The complex analytic topology and analytification

Let X be a reduced scheme which is of finite type over $\mathbb{C}$. We will see that, locally, the set $X(\mathbb{C})$ of $\mathbb{C}$-points of X** may be identified with a subset of $\mathbb{C}^n$ for some n, and we will use this to define a topology on $X(\mathbb{C})$ called the *complex analytic topology.* This construction will allow us to investigate the geometry of X using topological and (in the case where X is smooth) analytic techniques. We will denote this topological space

$\|$Note that this is not the standard definition of a non-hyperelliptic curve: Most authors define a hyperelliptic curve to be a curve of genus at least two admitting a degree two finite morphism $C \to \mathbb{P}^1$. However, for us it will be more convenient to define non-hyperelliptic curves in terms of the canonical map. The two definitions are equivalent by Proposition 5.2 of Section IV.5 of [34].

**Note that the $\mathbb{C}$-points of X correspond with the closed points of $|X|$

by X^{an}. We will sometimes adopt this viewpoint later in this thesis if it simplifies the discussion. The complex-analytic perspective will aid our understanding of abelian varieties in particular. Note that another way to deal with the deficiencies of the Zariski topology is to consider the *étale topology* which can be defined for schemes over arbitrary bases.

Let $X = \operatorname{Spec} \mathbb{C}[x_1, ..., x_n]/I$ be a reduced affine scheme of finite type over $\mathbb{C}$. By the discussion at the end of Section 2.1 we have

$$X(\mathbb{C}) = \{(a_1, ..., a_n) \in \mathbb{C}^n \mid f(a_1, ..., a_n) = 0, \forall f \in I\}$$

which we may topologise as a subspace of $\mathbb{C}^n$. The resulting topological space is denoted X^{an}. For arbitrary reduced finite-type $\mathbb{C}$-schemes we define the analytification by taking an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X, then taking the analytification U_i^{an} of each U_i and finally gluing the U_i^{an} together to produce a topological space X^{an}. There are some checks to be made here to ensure that the U_i^{an} do indeed glue and that this definition is independent of the choice of open affine cover, see Lemma 4.6.1 of [56].

The topological space X^{an} will not in general be a manifold, because the space of solutions to a set of polynomial equation may have singularities. However, it is an example of a more general object, a *complex analytic variety*, which is an object locally defined by the vanishing of holomorphic functions. More precisely, a complex analytic variety is a locally ringed space $(Y, \mathcal{O}_Y)$ where Y is a Hausdorff space which is locally isomorphic to the vanishing locus of finitely many holomorphic functions on an open subset of $\mathbb{C}^n$. Here $\mathcal{O}_Y$ is a sheaf of $\mathbb{C}$-algebras, such that if V is an open subset of Y which is isomorphic to a subset of $U \subset \mathbb{C}^n$ defined by the vanishing of holomorphic functions $(f_1, ..., f_n)$, then the functions on V are the restrictions to V of elements of the ring $\mathcal{O}^{\mathrm{hol}}(U)/(f_1, ..., f_n)$.

If $(X, \mathcal{O}_X)$ is a reduced finite-type $\mathbb{C}$-scheme, then we can also take the 'analytification' $\mathcal{O}_{X^{\mathrm{an}}}$ of the structure sheaf so that $(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$ is a complex analytic variety. Chow's Theorem 2.25 tells us when an analytic variety is in fact algebraic.

Theorem 2.25 (Chow's Theorem)**.** *Any closed analytic subvariety of the complex manifold $\mathbb{CP}^n$ is the analytification of an algebraic subvariety of the scheme $\mathbb{P}^n_{\mathbb{C}}$.*

We will not give a proof of this result, for further discussion see Appendix A of [3] or Theorem 1.1.3 of [56]. Using Chow's Theorem we will at times be able to construct certain objects in the complex analytic category and conclude that certain related objects exist in the category of complex schemes. The nature of the connection between algebraic and analytic varieties over $\mathbb{C}$ is made more precise by Serre's GAGA Theorem, which tells us that the functor which sends a coherent sheaf on $\mathbb{P}^n_{\mathbb{C}}$ to its analytification, a coherent analytic sheaf[††] on $\mathbb{CP}^n$ is an equivalence of categories. For a more in depth discussion of this result, we refer the reader to the book [56].

[††] An object which we will not define.

Chapter 3

Group schemes, algebraic groups and abelian varieties

3.1 Group Schemes

One of the advantages of the functorial point of view is that it allows a simple definition of an algebraic group. We define a *k-group functor* to be a functor $G : \mathbf{Alg}_k \to \mathbf{Grp}$. A morphism between k-group functors G and H is a natural transformation $f : G \to H$ such that $f_A : G(A) \to H(A)$ is a group homomorphism for all k-algebras A. We denote the set of all homomorphisms from G to H by $\mathrm{Hom}_k(G, H)$. A *k-subgroup functor* of G is a subfunctor H such that $H(A)$ is a subgroup of $G(A)$ for all k-algebras A. If each group $H(A)$ is a normal subgroup of $G(A)$, then H is *normal*.

We say that a k-group functor is an *affine k-group scheme* if it is representable by a k-algebra. Furthermore, we say that a k-group functor is a *k-group scheme* if the associated functor $\mathbf{Alg}_k \to \mathbf{Grp} \to \mathbf{Set}$ obtained by composing with the forgetful functor is a scheme. Note that the collection of all k-group schemes forms a category, as does the collection of all affine k-group schemes.

Our first example of an algebraic group is the *trivial group* $e_k : \mathbf{Alg}_k \to \mathbf{Grp}$ defined by $A \mapsto \{0\}$. The trivial group is an affine k-group, represented by the algebra k (because k is initial in the category of k-algebras). Next we have the *additive group* $\mathbb{G}_a$, defined by $\mathbb{G}_a(A) = (A, +)$. If we compose with the forgetful functor to $\mathbf{Set}$ then we obtain $\mathbb{A}^1$, so this functor is represented by the polynomial algebra $k[x]$. Similarly, we define the *multiplicative group* $\mathbb{G}_m$ by $\mathbb{G}_m(A) = (A^\times, \cdot)$. The multiplicative group is represented by the k-algebra $k[x, x^{-1}]$. The *kernel* $\ker\phi$ of a homomorphism of k-group functors $\phi : G \to H$ is the k-group functor $(\ker\phi)(R) = \ker\phi_R$. We observe that $\ker\phi$ is isomorphic as a k-group functor to the fibre product $G \times_H e_k$. If G and H are k-group schemes, then the inclusion $e_k \hookrightarrow H$ is a closed immersion and so the pullback $\ker\phi = G \times_H e_k \to G$ is a closed immersion. That is, $\ker\phi$ is a closed sub-group scheme of G. Furthermore, if G and H are affine then $\ker\phi = G \times_H e_k$ is also affine. We will not discuss the image of a group homomorphism until after we have introduced the notion of an algebraic group.

Now for any non-negative integer n we have a (faithfully flat) homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ defined by $x \mapsto x^n$. We call the kernel of this morphism μ_n, so that we have $\mu_n(A) = \{a \in A \mid a^n = 1\}$. We observe that μ_n is represented by the k-algebra $k[x]/(x^n - 1)$.

Let M be a k-module, then we define the *general linear group of M* to be the k-group functor $\mathrm{GL}_M : \mathbf{Alg}_k \to \mathbf{Grp}$ defined by

$$\mathrm{GL}_M(A) = \mathrm{Aut}_A(M \otimes_k A).$$

If $M = k^n$ then we write GL_n for GL_{k^n}. In this case $M \otimes_k A = A^n$ and $\mathrm{Aut}_A(A^n)$ may be identified with the group of $n \times n$ invertible matrices with coefficients in A. Then GL_n is the affine scheme obtained by inverting the determinant, so it is represented by the k-algebra,

$$k[x_{ij} \mid 1 \le i, j \le n]_d$$

where d is the polynomial $\det(x_{ij})$. Furthermore, by imposing additional equations on GL_n we can define other affine algebraic groups such as SL_n, Sp_{2n} and SO_n. Closed subgroups of GL_n such as these are known as *linear algebraic groups*.

In general, the category of group schemes is quite difficult to work with so in practice we will often impose the requirement that our group schemes are *algebraic* over a field k. Such a group scheme will be called an *algebraic group*.

3.1.1 Hopf Algebras

Let G be a k-group scheme, then we have morphisms of schemes

$$m : G \times G \to G, \quad i : G \to G, \quad \mathbf{1} : e_k \to G$$

such that m_A is the group multiplication on $G(A)$, i_A is the map $x \mapsto x^{-1}$ in $G(A)$ and $\mathbf{1}_A : e_k(A) \to G(A)$ sends the unique element of $e_k(A)$ to the identity in $G(A)$. If G is in fact an affine k-group scheme, then we can dualise these morphisms to obtain k-algebra homomorphisms

$$\Delta = m^* : \mathcal{O}(G) \to \mathcal{O}(G \times G) = \mathcal{O}(G) \otimes_k \mathcal{O}(G)$$

$$\sigma = i^* : \mathcal{O}(G) \to \mathcal{O}(G) \quad \text{and} \quad \varepsilon = \mathbf{1}^* : \mathcal{O}(G) \to \mathcal{O}(e_k) = k$$

called *comultiplication*, the *coinverse* and the *counit* respectively. These morphisms give $\mathcal{O}(G)$ the structure of a *Hopf algebra*. A k-Hopf algebra is an associative algebra A over k along with morphisms Δ, σ and ε, such that the dual morphisms give $\operatorname{Spec} A$ the structure of a k-group scheme. By dualising the definition of a group homomorphism, we see that a morphism of affine schemes $\varphi : G \to H$ is a group homomorphism if and only if $(\varphi^* \otimes \varphi^*) \circ \Delta_H = \Delta_G \circ \varphi^*$. In the case of the additive group $\mathbb{G}_a$, we have $\mathcal{O}(\mathbb{G}_a) = k[t]$ and it may be easily verified that the structure maps are given by

$$\Delta(t) = t \otimes 1 + 1 \otimes t, \quad \sigma(t) = -t, \quad \varepsilon(t) = 0. \tag{3.1}$$

Similarly, the multiplicative group has coordinate ring $k[t, t^{-1}]$ and the structure maps are given by

$$\Delta(t) = t \otimes t, \quad \sigma(t) = t^{-1}, \quad \varepsilon(t) = 1. \tag{3.2}$$

3.2 Algebraic groups over fields

In this section we will discuss the structure of algebraic groups over fields, summarising the results we will need to use to prove the Geometric Mordell Conjecture. Unfortunately, we will not have space to prove all of these results in full generality, and in many cases we will simply give references to the literature. For the rest of Section 3.2, we will use the notation k to denote a *field* rather than a commutative ring.

3.2.1 Quotients and subobjects

In this section we will discuss the (somewhat subtle) process of taking quotients in the category of algebraic groups. Given an algebraic group G with a normal subgroup H our first instinct is to define the quotient group to be the functor $R \mapsto G(R)/H(R)$. However, this is not necessarily an algebraic group and in general we will need to 'sheafify' this functor to obtain a scheme.

We say that a homomorphism $G \to H$ of algebraic groups is a *quotient homomorphism* if it is faithfully flat. Furthermore, we say that a sequence $1 \to G \to H \to K \to 1$ of algebraic groups is *exact* if $H \to K$ is faithfully flat and $G \to H$ is an isomorphism onto the kernel of $H \to K$. Theorem 3.1 allows us to take quotients in the category of algebraic groups.

Theorem 3.1 ([27] VI_A.3.2)**.** *Let H be a normal subgroup of a algebraic group G over a field k. Define G/H to be the fppf sheafification of the k-group functor*

$$R \mapsto G(R)/H(R).$$

Then G/H is an algebraic group and the morphism $p : G \to G/H$ obtained from the universal property of sheafification is faithfully flat.

We will not give a proof of this result here since it requires a great deal of technical background on quotients of schemes contained in Exposé V of [27] and Chapter 3 of [26]. Quotients of algebraic groups are discussed in Exposé VI_A of [27]. Next we note that in the category of algebraic groups we have the following criterion for checking whether a morphism is a closed immersion.

Proposition 3.2 ([27] VI_B.1.4.2)**.** *A homomorphism of algebraic groups is a closed immersion if and only if it has trivial kernel.*

Henceforth, if we write G/H we will always be referring to the algebraic group described in Theorem 3.1. This result allows us to define the *image* of a homomorphism of algebraic groups: if $\varphi : G \to H$ is such a homomorphism, then we begin by defining $\mathrm{im}' \varphi$ to be the k-functor image of φ. Since $G(R)/(\ker \varphi_R) \cong \mathrm{im}\, \varphi_R$ for all k-algebras R by the first isomorphism theorem for groups, we know that the fppf-sheafification of $\mathrm{im}' \varphi$ is the algebraic group $G/(\ker \varphi)$. We may therefore take this as the definition of the image of φ. That is, the algebraic group $\mathrm{im}\, \varphi$ is defined to be the fppf-sheafification of the k-functor image of φ.

Similarly, if we have two sub-group functors H and K of a group scheme G, then we can define a k-functor $R \mapsto H(R)K(R)$. However, even if G is an algebraic group over a field, this is not guaranteed to be an algebraic group. As in the case of quotients we need to take the fppf sheafification of this functor to obtain an algebraic group which we label HK (see page 89 of [49]). This allows us to formulate the following isomorphism theorem.

Theorem 3.3 (Theorem 6.19 of [49])**.** *Let H and N be algebraic subgroups of an algebraic group G, with N normal in G. Then HN is a normal subgroup of G, $H \cap N$ is a normal algebraic subgroup of H and the map*

$$H/(H \cap N) \to (HN)/N$$

defined by $x(H \cap N) \mapsto xN$, is an isomorphism of algebraic groups.

Proof. The group $H/(H \cap N)$ is the fppf sheafification of the functor $R \mapsto H(R)/(H(R) \cap N(R))$, whereas the group $(HN)/N$ is the fppf sheafification of the functor $R \mapsto H(R)N(R)/N(R)$. By the second isomorphism theorem, the groups $H(R)/(H(R) \cap N(R))$ and $H(R)N(R)/N(R)$ are isomorphic, so we have an isomorphism of the underlying k-group functors. Thus, upon sheafifying we obtain an isomorphism of algebraic groups. □

We have defined both the kernel and the image of a homomorphism of algebraic groups, so we might hope that the category algebraic groups is abelian. Although this is not the case, by restricting to the category of *commutative* algebraic groups we do in fact obtain an abelian category. This will allow us to apply homological algebra in Chapter 5.

Theorem 3.4 ([27] VI_A.5.4.2)**.** *Let k be a field. Then the category of commutative algebraic k-groups is abelian.*

3.2.2 Affinisation

Let X be a k-functor. Then we have a morphism $\psi_X : X \to \operatorname{Spec}\mathcal{O}(X)$ defined on R-points by sending $x \in X(R)$ to the morphism $\operatorname{Mor}(X, \mathbb{A}^1) \to R$ defined by $f \mapsto f_R(x)$. Note that although we use the notation $\mathcal{O}(X)$ here, there is no assumption that X is affine. Note that ψ_X induces canonical isomorphisms,

$$\operatorname{Mor}_k(X, \operatorname{Spec} A) = \operatorname{Hom}_k(A, \mathcal{O}(X)) = \operatorname{Mor}_k(\operatorname{Spec}\mathcal{O}(X), \operatorname{Spec} A). \tag{3.3}$$

defined by sending $f : \operatorname{Spec}\mathcal{O}(X) \to \operatorname{Spec} A$ to $f \circ \psi_X$. This adjunction follows from the fact that we can define a morphism out of X by choosing an open affine cover $\{\operatorname{Spec} B_i\}$ of X and defining ring homomorphisms $A \to B_i$ for each i which glue, see I, §1, 4.3 of [26].

If G is an algebraic group over a field, then Lemma 2.12 says that $\mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes_k \mathcal{O}(G)$ so the group multiplication $m_G : G \times G \to G$ induces a morphism $\mathcal{O}(G) \to \mathcal{O}(G) \otimes_k \mathcal{O}(G)$ which, along with the morphisms $i^* : \mathcal{O}(G) \to \mathcal{O}(G)$ and $\mathbf{1}^* : \operatorname{Spec} k \to \mathcal{O}(G)$, give $\mathcal{O}(G)$ the structure of a Hopf algebra, as in the case of affine groups. Therefore, $G^{\text{aff}} := \operatorname{Spec}\mathcal{O}(G)$ is an affine algebraic k-group called the *affinisation of* G. By the way we defined multiplication in $\operatorname{Spec}\mathcal{O}(G)$, to see that the morphism $\psi_G : G \to G^{\text{aff}}$ is a group homomorphism we need to prove that the morphisms $G \times G \xrightarrow{m_G} G \xrightarrow{\psi_G} \operatorname{Spec}\mathcal{O}(G)$ and $G \times G \xrightarrow{\psi_{G\times G}} \operatorname{Spec}\mathcal{O}(G \times G) \to \operatorname{Spec}\mathcal{O}(G)$ coincide. But under the adjunction 3.3 both of these morphisms correspond to the morphism $\operatorname{Spec}\mathcal{O}(G \times G) \to \operatorname{Spec}\mathcal{O}(G)$ induced by multiplication in G.

Proposition 3.5 (Proposition III, §3, 8.1 of [26]). *The morphism $\psi_G : G \to G^{aff}$ is universal among maps from G to affine group schemes.*

Proof. Let $H = \operatorname{Spec} A$ be an affine scheme and suppose we have a morphism $\varphi : G \to H$. Equation 3.3 tells us that there is a unique morphism $\varphi' : G^{\text{aff}} \to H$ such that $\varphi = \varphi' \circ \psi_G$. We need to prove that $m_H \circ (\varphi' \times \varphi') = \varphi' \circ m$, where m_H is the group law on H and m is the group law on G^{aff}. Now we note that by Lemma 2.12 we have $\mathcal{O}(G \times G) = \mathcal{O}(G) \otimes_k \mathcal{O}(G)$ which implies that $(G \times G)^{\text{aff}} = G^{\text{aff}} \times G^{\text{aff}}$. Therefore, by the adjunction 3.3 there is a correspondence between morphisms *of schemes* $G \times G \to H$ and morphisms of schemes $(G \times G)^{\text{aff}} = G^{\text{aff}} \times G^{\text{aff}} \to H$, which means that it will suffice to prove that the two compositions

$$m_H \circ (\varphi' \times \varphi') \circ (\psi_G \times \psi_G) \quad \text{and} \quad \varphi' \circ m \circ (\psi_G \times \psi_G)$$

agree, as shown in 3.4.

$$\begin{tikzcd} G \times G \arrow[r] & G^{\text{aff}} \times G^{\text{aff}} \arrow[r] \arrow[d] & H \times H \arrow[d] \\ & G^{\text{aff}} \arrow[r] & H \end{tikzcd} \tag{3.4}$$

Now we can show the required result by an explicit calculation,

$$\begin{aligned} m_H \circ (\varphi' \times \varphi') \circ (\psi_G \times \psi_G) &= m_H \circ (\varphi' \circ \psi_G \times \varphi' \circ \psi_G) \\ &= m_H \circ (\varphi \times \varphi) \\ &= \varphi \circ m_G && \text{(as } \varphi \text{ is a homomorphism)} \\ &= \varphi' \circ \psi_G \circ m_G \\ &= \varphi' \circ m \circ (\psi_G \times \psi_G) && \text{(as } \psi_G \text{ is a homomorphism.)} \end{aligned}$$

□

Theorem 3.6 (Theorem III, §3, 8.2 of [26])**.** *Let G be an algebraic k-group. Then G^{aff} is an algebraic k-group, the homomorphism $\psi : G \to G^{aff}$ is faithfully flat and the kernel C of ψ satisfies $\mathcal{O}(C) = k$.*

Remark 3.7. We will not give a proof of Theorem 3.6 but note that the kernel C of $\psi : G \to G^{\mathrm{aff}}$ is the smallest normal subgroup of G such that the quotient G/C is affine.

Remark 3.8. We call an algebraic k-group such that $\mathcal{O}(G) = k$ an *anti-affine group*. Any anti-affine group over a field k is smooth, connected and abelian (see Section III, §3, 8.3 of [26]). The properties of anti-affine groups are discussed extensively in [9].

3.2.3 Algebraic vector groups

In this section let k denote a field. Given a k-module V, we can define two k-group functors $\underline{V}$ and $V_{\mathfrak{a}}$ by

$$\underline{V}(R) = \mathrm{Hom}_{\mathbf{Mod}_k}(V, R) \quad \text{and} \quad V_{\mathfrak{a}}(R) = V \otimes_k R.$$

We have,

$$\underline{V}(R) = \mathrm{Hom}_{\mathbf{Mod}_k}(V, R) = \mathrm{Hom}_k(\mathrm{Sym}_k V, R)$$

and so $\underline{V} \cong \mathrm{Spec}(\mathrm{Sym}_k V)$ is an affine group scheme. The association $V \mapsto \underline{V}$ defines a contravariant functor from the category of k-vector spaces to affine algebraic groups. We say that an algebraic group is an *algebraic vector group* if it is isomorphic to $\underline{V}$ for some vector space V over k (not necessarily finite-dimensional). Similarly, if V is a finite-dimensional vector space then we have

$$V_{\mathfrak{a}}(R) = V \otimes_k R = \mathrm{Hom}_{\mathbf{Mod}_k}(V^{\vee}, R) = \mathrm{Spec}(\mathrm{Sym}_k V^{\vee})$$

which is again an algebraic vector group.

Next, we will turn our attention to the *Lie scheme* of an algebraic group. If X is a k-functor, then we can define a new k-functor $J^1(X)$ by $J^1(X)(R) = X(R[\tau])$ where $R[\tau] \coloneqq R[x]/(x^2)$. We will discuss this functor in much more detail in Chapter 4 but for now note that if X is a scheme then $J^1(X)$ is also a scheme by Proposition 4.8. The projection $R[\tau] \to R$ induces a morphism $J^1(X) \to X$. If $X = G$ is also a k-group functor, then the morphism $J^1(G) \to G$ is a group homomorphism and we define the k-group functor $\mathrm{Lie}(G) : \mathbf{Alg}_k \to \mathbf{Grp}$ to be the kernel of this morphism. If G is a group scheme, then $\mathrm{Lie}(G)$ is the kernel of a homomorphism of group schemes and therefore a group scheme in its own right. If $f : G \to H$ is a homomorphism of k-group schemes, then we have an induced morphism $\mathrm{Lie}(f) : \mathrm{Lie}\, G \to \mathrm{Lie}\, H$. Thus, $\mathrm{Lie}(-)$ defines an endofunctor of the category of k-group schemes.

We write $\mathscr{L}(G)$ (or $\mathfrak{g}$) for the group $\mathrm{Lie}(G)(k)$. By definition, $\mathscr{L}(G)$ is the set of morphisms $\gamma : \mathrm{Spec}\, k[\tau] \to G$ making Diagram 3.5 commute, where $p : \mathrm{Spec}\, k \to \mathrm{Spec}\, k[\tau]$ is induced by the projection $k[\tau] \to k$.

$$\begin{tikzcd} \mathrm{Spec}\, k \arrow[r, "p"] \arrow[dr, "e_k"'] & \mathrm{Spec}\, k[\tau] \arrow[d, "\gamma"] \\ & G \end{tikzcd} \tag{3.5}$$

Note that $\mathscr{L}(G)$ may be endowed with a vector space structure such that vector addition agrees with the group operation on $\mathrm{Lie}\, G(k) \subset G(k[\tau])$ (see Chapter 12 of [49]).

Now, given any k-algebra R, the functoriality of $\mathrm{Lie}\, G$ gives us a group homomorphism $\phi' : \mathscr{L}(G) = \mathrm{Lie}\, G(k) \to \mathrm{Lie}\, G(R)$. By the observation above, if we view $\mathrm{Lie}\, G(R)$ as a k-vector space via the homomorphism $k \to R$, then ϕ' is a homomorphism of k-vector spaces. The target of ϕ' is in fact an R-module, so we obtain an R-module homomorphism $\phi : R \otimes_k \mathscr{L}(G) \to \mathrm{Lie}\, G(R)$. In general ϕ will not be an isomorphism, but by Proposition 4.8 of Chapter II, Section §4,4 of [26], if $\Omega_{G/k}$ is locally-free and of finite type then ϕ becomes bijective. In particular, if G is an algebraic group we have an isomorphism

$$\mathrm{Lie}(G) = \mathscr{L}(G)_{\mathfrak{a}} \tag{3.6}$$

and so $\mathrm{Lie}(G)$ is an algebraic vector group.

Next we define a contravariant functor $X_a : \mathbf{GrSch}_k \to \mathbf{Mod}_k$ which assigns to a group scheme its *character group* $X_a(G) = \mathrm{Hom}_k(G, \mathbb{G}_a)$. Note that $X_a(G)$ forms a k-vector subspace of the k-algebra $\mathcal{O}(G) = \mathrm{Mor}_k(G, \mathbb{G}_a)$, and is therefore itself a k-vector space. Proposition 3.10 relates this functor with the functor $V \mapsto \underline{V}$ over a field of characteristic zero, thereby giving us conditions which will allow us to check whether a group is vectorial. First we will need to make the following definition.

Definition 3.9. A k-group U is *unipotent* if it is affine and if, for every non-trivial closed subgroup H of G, there is a non-zero homomorphism $H \to \mathbb{G}_a$.

