Aging, Secular Stagnation and the Business Cycle

Appendix

Not for Publication

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Contents

1 Time-Varying Demographic Parameters

Consider the alternative model to the the baseline model written in the paper, all agents are alive at time $t = 0$ and can trade claims to future consumption. The measure of agents of age s in period t is n_t^s.[1] Take an individual j belonging to the cohort born in period s. Rewrite her lifecycle problem as an infinite horizon problem with a preference process $\phi_t^{j,s}$ that proxies for the lifecycle. Instead of being born with zero wealth, allow her to trade consumption claims at $t = 0$. I remove for notational convenience the processes generating aggregate uncertainty, although they can be denoted simply as Markov processes in what follows. Her preferences are therefore:

$$\max_{\{c_t^{j,s},\ \ell_t^{j,s},\ a_t^{j,s}\}} \sum_{\tau=s}^{\infty} \beta^{\tau} \left[\prod_{r=s}^{\tau}(1-\gamma_r^s)\right] \phi_\tau^{j,s} u(c_\tau^{j,s}, \ell_\tau^{j,s}). \tag{1}$$

The individual's sequence of budget constraints is as in the paper. Let $\lambda_t^{j,s}$ be the marginal utility of wealth, or the Lagrange multiplier on the individual's budget constraint. Because of the redistribution of unintentional bequests, there is full insurance for mortality risk, so that the individual's savings decision implies the Euler equation:

$$\lambda_t^{j,s} = \lambda_{t+1}^{j,s} R_{t+1} \beta. \tag{2}$$

Optimizing (1) by choices of consumption and labor yields the first order conditions, first for the marginal utility of consumption: $\phi_t^{j,s} u_1(c_t^{j,s}, \ell_t^{j,s}) = \lambda_t^{j,s}$, and second, for the labor supply decision: $\phi_t^{j,s} u_2(c_t^{j,s}, \ell_t^{j,s}) = \lambda_t^{j,s} z^s w_t$. The Euler equation (2) implies that for any two individuals j, i, the ratios of their marginal utilities is constant for all time periods t, t', which implies:

$$\frac{\lambda_t^{j,s}}{\lambda_t^{i,s'}} = \frac{\lambda_{t'}^{j,s}}{\lambda_{t'}^{i,s'}} = \frac{\lambda^{i,s'}}{\lambda^{j,s}}, \tag{3}$$

where $\lambda^{j,s} = \frac{\lambda_t}{\lambda_t^{j,s}}$. Using this condition, we can rewrite the first order conditions for the individual's problem as:

$$\lambda^{j,s} \phi_t^{j,s} u_1(c_t^{j,s}, \ell_t^{j,s}) = \lambda_t, \tag{4}$$

and:

$$\lambda^{j,s} \phi_t^{j,s} u_2(c_t^{j,s}, \ell_t^{j,s}) = \lambda_t z^s w_t. \tag{5}$$

These two equations, together with each individual's budget constraints and aggregate definitions, characterize the decentralized economy.

Now consider a period-by-period problem of a social planner who chooses consumption and labor supply for each individual in each cohort to maximize the sum of individual utilities, weighted by welfare weights $\lambda^{j,s}$:

$$U(c_t, \ell_t) = \max_{\{c_t^{j,s}, \ell_t^{j,s}\}} \left\{ \sum_s \int \lambda^{j,s} \phi_t^{j,s} u(c_t^{j,s}, \ell_t^{j,s})\, \mathrm{d}j \right\}, \tag{6}$$

[1] Note, this notation is slightly different from the notation of the paper, where s refers to the birth date. Where possible, I keep the notation close to Maliar and Maliar (2003).

subject to the definitions for total consumption ($c_t = \sum_s n_t^s c_t^{j,s}$) and for total labor supplied in efficiency unit terms ($\ell_t = \sum_s n_t^s z^s \ell_t^{j,s}$). Because individuals within a cohort are identical and the measure of individuals within a cohort is given by n_t^s, we can write this problem as:

$$U(c_t, \ell_t) = \max_{\{c_t^{j,s}, \ell_t^{j,s}\}} \left\{ \sum_s n_t^s \lambda^{j,s} \phi_t^{j,s} u(c_t^{j,s}, \ell_t^{j,s}) \right\}. \tag{7}$$

Letting the Lagrange multiplier on the definitions of total consumption and labor be φ_t and ν_t respectively, the first order conditions of this static problem are:

$$n_t^s \lambda^{j,s} \phi_t^{j,s} u_1(c_t^{j,s}, \ell_t^{j,s}) = n_t^s \varphi_t, \tag{8}$$

and:

$$n_t^s \lambda^{j,s} \phi_t^{j,s} u_2(c_t^{j,s}, \ell_t^{j,s}) = n_t^s \nu_t z^s. \tag{9}$$

The envelope conditions for the problem (6) on c_t and ℓ_t are simply φ_t and ν_t, respectively, so that $U_1(c_t, \ell_t) = \varphi_t$ and $U_2(c_t, \ell_t) = \nu_t$.

Now consider the problem where the planner optimizes by choice of total consumption, total efficiency units of labor supplied ℓ_t, and total savings, the social utility function over time:

$$\max_{\{c_t, k_t, \ell_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t, \ell_t), \tag{10}$$

subject to the economy's resource constraint each period. Letting λ_t be the Lagrange multiplier on the resource constraint, the first order conditions of this problem imply the same expressions as those equations that characterize the decentralized economy's problem. In particular, we get for the choice of aggregate savings: $\lambda_t = \lambda_{t+1} R_{t+1} \beta$. For aggregate consumption, the first order condition implies the standard condition: $U_1(c_t, \ell_t) = \lambda_t$, and for the choice of labor, $U_2(c_t, \ell_t) = \lambda_t w_t$, where I have substituted in w_t for the marginal product of labor.

To derive expressions for the shocks, the particular setup of the individual's problem in the lifecycle model has $u(c_t^{j,s}, \ell_t^{j,s}) = \frac{(c_t^{j,s})^{1-\sigma}}{1-\sigma} - v^s \frac{(\ell_t^{j,s})^{1+\varphi}}{1+\varphi}$. The $\phi_t^{j,s}$ are one when the individual is alive and zero otherwise. The skill of each individual z^s is positive in periods the individual is alive and zero otherwise. The first order conditions for the optimization of the social utility function are:

$$\lambda^{j,s} \phi_t^{j,s} (c_t^{j,s})^{-\sigma} = \varphi_t, \tag{11}$$

and:

$$\lambda^{j,s} \phi_t^{j,s} v^s (\ell_t^{j,s})^{\varphi} = \nu_t z^s. \tag{12}$$

This implies:

$$c_t^{j,s} = \varphi_t^{-1/\sigma} (\lambda^{j,s} \phi_t^{j,s})^{1/\sigma}, \tag{13}$$

and:

$$z^s \ell_t^{j,s} = \nu_t^{1/\varphi} (z^s)^{1+1/\varphi} (v^s \phi_t^{j,s} \lambda^{j,s})^{-1/\varphi}. \tag{14}$$

Integrating (summing) these expressions with respect to individuals (cohorts) gives aggregate consumption:

$$c_t = \sum_s n_t^s c_t^{j,s} = \varphi_t^{-1/\sigma} \left(\sum_s n_t^s (\lambda^{j,s} \phi_t^{j,s})^{1/\sigma} \right). \tag{15}$$

For aggregate labor supplied in efficiency units:

$$\ell_t = \sum_s n_t^s z^s \ell_t^{j,s} = \nu_t^{1/\varphi} \left(\sum_s n_t^s (z^s)^{1+1/\varphi} (v^s \phi_t^{j,s} \lambda^{j,s})^{-1/\varphi} \right). \tag{16}$$

These expressions imply that individual consumption and labor supply are fractions of their respective aggregates:

$$c_t^{j,s} = \frac{(\lambda^{j,s}\phi_t^{j,s})^{1/\sigma}}{\sum_s n_t^s (\lambda^{j,s}\phi_t^{j,s})^{1/\sigma}} c_t, \qquad \text{and} \qquad \ell_t^{j,s} = \frac{(z^s)^{1/\varphi}(v^s\phi_t^{j,s}\lambda^{j,s})^{-1/\varphi}}{\sum_s n_t^s (z^s)^{1+1/\varphi}(v^s\phi_t^{j,s}\lambda^{j,s})^{-1/\varphi}} \ell_t. \tag{17}$$

More compactly, $c_t^{j,s} = \chi_1^{j,s} c_t$ and $\ell_t^{j,s} = \chi_2^{j,s} \ell_t$.

Substituting these expressions into the social utility function gives:

$$U = \sum_s n_t^s \lambda^{j,s} \phi_t^{j,s} \left((\chi_1^{j,s})^{1-\sigma} \frac{c_t^{1-\sigma}}{1-\sigma} - v^s (\chi_2^{j,s})^{1+\varphi} \frac{\ell_t^{1+\varphi}}{1+\varphi} \right), \tag{18}$$

which can be rearranged to get an aggregate utility function:

$$U = \phi_t \frac{c_t^{1-\sigma}}{1-\sigma} - v_t \frac{\ell_t^{1+\varphi}}{1+\varphi}. \tag{19}$$

In this representation, aggregate labor ℓ_t is expressed as efficiency units of labor. Reorganizing this gives:

$$U = \phi_t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{v_t}{\phi_t} \left(\frac{\ell_t^{1+\varphi}}{1+\varphi} \right) \right]. \tag{20}$$

The process ϕ_t can be interpreted as a preference shock while v_t/ϕ_t is the labor wedge on the efficiency units of labor supplied. Finally, if we know the welfare weights $\{\lambda^{j,s}\}_{j,s}$, all terms in ϕ_t and v_t are exogenous and can be computed.

