International Spillovers of Forward Guidance Shocks

Appendix

For Online Publication

Callum Jones Mariano Kulish Daniel M. Rees

Contents

1 Model Details

This section provides additional details of the model that we do not present in the text.

1.1 Large Economy

Labour unions A continuum of perfectly competitive labour aggregating firms combine the specialised labour types according to the technology:

$$N_t^* = \left[\int_0^1 N_t^*(j)^{\frac{\epsilon_w^*-1}{\epsilon_w^*}} \, \mathrm{d}j\right]^{\frac{\epsilon_w^*}{\epsilon_w^*-1}}.$$

Competition between the labour aggregating firms ensures that the nominal wage paid to aggregate labour and demand for individual varieties are given by:

$$W_t^* = \left[\int_0^1 W_t^*(j)^{1-\epsilon_w^*} \, \mathrm{d}j\right]^{\frac{1}{1-\epsilon_w^*}}, \text{ and } N_{t+s}^*(j) = \left(\frac{W_t^*(j)\Omega_{t,t+s}^{w*}}{W_{t+s}^*}\right)^{-\epsilon_w^*} N_{t+s}^*.$$

Workers of type j unionise in order take exploit their monopoly power. Like Erceg et al. (2000) we assume that these unions are subject to a Calvo-style friction such that each quarter only a fraction of unions, $1-\theta_w^*$, are able to set wages optimally. Unions that do *not* re-optimise follow an indexation rule that links wages growth to a weighted average of lagged wage inflation and steady-state wages growth:

$$\breve{W}_t^*(j) = \left(\Pi_{t-1}^{w*}\right)^{\chi_w^*} \left(\bar{\Pi}^*\mathcal{M}\right)^{1-\chi_w^*} W_{t-1}^*(j),$$

where $\breve{W}_t^*(j)$ is union j's wage conditional on *not* re-optimising in period t, $\Pi_t^{w*} = W_t^*/W_{t-1}^*$ is aggregate wage inflation and $\bar{\Pi}^*\mathcal{M}$ is steady state nominal wages growth, equal to the product of the central bank's inflation target, $\bar{\Pi}^*$, and steady state labour productivity growth, $\mathcal{M}$.

The wage setting problem for a union that is able to reset its wages at time t is:

$$\max_{W_t^*(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_w^*)^s \left[(1+\iota_w^*)\frac{\Lambda_{t+s}^*}{P_{t+s}^*} W_{t+s}^*(j)\Omega_{t,t+s}^{w*} N_{t+s}^*(j) - \frac{1}{1+\varphi} N_{t+s}(j)^{*1+\varphi}\right],$$

subject to the labour demand constraint given above. $\Omega_{t,t+s}^{w*}$ is the cumulative wage growth between period t and $t+s$ for a union that does not re-optimise, Λ_{t+s}^* is the shadow price of consumption in period $t+s$ and ι_w^* is a wage subsidy calibrated to offset the steady state distortion associated with monopolistic competition in the labour market.[1]

Firms The large economy's final good is produced by a representative firm that aggregates individual varieties according to the production function:

$$Y_t^* = \left[\int_0^1 Y_t^*(i)^{\frac{\epsilon_p^*-1}{\epsilon_p^*}} \, \mathrm{d}i\right]^{\frac{\epsilon_p^*}{\epsilon_p^*-1}}.$$

[1] Given the indexing rule, $\Omega_{t,t+s}^{w*} = \left(\bar{\Pi}^*\mathcal{M}\right)^{(1-\chi_w^*)s} \prod_{k=t}^{t+s-1} \left(\Pi_k^{w*}\right)^{\chi_w^*}$.

Perfect competition implies that the aggregate price index and demand functions for individual varieties are:

$$P_t^* = \left[\int_0^1 P_t^*(i)^{1-\epsilon_p^*}\ \mathrm{d}i\right]^{\frac{1}{1-\epsilon_p^*}},\ \text{and}\ Y_{t+s}^*(i) = \left(\frac{P_t^*(i)\Omega_{t,t+s}^*}{P_{t+s}^*}\right)^{-\epsilon_p^*} Y_{t+s}^*.$$

Intermediate goods are produced by monopolistically-competitive firms with the technology:

$$Y_t^*(i) = Z_t L_t^*(i),$$

where $Y_t^*(i)$ is the production and $L_t^*(i)$ is firm i's input of aggregate labour.[2] Z_t is the trend component of productivity, which in logs follows a random walk with drift that grows at the rate $\mathcal{M}$.

Intermediate firms face Calvo-style pricing frictions. Each quarter, a fraction of firms, $1-\theta^*$, sets prices optimally. The remaining firms that do not re-optimise their prices follow an indexation rule that links prices growth to a weighted average of lagged CPI inflation and the central bank's inflation target:

$$\breve{P}_t^*(i) = \left(\Pi_{t-1}^*\right)^{\chi_p^*}\left(\bar{\Pi}^*\right)^{1-\chi_p^*} P_{t-1}^*(i),$$

where $\breve{P}_t^*(i)$ is firm i's price condition on not re-optimising in period t and $\Pi_t^* = P_t^*/P_{t-1}^*$ is CPI inflation.

The pricing problem for a firm that re-optimises at time t is:

$$\max_{\{P_t^*(i)\}} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\theta_p^*)^s \left\{\Lambda_{t+s}^*\left[P_t^*(i)\Omega_{t,t+s}^* Y_{t+s}^*(i) - \frac{1}{1+\iota_p^*}\frac{W_{t+s}^*(i)}{P_{t+s}^* Z_{t+s}} Y_{t+s}^*(i)\right]\right\},$$

subject to the demand constraint given above. The term ι_p^* is a production subsidy that offsets the distortionary effects of monopolistic competition on the steady state and $\Omega_{t,t+s}^*$ is the cumulative price growth between t and $t+s$ is the firm does not re-optimise.[3].

1.2 Small Economy

Domestic Final Goods Retailers The domestically-produced final good, $Y_{H,t}$ is assembled by a perfectly competitive retailer that combines domestically-produced intermediate goods using the technology:

$$Y_{H,t} = \left[\int_0^1 Y_{H,t}(i)^{\frac{\epsilon_p-1}{\epsilon_p}}\ \mathrm{d}i\right]^{\frac{\epsilon_p}{\epsilon_p-1}}, \tag{1}$$

where ϵ_p is the elasticity of substitution between varieties of domestic intermediate goods. The price of the domestic final good and demand for individual varieties are given by:

$$P_{H,t} = \left[\int_0^1 P_{H,t}(i)^{1-\epsilon_p}\ \mathrm{d}i\right]^{\frac{1}{1-\epsilon_p}},\ \text{and}\ Y_{H,t}(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\epsilon_p} Y_{H,t}.$$

[2] Market clearing in the labour market requires that the amount of aggregate labour supplied by households equals the amount of aggregate labour demanded by firms, i.e. that $N_t^* = \int_0^1 L_t^*(i)di$.

[3] $\Omega_{t,t+s}^* = \left(\bar{\Pi}^*\right)^{(1-\chi_p^*)s}\prod_{k=t}^{t+s-1}\left(\Pi_k^*\right)^{\chi_p^*}$

Domestic Intermediate Goods Producers The pricing problem for a firm i that does re-optimise is

$$\max_{P_{H,t}(i)} \sum_{s=0}^{\infty} (\beta\theta_p)^s \mathbb{E}_t \left\{ \Lambda_{t+s} \left[\frac{P_{H,t}(i)\Omega_{t,t+s}\Gamma_{H,t+s}}{P_{H,t+s}} Y_{H,t+s}(i) - \frac{1}{1+\iota_p} \frac{W_t}{P_{H,t} Z_t} \Gamma_{H,t+s} Y_{H,t+s}(i) \right] \right\}, \quad (2)$$

subject to the domestic final goods demand condition given above. $\Gamma_{H,t} = P_{H,t}/P_t$ is the relative price of domestically-produced goods, $\Omega_{t,t+s}$ is the cumulative increase in prices for a firm that does not re-optimise between t and $t+s$ and ι_p is a production subsidy calibrated to offsets the distortionary effects of monopolistic competition on steady-state output.

Exporters An export retailer bundles export varieties before selling them overseas according to the technology:

$$X_t = \left[\int_0^1 X_t(i)^{\frac{\epsilon_x - 1}{\epsilon_x}} \, \mathrm{d}i \right]^{\frac{\epsilon_x}{\epsilon_x - 1}}, \quad (3)$$

where $\epsilon_x > 1$ is the elasticity of substitution between different varieties for export. The corresponding price index, in foreign currency terms, and demand function for total exports are given by:

$$P^*_{X,t} = \left[\int_0^1 P^*_{X,t}(i)^{1-\epsilon_x} \, \mathrm{d}i \right]^{\frac{1}{1-\epsilon_x}}, \text{ and } X_t = \alpha_X \left(\frac{P^*_{X,t}}{P^*_t} \right)^{-\tau} Y^*_t.$$

Exporters face Calvo-style pricing frictions, with only a fraction, $1 - \theta_x$, of firms able to adjust their prices each quarter. Firms that do not re-optimise index their prices to steady-state US inflation. The resulting pricing problem for firm i is:

$$\max_{P^*_{X,t}(i)} \sum_{s=0}^{\infty} (\beta\theta_x)^s \mathbb{E}_t \left\{ \Lambda_{t+s} \left[\frac{P^*_{X,t}(i)\Omega^x_{t,t+s}\Gamma_{x,t+s}}{P^*_{X,t+s}} X_{t+s}(i) - \frac{1}{1+\iota_x} \Gamma_{H,t+s} X_{t+s}(i) \right] \right\}, \quad (4)$$

subject to the usual demand constraint. $\Gamma_{x,t+s} = S_t P^*_{X,t}/P_t$ is the relative price between exports (in domestic currency terms) and the domestic CPI and ι_x is a production subsidy calibrated to offset the effect of imperfect competition on steady-state exports.

Importers The pricing problem for a representative firm i is:

$$\max_{P_{F,t}(i)} \sum_{s=0}^{\infty} (\beta\theta_f)^s \mathbb{E}_t \left\{ \Lambda_{t+s} \left[\frac{P_{F,t}(i)\Omega^F_{t,t+s}\Gamma_{F,t+s}}{P_{F,t+s}} Y_{F,t+s}(i) - \frac{1}{1+\iota_f} \frac{S_{t+s} P^*_{t+s}}{P_{t+s}} Y_{F,t+s}(i) \right] \right\}, \quad (5)$$

subject to the demand constraint above. $\Gamma_{F,t} = P_{F,t}/P_t$ is the price of imports (in domestic currency terms) relative to the domestic CPI, $\Omega^F_{t,t+s}$ is the cumulative price change for an importer that does not adjust its prices between t and $t+s$ and ι_f is a subsidy calibrated to offset the effect of imperfect competition on import volumes on steady state.

2 Model Solution with Two Fixed-rate Regimes

In our case, policy interest rates in the US and Canada can be fixed at different periods, or at the same time. The economy can therefore be in one of the following four possible regimes at a given point in our sample: (i) interest rates follow feedback rules, (ii) only the interest rate of the large economy is fixed, (iii) only the interest rate of the small economy is fixed, and (iv) both interest rates are fixed. Figure 1 illustrates one possibility, in which in an initial sub-sample conventional policy applies to both economies, then there is a period of time for which the interest rate is fixed only in the large economy. After that, interest rates are fixed in both economies, and eventually there is a return to conventional policy which takes place out-of-sample.

We first linearize the model around the steady state for which policy rates follow feedback rules, and write the resulting system of equations in matrix form as:

$$\mathbf{A}x_t = \mathbf{C} + \mathbf{B}x_{t-1} + \mathbf{D}\mathbb{E}_t x_{t+1} + \mathbf{F}\varepsilon_t, \tag{6}$$

where x_t is the state vector and ε_t is the vector of structural shocks, which we take to be i.i.d. without loss of generality. If it exists and is unique, the standard rational expectations solution to (6) is $x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}\varepsilon_t$.

When only the *foreign* interest rate is fixed the structural equations are given by:

$$\mathbf{A}^\star x_t = \mathbf{C}^\star + \mathbf{B}^\star x_{t-1} + \mathbf{D}^\star\mathbb{E}_t x_{t+1} + \mathbf{F}^\star\varepsilon_t, \tag{7}$$

where the only equation that has changed in the starred system relative to (6) is the equation defining the *foreign* policy interest rate rule, which is now specified such that the nominal interest rate is fixed.[4]

When only the *domestic* interest rate is fixed the structural equations are given by:

$$\bar{\mathbf{A}}x_t = \bar{\mathbf{C}} + \bar{\mathbf{B}}x_{t-1} + \bar{\mathbf{D}}\mathbb{E}_t x_{t+1} + \bar{\mathbf{F}}\varepsilon_t, \tag{8}$$

where the only equation that has changed relative to (6) is the equation defining the *domestic* policy interest rate rule, which is now specified such that the nominal interest rate is fixed.

And when both *foreign* and *domestic* interest rates are fixed the structural equations are given by:

$$\bar{\mathbf{A}}^\star x_t = \bar{\mathbf{C}}^\star + \bar{\mathbf{B}}^\star x_{t-1} + \bar{\mathbf{D}}^\star\mathbb{E}_t x_{t+1} + \bar{\mathbf{F}}^\star\varepsilon_t. \tag{9}$$

If interest rates follow feedback rules at time t, then $\mathbf{A}_t = \mathbf{A}$, $\mathbf{C}_t = \mathbf{C}$, $\mathbf{B}_t = \mathbf{B}$, and so. If both domestic and foreign policy interest rates are fixed at time t then $\mathbf{A}_t = \bar{\mathbf{A}}^\star$, $\mathbf{C}_t = \bar{\mathbf{C}}^\star$, $\mathbf{B}_t = \bar{\mathbf{B}}^\star$, and so on.

Assume then that at time $t = 1$ agents expect fixed interest rates in the large economy for $\mathbf{d}_1^*$ periods and fixed interest rates in the small economy for $\mathbf{d}_1$ periods. After $\max(\mathbf{d}_1^*, \mathbf{d}_1)$ both economies would have reverted back to (6) and the standard solution applies. From $t = 1$, $\mathbf{d}_1^*$ and $\mathbf{d}_1$ imply an expectation of which of the four possible regimes will be in place at each point in time. Let the expected regimes be summarised by the sequence

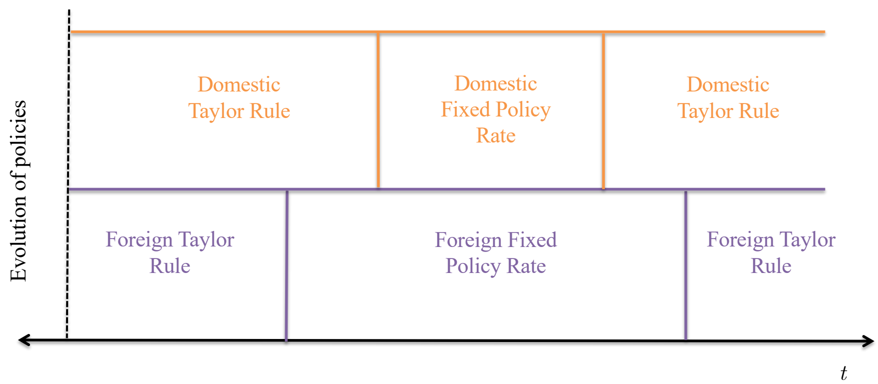
$$\{\mathbf{A}_t, \mathbf{C}_t, \mathbf{B}_t, \mathbf{D}_t, \mathbf{F}_t\}_{t=1}^{\max(\mathbf{d}_1^*, \mathbf{d}_1)}.$$

Following Kulish and Pagan (2017), the solution is a time-varying coefficient VAR of the form

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t\varepsilon_t, \tag{10}$$

[4] The notation accommodates additional structural changes which have to be accounted for if the expansion point of the approximation changes. In our application we work around the intended steady state.