Proposition 3.10 (Proposition IV, §2, 4.2 of [26])**.** *Let k be a field of characteristic zero. The functor $X_a : \mathbf{GrSch}_k \to \mathbf{Mod}_k$ is an equivalence of the category of commutative unipotent k-groups with $\mathbf{Mod}_k^{op}$. The functor $V \mapsto \underline{V}$ is an inverse.*

Corollary 3.11. *Any closed subgroup scheme of an algebraic vector group is an algebraic vector group.*

Proof. Let H be a subgroup scheme of an algebraic vector group $\underline{V}$. Then H is commutative and any closed subgroup of a unipotent group is clearly unipotent, so by Proposition 3.10 we conclude that H is an algebraic vector group. □

3.3 Abelian varieties

So far, all of our examples of algebraic groups have been affine. In this section we will investigate an important class of algebraic groups known as *abelian varieties*, which are not affine (except for a point, which is an affine abelian variety) but are still relatively well understood. Introductory references for the study of abelian varieties which we have used in this section are Mumford [52] and Milne [48].

Definition 3.12. Let k be a field. We say that a connected algebraic k-group is an *abelian variety* if it is smooth and proper over k. An *isogeny* of abelian varieties is a surjective homomorphism whose kernel is finite over k.

We do not have the space here to develop in full generality the theory of abelian varieties over arbitrary base fields. Instead we will state all results precisely, but in our proofs we will focus mainly on the theory of complex abelian varieties, following [3] and Chapter 1 of [52]. In our applications we will mainly be interested in abelian varieties over fields of characteristic zero, so the Lefschetz principle indicates that we are not making too severe a restriction.*

Let A be an abelian variety over $\mathbb{C}$ and consider the analytification $G := A^{\mathrm{an}}$. Because A is proper and connected over $\mathbb{C}$ we know that G is a compact, connected Lie group, so we have at our disposal many results in Lie theory and topology which will simplify the discussion. Firstly, we will prove that G is abelian, following pages 1-3 of [52]. For fixed $g \in G$, the map $\mathrm{Ad}_g : G \to G$ defined by $x \mapsto gxg^{-1}$ is an automomorphism of complex Lie groups. Differentating Ad_g at the origin we obtain an isomorphism $\mathrm{ad}_g := d(\mathrm{Ad}_g)_0 : \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g} = T_0G$ is the tangent space to G at the identity. Assume that $\dim \mathfrak{g} = n$. This defines the adjoint representation $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$. Since G is a complex Lie group, the adjoint representation is a holomorphic morphism of G into an open subset of $\mathbb{C}^{n^2}$, so because G is compact and connected we know that ad must be constant. That is, $\mathrm{ad}_g = \mathrm{id}_{\mathfrak{g}}$ for all $g \in G$. Therefore, by the naturality of the

*As stated in [31] the Lefschetz principle is as follows: 'If a statement A in the language of algebraic geometry holds over an algebraically closed field of characteristic zero, then it is true over any other algebraically closed field of characteristic zero'. This is based on the idea that any argument over an algebraically closed field k will involve at most countably many elements of k, and so they will generate a subfield isomorphic to $\mathbb{C}$. This means that we can prove the statement holds over $\mathbb{C}$, perhaps using analytic techniques (along as the result itself is stated in the language of algebraic geometry). However, we should note that we mean this as a statement of principle and *not* as a claim that for every result which follows, giving a proof over $\mathbb{C}$ is logically sufficient to establish the result in general.

exponential map $\exp : \mathfrak{g} \to G$ (see page 116 of [32]) we have

$$\mathrm{Ad}_g(\exp(v)) = \exp(\mathrm{ad}_g(v)) = \exp(v).$$

This means that $g \exp(v) = \exp(v) g$ for all $g \in G$, so $\exp(v) \in Z(G)$, the centre of G. By Proposition 8.33 of [32] the map $d(\exp)_0 : \mathfrak{g} \to \mathfrak{g}$ is the identity, so the inverse function theorem implies that exp sends a neighbourhood of 0 in $\mathfrak{g}$ diffeomorphically onto a neighbourhood of the identity in G. But any connected topological group is generated by a neighbourhood of the identity, so the fact that $\mathfrak{g}$ is abelian implies that G is also abelian.

Consider now the exponential map $\exp : \mathfrak{g} \to G$ and fix $x, y \in G$. Because G is abelian, this is a group homomorphism (see page 2 of [52]). Recall that the subgroup $\exp(\mathfrak{g})$ of G contains an open neighbourhood U of the identity in G. The subgroup $\langle U \rangle$ generated by U is non-empty, open and closed, so because G is connected we conclude that $G = \langle U \rangle \subset \exp(\mathfrak{g})$. That is, $\exp : \mathfrak{g} \to G$ is surjective and so $G \cong \mathfrak{g}/(\ker \exp)$ by the first isomorphism theorem. Finally, we see that $\ker \exp$ is a discrete subgroup of $\mathfrak{g}$, because the inverse function theorem allows us to choose a neighbourhood V of $0 \in \mathfrak{g}$ such that $\exp|_V : V \to G$ is injective. In particular, there is a neighbourhood of 0 which contains no other points of the kernel of exp. If v is another point of $\ker \exp$ then $v + V$ is an open neighbourhood of v such that $(\ker \exp) \cap (v + V) = \{v\}$ and so $\ker \exp$ is discrete. That is, $\ker \exp$ is a discrete subgroup of $\mathfrak{g}$ such that $\mathfrak{g}/\ker \exp$ is compact. This means that $\ker \exp$ is the free abelian group generated by a (real) basis for $\mathfrak{g}$. Such a subgroup is called a *lattice*.

Definition 3.13. If V is a complex vector space, and Λ is a lattice in V, then we say that the quotient space V/Λ is a *complex torus*. A morphism between two complex tori is a homomorphism of complex Lie groups.

The above discussion shows that if A is a complex abelian variety, then A^{an} is isomorphic to a complex torus. Now we note that if $X = V/\Lambda$ is a complex torus, then the action of Λ on V by translation is a covering space action in the sense of [35]. Therefore, Proposition 1.40 of [35] implies that $\pi : V \to X$ is the universal covering space of X and that $\pi_1(X) \cong \Lambda$.

Proposition 3.14 (Corollary 1.2 of [48])**.** *Any morphism of $A_1 \to A_2$ of abelian varieties factors as the composition of a translation and a homomorphism.*

Proof. We will prove this result for complex tori, following the proof of Proposition 1.2.1 of [3]. For a proof of this result for arbitrary abelian varieties, see Theorem 1.1 of [48]. Suppose that we have two complex tori $X_1 = V_1/\Lambda_1$ and $X_2 = V_2/\Lambda_2$, then we want to prove that every holomorphic map $f : X_1 \to X_2$ factors uniquely as $t_{f(0)} \circ \varphi$, where $t_{f(0)} : X_2 \to X_2$ is translation by $f(0)$, and φ is a homomorphism (of Lie groups).

Set $\varphi = t_{-f(0)} \circ f$ and consider the holomorphic map $V_1 \xrightarrow{\pi_1} X_1 \xrightarrow{f} X_2$, where $\pi_1 : X_1 \to V_1$ is the projection. Since V_1 is simply connected, this lifts to a holomorphic map $\widetilde{\varphi} : V_1 \to V_2$. Since φ sends $0 \mapsto 0$ we can choose the lift $\widetilde{\varphi}$ so that $\widetilde{\varphi}(0) = 0$. We will prove that $\widetilde{\varphi}$ is a linear map which sends Λ_1 to Λ_2.

$$\begin{tikzcd} V_1 \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\pi_1"] & V_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "\varphi"] & X_2 \end{tikzcd} \tag{3.7}$$

For any $\lambda \in \Lambda_1$ and $v \in V_1$ the two vectors v and $v + \lambda$ in V_1 have the same image in X_2 under $\varphi \circ \pi_1$, so $\widetilde{\varphi}(v + \lambda) - \widetilde{\varphi}(v) \in \ker \pi_2 = \Lambda_2$. Fixing λ, we have a continuous map $v \mapsto \widetilde{\varphi}(v + \lambda) - \widetilde{\varphi}(v)$ whose image is contained in Λ_2. Since Λ_2 is discrete, this map is constant. Because $\widetilde{\varphi}(0) = 0$, we have that $\widetilde{\varphi}(v + \lambda) = \widetilde{\varphi}(v) + \widetilde{\varphi}(\lambda)$, from which we can infer that all the $(2g)$ partial derivatives of $\widetilde{\varphi}$ are periodic in Λ_1. This implies that they are bounded and so Liouville's theorem implies that all the partial derivatives of $\widetilde{\varphi}$ are constant. As a holomorphic function $V_1 \to V_2$ which fixes 0 and has constant partial derivatives, $\widetilde{\varphi}$ is linear. Since the additive structure on X_i is induced by vector addition in V_i, this implies that φ is a homomorphism. □

Applying Proposition 3.14 to the inversion map $A \to A$ defined by $x \mapsto x^{-1}$ we obtain the following corollary, which we have already demonstrated for complex abelian varieties.

Corollary 3.15. *An abelian variety is a commutative algebraic group.*

Finally, we make note of the following important theorem which follows from the fact every abelian variety is endowed with an ample line bundle. For a proof of this result in the complex case, see Section 4, §5 of [3]. For a general proof, see Section I.6 of [48].

Theorem 3.16 (Theorem 6.4 of [48])**.** *Every abelian variety is projective.*

3.3.1 Families of abelian varieties

Let S be a k-variety. A *family of abelian varieties over* S (or an *abelian scheme*) is a proper smooth morphism $\pi : \mathcal{A} \to S$ with connected fibres, along with a multiplication morphism $\mathcal{A} \times_S \mathcal{A} \to S$. Note that the fibres of such a morphism are abelian varieties in the usual sense, see page 99 of [8]. A morphism of abelian schemes is a morphism of S-schemes which commutes with multiplication.

Proposition 3.17 (Proposition 16.3 of [48])**.** *Let A be an abelian variety over a field k and let S be a k-variety such that $S(k) \neq \emptyset$. For any injective homomorphism $\mathcal{B} \hookrightarrow A \times S$ of abelian schemes, there exists an abelian k-subvariety B of A such that $\mathcal{B} \cong B \times S$.*

Note that the main technical result needed to prove Proposition 3.17 is Mumford's 'Rigidity Lemma', Proposition 6.1 of [53].

Corollary 3.18. *Let F be a function field over an algebraically closed field k of characteristic zero. Suppose A is an abelian F-variety which descends to k and let B be an abelian subvariety of A. Then both B and A/B descend to k.*

Proof. If B is an abelian subvariety of $A = A_0 \times_{\operatorname{Spec} k} \operatorname{Spec} F$, then Proposition 3.17 implies directly that there exists a k-subvariety B_0 of A_0 along with an isomorphism $B_0 \times_{\operatorname{Spec} k} \operatorname{Spec} F \to B$. Note that this map is a homomorphism by Proposition 3.14 because it fixes the identity. Then A/B is a quotient of abelian varieties which are defined over k, so it is also defined over k. $\square$

3.3.2 The F/k-trace and the Lang–Néron Theorem

In this section we continue our study of families of abelian varieties. Fix an algebraically-closed field k and a function field F of one variable over k. By this we mean a finitely-generated field extension of transcendence degree one. Furthermore, let A be an abelian F-variety.

Definition 3.19. The *F/k-trace* of A is an abelian k-variety $\mathrm{Tr}_{F/k}(A)$ equipped with a morphism $\tau : \mathrm{Tr}_{F/k}(A)_F \to A$ of F-schemes, such that any other F-morphism $f : B_F \to A$ where B is an abelian k-variety, factors uniquely through τ.

$$\begin{tikzcd} B_F \arrow[rr, "f"] \arrow[dr, dashed, "\exists!"'] & & A \\ & \mathrm{Tr}_{F/k}(A)_K \arrow[ur, "\tau"'] & \end{tikzcd} \tag{3.8}$$

Intuitively, we think of $\mathrm{Tr}_{F/k}(A)$ as approximating A by an F-subvariety which descends to K. Now we may view $\mathrm{Tr}_{F/k}(A)(k)$ as sitting inside $A(F)$ via the composition,

$$\mathrm{Tr}_{F/k}(A)(k) \to \mathrm{Tr}_{F/k}(A)(F) \xrightarrow{\tau_F} A(F).$$

We will write $\tau(\mathrm{Tr}_{F/k}(A)(k))$ for the image of this map. We then have the following result, originally due to Lang–Néron [39]. For a proof of the Lang–Néron Theorem, we refer the reader to Theorem 6.2 of [40] and Theorem 7.1 of [23].

Theorem 3.20 (Lang–Néron [39], 1959)**.** *Let F be a function field of one variable over an algebraically closed field k of characteristic zero and let A be an abelian K-variety. Then the group*

$$A(F)/\tau(Tr_{F/k}(A)(k))$$

is finitely-generated.

Remark 3.21. If the (F/k)-trace of A is zero, then Theorem 3.20 implies that $A(F)$ is finitely-generated. This is the so-called 'Geometric Mordell–Weil Theorem'. Moreover, if $A \cong A_0 \times_{\operatorname{Spec} k} \operatorname{Spec} F$ is itself defined over k, then the the F/k-trace of A is A_0 and so we have that $A(F)/A_0(k)$ is finitely generated. In fact, these are the only two cases of the Lang–Néron which we will need in Chapter 6.

3.3.3 Chevalley's Theorem

The two types of group which have been studied in the greatest depth so far are the abelian varieties, which can often be thought of as complex tori, and the linear algebraic groups, which are subgroups of GL_n. We do not have as clear a picture of other algebraic groups. Fortunately, Chevalley's Theorem 3.22 tells us that every abelian variety is an extension of an abelian variety by a linear algebraic group. We refer the reader to [22] for a proof of this result.

Theorem 3.22 (Chevalley [19], 1960)**.** *Let G be an algebraic group over a field k of characteristic zero. Then there exists a unique normal linear algebraic closed subgroup $a_*(G)$ of G such that $G/a_*(G)$ is an abelian variety.*

We will call $a_*(G)$ the *Chevalley subgroup* of G.† The *Rosenlicht Decomposition* of Corollary 3.23 expresses the relationship between the subgroup $a_*(G)$ obtained from Chevalley's Theorem and the kernel of the affinisation morphism $\psi : G \to G^{\mathrm{aff}}$.

Corollary 3.23 (Corollary 5 of [59])**.** *Let G be an algebraic group. Let $a_*(G)$ be the Chevalley subgroup of G and let C be the kernel of the affinisation morphism $\psi : G \to G^{\mathrm{aff}}$. Then $a_*(G) \cdot C = G$.*

3.3.4 Jacobians and Albanese varieties

For us, abelian varieties will usually arise as the *Jacobian* of a smooth, projective curve, so in this section we will study this construction.

Let C be a proper, smooth, genus g curve over $\mathbb{C}$, then the set $Y := C^{\mathrm{an}}$ has the structure of a compact Riemann surface (see Lecture 1 of the appendix to [51]). Let ω be a differential p-form on Y, then we can define a linear functional $\iota(\omega) : H_p(Y;\mathbb{C}) \to \mathbb{C}$ by $[c] \mapsto \int_c \omega$. That is, we integrate the p-form along a differential chain.‡ This is well-defined by Stokes' Theorem: if $c = \partial b$ for some $(p-1)$-chain b then we have

$$\int_c \omega = \int_b d\omega = 0$$

because ω is a closed form. By the universal coefficient theorem, cohomology with $\mathbb{C}$ coefficients is dual to homology with $\mathbb{C}$ coefficients and so we have in fact defined

†Note that this terminology is not standard.

‡Here we are using that singular homology on a manifold can be computed using differentiable simplices rather than singular simplices.

a homomorphism $H^p_{\mathrm{dR}}(Y) \to H^p(Y;\mathbb{C})$. Moreover, this map is compatible with the algebra structure on $H^\bullet_{\mathrm{dR}}(Y)$ and $H^\bullet(Y;\mathbb{C})$ and so we have an algebra homomorphism

$$H^\bullet_{\mathrm{dR}}(Y) \xrightarrow{\sim} H^\bullet(Y;\mathbb{C}) \tag{3.9}$$

De Rham's Theorem implies that this is an isomorphism, and so at times we will conflate singular and de Rham cohomology.

Now let Ω^1 denote the sheaf of holomorphic 1-forms on Y. The *Hodge decomposition* of page 28 of [16] says that $H^1(Y;\mathbb{C}) = H^1_{\mathrm{dR}}(Y)$ decomposes as

$$H^1_{\mathrm{dR}}(Y) \cong \Gamma(Y,\Omega^1) \oplus \overline{\Gamma(Y,\Omega^1)}. \tag{3.10}$$

Here we note that $H^1(Y;\mathbb{C}) \cong H^1(Y;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ and so we may conjugate a holomorphic 1-form. Now we define a homomorphism $H_1(Y;\mathbb{Z}) \to H^1_{\mathrm{dR}}(Y)^\vee$ by

$$[\gamma] \mapsto \left([\omega] \mapsto \int_\gamma \omega\right).$$

This map is injective because $H_1(Y;\mathbb{Z})$ is a subgroup of $H_1(Y;\mathbb{C}) = H^1(Y;\mathbb{C})^\vee$. Moreover, the image of this homomorphism is contained in $\Gamma(Y,\Omega^1)$ (see Lemma 11.1.1 of [3]) and so we have defined an embedding of $H_1(Y,\mathbb{Z}) \cong \mathbb{Z}^{2g}$ into $\Gamma(Y,\Omega^1)$ as a lattice. Therefore, we have a complex torus

$$\mathrm{Jac}(Y) \coloneqq \Gamma(Y,\Omega^1)^\vee / H_1(Y;\mathbb{Z})$$

which we call the *Jacobian* of Y. If we fix a point $x_0 \in Y$ and choose another point y along with a path γ from x_0 to x then we can think of the integral $\int_{x_0}^x$ which integrates a holomorphic 1-form along the path γ as a linear functional on $\Gamma(Y,\Omega^1)$. However, if we were to choose a different path γ' from x_0 to x, then the integrals $\int_\gamma \omega$ and $\int_{\gamma'} \omega$ would differ by an element of $H_1(Y;\mathbb{Z})$ and so we can think of $\int_{x_0}^x$ as an element of $\Gamma(Y,\Omega^1)^\vee/H_1(Y;\mathbb{Z})$. In this way the association $x \mapsto \int_{x_0}^x$ defines a holomorphic map $\alpha : Y \to \mathrm{Jac}(Y)$ called the *Abel-Jacobi morphism* of Y. The Abel-Jacobi morphism sends x_0 to the identity of $\mathrm{Jac}(Y)$ and so we think of it as a pointed map $(Y, y_0) \to (\mathrm{Jac}(Y), 0)$.

The construction of the Jacobian may be algebraised (using the Picard functor, for example), allowing the construction of the Jacobian variety of a curve over any base field. Theorem 3.24 summarises the results we need about the Jacobian of a curve. For a proof of this result we refer the reader to Chapter III of [48].

Theorem 3.24 (Theorem 3.1.2, Proposition 3.2.3 and Proposition 3.6.1 of [48])**.** *Let C be a complete, non-singular, genus g curve over a field k with a k-rational point x_0. Then there exists an abelian variety* $\mathrm{Jac}(C)$ *along with a morphism $\alpha : C \to J(C)$ satisfying the following universal property: For any abelian variety A and morphism $f : C \to A$ sending x_0 to the identity of A, the morphism f factors uniquely through a group homomorphism* $\mathrm{Jac}(C) \to A$*. Moreover, if C is a curve of genus $g \geq 1$ then the "Abel-Jacobi morphism" α is a closed embedding.*

By Proposition 6.11 of [67], a Hodge decomposition holds for a class of objects known as *compact Kähler manifolds* which generalise smooth compact projective analytic varieties. As was the case for compact Riemann surfaces, for any compact Kähler manifold X, this Hodge decomposition allows us to define an associated complex torus, known as the *Albanese torus.* If X is further assumed to be projective, then the Albanese torus is projective by Corollary 12.12 of [67]. By Theorem 12.15 of [67], this torus satisfies a similar universal property to the Jacobian.

Theorem 3.25 (Theorem 5 of [61]). *Let X be a smooth projective pointed variety X over a field of characteristic zero and let $x_0 \in X(k)$ be a k-point of X. Then there exists an abelian variety $\mathrm{Alb}(X, x_0)$ along with a pointed morphism $\alpha : (X, x_0) \to (\mathrm{Alb}(X, x_0), 0)$ such that every pointed morphism $(X, x_0) \to (A, 0)$ from X to an abelian variety factors uniquely through α.*

Proof. In the case of varieties over $\mathbb{C}$, this follows from the discussion above and Chow's Theorem 2.25. An algebraic construction of the Albanese variety was given by Serre in [61]. $\square$

Remark 3.26. We call the abelian variety $\mathrm{Alb}(X, x_0)$ of Theorem 3.25, the *Albanese variety* of X.

Note that non-isomorphic curves may have Jacobians which are isomorphic as abelian varieties. However, the Jacobian of a curve always admits the additional structure of a *principal polarisation* (whose definition we omit, see Section 4.1 of [3]) and Torelli's Theorem states that a proper smooth curve over an algebraically closed field is determined up to isomorphism by its principally polarised Jacobian (see Theorem 12.1 of [48]). In Chapter 6 we will need the following corollary of Torelli's Theorem, whose proof may be found in [13].

Proposition 3.27 (Corollary 1.1.11 of [13]). *Let $k = \overline{k}$ be an algebraically closed field, let F/k be a field extension and let C be a smooth projective curve over k of genus at least 1. If the Jacobian of $Y \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \overline{F}$ descends to k, then $Y \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, \overline{F}$ itself also descends to k.*

3.3.5 The Gauss map

Let C be a curve of genus at least one over k. By Theorem 3.24 we may view C as being embedded within it Jacobian A via the Abel–Jacobi map α. Because $C \to A$ is a closed immersion, on an open affine it is described by a surjective map of rings, and so the sheaf of differentials $\Omega_{C/A}$ is zero. Therefore, by the exact sequence for differential sheaves 2.3 we have an exact sequence

$$\alpha^* \Omega_{A/k} \to \Omega_{C/k} \to 0 \tag{3.11}$$

Note that by 4.(iii) of [52] the tangent bundle of an abelian variety is trivialisable: we have a natural isomorphism $(\mathscr{L}A)^\vee \otimes_k \mathcal{O}_A$ and so the sequence 3.11 becomes

$$(\mathscr{L}A)^\vee \otimes_k \mathcal{O}_C \to \Omega_{C/k} \to 0.$$

By the defining property of projective space we thereby obtain a morphism $\Phi : C \to \mathbb{P}(\mathscr{L}A)$, which we call the *Gauss map*. Note that by Proposition 2.2 of [48], $\mathscr{L}A \cong (\Gamma(A, \Omega_{A/k}))^\vee \cong (\Gamma(C, \Omega_{C/k}))^\vee$ and so the Gauss map in fact coincides with the canonical map of Section 2.5. However, this alternative description of the canonical map will be useful in Chapter 6.

3.3.6 Extensions of abelian varieties

We conclude this chapter with a discussion of extensions of abelian varieties by algebraic vector groups, which will be central to our discussion in Chapter 5. All of the results we will discuss here were established by Rosenlicht in [60]. We will follow the treatment given by Serre in [62].

Let A and B be algebraic groups and consider an extension of A by B,

$$E : 0 \to B \xrightarrow{\alpha} C \xrightarrow{\beta} A \to 0.$$

We say that E is isomorphic to another extension $E' : 0 \to B \to C' \to A \to 0$ of A by B if we have a group homomorphism $f : C \to C'$ making the following diagram commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow f & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & C' & \longrightarrow & A & \longrightarrow & 0 \end{array} \tag{3.12}$$

The set of isomorphism classes of extensions of A by B is denoted $\mathrm{Ext}(A, B)$.We will use the notation $[C]$ to denote the isomorphism class of the extension $0 \to B \to C \to A \to 0$. Given $f : B \to B'$ we obtain a map $f_* : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B')$ which sends an extension $[C]$ to $[f_*C]$, where f_*C is the quotient of $C \times B'$ by the subgroup generated by elements of the form $(-b, f(b))$. Similarly, if we have $f : A' \to A$ then we obtain a map $f^* : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B)$ defined by sending an extension C to the extension f^*C which is the pullback of $f : A' \to A$ and $C \to A$. Now we have a diagonal map $d : A \to A \times A$ and a multiplication map $m : B \times B \to B$, so we can define a group structure on $\mathrm{Ext}(A, B)$ by setting

$$[C] + [C'] = [d^* m_*(C \times C')].$$

This group law is clearly abelian, and we omit the proof that it is well-defined and associative (see Proposition VII.1 of [62]). The identity is $[B \times A]$ and the inverse of an element $[C]$ is $[(-1)^*C]$, where $(-1) : A \to A$ is the morphism $a \mapsto -a$. The fo

Proposition 3.28 (Proposition VII.2 of [62]). *Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence of algebraic groups. Then we have homomorphism $\partial : \mathrm{Hom}(A', B) \to$*

$\operatorname{Ext}(A'', B)$ *which sends* $\varphi : A' \to B$ *to the extension* $[\varphi_*(A)] \in \operatorname{Ext}(A'', B)$. *The homomorphism* ∂ *is the connecting homomorphism in a long exact sequence,*

$$0 \to \operatorname{Hom}(A'', B) \to \operatorname{Hom}(A, B) \to \operatorname{Hom}(A', B) \xrightarrow{\partial} \operatorname{Ext}(A'', B) \to \operatorname{Ext}(A, B) \to \operatorname{Ext}(A', B).$$

Now we consider the group $\operatorname{Ext}(A, \mathbb{G}_a)$, where A is an abelian variety. In this case, any $\lambda \in k$ defines a homomorphism $(\lambda\times) : \mathbb{G}_a \to \mathbb{G}_a$. The scalar multiplication $\lambda \cdot [E] = [(\lambda\times)^* E]$ gives $\operatorname{Ext}(A, \mathbb{G}_a)$ the structure of a k-vector space.