The final expression to determine is the term which converts aggregate supply of units of labor, denoted by h_t, into ℓ_t, the aggregate supply of efficiency units of labor which enters the firm's production function. To get this shock, we integrate $\ell_t^{j,s}$ over individuals and cohorts:

$$\sum_s \int \ell_t^{j,s} \, \mathrm{d}j = \sum_s n_t^s \ell_t^{j,s}, \tag{21}$$

and compute $A_t = \ell_t / \ell_t = \sum_s n_t^s z_t^s \ell_t^{j,s} / \sum_s n_t^s \ell_t^{j,s}$, which becomes:

$$A_t = \frac{\sum_s n_t^s (z^s)^{1+1/\varphi} (v_t^{j,s} \phi_t^{j,s} \lambda^{j,s})^{-1/\varphi}}{\sum_s n_t^s (z^s)^{1/\varphi} (v_t^{j,s} \phi_t^{j,s} \lambda^{j,s})^{-1/\varphi}}. \tag{22}$$

The derivations show that the effect of heterogeneity by age can be approximated with exogenous and foreseen paths of aggregate productivity θ_t, to labor input A_t, to the consumption utility shifter ϕ_t, and to the labor disutility shifter v_t.

Discount Factor and Labor Tax Rate I also include a trend to the discount factor s_t to accout for how the increase in longevity affects the average discount factor. Finally, I include a trend to the tax rate computed from the perfect foresight solution.

1.1 Comparing Outcomes of the Two Models

Figure 1 plots the full nonlinear global solution of the lifecycle model with the calibrated demographic changes against the piecewise-linear solution with anticipated parameter changes arising from demographic changes. The two methods give very similar paths for the key variables in the model, including for the real interest rate. To compute these I use equal Pareto weights across all individuals.

Figure 2 compares the values of the real interest rate in simulations of the full lifecycle model and the aggregate model. For both models, I use the same set of productivity shocks, and compute the second order approximation of both models.

Figure 3 plots consumption by individuals of different ages against productivity shocks of different age, under a second order approximation of the full OLG model.

2 Full Set of Model Equations

2.1 Sticky Price Economy

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\phi_r} \left(\frac{\Pi_t}{\Pi^*}\right)^{(1-\phi_r)\phi_\pi} \left(\frac{y_t}{y_t^{\mathrm{F}}}\right)^{(1-\phi_r)\phi_y} \left(\frac{y_t/y_{t-1}}{y_t^{\mathrm{F}}/y_{t-1}^{\mathrm{F}}}\right)^{\phi_g} \exp\left(\frac{\sigma_R}{100}\varepsilon_{R,t}\right). \tag{23}$$

$$\frac{1}{\beta}\left[1+\phi_k\kappa_t\left(\frac{k_t}{k_{t-1}}\kappa_t-1\right)\right]\left[\frac{\phi_k}{2}\left(\frac{k_{t+1}^2}{k_t^2}\kappa_t^2-1\right)+r_{t+1}+1-\delta\right]^{-1} = \frac{1}{z}\frac{\lambda_{t+1}}{\lambda_t} \tag{24}$$

$$\chi_t\phi_t c^{-\sigma} = \lambda_t \tag{25}$$

$$\chi_t v_t\left(\mu_t n_t\right)^{\psi} = w_t\lambda_t\left(1-\tau_t\right) \tag{26}$$

$$y_t = \mu_t^{1-\alpha}\theta_t^{1-\alpha}\left(\frac{k_{t-1}}{z}\right)^{\alpha} n_t^{1-\alpha} \tag{27}$$

$$y_t = c_t + k_t + g_t - (1-\delta)\frac{k_{t-1}}{z} + \frac{\phi_p}{2}y_t\left(\frac{\pi_t}{\pi_{t-1}}-1\right)^2 + \frac{\phi_k}{2}\frac{1}{z}k_{t-1}\left(\frac{k_t}{k_{t-1}}-1\right)^2 \tag{28}$$

$$w_t = \mathrm{mc}_t(1-\alpha)\frac{y_t}{n_t} \tag{29}$$

$$r_t = \mathrm{mc}_t\alpha\frac{y_t}{k_{t-1}}z \tag{30}$$

$$\pi_{t+1}\lambda_t = \lambda_{t+1}R_t\beta\frac{1}{z} \tag{31}$$

$$\hat{\pi}_t = \beta\hat{\pi}_{t+1} + \epsilon_p\left(\hat{\mathrm{mc}}_t - e\right) \tag{32}$$

$$i_t = k_t - (1-\delta)\frac{k_t(-1)}{z} + \frac{\phi_k}{2}\frac{1}{z}k_{t-1}\left(\frac{k_t}{k_{t-1}}-1\right)^2 \tag{33}$$

When the ZLB binds, $R_t = 1$. Hat notation denotes deviation from steady-state.

2.2 Flexible Price Economy

Same set of equations as the sticky price economy, only with $\phi_p = 0$ ($\epsilon_p = \infty$).

2.3 Exogenous Processes

In addition to the monetary policy shock, preference shock:

$$\exp(\chi_t) = (1-\rho_\chi)\exp(\chi) + \rho_\chi \exp(\chi_{t-1}) + \frac{\sigma_\chi}{100}\varepsilon_{\chi,t} \tag{34}$$

Technology shock:

$$\exp(\mu_t) = (1-\rho_\mu)\exp(\mu) + \rho_\mu \exp(\mu_{t-1}) + \frac{\sigma_\mu}{100}\varepsilon_{\mu,t} \tag{35}$$

Markup shock:

$$\exp(\theta_t) = (1-\rho_\theta)\exp(\theta) + \rho_\theta \exp(\theta_{t-1}) + \frac{\sigma_\theta}{100}\varepsilon_{\theta,t} \tag{36}$$

Government spending shock:

$$\exp(g_t) = (1-\rho_g)\exp(g) + \rho_g \exp(g_{t-1}) + \frac{\sigma_g}{100}\varepsilon_{g,t} \tag{37}$$

Investment shock:

$$\exp(\kappa_t) = (1-\rho_\kappa)\exp(\kappa) + \rho_\kappa \exp(\kappa_{t-1}) + \frac{\sigma_\kappa}{100}\varepsilon_{\kappa,t} \tag{38}$$

2.4 Observation Equations

Output growth:

$$dy_t = \log\left(z\frac{y_t}{y_{t-1}}\right) \tag{39}$$

Consumption growth:

$$dc_t = \log\left(z\frac{c_t}{c_{t-1}}\right) \tag{40}$$

Investment growth:

$$di_t = \log\left(z\frac{i_t}{i_{t-1}}\right) \tag{41}$$

Fed Funds rate:

$$\hat{R}_t = \log R_t \tag{42}$$

Inflation:

$$\hat{\pi}_t = \log\left(\frac{\pi_t}{\pi}\right) \tag{43}$$

3 Estimation with Demographic Trends and the ZLB

This section details the estimation strategy tailored to my application with anticipated demographic shocks and where the zero lower bound is accounted for over the period 2009Q1 to 2015Q1.

3.1 Solution Method

A linear rational-expectations model can be written as:

$$\mathbf{A}x_t = \mathbf{C} + \mathbf{B}x_{t-1} + \mathbf{D}\mathbb{E}_t x_{t+1} + \mathbf{F}w_t, \tag{44}$$

where x_t is a $n \times 1$ vector of state and jump variables and w_t is a $l \times 1$ vector of exogenous variables. A solution to (44), following Binder and Peseran (1995), is:

$$x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}\varepsilon_t. \tag{45}$$

As in Binder and Peseran (1995), $\mathbf{Q}$ is solved by iterating on the quadratic expression:

$$\mathbf{Q} = [\mathbf{A} - \mathbf{DQ}]^{-1}\,\mathbf{B}. \tag{46}$$

With $\mathbf{Q}$ in hand, compute $\mathbf{J}$ and $\mathbf{G}$ with:

$$\mathbf{J} = [\mathbf{A} - \mathbf{DQ}]^{-1}\,(\mathbf{C} + \mathbf{DJ}) \tag{47}$$

$$\mathbf{G} = [\mathbf{A} - \mathbf{DQ}]^{-1}\,\mathbf{F}. \tag{48}$$

That is, $\mathbf{J}$, $\mathbf{Q}$ and $\mathbf{G}$ are conformable matrices which are functions of the structural matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$.

