MA20222 Coursework

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1 Section A - Theory

A1. Show that $p_1(x) = 2x$.

$$\begin{aligned} p_1(x) &= \frac{1}{sin(\theta)} sin(2\theta) \\ &= \frac{2sin(\theta)cos(\theta)}{sin(\theta)} \\ &= 2cos(\theta) \end{aligned}$$

As $\theta = cos^{-1}(x) \implies p_1(x) = 2cos(cos^{-1}(x)) = 2x$

By using a trigonometric identity, show that

$$p_n(x) = 2xp_{n-1}(x) - p_{n-2}(x), n = 2, 3, ...$$

$$\begin{aligned} p_n(x) &= 2xp_{n-1}(x) - p_{n-2}(x) \\ &= 2x \frac{1}{sin(\theta)} sin(n\theta) - \frac{1}{sin(\theta)} sin(n\theta - \theta) \end{aligned}$$

Using $sin(n\theta - \theta) = sin(n\theta)cos(\theta) - cos(n\theta)sin(\theta)$ and $x = cos(\theta)$ gives

$$\begin{aligned} p_n(x) &= 2cos(\theta) \frac{1}{sin(\theta)} sin(n\theta) - \frac{1}{sin(\theta)} [(sin(n\theta)cos(\theta) - cos(n\theta)sin(\theta)] \\ &= \frac{1}{sin(\theta)} cos(\theta)sin(n\theta) + \frac{1}{sin(\theta)} cos(n\theta)sin(\theta) \\ &= \frac{1}{sin(\theta)} [cos(\theta)sin(n\theta) + cos(n\theta)sin(\theta)] \\ &= \frac{1}{sin(\theta)} sin(n\theta + \theta) \\ &= \frac{1}{sin(\theta)} sin((n+1)\theta) \end{aligned}$$

Hence show that $p_n(x)$ is a polynomial of degree n and that $p_0, ..., p_n$ form a basis for $\mathcal{P}_n$

It is clear to see from the results given that $p_0(x) = 1 \in \mathcal{P}_0$ and $p_1(x) = 2x \in \mathcal{P}_1$. By using induction on $p_n(x) = 2xp_{n-1}(x) - p_{n-2}$ we can show that $p_n(x) \in \mathcal{P}_n$.

Base Case: $p_2(x) = 2xp_1(x) - p_0(x) = 4x^2 - 1 \in \mathcal{P}_2$

Inductive Step: Assume $p_i(x) \in \mathcal{P}_i \ \forall \ 0 \leq i \leq n-1$.

We have $p_n(x) = 2xp_{n-1}(x) - p_{n-2}(x)$. As $p_{n-1}(x) \in \mathcal{P}_{n-1} \implies 2x \times p_{n-1}(x) \in \mathcal{P}_n$. And this implies that $p_n(x) \in \mathcal{P}_n$, so we have proved the claim by induction.

As $p_n(x)$ is a polynomial of degree n $\implies$ the highest power of x is x^n. Thus, $p_N(x)$ has highest power of x as x^N for $N = 0, ..., n$ and therefore each $p_N(x)$ is going to be linearly independent from one another, explicitly $p_0, ..., p_n$ are linearly independent. $p_0, ..., p_n$ will also then span $\mathcal{P}_n$ as any linear combination of $p_0, ..., p_n$ will include every power of x^N for $N = 0, ..., n$ covering all possible elements in the vector space of polynomials up to and including degree n. As $p_0, ..., p_n$ are linearly independent and span $\mathcal{P}_n \implies p_0, ..., p_n$ is a basis for $\mathcal{P}_n$

A2. Let $x_1, ..., x_n$ denote the roots of $p_n(x)$. Show that the matrix below is non-singular.

$$J_n = \begin{pmatrix} p_0(x_1) & p_0(x_2) & \cdots & p_0(x_n) \\ p_1(x_1) & p_1(x_2) & \cdots & p_1(x_n) \\ p_2(x_1) & p_2(x_2) & \cdots & p_2(x_n) \\ \cdots & & & \vdots \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \cdots & p_{n-1}(x_n) \end{pmatrix}$$