Figure 1: Timing and Four Possible Regimes in the Model



Note: This figure shows the timing and 4 possible regimes that the model solution accounts for. During the sample period, the US interest rate can either be governed by a policy rule (foreign conventional policy) or fixed at its lower bound. For each of these two regimes, the Canadian interest rate can either be governed by a rule (domestic conventional policy) or at its lower bound.

where the reduced form matrices solve the recursions below:

$$\begin{aligned} \mathbf{J}_t &= \left[\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}(\mathbf{C}_t + \mathbf{D}_t\mathbf{J}_{t+1}) \\ \mathbf{Q}_t &= \left[\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}\mathbf{B}_t \\ \mathbf{G}_t &= \left[\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}\mathbf{F}_t. \end{aligned} \tag{11}$$

With the solution in hand for a given foreign and domestic duration, the likelihood is constructed following Kulish, Morley and Robinson (2017). We denote the estimated durations of the small economy by $\{\mathbf{d}_t\}_{t=1}^{T}$ and those of the large economy by $\{\mathbf{d}_t^*\}_{t=1}^{T}$ and adopt the convention that in periods where the Taylor rule is in operation in the small economy, $\mathbf{d}_t = 0$, and in the large econonmy, $\mathbf{d}_t^* = 0$.

3 Lower Bound Implementation

There are two occasionally binding lower bound constraints to impose in this model, one to the US nominal interest rate, and one to the Canadian nominal interest rate. A flexible algorithm is developed that relies on constructing a perfect foresight path of the nominal interest rate in both countries, and piecing together linear systems in a step-by-step way. These methods are based on the solution concepts developed in Cagliarini and Kulish (2013); Kulish and Pagan (2017), Guerrieri and Iacoviello (2015) and Jones (2017). As shown in these papers, the approximation does a good job at capturing the non-linear effects induced by the occasionally binding constraints.

3.1 Notation

Denote by x_t^* the vector of endogenous variables for the large country at time period t, one of which is the nominal interest rate in the large country, and x_t the vector of endogenous variables for the small country at time period t, one of which is the nominal interest rate R_t. The initial conditions are $[x^*_{t-1}{}'\ x_{t-1}{}']'$ and the initial vector of unanticipated exogenous variables, denoted by ε_t. The model is a system of n equations.

3.2 Initialization at t

We know, at period t:

- The shock that hits at period t: ε_t.
- The initial vector of variables x_{t-1}.

3.3 The Algorithm

The steps of the algorithm are:

0. Linearize the model around the non-stochastic steady state, ignoring the lower bounds in both countries.

1. For each period t:

   For the large country:

   (a) Solve for the path $\{x_\tau\}_{\tau=t}^{T}$ with T large, using the solution of the linearized economy from step (0), given ε_t and the initial vector of variables x_{t-1}, and assuming no future uncertainty. This gives a path for the nominal interest rate, $\mathbf{i}_t^k = \{i_\tau^k\}_{\tau=t}^{T}$.

   (b) Examine the path $\mathbf{i}_t^k$. If $\mathbf{i}_t^k \geq 0$, then the lower bound does not bind, so move onto step (2). If $\mathbf{i}_t^k < 0$, then move onto step (1c).

   (c) For the *first* time period where $\mathbf{i}_t^k < 0$, set the nominal interest rate in that period to zero. This changes the anticipated structure of the economy. Under this new structure, calculate the path of all variables, including the new path for the nominal interest rate $\mathbf{i}_t^{k+1} = \{i_\tau^{k+1}\}_{\tau=t}^{T}$.

   Iterate on steps 1a and 1c until convergence of $\mathbf{i}_t^*$.

   Repeat steps 1a to 1c for the small country.

2. Increment t by one. The initial vector of variables now becomes x_t, which was solved for in step 1. Draw a new vector of unanticipated shocks ε_{t+1} and return to step 1.

To compute the path $\{x_\tau\}_{\tau=t}^{T}$ under forward guidance, compute step (1c) first, imposing the sequence of structural matrices corresponding to the lower bound and non-lower bound periods. Then examine the path $\{i_\tau\}_{\tau=t}^{T}$ for subsequent violations of the lower bound.

3.4 Details of Each Step

At the following steps:

0. As above, write the n equations of the linearized structural model at t as:

$$\mathbf{A}x_t = \mathbf{C} + \mathbf{B}x_{t-1} + \mathbf{D}\mathbb{E}_t x_{t+1} + \mathbf{F}\varepsilon_t, \tag{SM}$$

where x_t is a $n \times 1$ vector of state and jump variables and ε_t is a $l \times 1$ vector of exogenous variables. Use standard methods to obtain the reduced form:

$$x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}\varepsilon_t. \tag{RF}$$

1. For each period t:

   (a) Using (RF), obtain the path $\{x_\tau\}_{\tau=t}^{T}$ given ε_t. Set T to be large. Assume $\{w_\tau\}_{\tau=t+1}^{T} = 0$ (no future uncertainty), so that $x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}\varepsilon_t$, and $x_{t+1} = \mathbf{J} + \mathbf{Q}x_t$, up to $x_T = \mathbf{J} + \mathbf{Q}x_{T-1}$.
   This step gives a path $\mathbf{i}_t = \{i_\tau\}_{\tau=t}^{T}$.

   (b) Examine the path $\{i_\tau\}_{\tau=t}^{T}$.

   - If $i_\tau \geq 0$ for all $t \leq \tau < T$, accept $\{x_\tau\}_{\tau=t}^{T}$. The i_t path does not violate lower bound today or in future.
   - If $i_\tau < 0$ for any $t \leq \tau < T$, move to step (1c).

   (c) Update the path of $\{i_\tau\}_{\tau=t}^{T}$ for the lower bound. For the first time period t^* where $i_{t^*} < 0$, set $i_{t^*} = 0$. The model system at t^* therefore becomes:

$$\mathbf{A}^* x_{t^*} = \mathbf{C}^* + \mathbf{B}^* x_{t^*-1} + \mathbf{D}^* \mathbb{E}_{t^*} x_{t^*+1} + \mathbf{F}^* w_{t^*}, \tag{12}$$

   Compute the new path $\{i_\tau\}_{\tau=t}^{T}$. This involves computing $\{x_\tau\}_{\tau=t}^{t^*}$ and $\{x_\tau\}_{\tau=t^*+1}^{T}$. At t^*, $\mathbb{E}_{t^*} x_{t^*+1}$ is computed using the the reduced form solution (RF) and $w_{t^*+1} = 0$. This expresses x_{t^*} as a function of x_{t^*-1}. Proceeding in this way with the correct structural matrices (either lower bound $*$ or no lower bound at each time period), compute the path $\{i_\tau\}_{\tau=t}^{T}$.
   A convenient way to compute the new path $\{i_\tau\}_{\tau=t}^{T}$ is to form the time varying matrices $\{\mathbf{J}_\tau, \mathbf{Q}_\tau, \mathbf{G}_\tau\}_{\tau=t}^{T}$ which satisfy the recursion (11), with the final set of reduced form matrices for the recursion being the non-lower bound matrices $\mathbf{J}$, $\mathbf{Q}$, $\mathbf{G}$ from (RF).
   These time-varying matrices are then used to compute the path $\{x_\tau\}_{\tau=t}^{T}$ by calculating $x_\tau = \mathbf{J}_\tau + \mathbf{Q}_\tau x_{\tau-1} + \mathbf{G}_\tau w_\tau$.

3.5 Output of the Algorithm

The algorithm yields a set of time-varying structural matrices:

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t \varepsilon_t, \tag{13}$$

from which we get the path of $\{x_\tau\}_{\tau=t}^{\infty}$ where the nominal interest rate is subject to the lower bound. Both the current value of the nominal interest rate, and expectations of the lower bound binding, affect current values of state variables.

3.6 Identifying Forward Guidance

Here, we explain how to use the algorithm in Section 3.3 to decompose an anticipated duration of the lower bound into a component due to structural shocks, and a component due to forward guidance. Assume that at period t, the lower bound binds and we have used procedures to estimate the model parameters and the anticipated length of the lower bound at period t. We have in hand at period t:

1. An estimated duration $\tilde{T}$ of the lower bound at t, so that the interest rate is expected to stay at zero until time period $t + \tilde{T}$.
2. An estimate of the history of the states $\{x_\tau\}_{\tau=0}^{t-1}$ and an estimate of the structural shocks $\{w_\tau\}_{\tau=1}^{t}$, computed using the Kalman smoother.

The estimated parameters, durations and shocks recover the observed series and give an estimate of the model's state variables x_t. To decompose the proportion of the estimated duration due to structural shocks, so that the remainder is due to forward guidance policies, at each point of time:

1. Use the state x_{t-1} and the structural shock ε_t to compute, using the lower bound algorithm of Section 3.3, the endogenous duration of the lower bound.
2. If the computed endogenous duration is less than the estimated duration, then the additional time is assigned to commitment forward guidance policy.

The endogenous duration is the duration that would have occurred had the central bank simply set the nominal interest rate to zero in periods where the policy rule would have specified that it be negative, and set the interest rate to its positive value when the policy rule specifies that it be positive.

3.7 Kalman Filter

The model in state space representation is:

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t \varepsilon_t \tag{State Eqn}$$
$$z_t = \mathbf{H}_t x_t. \tag{Obs Eqn}$$

The structural shocks are Gaussian, so that $\varepsilon_t \sim N(0, \mathbf{Q})$, where $\mathbf{Q}$ is the covariance matrix of ε_t. The Kalman filter recursion is given by the following equations. The state of the system is the state vector and its covariance matrix $(\hat{x}_t, P_{t-1})$. The 'predict step' involves using the structural matrices $\mathbf{J}_t$, $\mathbf{Q}_t$ and $\mathbf{G}_t$:

$$\hat{x}_{t|t-1} = \mathbf{J}_t + \mathbf{Q}_t \hat{x}_t \tag{14}$$
$$\mathbf{P}_{t|t-1} = \mathbf{Q}_t \mathbf{P}_{t-1} \mathbf{Q}_{t|t-1}^\top + \mathbf{G}_t \mathbf{Q} \mathbf{G}_t^\top. \tag{15}$$

Note that $\mathbf{H}_t$ is time-varying, reflecting that when the nominal interest rate is at its lower bound, we lose it as an observable variable. The update step involves computing forecast errors $\tilde{y}_t$ and its associated covariance matrix $\mathbf{S}_t$:

$$\tilde{y}_t = z_t - \mathbf{H}_t \hat{x}_{t|t-1} \tag{16}$$
$$\mathbf{S}_t = \mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^\top. \tag{17}$$

The Kalman gain matrix is given by:

$$\mathbf{K}_t = \mathbf{P}_{t|t-1}\mathbf{H}_t^\top \mathbf{S}_t^{-1}. \tag{18}$$

With $\tilde{y}_t$, $\mathbf{S}_t$ and $\mathbf{K}_t$ in hand, the optimal update of the state x_t and its associated covariance matrix is:

$$\hat{x}_t = \hat{x}_{t|t-1} + \mathbf{K}_t\tilde{y}_t \tag{19}$$

$$\mathbf{P}_t = (I - \mathbf{K}_t\mathbf{H}_t)\,\mathbf{P}_{t|t-1}. \tag{20}$$

The Kalman filter is initialized with x_0 and $\mathbf{P}_0$ computed from their unconditional moments. The recursion is computed until the final time period T of data.

3.8 Kalman Smoother

With the estimates of the parameters and durations in hand at time period T, the Kalman smoother gives an estimate of $x_{t|T}$, or an estimate of the state vector at each point in time given all available information. With $\hat{x}_{t|t-1}$, $\mathbf{P}_{t|t-1}$, $\mathbf{K}_t$ and $\mathbf{S}_t$ in hand from the filter, the vector $x_{t|T}$ is computed by:

$$x_{t|T} = \hat{x}_{t|t-1} + \mathbf{P}_{t|t-1}r_{t|T}, \tag{21}$$

where the vector $r_{T+1|T} = 0$ and is updated with the recursion:

$$r_{t|T} = \mathbf{H}_t^\top \mathbf{S}_t^{-1}\left(z_t - \mathbf{H}_t\hat{x}_{t|t-1}\right) + (I - \mathbf{K}_t\mathbf{H}_t)^\top\,\mathbf{P}_{t|t-1}^\top r_{t+1|T}. \tag{22}$$

Finally, to get an estimate of the shocks to each state variable, denoted by e_t, we compute:

$$e_t = \mathbf{G}_t\varepsilon_t = \mathbf{G}_t r_{t|T}. \tag{23}$$

3.9 Sampler

We use the same sampler as in Kulish, Morley and Robinson (2017). Denote by ϑ the vector of parameters to be estimated and $\mathbf{T}$ the vector of durations to be estimated. Contained in $\mathbf{T}$ are a set of durations for both the foreign and domestic countries. Denote by $Z = \{z_\tau\}_{\tau=1}^T$ the sequence of observable vectors. The posterior $\mathcal{P}(\vartheta, \mathbf{T} \mid Z)$ satisfies:

$$\mathcal{P}(\vartheta, \mathbf{T} \mid Z) \propto \mathcal{L}(Z \mid \vartheta, \mathbf{T}) \times \mathcal{P}(\vartheta, \mathbf{T}). \tag{24}$$

With Gaussian errors, the likelihood function $\mathcal{L}(Z \mid \vartheta, \mathbf{T})$ is computed using the appropriate sequence of structural matrices and the Kalman filter:

$$\log\mathcal{L}(Z \mid \vartheta, \mathbf{T}) = -\left(\frac{N_z T}{2}\right)\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\log\det\mathbf{H}_t\mathbf{S}_t\mathbf{H}_t^\top - \frac{1}{2}\sum_{t=1}^{T}\tilde{y}_t^\top\left(\mathbf{H}_t\mathbf{S}_t\mathbf{H}_t^\top\right)^{-1}\tilde{y}_t.$$

For practical convenience, we require that each estimated duration lies below some maximum value T^* which, in practice, is rarely visited by the sampler.