In a model where agents have time-varying beliefs about the evolution of the model's structural parameters $\mathbf{A}_t$, $\mathbf{B}_t$, $\mathbf{C}_t$, $\mathbf{D}_t$ and $\mathbf{F}_t$, the solution becomes:

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t w_t, \tag{49}$$

where $\mathbf{J}_t$, $\mathbf{Q}_t$ and $\mathbf{G}_t$ are conformable matrices which are functions of the evolution of beliefs about the time-varying structural matrices $\mathbf{A}_t$, $\mathbf{B}_t$, $\mathbf{C}_t$, $\mathbf{D}_t$ and $\mathbf{F}_t$ (Kulish and Pagan, 2016). They satisfy the recursion:

$$\mathbf{Q}_t = [\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}]^{-1}\,\mathbf{B}_t \tag{50}$$

$$\mathbf{J}_t = [\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}]^{-1}\,(\mathbf{C}_t + \mathbf{D}_t\mathbf{J}_{t+1}) \tag{51}$$

$$\mathbf{G}_t = [\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}]^{-1}\,\mathbf{E}_t, \tag{52}$$

where the final structures $\mathbf{Q}_T$ and $\mathbf{J}_T$ are known and computed from the time invariant structure above under the terminal period's structural parameters.

Anticipated changes in the path of demographic shocks and the zero lower bound are anticipated changes to the model's structural parameters which can be handled by solution (49).

3.2 Kalman Filter

Likelihood methods are used to estimate the parameters of the monetary policy rule and the parameters of the transitory shocks. For that, we need to filter the data, and owing to the linear structure of (49), we can use the Kalman filter, and exploit its computational advantages.

The model in its state space representation is:

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t\varepsilon_t \tag{53}$$

$$z_t = \mathbf{H}_t x_t. \tag{54}$$

The error is distributed $\varepsilon_t \sim N(0, \Omega)$ where Ω is the covariance matrix of ε_t. By assumption, there is no observation error of the data z_t. The Kalman filter recursion is given by the following equations, conceptualized as the predict and update steps. The state of the system is $(\hat{x}_t, \mathbf{P}_{t-1})$. In the predict step, the structural matrices $\mathbf{J}_t$, $\mathbf{Q}_t$ and $\mathbf{G}_t$ are used to compute a forecast of the state $\hat{x}_{t|t-1}$ and the forecast covariance matrix $\mathbf{P}_{t|t-1}$ as:

$$\hat{x}_{t|t-1} = \mathbf{J}_t + \mathbf{Q}_t \hat{x}_t \tag{55}$$

$$\mathbf{P}_{t|t-1} = \mathbf{Q}_t \mathbf{P}_{t-1} \mathbf{Q}_{t|t-1}^{\top} + \mathbf{G}_t \Omega \mathbf{G}_t^{\top}. \tag{56}$$

This formulation differs from the time-invariant Kalman filter because in the forecast stage the matrices $\mathbf{J}_t$, $\mathbf{Q}_t$ and $\mathbf{G}_t$ can vary over time. We update these forecasts with imperfect observations of the state vector. This update step involves computing forecast errors $\tilde{y}_t$ and its associated covariance matrix $\mathbf{S}_t$ as:

$$\tilde{y}_t = z_t - \mathbf{H}_t \hat{x}_{t|t-1} \tag{57}$$

$$\mathbf{S}_t = \mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^{\top}. \tag{58}$$

The Kalman gain matrix is given by:

$$\mathbf{K}_t = \mathbf{P}_{t|t-1} \mathbf{H}_t^{\top} \mathbf{S}_t^{-1}. \tag{59}$$

With $\tilde{y}_t$, $\mathbf{S}_t$ and $\mathbf{K}_t$ in hand, the optimal filtered update of the state x_t is

$$\hat{x}_t = \hat{x}_{t|t-1} + \mathbf{K}_t \tilde{y}_t, \tag{60}$$

and for its associated covariance matrix:

$$\mathbf{P}_t = (I - \mathbf{K}_t \mathbf{H}_t)\, \mathbf{P}_{t|t-1}. \tag{61}$$

The Kalman filter is initialized with x_0 and $\mathbf{P}_0$ determined from their unconditional moments, and is computed until the final time period T of data.

3.3 Kalman Smoother

With the estimates of the parameters and durations in hand at time period T, the Kalman smoother gives an estimate of $x_{t|T}$, or an estimate of the state vector at each point in time given all available information (see Hamilton, 1994). With $\hat{x}_{t|t-1}$, $\mathbf{P}_{t|t-1}$, $\mathbf{K}_t$ and $\mathbf{S}_t$ in hand from the Kalman filter, the vector $x_{t|T}$ is computed by:

$$x_{t|T} = \hat{x}_{t|t-1} + \mathbf{P}_{t|t-1} r_{t|T}, \tag{62}$$

where the vector $r_{T+1|T} = 0$ and is updated with the recursion:

$$r_{t|T} = \mathbf{H}_t^{\top} \mathbf{S}_t^{-1} \left(z_t - \mathbf{H}_t \hat{x}_{t|t-1} \right) + (I - \mathbf{K}_t \mathbf{H}_t)^{\top} \mathbf{P}_{t|t-1}^{\top} r_{t+1|T}. \tag{63}$$

Finally, to get an estimate of the shocks to each state variable under this model's shock structure, denoted by e_t, we compute:

$$e_t = \mathbf{G}_t \varepsilon_t = \mathbf{G}_t r_{t|T}. \tag{64}$$

These are the estimates of the structural shocks.

3.4 Sampler

This section describes the sampler used to obtain the posterior distribution of interest. Denote by ϑ the vector of parameters to be estimated and by $\mathbf{T}$ the vector of ZLB durations that are observed each period. Denote by $\mathcal{Z} = \{z_\tau\}_{\tau=1}^{T}$ the sequence of vectors of observable variables. The posterior $\mathcal{P}(\vartheta \mid \mathbf{T}, \mathcal{Z})$ satisfies:

$$\mathcal{P}(\vartheta \mid \mathbf{T}, \mathcal{Z}) \propto L(\mathcal{Z}, \mathbf{T} \mid \vartheta) \times \mathcal{P}(\vartheta). \tag{65}$$

With Gaussian errors, the likelihood function $L(\mathcal{Z}, \mathbf{T} \mid \vartheta)$ is computed using the appropriate sequence of structural matrices and the Kalman filter:

$$\log L(\mathcal{Z}, \mathbf{T} \mid \vartheta) = -\left(\frac{N_z T}{2}\right) \log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log \det \mathbf{H}_t \mathbf{S}_t \mathbf{H}_t^\top - \frac{1}{2} \sum_{t=1}^{T} \tilde{y}_t^\top \left(\mathbf{H}_t \mathbf{S}_t \mathbf{H}_t^\top\right)^{-1} \tilde{y}_t. \tag{66}$$

The prior is simply computed using priors over ϑ which are consistent with the literature.

The Markov Chain Monte Carlo posterior sampler has a single block, corresponding to the parameters ϑ.[2] The sampler at step j is initialized with the last accepted draw of the structural parameters ϑ_{j-1}.

4 Data Sources

4.1 Data Sources for OLG Model

This section details the data series used for calibration of the lifecycle model.

Current Population Survey I use the Current Population Survey to get estimates of labor force participation rates by age in the calibration of the disutility of providing labor.

Census/American Community Survey I use Census and American Community Survey extracts from IPUMS-USA to compute the experience-productivity profiles following Elsby and Shapiro (2012).

Social Security Administration I use Social Security Administration estimates and forecasts for mortality rates γ_t^s.

BLS-Multifactor Productivity Program I use the BLS-MFP data to construct a measure of the real interest rate from observed capital-output ratios.

The following description is taken from the BLS website: "Capital input data–service-flows of equipment, structures, intellectual property products, inventories, and land. BLS measures of capital service inputs are prepared using NIPA data on real gross investment in depreciable assets and inventories. Labor input data–hours worked by all persons engaged in a sector–is based on information on employment and average weekly hours collected in the monthly BLS survey of establishments and the hours at work survey. Labor composition data are based on March supplements to the Current Population Survey."

[2] It is worth noting that one can estimate in addition to the structural parameters ϑ, the expected zero lower bound durations can be estimated together with the structural parameters (Kulish, Morley and Robinson, 2017), in which case an additional block is needed in the posterior sampler.

4.2 Data Sources for Estimating the New Keynesian Model

I use data on output, consumption, investment, inflation, and the Fed Funds rate, available at `https://research.stlouisfed.org/pdl/803`. Construction of the data series follows Smets and Wouters (2007). The codes for each raw data series are as follows:

- Real Gross Domestic Product, 3 Decimal (GDPC96). Current, \$.
- Gross Domestic Product: Implicit Price Deflator (GDPDEF). Index, 2009=100.
- Personal Consumption Expenditures (PCEC). Current, \$.
- Fixed Private Investment (FPI). Current, \$.
- Total population (CNP16OV), Thousands of Persons.
- Civilian Noninstitutional Population. (CNP16OV). Thousands of Persons.

To map these data series to the model variables, I do the following transformations.