Theorem 3.29 (Theorems VII.7 and VII.10 of [62])**.** *The vector space* $\operatorname{Ext}(A, \mathbb{G}_a)$ *is finite-dimensional over* k*.*

Proof. In [62] Serre establishes this result by showing that an extension belonging to $\operatorname{Ext}(A, \mathbb{G}_a)$ is a $\mathbb{G}_a$-torsor and showing that $\mathbb{G}_a$-torsors are classified by the sheaf cohomology group $H^1(A, \mathcal{O}_A)$.[§] Then the finite-dimensionality of $H^1(A, \mathcal{O}_A)$ follows from the fact that A is projective (by Theorem 3.16). The main technical result used to establish the correspondence between extensions and $\mathbb{G}_a$-torsors is Rosenlicht's rational cross-section theorem (Theorem 10 of [59]) which implies that for any extension $0 \to \mathbb{G}_a \to E \to A \to 0$, the homomorphism $E \to A$ has a rational section. $\square$

[§]Note that we have not even defined sheaf cohomology in this thesis, so we mean this as an informal sketch.

Chapter 4

Differential algebraic geometry

In this chapter we will discuss the foundations of *differential algebraic geometry*, the study of schemes which are locally isomorphic to the spectrum of a ring in which we can take derivatives. Broadly speaking, if algebraic geometry is concerned with finding solutions to systems of polynomial equations, then differential algebraic geometry is concerned with finding solutions to systems of ordinary differential equations. In [13] Buium establishes differential algebraic geometry as a generalisation of ordinary algebraic geometry, in which we allow objects and morphisms defined by differential equations rather than just polynomial equations. That is, the basic object is the vanishing locus of differential equations over a field endowed with a derivative, and a morphism $X \to Y$ is one which is locally defined by differential functions. Just as in classical algebraic geometry we study k-varieties by studying their $\overline{k}$-points, within Buium's approach to differential algebraic geometry we understand a 'differential variety' by considering its points in a *differentially closed field*. That is, a field over which all ordinary differential equations in one variable have a solution.

We will take a different approach in this thesis, using *jet schemes* to integrate differential algebraic geometry as a part of the standard theory of algebraic geometry. The trade-off will be that we will often have to consider schemes which are not of finite type. Moreover, we will try to minimise our reliance on differentially-closed fields, since these objects are difficult to specify. Instead we will emphasise a 'Yoneda' style approach, considering all differential extensions of our field without specifying one in particular.

4.1 Jet schemes and arc schemes

To see how differential algebraic geometry arises in standard algebraic geometry, we will first discuss *jet schemes* and *arc schemes*, objects which generalise the tangent bundle of a scheme. Jet schemes were introduced by Nash in 1968 [55], where they were used to study singularities of complex varieties. For an up-to-date account of the use of jet schemes in the study of singularities, see [54]. Furthermore, jet and arc schemes are important in the field of birational geometry: In a 1995 lecture at Orsay, Kontsevich used arc schemes to construct the *motivic integral* and used this concept to prove that the Hodge numbers of a Calabi-Yau manifold are birationally invariant. For an account of this proof, see Section 1, §2 of [18] or Section 4 of [65]. Note that there is also a theory of arithmetic jet schemes, see [5]. In addition to the above references, we have also consulted [24], [58] and [41].

In this section we will work over the base $\operatorname{Spec}\mathbb{Q}$. If X is a $\mathbb{Q}$-scheme, then for any $r \in \mathbb{N} \cup \{\infty\}$ we define an *r-jet* in X to be a morphism of k-schemes

$$\gamma : \operatorname{Spec}\mathbb{Q}[t]/(t^{r+1}) \to X.$$

We make the convention that when $r = \infty$, the notation $\mathbb{Q}[t]/(t^{r+1})$ refers to the power series ring $\mathbb{Q}[\![t]\!]$. More generally, if A is a $\mathbb{Q}$-algebra, then we define an *A-valued r-jet in X* to be a morphism

$$\operatorname{Spec}\mathbb{Q}[t]/(t^{r+1}) \times_{\mathbb{Q}} \operatorname{Spec} A = \operatorname{Spec} A[t]/(t^{r+1}) \to X.$$

Now we observe that by the Yoneda Lemma, an A-valued r-jet may be thought of as an element of $X\left(A[t]/(t^{r+1})\right)$, which suggests that we define a $\mathbb{Q}$-functor $J^r X$ by

$$J^r X(A) = X\left(A[t]/(t^{r+1})\right),$$

for any $\mathbb{Q}$-algebra A. It is immediate from this definition that for any $r \in \mathbb{N} \cup \{\infty\}$, the canonical morphism $J^r(X \times_S Y) \to J^r(X) \times_{J^r(S)} J^r(Y)$ is an isomorphism

As a first example, we will compute $J^r(\mathbb{A}^m)$. For a $\mathbb{Q}$-algebra B we have

$$\begin{aligned}(J^r\mathbb{A}^m_{\mathbb{Q}})(B) &= \mathbb{A}^m_{\mathbb{Q}}\left(B[t]/(t^{r+1})\right) \\ &= \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}[x_1, ..., x_n], B[t]/(t^{r+1})) \\ &\overset{1}{=} \operatorname{Hom}_{\mathbb{Q}}\left(\mathbb{Q}[x_1, x_1', ..., x_1^{(r)}; ...; x_n, ..., x_n^{(r)}], B\right).\end{aligned}$$

To see the equality marked 1, consider a map of $\mathbb{Q}$-algebras $\varphi : \mathbb{Q}[x_1, ..., x_n] \to B[t]/(t^{r+1})$. Then φ is defined by where it sends the symbols x_i. We can write the image of x_i as,

$$\varphi(x_i) = x_i^{(0)} + \frac{x_i^{(1)}}{1!}t + \cdots + \frac{x_i^{(r)}}{r!}t^r.$$

for $x_i^{(l)} \in B$. We thereby obtain a map $R \coloneqq \mathbb{Q}\left[x_1^{(0)}, x_1^{(1)}, \cdots, x_1^{(r)}; \cdots; x_n^{(0)}, \cdots, x_n^{(r)}\right] \to B$ in the obvious way. Conversely, given a homomorphism $R \to \mathbb{Q}$ we can construct a homomorphism $\mathbb{Q}[x_1, ..., x_n] \to \mathbb{Q}[t]/(t^{r+1})$. Since these two constructions are inverse to one another, we conclude that $J^r\mathbb{A}^m_{\mathbb{Q}} \cong \mathbb{A}^{(r+1)m}$. This means that all the 'jet bundles' of affine space are trivial, generalising the fact that the tangent bundle of $\mathbb{A}^n$ is trivial. Now we will extend this result to show that if $X = \operatorname{Spec} R$ is an affine $\mathbb{Q}$-scheme, then $J^r X$ is again an affine scheme. We can write $R = \mathbb{Q}[x_j \mid j \in J]/I$ for some set J and an ideal I of $\mathbb{Q}[x_j \mid j \in J]$. Then if A is another $\mathbb{Q}$-algebra we have,

$$\begin{aligned} J^r(X)(A) &= X(A[t]/(t^{r+1})) = \mathrm{Hom}_{\mathbb{Q}}(R, A[t]/(t^{r+1})) \\ &= \mathrm{Hom}_{\mathbb{Q}}\left(\mathbb{Q}[x_j \mid j \in J]/I, A[t]/(t^{r+1})\right) \\ &= \{\varphi : \mathbb{Q}[x_j] \to A[t]/(t^{r+1}) \mid \varphi(f) = 0, \forall f \in I\} \end{aligned}$$

We've already shown that maps $\mathbb{Q}[x_j \mid j \in J] \to A[t]/(t^{r+1})$ are the same as maps $\mathbb{Q}[x_j^{(l)} \mid j \in J, 0 \leq l \leq r] \to A$, and the condition that $\varphi(f) = 0$ for all $f \in I$ translates to the condition that

$$0 = \varphi(f(x_1, ..., x_n) = f(\varphi(x_1), ..., \varphi(x_n)) = f(\mathbf{x}_1, ..., \mathbf{x}_n) \tag{4.1}$$

where

$$\mathbf{x}_i := x_i^{(0)} + \frac{x_i^{(1)}}{1!}t + \cdots + \frac{x_i^{(r)}}{r!}t^r. \tag{4.2}$$

Comparing coefficients of t on either side of Equation 4.1 gives r polynomial conditions on the variables $x_j^{(l)}$. If we define I_r to be the ideal generated by these polynomial equations then we have demonstrated that

$$J^r(X)(A) = \mathrm{Hom}_{\mathbb{Q}}\left(\mathbb{Q}[x_j^{(l)} \mid j \in J, 0 \leq l \leq r]/I_r, A\right). \tag{4.3}$$

Next we will describe a useful alternative description of the ideal I_r in terms of formal derivatives. Before we can give this alternative description we must introduce the notion of a *δ-ring.*

Definition 4.1. Let R be a $\mathbb{Q}$-algebra. A $\mathbb{Q}$-module homomorphism $\delta : R \to R$ is called a *derivation* if it satisfies the Leibniz law:

$$\delta(xy) = x\,\delta(y) + y\,\delta(x)$$

for all $x, y \in R$. A *δ-ring* (A, δ) is a ring along with a fixed derivation δ.

Returning to our example, we see that the map $\delta : \mathbb{Q}[x_j^{(l)}] \to \mathbb{Q}[x_j^{(l)}]$ defined by sending $x_j^{(l)}$ to $x_j^{(l+1)}$ if $l < r$ and sending $x_j^{(r)}$ to zero is a derivation on $\mathbb{Q}[x_j^{(l)}]$. View $\mathbb{Q}[x_j]$ as a sub-$\mathbb{Q}$-algebra of $\mathbb{Q}[x_j^{(l)}]$ via the map $x_i \mapsto x_i^{(0)}$. Then for a polynomial $f \in \mathbb{Q}[x_j]$ we set $f^{(j)}$ to be the jth formal derivative $\delta^j(f) \in \mathbb{Q}[x_j^{(l)}]$. Now we claim that

$$I_r = \left(f^{(j)}, j = 0, ..., r\right), \tag{4.4}$$

where I_r was defined in Equation 4.3. To prove this we begin with the following lemma.

Lemma 4.2. *Let $f, g \in \mathbb{Q}[x_j \mid j \in J]$ then we have*

$$(fg)+(fg)'t+\cdots+\frac{1}{n!}(fg)^{(n)}t^n = \left(f + ft + \cdots + \frac{1}{n!}f^{(n)}t^n\right)\left(g + gt + \cdots + \frac{1}{n!}g^{(n)}t^n\right)$$

Proof. This lemma follows from the calculus result which says that taking Taylor series expansions is multiplicative. However, here we prove this identity using a direct calculation:

$$\begin{aligned}\left(\sum_{i=0}^{n}\frac{1}{i!}f^{(i)}t^{i}\right)\left(\sum_{j=0}^{n}\frac{1}{j!}g^{(j)}t^{j}\right) &= \sum_{l=0}^{n}\left(\sum_{i+j=l}\frac{1}{i!j!}f^{(i)}g^{(j)}\right)t^{l}\\ &= \sum_{l=0}^{n}\frac{1}{l!}\left(\sum_{i=0}^{l}\binom{l}{i}f^{(i)}g^{(l-i)}\right)t^{l}\\ &= \sum_{l=0}^{n}\frac{1}{l!}(fg)^{(l)}t^{l}.\end{aligned}$$

□

By an induction argument using Lemma 4.2 we see that

$$\left(x_i + \cdots + \frac{1}{r!}x_i^{(r)}t^r\right)^m = x_i^m + (x_i^m)' + \cdots + \frac{1}{r!}(x_i^m)^{(r)}t^r. \tag{4.5}$$

If the ideal I which defines the $\mathbb{Q}$-algebra R is generated by x_i^m, then by definition the coefficients of t^i on the left of Equation 4.5 are the generators for the ideal I_r (which was defined in Equation 4.3). The coefficients of each power of t on the right of Equation 4.5 are the elements which we claimed generate I_r. That is, we have shown that if $I = (x_i^m)$, then $I_r = (x_i^m, (x_i^m)', ..., (x_i^m)^{(r)})$. To extend to arbitrary I, we note that every polynomial in I can be written as a linear combination of products of monomials of the form x_i^m and so Equation 4.5 and Lemma 4.2 give the following result.

Proposition 4.3. *Let $r \in \mathbb{N} \cup \{\infty\}$ and let $R = \mathbb{Q}[x_j \mid j \in J]/I$ be a $\mathbb{Q}$-algebra. If we define $J^r(R)$ to be the ring*

$$J^r(R) := \mathbb{Q}[x_j^{(l)} \mid j \in J, 0 \le j \le r]/(f^{(j)}, f \in I, j = 0, ..., r)$$

then we have $J^r(\operatorname{Spec} R) = \operatorname{Spec} J^r(R)$.

Remark 4.4. Note that we can express $J^r(B)$ without choosing a set of $\mathbb{Q}$-algebra generators for B by simply adjoining the formal derivatives of every element of B. We have

$$J^r(B) = B\left[\{b^{(j)} \mid b \in B, 1 \le j \le r\}\right]/I$$

where I is the ideal generated by the relations $(a+b)^{(1)} = a^{(1)} + b^{(1)}$ and $(ab)^{(1)} = a^{(0)}b^{(1)} + a^{(1)}b^{(0)}$ for all $a, b \in B$. The derivation on this ring is defined by $\delta(a^{(n)}) = a^{(n+1)}$ for $n < r$ and $\delta(a^{(r)}) = 0$.

Proposition 4.3 implies that the jet space functor $J^r(-)$ defined above restricts to a functor $\mathbf{Aff}_{\mathbb{Q}} \to \mathbf{Aff}_{\mathbb{Q}}$.

Proposition 4.5 (Proposition 2.2 of [4])**.** *Let $f : X \to Y$ be a formally étale morphism* *of $\mathbb{Q}$-functors. Then for any $m \in \mathbb{N}$ the square*

$$\begin{tikzcd} J^m(X) \arrow[r, "J^m(f)"] \arrow[d, "p_1"] & J^m(Y) \arrow[d, "p_2"] \\ X \arrow[r, "f"] & Y. \end{tikzcd} \tag{4.6}$$

is cartesian (in the category of $\mathbb{Q}$-functors). That is, the natural morphism $J^m(X) \to J^m(Y) \times_Y X$ is an isomorphism for any $m \geq 0$.

Proof. By the universal property of the fibre product we have a morphism $g : J^m(X) \to J^m(Y) \times_Y X$. We need to prove that for any k-algebra A, the induced map

$$g_A : X\left(A[t]/(t^{m+1})\right) \to Y\left(A[t]/(t^{m+1})\right) \times_{Y(A)} X(A) \tag{4.7}$$

is a bijection. Let $\theta : \operatorname{Spec} A \to \operatorname{Spec} A[t]/(t^{m+1})$ be the morphism induced by truncation, then g_A sends a point $x : \operatorname{Spec} A[t]/(t^{m+1}) \to X$ to the maps $f \circ x : \operatorname{Spec} A[t]/(t^{m+1}) \to Y$ and $x \circ \theta : \operatorname{Spec} A \to X$. Conversely, suppose we are given $(y, y') \in Y\left(A[t]/(t^{m+1})\right) \times_{Y(A)} X(A)$ making the following diagram commute:

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "y'"] \arrow[d, "\theta"] & X \arrow[d, "f"] \\ \operatorname{Spec} A[t]/(t^{m+1}) \arrow[r, "y"] \arrow[ur, dashed] & Y \end{tikzcd} \tag{4.8}$$

Because $f : X \to Y$ is formally étale and the ideal (t) of $A[t]/(t^{m+1})$ is nilpotent, given any square of the form of Diagram 4.8 there exists a unique morphism $x : \operatorname{Spec} A[t]/(t^{m+1}) \to X$ agreeing with the other maps in the diagram. This association defines a two-sided inverse to g_A. $\square$

Remark 4.6. Note that the argument of Proposition 4.5 does not apply to the $\mathbb{Q}$-functor $A \mapsto X(A[\![t]\!])$ because the ideal (t) of $A[\![t]\!]$ is not nilpotent.

Corollary 4.7. *Let $f : X \to Y$ be a morphism of $\mathbb{Q}$-functors which is formally étale (resp. formally smooth, formally unramified). Then $J^r(f) : J^r(X) \to J^r(Y)$ is also formally étale (resp. formally smooth, formally unramified) for any $r \in \mathbb{N}$.*

Proof. If $f : X \to Y$ is formally étale, then it is immediate from Proposition 4.5 that $J^r(f)$ is formally étale, since formally étale morphisms are stable under base change. Let I be a nilpotent ideal of a $\mathbb{Q}$-algebra A. To prove that $J^r(f)$ is formally unramified (resp. formally smooth) we need to prove that the associated morphism

$$J^r(X)(A) \to J^r(X)(A/I) \times_{J^r(Y)(A/I)} J^r(Y)(A)$$

is injective (resp. surjective). Expanding the definition of the functor J^r we obtain

$$X(A[t]/(t^{r+1})) \to X(A/I[t]/(t^{r+1})) \times_{Y(A/I[t]/(t^{r+1}))} Y(A[t]/(t^{r+1}))$$

which is injective if f is formally unramified and surjective if f is formally smooth, because the ideal I of $A/I[t](t^{r+1})$ is nilpotent. $\square$

The étale invariance of the jet functor, along with the explicit description of the closed subschemes $J^r(V(I))$ given by Proposition 4.3 will be our most important tools for studying jet schemes. For instance, we can now prove easily that if X is a scheme, then $J^r X$ is also a scheme.

Proposition 4.8 (Proposition 3.2.1.3 of [18])**.** *If X is a scheme then $J^r X$ is a scheme for any $r \in \mathbb{N}$.*

Proof. First we will prove that $J^r X$ has an affine open cover. Let $\{U_i \to X\}$ be an affine open cover of X. By the second axiom of a Grothendieck topology we have that $\{U_i \times_X J^r(X) \to J^r(X)\}$ is an open covering of $J^r(X)$. But, by Proposition 4.5 we have that $U_i \times_X J^r(X) = J^r(U_i)$, which is affine, so we have constructed an open affine covering of $J^r(X)$.

Next, we need to prove that $J^r(X)$ is a local functor. That is, we need to prove that if R is a ring and $(f_1, ..., f_n) = R$ then the sequence

$$J^r(X)(R) \to \prod_i J^r(X)(R_{f_i}) \rightrightarrows \prod_{i,j} J^r(X)(R_{f_i f_j}) \tag{4.9}$$

is an equaliser diagram. But

$$J^r(X)(R_{f_i}) = X(R_{f_i}[t]/(t^{r+1})) = X\left((R[t]/(t^{r+1}))_{f_i}\right).$$

Since $(f_1, ..., f_n)$ also generate the unit ideal in $R[t]/(t^{r+1})$, the exactness of Sequence 4.9 follows from the fact that X is a local functor. □

Now that we have proved that $J^r(X)$ is a scheme, the following universal property is immediate from the definition of $J^r(X)$ and the Yoneda lemma.

Proposition 4.9 (Universal property of $J^r(-)$)**.** *Let X be a $\mathbb{Q}$-scheme, then J^r : $\mathbf{Sch}_{\mathbb{Q}} \to \mathbf{Sch}_{\mathbb{Q}}$ is right adjoint to the functor $Z \mapsto Z \times_{\mathbb{Q}} \operatorname{Spec} \mathbb{Q}[t]/(t^{r+1})$. That is, we have an isomorphism*

$$\operatorname{Mor}\left(Z \times_{\mathbb{Q}} \operatorname{Spec} \mathbb{Q}[t]/(t^{r+1}), X\right) = \operatorname{Mor}(Z, J^r X),$$

natural in Z and X.

Remark 4.10. Taking the rth jet scheme of a scheme is an instance of a more general phenomenon known as *Weil restriction.* If we have a morphism $h : S' \to S$ and an S'-scheme X', then we can define a functor $R_{S'/S} : (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Set}$ by

$$T \mapsto \operatorname{Mor}_{S'}(T \times_S S', X')$$

If this functor is represented by a scheme, then we call that scheme the *Weil restriction of X' along h.* Many of the results of this section are consequences of more general results proved about the Weil restriction in Section 7.6 of [8], where the concept is discussed in greater depth.

For $m \geq n$ and any $\mathbb{Q}$-algebra R, reduction modulo t^{n+1} induces a truncation map $R[t]/(t^{m+1}) \to R[t]/(t^{n+1})$. We thereby obtain a map of schemes $\pi_n^m : J^m(X) \to J^n(X)$.

Lemma 4.11 (Corollary 3.2.2.4 of [18])**.** *For any $m \geq n$, the morphism $\pi_n^m : J^m(X) \to J^n(X)$ is affine.*

Proof. Choose an affine open cover $\{U_i\}$ for X. Then we obtain an affine open cover $\{J^n(U_i)\}$ for $J^n(X)$ and so it will suffice to prove that each pullback $J^n(U_i) \times_{J^n(X)} J^m(X)$ is affine. We have

$$J^n(U_i) \times_{J^n(X)} J^m(X) = (U_i \times_X J^n(X)) \times_{J^n(X)} J^m(X) = U_i \times_X J^m(X) = J^m(U_i)$$

where $J^m(U_i)$ is affine by Proposition 4.3. $\square$

Therefore, we have an inverse system,

$$\cdots \to J^{n+1}X \to J^n X \to \cdots \to J^2 X \to J^1 X \to X$$

where each morphism is affine by Lemma 4.11. Therefore, [63, Tag 01YX] allows us to take the limit of this diagram in the category of $\mathbb{Q}$-schemes to obtain the *arc scheme* of X.

Remark 4.12. Recall that we defined a functor $J^\infty(-)$ by $J^\infty(X)(A) = X(A[\![t]\!])$. Since we have truncation homomorphisms $R[\![t]\!] \to R[t]/(t^{m+1})$ for any $\mathbb{Q}$-algebra R, we obtain morphisms $\pi^m : J^\infty(X) \to J^m(X)$ (in the category of $\mathbb{Q}$-functors) which are compatible with the morphisms π_n^m defined above. From the universal property of the limit we thereby obtain a canonical morphism $\phi : J^\infty(X) \to \varprojlim_r J^r(X)$ of $\mathbb{Q}$-functors. In fact this morphism is an isomorphism, as follows from a result of Bhatt (Theorem 1.1 of [2]), so we will use the notation $J^\infty(X)$ to refer to the arc scheme of X. Bhatt's theorem is technical and makes use of techniques in derived algebraic geometry which we will not discuss in this thesis. However, in many of the cases which arise in practice we do not require the full strength of Bhatt's theorem to prove that ϕ is an isomorphism. Indeed, Proposition 4.3 shows that ϕ is an isomorphism when X is affine and the discussion of Section 3.3.8 of Chapter 3 of [18] demonstrates that ϕ is an isomorphism when X is a projective scheme. In these cases we will use the description of $J^\infty(X)$ in terms of power series rings. Therefore, our argument in Chapters 5 and 6 does not rely on Bhatt's theorem. Moreover, the description of $J^\infty(X)$ as the limit $\varprojlim_r J^r(X)$ will usually be more useful to us than the description of $J^\infty(X)$ as a functor.

Remark 4.13. By Part (2) of [63, Tag 01YX] we have that if U is an open in X then $U \times_X J^\infty(X) = \varprojlim_r (U \times_X J^r(X))$. However, we also know that $U \times_X J^r(X) = J^r(U)$ for all $r \geq 1$ by Proposition 4.5 and so we have

$$U \times_X J^\infty(X) = \varprojlim_r (U \times_X J^r(X)) = \varprojlim_r J^r(U) = J^\infty(U).$$

In particular, this implies that if $\{U_i\}$ is an affine open cover of X, then $\{J^\infty(U_i)\}$ is an affine open cover for $J^\infty(X)$.

Proposition 4.14 collates a number of properties of morphisms which are preserved by the jet space functor. Many of these results are discussed in Proposition 3.2.1.3 of [18].

Proposition 4.14. *The following classes of morphisms are preserved by $J^m(-)$ for any $m \in \mathbb{N} \cup \{\infty\}$:*

(a) quasi-compact morphisms;

(b) quasi-separated morphisms;

(c) affine morphisms;

(d) closed immersions;

(e) separated morphisms.

The following classes of morphisms are preserved by $J^m(-)$ for any $m \in \mathbb{N}$:

(i) morphisms locally of finite presentation, locally of finite type, of finite presentation or of finite type;

(ii) étale, smooth and unramified morphisms.

Proof. Let $\{U_i \to Y\}$ be an affine open cover of Y. Then $\{J^m(U_i) \to J^m(Y)\}$ is an affine open cover of $J^m(Y)$. All of the properties in the statement of the proposition are Zariski local on the base, so it will suffice to prove that they hold for each of the morphisms $\{J^m(U_i \times_Y X) \to J^m(U_i)\}$ obtained by base change. Thus, we've reduced to the case where Y is affine.

(a): We need to show that if $f : X \to \operatorname{Spec} A$ is quasi-compact, then $J^m(X) \to J^m(\operatorname{Spec} A) = \operatorname{Spec} J^m(A)$ is quasi-compact. Since $\operatorname{Spec} J^m(A)$ is affine it will suffice to prove that $J^m(X)$ is covered by finitely many open affines. But since $f : X \to \operatorname{Spec} A$ is quasi-compact we know that X is covered by finitely many open affines $\{U_i \to X\}$ and so $J^m(X)$ has a finite affine open cover $\{J^m(U_i)\}$. (In the case where m is finite this follows by Proposition 4.5 and when $m = \infty$ this follows from Remark 4.13.)