The Markov Chain Monte Carlo posterior sampler has two blocks, corresponding to ϑ and $\mathbf{T}$. Initialize the sampler at step j with the last accepted draw of the structural parameters, the period of the breaking parameters and durations, denoted by ϑ_{j-1} and $\mathbf{T}_{j-1}$ respectively. The blocks are, in order of computation:

1. In the first block, randomly choose up to $\bar{T}$ durations to test in each country, corresponding to up to $\bar{T}$ time periods that each economy is at the lower bound. For each of those time periods, randomly choose a duration in the interval $[1, T^*]$ for each country to generate a new $\mathbf{T}_j$ proposal. Recompute the sequence of structural matrices associated with $(\vartheta_{j-1}, \mathbf{T}_j)$, compute the posterior $\mathcal{P}(\vartheta_{j-1}, \mathbf{T}_{j-1} \mid Z)$, and accept the proposal $(\vartheta_{j-1}, \mathbf{T}_j)$ with probability $\frac{\mathcal{P}(\vartheta_{j-1}, \mathbf{T}_j|Z)}{\mathcal{P}(\vartheta_{j-1}, \mathbf{T}_{j-1}|Z)}$. If $(\vartheta_{j-1}, \mathbf{T}_j)$ is accepted, then set $\mathbf{T}_{j-1} = \mathbf{T}_j$.

2. The second block is a more standard Metropolis-Hastings random walk step. Start by selecting which structural parameters to propose a new value for. For those parameters, draw a new proposal ϑ_j from a proposal density centered at ϑ_{j-1} chosen to ensure sufficient coverage of the parameter space. The proposal ϑ_j is accepted with probability $\frac{\mathcal{P}(\vartheta_j, \mathbf{T}_{j-1}|Z)}{\mathcal{P}(\vartheta_{j-1}, \mathbf{T}_{j-1}|Z)}$. If $(\vartheta_j, \mathbf{T}_{j-1})$ is accepted, then set $\vartheta_{j-1} = \vartheta_j$.

3.10 A Worked Example of the Algorithm

Consider the simple example, log-linearized around steady-state where y_t is output and the nominal interest rate i_t ignores the ZLB:

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - \bar{i}) + \varepsilon_t \\ i_t - \bar{i} &= \rho \left(i_{t-1} - \bar{i}\right) + \gamma y_t. \end{aligned}$$

Putting this model in the form of (SM) requires $x_t = [\; i_t \quad y_t \;]'$ and:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -\gamma \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ \rho & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \bar{i} \\ \bar{i}(1-\rho) \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The routines of Sims (2002) are used to obtain:

$$x_t = \mathbf{J} + \mathbf{Q} x_{t-1} + \mathbf{G} \varepsilon_t. \tag{25}$$

The algorithm proceeds as follows. Given a shock ε_t:

1. Using the reduced form system without the ZLB (25), obtain the path $\mathbf{y}_t$ up to some large T. Assume no future shocks:

$$\begin{aligned} x_t &= \mathbf{J} + \mathbf{Q} x_{t-1} + \mathbf{G} \varepsilon_t \\ x_{t+1} &= \mathbf{J} + \mathbf{Q} x_t \\ &\vdots \\ x_T &= \mathbf{J} + \mathbf{Q} x_{T-1}. \end{aligned}$$

2. Examine $\{i_\tau\}_{\tau=t}^{T}$. If $i_\tau > 0 \;\forall\; \tau$, then stop the algorithm. Otherwise, move to the next step.

3. Find the first time period where $i_\tau < 0$. Suppose $i_{t+1} < 0$ under the shock ε_t. Then, we want the following system to apply at time period $t+1$:

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - \bar{i}) + \varepsilon_t \\ i_t &= 0, \end{aligned}$$

and the non-ZLB system to apply for t and time periods $\tau > t+1$. The system at $t+1$ translates into the following structural matrices:

$$\mathbf{A}^*_{t+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B}^*_{t+1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C}^*_{t+1} = \begin{bmatrix} \bar{i} \\ 0 \end{bmatrix}, \quad \mathbf{D}^*_{t+1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{F}^*_{t+1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

while $\Pi^*_{t+1} = \Pi$. Then use the solution to the non-ZLB system to obtain x_{t+j} for $j > t+1$ again assuming no future shocks.

Return to step 2 with the the new path of $\{i_\tau\}_{\tau=t}^T$.

4. Examine the new path of $\{i_\tau\}_{\tau=t}^T$. If $i_\tau > 0 \ \forall \ \tau$, then stop the algorithm: the ZLB applies only for time period $t+1$. Otherwise, move to the next step having already imposed the ZLB at time period $t+1$.

5. Continue iterating until the nominal interest rate satisfies the ZLB across the forecast horizon.

4 Data Sources

The model is estimated using 15 macroeconomic time series.

- **US real output growth**: The quarterly log change in US real GDP per capita. We construct the latter series by dividing US real GDP, seasonally adjusted (FRED GDPC1) by the US civilian population aged over 16 years (FRED CNP16OV).

- **US real consumption growth**: The quarterly log change in real US personal consumption expenditures (Fred code PCE) divided by the population (FRED CNP16OV).

- **US inflation**: The quarterly log change in the US core PCE price index, seasonally adjusted (FRED PCEPILFE).

- **US policy rate**: The quarterly average of the target US Federal Funds rate (FRED DFF).

- **US 2 year bond yield**: The quarterly average of the US 2-year constant maturity treasury bond yield (FRED GS2).

- **US Nominal Wages**: The quarterly log change in average hourly earnings of private sector production and non-supervisory employees, seasonally adjusted, (FRED code AHETPI).

- **Canada real GDP growth**: The quarterly log change in Canadian real GDP per capita. We construct the latter series by dividing Canadian real GDP, seasonally adjusted (CANSIM 380-0064) by the Canadian Working Age Population (CANSIM 051-0001).

- **Canada real consumption growth**: Quarterly log change in household consumption per capita (Statcan Table 37-10-0107-01).

- **Canada inflation**: The quarterly log change in the Canadian CPI excluding food, energy and indirect taxes, seasonally adjusted (CANSIM 326-0022).

- **Canada policy rate**: The quarterly average of the Bank of Canada target rate rate (CANSIM v122530).
- **Canada nominal wages**: The quarterly log change in total compensation of employees divided by number of workers (Fred Codes, CANCOMPQDSNAQ / LFEMTTTTCAQ647S).
- **Canada 2 year bond yield**: The quarterly average of the Canadian 2-year constant maturity treasury bond yield (CANSIM v122538).
- **Canada-US exchange rate**: The log change in the quarterly average level of the Canada-US exchange rate (CANSIM 176-0064).
- **Canada import volumes per capita, growth**: The log change in quarterly import volumes per capita (CANSIM 380-0064).
- **Canada export growth**: The log change in quarterly export volumes per capita (CANSIM 380-0064).

Our economic model implies some relatively strong restrictions on the joint behavior of the observed variables. For example, balanced growth requires that wages grow at the product of labour productivity growth and inflation, while exports, imports, consumption and GDP should all grow at the same rate. In practice, the relative growth rates of many of the observed variables differs from that implied by the model. Because many of these differences reflect economic forces, including changes in worker bargaining power, or in tariff barriers, that are outside the scope of our empirical exercise, we accommodate them by transforming the data. The specific transformations we make are:

1. We calculate the average growth rate of US and Canadian GDP per capita prior to 2009. Define this value as $\bar{Y}$.
2. We adjust the means of US and Canadian GDP, consumption and trade data so that they equal $\bar{Y}$ prior to 2009.
3. We adjust the mean of wages growth so that it equals $\bar{Y}$ plus the country-specific average inflation rate prior to 2009.
4. We normalize the change in the Canadian nominal exchange rate so that it equals 0 prior to 2009.

Figure 2 plots the data we use in the estimation.

5 Estimation Diagnostics

Figure 3 plots the Gelman diagnostic statistics on the convergence of the MCMC chains. The R^2 diagnostic converges to 1 over the length of the chain for both parameters and durations.

Figure 2: Quarterly Data

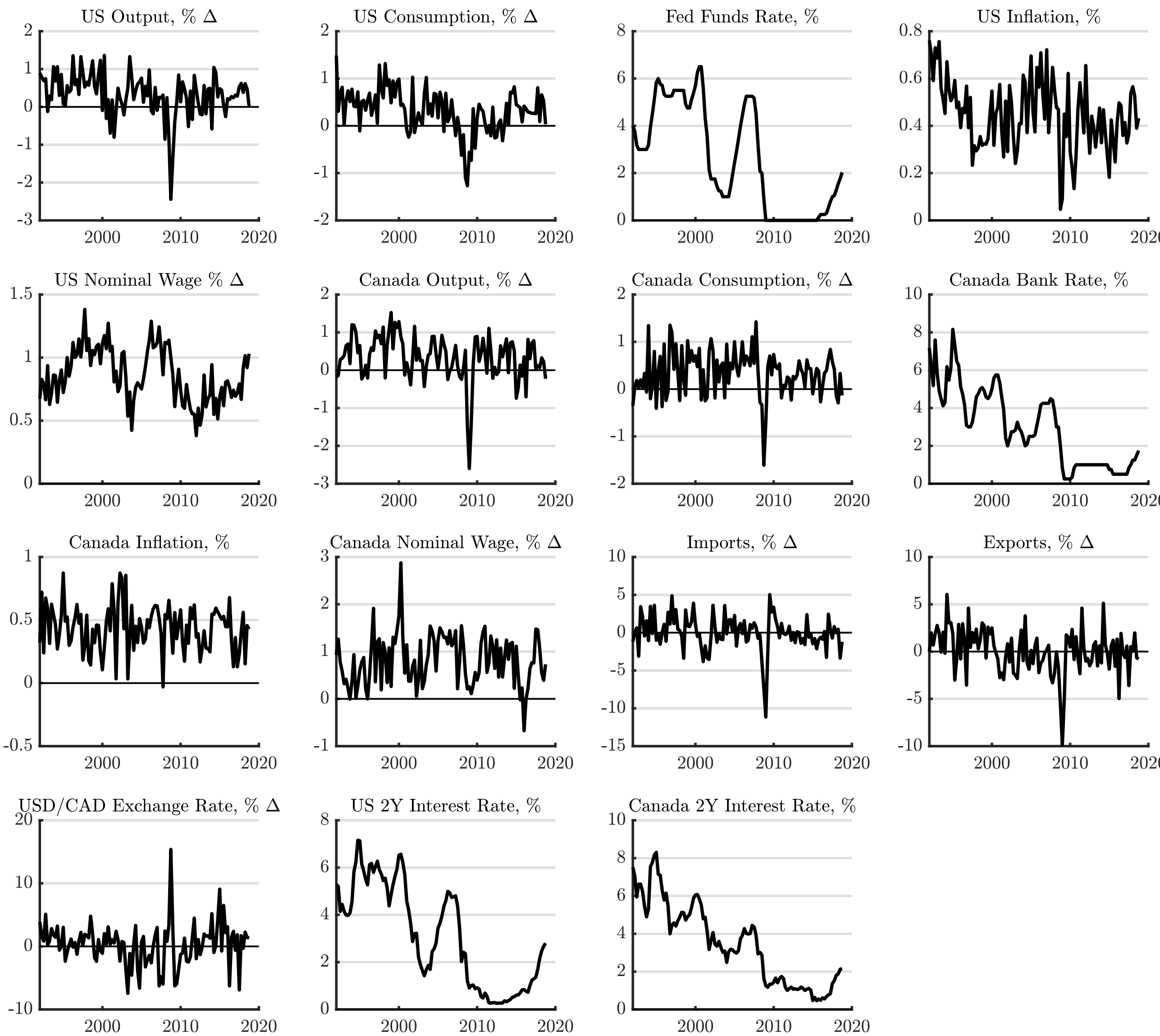
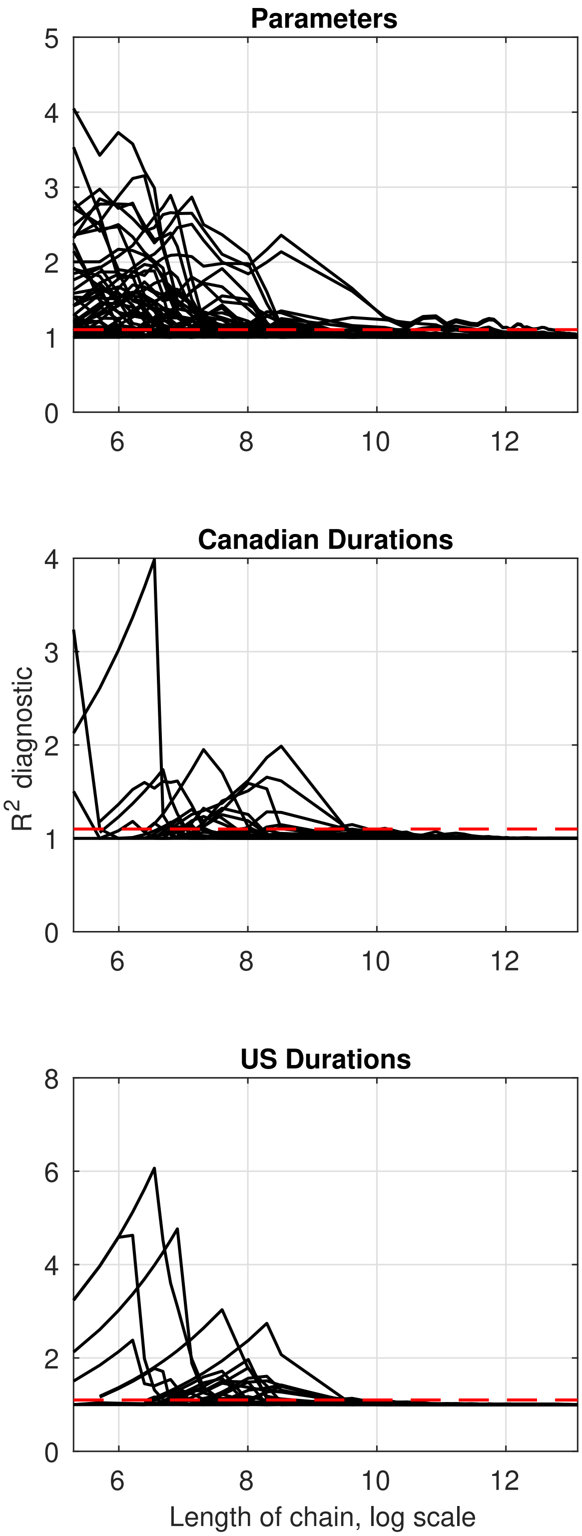


Figure 3: Gelman Chain Diagnostics



6 GIRFs to a Forward Guidance Shock

A forward guidance shock is an unanticipated change in forward guidance component of the expected duration. This gives rise to a change in $\mathbf{d}_t$ orthogonal to the structural shocks. A foreign forward guidance shock at t changes the reduced form matrices that prevail at t as well as those that agent expected to prevail in the forecast horizon according to the solution given in (10) and (11)

We select a base duration, $\mathbf{d}^{\text{base}}$, a quarter of the fixed interest rate regime, t and compute generalized impulse responses conditional on the history of the observed variables. We take draws from the posterior and keep draws for which the foreign duration at t corresponds to our base duration, that is $\mathbf{d}_t^* = \mathbf{d}^{\text{base}}$. For admissible draws from the posterior, the Kalman smoother gives estimates of the state, $\hat{x}_{t-1|T}$ and structural shocks, $\hat{\varepsilon}_{t|T}$. We then assume no future structural shocks, $\varepsilon_{t+j} = 0$, and no future forward guidance shocks, to obtain forecasts for the state

$$\mathbb{E}_t(x_{t+n}|\mathbf{d}_t^* = \mathbf{d}^{\text{base}}, \hat{x}_{t-1|T}, \hat{\varepsilon}_{t|T}), \quad \text{for} \quad n = 0, 1, 2, \ldots$$

where for $n = 0$ we recover the smooth estimate $\hat{x}_{t|T}$. We then consider a foreign forward guidance shock, $\varepsilon_t^{\text{fg}^*}$, which changes the duration from $\mathbf{d}_t^*$ to $\mathbf{d}_t^* + \varepsilon_t^{\text{fg}^*}$. This changes the reduced-form matrices at t and in the forecast horizon as well. We forecast x_t under the forward guidance shock to obtain

$$\mathbb{E}_t(x_{t+n}|\mathbf{d}_t^* = \mathbf{d}^{\text{base}} + \varepsilon_t^{\text{fg}^*}, \hat{x}_{t-1|T}, \hat{\varepsilon}_{t|T}), \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Generalized impulse response are given by

$$\text{GIRF}(x_{t+n}) = \mathbb{E}_t(x_{t+n}|\mathbf{d}_t^* = \mathbf{d}^{\text{base}} + \varepsilon_t^{\text{fg}^*}, \hat{x}_{t-1|T}, \hat{\varepsilon}_{t|T}) - \mathbb{E}_t(x_{t+n}|\mathbf{d}_t^* = \mathbf{d}^{\text{base}}, \hat{x}_{t-1|T}, \hat{\varepsilon}_{t|T}).$$

Figures 4 to 7 plots four generalized impulse responses to a two quarter US forward guidance shock in 2010Q3 and 2013Q3, for base US lower bound durations of two quarters and 6 quarters.