We prove this by contradiction:

Assume $\exists$ a $\mathbf{c} = (c_0, ..., c_{n-1})^T \in \mathbb{R}$ with $\mathbf{c} \neq 0$ s.t. $\mathbf{c}^T J_n = \mathbf{0}$ That is, J_n is a singular matrix. On multiplying $\mathbf{c}^T J_n$ we get the matrix below:

$$\mathbf{c}^T J_n = \begin{pmatrix} c_0p_0(x_1) & c_0p_0(x_2) & \cdots & c_0p_0(x_n) \\ c_1p_1(x_1) & c_1p_1(x_2) & \cdots & c_1p_1(x_n) \\ c_2p_2(x_1) & c_2p_2(x_2) & \cdots & c_2p_2(x_n) \\ \cdots & & & \vdots \\ c_{n-1}p_{n-1}(x_1) & c_{n-1}p_{n-1}(x_2) & \cdots & c_{n-1}p_{n-1}(x_n) \end{pmatrix}$$

We can write every column of this matrix in the form of the polynomial

$$q(x) = \sum_{i=0}^{n-1} c_i p_i(x)$$

with $degree(p) < n$. This polynomial has n distinct roots, $x_1, ..., x_n$ and so must vanish. Since the polynomials $p_i(.)$ are linearly independent, then if $q(x) = \sum_{i=0}^{n-1} c_i p_i(x) = 0 \implies c_i = 0 \ \forall \ i$ so $\mathbf{c} = 0$. This contradicts the assumption that $\mathbf{c} \neq 0$ and so we have $\exists$ a $\mathbf{c} = (c_0, ..., c_{n-1})^T \in \mathbb{R}$ with $c_i = 0 \quad \forall i = 0, ..., n-1$ s.t. $\mathbf{c}^T J_n = \mathbf{0}$. That is J_n is non-singular.

Hence, show that there exist $w_1, ..., w_n$ such that

$$\sum_{i=1}^{n} p_k(x_i) w_i = \begin{cases} \int_{-1}^{1} \sqrt{1-x^2} \text{ dx} & \text{if k} = 0 \\ 0 & \text{if k} = 1, ..., n-1 \end{cases}$$

Setting $g(x) = p_k(x)$ in the equation given in the introduction information gives

$$\sum_{i=1}^{n} p_k(x_i)w_i = \int_{-1}^{1} p_k(x)\sqrt{1-x^2}\ \mathrm{dx}$$

and as $p_0(x) = 1$ we can also write this as

$$\sum_{i=1}^{n} p_k(x_i)w_i = \int_{-1}^{1} p_0(x)p_k(x)\sqrt{1-x^2}\ \mathrm{dx}$$

For $k = 0$ we have

$$\begin{aligned}\sum_{i=1}^{n} p_0(x_i)w_i &= \int_{-1}^{1} p_0(x)p_0(x)\sqrt{1-x^2}\ \mathrm{dx} \\ &= \int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{dx}\end{aligned}$$

For $k = 1, ..., n-1$ we have by A3 that

$$\begin{aligned}\sum_{i=1}^{n} p_k(x_i)w_i &= \int_{-1}^{1} p_0(x)p_k(x)\sqrt{1-x^2}\ \mathrm{dx} \\ &= 0\end{aligned}$$

A3. By making a substitution, show that

$$\int_{-1}^{1} p_n(x)p_m(x)\sqrt{1-x^2}\ \mathrm{dx} = 0 \quad \text{for} \quad n \neq m$$

We will use the substitution $x = cos(\theta)$. This means that the limits of the integral change and we get:

