MA30287: Mathematics of Planet Earth
Assessed Coursework

Date set: Tuesday 28 March 2023

Submission deadline: 14:00 Friday 28 April 2023

Time required for average submission: Approx. 15 hours per person.

Length: Prepare a typed report consisting of no more than eight sides of A4 including any graphical material and references, but not including the cover page and contents page (if any). Text must be 11pt and in a standard font e.g. Computer Modern or Times. Margins must be at least 2cm. Figures must be at least 7cm $\times$ 7cm. Any material that exceeds eight pages will be ignored. You are encouraged to use LaTeX to type your report (see for example `www.mathcentre.ac.uk/bathmash/LaTeX/index.html`) but Word or similar software is also acceptable. If you work on your own computer or use a cloud service, make sure you regularly save a backup copy of your work to the university system.

Submission: Each group should submit on the MA30287 Moodle page with the following components: (i) signed group coursework coversheet, available on the Moodle page; (ii) the group report in PDF form; (iii) a compressed archive of all `ipynb` files used.

The Python `ipynb` files are not marked and should be seen as supplementary appendices to ensure the reproducibility of the results presented in the main report.

Marks: 25% of the final grade for MA30287.

Groups: This is group coursework. Each member of the group will normally be given the same mark unless there are special circumstances that would make such a distribution inappropriate. One example would be where one member had failed to cooperate with the rest of the group. It is the responsibility of the group to bring any such case to my attention. Please let me know as soon as possible if, for example, a student drops out of your group. The Department of Mathematical Sciences Guidelines for Group Coursework are available on the MA30287 Moodle page. Before starting on the project itself you may like to think individually and as a group about how to ensure you work together effectively. You may find some useful advice here `www.birmingham.ac.uk/schools/metallurgy-materials/about/cases/group-work/tips.aspx`

Plagiarism: There are severe penalties for unfair practices such as plagiarism in assessed coursework. See the UG Programmes Handbook, available on the Moodle Maths Student Zone.

Extensions: Extensions may be granted by the DoS Team if there are valid mitigating circumstances affecting your ability to meet the deadline. Please let me know as soon as possible of any such circumstances. If no extension has been granted and a piece of coursework is submitted after the deadline it will be assessed at a maximum mark of 40%. If it is submitted more than five days after the deadline it will normally receive a mark of zero.

Analysis of a latitude-dependent EBM

A latitude-dependent energy balance model (EBM) is given by the following equation for a temperature, $T = T(y,t)$:

$$C\frac{\partial T}{\partial t} = E_{\text{in}} - E_{\text{out}} + E_{\text{transport}}, \qquad t \geq 0,\ y \in [0,1], \tag{1a}$$

where $y = \sin\varphi$ with $\varphi = 0$ at the equator and $\varphi = \pi/2$ at the North Pole. The energy terms are defined as:

$$E_{\text{in}}(y,t) = Qs(y)(1 - a(y)), \tag{1b}$$
$$E_{\text{out}}(y,t) = A + BT, \tag{1c}$$
$$E_{\text{transport}}(y,t) = k(\bar{T} - T). \tag{1d}$$

We assume there exists a single ice line at $y = y_s$ where the temperature reaches T_C. The default parameters used are

$$Q = 342\text{W m}^{-2}, \quad A = 202\text{W m}^{-2}, \quad B = 1.9\text{W m}^{-2}\,{}^\circ\text{C}^{-1},$$
$$T_C = -10^\circ\text{C}, \quad k = 1.6B,$$
$$s(y) = 1 - \frac{0.482}{2}(3y^2 - 1), \qquad a(y) = \begin{cases} a_i = 0.62 & y > y_s, \\ a_w = 0.32 & y < y_s, \\ 0.5(a_w + a_i) & y = y_s. \end{cases}$$

The definitions and interpretations of the above components can be found in your lecture notes.

A complicated bifurcation space. Solutions of the above model can be considered as points in a high-dimensional bifurcation space, where perturbations of each of the parameters such as Q, A, B, k, ... can impact the resultant solutions (and predicted climate). For example, the following bifurcation diagrams show how the ice line y_s (left) or mean temperature $\bar{T}$ (right) is affected by changes in the solar constant, Q.