7 Alternative Estimation Specifications: Robustness

7.1 With Discounting in Euler Equation

We estimate the model with discounting in the Euler Equation, with a discount factor of $\gamma = 0.99$. Tables 1 and 2 contain the estimates of the structural parameters and the shock processes. Figure 8 shows the mean of the estimated durations and mean of the decomposed lower bound durations for the US and Canada under these estimates. Compared to our baseline estimates the estimated US interest rate durations are longer by, on average, one quarter, while the decomposition is largely the same.

Figures 9 and 10 plot contemporaneous, and expected 1Y and 2Y-ahead monetary policy shocks, as in Del Negro, Giannoni and Patterson (2012). The impulse responses compared to our baseline specification and with discounting in the Euler equation generates similar impulse responses for the expected monetary policy shocks, with the paths under discounting slightly damped compared to our baseline estimates.

Figure 4: GIRF of 2 Quarter Forward Guidance Shock in 2010Q3, 2Q Base Duration

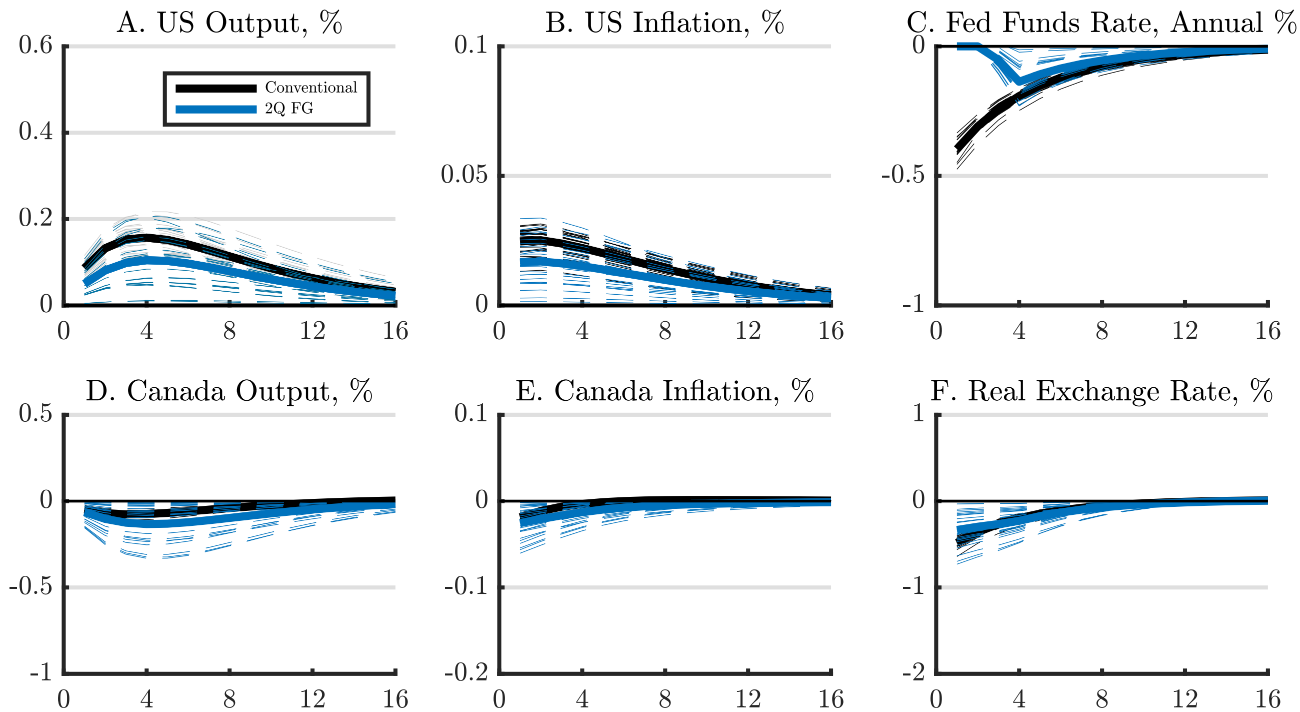


Figure 5: GIRF of 2 Quarter Forward Guidance Shock in 2010Q3, 6Q Base Duration

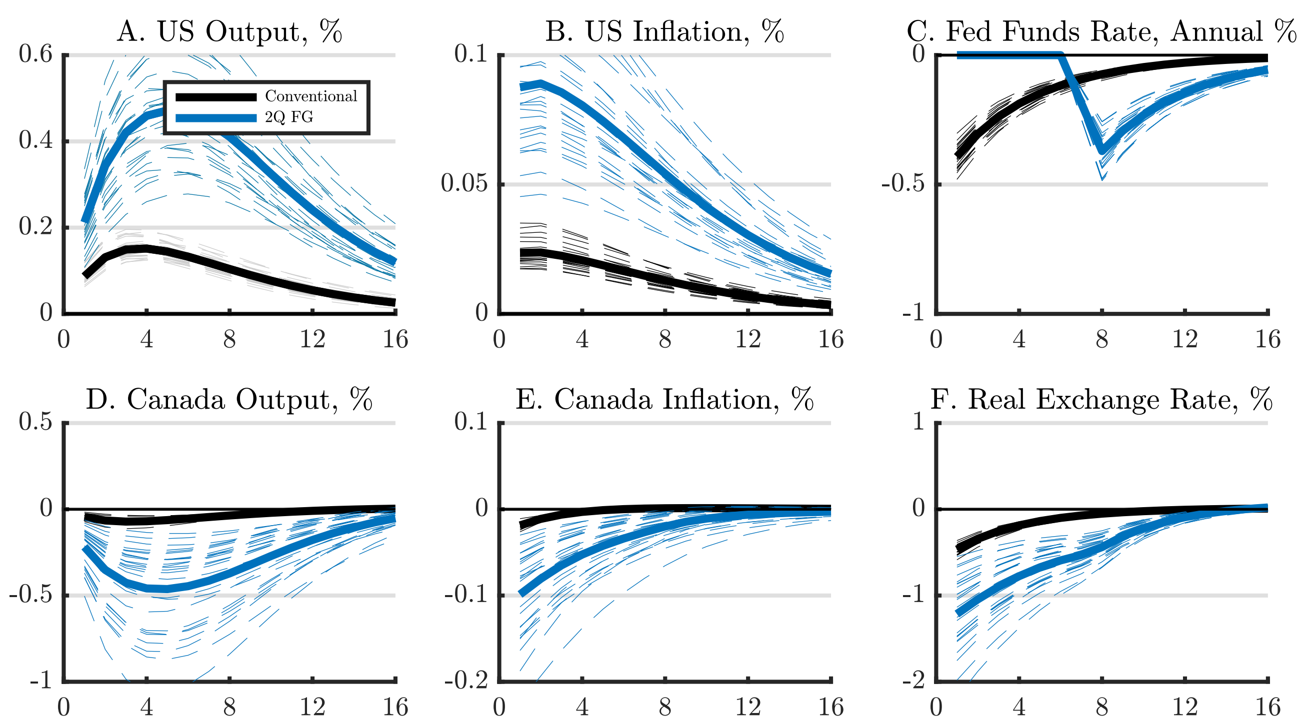


Figure 6: GIRF of 2 Quarter Forward Guidance Shock in 2013Q3, 2Q Base Duration

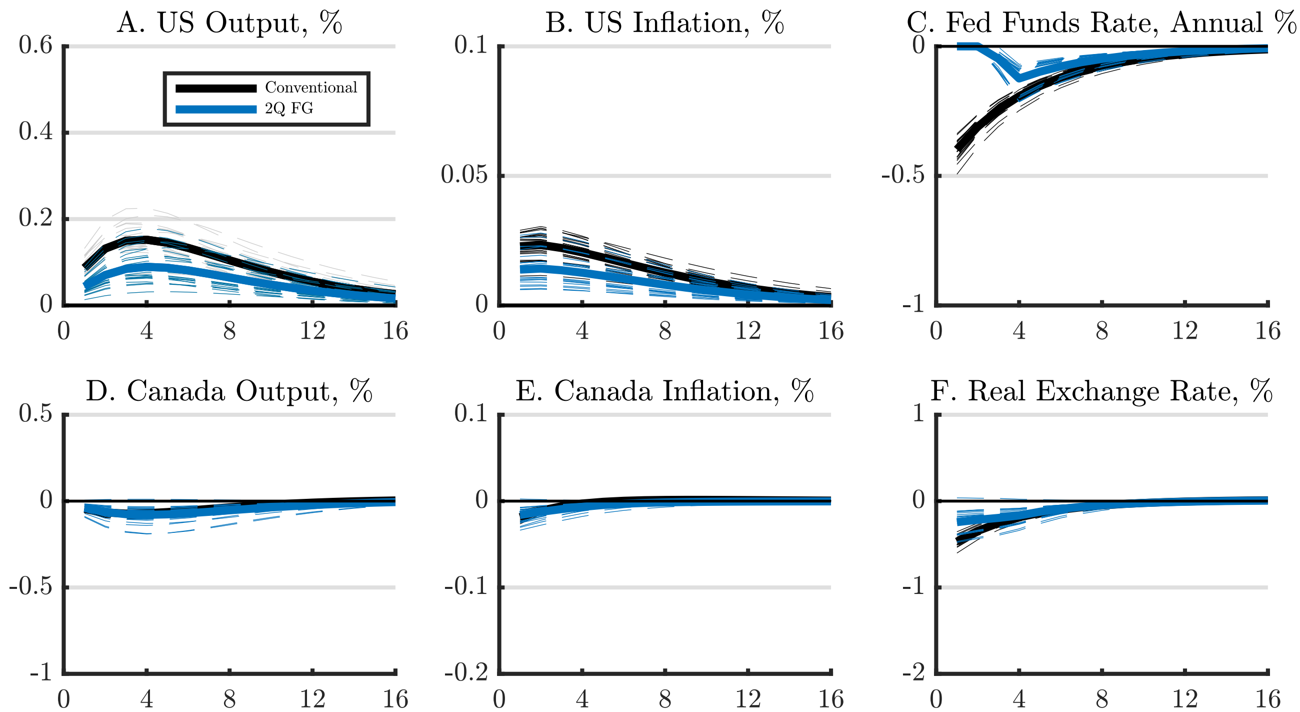


Figure 7: GIRF of 2 Quarter Forward Guidance Shock in 2013Q3, 6Q Base Duration

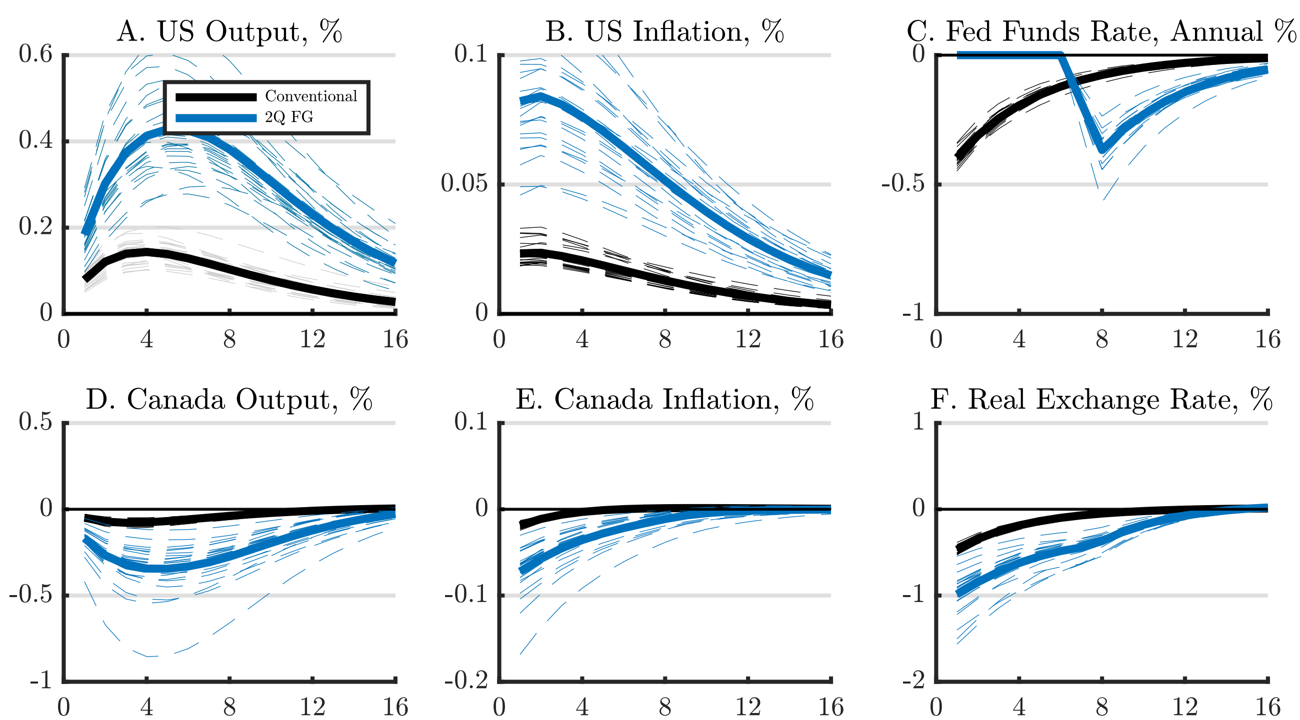


Table 1: Estimated Structural Parameters, $\gamma = 0.99$

	Prior				Posterior			
Parameter	Dist	Median	10%	90%	Mode	Median	10%	90%
				US				
h^*	B	0.7	0.6	0.8	0.70	0.70	0.64	0.76
θ_p^*	B	0.7	0.7	0.8	0.76	0.75	0.73	0.78
θ_w^*	B	0.7	0.7	0.8	0.79	0.79	0.76	0.81
ρ_r^*	B	0.5	0.2	0.8	0.89	0.89	0.87	0.91
ϕ_π^*	N	2.0	1.7	2.3	1.65	1.72	1.46	2.02
ϕ_g^*	G	0.5	0.3	0.7	0.09	0.09	0.07	0.12
ϕ_y^*	G	0.5	0.3	0.7	0.09	0.09	0.07	0.11
c_8^*	N	0.3	0.1	0.8	0.10	0.10	0.06	0.15
				Canada				
h	B	0.7	0.6	0.8	0.81	0.81	0.75	0.84
τ	N	1.0	0.4	1.6	3.01	3.02	2.73	3.38
θ_p	B	0.7	0.7	0.8	0.79	0.79	0.76	0.83
θ_w	B	0.7	0.7	0.8	0.78	0.77	0.74	0.80
θ_x	B	0.7	0.7	0.8	0.78	0.78	0.75	0.80
θ_F	B	0.7	0.7	0.8	0.76	0.76	0.72	0.80
ρ_r	B	0.5	0.2	0.8	0.91	0.91	0.87	0.93
ϕ_π	N	2.0	1.7	2.3	2.16	2.15	1.85	2.45
ϕ_g	G	0.5	0.3	0.7	0.11	0.11	0.09	0.14
ϕ_y	G	0.5	0.3	0.7	0.18	0.17	0.09	0.24
c_8	N	0.3	0.1	0.9	0.12	0.13	0.06	0.19