1. Construct the series LNSindex, which is an index of CNP16OV where 1992Q3=1. I adjust the CNP16OV series to account for breaks in the series each January, due to revisions from updated Census reports, which can be substantial. To do this, I impute an estimate of each January's monthly change in population and construct an estimate of the revised change in population from the actual change to the imputed change. I then distribute that revised change in population across the preceding 12 months.
2. Construct the series CE16OVIndex, which is an index of CE16OV where 1992Q3=1.
3. Output = Y_t = ln(GDPC96 / LNSindex) * 100. Then compute the percentage change in output as an observable, $\ln Y_t - \ln Y_{t-1}$.
4. Inflation = Π_t = ln(GDPDEF / GDPDEF(-1)) * 100.
5. Consumption = C_t = ln((PCEC / GDPDEF) / LNSindex) * 100. Then compute the percentage change in consumption as an observable, $\ln C_t - \ln C_{t-1}$.
6. Investment = I_t = ln((FPI / GDPDEF) / LNSindex) * 100. Then compute the percentage change in investment as an observable, $\ln I_t - \ln I_{t-1}$.

The interest rate is the Fed Funds Rate, taken from the Federal Reserve Economic Database.

For the expected ZLB durations, I use survey data from Blue Chip, from 2009 to 2010, and the NY Fed's Survey of Primary Dealers from 2011 to 2015. These six series are plotted in Figure 4. The expected ZLB durations are hump-shaped pattern, and peak in 2012-13.

5 Estimation Results

Figure 5 plots R^2 Gelman chain diagnostics for the baseline estimates from 1984Q1 to 2015Q1, and illustrate that the estimated posterior distributions lie around or below 1.1, commonly used as a value indicating convergence of the posterior distributions (Bianchi, 2013).

The prior and posterior distributions for the estimated parameters are plotted in Figure 6.

The shocks are plotted in Figure 7. Around 2008-09, I estimate a large negative preference shock, a large negative TFP shock, large shocks to the efficiency of investment, large negative markup shocks, and positive government spending shocks. The shocks also show how the monetary policy shock becomes zero during the ZLB period, as the nominal interest rate is fixed at zero.

6 Lifecycle Model Calibration

6.1 Disutility of Labor Calibration

As in Kulish et al (2006), the equation for the disutility of labor supply is:

$$v^s = \kappa_0 + \left(\kappa_1 \frac{s}{70}\right) \int_{-\infty}^{s} \frac{1}{70\sqrt{2\pi}\kappa_3} \exp\left(-\frac{1}{2}\left[\frac{x - 70\kappa_2}{70\kappa_3}\right]^2\right) \mathrm{d}x. \tag{67}$$

The function is a scaled version of the cumulative density function of a normal distribution. There are four parameters governing v^s: (i) κ_0 implies a baseline level of disutility from labor, (ii) κ_1 scales the entire disutility function, (iii) κ_2 scales the mean of the distribution, or the age at which the disutility from work is increasing the most, and (iv) κ_3 scales the standard deviation of the function, or the slope for which the disutility increases.

6.2 Labor Force Participation by Age

In Figure 8, I plot the labor force participation rate by age from censuses and the American Community Surveys against their predictions under the demographic changes. The model broadly matches the values of the LFP by age and does a decent job at matching the dynamics of the labor force participation rate, particularly for those workers on the edge of retirement.

7 Productivity Decomposition

Here, I discuss in more detail how demographic trends alone affect the growth rates of aggregate quantities and productivity. Panel A of Figure 10 shows the growth rates of total output, output per capita and output per worker. Total output growth due to demographics peaks at 1 percentage point just before 1980. The growth rate then steadily declines, until demographics becomes a drag on total output growth, which occurs in 2012. Demographic changes are then a substantial drag on total output growth, with the contribution to overall growth from demographics staying negative throughout the forecast horizon to 2070. In total, output growth declines from peak-to-trough by just less than 2 percentage points.

Output per capita and output per worker show a very different pattern to the total growth rate. This difference is due to labor supply, in particular, the entrance of the baby boomer generation into the workforce in the 1960s. Per worker and per capita output growth due to demographic changes is initially negative between 1960 and 1980, before becoming positive between 1980 and 2010. From then on, demographics causes per capita output growth to turn negative until at least 2040, while per worker output growth stays slightly negative over the forecast horizon. In total, per capita output growth declines from peak-to-trough by just over 1 percentage point between 1990 and 2025, while per worker output growth declines by about 0.7 percentage points over the same period.[3]

In the model, there are three ways that these measures of output growth can change over time. Individuals can supply more hours, affecting both output and aggregate labor. There are also changes in physical capital, as individuals save and consume out of accumulated savings in retirement. Consumption smoothing motives ensure that the level of savings changes at a different rate to the supply of labor. Third, the *quality* of labor can change. In particular, changes in the distribution of workers resulting from demographic changes alters the average skill-level of the workforce, which shows up in the productivity decomposition as fluctuations in the quality of labor.

Formally, I decompose the model's predictions for output growth and labor productivity growth into their component parts following a standard growth accounting exercise. The production function in the model is $y_t = k_t^{\alpha} \ell_t^{1-\alpha}$, where ℓ_t is aggregate efficiency units of labor. The total derivative of the production function decomposes the change in output into: $\frac{\mathrm{d}y_t}{y_t} = \alpha \frac{\mathrm{d}k_t}{k_t} + (1-\alpha)\frac{\mathrm{d}\ell_t}{\ell_t}$. In the lifecycle model, growth in output per efficiency unit of labor ℓ_t arises from changes in aggregate labor supply or from changes in the labor quality of the workforce, as individual workers become more productive with age: $\frac{\mathrm{d}\ell_t}{\ell_t} = \frac{\mathrm{d}h_t}{h_t} + \frac{\mathrm{dLQ}_t}{\mathrm{LQ}_t}$, where h_t is aggregate hours and LQ_t is labor quality.

Panel B of Figure 10 plots the decomposition for labor productivity growth, while Panel C plots the decomposition for total growth. Accelerating capital accumulation increases the growth rate of both labor productivity and total output up to 1995, after which the growth rate starts to decline. The change in labor supply has a large negative effect on productivity growth, but a positive effect on total growth, when the baby boomer cohorts enter the labor force around 1960.

A key component of both labor productivity and total growth is the change in the quality of the workforce which arises as the composition of the workforce interacts with the age-productivity profile. The decomposition implies that the contribution of the change in average labor quality to the growth rate of output and output per worker peaks around 1990, adding roughly 0.35 percentage points to total growth and productivity growth. The contribution of labor quality turns negative in 2000 as the mass of workers reaches the peak of the age-productivity profile, exhausting the potential for further growth in average human capital across the workforce. This force remains a drag on productivity growth until 2030.

8 Estimation Without Demographic Trends

Here, I discuss an estimation of the baseline model using the same quarterly data series on output, consumption, investment, the Fed Funds rate, inflation, and the expected ZLB durations, but without the exogenous demographic shocks, to study how the estimation procedure assigns variation in the data to each shock without the demographic trend. To do this, I hold the time-varying demographic parameters constant at their values in 1984. I run two chains of length 150000 draws, and discard the first third of the chain. Table 1 presents moments of the posterior distribution, as well as the moments of the results with demographics reported in the paper.

Without demographics, the estimation prefers higher trend growth at around 1.7% annually, rather than 1.3% annual in the estimation with demographics. This is because

[3] The magnitude of the decline in per capita growth accords with the results in Antolin et al (2014).

Table 1: Estimated Parameters With and Without Demographics

	Prior				Post (Demog.)			Post (No Demog.)		
Parameter	Dist	Median	10%	90%	Mode	10%	90%	Mode	10%	90%
ϵ_p	U	0.1	0.0	0.2	0.01	0.01	0.02	0.04	0.03	0.05
$400 \times (z-1)$	N	2.0	1.1	3.0	1.34	1.24	1.46	1.68	1.58	1.77
ρ_χ	B	0.5	0.3	0.7	0.94	0.93	0.95	0.85	0.84	0.87
ρ_μ	B	0.5	0.3	0.7	0.76	0.55	0.83	0.97	0.96	0.98
ρ_θ	B	0.5	0.3	0.7	0.96	0.95	0.97	0.97	0.95	0.97
ρ_g	B	0.5	0.3	0.7	0.95	0.94	0.97	0.94	0.93	0.96
ρ_κ	B	0.5	0.3	0.7	0.94	0.93	0.95	0.94	0.91	0.95
$100 \times \sigma_\chi$	IG	1.2	0.5	3.7	2.15	1.95	2.47	1.19	1.08	1.34
$100 \times \sigma_\mu$	IG	1.2	0.5	3.7	0.48	0.33	0.73	0.65	0.54	0.78
$100 \times \sigma_\theta$	IG	1.2	0.5	3.7	3.51	3.20	4.03	2.34	2.09	2.83
$100 \times \sigma_i$	IG	1.2	0.5	3.7	0.16	0.15	0.18	0.15	0.13	0.17
$100 \times \sigma_g$	IG	1.2	0.5	3.7	1.05	0.98	1.15	0.96	0.90	1.06
$100 \times \sigma_\kappa$	IG	1.2	0.5	3.7	1.00	0.86	1.25	0.88	0.77	1.22

demographic trends contribute positively to productivity growth through human capital accumulation, so that a lower value of trend growth is needed to match the data.