By the same argument as in the previous paragraph, to prove part (b) it will suffice to prove that if X is quasi-separated then so is $J^m(X)$. Choose an affine open cover $\{U_i \to X\}$ such that each $U_i \times_X U_j$ is covered by finitely many affine opens. Then $\{J^m(U_i) \to J^m(X)\}$ is an open affine cover of $J^m(X)$ and each $J^m(U_i) \times_{J^m(X)} J^m(U_j) = J^m(U_i \times_X U_j)$ is covered by $J^m(-)$ applied to the finite affine open cover of $U_i \times_X U_j$.

(c): If $X \to Y = \operatorname{Spec} A$ is an affine morphism, then $X = \operatorname{Spec} B$ and so the morphism $J^m(\operatorname{Spec} B) \to J^m(\operatorname{Spec} A)$ is affine as a morphism of affine schemes. In particular, if $X = \operatorname{Spec} B/I$ is a closed subscheme of $Y = \operatorname{Spec} B$, then we've shown that $J^m(X) = \operatorname{Spec} J^m(B)/I_m$ which is a closed subscheme of $J^m(Y) = \operatorname{Spec} J^m(B)$. This proves part (d).

(e): We need to prove that the morphism $J^m(X) \to J^m(X) \times_{J^m(Y)} J^m(X) = J^m(X \times_Y X)$ is a closed immersion. But this is $J^m(X \to X \times_Y X)$ and $X \to X \times_Y X$ is a closed immersion by assumption, so the result follows immediately from part (d).

(i): In Proposition 4.3 we proved that if B is a finitely-generated (resp. finitely-presented) A-algebra, then $J^m(B)$ is a finitely-generated (resp. finitely-presented) $J^m(A)$-algebra. Assume that $X \to \operatorname{Spec} A$ is locally of finite type, so that X has an open affine cover $\{U_i = \operatorname{Spec} B_i\}$ where each B_i is a finitely-generated A-algebra. Then $J^m(X)$ is covered by $\{J^m(U_i) = \operatorname{Spec} J^m(B_i)\}$ where $J^m(B_i)$ is a finitely-generated $J^m(A)$-algebra. If we had chosen the B_i to be finitely-presented then the $J^m(\operatorname{Spec} B_i)$ would have been finitely-presented. Thus, we have established the first two properties of (i). The latter two properties then follow from (a) and (b).

Part (ii) now follows from part (i) and Corollary 4.7. □

Remark 4.15. Note that finiteness is not preserved by taking $J^m(-)$. Consider the finite $\mathbb{Q}$-algebra $A = \mathbb{Q}[x]/(x^2)$. Then $J^1(A) = \mathbb{Q}[x, x']/(x^2, xx')$, which is not a finite $\mathbb{Q}$-vector space because we have the linearly independent elements $\{x', (x')^2, (x'')^3, \cdots\}$.

Proposition 4.16 (Theorem 1.10 of [58]). *Let k be a field and let X be a smooth k-scheme. Then there exists an integer $n \geq 0$ such that for any $m, e \geq 0$ the morphism $J^{m+e}(X) \to J^e(X)$ is locally an $\mathbb{A}^{mn}$-bundle. That is, there exists an open cover $\{U_i\}$ of X such that $J^{m+e}(X) \times_{J^e(X)} J^e(U_i) \cong J^e(U_i) \times_{\operatorname{Spec} k} \mathbb{A}_k^{mn}$ for all i.*

Proof. First of all assume that $e = 0$. Here we will use the fact that smooth morphisms are 'locally étale'. That is, by [63, Tag 039P] we can choose an open cover $\{U_i\}$ of X along with étale morphisms $\pi_i : U_i \to \mathbb{A}_k^n$, for some n. Then, by the étale invariance of the jet functor we have

$$J^m(X) \times_X U_i = J^m(U_i) = U_i \times_{\mathbb{A}_k^n} J^m(\mathbb{A}_k^n)$$

By the example at the start of this section we know that $J^m(\mathbb{A}_k^n) \cong \mathbb{A}_k^{n(m+1)}$ and so

$$U_i \times_{\mathbb{A}_k^n} J^m(\mathbb{A}_k^n) \cong U_i \times_{\mathbb{A}_k^n} \mathbb{A}_k^{(m+1)n} \cong U_i \times_k \mathbb{A}_k^{mn}.$$

For arbitrary e we consider the composition $J^{m+e}(X) \to J^e(X) \to X$. The map $J^e(X)$ and $J^{m+e}(X) \to X$ are both locally trivial, so the map $J^{m+e}(X) \to J^e(X)$ must also be locally trivial. □

4.2 Differential algebra

In Section 4.1 we saw how δ-rings* emerge when we take jet spaces of affine schemes. Informally, this is because the functions on $J^r(X)$ are the functions on X along with all their order r derivatives. Before we can understand differential structures on schemes more generally, we will need to develop the theory of *differential algebra*, by which we mean the study of commutative δ-rings. A comprehensive reference for the study of differential algebra is Kolchin's book [38], although many of the results we need are also discussed in Chapter 2 of [13].

Recall that we defined a δ-ring to be a ring R along with a $\mathbb{Q}$-linear derivation δ. A δ-ring which is also a field will be called a *δ-field.* By the assumption that all δ-rings are $\mathbb{Q}$-algebras, we know that all δ-fields are of characteristic zero. The *ring of constants* of a δ-ring R is the subring $R^\delta = \{x \in R \mid \delta(x) = 0\}$. Note that if R is a field, then R^δ is too, because if $x \in R^\delta$ and $xy = 1$, then

$$0 = \delta(1) = \delta(xy) = x\delta(y) + y\delta(x) = x\delta(y)$$

and hence the inverse y of x also belongs to R^δ.

A *δ-homomorphism* between δ-rings (R, δ_R) and (S, δ_S) is a homomorphism $\varphi : R \to S$ such that $\varphi \circ \delta_R = \delta_S \circ \varphi$. If we have such a map then we call S a *δ-R-algebra.* We write $\mathbf{DAlg}_R$ for the category of δ-R-algebras. Note that the kernel of a δ-homomorphism is an ideal I such that if $x \in I$ then $\delta(x) \in I$. This observation leads us to define a *δ-ideal* to be an ideal which is closed under taking derivations. A δ-ideal is called *radical* if $\sqrt{I} = I$. If T is any subset of R then we will write $\{T\}$ for the δ-ideal generated by T.

If we have two δ-R-algebras S_1 and S_2, then we may give $S_1 \otimes_R S_2$ the derivation

$$\delta(x \otimes y) = \delta x \otimes y + x \otimes \delta y.$$

A simple calculation shows that this is the unique δ-R-algebra structure on $S_1 \otimes_R S_2$ making both the homomorphism $S_i \to S_1 \otimes_R S_2$ homomorphisms of δ-R-algebras. Similarly, if $S \subset R$ is multiplicative, then $S^{-1}R$ has a derivation

$$\delta(x/s) = (s\delta x - x\delta s)/s^2.$$

Again this is the unique δ-R-algebra structure on $S^{-1}R$ making the natural map $R \to S^{-1}R$ a δ-homomorphism. The following Proposition is a consequence of the discussion of Section I.2 of [38]. See also Section 2.1.3 of [13].

Proposition 4.17. *Let I be a δ-ideal of a δ-ring R. Then $\sqrt{I}$ is a δ-ideal.*

Proof. If we have a δ-ring homomorphism $\varphi : R \to S$ and I is a δ-ideal of S, then $\varphi^{-1}(I)$ is a δ-ideal of R and $\varphi^{-1}(\sqrt{I}) = \sqrt{\varphi^{-1}I}$. Note that both of these facts are immediate from the definitions. Now consider the quotient map $\pi : R \to R/I$. We have, $\pi^{-1}(\sqrt{(0)}) = \sqrt{\pi^{-1}(0)} = \sqrt{I}$ so it will suffice to prove that the radical of the zero ideal

*We defined these objects in Definition 4.1

is always a δ-ideal. To prove this we first of all claim that for $m, n \geq 1$, if $x^m(\delta x)^n = 0$, then $x^{m-1}(\delta x)^{n+2} = 0$. To see this we take the derivative of $x^m(\delta x)^n$ which gives

$$\begin{aligned} 0 &= \delta\left(x^m(\delta x)^n\right) \\ &= \delta(x^m)(\delta x)^{n+1} + x^m \delta\left(\delta(x)^n\right) \\ &= m x^{m-1}(\delta x)^{n+1} + x^m n \delta(x)^{n-1}\delta^2(x). \end{aligned}$$

Multiplying both sides by δx yields

$$0 = m x^{m-1}(\delta x)^{n+2} + n x^m \delta(x)^n \delta^2(x) = m x^{m-1}(\delta x)^{n+2}$$

because $x^m\delta(x)^n = 0$ by assumption. This establishes the claim. Now, suppose we have $x \in \sqrt{(0)}$, then we can choose m such that $x^m = 0$. This means that

$$0 = \delta(0) = \delta(x^m) = x\delta(x^{m-1}) + x^{m-1}\delta x = \cdots = m x^{m-1}\delta(x).$$

That is, $x^{m-1}\delta(x) = 0$. Applying the claim repeatedly shows that

$$0 = x^{m-1}\delta(x) = x^{m-2}(\delta(x))^3 = \cdots = (\delta(x))^{2m-1}.$$

Therefore, $\delta(x) \in \sqrt{(0)}$ and so $\sqrt{(0)}$ is a δ-ideal. $\square$

If R is a δ-ring then we will write $R\{x_1, \cdots, x_n\} = R[x_j^{(i)} \mid i = 1, ..., n, j \geq 0]$ for the ring obtained by adjoining formal derivatives of each x_i of all orders to the polynomial ring $R[x_1, ..., x_n]$. The derivation on R extends to $R\{x_1, ..., x_n\}$ by setting $\delta(x_i^{(j)}) = x_i^{(j+1)}$. The elements of this ring are called *δ-polynomials*, and the *order* of a δ-polynomial f is the highest order derivative which appears in f. For instance, $f(y) = y^2 - 2y''y + y^{(3)}x'$ is a δ-polynomial of order 3.

Example 4.18. Let A be a $\mathbb{Q}$-algebra, then we can write $A = \mathbb{Q}[x_j \mid j \in J]/I$. We proved in Proposition 4.3 that $J^\infty(\operatorname{Spec} A) = \operatorname{Spec} J^\infty(A)$ where

$$J^\infty(A) = \mathbb{Q}[x_i^{(j)}, i = 1, ..., n, j \geq 0]/I_\infty = \mathbb{Q}\{x_1, ..., x_n\}/I_\infty.$$

The ring $\mathbb{Q}\{x_1, ..., x_n\}$ is a δ-ring with derivation defined by $\delta(x_i^{(j)}) = x_i^{(j+1)}$ and I_∞ is the δ-ideal generated by I. We observe that to define a δ-homomorphism out of $J^\infty(A)$ we only need to specify the image of the generators $x_i^{(0)}$, and so giving a δ-homomorphism out of $J^\infty(A)$ is equivalent to specifying a homomorphism out of A. Therefore, the functor $J^\infty(-) : \mathbf{Alg}_{\mathbb{Q}} \to \mathbf{DAlg}_{\mathbb{Q}}$ is left adjoint to the forgetful functor $\mathbf{DAlg}_{\mathbb{Q}} \to \mathbf{Alg}_{\mathbb{Q}}$.

Just as in algebra we adjoin solutions to polynomial equations to rings, differential algebra gives us a framework in which we can formally adjoin solutions to differential equations to our ring. Consider the ring $\mathbb{C}(t)$ along with the derivation $\delta = \frac{d}{dt}$. There is no rational function in $\mathbb{C}(t)$ satisfying the equation $y' = y$. To adjoin a solution to this equation, we quotient the ring $\mathbb{C}(t)\{y\}$ by the δ-ideal $\{y' - y\}$. We can express this ring more succinctly as $\mathbb{C}(t)[y]$, where the derivation on $\mathbb{C}(t)$ extends to $\mathbb{C}(t)[y]$ by setting $\delta(y) = y$. Note that y satisfies the same differential relations as the function e^t and so we write this ring as $\mathbb{C}(t)[e^t]$. At times it will be convenient to assume that our δ-field already contains solutions to all ordinary differential equations, so we make the following definition:

Definition 4.19. A δ-field $\mathcal{F}$ is *δ-closed* if for any δ-polynomials $f(y), g(y) \in \mathcal{F}\{y\}$ with $g(y) \neq 0$ and $\operatorname{ord} g < \operatorname{ord} f$, there exists $\alpha \in F$ such that $f(\alpha) = 0$ but $g(\alpha) \neq 0$. We will always use the notation $\mathcal{F}$ to refer to a δ-closed field.

Remark 4.20. If $\mathcal{F}$ is a δ-closed field, then the *Kolchin topology* on $\mathcal{F}^n$ has closed sets defined by the vanishing loci of δ-polynomials in $\mathcal{F}\{y_1, ..., y_n\}$. One approach to differential algebraic geometry (which we might call the 'classical' approach) takes closed subsets of $\mathcal{F}^n$ in the Kolchin topology as the basic geometric object. However, we will not take this point of view, for more information see Chapter IV of [38].

Note that none of the fields we have discussed so far are δ-closed, and in fact we will never give a concrete example of a δ-closed field. Because differentially closed fields do not arise naturally, one of our goals in this thesis has been to minimise the use of δ-closed fields within the proof of the geometric Mordell conjecture. However, at times it will be useful to invoke the 'Differential Nullstellensatz', Theorem 4.21, and so we have not been able to eliminate the use of δ-closed fields entirely from our argument. For a proof of the Differential Nullstellensatz, see Theorem 2.5.1 of [13] or Theorem 2.2 of [45].

Theorem 4.21 (The Differential Nullstellensatz, Theorem 2.5.1 of [13])**.** *A δ-field $\mathcal{F}$ is differentially closed if and only if for any $n \geq 1$ and any prime δ-ideal $\mathfrak{p} \subset \mathcal{F}\{y_1, ..., y_n\}$, the ideal $\mathfrak{p}$ has a zero in $\mathcal{F}^n$.*

The following result may be established using the differential Nullstellensatz, see Theorem 5.2 of Chapter 2 of [13].

Corollary 4.22 (Theorem 2.5.2 of [13], Corollary 3.3 of [45])**.** *Let F be a δ-field. Then there exists a δ-field extension $\mathcal{F}/F$, where $\mathcal{F}$ is δ-closed.*

4.2.1 The theory of D-modules

Let R be a δ-ring with derivation δ. The ring $D_R = R[\delta]$ of *linear differential operators on R* is the non-commutative ring whose elements are polynomials in the symbol δ and with multiplication defined by

$$\lambda \cdot \delta^i = \lambda\delta^i \quad \text{and} \quad \delta^i \cdot \lambda = \sum_{j=0}^{i} \binom{i}{j} (\delta^j \lambda)\delta^{i-j}, \quad \forall \lambda \in F,\ i \geq 0.$$

If the δ-ring in question is clear from the context then we will sometimes write D instead of D_R. By a *D_R-module* we will always mean a left D_R-module. Note that R is a

D_R-module with scalar multiplication defined by $\delta^i \cdot \lambda = \delta^i(\lambda)$. If M is a R^δ-module, then the R-module $R \otimes_{R^\delta} M$ is a D_R-module where δ acts by

$$\delta \cdot (r \otimes v) = \delta(r) \otimes v.$$

We say that a D_R-module is *split* if it is isomorphic to a D_R-module of this form. Note that if a free R-module M has an R-basis $\{e_i\}$ such that $\delta \cdot e_i = 0$ for all i, then M is split. The following result is discussed in Section 2.6.1 of [13].

Lemma 4.23. *Let F be a δ-field, then every left ideal in D_F is principal.*

Proof. We can perform polynomial division in D_F, taking care to always multiply on the left, so the usual argument that a polynomial ring $F[x]$ is a principal ideal domain still applies in D_F. □

Proposition 4.24 (Proposition 2.6.3 of [13])**.** *Let F be a δ-field with field of constants C and let M be a split D_F-module. Then any D_F-submodule N of M is also split.*

Proof. Since M is a split D-module we know that M can be written as $F \otimes_C V$ for some C-vector space V. If $\{e_i\}$ is a basis for V, then the basis $\{x_i \coloneqq 1 \otimes e_i\}$ for $F \otimes_C V$ satisfies

$$\delta \cdot x_i = \delta \cdot (1 \otimes e_i) = \delta(1) \otimes e_i = 0. \tag{4.10}$$

Now let $N_0 = \{x \in V \mid 1 \otimes x \in N\}$. Choose a basis $\{y_i\}$ for N_0, then we can write y_i as $\sum_j a_{ij} e_j$. We observe that $\delta(1 \otimes y_i) = 0$ and so the vectors $\{1 \otimes y_i = \sum_j a_{ij} x_j\}$ generate a split D-submodule N' of M. Thus, it will suffice to show that $N = N'$.

Evidently $N' \subset N$, so assume that we can find some $x \in N \setminus N'$. We can write x in terms of our basis for N as $x = \sum_{i=1}^n b_i x_i$. We assume that n is minimal among elements of $N \setminus N'$. Because $x \neq 0$ some coefficient b_i is non-zero and so we can rescale this coefficient to be equal to 1. By reordering we can write $x = x_1 + b_2 x_2 + \cdots + b_n x_n$. We know that N is a D-submodule of M, so $\delta x \in N$. We have

$$\begin{aligned}
\delta x &= \delta\left(x_1 + b_2 x_2 + \cdots + b_n x_n\right) \\
&= (\delta \cdot x_1 + b_2(\delta \cdot x_2) + \cdots + b_n(\delta \cdot x_n)) + (\delta(b_2) x_2 + \cdots + \delta(b_n) x_n) \\
&= \delta(b_2) x_2 + \cdots + \delta(b_n) x_n,
\end{aligned}$$

where we have applied Equation 4.10. That is, δx is an element of N which can be written using fewer than n basis vectors, so we conclude that $\delta x \in N'$. Furthermore, we know that $\delta(b_i) \neq 0$ for some i, because otherwise all the b_i would belong to C which could mean that x comes from an element of V and would therefore belong to N. By reordering if necessary we can assume that $\delta b_2 \neq 0$. Therefore,

$$x - \frac{b_2}{\delta b_2} \delta x = x_1 + b_2 x_2 \cdots + b_n x_n - \left(b_2 x_2 + \cdots + \frac{b_2 \delta b_n}{\delta b_2} x_n \right)$$

belongs to N because it belongs to N' and can be written as a sum of strictly fewer than n basis vectors (the x_2 term vanishes). This is a contradiction. □

Proposition 4.25 (Proposition 2.6.2 of [13]). *Let F be a δ-field and let V be a D_F-module which is finite dimensional over F. Then there exists a δ-extension K/F such that $V \otimes_F K$ is split.*

Proof. To prove this result we may assume that $F = \mathcal{F}$ is a δ-closed field. Choose an $\mathcal{F}$-basis $\{x_1, ..., x_n\}$ for V. We define a matrix $A = (a_{ij}) \in M_n(\mathcal{F})$ by $\delta(x_i) = \sum_{j=1}^n a_{ij}x_j$. We need to prove that there is an invertible matrix $B = (b_{ij}) \in \mathrm{GL}_n(\mathcal{F})$ such that

$$0 = \sum_j \delta(b_{ij}x_j) = \sum_j \delta(b_{ij})x_j + \sum_j b_{ij}\delta(x_j)$$

for all i. That is, we want to find a matrix (b_{ij}) satisfying the equation

$$\begin{bmatrix} \delta b_{11} & & \delta b_{1n} \\ & \ddots & \\ \delta b_{n1} & & \delta b_{nn} \end{bmatrix} = - \begin{bmatrix} b_{11} & & b_{1n} \\ & \ddots & \\ b_{n1} & & b_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & & a_{1n} \\ & \ddots & \\ a_{n1} & & a_{nn} \end{bmatrix}. \tag{4.11}$$

To find such a matrix we consider the ring $R = F[t_{ij} \mid 1 \leq i, j \leq n]_d$ where $d = \det(t_{ij})$ along with the derivation defined by

$$\delta(t_{ij}) = -\sum_k a_{kj}t_{ik}.$$

We have a natural (surjective) homomorphism $\phi : \mathcal{F}\{y_{ij}, z\} \to \mathcal{F}[t_{ij}]_d$ defined by $y_{ij} \mapsto t_{ij}$ and $z \mapsto 1/d$ and by construction a solution to Equation 4.11 is precisely a zero of the kernel of ϕ. Since $\mathcal{F}[t_{ij}]_d$ is an integral domain we know that $\ker \phi$ is a prime ideal and so such a solution is guaranteed by the Differential Nullstellensatz 4.21. □

Proposition 4.26 (Proposition 2.6.4 of [13]). *Let F be a δ-field. For any finitely-generated D_F-module M we may choose a δ-extension K of F such that there is an exact sequence*

$$0 \to D_K^n \to K \otimes_F M \to K^m \to 0.$$

Proof. Let $\{x_1, ..., x_N\}$ be a set of D_F-module generators for M. Then we can choose a maximal D_F-linearly independent subset $\{x_1, ..., x_n\}$ of our generating set. The elements $\{x_1, ..., x_n\}$ generate a copy of D_F^n and we claim that the quotient M/D_F^n is finite dimensional over F. To see that this is true note that M/D_F^n is generated as an F-vector space by all the derivatives of the elements of the set $\{x_{n+1}, ..., x_N\}$. But, by the linear dependence of the set $\{x_1, ..., x_n, x_j\}$ for each $n+1 \leq j \leq N$, we can choose large enough l such that $\delta^l(x_j) = 0$ in M/D_F^n and so our generating set for M/D_F^n is finite. Therefore, by Proposition 4.25 we can choose a δ-field extension K/F so that $(M/D_F^n) \otimes_F K \cong K^m$ for some m, giving us the required exact sequence. □

Corollary 4.27 (Corollary 2.6.5 of [13]). *Let F be a δ-field and let M be a finitely generated D_F-module. If $\mathrm{Hom}_{D_K}(M \otimes_F K, K) = 0$ for all δ-field extensions K/F then*

$M = 0$. Moreover, if $F = \mathcal{F}$ is δ-closed and M is a finitely-generated $D_{\mathcal{F}}$-module, then $\mathrm{Hom}_{D_{\mathcal{F}}}(M, \mathcal{F}) = 0$ *if and only if* $M = 0$.

Proof. Let M be a non-zero finitely-generated D_F-module. By the preceding discussion, we may choose a δ-field extension K/F so that we have an exact sequence of D_K-modules

$$0 \to D_K^n \to M \otimes_F K \to K^m \to 0.$$

Since $M \otimes_F K$ is non-zero we know that either m or n is greater than zero in the previous exact sequence. But both D_K^n and K^m admit non-zero D_K-module homomorphisms to K when m or n are greater than zero, so $\mathrm{Hom}_{D_K}(M \otimes_F K, K)$ cannot be zero. $\square$

4.3 D-Schemes

In Chapter 2 we defined schemes as geometric objects which are modelled on the spectra of rings. In this section we will instead consider objects known as 'D-schemes' which locally look like δ-rings and investigate the geometry of these objects. The approach we take to D-schemes in the section is slightly different to the approach taken by Buium in Chapter 3 of [13]. Indeed the philosophy behind our construction of the category of D-schemes is more closely related to the approach taken in [7], where arithmetic (rather than geometric) δ-structures are defined.

Firstly, we want the category of affine D-schemes to be dual to the category of δ-rings, so that an affine D-scheme is simply the spectrum of a δ-ring. The morphisms in the category of affine D-schemes should be the same as δ-homomorphisms. Since a derivation $\delta : A \to A$ is not a morphism in the category of rings, it is not immediately clear which morphisms between affine D-schemes are D-scheme homomorphisms. In order to find an interpretation of δ-homomorphisms in terms of morphisms in the category of affine schemes, we will first need to understand the concept of a *monad*. We will follow [42] in our treatment of monads and comonads.

Definition 4.28. Let $\mathscr{C}$ be a category. A *monad* (T, η, μ) on $\mathscr{C}$ consists of an endofunctor $T : \mathscr{C} \to \mathscr{C}$ along with natural transformations $\eta : \mathrm{id}_X \to T$ (the *unit*) and $\mu : T^2 \to T$ (the *multiplication*) which make the following two diagrams commute.

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[d, "T\eta"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"] & T \end{tikzcd} \tag{4.12}$$

Here $T\mu$ is the natural transformation defined by $(T\mu)_X = T(\mu_X)$ and μT is defined by $(\mu T)_X = \mu_{T(X)}$. A *comonad* is defined by dualising this definition.

Example 4.29. Here we will show that the functor $J : \mathbf{Alg}_{\mathbb{Q}} \to \mathbf{Alg}_{\mathbb{Q}}$ defined in Proposition 4.3 gives rise to a monad in the category of $\mathbb{Q}$-algebras. The unit η of the monad is defined by setting $\eta_A : A \to J(A)$ to be the evident inclusion $a \mapsto a^{(0)}$. The naturality in A is clear here. To define the multiplication μ we observe that for any $\mathbb{Q}$-algebra A we have a homomorphism $J(J(A)) \to J(A)$ defined by $(a^{(j)})^{(i)} \mapsto a^{(i+j)}$. If we have a homomorphism $\varphi : A \to B$ then $J(\varphi) : J(A) \to J(B)$ is defined by $J(\varphi)(a^{(i)}) = \varphi(a)^{(i)}$, from which the naturality of μ in A is apparent.

To see that the first diagram of Definition 4.28 commutes, let A be a $\mathbb{Q}$-algebra. Then the composition $\mu_A \circ J(\mu_A)$ is defined by

$$((a^{(l)})^{(j)})^{(i)} \mapsto (a^{(j+l)})^{(i)} \mapsto a^{(i+j+l)}$$

whereas the composition $\mu_A \circ \mu_{J(A)}$ is defined by

$$((a^{(l)})^{(j)})^{(i)} \mapsto (a^{(l)})^{(i+j)} \mapsto a^{(i+j+l)}$$

and so the diagram commutes. To see that the second diagram commutes, note that $(\eta J)_A$ sends $a^{(i)} \mapsto (a^{(i)})^{(0)}$ whereas $(J\eta)_A$ sends $a^{(i)} \mapsto (a^{(0)})^{(i)}$. These two elements agree when we apply the multiplication map to return to $J(A)$.