Figure 8: Fixed Interest Rate Duration and Forward Guidance, Mean Across Draws, $\gamma = 0.99$

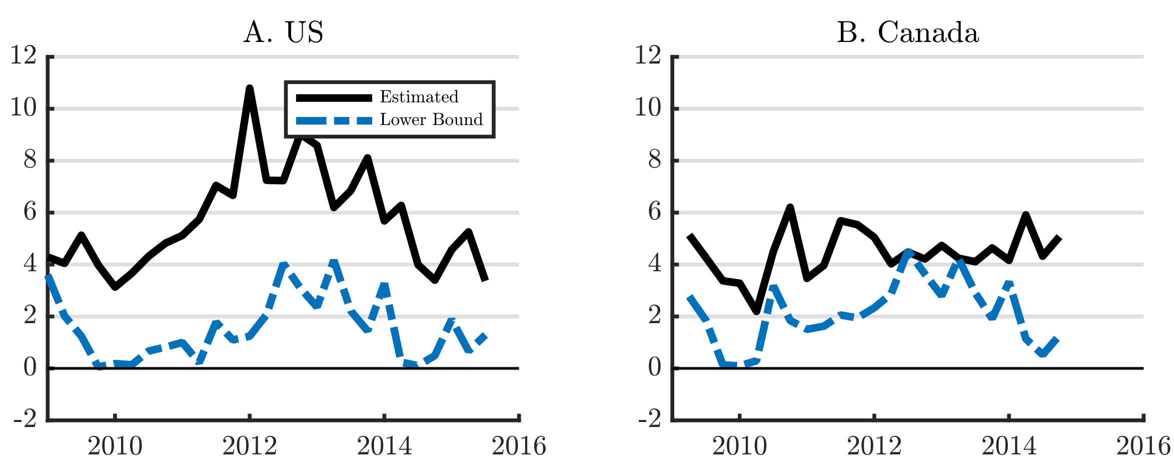


Table 2: Estimated Parameters, Exogenous Processes, $\gamma = 0.99$

	Prior				Posterior			
Parameter	Dist	Median	10%	90%	Mode	Median	10%	90%
				US				
ρ^*_ξ	B	0.5	0.2	0.8	0.95	0.95	0.93	0.96
ρ^*_g	B	0.5	0.2	0.8	0.95	0.95	0.93	0.97
$\rho^*_{\xi_p}$	B	0.5	0.2	0.8	0.99	0.98	0.97	0.99
$\rho^*_{\xi_w}$	B	0.5	0.2	0.8	0.80	0.78	0.67	0.86
ρ^*_{tp}	B	0.5	0.2	0.8	0.73	0.73	0.64	0.82
$100 \times \sigma_z$	IG	0.3	0.1	2.6	0.11	0.11	0.09	0.14
$100 \times \sigma^*_r$	IG	0.3	0.1	0.6	0.11	0.11	0.10	0.12
$100 \times \sigma^*_\xi$	IG	0.3	0.1	0.7	0.25	0.28	0.21	0.41
$10 \times \sigma^*_g$	IG	0.3	0.1	0.7	0.13	0.13	0.12	0.14
$100 \times \sigma^*_{\xi_p}$	IG	0.1	0.1	0.3	0.15	0.15	0.13	0.17
$100 \times \sigma^*_{\xi_w}$	IG	0.3	0.1	0.9	0.11	0.11	0.08	0.13
$100 \times \sigma^*_{r,8}$	IG	0.3	0.1	0.9	0.09	0.09	0.09	0.10
				Canada				
ρ_{rp}	B	0.5	0.2	0.8	0.96	0.95	0.79	0.97
ρ_ξ	B	0.5	0.2	0.8	0.69	0.72	0.58	0.93
ρ_g	B	0.5	0.2	0.8	0.94	0.93	0.90	0.96
ρ_{ξ_H}	B	0.5	0.2	0.8	0.86	0.83	0.71	0.91
ρ_{ξ_w}	B	0.5	0.2	0.8	0.24	0.26	0.16	0.37
ρ_{ξ_X}	B	0.5	0.2	0.8	0.91	0.91	0.87	0.95
ρ_{ξ_F}	B	0.5	0.2	0.8	0.99	0.96	0.92	0.99
ρ_{tp}	B	0.5	0.2	0.8	0.75	0.74	0.64	0.83
$100 \times \sigma_r$	IG	0.3	0.1	0.7	0.17	0.17	0.15	0.19
$100 \times \sigma_{rp}$	IG	0.1	0.1	0.2	0.29	0.34	0.27	0.58
$10 \times \sigma_g$	IG	0.3	0.1	0.9	0.14	0.14	0.13	0.15
$100 \times \sigma_\xi$	IG	0.3	0.1	0.9	0.23	0.24	0.21	0.29
$100 \times \sigma_{\xi_H}$	IG	0.2	0.1	0.3	0.36	0.37	0.33	0.41
$100 \times \sigma_{\xi_w}$	IG	0.3	0.1	0.8	0.49	0.50	0.45	0.55
$100 \times \sigma_{\xi_X}$	IG	0.3	0.1	0.9	1.31	1.34	1.15	1.57
$100 \times \sigma_{\xi_F}$	IG	0.1	0.1	0.2	1.03	1.08	0.89	1.34
$100 \times \sigma_{r,8}$	IG	0.3	0.1	0.7	0.12	0.12	0.11	0.13

Figure 9: Contemporaneous and Anticipated Monetary Policy Shocks

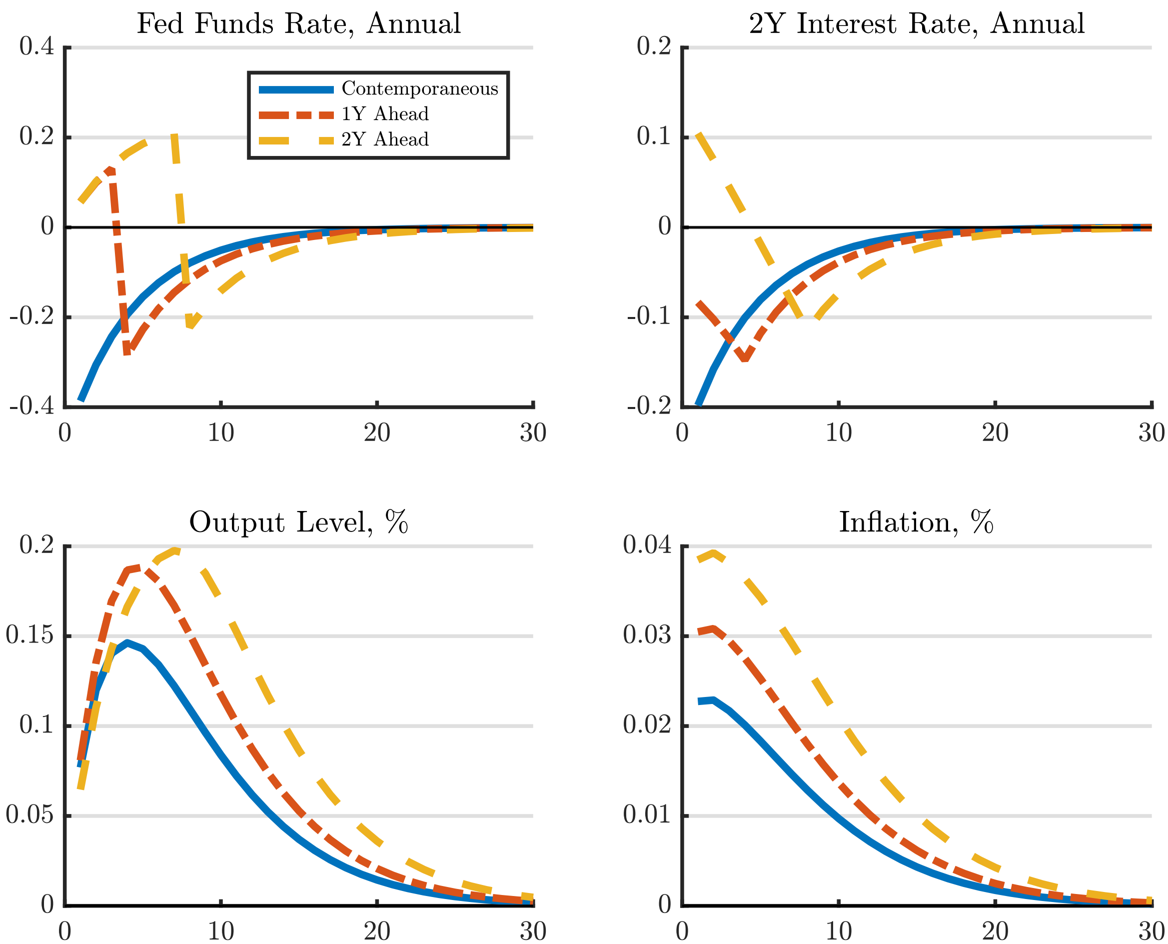


Figure 10: Contemporaneous and Anticipated Monetary Policy Shocks, $\gamma = 0.97$

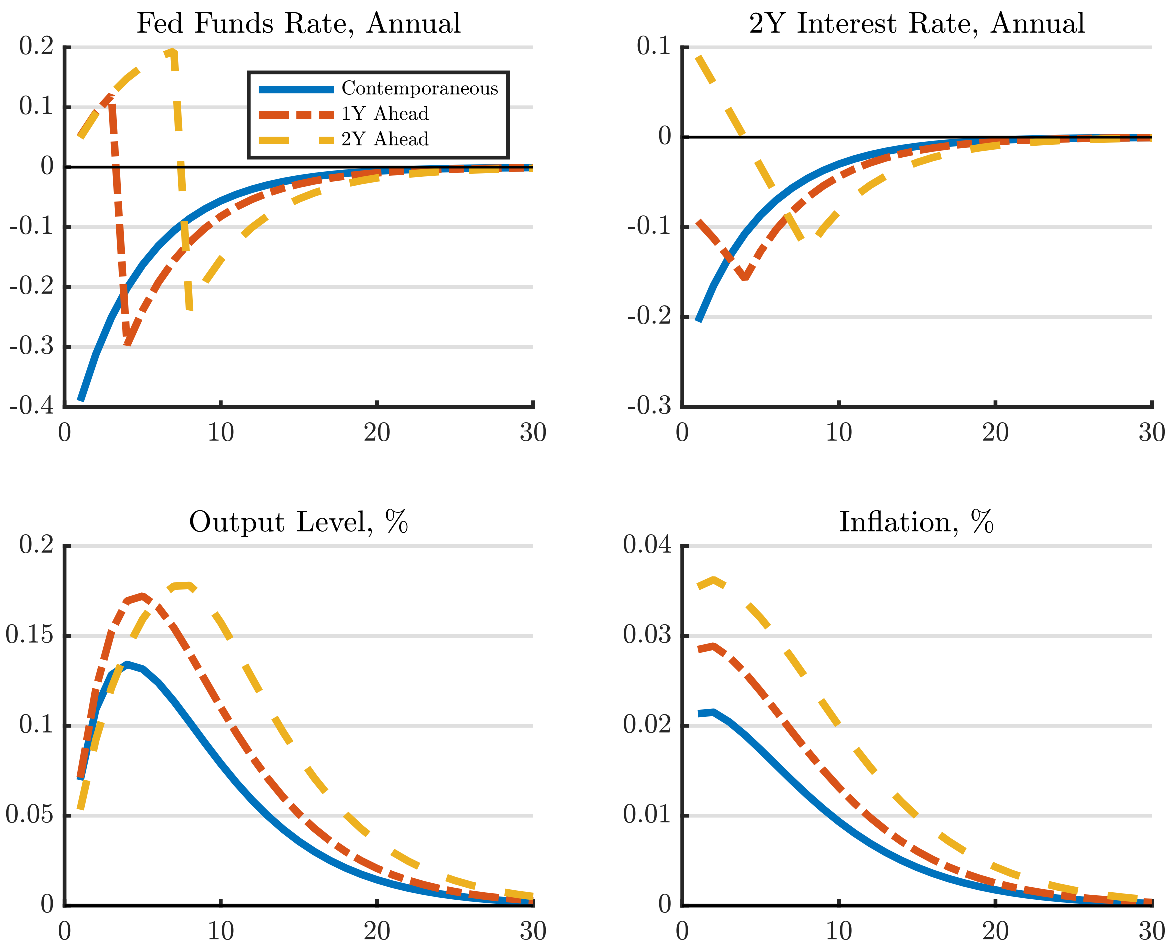
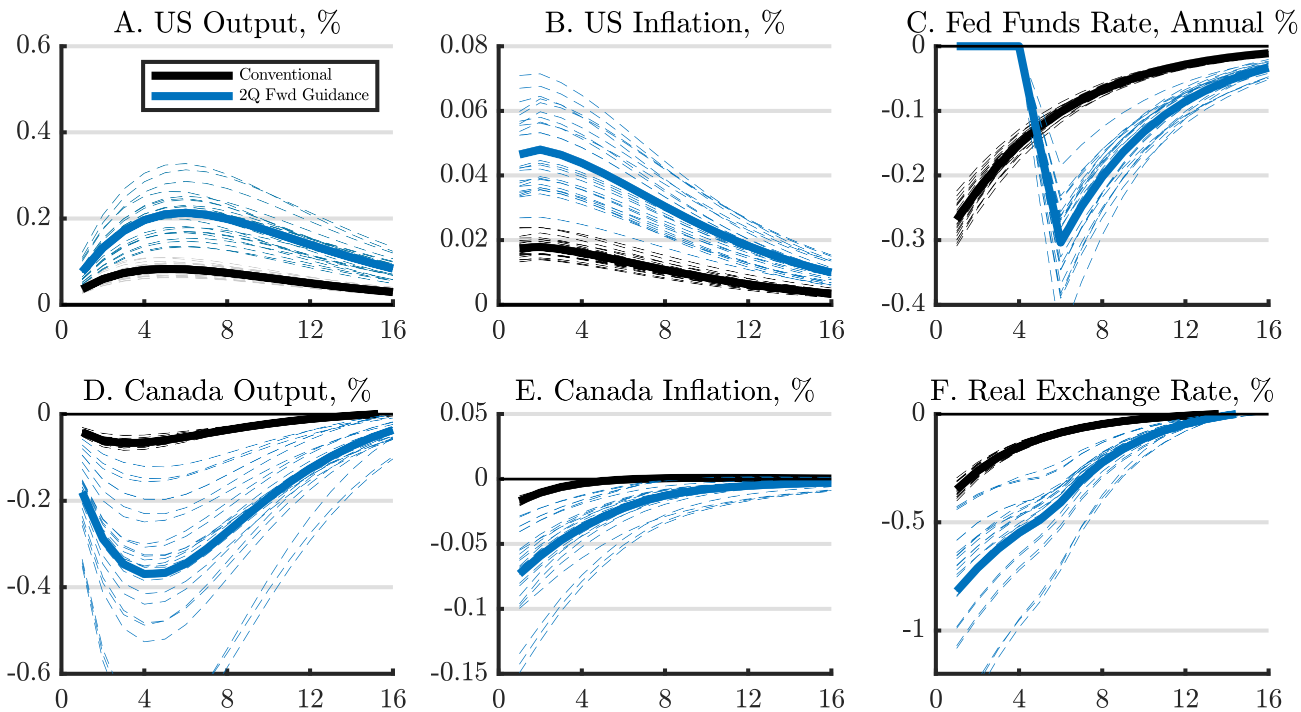