Looking at the shocks, productivity shocks are a lot more persistent and have higher variance. To interpret what this implies for what shocks drive the business cycle, Table 2 presents the 4 quarter ahead, and unconditional, forecast error variance decompositions of the observable variables (and wages). I find that technology shocks are a lot more important in driving variation in the short-run and in the long-run, particularly for consumption, as compared to an estimation that does not incorporate demographic trends. This suggests that demographic trends generate substantial, but positively correlated, trends in real variables, including output, consumption, and the capital stock, that are similar to the paths that would be generated from highly persistent technology shocks, so that the estimation procedure assigns more of a role to persistent technology shocks.

More generally, this illustrates the importance of including trends like demographic trends in estimating models of the business cycle, as they can affect the interpretation of structural parameters and shocks substantially.

9 Algorithm to Solve Path of OLG Model

I use a perfect foresight, deterministic shooting algorithm to solve for the path of the OLG model under the exogenous demographic forces and no idiosyncratic shocks. I specify the year 2200, well beyond the year 2070 that exogenous demographics stay unchanged, as the period for which the economy is assumed to return to the steady-state associated with the final demographic structure. Computationally, the system is solved by taking the set of model equations at each point in time, and stacking them. Repeated Newton-type iterations are then done on the stacked system. A step in the Newton method is to compute the Jacobian of the full system. The shape of the Jacobian is triangular, and relaxation and

Table 2: Variance Decomposition Due to Shocks Without Demographics, %

Variable \ Shock	Preference	Technology	Markup	Policy	Government	Investment
A. Conditional, 4 Quarter Ahead						
Fed Funds Rate	13.9	13.7	23.5	4.7	13.1	31.2
Inflation	11.7	6.4	41.8	11.3	7.7	21.1
Wages	1.3	8.1	86.7	1.4	0.3	2.2
Output	0.3	21.4	55.9	0.4	5.1	17.0
Consumption	1.7	40.8	40.3	0.4	9.6	7.1
Investment	1.0	10.2	52.1	0.3	1.8	34.6
B. Unconditional						
Fed Funds Rate	27.3	4.8	2.4	15.9	11.2	38.3
Inflation	16.7	3.5	29.7	16.8	7.8	25.5
Wages	6.7	4.5	77.6	7.7	0.4	3.2
Output	0.2	16.2	38.3	2.5	16.9	26.0
Consumption	10.4	39.3	16.6	2.8	18.7	12.2
Investment	2.3	4.8	32.6	1.4	1.6	57.3

block decomposition methods solve the problem efficiently.[4]

10 Robustness Exercises

Here, I report a number of robustness exercises in the individual calibration and compute the perfect foresight path of the economy. The paths of the real interest rate, employment-population ratios, and output growth are similar. These show that the main demographic effects manifest through changes in the size of the population.

10.1 Borrowing Constraints

In this robustness exercise, I set $a_t^s \geq 0$ for all t, s. The effect of this change is to cause the young to supply less labor for the periods when borrowing is constrained. The consumption profile is steep for those periods the young are constrained. In the aggregate, with the other calibrated parameters kept constant, there is more aggregate savings and less aggregate labor supplied, resulting in a higher capital-output ratio and lower real interest rate. The movements in the interest rate and labor force participation rate are largely unaffected as compared to the baseline results. In Figure 11 it is labeled as the A series.

10.2 Time-varying Productivity Profiles

In this exercise, I calibrate the productivity profiles to be time-varying and fully anticipated, so that there are anticipated changes to the slope and magnitude of the productivity profiles

[4]These methods are implemented in Dynare.

faced by workers. I recalculate the age-productivity profile by recomputing the Census/ACS age-earning profile for full-time workers and rescaling the profile to those entering the workforce. Earnings are deflated by the GDP deflator. In Figure 11 it is labeled as the B series.

10.3 Gender-based Calibration

In this exercise, I divide workers into genders, recalibrating the age-productivity profile to male workers and female workers separately. I also choose the age-disutilities separately so that female labor force participation increases since the 1950s, pushing up the overall increase in the employment-population ratio. In Figure 11 it is labeled as the C series.

10.4 Skill-based Calibration

In this exercise, I divide workers into two skills, recalibrating the age-productivity profile to those of less than college-education and those with at least some college education. In Figure 11 it is labeled as the D series.

10.5 Exogenous Labor Supply

In this exercise, households have no disutility of supplying labor, and are forced to enter retirement full-time at age 65. This exercise changes substantially the profiles for aggregate labor force participation and for the real interest rate.

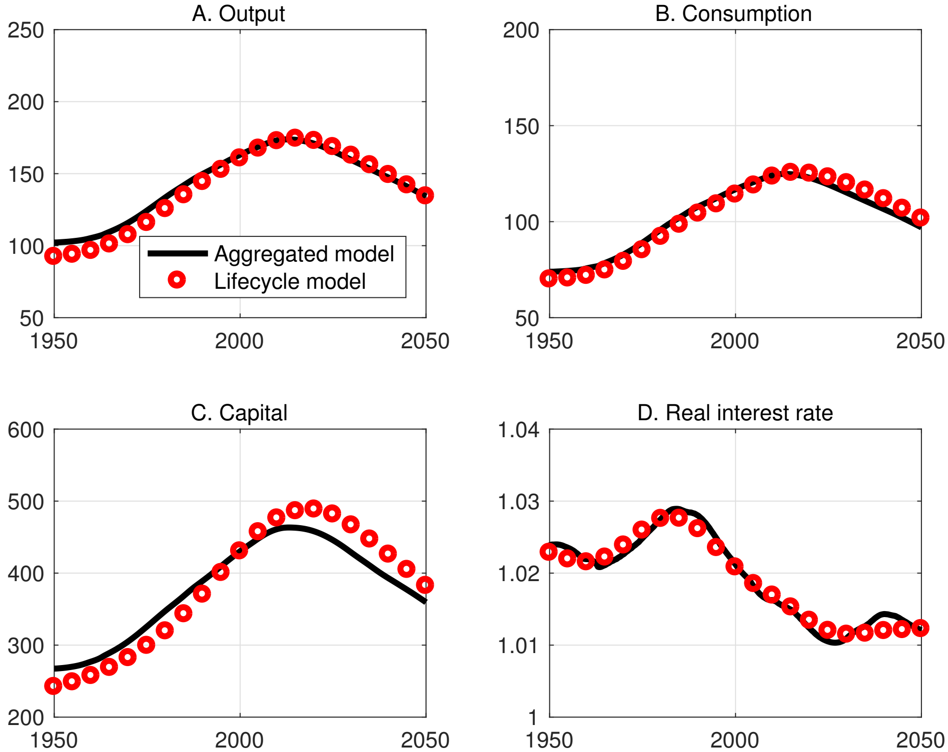

Figure 1: Comparison of Trends Computed in Model With Time-Varying Parameters and Perfect Foresight OLG Model

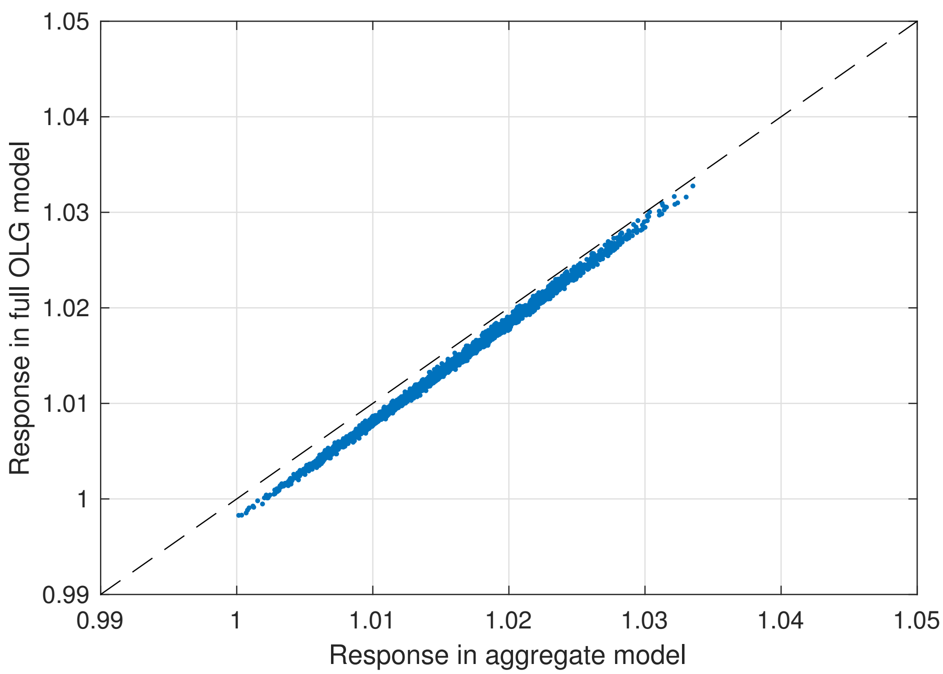

Figure 2: Response of the Real Interest Rate in Simulations of the Lifecycle Model and the Aggregate Model Under the Same Shocks. Second order approximation for both models