Definition 4.30. Let (T, η, μ) be a monad in a category $\mathscr{C}$. A *T-algebra* is an object X of $\mathscr{C}$ along with a morphism $h : T(X) \to X$ making the following two diagrams commute.

$$\begin{tikzcd} T^2X \arrow[r, "Th"] \arrow[d, "\mu_X"] & TX \arrow[d, "h"] \\ TX \arrow[r, "h"] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\mathrm{id}_X"'] & TX \arrow[d, "h"] \\ & X \end{tikzcd} \tag{4.13}$$

A *morphism* of T-algebras $(X, h) \to (X', h')$ is a morphism $f : X \to X'$ which makes the following square commute.

$$\begin{tikzcd} TX \arrow[r, "T(f)"] \arrow[d, "h"] & TX' \arrow[d, "h'"] \\ X \arrow[r, "f"] & X' \end{tikzcd} \tag{4.14}$$

If (T, η, μ) is a comonad, then a *T-coalgebra* is defined analogously.

The following proposition explains how monads give a categorical interpretation of δ-rings.

Proposition 4.31. *There is an equivalence of categories between the category of δ-rings and the category of algebras over the monad (J, η, μ) defined in Example 4.29.*

Proof. Let (A, δ) be a δ-ring. An object of $J(A)$ is a symbol $a^{(i)}$ where $a \in A$. Because A is a δ-ring, we have a 'derivative evaluation' map $h_\delta : J(A) \to A$ defined by $a^{(i)} \mapsto \delta^i(a) \in A$. To see that (A, h_δ) is a J-algebra we observe that for any $(a^{(i)})^{(j)} \in J(J(A))$

and $b \in A$ the two diagrams of Definition 4.30 applied to these elements give:

$$\begin{tikzcd} (a^{(i)})^{(j)} \arrow[r, mapsto] \arrow[d, mapsto] & (\delta^i(a))^{(j)} \arrow[d, mapsto] & & b \arrow[r, mapsto] \arrow[dr, mapsto] & b^{(0)} \arrow[d, mapsto] \\ a^{(i+j)} \arrow[r, mapsto] & \delta^{i+j}(a) = \delta^j(\delta^i(a)) & & & b = \delta^0(b) \end{tikzcd} \tag{4.15}$$

Therefore, (A, h_δ) is a J-algebra. Moreover, a morphism $f : (A, \delta) \to (A', \delta')$ of δ-rings defines a morphism $(A, h_\delta) \to (A', h_{\delta'})$ because for any $a^{(i)}$ in $J(A)$ we have,

$$h_\delta \circ Jf(a^{(i)}) = h_\delta(f(a)^{(i)}) = \delta^i(f(a)) = f(\delta^i(a)) = f \circ h_\delta(a^{(i)}).$$

Conversely, let (A, h) be a J-algebra, then we may define a derivation δ_h on A by setting $\delta_h(a) = h(a^{(1)})$. Because $h(a^{(0)}) = a$ by the second commutative diagram of Definition 4.30 we have that,

$$\delta_h(ab) = h((ab)^{(1)}) = h(a^{(0)}b^{(1)} + a^{(1)}b^{(0)}) = a\,h(b^{(1)}) + b\,h(a^{(1)}) = a\,\delta_h(b) + b\,\delta_h(a)$$

and so δ_h is indeed a derivation. Moreover, given a morphism $(A, h) \to (A', h')$ of J-algbras we know that $h' \circ J(f) = f \circ h$. This implies that the associated map of rings $(A, \delta_h) \to (A', \delta_{h'})$ is a δ-homomorphism because we have

$$\delta_{h'}(f(a)) = h'(f(a)') = h' \circ Jf(a') = f \circ h(a') = f(\delta_h(a)).$$

Thus, we have defined a functor from J-algebras to δ-rings.

To prove that we have defined an equivalence of categories we need to prove that for any δ-ring (A, δ), $(A, \delta) \cong (A, \delta_{h_\delta})$ as δ-rings and that for any J-algebra (A, h) we have $(A, h) \cong (A, h_{\delta_h})$ as J-algebras. The first of these isomorphisms is clear because $\delta_{h_\delta}(a) = h_\delta(a^{(1)}) = \delta(a)$. To see the latter isomorphism, note that because (A, h) is a J-algebra we have that $h(h(a^{(i)})^{(j)}) = h(a^{(i+j)})$ by the first diagram of Definition 4.30. Therefore, we have

$$h_{\delta_h}(a^{(i)}) = \delta_h^i(a) = \delta_h^{i-1}(h(a')) = \delta_h^{i-2}(h(h(a')')) = \delta_h^{i-2}(h(a'')) = \cdots = h(a^{(i)})$$

That is, h and h_{δ_h} agree as maps $J(A) \to A$ and hence $(A, h) \cong (A, h_{\delta_h})$. $\square$

D-schemes

Now that we have the categorical description of δ-rings of Proposition 4.31 we are able to extend the notion of a differential structure to the category of $\mathbb{Q}$-schemes. First of all, note that the category of affine $\mathbb{Q}$-schemes inherits a comonad structure from the monad structure on $\mathbf{Alg}_\mathbb{Q}$ described in Example 4.29. To extend this comonad structure to the category of $\mathbb{Q}$-schemes, let X be a scheme with an open affine cover $\{U_i\}$. Then we have morphisms $J(U_i) \to U_i$ and $J(U_i) \to J(J(U_i))$. Because $J(U_i \times_X U_j) = J(U_i) \times_{J(X)} J(U_j)$ we may glue these morphisms to obtain morphisms $J(X) \to X$ and $J(X) \to J(J(X))$. The natural transformations $\mu : J \to J^2$ and $p : J \to \mathrm{id}_{\mathbf{Sch}_\mathbb{Q}}$ thus obtained define a comonad structure on $\mathbf{Sch}_\mathbb{Q}$. However, we may also describe the counit and comultiplication maps on the level of functors.

Firstly, the morphism $p_X : J(X) \to X$ is the limit of the morphisms $J^r(X) \to X$ induced by truncation. Furthermore, let A be a $\mathbb{Q}$-algebra and let $m, n \geq 0$. Then we have a morphism

$$A[z]/(z^{m+n+1}) \to A[z,t]/(z^{m+1}, t^{n+1})$$

defined by $z \mapsto z + t$. Therefore, for any scheme X we have a morphism $J^{m+n}(X) \to J^m(J^n(X))$. Taking the limit of these morphisms we obtain a morphism $\mu_X : J(X) \to J(J(X))$, which we think of as taking the Taylor series of a power series with respect to another variable. The associated natural transformation μ is the counit of the comonad. Using this comonad structure on $\mathbf{Sch}_{\mathbb{Q}}$ we are now able to define what it means for a scheme to have a differential structure.

Definition 4.32. A *D-scheme* is a coalgebra for the comonad (J, p, μ). Explicitly, this means that a D-scheme is a scheme X along with a morphism $a_X : X \to J(X)$ such that the following two diagrams commute:

$$\begin{tikzcd} X \arrow[r, "a_X"] \arrow[d, "a_X"] & J(X) \arrow[d, "\mu_X"] \\ J(X) \arrow[r, "J(a_X)"] & J(J(X)) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "a_X"] \arrow[dr, "\mathrm{id}_X"'] & J(X) \arrow[d, "p_X"] \\ & X \end{tikzcd} \tag{4.16}$$

Moreover, a morphism $(X, a_X) \to (Y, a_Y)$ of D-schemes is a morphism $\pi : X \to Y$ making Diagram 4.17 commute. We will often use the notation $X \underset{D}{\to} Y$ for a morphism of D-schemes.

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "a_X"] & Y \arrow[d, "a_Y"] \\ J(X) \arrow[r, "J(\pi)"] & J(Y) \end{tikzcd} \tag{4.17}$$

Example 4.33. Let X be a scheme. Then, the two axioms of a comonad imply that the morphism $\mu_X : J(X) \to J(J(X))$ gives $J(X)$ the structure of a D-scheme. Moreover, the naturality of μ means that for any morphism $f : X \to Y$, the associated morphism $J(f) : J(X) \to J(Y)$ is a D-scheme homomorphism.

Remark 4.34. For any $\mathbb{Q}$-algebra A, we have an inclusion $A \hookrightarrow A[t]/(t^{r+1})$ and so for any scheme X we obtain a morphism of schemes $X \to J^r(X)$. Taking the limit of these morphisms we obtain a morphism $X \to J(X)$ which gives X the structure of D-scheme. However, we will not be able to glean any information about X from this construction, just as we gain nothing by endowing a ring with the zero derivation.

Proposition 4.35. *Let X be a D-scheme, let Y be a scheme and let $f : Y \to X$ be a formally étale morphism. Then there is a unique choice of D-scheme structure on Y making the morphism $Y \to X$ a morphism of D-schemes. In particular, if U is an open* *subscheme of a D-scheme X, then there is an induced D-scheme structure on U such that the inclusion $U \hookrightarrow X$ is a morphism of D-schemes.*

Proof. By Proposition 4.5 we know that $J(Y) = J(X) \times_X Y$. We have morphisms $Y \xrightarrow{\mathrm{id}} Y$ and $Y \xrightarrow{f} X \xrightarrow{a_X} J(X)$ which satisfy $\mathrm{id}_Y \circ f = f = p_X \circ a_X \circ f$, as in the following diagram:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[dr, dashed, "a_Y"] \arrow[ddr, bend right, "\mathrm{id}"'] & X \arrow[dr, bend left, "a_X"] & \\
& J(Y) \arrow[r, "J^m(f)"] \arrow[d, "p_Y"] & J(X) \arrow[d, "p_X"] \\
& Y \arrow[r, "f"] & X
\end{tikzcd} \tag{4.18}$$

By the universal property of the fibre product we obtain a unique morphism $a_Y : Y \to J(Y)$ such that $Y \xrightarrow{a_Y} J(Y) \xrightarrow{p_Y} Y = \mathrm{id}_Y$. To show that a_Y gives Y the structure of a D-scheme we need to prove that $J(a_Y) \circ a_Y = \mu_Y \circ a_Y$. To see this, note that because (X, a_X) is a J-algebra, we have that $J(a_X) \circ a_X = \mu_X \circ a_X$. Therefore, the map

$$Y = X \times_X Y \xrightarrow{a_X \times \mathrm{id}_Y} J(X) \times_X Y = J(Y) \xrightarrow{J(a_X) \times \mathrm{id}_Y} J(J(X)) \times_X Y = J(J(Y))$$

coincides with the map

$$Y = X \times_X Y \xrightarrow{a_X \times \mathrm{id}_Y} J(X) \times_X Y = J(Y) \xrightarrow{\mu_X \times \mathrm{id}_Y} J(J(X)) \times_X Y = J(J(Y))$$

But $J(a_X) \times \mathrm{id}_Y = J(a_Y)$ and $\mu_X \times \mathrm{id}_Y = \mu_Y$ and so we've shown that $J(a_Y) \circ a_Y = \mu_Y \circ a_Y$, as required. Now from Diagram 4.18 we observe that $J(f) \circ a_Y = a_X \circ f$ and so f is a morphism of D-schemes. □

Corollary 4.36. *Let F be a δ-ring and let K/F be a finite separable extension. Then there is a unique choice of derivation on K such that K/F is a δ-field extension.*

Proof. By Proposition 2.18 we know that $\operatorname{Spec} K \to \operatorname{Spec} F$ is étale, so the result follows from Proposition 4.35. □

Proposition 4.35 also has the following useful corollary, which indicates why the theory of D-schemes is relevant to the Geometric Mordell Conjecture. Buium states this result in Chapter 7 of [13].

Corollary 4.37. *Let k be an algebraically closed field of characteristic zero. If F/k is any function field of one variable (a finitely generated field extension of transcendence degree one), then we can choose a derivation δ on F such that $F^\delta = k$.*

Proof. Note that we can write our field extension as $F/k(t)/k$, where t is transcendental over k and $F/k(t)$ is algebraic. Because $k(t)$ is of characteristic zero, this means that $F/k(t)$ is finite separable. The zero derivation on k extends to the derivation $\frac{d}{dt}$ on $k(t)$ and by Corollary 4.36 this derivation extends uniquely to a derivation δ on F.

Now we can write $F = k(t)(\alpha)$ for some element $\alpha \in F$ which is algebraic over $k(t)$. Let $f(x) = x^n + \cdots + a_0 \in k(t)[x]$ be the minimal polynomial of F. Then we claim that $\delta(\alpha) \neq 0$. Indeed we have

$$\begin{aligned} 0 &= \delta(\alpha^n + \cdots + a_0) \\ &= \sum_j \delta(a_j \alpha^j) = \sum_j \delta(a_j)\alpha^j + \sum_j j a_j \alpha^{j-1}\delta(\alpha) \\ &= \big(\delta(a_{n-1})\alpha^{n-1} + \cdots + \delta(a_0)\big) + \delta(\alpha) f'(\alpha). \end{aligned}$$

Suppose, in order to gain a contradiction, that the first summand is zero. By assumption we know that α is not the root of any non-zero polynomial of degree less than n and so $\delta(a_0) = \cdots = \delta(a_{n-1}) = 0$. This implies that $a_0, \cdots, a_{n-1}$ belong to k and so α satisfies a polynomial equation over k, which is a contradiction because k is algebraically closed. Furthermore, $f'(\alpha)$ cannot be zero because $F/k(t)$ is separable. Therefore, $\delta(\alpha) \neq 0$. This demonstrates that the field of constants of (F, δ) is precisely k. □

Proposition 4.38. *Let A be a δ-ring and let X be a D-scheme. Then to give a D-scheme homomorphism $X \to \operatorname{Spec} A$ is equivalent to giving a δ-homomorphism $A \to \Gamma(X, \mathcal{O}_X)$.*

Proof. Let $\{U_i\}$ be an open cover of X by the spectra of δ-rings. Then to give a morphism $X \xrightarrow{D} \operatorname{Spec} A$ is equivalent to giving δ-homomorphisms $A \to \mathcal{O}(U_i)$ which glue as maps of schemes. But this is equivalent to giving a morphism of δ-rings $A \to \Gamma(X, \mathcal{O}_X)$. □

Proposition 4.39. *The category of D-schemes contains fibre-products. That is, given D-scheme morphisms $X \underset{D}{\longrightarrow} S$ and $Y \underset{D}{\longrightarrow} S$ there is a unique D-scheme structure on $X \times_S Y$ so that the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$ are morphisms of D-schemes.*

Proof. The projections $X \times_S Y \to X$ and $X \times_S Y \to Y$ along with the morphisms $a_X : X \to J(X)$ and $a_Y : Y \to J(Y)$ give us the following diagram:

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & J(X) \arrow[dd] \\ Y \arrow[r] \arrow[d] & S \arrow[dr, "a_S"] & \\ J(Y) \arrow[rr] & & J(S) \end{tikzcd} \tag{4.19}$$

The three smaller squares in Diagram 4.19 commute by our assumptions, so the larger square commutes and hence the universal property of the fibre product gives us a morphism

$$a_{X \times_S Y} : X \times_S Y \to J(X) \times_{J(S)} J(Y) = J(X \times_S Y).$$

That $a_{X\times_S Y}$ satisfies the two properties required to define a J-coalgebra structure is immediate from the observation that $a_{X\times_S Y} = a_X \times a_Y$. □

4.3.1 Relative D-schemes

Given a fixed base D-scheme S, we write $\mathbf{DSch}_S$ for the slice category whose objects are D-scheme morphisms $X \to S$, and whose morphisms are D-scheme morphisms $X \to Y$ commuting with the maps to S. If the base scheme is $\operatorname{Spec} R$ for a δ-ring R, then we label the category $\mathbf{DSch}_R$. Let S be a D-scheme and let X be an S-scheme (not necessarily an object of $\mathbf{DSch}_S$). Both $a_S : S \to J(S)$ and $J(X) \to J(S)$ are morphisms of D-schemes, the latter by Example 4.33, so by Proposition 4.39 we know that $J(X)\times_{J(S)} S$ is a D-scheme and that the projection $J(X)\times_{J(S)} S \to S$ is a morphism of D-schemes. Therefore, we may define a *relative jet functor* $J_S : \mathbf{Sch}_S \to \mathbf{DSch}_S$ by $J_S(X) = J(X) \times_{J(S)} S$.

Proposition 4.40. *The functor $J_S : \mathbf{Sch}_S \to \mathbf{DSch}_S$ is right adjoint to the forgetful functor $\mathbf{DSch}_S \to \mathbf{Sch}_S$. That is, we have a natural isomorphism*

$$\operatorname{Mor}_{D,S}(Y, J_S(X)) = \operatorname{Mor}_S(Y, X).$$

Proof. We will begin by proving that if Y is a D-scheme, then

$$\operatorname{Mor}(Y, X) = \operatorname{Mor}_D(Y, J(X)). \tag{4.20}$$

First of all, if we have a D-scheme morphism $Y \to J(X)$ then we obtain a morphism $Y \to J(X) \xrightarrow{p_X} X$. On the other hand if we have a morphism $f : Y \to X$ then we have a morphism $Y \xrightarrow{a_Y} J(Y) \xrightarrow{J(f)} J(X)$. Conversely, to see that this is a D-scheme morphism consider the following diagram:

$$\begin{tikzcd} Y \arrow[r, "a_Y"] \arrow[d, "a_Y"] & J(Y) \arrow[r, "J(f)"] \arrow[d, "\mu_Y"] & J(X) \arrow[d, "\mu_X"] \\ J(Y) \arrow[r, "J(a_Y)"] & J(J(Y)) \arrow[r, "J(J(f))"] & J(J(X)) \end{tikzcd} \tag{4.21}$$

The square on the left commutes because (Y, a_Y) is a D-scheme and the square on the right commutes because $J(f) : J(Y) \to J(X)$ is a morphism of D-schemes by Example 4.33. Therefore, the whole diagram commutes and so $J(f) \circ a_Y$ is a morphism of D-schemes. It remains to be shown that the map $\operatorname{Mor}_D(Y, J(X)) \to \operatorname{Mor}(Y, X)$ defined by $g \mapsto p_X \circ g$ and the map $\operatorname{Mor}(Y, X) \to \operatorname{Mor}_D(Y, J(X))$ defined by $f \mapsto J(f) \circ a_Y$ are mutually inverse to one another. That is, we need to prove that for any $f : Y \to X$ we have $p_X \circ J(f) \circ a_Y = f$ and for any $g : Y \underset{D}{\to} J(X)$ we have $J(p_X \circ g) \circ a_Y = g$.

To prove the first equality we note that, because p is a natural transformation, we have $f \circ p_Y = p_X \circ J(f)$ and hence $p_X \circ J(f) \circ a_Y = f \circ p_Y \circ a_Y = f \circ \mathrm{id}_Y = f$ (we've used that $p_Y \circ a_Y = \mathrm{id}_Y$ because (Y, a_Y) is a J-coalgebra). To prove the second identity we observe that

$$\begin{aligned} J(p_X \circ g) \circ a_Y &= J(p_X) \circ J(g) \circ a_Y \\ &\stackrel{1}{=} J(p_X) \circ \mu_X \circ g \\ &\stackrel{2}{=} p_{J(X)} \circ \mu_X \circ g \\ &\stackrel{3}{=} \mathrm{id}_{J(X)} \circ g = g. \end{aligned}$$

Here equality 1 follows because g is a D-scheme morphism and equality 2 is the first axiom of a comonad. Finally, equality 3 is a consequence of the fact that $J(X)$ is a J-algebra with structure map μ_X.

Now we consider the general case. By the universal property of the fibre product we know that D-morphisms $Y \to J_S(X)$ over S are the same as maps $Y \to J(X)$ making the diagram on the left commute.

$$\begin{tikzcd} Y \arrow[r, "D"] \arrow[d, "D"] & J(X) \arrow[d, "D"] \\ S \arrow[r, "D"] & J(S) \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r] & S. \end{tikzcd} \tag{4.22}$$

By the adjunction $\mathrm{Mor}_D(Y, J(X)) = \mathrm{Mor}(Y, X)$, such morphisms are in bijection with morphisms $Y \to X$ making the square on the right commute. Note that the lower horizontal map comes from $S \xrightarrow{a_S} J(S) \xrightarrow{p_S} S$ and is therefore the identity, so these are exactly the morphisms belonging to $\mathrm{Mor}_S(Y, X)$. □

Here we remark that for any $m \in \mathbb{N}$ we have a *truncated relative jet functor* defined by $J^m_S(X) = J^m(X) \times_{J^m(S)} S$. When $S = \operatorname{Spec} R$ is the spectrum of a δ-ring, the relative jet functor $J^r_R(X)$ enjoys the following universal property:

Proposition 4.41 (Proposition 3.3.16 of [13])**.** *Let R be a δ-ring and let X be an R-scheme. Then there is an adjunction,*

$$\mathrm{Mor}_R(S \times_{\operatorname{Spec} R} \operatorname{Spec} R[t]/(t^{r+1}), X) = \mathrm{Mor}_R(S, J^r_R X).$$

Proof. Specifying a morphism $S \to J^r_R(X) = J^r(X) \times_{\operatorname{Spec} J^r(R)} \operatorname{Spec} R$ over R correspond to giving a morphism $S \to J^r(X)$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & J^r(X) \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \operatorname{Spec} J^r(R) \end{tikzcd} \tag{4.23}$$

Under the adjunction of Proposition 4.9 these are equivalent to morphisms $S \times_R$

$\operatorname{Spec} R[t]/(t^{r+1}) \to X$ making the following diagram commute:

$$\begin{array}{ccc} S \times_R \operatorname{Spec} R[t]/(t^{r+1}) & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R[t]/(t^{r+1}) & \longrightarrow & \operatorname{Spec} R. \end{array} \tag{4.24}$$

That is, to morphisms $S \times_R \operatorname{Spec} R[t]/(t^{r+1}) \to X$ over R, where we note that $S \times_R \operatorname{Spec} R[t]/(t^{r+1})$ is an R-scheme via the lower composition in the previous diagram. $\square$

Remark 4.42. When we are working over a base δ-field F, we will often write X^r for the truncated relative jet scheme $J^r_F(X)$.

4.4 $\mathcal{D}_X$-modules

Let (X, a) be a D-scheme, then the morphism of schemes $a : X \to J(X)$ induces a morphism of sheaves $\mathcal{O}_{J(X)} \to a_*\mathcal{O}_X$. For each open subset U of X, this morphism gives $\mathcal{O}_X(U)$ the structure of a δ-ring and so $a : X \to J(X)$ defines a *derivation* on $\mathcal{O}_X$. By a derivation on $\mathcal{O}_X$ we mean a map of sheaves of $\mathbb{Q}$-modules $\delta : \mathcal{O}_X \to \mathcal{O}_X$ such that $\delta_U : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is a derivation for all opens U of X. Note that this implies that the restriction maps $\rho_{U,V}$ are δ-homomorphisms. In this approach to the theory of D-schemes, a morphism of D-schemes $X \to Y$ is a morphism of schemes such that the associated morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a homomorphism of sheaves of δ-rings. Indeed, this is how Buium defines a D-scheme in [13].

By the universal property of the sheaf of relative differentials, a derivation $\delta : \mathcal{O}_X \to \mathcal{O}_X$ is equivalent to an $\mathcal{O}_X$-module homomorphism $\Omega_X \to \mathcal{O}_X$. By applying $\mathcal{H}om(-, \mathcal{O}_X)$ we obtain a morphism $\mathcal{O}_X \to \mathcal{H}om(\Omega_X, \mathcal{O}_X)$, or a global section of the tangent sheaf $\mathcal{T}_X$ of X. Thus, we may think of a derivation $\delta : \mathcal{O}_X \to \mathcal{O}_X$ as a 'vector field' on X. Extending this analogy, a D-scheme is a scheme along with fixed vector field and if $X \underset{D}{\to} S$ is a D-scheme over S, then the derivation on X must 'lie over' the vector field on S. Note that we intend this description as an aid to conceptualising D-schemes, we will not phrase any of our arguments in terms of vector fields.

Now, because each ring $\mathcal{O}_X(U)$ is a δ-ring we can define a sheaf of (non-commutative) rings $\mathcal{D}_X$ on X by $\mathcal{D}_X(U) = D_{\mathcal{O}_X(U)}$. Note that $\mathcal{D}_X$ is indeed a sheaf, because it is isomorphic as an $\mathcal{O}_X$-module to the direct sum $\bigoplus_{i \geq 0} \mathcal{O}_X \delta^i = \mathcal{O}_X[\delta]$. We call this the sheaf of *linear differential operators on* X. We then have the following definition.

Definition 4.43. Let X be a smooth D-scheme. A $\mathcal{D}_X$*-module* on X is an $\mathcal{O}_X$-module $\mathcal{F}$ on X such that for any open set U of X, $\mathcal{F}(U)$ is a left $\mathcal{D}_X(U)$-module and the restriction morphisms are compatible with the D-module structure on each $\mathcal{F}(U)$. By this we mean that for any open inclusion $V \subset U$, Diagram 4.25 commutes, where the horizontal maps are defined by the action of $\mathcal{D}_X$ on $\mathcal{F}$.

$$\begin{array}{ccc} \mathcal{D}_X(U) \times \mathcal{F}(U) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{D}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array} \tag{4.25}$$

Remark 4.44. Note that we have not given the standard definition of a $\mathcal{D}_X$-module here, but a modified version which will suit our purposes. Indeed, the study of $\mathcal{D}_X$-modules is a rich area of mathematics, but we will only need a fairly superficial understanding of these modules in this thesis, so we will not go very far into the theory. For a more comprehensive introduction to the study of $\mathcal{D}_X$-modules, see the unpublished online notes [1, 33, 47]. See also Example 1.3.6 of [20].