Figure 11: GIRF of 2 Quarter Forward Guidance Shock in 2011Q3, 4Q Base Duration, U.S. Preference Shock in Canada Preference Process



7.2 U.S. Preference Shock Entering Canadian Preference Shock Directly

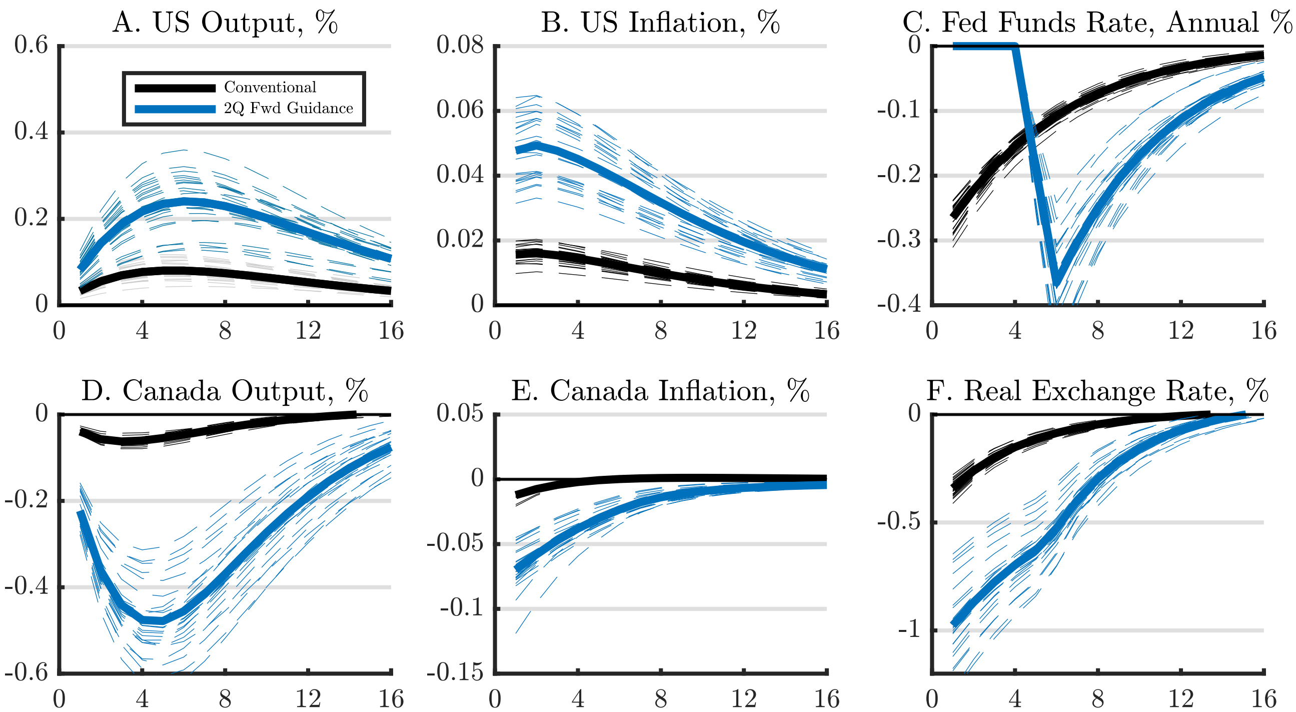
Next, we studied the robustness of our findings to a model specification when we directly augment the Canadian economy's preference shock with the U.S. preference shock, that is, where now Canadian preferences are:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t e^{\xi_t + \xi_t^*} \left[\log(C_t - hC_{t-1}) - \frac{1}{1+\varphi} \int_0^1 (N_t(j))^{1+\varphi} \, \mathrm{d}j \right]$$

where ξ_t^* is the exogenous U.S. preference process subject to shocks. Table 3 presents the unconditional variance decomposition in an estimation with this specification. Adding the U.S. preference shock to Canada increases the contribution of U.S. preference shocks to explaining Canadian output and consumption growth to about 10 percent and about 20 percent of the variation in interest rates (compared to about 2 percent and 5 percent in the baseline). This specification also pushes up the contribution of U.S. preference shocks in explaining U.S. variables by about 10 to 15 percentage points. In Canada, compared to the baseline, there is a reduction in the contribution of the exchange rate risk premium shock for interest rates and output growth, and a reduction in the Canadian preference shock in explaining consumption growth.

Importantly, the implications of this setup for the spillovers of a U.S. forward guidance shock to Canada are similar, and in some cases slightly larger, than the spillovers in our baseline model. Figure 11 plots the same generalized impulse response as in our baseline exercise: an extension of the lower bound duration by 2 quarters in 2011Q3, on a base duration of 4 quarters.

Figure 12: GIRF of 2 Quarter Forward Guidance Shock in 2011Q3, 4Q Base Duration, U.S. Preference Shock in Canada Preference Process



7.3 U.S. Preference/Gov Shocks Entering Canadian Preference/Gov Shocks

We next consider a specification where, in addition to the U.S. preference shock entering the Canadian preference shock as in the previous subsection, we also allow the U.S. government spending shock to enter directly into the Canadian government spending shock, which broadly captures additional direct demand channels from the U.S. to Canada.

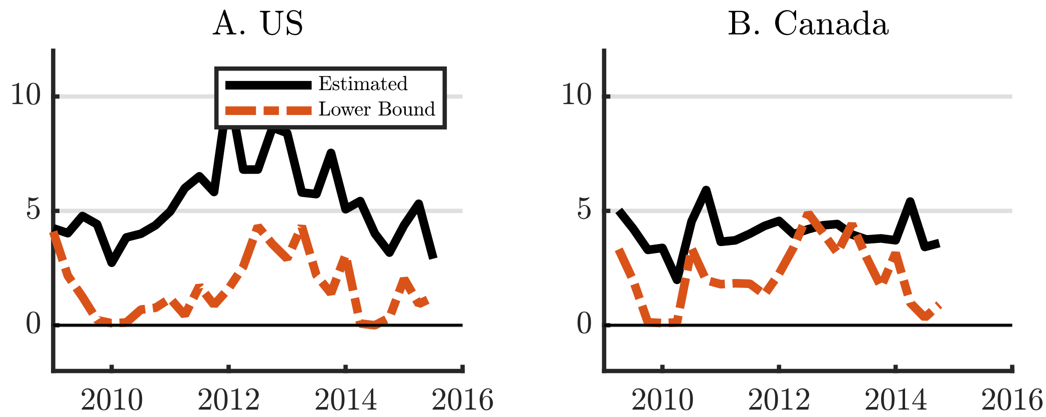
As Figure 12 shows, the conventional impulse responses and forward guidance generalized impulse response is very similar to our baseline model, indicating that the spillovers of monetary policy shocks are similar as we add the U.S. shocks directly in the Canadian exogenous processes that increases the co-movements between the U.S. and Canadian variables.

The variance decompositions of the model at the mode of the estimated posterior distributions for this specification is given in Table 4. First, the contribution of the U.S. preference shock in Canadian variables falls compared to the specification when we have only the U.S. preference shock in the Canadian preference shock. The U.S. government spending shock now explains 10 percentage points more of Canadian output growth, primarily at the expense of the import markup shock, while the US government spending shock explains about 10% more of imports growth, mostly at the expense of the risk premium shock.

7.4 Estimation without Yield Curve

We also studied whether removing long-rates from our estimation changes our results. In this exercise, we remove the 2 year yields in the U.S. and Canada. The variance decomposition implied by the estimated parameters is shown in Table 5. The variance decomposition is similar to our baseline model. The main reason for this is that the contribution of the term premia

Figure 13: Fixed Interest Rate Durations, No Long-Rates in Estimation, Mean Across Draws



shocks to the observables is relatively small. The contribution in our baseline model of these term premia shocks to long-rates is the residual variation in our main variance decomposition table (Table 3 in the text) variance decomposition table - for the U.S., for example, 11.8 percent of the variation of the 2Y yield is explained by these shocks, while the same number for Canada is 10.7 percent.

The mean estimates of the fixed interest rate durations when we do not use long-rates in estimation is shown in Figure 13. We estimate similar durations and a similar decomposition to our baseline results when 2-year yields are not used. However, we do estimate the lower-bound duration to be slightly higher than the overall duration for Canada for some time periods, in contrast to our baseline estimates. This suggests the addition of the 2-year yields in estimation helps in more accurately identifying lower bound durations.

The impulse responses to a conventional US policy shock and the generalized impulse response to a 2Q US forward guidance shock in our estimation without a yield curve is shown in Figure 14, and is comparable to the impulse responses plotted in Figure 7 of the paper. A comparison of the two figures shows the spillovers of conventional and forward guidance shocks are very similar if we remove the 2-year yield from the set of observables in estimation.

8 Lower Bound Value and Expected Durations

We next explore the interaction between the value of the lower bound and the lower bound duration. To do so, we focus on the US and plot in blue in Figure 15 the impulse response to a large negative preference shock that causes the lower bound (at a value of .1%) to bind for 4 quarters. For the *same* shock, we plot in red the impulse response when the value of the lower bound is instead .5%. When the lower bound is higher, it binds for an additional 3 quarters (and is reached one quarter earlier), and output would fall by an additional .5% at its trough.

We next ask what calibration of forward guidance policy when the lower bound value is .5% would generate similar outcomes as the path of the economy under the preference shock and a lower bound value of .1%. The central bank would need to announce an additional 3 quarter extension of the lower bound at .5% when it is reached in order to generate a similar profile for output. This path is plotted in yellow in Figure 15. Thus the durations must be interpreted as durations at a certain peg. In estimation what this peg is, however, is given by the data.

Figure 14: IRF of Conventional US Policy Shock and GIRF of 2Q US Forward Guidance Shock, No Long-Rates in Estimation

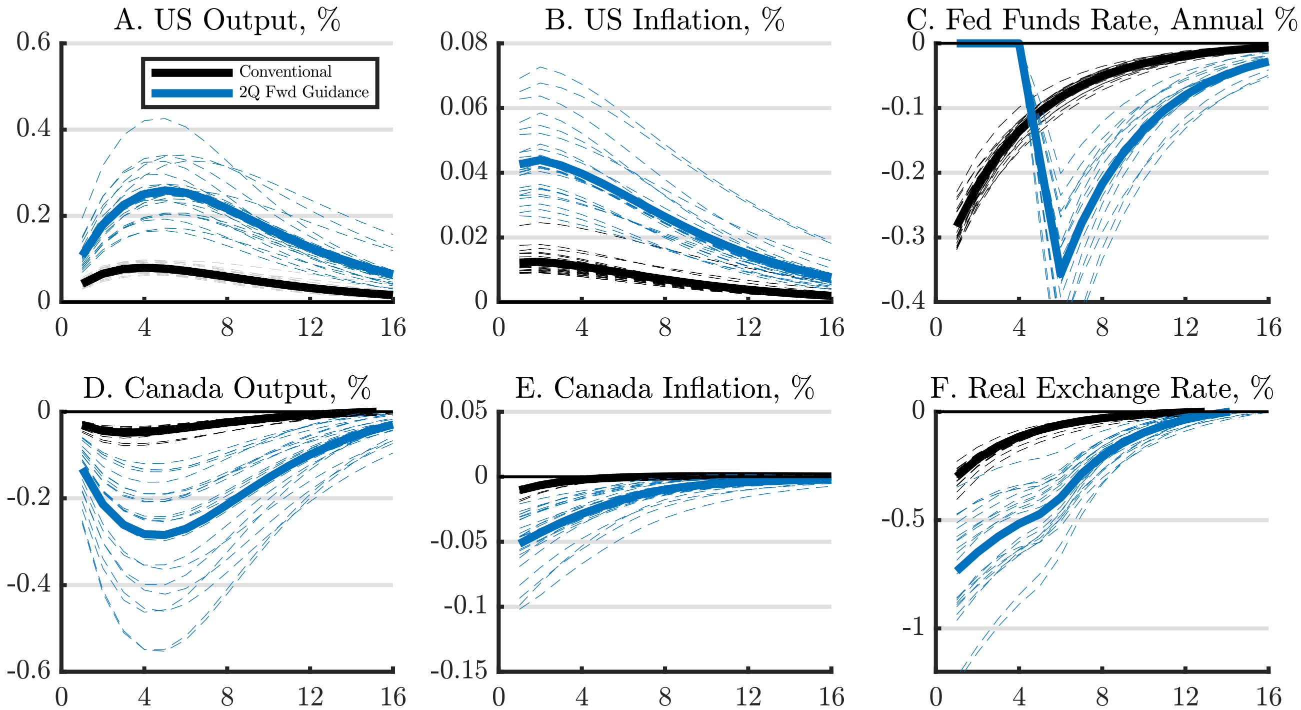


Figure 15: US Preference Shock: Different Lower Bounds

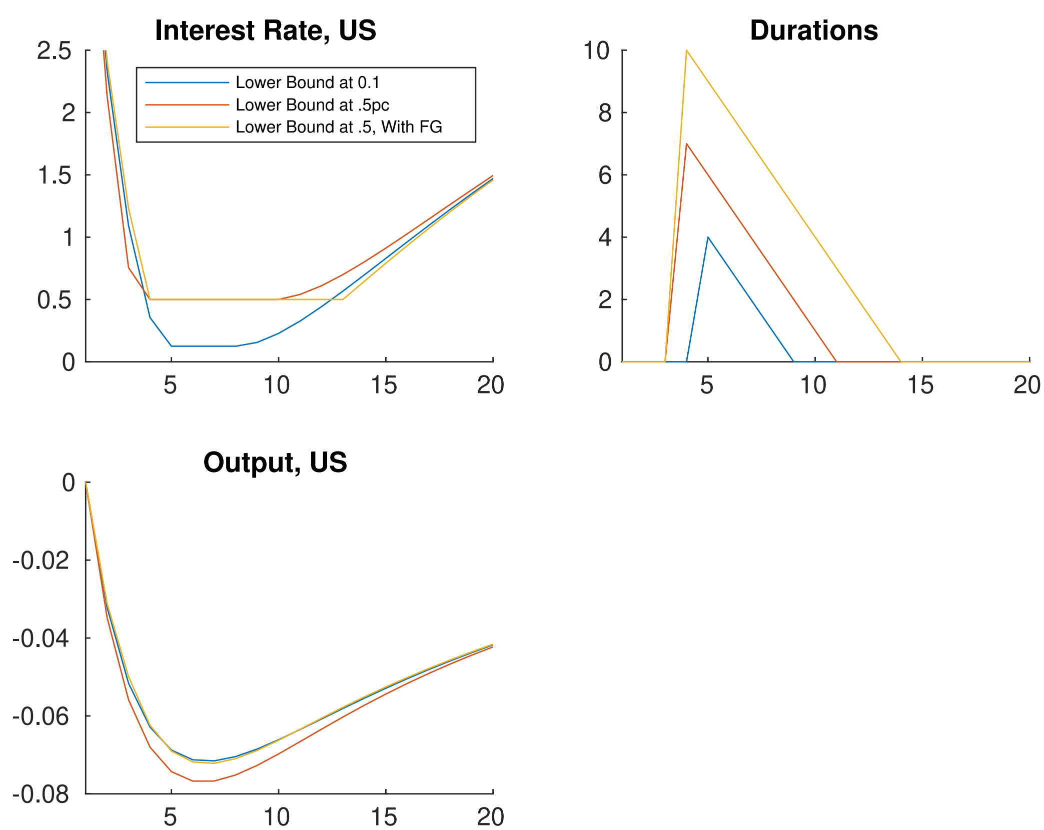


Table 3: Variance Decomposition Due to Shocks, %, With Foreign Preference Shocks in Domestic Preference