- $1 = cos(\theta) \implies \theta = arccos(1) = 0$
- $-1 = cos(\theta) \implies \theta = arccos(-1) = \pi$

$$\begin{aligned}&\int_{-1}^{1} p_n(x)p_m(x)\sqrt{1-x^2}\ \mathrm{dx} \\ &= \int_{\pi}^{0} \frac{1}{sin(\theta)}sin((n+1)\theta)\frac{1}{sin(\theta)}sin((m+1)\theta)sin(\theta) \times -sin(\theta)\ \mathrm{d}\theta \\ &= -\int_{\pi}^{0} sin((n+1)\theta)sin((m+1)\theta)\ \mathrm{d}\theta \\ &= \int_{0}^{\pi} sin((n+1)\theta)sin((m+1)\theta)\ \mathrm{d}\theta\end{aligned}$$

We now use the product to sum formula:

$$sin(x)sin(y) = \frac{1}{2}\left(cos(y-x) - cos(y+x)\right)$$

$$\int_{-1}^{1} p_n(x)p_m(x)\sqrt{1-x^2}\ \mathrm{dx}$$
$$= \int_0^{\pi} \frac{1}{2}\left(cos((m+1-(n+1))\theta) - cos((m+1+n+1)\theta)\right)\ \mathrm{d}\theta$$
$$= \frac{1}{2}\int_0^{\pi} cos((m-n)\theta)\ \mathrm{d}\theta - \frac{1}{2}\int_0^{\pi} cos((m+n+2)\theta)\ \mathrm{d}\theta$$
$$= \frac{1}{2}\left[\frac{1}{m-n}sin((m-n)\theta)\right]_0^{\pi} - \frac{1}{2}\left[\frac{1}{m+n+2}sin((m+n+2)\theta)\right]_0^{\pi}$$
$$= \frac{1}{2}\frac{1}{m-n}\Bigg[sin((m-n)\pi) - sin((m-n)0)\Bigg]$$
$$- \frac{1}{2}\frac{1}{m+n+2}\Bigg[sin((m+n+2)\pi) - sin((m+n+2)0)\Bigg]$$
$$= \frac{1}{2}\frac{1}{m-n}[0-0] - \frac{1}{2}\frac{1}{m+n+2}[0-0] = 0$$

This is because $n, m \in \mathbb{N}$ which $\implies$ $m-n \in \mathbb{N}$ and $m+n+2 \in \mathbb{N}$ so:

$sin((n-m)\pi) = sin((n-m)0) = sin((m+n+2)\pi) = sin((m+n+2)0) = 0$

A4. By using the basis $p_0, ..., p_n$ prove that

$$\int_{-1}^{1} p(x)\sqrt{1-x^2}\ \mathrm{dx} = \sum_{i=1}^{n} p(x_i)w_i \quad \forall p \in \mathcal{P}_n$$

for the x_i and w_i defined.

Consider an arbitrary polynomial $p \in \mathcal{P}_n$. We can write $p(x)$ in terms of the basis. So we get:

$$p(x) = \sum_{i=0}^{n} a_i p_i(x)$$

Thus, the LHS of the equation becomes

$$\int_{-1}^{1} \sum_{i=0}^{n} a_i p_i(x)\sqrt{1-x^2}\ \mathrm{dx}$$

From the introduction paragraph, we are told that

$$\int_{-1}^{1} g(x)\sqrt{1-x^2}\ \mathrm{dx} \approx \sum_{i=1}^{n} g(x_i)w_i$$

for integrals with a weight function $\sqrt{1-x^2}$, for a smooth function $g : [-1,1] \to \mathbb{R}$, using weights w_i and nodes $x_i \in [-1,1]$.

It is clear to see that the $\sum_{i=1}^{n} w_i = \int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{dx}$. As $p_i(x)$ are all polynomials of degree i, they are smooth functions that take inputs $x \in [-1,1]$ and map them to $\mathbb{R}$. And each $p_i(x)\ \forall\ 1 \leq i \leq n$ is being evaluated at the root $x_1, ..., x_n$ of $p_n(x)$ respectively.