Remark 4.45. A quasi-coherent $\mathcal{D}_X$-module on $X = \operatorname{Spec} A$ is of the form $\widetilde{M}$ for some A-module M. But M is also a $D_{\mathcal{O}_X(X)} = D_A$-module by assumption, so there is an equivalence of categories between quasi-coherent $\mathcal{D}_{\operatorname{Spec} A}$-modules and D_A-modules.

Note that if $\varphi : B \to A$ is a δ-homomorphism, then the A-module $\Omega_{A/B}$ is given the structure of a D_A-module by the action $\delta(da) = d(\delta a)$. Therefore, if $f : X \to Y$ is a morphism of D-schemes, the sheaf $\Omega_{X/Y}$ of relative differentials has the structure of a $\mathcal{D}_X$-module. Moreover, if we have δ-ring homomorphisms $A \to B$ and $B \to C$ then we claim that the induced morphisms $\Omega_{C/A} \to \Omega_{C/B}$ and $\psi : \Omega_{B/A} \otimes_B C \to \Omega_{C/A}$ are morphisms of D_C-modules. That the former map is a D_C-module homomorphism is clear, and to see that ψ is a D_C-module homomorphism we observe that for $b \in B$ and $c \in C$ we have

$$\psi(\delta(db \otimes c)) = g\,(d(\delta b) \otimes c + db \otimes \delta(c)) = c\, d(\delta b) + \delta(c)\, db = \delta(\psi(db \otimes c)).$$

Therefore, if we have D-scheme morphisms $X \xrightarrow{f} Y$ and $Y \to Z$ the induced morphisms $f^*\Omega_{Y/Z} \to \Omega_{X/Z}$ and $\Omega_{X/Z} \to \Omega_{X/Y}$ are $\mathcal{D}_X$-module homomorphisms. This observation, in conjunction with the exact sequence 2.3, demonstrates the following result.

Proposition 4.46. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of D-schemes. Then we have an exact sequence of $\mathcal{D}_X$-modules,*

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

Note that the affine case of the following result is related to Proposition 3.6.1 of [13].

Proposition 4.47. *If Y is a D-scheme and X is any Y-scheme (not necessarily an object of $\mathbf{DSch}_Y$), then*

$$\Omega_{J_Y^\infty(X)/Y} \cong \mathcal{D}_{J_Y^\infty(X)} \otimes_{\mathcal{O}_X} \Omega_{X/Y}.$$

In particular, if B is a δ-ring and A is a B-algebra then we have an isomorphism of $D_{J_B^\infty(A)}$-modules,

$$\Omega_{J_B^\infty(A)/B} \cong D_{J_B^\infty(A)} \otimes_A \Omega_{A/B}.$$

Proof. By the definition of the sheaf of relative differentials, it will suffice to prove the affine case. Let $D = D_{J^\infty(B)/A}$ and let M be a D-module. By the tensor-hom adjunction, to give a D-module homomorphism $D \otimes_A \Omega_{A/B} \to M$ is equivalent to giving an A-module homomorphism $\Omega_{A/B} \to M$. Now suppose we have a D-module homomorphism $\phi : \Omega_{J_B^\infty(A)/B} \to M$. Then for any $a^{(i)} \in J_B^\infty(A)$ we have,

$$\phi(da^{(i)}) = \phi\left(\delta^i \cdot da\right) = \delta^i \cdot \phi(da).$$

That is, ϕ is determined by where it sends da, for $a \in A \subset J_B^\infty(A)$. Therefore, to give a D-module homomorphism $\Omega_{J_B^\infty(A)/B} \to M$ is also equivalent to giving an A-module homomorphism $\Omega_{A/B} \to M$. We conclude by the Yoneda lemma. $\square$

Morphisms of D-schemes

Let $X : \mathbf{Alg}_k \to \mathbf{Set}$ be a D-scheme, then we have an associated functor $X^\delta : \mathbf{DAlg}_k \to \mathbf{Set}$ defined by $A \mapsto \mathrm{Mor}_D(\operatorname{Spec} A, X)$. The following result tells us that we can recover X from X^δ. This will be useful when we want to define morphisms of D-schemes, because it will allow us to restrict to δ-algebras. At times we will do this implicitly without invoking Proposition 4.48.

Proposition 4.48. *Let R be a δ-ring and let $X : \mathbf{Alg}_R \to \mathbf{Set}$ be a D-scheme over R. Then the functor $\mathbf{DSch}_R \to [\mathbf{DAlg}_R, \mathbf{Set}]$ defined by $X \mapsto X^\delta$ is fully faithful.*

Proof. This follows from an argument similar to the one we used in the proof of Proposition 2.2: Suppose we have a morphism $\varphi : X^\delta \to Y^\delta$ and let $\{\operatorname{Spec} A_i\}$ be a cover of X by the spectra of δ-rings. Then for each i, the map φ_{A_i} sends the inclusion $\operatorname{Spec} A_i \hookrightarrow X$ to a D-scheme morphism $\operatorname{Spec} A_i \to Y$, and we may glue these to obtain a D-scheme morphism $X \to Y$. $\square$

Definition 4.49. A morphism of D-schemes $Z \to X$ is called a *closed D-immersion* if it is both a closed immersion and a morphism of D-schemes. Given a subfunctor Z of X, Z is called a *closed D-subscheme* if the inclusion $Z \to X$ is a closed D-immersion.

Suppose $f : Z \to X$ is a closed D-immersion. Then for any D-morphism $\operatorname{Spec} A \underset{D}{\to} X$, the pullback $\operatorname{Spec} A \times_X Z$ is isomorphic to $\operatorname{Spec} A/I$ for some ideal I of A. But since the morphism $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is a D-scheme morphism by Proposition 4.39 we know that I is a δ-ideal of A. Conversely, because X has an open cover by the spectra of δ-rings, to check that $Z \to X$ is a closed D-immersion, it is enough to check that for any D-scheme morphism $\operatorname{Spec} A \underset{D}{\to} X$, the pullback $\operatorname{Spec} A \times_X Z$ is isomorphic to $\operatorname{Spec} A/I$ for a δ-ideal I of A. That is, a closed subscheme of a D-scheme X is defined by a quasi-coherent sheaf of δ-ideals.

Proposition 4.50 (Proposition I.2.2 of [10])**.** *(a) Let X be a D-scheme, then the reduction X_{red} of X is a closed D-subscheme of X.*

(b) *Let $f : X \underset{D}{\to} Y$ be a quasi-compact and quasi-separated morphism of D-schemes. Then both the closed image and the reduced closed image of f are closed D-subschemes of Y.*

(c) *Let Z be an irreducible component of a noetherian D-scheme X. Then there is an induced D-scheme structure on Z.*

Proof. The second part of statement (b) follows from part (a) and the first part of (b), so it will suffice to prove these two results.

(a): Let X be a D-scheme. Then X_{red} is defined by the quasi-coherent sheaf of ideals defined on affine opens of X by $\mathcal{J}(\operatorname{Spec} A) = A/\sqrt{(0)}$. But by Proposition 4.17 we know that $\sqrt{(0)}$ is a δ-ideal of A, which implies that X_{red} is a closed D-subscheme.

(b): Since $f : X \to Y$ is quasi-compact and quasi-separated, the discussion of Section 2.4.3 implies that the ideal sheaf of $\operatorname{im} f$ is $\ker[\mathcal{O}_Y \to f_*\mathcal{O}_X]$. But because f is a morphism of D-schemes we know that for each open $U \subset Y$, the homomorphism $\mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U)$ is a morphism of δ-rings and hence the kernel of $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is a sheaf of δ-ideals. We refer the reader to Section I.2 [10] for the proof of part (c). □

Remark 4.51. Let R be a δ-ring with ring of constants C. Note that the morphism $J^m(X) \to X$ always has a section, induced by the inclusion $B \to B[t]/(t^{m+1})$ for any B. It is less clear when the morphism $J^m_R(X) \to X$ has a section over R. However, if X descends to C, so that $X = X_0 \times_{\operatorname{Spec} C} \operatorname{Spec} R$ for some C-scheme X_0, then we have

$$J^m_R(X) = \big(J^m(X_0) \times_{\operatorname{Spec} J^m(C)} \operatorname{Spec} J^m(R)\big) \times_{\operatorname{Spec} J^m(R)} \operatorname{Spec} R = J^m(X_0) \times_{\operatorname{Spec} J^m(C)} \operatorname{Spec} R.$$

Therefore, the trivial section $X_0 \to J^m(X_0)$ defines a section of the morphism $J^m_R(X) \to X$ over R.

4.4.1 D-group schemes

Suppose $G : \mathbf{Alg}_{\mathbb{Q}} \to \mathbf{Grp}$ is a group scheme, then $J(G) : \mathbf{Alg}_{\mathbb{Q}} \to \mathbf{Grp}$ is a scheme by Proposition 4.8 and is therefore a group scheme. The structure morphisms of $J(G)$ are given by $J(m) : J(G) \times_{\mathbb{Q}} J(G) \to J(G)$, $J(i) : J(G) \to J(G)$ and $J(e_{\mathbb{Q}}) : J(\operatorname{Spec} \mathbb{Q}) = \operatorname{Spec} \mathbb{Q} \to J(G)$. If, in addition, G is a D-scheme, then we have a morphism of schemes $a_G : G \to J(G)$. We say that G is a *D-group scheme* if a_G is also a homomorphism of group schemes, so that Diagram 4.26 commutes.

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "(a_G, a_G)"] & G \arrow[d, "a_G"] \\ J(G) \times J(G) \arrow[r, "J(m)"] & J(G) \end{tikzcd} \tag{4.26}$$

Because $J(m)$ is the multiplication morphism in $J(G)$, Diagram 4.26 implies directly that the multiplication morphism $m : G \times G \to G$ is a morphism of D-schemes. Similarly, because a_G is group homomorphism we have that $a_G \circ i = J(i) \circ a_G$ which also means that i is a D-scheme morphism, and finally $a_G \circ e_{\mathbb{Q}} = J(e_{\mathbb{Q}})$, which means that $e_{\mathbb{Q}}$ is a D-scheme morphism. Therefore, we might equivalently have defined a D-group scheme to be a group object in the category of D-schemes.

If $G = \operatorname{Spec} H$ is an affine D-group scheme over a δ-ring R, then H is a Hopf algebra because G is a group scheme, and a δ-R-algebra because G is a D-scheme. Moreover, the discussion above implies that the morphisms

$$\Delta : H \to H \otimes_R H, \quad \sigma : H \to H \quad \text{and} \quad \varepsilon : \operatorname{Spec} R \to H$$

are all morphisms of δ-R-algebras. We call such an object a *δ-Hopf R-algebra*. By construction, the category of δ-Hopf R-algebras is equivalent dual to the category of affine D-schemes over R. Furthermore, we define a *D-algebraic group* to be a D-group scheme which is also an algebraic group.

4.4.2 The support of a finite type $\mathcal{D}_X$-module

Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. As a set, we define the support of $\mathcal{F}$ to be

$$\operatorname{Supp} \mathcal{F} = \{p \in |X| \mid \mathcal{F}_p \neq 0\}.$$

By Nakayama's lemma [63, Tag 07RC] we know that

$$\mathcal{F}_p \neq 0 \iff \mathcal{F}|_p = \mathcal{F}_p \otimes_{\mathcal{O}_{X,x}} \kappa(x) \neq 0.$$

and so we could have defined the support of $\mathcal{F}$ to be the points such that $\mathcal{F}|_p \neq 0$. In this section we will show that if $\mathcal{F}$ is a finite-type quasi-coherent sheaf, then we can give the support of $\mathcal{F}$ the structure of a closed subscheme of X. The following result is stated in [63, Tag 00L2], and we will follow the proof given there.

Lemma 4.52. *Let R be a ring and let M be a finitely generated R-module. Then*

$$V(\operatorname{Ann}_R(M)) = \operatorname{Supp}(\widetilde{M}) = \{\mathfrak{p} \in |\operatorname{Spec} R| \mid M_{\mathfrak{p}} \neq 0\},$$

where $\operatorname{Ann}_R(M)$ *is the* annihilator of M*, the ideal of all $x \in R$ such that $xm = 0$ for all $m \in M$.*

Proof. To prove that $\operatorname{Supp}(M) \subset V(\operatorname{Ann}_R(M))$ suppose $[\mathfrak{p}] \in \operatorname{Supp} M$, so that $M_{\mathfrak{p}} \neq 0$. Therefore, we can choose non-zero m/s in $M_{\mathfrak{p}}$, which means that there is no $r \in A \setminus \mathfrak{p}$ such that $rm = 0$. In particular, the annihilator $\operatorname{Ann}_R(M)$ must be contained in $\mathfrak{p}$.

To prove the other inclusion, assume $\mathfrak{p} \notin \operatorname{Supp}(M)$, so that $M_{\mathfrak{p}} = 0$. By assumption, we can choose generators $m_1, ..., m_r$ for M. From commutative algebra we know that

$$0 = M_{\mathfrak{p}} = \varinjlim_{s \in A \setminus \mathfrak{p}} M_s$$

which means that there exists $f \in R \setminus \mathfrak{p}$ such that $m_i/1 = 0/1$ in R_f for all $i = 1, ..., r$.

Therefore, we can choose $n_i \geq 1$ such that $f_i^{n_i} m_i = 0$. Let $n = \max_i\{n_i\}$, then we have $f^n m_i = 0$ for all i. That is, $f^n \in \mathrm{Ann}_R(M)$ but f^n is not in $\mathfrak{p}$, so $\mathfrak{p} \notin V(\mathrm{Ann}_R(M))$. $\square$

We now extend this result to quasi-coherent $\mathcal{O}_X$-modules and $\mathcal{D}_X$-modules. The $\mathcal{O}_X$-module case of the following theorem is stated and proved in [63, Tag 05JU].

Theorem 4.53. *Let X be a scheme. Then the support Z of a finite-type quasi-coherent sheaf $\mathcal{F}$ on X has the structure of a closed subscheme of X. Furthermore, if X is a D-scheme and $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{D}_X$-modules which is of finite type as an $\mathcal{O}_X$-module, then the support of $\mathcal{F}$ is a closed D-subscheme of X.*

Proof. Let $i : \mathrm{Spec}\, A \hookrightarrow X$ be an open affine of X. Then $i^*\mathcal{F} \cong \widetilde{M}$ for some A-module M, because $\mathcal{F}$ is quasi-coherent. By Lemma 4.52 we know that $Z \times_X \mathrm{Spec}\, A = V(\mathrm{Ann}_A(M))$, so to prove that Z is a closed subscheme of X it will suffice to prove that the sheaf of ideals on X defined by $\mathcal{I}(\mathrm{Spec}\, A) = \mathrm{Ann}_A(M)$ is quasi-coherent. To prove this it will suffice (by Exercise 8.1.H of [64]) to show that for any $f \in A$, $\mathrm{Ann}_A(M)_f = \mathrm{Ann}_{A_f}(M_f)$ as ideals of A_f, but this fact may be verified directly.

To prove the second statement of the theorem, we need to prove that if $\mathcal{I}$ is a sheaf of δ-ideals. That is, we need to prove that if A is a δ-ring and M is a D_A-module, then $\mathrm{Ann}_A(M)$ is a δ-ideal. To see this, suppose $x \in \mathrm{Ann}_A(M)$ and $m \in M$. Then $x\delta(m) = 0$ and so we have,

$$0 = \delta(xm) = x\delta(m) + \delta(x)m = \delta(x)m.$$

Therefore, $\delta(x)m = 0$ for all $m \in M$. $\square$

Definition 4.54. Let $f : X \to Y$ be a finite type morphism of D-schemes, then $\Omega_{X/Y}$ is a finite-type quasi-coherent sheaf by construction. We define the *ramification locus* of f to be the support of $\Omega_{X/Y}$. If f is also a D-scheme morphism, then the ramification locus of f is a closed D-subscheme of X, by Theorem 4.53.

4.5 Descent for D-schemes

To begin with we will study descent in the affine case. Let F be a δ-field with field of constants C. If A_0 is a C-algebra, then $A_0 \otimes_C F$ is given the structure of a δ-F-algebra by the derivation $\delta(a \otimes \lambda) = a \otimes \delta(\lambda)$. We say that a δ-F-algebra B is *split* or *descends to C* if B is isomorphic as a δ-F-algebra to a δ-F-algebra of this form.

Lemma 4.55 (Lemma 1.3.4 of [11])**.** *Let A be a δ-F-algebra (resp. δ-F-Hopf algebra). Then A is split as a δ-F-algebra (resp. Hopf algebra) if and only if A is split as a D_F-module.*

Proof. Let A be a split δ-F-algebra and let $\{x_i\}$ be a C-basis for A_0. Then $\{x_i \otimes 1\}$ is an F-basis for $A_0 \otimes_C F$ which is annihilated by δ, so $A_0 \otimes_C F$ is split as a D-module.

To prove the converse, suppose A is split as a D-module, so that $A \cong \bigoplus_{i\in I} F$. This allows us to write an element of A as $(a_i)_{i\in I}$ so that $\delta : A \to A$ is given by $\delta\left((a_i)_{i\in I}\right) = (\delta a_i)_{i\in I}$. We have a δ-F-algebra $A^\delta \otimes_C F \to F$ homomorphism which, from our description of A as a D-module, is bijective.

To extend to the case of a Hopf algbera we will prove that if H is a δ-Hopf algebra over F, then H^δ is a Hopf algebra over $F^\delta = C$. First of all we define δ-Hopf F-algebra homomorphisms $\phi_1, \phi_2 : H \otimes_F H \to H$ by $x \otimes y \mapsto \varepsilon(y)x$ and $x \otimes y \mapsto \varepsilon(x)y$. Both compositions

$$H \xrightarrow{\Delta} H \otimes_F H \xrightarrow{\phi_i} H$$

are the identity, because if we consider the dual morphism of group schemes we get the identity. Since Δ, ϕ_1 and ϕ_2 are δ-homomorphisms we conclude that the image of H^δ under Δ is in fact contained in $H^\delta \otimes_C H^\delta$. Therefore, we have algebra homomorphisms

$$\Delta : H^\delta \to H^\delta \otimes_C H^\delta,\ \varepsilon : F^\delta = C \to H^\delta \ \text{ and }\ \sigma : H^\delta \to H^\delta$$

which give H^δ the structure of a Hopf C-algebra. Now the same argument as before proves that $H^\delta \otimes_C F$ is isomorphic to H as a δ-Hopf F-algebra. $\square$

If instead we are given a C-scheme X_0, then we can endow X_0 with the trivial D-scheme structure described in Remark 4.34, thereby giving $X_0 \times_{\operatorname{Spec} C} \operatorname{Spec} F$ the structure of a D-F-scheme.

Definition 4.56. Let X be a D-F-scheme (resp. D-F-group scheme). We say that X is *split* if it is isomorphic to a D-F-scheme (resp. D-F-group scheme) of the form $X_0 \times_{\operatorname{Spec} C} \operatorname{Spec} F$, with the derivation described above.

From Lemma 4.55 we know that an affine D-scheme over a δ-field F (or affine D-group scheme) is split if and only if its coordinate ring is split as a D_F-module. The following proposition will be important in the proof of the Mordell-Weil conjecture, where our analysis will depend on whether or not certain schemes to descend to the field of constants.

Proposition 4.57 (Section 1.3.6 of [11]). *(a) Let $f : X = X_0 \times_{\operatorname{Spec} C} \operatorname{Spec} F \to Y \times_{\operatorname{Spec} C} \operatorname{Spec} F$ be a morphism of D-schemes over F. Then f is of the form $f_0 \times \mathrm{id}$ for some C-scheme morphism $f_0 : X_0 \to Y_0$.*

(b) Every closed D-subscheme of a split D-scheme which is separated over F is split.

(c) Suppose a D-scheme X which is separated over F is covered by split affine D-schemes U such that the inclusion of each $X \setminus U$ is a D-scheme morphism. Then X is split.

Proof. We follow the proof outlined in [11]. First we will prove part (a) in the case where $X = \operatorname{Spec}(A_0 \otimes_C F)$ and $Y = \operatorname{Spec}(B_0 \otimes_C F)$ are affine. Consider the morphism

$$\phi^* : B_0 \otimes_C F \to A_0 \otimes_C F.$$

Since ϕ is a D-scheme morphism, we know that ϕ^* is a morphism of δ-F-algebras and hence ϕ^* induces a map $\phi_0^* : B_0 = (B_0 \otimes_C F)^\delta \to (A_0 \otimes_C F)^\delta = A_0$. Therefore, $\phi^* = \phi_0^* \otimes \mathrm{id}_F$ and and so $\phi = \phi_0 \times \mathrm{id}_{\operatorname{Spec} F}$.

To prove the affine case of part (b) we need to prove that if A is a split δ-F-algebra and I is a δ-ideal of A, then A/I is a split δ-F-algebra. To see this we note that A is split as a D-module by Lemma 4.55 and so I is split as a D-module by Proposition 4.24. Therefore, we I is isomorphic to $I_0 \otimes_C F$, for some C-vector space I_0. In fact, I_0 is an ideal: if we have $r \in A_0$ and $x \in I_0$ then

$$rx \otimes 1 = (r \otimes 1)(x \otimes 1) \in I_0 \otimes_C F$$

so $rx \in I_0$. Therefore, we have that

$$A/I \cong (A_0/I_0) \otimes_C F,$$

which is what we wanted to prove.

(c): Let $\{U_i = \operatorname{Spec}(A_i \otimes_C F) \underset{D}{\to} X\}$ be an affine open covering of X by split affine opens. Since X is separated, $U_i \cap U_j$ is affine, say isomorphic to $\operatorname{Spec} B_{ij}$. To see that B_{ij} is split as a δ-F-algebra, we observe that its complement in U_i is a closed D-subscheme of U_i and is therefore split by the affine case of (b). Because U_i is split, the splitting of $U_i \setminus \operatorname{Spec} B_{ij}$ defines a splitting of $\operatorname{Spec} B_{ij}$ and so we conclude that B_{ij} is split as a δ-F-algebra.

Thus, $U_i \cap U_j$ is isomorphic as a D-scheme to $\operatorname{Spec}(A_{ij} \otimes_C F)$ for some C-algebra A_{ij}. Therefore, we can glue together the schemes $\operatorname{Spec} A_i$ (by identifying $\operatorname{Spec} A_i$ and $\operatorname{Spec} A_j$ along $\operatorname{Spec} A_{ij}$) to form a scheme which we label X_0. Now we note that X and $X_0 \times_{\operatorname{Spec} C} \operatorname{Spec} F$ are covered by the same affine opens glued together the same way, so they are isomorphic as D-schemes.

Now (a) follows by gluing together morphisms on an open affine cover. To see (b) suppose we have a split D-scheme $X = X_0 \times_{\operatorname{Spec} C} \operatorname{Spec} F$ with a closed sub D-scheme Z. Choose an open cover $\{\operatorname{Spec} A_i \to X_0\}$, then we have a cover $\{Z \times_X \operatorname{Spec}(A_i \otimes_C F) \underset{D}{\to} Z\}$ where $Z \cap \operatorname{Spec}(A_i \otimes_C F)$ is again a split affine D-scheme because Z is a closed D-subscheme. By (c) we conclude that Z is split. $\square$

Remark 4.58. Compare this to the behaviour of normal (not differential) schemes: if we have a scheme X over F which descends to C, then it is not necessarily the case that any closed subscheme of X descends to C. For instance, let $X = \mathbb{A}^2_{\mathbb{C}(t)}$ and let Y be the curve $\{y^2 = x(x-1)(x-t)\}$. (Here Y is the set of affine points of a 'generic' elliptic curve over $\mathbb{C}(t)$.) In this case X descends to $\mathbb{C} = \mathbb{C}(t)^\delta$ (indeed X descends all the way to $\mathbb{Z}$), but Y does not descend to $\mathbb{C}$.

Corollary 4.59. *A separated*† *D-F-group scheme is split as a D-F-group scheme if and only if it is split as a D-F-scheme.*

Proof. Suppose $G \cong X_0 \times_C \operatorname{Spec} F$ as a scheme. By Part (a) of Proposition 4.57 the structure morphisms $m : G \times G \to G$, $e : \operatorname{Spec} k \to G$ and $i : G \to G$ all descend to X_0, giving X_0 the structure of a group such that $G \to X_0 \times_C \operatorname{Spec} F$ is a group isomorphism. □

4.6 Differential algebraic vector groups

We conclude this chapter with a discussion of the structure of differential algebraic vector groups. Both results of this section are stated in Section 3.6.3 of [13]. Fix a δ-field F of characteristic zero. For the rest of this section we will write D for D_F and D^r for the truncated ring $F[\delta]/(\delta^{r+1})$. Recall that for any F-vector space V we have defined an algebraic group $\underline{V}$ by,

$$\underline{V}(R) = \operatorname{Hom}_{\mathbf{Mod}_F}(V, R) = \operatorname{Hom}_F(\operatorname{Sym}_F V, R)$$

so that $\underline{V} = \operatorname{Spec}(\operatorname{Sym}_F V)$.

Proposition 4.60. *Let F be a δ-field of characteristic zero, then*

$$\underline{V}^r = \underline{D^r \otimes_F V}$$

for all $r \in \mathbb{N} \cup \{\infty\}$.