Shock / Variable	Common	US Shocks					Canadian Shocks							
	Prod.	Pref.	Policy	Gov.	Price	Wage	Pref.	Policy	Gov.	Risk Pr.	Price	Wage	Exports	Imports
A. US Variables														
Policy Rate	0.0	76.1	10.6	2.5	5.0	5.7	-	-	-	-	-	-	-	-
2Y Interest Rate	0.0	77.8	3.6	1.6	4.4	5.1	-	-	-	-	-	-	-	-
Output Growth	0.6	53.2	1.1	42.9	1.0	1.2	-	-	-	-	-	-	-	-
Consumption Growth	0.2	92.1	1.9	2.1	1.6	2.1	-	-	-	-	-	-	-	-
Inflation	1.1	16.8	5.0	0.1	38.1	38.8	-	-	-	-	-	-	-	-
Wage Growth	12.8	0.5	0.4	0.0	67.2	19.1	-	-	-	-	-	-	-	-
B. Canadian Variables														
Policy Rate	0.0	20.3	0.8	0.4	0.3	0.7	0.9	8.0	0.8	41.7	1.4	0.8	8.5	15.5
2Y Interest Rate	0.0	22.2	0.5	0.2	0.3	0.7	0.3	1.5	0.5	39.0	0.6	0.6	7.6	17.4
Output Growth	0.3	10.7	1.0	2.0	0.2	0.2	8.7	10.0	8.0	16.7	13.0	2.2	25.7	1.3
Consumption Growth	0.3	9.7	0.3	0.2	0.3	0.6	33.3	4.4	2.8	37.9	2.8	0.8	6.5	0.2
Inflation	0.2	0.1	0.4	0.0	0.1	0.1	0.0	4.4	0.0	11.7	36.4	8.1	3.6	34.8
Wage Growth	1.7	0.1	0.2	0.0	0.0	0.0	0.1	0.1	0.0	4.4	32.2	50.0	5.0	6.3
Imports Growth	0.0	0.7	1.0	0.1	0.1	0.2	2.9	0.3	25.4	30.9	3.5	0.3	11.4	23.4
Exports Growth	0.0	1.5	0.2	1.4	0.2	0.2	0.0	0.7	0.0	9.8	0.4	0.1	65.1	20.4
Nominal Ex Rate, Δ	0.0	0.2	3.0	0.1	0.3	0.3	0.2	6.0	0.0	38.3	0.7	0.3	20.3	30.3

Table 4: Variance Decomposition Due to Shocks, %, With Foreign Preference/Gov. Shocks in Domestic Preference/Gov. Shocks

Shock / Variable	Common	US Shocks					Canadian Shocks							
	Prod.	Pref.	Policy	Gov.	Price	Wage	Pref.	Policy	Gov.	Risk Pr.	Price	Wage	Exports	Imports
A. US Variables														
Policy Rate	0.5	47.7	13.7	1.8	4.9	31.4	-	-	-	-	-	-	-	-
2Y Interest Rate	0.4	48.0	5.0	0.8	4.4	32.4	-	-	-	-	-	-	-	-
Output Growth	5.2	39.4	1.3	48.3	0.7	5.0	-	-	-	-	-	-	-	-
Consumption Growth	2.4	82.7	2.8	0.2	1.5	10.4	-	-	-	-	-	-	-	-
Inflation	5.5	0.8	4.3	0.1	17.7	71.6	-	-	-	-	-	-	-	-
Wage Growth	63.4	0.1	0.3	0.0	32.7	3.5	-	-	-	-	-	-	-	-
B. Canadian Variables														
Policy Rate	0.2	14.0	0.8	2.9	0.2	0.6	1.2	9.1	0.8	52.7	0.4	2.7	6.5	7.8
2Y Interest Rate	0.2	15.2	0.6	2.0	0.2	0.7	0.5	2.2	0.1	53.2	0.2	2.2	6.0	6.5
Output Growth	1.9	6.6	1.0	10.3	0.1	0.4	8.7	10.3	16.8	17.9	7.7	2.3	15.0	0.9
Consumption Growth	1.5	8.2	0.3	3.6	0.1	0.6	28.8	3.2	0.2	43.2	0.7	0.7	6.2	2.8
Inflation	2.1	1.6	0.2	0.2	0.0	0.1	0.0	3.0	0.0	7.8	36.2	28.4	6.5	13.9
Wage Growth	12.0	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	3.6	24.0	49.5	3.5	7.2
Imports Growth	0.1	0.7	0.4	13.3	0.0	0.2	2.1	0.1	29.3	20.5	2.4	0.3	6.6	23.9
Exports Growth	0.1	0.8	0.4	1.4	0.2	0.1	0.1	1.4	0.0	20.3	0.2	0.2	64.9	9.9
Nominal Ex Rate, Δ	0.0	0.3	3.2	0.1	0.3	1.3	0.3	6.0	0.1	44.9	1.5	2.0	27.3	12.7

Table 5: Variance Decomposition Due to Shocks, %, No Long-Rates in Estimation

Shock / Variable	Common	US Shocks					Canadian Shocks							
	Prod.	Pref.	Policy	Gov.	Price	Wage	Pref.	Policy	Gov.	Risk Pr.	Price	Wage	Exports	Imports
A. US Variables														
Policy Rate	0.1	64.4	12.8	4.7	8.6	9.4	-	-	-	-	-	-	-	-
2Y Interest Rate	0.1	74.1	4.1	3.1	8.9	9.8	-	-	-	-	-	-	-	-
Output Growth	1.7	38.5	2.0	53.2	2.1	2.6	-	-	-	-	-	-	-	-
Consumption Growth	0.9	80.6	4.1	4.5	4.4	5.5	-	-	-	-	-	-	-	-
Inflation	2.0	12.7	2.8	0.3	41.8	40.4	-	-	-	-	-	-	-	-
Wage Growth	20.8	0.1	0.3	0.0	62.7	16.1	-	-	-	-	-	-	-	-
B. Canadian Variables														
Policy Rate	0.0	6.0	0.4	0.4	0.4	0.7	2.9	8.0	1.1	58.3	1.4	1.0	9.8	9.7
2Y Interest Rate	0.1	6.8	0.3	0.3	0.4	0.8	2.0	1.9	0.7	64.7	0.6	0.8	9.9	10.7
Output Growth	0.6	2.6	0.6	2.3	0.3	0.2	10.7	6.0	9.5	23.7	13.3	2.4	27.6	0.4
Consumption Growth	0.4	2.7	0.1	0.1	0.1	0.3	54.7	1.4	2.0	30.4	1.3	0.6	3.7	2.1
Inflation	0.5	1.0	0.2	0.0	0.1	0.1	0.0	3.3	0.1	14.1	44.4	11.0	3.8	21.3
Wage Growth	3.0	0.1	0.1	0.0	0.0	0.0	0.1	0.0	0.0	3.6	30.6	50.0	4.1	8.4
Imports Growth	0.0	0.9	0.3	0.0	0.1	0.1	4.7	0.2	24.5	21.4	4.6	0.4	10.0	32.6
Exports Growth	0.0	2.5	0.1	1.5	0.2	0.1	0.2	0.8	0.0	13.5	0.5	0.1	62.2	18.3
Nominal Ex Rate, Δ	0.0	0.7	2.3	0.1	0.3	0.3	0.8	5.5	0.1	39.0	0.7	0.3	24.5	25.4

Model Equations

9 Full Equations: Large Economy

9.1 Normalized Equilibrium Conditions

With the following normalizations: $\tilde{\Lambda}^*_t = \Lambda^*_t Z_t$, $\tilde{C}^*_t = C^*_t/Z_t$, $\tilde{W}^*_t = W^*_t/(Z_t P_t)$, $\tilde{Y}^*_t = Y_t/Z_t$, $\tilde{G}^*_t = G^*_t/Z_t$, we can write the stationary model as:

Consumption decision:

$$\tilde{\Lambda}^*_t = \frac{e^{\xi^*_t}\mathcal{M}_t}{\tilde{C}^*_t\mathcal{M}_t - h\tilde{C}^*_{t-1}} - \beta h \mathbb{E}_t \frac{e^{\xi^*_{t+1}}}{\mathcal{M}_{t+1}\tilde{C}^*_{t+1} - h^*\tilde{C}^*_t} \tag{26}$$

Euler equation:

$$1 = \beta R^*_t \mathbb{E}_t \left\{ \frac{\tilde{\Lambda}^*_{t+1}}{\mathcal{M}_{t+1}\tilde{\Lambda}^*_t \Pi^*_{t+1}} \right\} \tag{27}$$

$H^*_{w1,t}$:

$$H^*_{w1,t} = \nu N_t^{*1+\varphi} + \beta\theta^*_w \mathbb{E}_t \left(\frac{\Omega^{w*}_{t,t+1}}{\Pi^{w*}_{t+1}} \right)^{-\epsilon^*_w(1+\varphi)} H^*_{w1,t+1} \tag{28}$$

$H^*_{w1,t}$:

$$\tilde{H}^*_{w2,t} = \tilde{\Lambda}^*_t \tilde{W}^*_t N^*_t + \beta\theta^*_w \mathbb{E}_t \left(\frac{\Omega^{w*}_{t,t+1}}{\Pi^{w*}_{t+1}} \right)^{1-\epsilon^*_w} \tilde{H}^*_{w2,t+1} \tag{29}$$

$\Omega^{w*}_{t,t+1}$:

$$\Omega^{w*}_{t,t+1} = (\bar{\Pi}^*\mathcal{M})^{1-\chi^*_w} (\Pi^{w*}_t)^{\chi^*_w} \tag{30}$$

Wage index:

$$1 = (1-\theta^*_w) \left(\frac{\epsilon^*_w}{(1+\iota_w)(\epsilon^*_w - 1)} \frac{H^*_{w1,t}}{H^*_{w2,t}} \right)^{\frac{1-\epsilon^*_w}{1+\epsilon^*_w\varphi}} + \theta^*_w \left(\frac{(\Pi^{w*}_{t-1})^{\chi^*_w} (\bar{\Pi}^*\mathcal{M})^{1-\chi^*_w}}{\Pi^{w*}_t} \right)^{1-\epsilon^*_w} \tag{31}$$

Wage inflation:

$$\Pi^{w*}_t = \frac{\tilde{W}^*_t}{\tilde{W}^*_{t-1}} \Pi^*_t \mathcal{M}_t \tag{32}$$

Aggregate production function:

$$\tilde{Y}^*_t = A^*_t N^*_t \tag{33}$$

Market clearing:

$$\tilde{Y}^*_t = \tilde{C}^*_t + \tilde{G}^*_t \tag{34}$$

$H_{p1,t}$:

$$H^*_{p1,t} = \tilde{\Lambda}_t MC^*_t \tilde{Y}^*_t + \beta\theta^*_p \mathbb{E}_t \left(\frac{\Omega^*_{t,t+1}}{\Pi^*_{t+1}} \right)^{-\epsilon^*_p} H^*_{p1,t+1} \tag{35}$$

$H_{p2,t}$:

$$H^*_{p2,t} = \tilde{\Lambda}^*_t \tilde{Y}^*_t + \beta\theta^*_p \mathbb{E}_t \left(\frac{\Omega^*_{t,t+1}}{\Pi^*_{t+1}} \right)^{1-\epsilon^*_p} H^*_{p2,t+1} \tag{36}$$

Price index:

$$1 = (1-\theta_p^*)\left(\frac{\epsilon_p^*}{(1+\iota_p^*)(\epsilon_p^*-1)}\frac{H_{p1,t}^*}{H_{p2,t}^*}\right)^{1-\epsilon_p^*} + \theta_p^*\left(\frac{(\bar{\Pi}^*)^{1-\chi_p^*}(\Pi_{t-1}^*)^{\chi_p^*}}{\Pi_t^*}\right)^{1-\epsilon_p^*} \tag{37}$$

Marginal costs:

$$MC_t^* = \frac{\tilde{W}_t^*}{A_t^*} \tag{38}$$

Monetary policy rule:

$$\frac{R_t^*}{\bar{R}^*} = \left[\frac{R_{t-1}^*}{\bar{R}^*}\right]^{\rho_R^*}\left[\left(\frac{\Pi_t^*}{\bar{\Pi}_t{}^*}\right)^{\phi_\pi^*}\left(\frac{\tilde{Y}_t^*}{\tilde{Y}_{t-1}^*}\right)^{\phi_g^*}\left(\tilde{Y}_t^*\right)^{\phi_y^*}\right]^{1-\rho_R^*} e^{\varepsilon_{R,t}^*} \tag{39}$$

9.2 Steady State

Consumption decision:

$$\bar{\Lambda}^* = \frac{\mathcal{M}-\beta h^*}{\bar{C}^*(\mathcal{M}-h^*)} \tag{40}$$

Discounted Euler equation:

$$\bar{R}^* = \frac{\mathcal{M}\bar{\Pi}^*}{\beta} \tag{41}$$

H_{w1}^*:

$$\bar{H}_{w1}^* = \frac{\nu\bar{N}^{*1+\varphi}}{1-\beta\theta_w^*} \tag{42}$$

H_{w2}^*:

$$\bar{H}_{w2}^* = \frac{\bar{\Lambda}^*\bar{W}^*\bar{N}^*}{1-\beta\theta_w^*} \tag{43}$$