So we have,

$$\int_{-1}^{1} \sum_{i=0}^{n} a_i p_i(x)\sqrt{1-x^2}\ \mathrm{dx} = \sum_{i=1}^{n} w_i \sum_{i=0}^{n} a_i p_i(x_i)$$

which is equivalent to

$$\sum_{i=1}^{n} w_i p(x_i)$$

when not writing p in terms of a linear combination of the basis. Thus, the equation holds $\forall\ p \in \mathcal{P}_n$.

Now show that the relationship also holds for $p \in \mathcal{P}_{2n-1}$ to show the quadrature rule has degree of precision at least $2n-1$.

Let $p \in \mathcal{P}_{2n-1}$.

Using the polynomial remainder theorem we get:

$$p(x) = q(x)p_n(x) + r(x)$$

where $q(x), r(x) \in \mathcal{P}_{n-1}$ as we are dividing $p(x)$ by $p_n(x) \in \mathcal{P}_n$. We can express $q(x)$ and $r(x)$ respectively in the form:

$$q(x) = \sum_{i=0}^{n-1} a_i p_i(x)$$

$$r(x) = \sum_{i=0}^{n-1} b_i p_i(x)$$

We can express the LHS as

$$\int_{-1}^{1} p(x)\sqrt{1-x^2}\ \mathrm{dx} = \int_{-1}^{1} q(x)p_n(x)\sqrt{1-x^2}\ \mathrm{dx} + \int_{-1}^{1} r(x)\sqrt{1-x^2}\ \mathrm{dx}$$

Since $p_0(x) = 1$ and writing $q(x)$ in terms of the basis gives us

$$\int_{-1}^{1} p(x)\sqrt{1-x^2}\ \mathrm{dx} = \int_{-1}^{1} \sum_{i=0}^{n-1} a_i p_i(x) p_n(x)\sqrt{1-x^2}\ \mathrm{dx} + \int_{-1}^{1} p_0(x)r(x)\sqrt{1-x^2}\ \mathrm{dx}$$

By A3,

$$\int_{-1}^{1} \sum_{i=0}^{n-1} a_i p_i(x) p_n(x)\sqrt{1-x^2}\ \mathrm{dx} = 0$$

and

$$\begin{aligned}
\int_{-1}^{1} p_0(x)r(x)\sqrt{1-x^2}\ \mathrm{dx} &= \int_{-1}^{1} p_0(x)\sum_{i=0}^{n-1} b_i p_i(x)\sqrt{1-x^2}\ \mathrm{dx} \\
&= \int_{-1}^{1} b_0 p_0(x)p_0(x)\sqrt{1-x^2}\ \mathrm{dx} \\
&= b_0 \int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{dx}
\end{aligned}$$

Now observing the RHS we have:

$$\begin{aligned}
\sum_{i=1}^{n} w_i p(x_i) &= \sum_{i=1}^{n} w_i \Big(q(x_i)p_n(x_i) + r(x_i) \Big) \\
&= \sum_{i=1}^{n} w_i q(x_i) p_n(x_i) + \sum_{i=1}^{n} w_i r(x_i) \\
&= \sum_{i=1}^{n} w_i p_n(x_i) \sum_{k=0}^{n-1} a_k p_k(x_i) + \sum_{i=1}^{n} w_i \sum_{k=0}^{n-1} b_k p_k(x_i)
\end{aligned}$$

As $x_1, .., x_n$ are roots of $p_n(x) \implies p_n(x_i) = 0$ for $i = 1, ..., n$ and so

$$\sum_{i=1}^{n} w_i p_n(x_i) \sum_{k=0}^{n-1} a_k p_k(x_i) = 0$$

so we have

$$\sum_{i=1}^{n} w_i p(x_i) = \sum_{i=1}^{n} w_i \sum_{k=0}^{n-1} b_k p_k(x_i)$$

From A2, we know that

$$\sum_{i=1}^{n} p_k(x_i) w_i = \begin{cases} \int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{dx} & \text{if k} = 0 \\ 0 & \text{if k} = 1, ..., n-1 \end{cases}$$

so we end up with

$$\sum_{i=1}^{n} w_i p(x_i) = b_0 \sum_{i=1}^{n} p_0(x_i) w_i = b_0 \int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{dx}$$