Proof. Firstly, note that because $\delta(t) = 1$, δ can be thought of as an element of the dual space to the vector space $F[t]/(t^{r+1})$. Therefore, for any $r \in \mathbb{N} \cup \{\infty\}$ we have that $\big(F[t]/(t^{r+1})\big)^\vee \cong D^r$ as an F-vector space, where we observe the convention that when $r = \infty$ we are referring to the ring $F[\![t]\!]$. Then for any F-algebra R we have,

$$\begin{aligned} J_F^r(\underline{V})(R) &= \operatorname{Hom}_{\mathbf{Mod}_F}(V, R[t]/(t^{r+1})) \\ &\overset{1}{=} \operatorname{Hom}_{\mathbf{Mod}_F}(\big(F[t]/(t^{r+1})\big)^\vee \otimes_F V, R) \\ &= \operatorname{Hom}_{\mathbf{Mod}_F}(D^r \otimes_F V, R) \end{aligned}$$

Equality 1 follows from the fact that $R[t]/(t^{r+1}) = R \otimes_F F[t]/(t^{r+1}) = \operatorname{Hom}_{\mathbf{Mod}_F}((F[t]/(t^{r+1}))^\vee, R)$ as well as the tensor-hom adjunction. □

Proposition 4.61. *Let H be a closed sub-D-group scheme of $J_F(\underline{V})$. Then there exists a finite dimensional F-vector space W and a homomorphism $\lambda : J_F(\underline{V}) \to J_F(\underline{W})$ such that $H = \ker(\lambda)$.*

Proof. By Proposition 4.60 we know that $J_F(\underline{V}) = \underline{D_F \otimes_F V}$ and by assumption we have that H is a closed subgroup scheme of $\underline{D_F \otimes_F V}$. Therefore, Corollary 3.11

†Note that a group scheme over a field is always separated by [63, Tag 047L] and so we will not need to check this fact in Chapters 5 and 6

implies that H is an algebraic vector group, isomorphic to $\underline{U}$ for some F-vector space U. Because H is a sub-D-scheme we know that U is also a D-module. The closed immersion $H \hookrightarrow \underline{D_F \otimes_F V}$ induces (by definition) a surjective δ-homomorphism $\mathrm{Sym}_F(D_F \otimes_F V) \to \mathrm{Sym}_F(U)$ and hence a surjective D_F-module homomorphism $\alpha : D_F \otimes_F V \to U$.

Since every left ideal in D is principal by Lemma 4.23 we have that $D_F \otimes_F V$ is a Noetherian module and hence $\ker(\alpha)$ is finitely generated over D. Therefore, for some $M \geq 0$ we have an exact sequence,

$$D_F^M \to D_F \otimes_F V \xrightarrow{\alpha} U \to 0.$$

Let $\lambda : \underline{D_F \otimes_F V} \to \underline{D_F^M}$ be the map on D-schemes induced by the first homomorphism in this sequence. Then the duality of Proposition 3.10 gives us,

$$\ker\left[\underline{D_F \otimes_F V} \xrightarrow{\lambda} \underline{D_F^M}\right] = \underline{\mathrm{coker}[D_F^M \to D_F \otimes_F V]} = \underline{U}.$$

□

Chapter 5

Differential characters of abelian varieties

Recall that an *additive character* of an algebraic k-group G is a group homomorphism $G \to \mathbb{G}_a$. We denote the k-module of all (additive) characters of a group by $X_a(G) = \mathrm{Hom}(G, \mathbb{G}_a)$. Note that because a character of a group is necessarily also a regular function we have $\mathrm{Hom}(G, \mathbb{G}_a) \subset \mathrm{Mor}(G, \mathbb{G}_a) = \mathcal{O}(G)$. Now if $G = \operatorname{Spec} H$ is an affine algebraic group, then additive characters correspond to morphisms of Hopf algebras $\mathcal{O}(\mathbb{G}_a) = k[x] \to H$. That is, morphisms $\varphi : k[x] \to H$ such that

$$\Delta_H \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_{\mathbb{G}_a}.$$

If we let $\varphi(x) = f \in H$, then these are the k-algebra morphisms $\varphi : k[x] \to H$ such that

$$\Delta_H(f) = (\varphi \otimes \varphi) \circ \Delta_{\mathbb{G}_a}(x) = (\varphi \otimes \varphi)(1 \otimes x + x \otimes 1) = 1 \otimes f + f \otimes 1. \tag{5.1}$$

We call a function f satisfying Equation 5.1 a *primitive element* of $\mathcal{O}(G)$. Thus, the character group $X_a(G)$ is isomorphic to the group of primitive elements of $\mathcal{O}(G)$ with respect to addition,

$$X_a(G) \cong \{f \in \mathcal{O}(G) \mid \Delta_G(f) = 1 \otimes f + f \otimes 1\}.$$

Therefore, giving a homomorphism $\mathbb{G}_m \to \mathbb{G}_a$ is equivalent to specifying a polynomial $f \in k[x, x^{-1}]$ such that $f(xy) = f(x) + f(y)$ when viewed as an element of $k[x, y, x^{-1}, y^{-1}]$. By comparing coefficients on either side of this equation, we see that the only element of $k[x, x^{-1}]$ satisfying this equation is $f(x) = 0$. Therefore, the multiplicative group $\mathbb{G}_m$ does not admit non-zero additive characters. However, we do have a non-zero homomorphism $\psi : J^1(\mathbb{G}_m) \to \mathbb{G}_a$, where $\psi_R : \mathbb{G}_m(R[t]/(t^2)) = \big(R[t]/(t^2)\big)^\times \to \mathbb{G}_a(R) = R^+$ is defined by

$$f(t) = (a + bt) \mapsto \frac{b}{a} = \frac{f'(0)}{f(0)}.$$

We think of this as the 'logarithmic derivative', and the fact that it is a homomorphism can be verified directly (or using the product rule for differentiation). From this example we see that although an algebraic group might not have any characters, we may be able to produce characters by passing to a higher jet scheme. This observation motivates the following definition.

Definition 5.1. Let G be a group scheme over a δ-field F. Then we will call a homomorphism $J^r_F G \to \mathbb{G}_a$ a *δ-character of G of order (at most) r*. The k-module of all δ-characters of order at most r is denoted $X^r(G)$.

Returning to the case of $\mathbb{G}_m$, we see that we can lift our character $J^1(\mathbb{G}_m) \to \mathbb{G}_a$ along the derivation to construct a universal character $\widetilde{\psi} : J^\infty(\mathbb{G}_m) \to J^\infty(\mathbb{G}_a)$. On R-points this is the map $R[\![z]\!]^\times \to R[\![t]\!]$ given by

$$f(z) \mapsto \sum_{n\geq 0} \frac{1}{n!} \frac{d^n(f'/f)}{dz^n}(0)\, t^n.$$

Using an induction argument (which we omit) one can prove that the kernel of this homomorphism is the set of power series in $R[\![z]\!]^\times$ for which all non-constant terms vanish. That is, $\ker \widetilde{\psi}_R = \mathbb{G}_m(R)$ for any R and so we have an exact sequence

$$0 \to \mathbb{G}_m \to J^\infty(\mathbb{G}_m) \to J^\infty(\mathbb{G}_a) \to 0.$$

In Section 5.1 we will show that a similar phenomenon holds for abelian varieties. More specifically, we will construct a finite type D-group homomorphism $\psi : J^\infty_F(A) \to J^\infty_F(\underline{V})$, where V is a finite-dimensional k-vector space. As another example of this phenomenon, in Appendix A we derive the Picard–Fuchs equation for elliptic curves and discuss how this equation may be used to construct characters.

5.1 Constructing δ-characters of abelian varieties

In this section we will establish the following result, following Section 6 of Buium's paper [12].

Theorem 5.2 (Theorem 6.1(3) of [12])**.** *Let F be a δ-field and let A be an abelian variety of dimension g over F. Then there exists a finite-dimensional F-vector space V along with a homomorphism of D-group schemes $\psi : A^\infty \to \underline{V}^\infty$ which is of finite type.*

Let A be an abelian variety of dimension g. By Proposition 4.16 we know that for $m \geq n \geq 0$ the kernel of the homomorphism $A^m \to A^n$ is isomorphic to $\mathbb{A}^l_F$ for some l. Because we are working over a field of characteristic zero, this means that the kernel is an algebraic vector group (by the duality of Proposition 3.10). Furthermore, Proposition 4.16 implies that the cokernel of $A^n \to A$ is trivial and so, because the category of commutative algebraic groups is abelian, we have an extension,

$$0 \to H^n \to A^n \to A \to 0$$

where H^n is an algebraic vector group. By the existence of this sequence we know that H^n is the Chevalley subgroup of A^n. Now, by the Affinisation Theorem 3.6, for all n we have a subgroup C^n of A^n along with an exact sequence

$$0 \to C^n \to A^n \to (A^n)^{\mathrm{aff}} \to 0 \tag{5.2}$$

and we observe that $(A^n)^{\mathrm{aff}} = \operatorname{Spec}\mathcal{O}(A^n) = \operatorname{Spec}\mathcal{O}(A)^n = (\operatorname{Spec}\mathcal{O}(A))^n = (A^{\mathrm{aff}})^n$. We will denote this group by L^n. By Rosenlicht's Decomposition 3.23 we know that $A^n = C^n H^n$ and so by the second isomorphism theorem 3.3 we have

$$A = A^n/H^n = (C^n H^n)/H^n = C^n/(C^n \cap H^n).$$

That is, $C^n/(C^n \cap H^n)$ is an abelian variety and so $C^n \cap H^n$ is the Chevalley subgroup $a_*(C^n)$ of C^n. Now because $a_*(C^n)$ is a closed subgroup of H^n it is itself an algebraic vector group. We may therefore take the quotient of H^n by $a_*(C^n)$ which yields

$$H^n/a_*(C^n) = H^n/(C^n \cap H^n) = (C^n H^n)/C^n = A^n/C^n = L^n.$$

Thus, we have Diagram 5.3 in which all rows and columns are exact. Note that all the groups present here are commutative algebraic groups and so we are working in an abelian category by Theorem 3.4.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & a_*(C^n) \arrow[r, hook] \arrow[d, hook] & H^n \arrow[r] \arrow[d] & H^n/a_*(C^n) \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & C^n \arrow[r, hook] \arrow[d] & A^n \arrow[r] \arrow[d] & L^n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, equal] \arrow[d] & A \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd} \tag{5.3}$$

Note that the top row of Diagram 5.3 is split, as a sequence of algebraic vector groups. Therefore, we have a section $s : H^n/a_*(C^n) \to H^n$ and so the homomorphism $L^n \xrightarrow{\sim} H^n/a_*(C^n) \xrightarrow{s} H^n \hookrightarrow A^n$ defines a splitting of the middle row.

For each n, the homomorphism $\beta : A^{n+1} \to A^n$ induces a morphism $\gamma : (A^{n+1})^{\mathrm{aff}} \to (A^n)^{\mathrm{aff}}$ and therefore a morphism $\alpha : C^{n+1} \to C^n$. By Proposition 4.16 we know that β is surjective and we claim that α is also surjective. To see this, let $\tau : a_*(C^{n+1}) = H^{n+1} \cap C^{n+1} \to a_*(C^n) = C^n \cap H^n$ be the morphism induced by β and consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & a_*(C^{n+1}) \arrow[r] \arrow[d, "\tau"] & C^{n+1} \arrow[r] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & a_*(C^n) \arrow[r] & C^n \arrow[r] & A \arrow[r] & 0
\end{tikzcd} \tag{5.4}$$

By Theorem 3.4 we know that the category of commutative algebraic groups is abelian, so the snake lemma gives us an exact sequence

$$0 \to \ker \tau \to \ker \alpha \to 0 \to \operatorname{coker} \tau \to \operatorname{coker} \alpha \to 0.$$

Now $\operatorname{coker} \tau$ is the cokernel of a homomorphism of algebraic vector groups so it is an algebraic vector group. Therefore, $\operatorname{coker} \alpha \cong \operatorname{coker} \tau$ is also an algebraic vector group. Since C^n is anti-affine, it has no homomorphisms to a vector group and so we conclude that the homomorphism $C^n \to \operatorname{coker} \alpha$ is the zero homomorphism. Thus, we have established the claim.

Lemma 5.3. *The sequence $C^0 \twoheadleftarrow C^1 \twoheadleftarrow C^2 \twoheadleftarrow \cdots$ is eventually constant. That is, there is a group C^∞ and a positive integer n_0 such that $C^n \cong C^\infty$ for $n \geq n_0$.*

Proof. By Proposition 3.28 we have an exact sequence of abelian groups,

$$\operatorname{Hom}(C^n, \mathbb{G}_a) = X_a(C^n) \to \operatorname{Hom}(B^n, \mathbb{G}_a) = X_a(B^n) \to \operatorname{Ext}(A, \mathbb{G}_a). \tag{5.5}$$

But C^n is an anti-affine group by the Affinisation Theorem 3.6, so $\mathcal{O}(C^n) = F$ and therefore $X_a(C^n) = 0$. By the exact sequence 5.5 we thereby obtain an injection $X_a(B^n) \hookrightarrow \operatorname{Ext}(A, \mathbb{G}_a)$, where the latter space is a finite-dimensional vector space by Theorem 3.29.

The diagram $B^0 \twoheadleftarrow B^1 \twoheadleftarrow \cdots$ induces a sequence of F-vector spaces

$$X_a(B^0) \to X_a(B^1) \to X_a(B^2) \to \cdots$$

which is a non-decreasing sequence of subspaces of $\operatorname{Ext}(A, \mathbb{G}_a)$. Since $\operatorname{Ext}(A, \mathbb{G}_a)$ is finite dimensional we conclude that the sequence $X_a(B^0) \subset X_a(B^1) \subset \cdots$ eventually stabilises. By the duality between algebraic vector groups and vector spaces of Proposition 3.10, the sequence $B^0 \to B^1 \to \cdots$ also eventually stabilises. Then, from the series of exact sequences

$$0 \to B^n \to C^n \to A \to 0$$

we conclude that the sequence $C^0 \twoheadleftarrow C^1 \twoheadleftarrow \cdots$ must also stabilise. $\square$

By Lemma 5.3 we can choose n_0 such that $C^{n+1} \to C^n$ is an isomorphism for all $n \geq n_0$.

Lemma 5.4. *For $n \geq n_0$ the group homomorphism $\phi : A^{n+1} \to A^n$ has a section.*

Proof. By the discussion following Diagram 5.3, the sequence

$$0 \to C^n \to A^n \to (A^n)^{\text{aff}} = L^n \to 0$$

is split. Therefore, $A^n \cong C^n \times L^n = C^\infty \times L^n$. The homomorphism $A^{n+1} = C^\infty \times L^{n+1} \to A^n = C^\infty \times L^n$ is given as a matrix by $(\phi_{ij})_{i,j=1}^2$ where we have

$$\phi_{11} : C^{n+1} \to C^n, \quad \phi_{21} : C^{n+1} \to L^n, \quad \phi_{12} : L^{n+1} \to C^n \quad \text{and} \quad \phi_{22} : L^{n+1} \to L^n$$

We proved in Lemma 5.3 that $\phi_{11}C^{n+1} \to C^n$. Moreover, since C^{n+1} is anti-affine and L^n is an algebraic vector group, we know that $\phi_{21} : C^{n+1} \to L^n$ is the zero homomorphism. Since the homomorphism $L^{n+1} \to L^n$ is surjective, the morphism of vector spaces $X_a(L^n) \to X_a(L^{n+1})$ is injective and so it has a left inverse. Since L^{n+1} and L^n are vectorial, this means that the morphism $L^{n+1} \to L^n$ has a right inverse, ψ_{22}. Therefore, we can define a homomorphism $\psi : C^n \times L^n \to C^{n+1} \times L^{n+1}$ by

$$\psi = \begin{pmatrix} \phi_{11}^{-1} & -\phi_{11}^{-1} \circ \phi_{12} \circ \psi_{22} \\ 0 & \psi_{22} \end{pmatrix}.$$

A direct computation shows that ψ is a right inverse to $\phi : A^{n+1} \to A^n$. □

Proposition 5.5. *The D_F-module $X_a(A^\infty)$ is finitely-generated.*

Proof. By Lemma 5.4 we know that there exists a finite-dimensional F-vector space W such that for any $n \geq n_0$, $A^{n+1} \cong A^n \times \underline{W}$ (in particular $\underline{W} \cong \operatorname{Lie} A$). We claim that $X_a(\underline{W}^1) = W \oplus \delta \cdot W$, where the derivation is obtained by viewing $X_a(\underline{W}^1)$ as a subspace of $X_a(\underline{W}^\infty)$. This follows from the fact that $\underline{W}^1 \cong \underline{W} \times \underline{W}$ by our computation of $J^m(\mathbb{A}^n)$ in Section 4.1. Therefore, for any $n \geq n_0$ we have

$$X_a(A^{n+2}) = X_a(A^n \times \underline{W} \times \underline{W}) = X_a(A^n \times \underline{W}^1) \cong X_a(A^n) \otimes X_a(\underline{W}^1) \cong X_a(A^n) \otimes (W \oplus \delta W).$$

Therefore, $X_a(A^\infty)$ is generated as a D-module by $X_a(A^{n_0}) = X_a(L^{n_0})$, which is a finite-dimensional F-vector space (as L^{n_0} is an algebraic vector group of finite type over F). □

Proof of Theorem 5.2. Consider the group A^∞. Recall that by Theorem 3.4 the category of commutative algebraic groups is abelian. The morphisms $f_{ij} : C^i \to C^j$ are surjective, so this sequence satisfies the Mittag-Leffler condition and we can take the limit of the sequences $0 \to C^n \to A^n \to L^n \to 0$ in the category of commutative algebraic groups to obtain an exact sequence

$$0 \to C^\infty \to A^\infty \to L^\infty \to 0. \tag{5.6}$$

Note that the second map coincides with the affinisation map of A^∞, which is a morphism of D-schemes by Proposition 4.38. Therefore, Sequence 5.6 is an exact sequence of D-group schemes. Let V be the finite-dimensional F-vector space $X_a(L^{n_0})$, then Proposition 5.5 implies that we have a surjective D-module homomorphism

$$D_F \otimes_F V \twoheadrightarrow X_a(A^\infty)$$

and so by Proposition 3.10 we obtain a closed D-immersion

$$\underline{X_a(A^\infty)} = L^\infty \to \underline{D_F \otimes_F V} = \underline{V}^\infty.$$

Thus, we've constructed a δ-homomorphism $A^\infty \to L^\infty \hookrightarrow \underline{V}^\infty$ with kernel C^∞, which is of finite type over F. □

Chapter 6

Proof of the Geometric Mordell Conjecture

Let k be a field of characteristic zero and let F be a function field of one variable over k. Let X be a smooth projective non-hyperelliptic curve of genus at least 2 over F such that $X \times_F \operatorname{Spec} \overline{F}$ does not descend to k. To prove the Geometric Mordell Conjecture (Theorem 1.1) we need to prove that the set $X(F)$ is finite. First of all note that we may assume that k is algebraically closed: otherwise we can choose a function field of one variable K over $\overline{k}$ such that $X(F) \subset X_K(K)$. Therefore, by Corollary 4.37 we can choose a derivation δ on F whose field of constants is k. We fix this derivation for the remainder of this chapter. Furthermore, let A be the Jacobian of X. By Theorem 3.24 we know that X embeds in A via the Abel–Jacobi map α. By Proposition 4.14 and the stability of closed embeddings under base change we know that $\alpha^\infty : X^\infty \hookrightarrow A^\infty$ is a closed embedding, so we will view X^∞ as a closed subscheme of A^∞.

6.1 The ramification locus of a finite type delta character

Let W be a finite-dimensional F-vector space and let $\psi : A^\infty \to \underline{W}^\infty$ be a group homomorphism of finite type with kernel Σ. Composing with the Abel-Jacobi map we obtain a morphism

$$f : X^\infty \xhookrightarrow{\alpha^\infty} A^\infty \xrightarrow{\psi} \underline{W}^\infty$$

and we define Z to be the ramification locus of this morphism. By Theorem 4.53 we know that Z is a closed D-subscheme of X^∞. We will prove that the set $Z^\delta(F)$ of δ-F-points x such that $\Omega_{A^\infty/\underline{W}^\infty}\big|_x \neq 0$ is finite. Recall that if R is a δ-F-algebra then we have a bijection between D-scheme morphisms $x : \operatorname{Spec} R \underset{D}{\to} X^\infty$ and F-scheme homomorphisms $\operatorname{Spec} R \to X$, allowing us to think of a morphism $x : \operatorname{Spec} R \underset{D}{\to} X^\infty$ as being both an element of $X^\infty(R)$ and of $X(R)$. In the sequel we will not always comment when we are making such an identification.

Theorem 6.1. *The set $Z^\delta(F)$ is finite. That is, there are finitely many δ-F-points $x \in (X^\infty)^\delta(F) = X(F)$ such that $\Omega_{X^\infty/\underline{W}^\infty}|_x \neq 0$.*

Before taking up the proof of Theorem 6.1 we will digress briefly to comment on why the statement of the theorem takes this form. A similar statement (cf. Lemma 2.14) with a more geometric character might be that "$\mathcal{O}(Z)$ is finitely generated as a D_F-module". However, this is in fact weaker than the conclusion of Theorem 6.1 as Example 6.2 will demonstrate. Moreover, we know of no algebraic condition on a δ-F-algebra which would ensure that $\mathrm{Hom}_\delta(\mathcal{O}(Z), F)$ is finite, other than assuming that $\mathcal{O}(Z)$ is finite over F. Therefore, in Theorem 6.1 we 'forget' the scheme structure on Z and consider the set $Z^\delta(F)$.

Example 6.2. Let F be a δ-field. Then the derivation determined by $\delta(x) = x^2$ endows the polynomial algebra $F[x]$ with the structure of a δ-ring. We have

$$\delta(x^m) = mx^{m-1}\delta(x) = mx^{m+1}$$

which means that $F[x]$ is generated as a D_F-module by $\{1, x\}$. However, the set $Y^\delta(F)$ is not necessarily finite: If $F = \mathbb{C}(t)$ then to define a δ-homomorphism $\mathbb{C}(t)[x] \to \mathbb{C}(t)$ is equivalent to choosing $f(t)$ in $\mathbb{C}(t)$ such that $f'(t) = f(t)^2$. But $f(t) = \frac{1}{c-t}$ satisfies this equation for any $c \in \mathbb{C}$, so $Y^\delta(\mathbb{C}(t))$ is not finite.

Proof of Theorem 6.1. By Corollary 4.22 we may embed F into a δ-closed field $\mathcal{F}$, and so it will suffice to prove Theorem 6.1 after base-change to the δ-closure $\mathcal{F}$ of F. That is, we need to prove that the set of points $x \in (X^\infty_{\mathcal{F}})^\delta(\mathcal{F})$ such that $\Omega_{X^\infty_{\mathcal{F}}/\mathcal{F}}\Big|_x \neq 0$ is finite. For the rest of this proof we will be working over $\mathcal{F}$, so we write X^∞ rather than $X^\infty_{\mathcal{F}}$ for the base change to $\mathcal{F}$. Furthermore, we will write $\mathcal{D}$ for the sheaf $\mathcal{D}_{X^\infty}$.

By Proposition 4.46, the D-scheme homomorphism $f : X^\infty \to \underline{W}^\infty$ induces an exact sequence of $\mathcal{D}$-modules

$$f^*\Omega_{\underline{W}^\infty/\mathcal{F}} \to \Omega_{X^\infty/\mathcal{F}} \to \Omega_{X^\infty/\underline{W}^\infty} \to 0.$$

Let $x \in (X^\infty)^\delta(\mathcal{F})$ be a δ-$\mathcal{F}$-point of X^∞. Since taking the fibre is right exact we obtain an exact sequence

$$\Omega_{\underline{W}^\infty/\mathcal{F}}\big|_{f(x)} \to \Omega_{X^\infty/\mathcal{F}}\big|_x \to \Omega_{X^\infty/\underline{W}^\infty}\big|_x \to 0.$$

We then apply the contravariant left-exact functor $\mathrm{Hom}_{D_{\mathcal{F}}}(-, \mathcal{F})$ to our sequence to obtain

$$0 \to \mathrm{Hom}_{D_{\mathcal{F}}}\left(\Omega_{X^\infty/\underline{W}^\infty}\big|_x, \mathcal{F}\right) \to \mathrm{Hom}_{D_{\mathcal{F}}}\left(\Omega_{X^\infty/\mathcal{F}}\big|_x, \mathcal{F}\right) \to \mathrm{Hom}_{D_{\mathcal{F}}}\left(\Omega_{\underline{W}^\infty/\mathcal{F}}\big|_{f(x)}, \mathcal{F}\right)$$

Now we observe that by Proposition 4.47 we have $\Omega_{X^\infty/\mathcal{F}} \cong \mathcal{D}_{X^\infty} \otimes_{\mathcal{O}_X} \Omega_{X/\mathcal{F}}$ and $\Omega_{\underline{W}^\infty/\mathcal{F}} \cong \mathcal{D}_{\underline{W}^\infty} \otimes_{\mathcal{O}_{\underline{W}}} \Omega_{\underline{W}/\mathcal{F}}$, so by applying the tensor-hom adjunction our exact sequence may be written as

$$0 \to \mathrm{Hom}_{D_{\mathcal{F}}}\left(\Omega_{X^\infty/\underline{W}^\infty}\big|_x, \mathcal{F}\right) \to \mathrm{Hom}_{\mathcal{F}}\left(\Omega_{X/\mathcal{F}}\big|_x, \mathcal{F}\right) \xrightarrow{T_x f} \mathrm{Hom}_{\mathcal{F}}\left(\Omega_{\underline{W}/\mathcal{F}}\big|_{f(x)}, \mathcal{F}\right).$$

That is, even though there is no morphism $X \to \underline{W}$, we have produced a map of *tangent spaces* $T_x f : T_x X \to T_{f(x)}\underline{W}$ with kernel $\mathrm{Hom}_{D_{\mathcal{F}}}\left(\Omega_{X^\infty/\underline{W}^\infty}\big|_x, \mathcal{F}\right)$. Since $\Omega_{X^\infty/\underline{W}^\infty}|_x$ is a finitely generated $D_{\mathcal{F}}$-module and $\mathcal{F}$ is δ-closed, Corollary 4.27 implies that $\mathrm{Hom}_{D_{\mathcal{F}}}\left(\Omega_{X^\infty/\underline{W}^\infty}\big|_x, \mathcal{F}\right) = 0$ if and only if $\Omega_{X^\infty/\underline{W}^\infty}\big|_x = 0$. That is, $Z^\delta(\mathcal{F})$ is equal to the set of δ-$\mathcal{F}$-points x such that $T_x f$ is not injective. Therefore, it will suffice to prove that this latter set is finite. Thus, we have reduced to the case of Theorem 4.1 of [13] and now we follow this proof to establish the result.