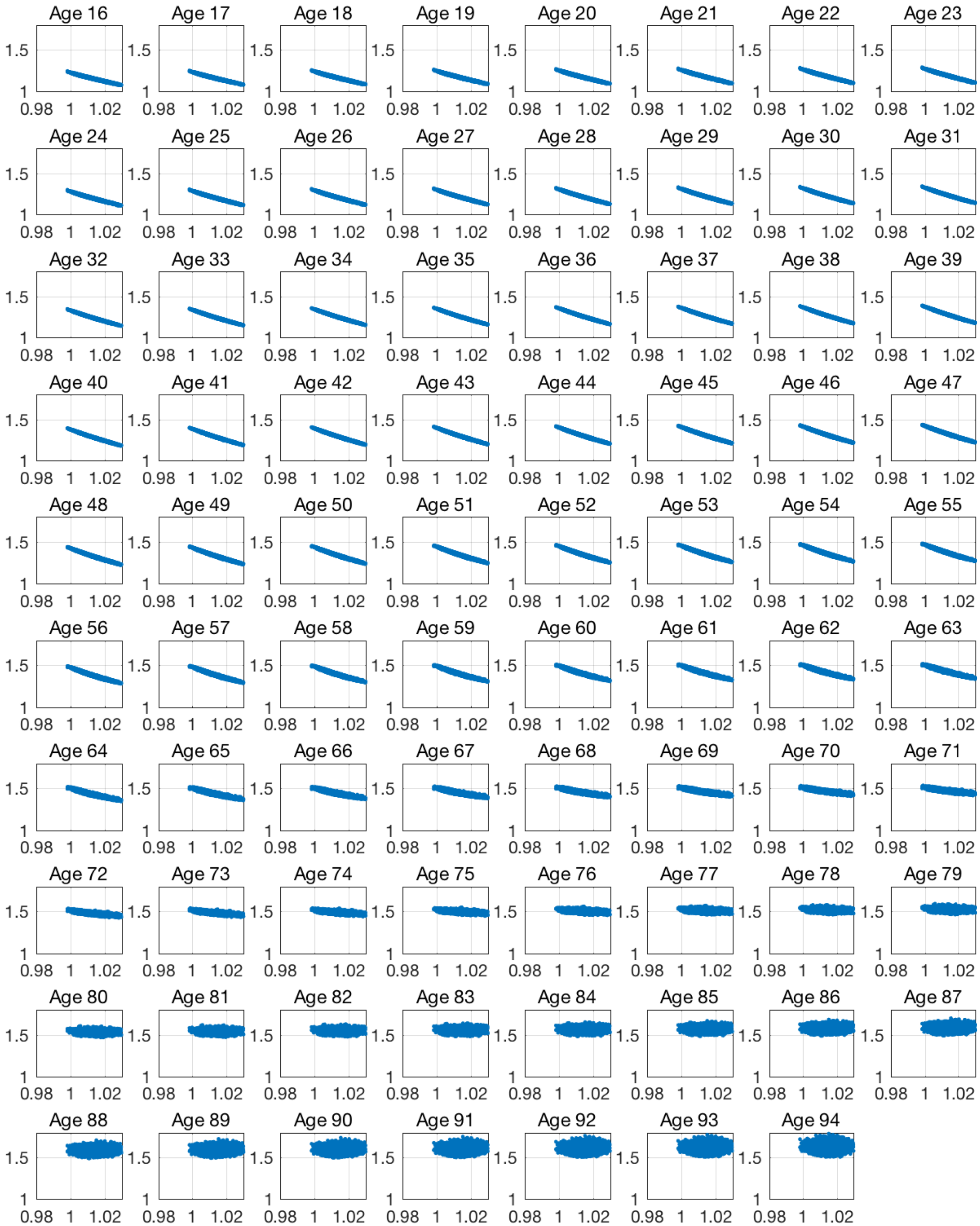

Figure 3: Individual Consumption Under Productivity Shock

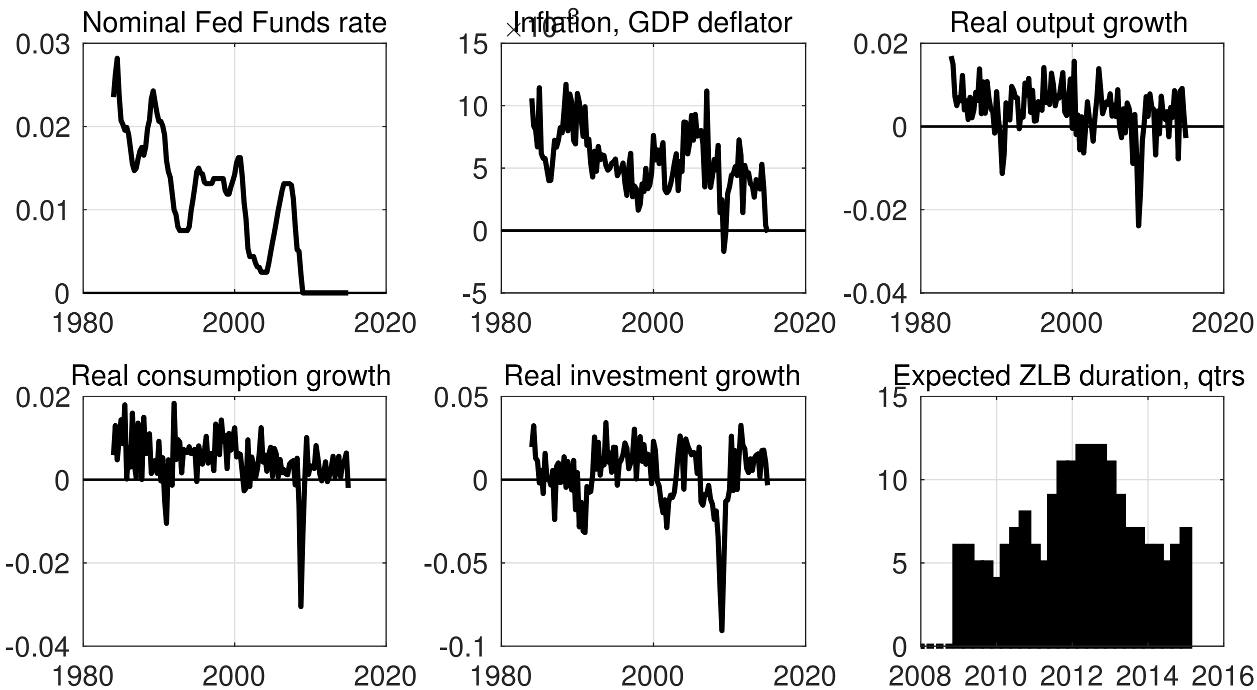

Figure 4: Quarterly Data Used in Estimation

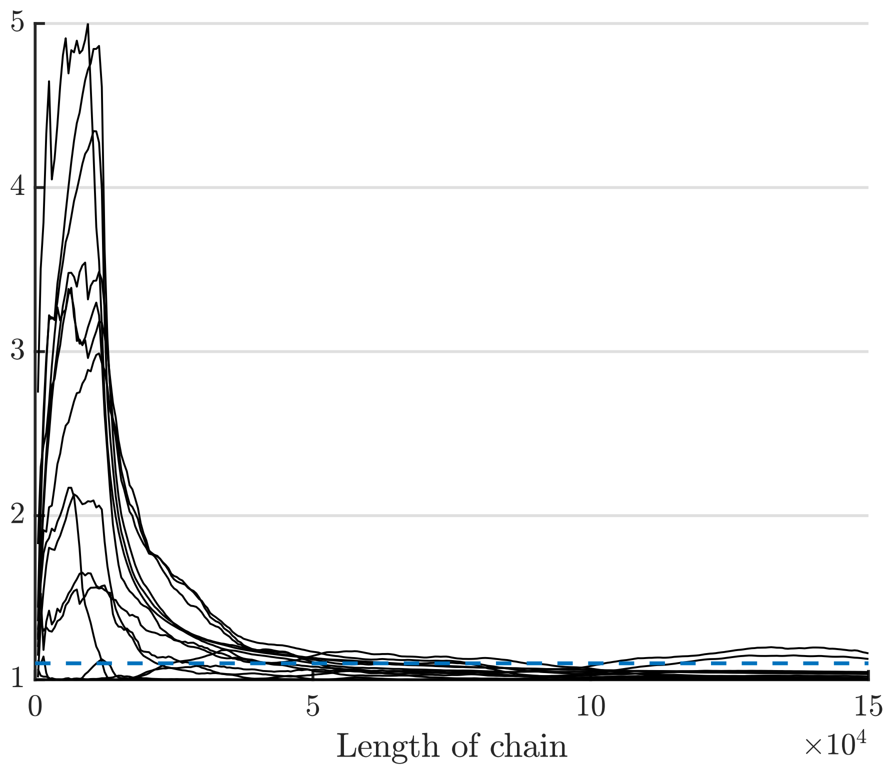

Figure 5: Gelman Chain Diagnostic For Each Parameter

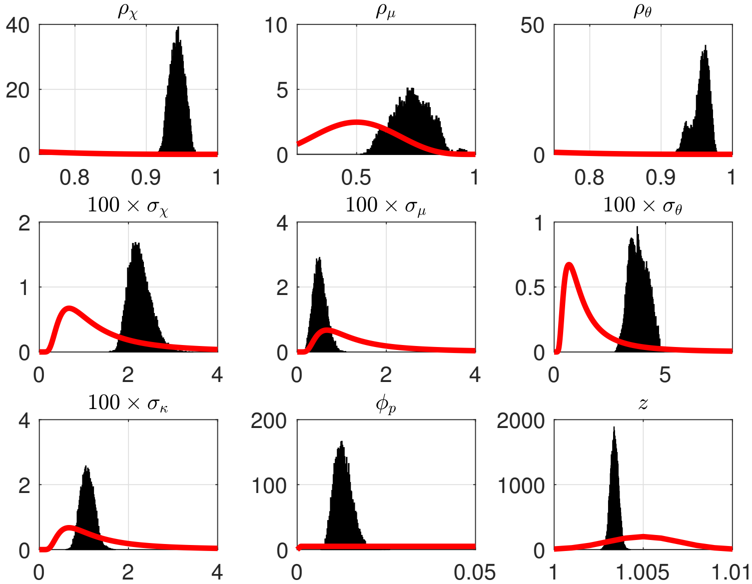

Figure 6: Estimated Posterior Distributions. Priors (red) and posteriors (black) of the estimated parameters