Hence,

$$\int_{-1}^{1} p(x)\sqrt{1-x^2}\ \mathrm{dx} = \sum_{i=1}^{n} p(x_i) w_i \quad \forall p \in \mathcal{P}_{2n-1}$$

and the quadrature rule holds for $p \in \mathcal{P}_{2n-1}$.

A5. Show that $q_n = p_n$ for $n = 1, 2, ...$

We are given that

$$\begin{aligned} q_n(x) &= 2^n det(xI - A_n) \\ &= det(2(xI - A_n)) \end{aligned}$$

The matrix $Q_n = 2(xI - A_n)$ is: $\begin{pmatrix} 2\mathrm{x} & -1 & 0 & \cdots & & \\ -1 & 2\mathrm{x} & -1 & \ddots & & \\ 0 & -1 & 2\mathrm{x} & -1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & & -1 \\ & & & & -1 & 2\mathrm{x} \end{pmatrix}$

We calculate the determinant of this matrix by expanding along the first row and then by the second column. This gives us:

$$\begin{aligned} q_n(x) = det(Q_n) &= 2x det(Q_{n-1}) - (-1)(-1) det(Q_{n-2}) \\ &= 2x det(2(xI - A_{n-1})) - 1 det(2(xI - A_{n-2}) \\ &= 2x q_{n-1}(x) - q_{n-2}(x) \end{aligned}$$

and thus $\quad p_n(x) = q_n(x)$

Hence, show that eigenvalues of A_n equal the quadrature nodes x_i

The eigenvalues of A_n are λ where $0 = det(A_n - \lambda I)$

Multiplying both sides by $(-2)^n$ gives

$$0 = (-2)^n det(A_n - \lambda I) = det(-2(A_n - \lambda I)$$

$$-2(A_n - \lambda I) = \begin{pmatrix} 2\lambda & -1 & 0 & \cdots & & \\ -1 & 2\lambda & -1 & \ddots & & \\ 0 & -1 & 2\lambda & -1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & & -1 \\ & & & & -1 & 2\lambda \end{pmatrix}$$

This matrix is of the exact form as $Q_n(x)$ stated earlier in the question where $\lambda = x$. Thus each eigenvalue $\lambda_i = x_i$ respectively, which are the quadrature nodes.

A6. Show that $\mathbf{v}^i$ is an eigenvector of A_n corresponding to the eigenvalue x_i. We need to show that $A_n\mathbf{v}^i = x_i\mathbf{v}^i$

$$A_n\mathbf{v}^i = \begin{pmatrix} 0 & 1/2 & 0 & \cdots & & \\ 1/2 & 0 & 1/2 & \ddots & & \\ 0 & 1/2 & 0 & 1/2 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & & 1/2 \\ & & & & 1/2 & 0 \end{pmatrix} \begin{pmatrix} p_0(x_i) \\ p_1(x_i) \\ p_2(x_i) \\ \vdots \\ \vdots \\ p_{n-1}(x_i) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}p_1(x_i) \\ \frac{1}{2}p_0(x_i) + \frac{1}{2}p_2(x_i) \\ \frac{1}{2}p_1(x_i) + \frac{1}{2}p_3(x_i) \\ \vdots \\ \vdots \\ \frac{1}{2}p_{n-2}(x_i) \end{pmatrix} = \begin{pmatrix} x_i \\ \frac{1}{2} + \frac{1}{2}\Big(2x_ip_1(x_i) - p_0(x_i)\Big) \\ \frac{1}{2}p_1(x_i) + \frac{1}{2}\Big(2x_ip_2(x_i) - p_1(x_i)\Big) \\ \vdots \\ \vdots \\ \frac{1}{2}\Big(2x_ip_{n-1}(x_i) - p_n(x_i)\Big) \end{pmatrix}$$