Recall from Section 3.3.5 that we have the Gauss map $\Phi : X \hookrightarrow \mathbb{P}(\mathscr{L}A)$, which is a closed embedding because X is non-hyperelliptic. By Proposition 4.14 and the stability of closed embeddings under base change we thereby obtain a closed embedding

$$X^\infty \hookrightarrow \mathbb{P}(\mathscr{L}A)^\infty$$

To avoid confusion, we will write $\widehat{X}^\infty$ for the closed subscheme of $\mathbb{P}(\mathscr{L}A)^\infty$ isomorphic to X^∞.

$$\begin{tikzcd}
Z \arrow[r, hook] & X^\infty \arrow[r, hook, "\alpha^\infty"] \arrow[d, hook, "\Phi^\infty"] & A^\infty \arrow[r, "\psi"] & \underline{W}^\infty \\
& \mathbb{P}(\mathscr{L}A)^\infty & &
\end{tikzcd} \tag{6.1}$$

The Gauss map Φ^∞ sends a point $x \in |X^\infty|$ to $\Phi^\infty(x) \in \mathbb{P}(\mathscr{L}A)^\infty$. Here we will explain how we may instead view $\Phi^\infty(x)$ as a closed subscheme of $\mathrm{Lie}\, A^\infty$. First of all, we have

$$\mathrm{Lie}\, A^\infty \overset{1}{\cong} (\mathrm{Lie}\, A)^\infty \overset{2}{\cong} \left(\underline{\mathscr{L}(A)^\vee}\right)^\infty \overset{3}{\cong} \underline{D_{\mathcal{F}} \otimes_{\mathcal{F}} \mathscr{L}(A)^\vee}. \tag{6.2}$$

The isomorphism marked 1 follows from the definition of $\mathrm{Lie}\, A$, the isomorphism marked 2 follows from Equation 3.6 and the isomorphism marked 3 follows from Proposition 4.60. By the discussion of Section 3.3.5 we know that $\Phi(x)$ can be viewed as the tangent space $T_x X$ transported to the origin of A and then viewed as a subscheme of $\mathrm{Lie}\, A$. Therefore, by the isomorphisms of 6.2 we may view our 'infinite' Gauss map Φ^∞ as sending x to the tangent space $D_{\mathcal{F}} \otimes_{\mathcal{F}} T_x X$ viewed as a subscheme of $\mathrm{Lie}\, A^\infty$.

We claim that if we translate a tangent vector $\gamma \in \ker T_x f$ to the origin of A^∞, the resulting vector γ' will belong to the intersection $\Phi^\infty(x) \cap \mathrm{Lie}\,\Sigma$ in $\mathrm{Lie}\, A^\infty$. The fact that γ' belongs to $\Phi^\infty(x)$ is immediate from the definition of the Gauss map, so it remains to be shown that γ' belongs to $\mathrm{Lie}\,\Sigma$. Because the tangent bundle of A is trivial by the discussion of Section 3.3.5 it will suffice to establish the result when $x = 0$, so that $\gamma' = \gamma$. That is, we need to prove that if we have a δ-tangent vector $\gamma : \Omega_{A^\infty/\mathcal{F}}|_0 \underset{\delta}{\to} \mathcal{F}$ such that the pullback tangent vector $\Omega_{\underline{W}^\infty/\mathcal{F}}|_0 \underset{\delta}{\to} \mathcal{F}$ is zero, then

$\Omega_{A^\infty/\mathcal{F}}|_0 \to \mathcal{F}$ factors through $\Omega_{\Sigma/\mathcal{F}}|_0$, as shown by the dashed arrow in Diagram 6.3.

$$\begin{tikzcd} \Omega_{\underline{W}^\infty/\mathcal{F}}|_0 \arrow[r] \arrow[dr, "0"] & \Omega_{A^\infty/\mathcal{F}}|_0 \arrow[r] \arrow[d, "\gamma"] & \Omega_{\Sigma/\mathcal{F}}|_0 \arrow[dl, dashed] \\ & \mathcal{F} & \end{tikzcd} \tag{6.3}$$

By the first exact sequence for the cotangent sheaf we know that

$$\Omega_{A^\infty/\underline{W}^\infty} = \operatorname{coker}\left[\psi^*\Omega_{\underline{W}^\infty/\mathcal{F}} \to \Omega_{A^\infty/\mathcal{F}}\right]$$

and hence

$$\Omega_{A^\infty/\underline{W}^\infty}|_0 = \operatorname{coker}\left[\Omega_{\underline{W}^\infty/\mathcal{F}}|_0 \to \Omega_{A^\infty/\mathcal{F}}|_0\right]$$

because taking fibres is right exact. But, by the stability of differentials under pullback we know that $\Omega_{A^\infty/\underline{W}^\infty}|_0 = \Omega_{\Sigma/\mathcal{F}}|_0$. Therefore, the existence of the dashed arrow in Diagram 6.3 follows from the universal property of the cokernel. Thus, we have established the claim.

Define V to be the closed subscheme $\bigcup_{x\in(X^\infty)^\delta} \Phi^\infty(x) \subset \operatorname{Lie} A^\infty$ of $\operatorname{Lie} A^\infty$, the affine cone over $\widehat{X}^\infty \subset \mathbb{P}(\mathscr{L}A)^\infty$. Note that V is a closed subscheme of the algebraic vector group $\operatorname{Lie} A^\infty$. Define an open subscheme $V^* = V \setminus \{0\}$ of V so that we have a projectivisation morphism $V^* \to \mathbb{P}(\mathscr{L}A)^\infty$. By construction, this factors through a map $V^* \to \widehat{X}^\infty$ and we define h to be the composition

$$h : V^* \cap \operatorname{Lie}\Sigma \hookrightarrow V^* \to \widehat{X}^\infty \xrightarrow{p_X} X.$$

Assume, in order to gain a contradiction, that there are infinitely many δ-$\mathcal{F}$-points x in X^∞ such that $\ker T_x f$ is not injective. We have shown that each such point gives rise to a δ-$\mathcal{F}$-point of the intersection $\Phi^\infty(x) \cap \operatorname{Lie}\Sigma$ and hence an $\mathcal{F}$-point in the image of h. That is, the closed image of h contains infinitely many $\mathcal{F}$-points and so we conclude that h is surjective (because X is a curve). Now choose an irreducible component Y^* of $V^* \cap \operatorname{Lie}\Sigma$ so that the closed image of Y^* in $\widehat{X}$ is still all of X. Define Y to be the closure of Y^* in $\operatorname{Lie}\Sigma \cap V$. We observe that Y is an irreducible component of $\operatorname{Lie}\Sigma$ and so in particular Y is a closed D-subscheme of $\operatorname{Lie}\Sigma$ by Proposition 4.50. But Σ is of finite type over $\mathcal{F}$ so $\operatorname{Lie}\Sigma$ is a finite-dimensional D-vector group. Therefore, by Proposition 4.25 $\operatorname{Lie}\Sigma$ descends to k and hence Proposition 4.57 implies that Y descends to k.

By de Jong's Theorem 2.24 we can choose a smooth variety $\overline{Y}$ along with a surjective morphism $\overline{Y} \to Y$ which means that the composition $\overline{Y} \to Y \to X$ is surjective. By the universal property of the Albanese variety the morphism $\overline{Y} \to Y \to X \to \operatorname{Jac}(X)$ factors through a morphism $\operatorname{Alb}(\overline{Y}) \to \operatorname{Jac}(X)$ making Diagram 6.4 commute.

$$\begin{tikzcd} \overline{Y} \arrow[r] \arrow[d] & X \arrow[d, hook] \\ \operatorname{Alb}(\overline{Y}) \arrow[r] & \operatorname{Jac}(X) \end{tikzcd} \tag{6.4}$$

Let Λ denote the scheme-theoretic image of the morphism $\mathrm{Alb}(\overline{Y}) \to \mathrm{Jac}(X)$, a sub-abelian variety of $\mathrm{Jac}(X)$. Diagram 6.4 implies that X is contained in Λ. By the universal property of the Albanese variety, the inclusion $\widehat{X} \hookrightarrow \Lambda$ must factor through $\mathrm{Jac}(X)$ and so $\Lambda = \mathrm{Jac}(X)$. That is, the group homomorphism $\mathrm{Alb}(\overline{Y}) \to \mathrm{Jac}(X)$ is scheme-theoretically surjective and so by Proposition 3.18 we conclude that $\mathrm{Jac}(X)$ descends to k. This implies that $\mathrm{Jac}(\overline{X})$ also descends to k and so by Corollary 3.27 we have that $\overline{\widehat{X}} \cong \overline{X}$ descends to k, a contradiction. $\square$

Remark 6.3. Here we will sketch a possible alternative approach to proving Theorem 6.1. Unfortunately, we were unable to develop all the details of this approach. First, note that it will suffice to prove that there exists a finite-dimensional D_F-module V such that Z is a closed D-subscheme of $\mathbb{P}(\underline{V})$, because then we would able to apply a similar argument using the Albanese variety to complete the proof. By Proposition 4.46 we have an exact sequence of $\mathcal{D}_{X^\infty}$-module homomorphism

$$\mathcal{D}_{X^\infty} \otimes_{\mathcal{O}_X} \Omega_{X/F} = \Omega_{X^\infty/F} \to \Omega_{X^\infty/\underline{W}^\infty} \to 0$$

where the first module is locally free of rank one over $\mathcal{D}_{X^\infty}$ and the second is a coherent $\mathcal{O}_{X^\infty}$-module. Because $\Omega_{X^\infty/\underline{W}^\infty}$ is supported on Z, this quotient should define a D-scheme morphism ϕ from Z to a 'δ-Quot scheme', a projective D-variety. Using the canonical embedding $X^\infty \hookrightarrow \mathbb{P}(\Gamma(X, \Omega_{X/F})^\vee)^\infty$ it should be possible to prove that ϕ is a closed embedding, which would complete the proof.

Corollary 6.4. *The set $(X^\infty \cap \Sigma)^\delta(F)$ is finite.*

Proof. Let U be the complement of the ramification locus Z in X^∞. Then U is an open subscheme of X^∞ and $\Omega_{X^\infty/\underline{W}^\infty}|_U = 0$. That is, the morphism $U \to \underline{W}^\infty$ is unramified. By the stability of differentials under pullback we know that $\Omega_{X^\infty\cap\Sigma/F}|_{\Sigma\cap U}$ is also zero. But $U \cap \Sigma$ is of finite type over F and so we conclude by Proposition 2.19 that $(U \cap \Sigma)(F)$ is a finite set. By Theorem 6.1 we also know that $(Z \cap \Sigma)^\delta(F)$ is finite and so we conclude that $(X^\infty \cap \Sigma)^\delta(F)$ is finite. $\square$

6.2 The δ-closure of a subgroup

Let G be an algebraic group over F. If Γ is a subgroup of $G(F)$ then each point $g \in \Gamma$ defines a map $\operatorname{Spec} F \to G$. Using the adjunction 4.40 this gives us a morphism of D-schemes $\overline{g} : \operatorname{Spec} F \to G^\infty$. Each $g \in \Gamma$ defines such a morphism, so we have a D-scheme morphism

$$\coprod_{g\in\Gamma} \operatorname{Spec} F \xrightarrow{\overline{g}} G^\infty. \tag{6.5}$$

We define the *δ-closure* Γ^* of Γ to be the reduced closed image of this map in G^∞. Note that Γ^* is a closed D-scheme by Proposition 4.50.

Example 6.5. Here we consider a simple example to illustrate the idea of a δ-closure. Let $G = \mathbb{G}_m$ and let $\Gamma = \langle t \rangle \subset \mathbb{G}_m(\mathbb{C}(t)) = \mathbb{C}(t)^\times$. Then the δ-closure Γ_r^* of Γ in $J^r(\mathbb{G}_m)$ is the Zariski closure of the set

$$T \coloneqq \left\{ \left(t^n, nt^{n-1}, \cdots, \frac{n!}{(n-r)!} t^{n-r} \right) \,\middle|\, n \in \mathbb{Z} \right\}$$

in $J^r(\mathbb{G}_m)$. Suppose we have $(x_0, ..., x_{r-1})$ belonging to this set. Since $x_0 \neq 0$, we have

$$n = \frac{nt^{n-1}}{t^n} = \frac{x_1}{x_0}$$

and so

$$x_{j+1} = \frac{(n-j)}{t} x_j = \frac{x_1 x_j}{x_0} - \frac{j}{t} x_j$$

for all $j \geq 2$ and so all the higher variables may be written in terms of x_0 and x_1. Therefore we have that $\Gamma_r^* \subset J^1(\mathbb{G}_m) \subset J^r(\mathbb{G}_m)$. Since this does not depend on r, by taking a limit we conclude that $\Gamma^* \subset J^1(\mathbb{G}_m) \subset J^\infty(\mathbb{G}_m)$ also and so the δ-closure Γ^* of Γ is of finite type over $\mathbb{C}(t)$. A similar argument applies to any finitely generated subgroup of $\mathbb{C}(t)^\times$.

Now let Λ be the δ-closure of $A(F)$ in A^∞, the reduced closed image of the map

$$\coprod_{A(F)} \operatorname{Spec} F \underset{D}{\rightarrow} A^\infty.$$

Then Λ is a closed D-subscheme of A^∞ by Proposition 4.50 and $A(F) \subset \Lambda^\delta(F)$ by construction.

Lemma 6.6 (Proposition 7.2.4 of [13]). *The scheme Λ is of finite type over F.*

Proof. By the Poincaré complete reducibility theorem (Theorem 19.1 of [52]) we know that the category of abelian varieties up to isogeny is a semi-simple abelian category. Therefore, A is isogenous to a product $A_1 \times \cdots \times A_n$, where each A_i is a *simple* abelian F-variety in that A_i contains no non-trivial abelian subvariety. Therefore, it will suffice to prove this result when A is an abelian variety which contains no non-trivial F-subvariety. We claim that either A has F/k-trace zero (see Definition 3.19) or A descends to k. Indeed, if A does not have F/k-trace zero, then the image of $\mathrm{Tr}_{F/k}(A)_F$ in A is a non-trivial abelian subvariety and so the simplicity of A implies that $\mathrm{Tr}_{F/k}(A)$ is all of A. By Corollary 3.18 we then conclude that A is defined over k.

We will deal with these two cases separately. To begin with, assume that A does not descend to k. Then by Remark 3.21, the group $A(F)$ is finitely-generated. Therefore, the image of the morphism

$$\coprod_{A(F)} \operatorname{Spec} F \rightarrow A^\infty \xrightarrow{\psi} \underline{V}^\infty.$$

is contained within a finite-dimensional subgroup $\underline{W}$ of $\underline{V}^\infty$ (where W is a finite-dimensional F-vector space). Since ψ is of finite type, the inverse image $\psi^{-1}(\underline{W})$ is a subscheme of A^∞ which is of finite type over F and contains Λ. Therefore, Λ is of finite type.

If instead A does descend to k, then by the discussion of Section 5.1 we have a short exact sequence

$$0 \to \operatorname{Lie} A \to A^1 \to A \to 0. \tag{6.6}$$

By Remark 4.51, the inclusions $R \to R[t]/(t^{r+1})$ induce morphisms $A \to A^r$ which are sections of $\pi_r : A^r \to A$. Passing to the limit we also obtain a section $A \to A^\infty$ of $p_A : A^\infty \to A$. Therefore, Sequence 6.6 is split and so we obtain a homomorphism $\psi : A^1 \to \operatorname{Lie} A$. We claim that if we lift a point x from $A_0(k)$ to $A(F)$ and then to $A^\infty(F)$ then it lies in the image of the trivial section $A^1(F) \to A^\infty(F)$. This is true because x comes from $A_0(k)$ and so when we lift x along each exponential morphism $F \to F[t]/(t^{r+1})$ we get the same element as if we had lifted along the inclusion $F \hookrightarrow F[t]/(t^{r+1})$. Let x' be the corresponding element of $A^1(F)$.

By the previous paragraph, $\overline{x}$ gets sent to the image of $\psi(x')$ under the trivial section $\operatorname{Lie} A(F) \to \operatorname{Lie} A^\infty(F)$. But $\psi(x') = 0$ because the sequence 6.6 is exact and so $\overline{x}$ is sent to zero in $\operatorname{Lie} A^\infty(F)$. Therefore, the image of Λ in $\operatorname{Lie} A^\infty$ only depends on $A(F)/A_0(k)$, which is finitely generated by the Lang–Néron Theorem (see Remark 3.21). By a similar argument to the first case, we conclude that Λ is of finite type over F.

$$\begin{tikzcd}
 & \operatorname{Spec} F \arrow[rr, "\overline{x}"] \arrow[dl] \arrow[dr, "x"] & & A^\infty \arrow[r] \arrow[dl] \arrow[d] & \operatorname{Lie} A^\infty \arrow[d] & \\
\operatorname{Spec} k \arrow[dr, "x"] & & A \arrow[r] \arrow[dl] & A^1 \arrow[r, "\psi"] & \operatorname{Lie} A \arrow[r] & 0 \\
 & A_0 & & & &
\end{tikzcd} \tag{6.7}$$

$\square$

Proof of Theorem 1.1. Let $\psi : A^\infty \to \underline{V}^\infty$ be the universal additive δ-character of A guaranteed by Theorem 5.2. Because $\Lambda = A(F)^*$ is an algebraic group so its image under ψ is a closed subgroup of $\underline{V}^\infty$ and so Theorem 4.61 allows us to choose an F-group homomorphism $\lambda : \underline{V}^\infty \to \underline{W}^\infty$ such that $\ker \lambda = \psi(\Lambda)$. Now we have a finite-type group homomorphism

$$\widetilde{\psi} : A^\infty \xrightarrow{\psi} \underline{V}^\infty \xrightarrow{\lambda} \underline{W}^\infty.$$

Let $\Sigma = \ker \widetilde{\psi}$, then by construction we have $\Lambda(F) \subset \Sigma(F)$. We also know that $X(F) = (X^\infty)^\delta(F)$ by Proposition 4.40 and because X embeds in A we have $X(F) \subset A(F) \subset \Lambda^\delta(F) \subset \Sigma^\delta(F)$. Therefore, we have

$$X(F) \subset (X^\infty)^\delta(F) \cap \Lambda^\delta(F) \subset (X^\infty)^\delta(F) \cap \Sigma^\delta(F) = (X^\infty \cap \Sigma)^\delta(F)$$

where the final set is finite by Corollary 6.4. $\square$

Remark 6.7. Theorem 1.1 may be extended to hyperelliptic curves using the Chevalley–Weil Theorem, for details see [13].

Appendix A

The Picard–Fuchs equation

In this appendix we will give a complex analytic approach to the construction of differential characters of elliptic curves, developed by Manin in [44]. The contents of this appendix are not a part of our main argument, and we include this section to demonstrate a more 'concrete' approach to constructing differential characters. Let E be the elliptic curve over $\mathbb{C}(t)$ defined by the Legendre equation $y^2 = x(x-1)(x-t)$. We will construct a character of order two $\Theta : J^2(E) \to \mathbb{G}_a$. To begin with we will discuss the *Picard–Fuchs equation*, following the book [16].

Consider the elliptic curve E_λ over $\mathbb{C}$ defined by the equation $y^2 = x(x-1)(x-\lambda)$. (More specifically E_λ is the projective closure of the curve in $\mathbb{A}^2$ defined by the Legendre equation.) The function $y = \sqrt{x(x-1)(x-\lambda)}$ has two possible holomorphic branches on any simply-connected open subset of the Riemann sphere $\mathbb{CP}^1$ which does not contain $0, 1, \infty$ or λ. The (compact) Riemann surface C_λ defined by this equation is topologically a torus and so we can choose generators γ_1 and γ_2 for the singular homology group $H_1(C_\lambda; \mathbb{Z})$. Concretely, these homology classes are represented by the two loops which generate the fundamental group of $S^1 \times S^1$.

Now consider the holomorphic differential 1-form $\omega = \frac{dx}{y}$. By the de Rham isomorphism 3.9 this corresponds to a class $[\omega]$ in the singular cohomology group $H^1(C_\lambda; \mathbb{C})$. Because γ_1^* and γ_2^* form a basis for $H^1(C_\lambda; \mathbb{Z})$ (and hence also for cohomology with complex coefficients) we can write $[\omega]$ in terms of this basis as,

$$[\omega] = \left(\int_{\gamma_1} \omega \right) \gamma_1^* + \left(\int_{\gamma_2} \omega \right) \gamma_2^*.$$

Here the quantities $\omega_i = \int_{\gamma_i} \omega$ are called the *periods* of ω. The periods of an elliptic curve are important because the Abel-Jacobi map gives us an isomorphism

$$C_\lambda \cong \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2).$$

The two topological generators γ_1 and γ_2 of $H_1(C_\lambda; \mathbb{Z})$ also define generators for $H_1(C_\mu; \mathbb{Z})$ for nearby $\mu \in \mathbb{C}$. Therefore, as λ varies in a small open neighbourhood we can consider the way in which the periods vary as we integrate the differential form ω_λ

over the same homology generators. That is, we define functions $A(\lambda)$ and $B(\lambda)$ by

$$\omega_\lambda = A(\lambda)\gamma_1^* + B(\lambda)\gamma_2^*.$$

We can then differentiate this function with respect to λ to obtain

$$\omega'_\lambda = A'(\lambda)\gamma_1^* + B'(\lambda)\gamma_2^*.$$

Now the two cohomology classes ω_λ and ω'_λ are linearly independent in the two-dimensional vector space $H^1(C_\lambda;\mathbb{C})$ (this is because the integral of $[\omega_\lambda] \smile [\omega'_\lambda]$ is non-zero, see page 14 of [16]), so there exist coefficients $a(\lambda), b(\lambda), c(\lambda) \in \mathbb{C}$ such that

$$a(\lambda)[\omega''_\lambda] + b(\lambda)[\omega'_\lambda] + c(\lambda)[\omega_\lambda] = 0$$

in cohomology. Integrating both sides of this expression with respect to $\int_\xi$ for some 1-cycle ξ and setting $\pi(\lambda) = \int_\xi \omega_\lambda$ we obtain a differential equation

$$a(\lambda)\pi''(\lambda) + b(\lambda)\pi'(\lambda) + c(\lambda)\pi(\lambda) = 0.$$

We would like to determine these coefficients explicitly. Since $\omega = x^{-1/2}(x-1)^{-1/2}(x-\lambda)^{-1/2}$, differentiating with respect to λ gives

$$\omega' = \frac{1}{2}\frac{dx}{\sqrt{x(x-1)(x-\lambda)^3}} = \frac{1}{2(x-\lambda)}\omega$$

and

$$\omega'' = \frac{3}{4}\frac{dx}{\sqrt{x(x-1)(x-\lambda)^5}} = \frac{3}{2(x-\lambda)}\omega'$$

A direct computation using these relations shows that

$$-\frac{1}{2}d\left(\frac{y}{(x-\lambda)^2}\right) = \frac{1}{4}\omega + (2\lambda-1)\omega' + \lambda(\lambda-1)\omega''. \tag{A.1}$$

Equation A.1 gives the identity in cohomology we discussed above. We can integrate both sides of Equation A.1 along the 1-cycle ξ. Swapping the order of integration and differentation and using the fact that the integral of an exact form around a loop is 0 we get,

$$0 = \frac{1}{4}\pi + (2\lambda-1)\pi' + \lambda(\lambda-1)\pi'' \tag{A.2}$$

the so-called *Picard-Fuchs equation.* The solutions of this equation are the periods of our elliptic curve.

Now we replace the complex number λ with a formal variable t and consider the curve E defined by $y^2 = x(x-1)(x-t)$ over $\mathbb{C}(t)$. Furthermore, we replace differentiation with respect to λ by the derivation $\frac{d}{dt}$. Note that because Equation A.1 was derived algebraically, the same equation holds with t replacing λ. Furthermore, *Euler's addition formula* for elliptic integrals states that

$$\int_0^p \omega + \int_0^q \omega = \int_0^{p+q} \omega \tag{A.3}$$

where the sum is taken with respect to the group law on E. In [44], Manin uses the differential equation A.1 of order two along with Euler's addition formula to construct a differential character of E of order 2, $\Theta : J^2(E) \to \mathbb{G}_a$. In coordinates this homomorphism is given by the formula

$$\Theta(x, y, x', y', x'', y'') = \frac{y}{2(x-t)^2} - \frac{d}{dt}\left[2t(t-1)\frac{x'}{y}\right] + 2t(t-1)\frac{x'y'}{y^2}.$$

Finally, we note that an analogue of the Picard-Fuchs equation may be derived in a more general setting using the notion of the *Gauss-Manin connection* on a vector bundle. By differentiating with respect to the Gauss-Manin connection enough times we eventually produce linearly dependent classes in de Rham cohomology. The linear relation between these classes is also called the *Picard-Fuchs equation* and was used by Manin in [44] to construct characters of abelian varieties.

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