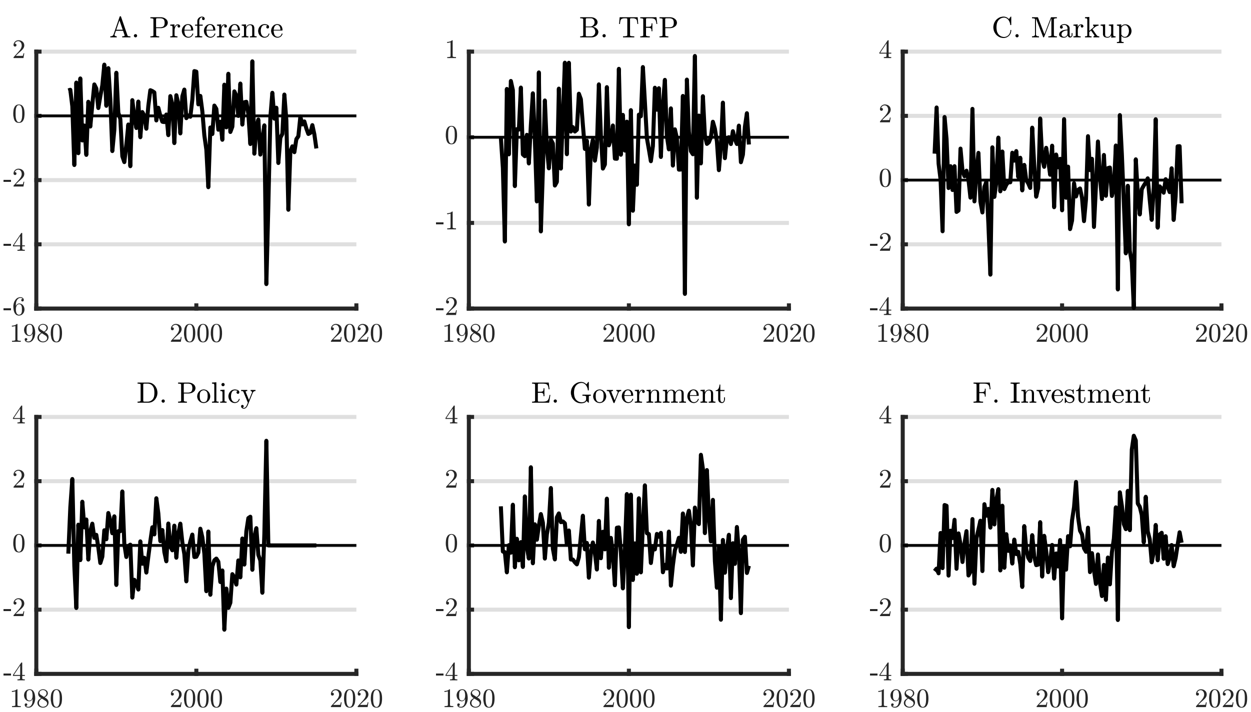

Figure 7: Structural Shocks

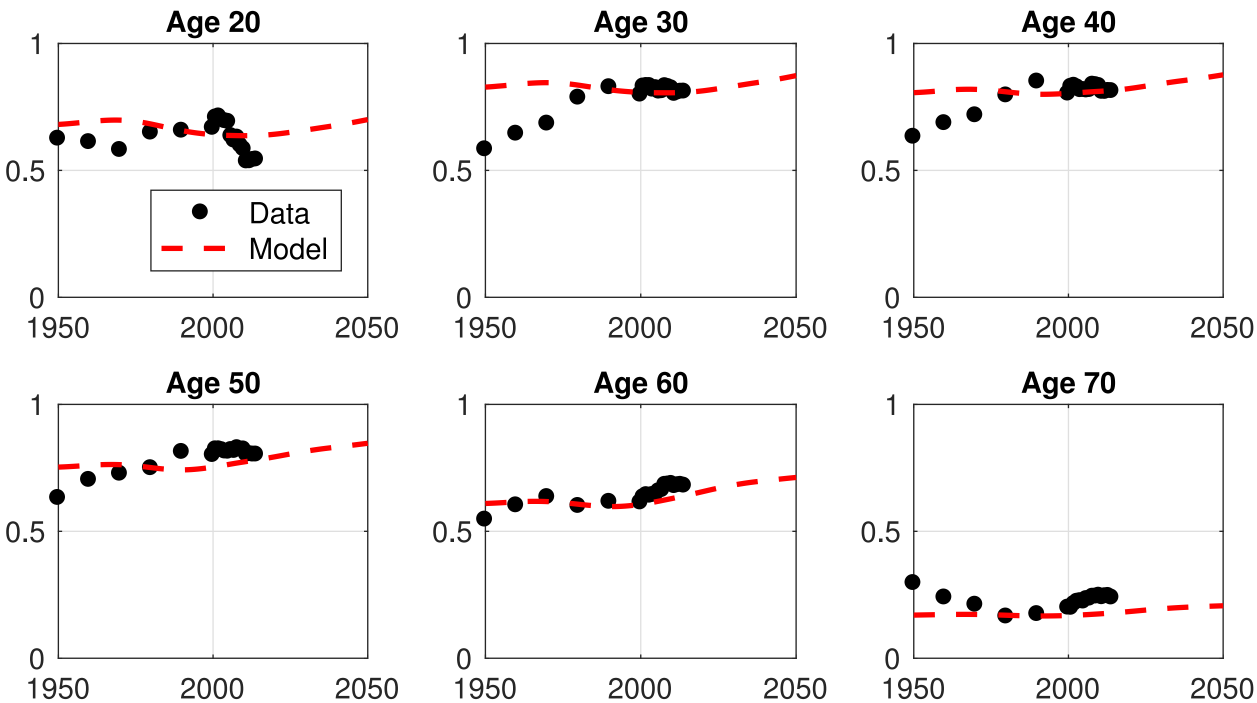

Figure 8: Labor Force Participation by Age. This figure shows the fraction of the labor endowment chosen by workers at each age in the model against the labor force participation rates observed in censuses and American Community Surveys.