As x_i is a root of $p_n \implies p_n(x_i) = 0$
And as $p_0(x_i) = 1 \implies x_i = p_0(x_i)x_i$

$$A_n\mathbf{v}^i = \begin{pmatrix} p_0(x_i)x_i \\ p_1(x_i)x_i \\ p_2(x_i)x_i \\ \vdots \\ \vdots \\ p_{n-1}(x_i)x_i \end{pmatrix} = x_i\mathbf{v}^i$$

Hence, using the fact that eigenvectors of a symmetric matrix are orthogonal, show that the quadrature weight satisfy:

$$w_i = \frac{1}{2}\pi\frac{1}{\parallel \mathbf{v}^i \parallel}(v_1^i)^2$$

We can summarise the result from A2 in the matrix below:

$$\begin{pmatrix} p_0(x_1) & p_0(x_2) & \cdots & p_0(x_n) \\ p_1(x_1) & p_1(x_2) & \cdots & p_1(x_n) \\ \vdots & \vdots & & \vdots \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \cdots & p_{n-1}(x_n) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{dx} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Using the fact that eigenvectors of a symmetric matrix are orthogonal, we know that when we multiply any column by its transpose we get

$$((\mathbf{v}^i)^T)\mathbf{v}^i w_i = \int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{dx}$$

and as $\mathbf{v}_1^i = p_0(x_i) = 1\ \forall i$ we can thus write

$$((\mathbf{v}^i)^T)\mathbf{v}^i w_i = (\mathbf{v}_1^i)^2 \int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{dx}$$

We know that

$$\int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{dx} = \frac{\pi}{2}$$

and

$$\begin{aligned} (\mathbf{v}^i)^T\mathbf{v}^i &= (p_0(x_i), ..., p_{n-1}(x_i)) \begin{pmatrix} p_0(x_i) \\ \vdots \\ p_{n-1}(x_i) \end{pmatrix} \\ &= \sum_{j=0}^{n-1} (p_j(x_i))^2 \\ &= \parallel (\mathbf{v}^i) \parallel^2 \end{aligned}$$

Thus

$$\begin{aligned} ((\mathbf{v}^i)^T)\mathbf{v}^i w_i &= (\mathbf{v}_1^i)^2 \int_{-1}^{1} \sqrt{1-x^2}\ \mathrm{dx} \\ w_i &= \frac{1}{((\mathbf{v}^i)^T)\mathbf{v}^i} \frac{\pi}{2} (\mathbf{v}_1^i)^2 \\ &= \frac{1}{2}\pi \frac{1}{\parallel (\mathbf{v}^i) \parallel^2} (\mathbf{v}_1^i)^2 \end{aligned}$$

2 Section B - Computing

B1. Write a MATLAB code to compute the quadrature nodes $x_1, ..., x_n$ and weights $w_1, .., w_n$ for the quadrature rule developed in Section A.

```
function [x,w]=getquad(n)
% This function returns a vector of the quadrature nodes x and
    weights w of
% dimension n
% Creating the matrix An
An=diag(0.5*ones(1,n-1),1)+diag(0.5*ones(1,n-1),-1);
% Working out the Eigenvectors and Eigenvalues of An
[Eigenvector,Eigenvalue]=eig(An,'vector');
x=zeros(1,n);
```

```
% Transposing the Eigenvalue into a row vector
Eigenvalue=transpose(Eigenvalue);
for i=1:n
    % Working out the quadrature nodes
    x(i)=Eigenvalue(i);
end
w=zeros(1,n);
for j=1:n
    % Working out the weights
    w(j)=(pi/2)*(1/(sum(Eigenvector(:,j).^2)))*(Eigenvector(1,j))
        ^2;
end
end
```