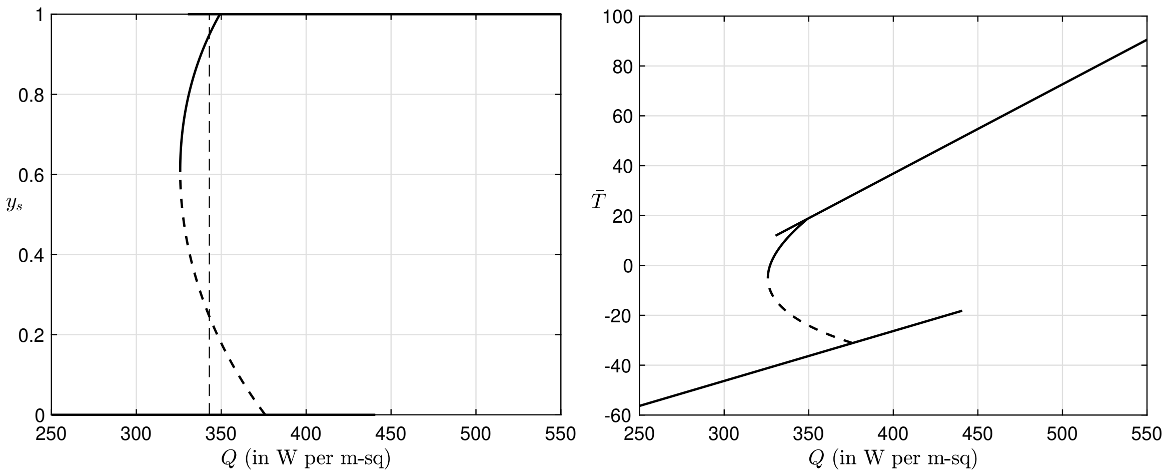

Figure 1: Examples of bifurcation diagrams

Your challenge in the following exercises is to investigate how the system (1) responds to other parameter choices, and to discuss the resultant implications on the climate. Your report should investigate the model using a combination of numerical methods (solutions of differential equations, solutions of nonlinear problems, etc.) and analytical methods (qualitative analysis of ODEs, asymptotic analysis, etc.). Use numerical computations to probe and form an initial study; then where appropriate, use analytical methods, like asymptotic analysis, in order to offer deeper insight.

Investigations 1–4 relate to the steady-state solutions.

Investigation 1. (Steady state) Effect of parameters on the ice-age state.

(a) Assume that the planet begins in a partially-iced state with an ice line near the North Pole. Using the default values of albedo, k, A, and B, determine what decrease in the solar constant, Q, is required just to glaciate the Earth completely (ice edge at the equator). Once the Earth is fully glaciated, begin to increase the solar constant—how much of an increase is required before the ice retreats from the equator?

(b) We are interested in understanding how the critical values of Q, derived above, change in response to other values of A, B, k, and the albedo formulation. Investigate the solution space and summarise your findings.

Investigation 2. (Steady state) On the energy transport.

(a) Various authors have suggested different values for the transport coefficient k. For example, Budyko (1969) used $k = 3.91\mathrm{W\,m^{-2}\,{}^\circ C^{-1}}$ while Warren and Schneider (1979) used $k = 3.74\mathrm{W\,m^{-2}\,{}^\circ C^{-1}}$. Explore how sensitive the model's climate prediction is to the particular value of k.

(b) Investigate the climate that results when using very small ($k \ll 1$) or very large ($k \gg 1$) values of k. How sensitive are these different climates to changes in the solar constant? Can the behaviour be described intuitively based on physical principles?

Investigation 3. (Steady state) On the solar constant.

(a) Observations show that land will be totally snow-covered during winter for an annual mean surface temperature of 0°C, and oceans totally ice-covered all year at −13°C. Alter the 'critical' temperature, T_C, and investigate the change in the climate and the climatic sensitivity to changing the solar constant about its default value of $Q = 342\mathrm{W\,m^{-2}}$.

(b) The albedo over snow-covered areas can vary within the limits of 0.5–0.8 depending on the vegetation type, cloud cover, and snow/ice condition. Investigate the sensitivity of the simulated climate to changing snow/ice albedo.

Investigation 4. (Steady state) On the outgoing radiation

(a) There have been many suggestions for the values of the constants A and B determining the longwave outgoing radiation from the planet—some estimates are dependent on modelling the amount of cloud present. Budyko (1969) originally used $A = 202\mathrm{W\,m^{-2}}$ and $B = 1.45\mathrm{W\,m^{-2}\,{}^\circ C^{-1}}$. Cess (1976) suggested $A = 212\mathrm{W\,m^{-2}}$ and $B = 1.6\mathrm{W\,m^{-2}\,{}^\circ C^{-1}}$. How do these different constants influence the climate and its sensitivity?

(b) Holding A constant, just vary B and investigate the effect on the climate. What does a variation of B correspond to physically?

Investigation 5. (Steady state and unsteady; challenging)

(a) By modifying the model, investigate the possibility of non-symmetric steady states about the equator, where $y \in [-1, 1]$.

(b) By modifying the model, investigate the use of a smooth, continuous albedo function, a, instead of the discontinuous one given. Does the use of a continuous albedo significantly change the typical steady-state solutions derived in the lectures?

(c) Return to Investigation 2 and consider the **unsteady** solutions. What are the stability properties of the solution(s) for different values of k?

Marking scheme - total 100 marks

Reports should begin with a summary introduction and conclusion that ties together all of the five investigations.

A *first-class* report will demonstrate these qualities:

1. **Presentation and overall quality.** Elegantly presented and text free of errors; figures and diagrams clear, informative, and appropriately annotated; well-conceived structure with excellent use of layout; good bibliography (where appropriate); writing clear, concise, elegant, and concise; tone appropriate for non-specialist mathematicians and scientists. Introduction is coherent, provides insightful background, discusses sources and outlines structure. 20 marks.

2. **Numerical and analytical analysis.** Systematic, accurate, rigorous, and thorough. The analysis goes beyond a routine treatment. Numerical computations are supplemented with analytical insights. The solution space and mathematical structure is investigated. A broad range of different tools and approaches are used appropriately. 50 marks.

3. **Synthesis and interpretation.** Accurate and insightful. Demonstrates and communicates a deep understanding of mathematical, scientific, and societal implications. Concise, engaging presentation accessible for non-specialist mathematicians and scientists. Results are well-summarised and placed in context. Good discussion of model limitations. 30 marks.