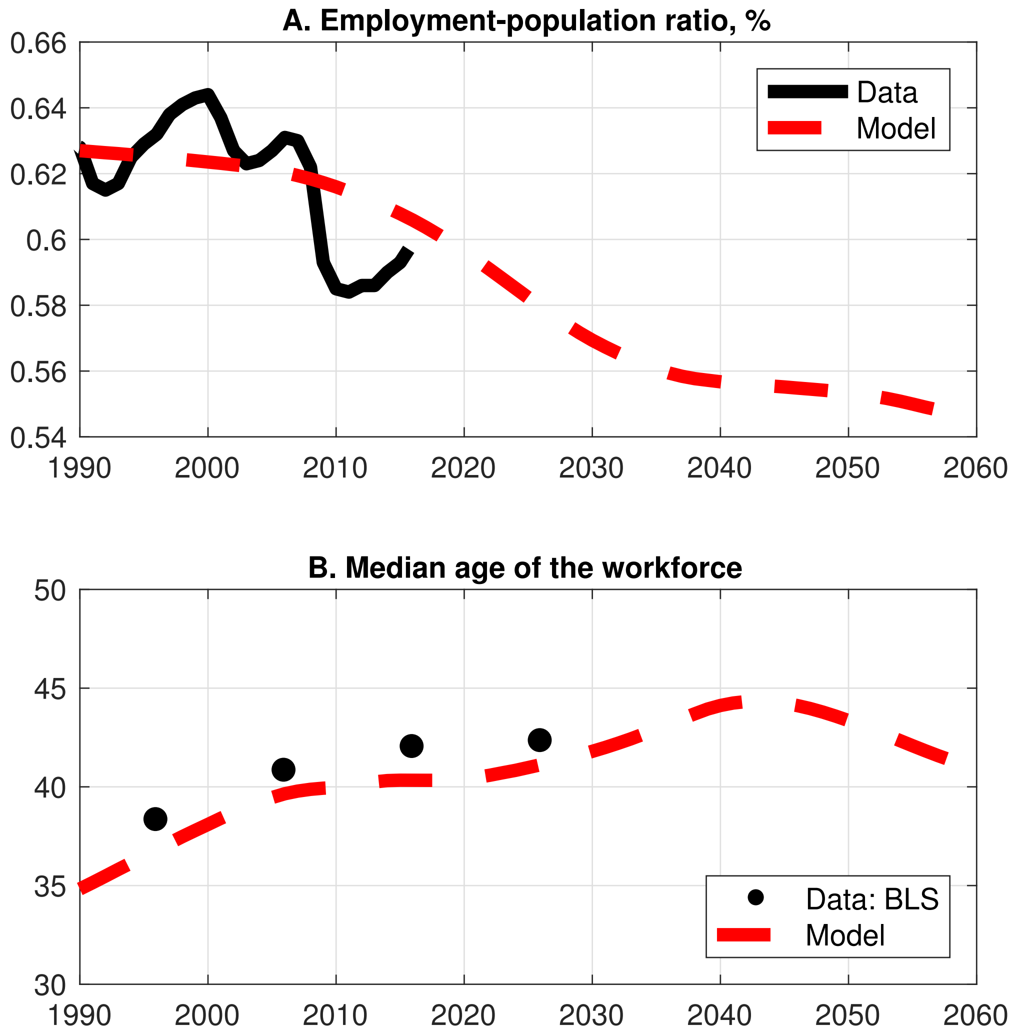

Figure 9: Employment-Population Ratio. This figure shows total employment-population ratios in the model and as observed (Panel A), and the median age of the workforce (Panel B).

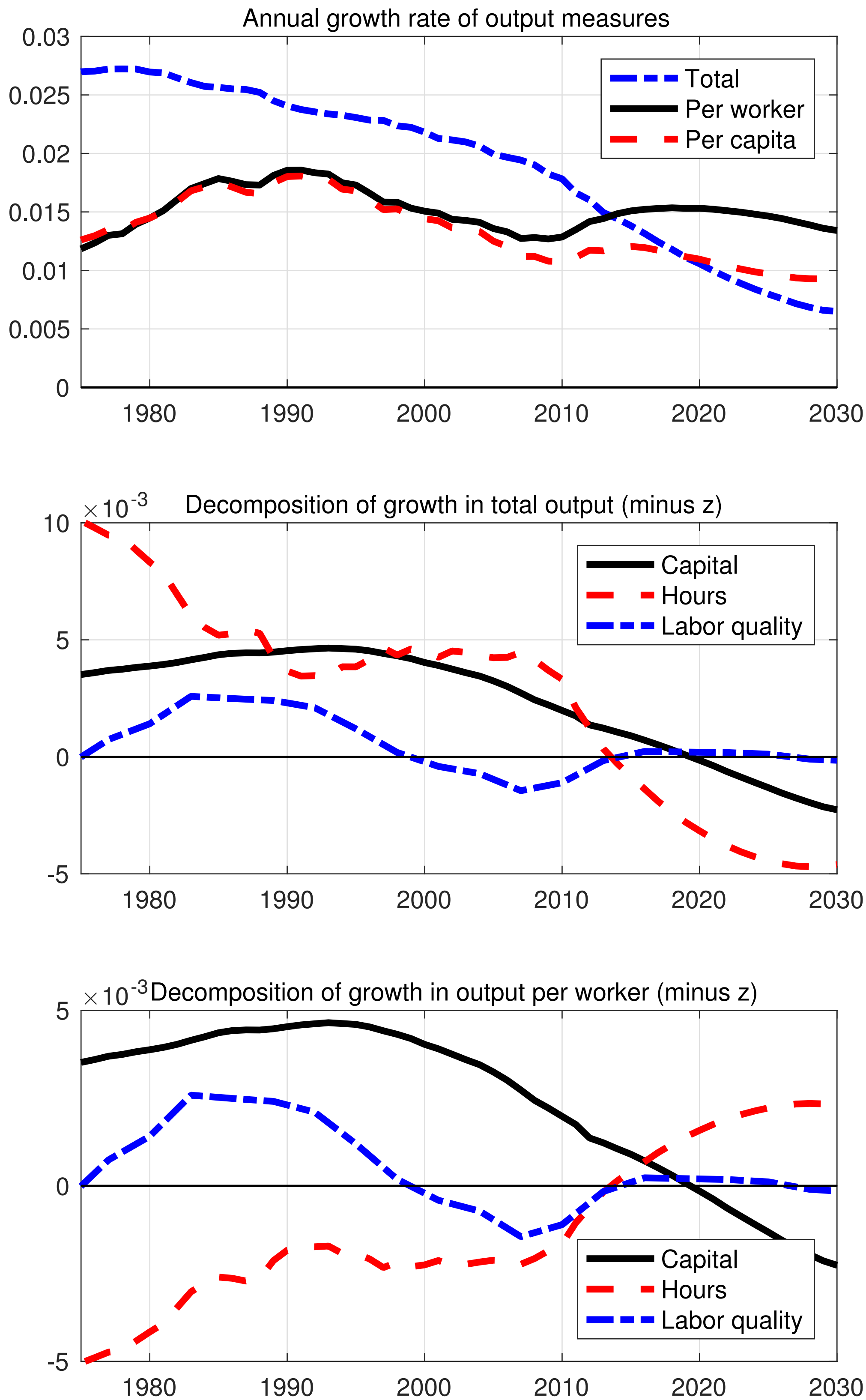

Figure 10: Output Growth Due to Demographics Only

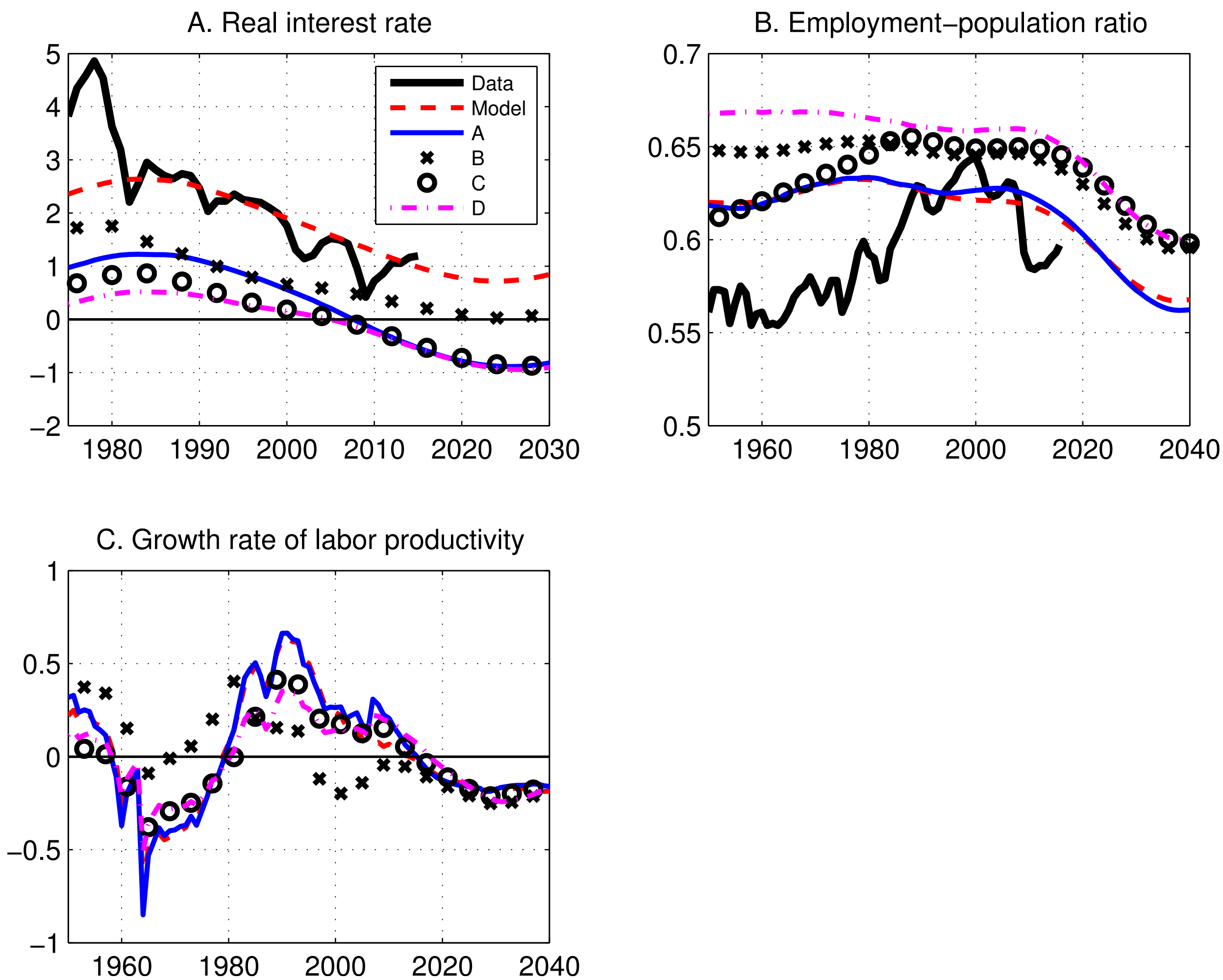

Figure 11: Robustness Exercises. Labor productivity is net of z.