B2. Write a routine to evaluate the quadrature for a given function g

```
function out=myquad(g,x,w)
% This function evaluates the quadrature for a given function g i.
    e. it
% evaluates the sum_{i=1,...,n}w_ig(x_i)
if length(x)~=length(w)
    % Checking to see if the number of quadrature nodes = number
        of weights
    error('There is not the same number of quadrature nodes as
        weights\n%s','so we cannot work out an approximation for
        the integral')
else
    out=sum(g(x).*w);
end
end
```

Creating code to verify that the degree of precision is $2n-1$ for $n=10$

```
% Setting a zeros vector
quadvec=zeros(1,21);
for i=1:21
    % Working out the quadrature nodes and weights when n=10
    [x,w]=getquad(10);
    % Using the quadrature rule to work out the output when x^(i
        -1) for
    % i=1,...,21
    quadvec(i)=myquad(@(x) x.^(i-1),x,w);
end
nvec=0:20;
intvec=1:21;
for j=1:21
    % Working out the actual integral for x^(j-1)sqrt(1-x^2) when
    % j=1,..,21
    syms x
    intvec(j)=int(x^(j-1)*sqrt(1-x*x),-1,1);
end
% Creating a table that stores quadrature rule, the actual
    integral and
% the error when x^(i-1) for i=1,...,21
T1=table(transpose(nvec),transpose(quadvec),transpose(intvec),
    transpose(intvec)-transpose(quadvec));
T1=renamevars(T1,'Var1','x^i');
T1=renamevars(T1,'Var2','Quadrature Rule');
T1=renamevars(T1,'Var3','Actual Integral');
T1=renamevars(T1,'Var4','Error');
T1
```

```
    >> ScriptforB2

T1 =
```

21×4 table

x^i	Quadrature Rule	Actual Integral	Error
0	1.5708	1.5708	2.2204e-16
1	-1.3878e-17	0	1.3878e-17
2	0.3927	0.3927	3.3307e-16
3	2.7756e-17	0	-2.7756e-17
4	0.19635	0.19635	3.3307e-16
5	1.3878e-17	0	-1.3878e-17
6	0.12272	0.12272	3.0531e-16
7	3.4694e-18	0	-3.4694e-18
8	0.085903	0.085903	2.6368e-16
9	-1.7347e-18	0	1.7347e-18
10	0.064427	0.064427	2.498e-16
11	-1.2143e-17	0	1.2143e-17
12	0.050621	0.050621	2.0817e-16
13	-1.9082e-17	0	1.9082e-17
14	0.04113	0.04113	1.7347e-16
15	-2.2551e-17	0	2.2551e-17
16	0.034275	0.034275	1.5959e-16
17	-2.7756e-17	0	2.7756e-17
18	0.029134	0.029134	1.3184e-16
19	-3.1225e-17	0	3.1225e-17
20	0.025159	0.025161	1.498e-06

Due to floating point precision we don't see exactly zero for the error as MATLAB works with a precision of 16 significant figures. To check numerically whether something is zero, set a tolerance of 1×10^{-14}. With this in mind, the error for x^i when $i = 1, ..., 19$ is thus zero. And as the error for $x^{20} = 1.498e-06 \neq 0$ we thus have verified that the degree of precision (which is the highest power of x^i s.t. $Error(x^i) \neq 0$) is 19 which is $2(10) - 1 = 19$

B3. Use the built-in MATLAB routines to find reference values for

$$\int_{-1}^{1} g(x)\sqrt{1-x^2} \quad \mathrm{dx}$$

in the case $g_1(x) = exp(x)$ and $g_2(x) = xsin(x)$. We use these values in the next question to evaluate error.

```
syms x
% Working out the actual integral with g_1
I1=int(exp(x)*sqrt(1-x*x),-1,1);
I1=eval(I1);
% Working out the actual integral with g_2
I2=int(x*sin(x)*sqrt(1-x*x),-1,1);
I2=eval(I2);
% MATLAB cannot give any useful information so we will have to use
    it's own
% quadrature rule to evaluate the integral with g_2
% Using MATLAB's quadrature rule to evaluate the integral
tol=1e-10; % defining the tolerance or accuracy
Iq2 = integral(@(x)x.*sin(x).*sqrt(1-x.^2),-1,1,'AbsTol',tol);
```

```
>> ScriptforB3

I1 =

    1.7755


I2 =

int(x*sin(x)*(1 - x^2)^(1/2), x, -1, 1)


Iq2 =

    0.3610
```

B4. Using the quadrature method developed in **B2.**, evaluate

$$\int_{-1}^{1} g_i(x)\sqrt{1-x^2} \quad \mathrm{dx} \qquad \text{for } i = 1, 2$$

and give a plot of the error against n for $n = 1, 2, ..., 20$.

```
quadvec1=zeros(1,20);
quadvec2=zeros(1,20);
syms y
% Working out the actual integral with g_1
I1=int(exp(y)*sqrt(1-y*y),-1,1);
% Working out the actual integral with g_2
I2=int(y*sin(y)*sqrt(1-y*y),-1,1);
errorvec1=zeros(1,20);
errorvec2=zeros(1,20);
for i=1:20
    % Finding the quadrature nodes and the weights
    [x,w]=getquad(i);
    % Using the quadrature rule with g_1
    quadvec1(i)=myquad(@(x) exp(x),x,w);
    % Using the quadrature rule with g_2
    quadvec2(i)=myquad(@(x) x.*sin(x),x,w);
    % Working out the error between the actual integral and the
    % quadrature rules
    errorvec1(i)=abs(quadvec1(i)-I1);
    errorvec2(i)=abs(quadvec2(i)-I2);
end
% Plotting a graph of the error
semilogy(1:20,errorvec1,'g')
hold on
semilogy(1:20,errorvec2,'b')
title1=title('Plot of the Error for each Quadrature rule');
leg1=legend('Error with $g_1(x)=exp(x)$','Error with $g_2(x)=xsin(
    x)$');
set(leg1,'Interpreter','latex');
set(title1,'Interpreter','latex');
```

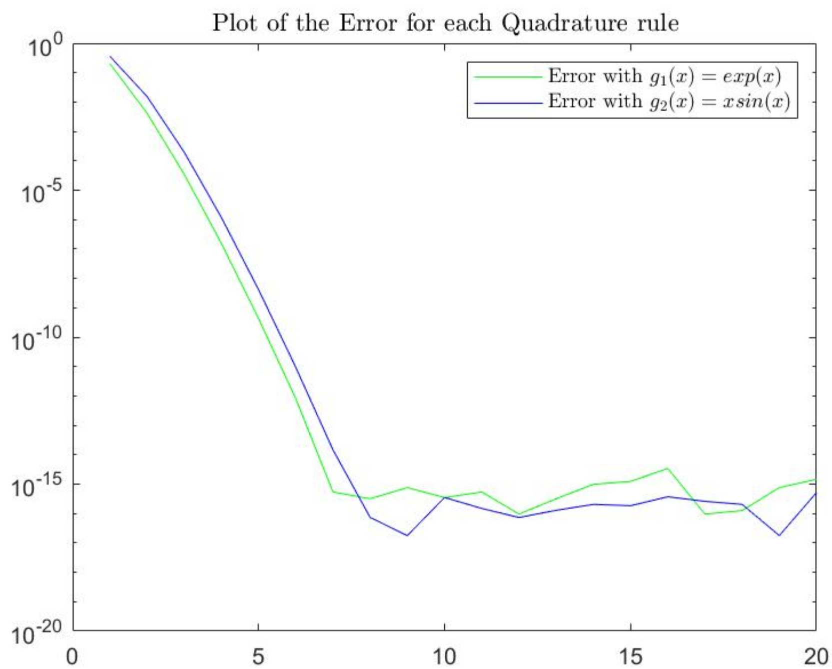

Figure 1: Here is a plot of the error of the first and second quadrature rules