Ω^{w*}:

$$\bar{\Omega}^{w*} = \bar{\Pi}^*\mathcal{M} \tag{44}$$

Wage index:

$$\bar{H}_{w1}^* = \bar{H}_{w2}^* \tag{45}$$

Wage inflation:

$$\bar{\Pi}^{w*} = \bar{\Pi}^*\mathcal{M} \tag{46}$$

Production function:

$$\bar{Y}^* = \bar{N}^* \tag{47}$$

Marginal costs:

$$M\bar{C}^* = \bar{W}^* \tag{48}$$

H_{p1}^*:

$$\bar{H}_{p1}^* = \frac{\bar{\Lambda}M\bar{C}\bar{Y}}{1-\beta\theta_p^*} \tag{49}$$

H^*_{p2}:

$$\bar{H}^*_{p2} = \frac{\bar{\Lambda}^*\bar{Y}^*}{1-\beta\theta^*_t} \tag{50}$$

Price index:

$$\bar{H}^*_{p1} = \bar{H}^*_{p2} \tag{51}$$

Market clearing:

$$\bar{Y}^* = \bar{C}^* + \bar{G}^* \tag{52}$$

Monetary policy rule:

$$\bar{\Pi}^* = \bar{\Pi}^* \tag{53}$$

Discounted Euler equation

$$1 = \beta R^*_t \mathbb{E}_t \left\{ \frac{(\tilde{\Lambda}^*_{t+1})^\kappa}{\mathcal{M}_{t+1}\tilde{\Lambda}^*_t \Pi^*_{t+1}} \right\} \tag{54}$$

9.3 Log-linearized Equilibrium Conditions

Consumption decision:

$$\begin{aligned}(\mathcal{M}^2 + \beta h^{*2})c^*_t =& \beta h^*\mathcal{M}E_t\{c^*_{t+1}\} + h^*\mathcal{M}c^*_{t-1} - (\mathcal{M}-\beta h^*)(\mathcal{M}-h^*)\lambda^*_t \\ & - h^*\mathcal{M}(\mu_t - \beta E_t\{\mu_{t+1}\}) + \mathcal{M}((\mathcal{M}-h^*)\xi^*_t - \beta h^* E_t\{\xi^*_{t+1}\})\end{aligned} \tag{55}$$

Discounted Euler equation:

$$\lambda^*_t = \kappa\lambda^*_{t+1} - E_t\{\mu_{t+1}\} + (r^*_t - E_t\{\pi^*_{t+1}\}) \tag{56}$$

$H^{w*}_{w1,t}$:

$$h^*_{w1,t} = (1-\beta\theta^*_w)(1+\varphi)n^*_t + \beta\theta^*_w\left[h^*_{w1,t+1} + (1+\varphi)\epsilon^*_w(\pi^{w*}_{t+1} - \chi^*_w\pi^{w*}_t)\right] \tag{57}$$

$H^*_{w2,t}$:

$$h^*_{w2,t} = (1-\beta\theta^*_w)(\lambda^*_t + w^*_t + n^*_t) + \beta\theta^*_w\left[h^*_{w2,t+1} + (\epsilon^*_w - 1)(\pi^{w*}_{t+1} - \chi^*_w\pi^{w*}_t)\right] \tag{58}$$

Wage index:

$$\pi^{w*}_t = \frac{1-\theta^*_w}{\theta^*_w}\frac{1}{1+\epsilon^*_w\varphi}(h^*_{w1,t} - h^*_{w2,t} + \xi^*_{w,t}) + \chi^*_w\pi^{w*}_{t-1} \tag{59}$$

Wage inflation:

$$\pi^{w*}_t = w^*_t - w^*_{t-1} + \pi^*_t + \mu_t \tag{60}$$

Aggregate production function:

$$y^*_t = a^*_t + n^*_t \tag{61}$$

Marginal costs:

$$mc^*_t = w^*_t - a^*_t \tag{62}$$

$H^*_{p1,t}$:

$$h^*_{p1,t} = (1-\beta\theta^*_p)(\lambda^*_t + mc^*_t + y^*_t) + \beta\theta^*_t(h^*_{p1,t+1} + \epsilon^*_p(\pi^*_{t+1} - \chi^*_p\pi^*_t)) \tag{63}$$

$H^*_{p2,t}$:

$$h^*_{p2,t} = (1-\beta\theta^*_p)(\lambda^*_t + y^*_t) + \beta\theta^*_t(h^*_{p2,t+1} + (\epsilon^*_p - 1)(\pi^*_{t+1} - \chi^*_p\pi^*_t)) \tag{64}$$

Price index:

$$\pi^*_t = \frac{1-\theta^*_p}{\theta^*_p}(h^*_{p1,t} - h^*_{p2,t} + \xi^*_{p,t}) + \chi^*_p\pi^*_{t-1} \tag{65}$$

Market clearing

$$\bar{Y}y^*_t = \bar{C}c^*_t + \bar{G}g^*_t \tag{66}$$

Monetary policy rule:

$$r^*_t = \rho^*_R r^*_{t-1} + (1-\rho^*_R(\phi^*_\pi\pi^*_t + \phi^*_g(y^*_t - y^*_{t-1}) + \phi^*_y y^*_t) + \varepsilon^*_{R,t} \tag{67}$$

This is a system of 13 equations in 13 unknowns r^*_t, y^*_t, π^*_t, π^{w*}_t, λ^*_t, n^*_t, c^*_t, mc^*_t, $h^*_{p1,t}$, $h^*_{p2,t}$, $h^*_{w1,t}$, $h^*_{w2,t}$, w_t plus shocks.

10 Full Equations: Small Economy

10.1 Normalized Equilibrium Conditions

With the following normalizations, we can express the system of equations in terms of stationary variables: $\tilde{\Lambda}_t = \Lambda_t Z_t$, $\tilde{C}_t = C_t/Z_t$, $\tilde{C}_{H,t} = C_{H,t}/Z_t$, $\Delta S_t = S_t/S_{t-1}$, $\tilde{X}_t = Y_{X,t}/Z_t$, $\tilde{Y}_{F,t} = Y_{F,t}/Z_t$, $\tilde{W}_t = W_t/(P_t Z_t)$, $\tilde{B}_{F,t+1} = S_t B_{F,t+1}/(P_t Z_t)$

The stationary equations are:

Consumption decision:

$$\tilde{\Lambda}_t = \frac{e^{\xi_t}\mathcal{M}_t}{\tilde{C}_t\mathcal{M}_t - h\tilde{C}_{t-1}} - \beta h\mathbb{E}_t\left\{\frac{e^{\xi_{t+1}}}{\mathcal{M}_{t+1}\tilde{C}_{t+1} - h\tilde{C}_t}\right\} \tag{68}$$

Euler equation:

$$1 = \beta R_t\mathbb{E}_t\left\{\frac{\tilde{\Lambda}_{t+1}}{\mathcal{M}_{t+1}\tilde{\Lambda}_t\Pi_{t+1}}\right\} \tag{69}$$

UIP:

$$R_t = R_{F,t}\mathbb{E}_t\left\{\Delta S_{t+1}\right\} \tag{70}$$

Real exchange rate:

$$Q_t = \frac{Q_{t-1}\Delta S_t\Pi^*_t}{\Pi_t} \tag{71}$$

Interest rate on foreign borrowing:

$$R_{F,t} = R^*_t(\tilde{B}_{F,t+1}/\tilde{Y}_t) - b_F)^{-\psi_H}e^{\psi_t} \tag{72}$$

Demand for imports:

$$\tilde{Y}_{F,t} = \alpha\left(\Gamma_{F,t}\right)^{-\tau}\tilde{C}_t \tag{73}$$

Demand for home-produced goods:

$$\tilde{C}_{H,t} = (1-\alpha)\left(\Gamma_{H,t}\right)^{-\tau}\tilde{C}_t \tag{74}$$

Consumer price index:

$$\Pi_t = \left[\alpha \left(\Pi_{F,t}\Gamma_{F,t-1}\right)^{1-\tau} + (1-\alpha)\left(\Pi_{H,t}\Gamma_{H,t-1}\right)^{1-\tau}\right]^{\frac{1}{1-\tau}} \tag{75}$$

$H_{X1,t}$:

$$H_{X1,t} = \tilde{\Lambda}_t \Gamma_{H,t} \tilde{X}_t + \beta\theta_X \mathbb{E}_t \left\{ \left(\frac{(\Pi^*_{X,t})^{\chi_X} (\bar{\Pi}^*)^{1-\chi_X}}{\Pi^*_{X,t+1}} \right)^{-\epsilon_X} H_{X1,t+1} \right\} \tag{76}$$

$H_{X2,t}$:

$$H_{X2,t} = \tilde{\Lambda}_t \Gamma_{X,t} \tilde{X}_t + \beta\theta_X \mathbb{E}_t \left\{ \left(\frac{(\Pi^*_{X,t})^{\chi_X} (\bar{\Pi}^*)^{1-\chi_X}}{\Pi^*_{X,t+1}} \right)^{1-\epsilon_X} H_{X2,t+1} \right\} \tag{77}$$

Exports inflation:

$$1 = \theta_X \left(\frac{(\Pi^*_{X,t-1})^{\chi_X} (\bar{\Pi}^*)^{1-\chi_X}}{\Pi^*_{X,t}} \right)^{1-\epsilon_X} + (1-\theta_X) \left(\frac{\epsilon_X}{(1+\tau_X)(\epsilon_X - 1)} \frac{H_{X1,t}}{H_{X2,t}} \right)^{1-\epsilon_X} \tag{78}$$

Exports demand:

$$\tilde{X}_t = \alpha \left(\frac{\Gamma_{X,t}}{Q_t} \right)^{-\tau} Y^*_t \tag{79}$$

$H_{F1,t}$:

$$H_{F1,t} = Q_t \tilde{\Lambda}_t \tilde{Y}_{F,t} + \beta\theta_{p,F} \mathbb{E}_t \left\{ \left(\frac{\Pi_{F,t}^{\chi_F} (\bar{\Pi})^{1-\chi_F}}{\Pi_{F,t+1}} \right)^{-\epsilon_F} H_{F1,t+1} \right\} \tag{80}$$

$H_{F2,t}$:

$$H_{F2,t} = \tilde{\Lambda}_t \Gamma_{F,t} \tilde{Y}_{F,t} + \beta\theta_{p,F} \mathbb{E}_t \left\{ \left(\frac{\Pi_{F,t}^{\chi_F} (\bar{\Pi})^{1-\chi_F}}{\Pi_{F,t+1}} \right)^{1-\epsilon_F} H_{F2,t+1} \right\} \tag{81}$$

Imports inflation:

$$1 = \theta_{p,F} \left(\frac{\Pi_{F,t-1}^{\chi_F} (\bar{\Pi})^{1-\chi_F}}{\Pi_{F,t}} \right)^{1-\epsilon_F} + (1-\theta_{p,F}) \left(\frac{\epsilon_F}{(1+\tau_F)(\epsilon_F - 1)} \frac{H_{F1,t}}{H_{F2,t}} \right)^{1-\epsilon_F} \tag{82}$$

Production of domestically-produced goods:

$$\tilde{Y}_t = A_t N_t \tag{83}$$

Marginal costs:

$$MC_t = \frac{\tilde{W}_t}{\Gamma_{H,t} A_t} \tag{84}$$

$H_{p1,t}$:

$$H_{p1,t} = \tilde{\Lambda}_t MC_t \Gamma_{H,t} \tilde{Y}_t + \beta\theta_p \mathbb{E}_t \left\{ \left(\frac{(\Pi_{H,t})^{\chi_p} (\bar{\Pi})^{1-\chi_p}}{\Pi_{H,t+1}} \right)^{-\epsilon_p} H_{p1,t+1} \right\} \tag{85}$$

$H_{p2,t}$:

$$H_{p2,t} = \tilde{\Lambda}_t \Gamma_{H,t} \tilde{Y}_t + \beta\theta_p \mathbb{E}_t \left\{ \left(\frac{(\Pi_{H,t})^{\chi_p} (\bar{\Pi})^{1-\chi_p}}{\Pi_{H,t+1}} \right)^{1-\epsilon_p} H_{p2,t+1} \right\} \tag{86}$$

Domestic price inflation:

$$1 = (1-\theta_p) \left(\frac{\epsilon_p}{(1+\tau_p)(\epsilon_p - 1)} \frac{H_{p1,t}}{H_{p2,t}} \right)^{1-\epsilon_p} + \theta_p \left(\frac{\bar{\Pi}^{1-\chi_p} (\Pi_{H,t-1})^{\chi_p}}{\Pi_{H,t}} \right)^{1-\epsilon_p} \tag{87}$$

$H_{w1,t}$:

$$H_{w1,t} = \nu N_t^{1+\varphi} + \beta\theta_w \mathbb{E}_t \left\{ \left(\frac{\Omega^w_{t,t+s}}{\Pi^w_{t+1}} \right)^{-\epsilon_w(1+\varphi)} H_{w1,t+1} \right\} \tag{88}$$

$H_{w2,t}$:

$$H_{w2,t} = \tilde{\Lambda}_t \tilde{W}_t N_t + \beta\theta_w \mathbb{E}_t \left\{ \left(\frac{\Omega^w_{t,t+1}}{\Pi^w_{t+1}} \right)^{1-\epsilon_w} H_{w1,t+1} \right\} \tag{89}$$

Wage index:

$$1 = (1-\theta_w) \left(\frac{\epsilon_w}{(1+\tau_w)(\epsilon_w - 1)} \frac{H_{w1,t}}{H_{w2,t}} \right)^{\frac{1-\epsilon_w}{1+\epsilon_w\varphi}} + \theta_w \left(\frac{(\Pi^w_{t-1})^{\chi_w} (\bar{\Pi}\mathcal{M})^{1-\chi_w}}{\Pi^w_t} \right)^{1-\epsilon_w} \tag{90}$$

Wage inflation

$$\Pi^w_t = \frac{\tilde{W}_t}{\tilde{W}_{t-1}} \Pi_t \mathcal{M}_t \tag{91}$$

Domestic market clearing:

$$\tilde{Y}_t = \tilde{C}_{H,t} + \tilde{X}_t \tag{92}$$

Financial market clearing:

$$\frac{\tilde{B}_{F,t+1}}{R_{F,t}} = \frac{\Delta S_t \tilde{B}_{F,t}}{\mathcal{M}_t \Pi_t} + \Gamma_{X,t} X_t - Q_t Y_{F,t} \tag{93}$$

Monetary policy:

$$\frac{R_t}{\bar{R}} = \left[\frac{R_{t-1}}{\bar{R}} \right]^{\rho_R} \left[\left(\frac{\Pi_t}{\bar{\Pi}_t} \right)^{\phi_\pi} \left(\frac{\tilde{Y}_t}{\tilde{Y}_{t-1}} \right)^{\phi_g} \right]^{1-\rho_R} e^{\varepsilon_{R,t}} \tag{94}$$

Relative prices:

$$\Gamma_{X,t} = \Gamma_{X,t-1} \frac{\Delta S_t \Pi_{X,t}}{\Pi_t} \tag{95}$$

$$\Gamma_{F,t} = \Gamma_{F,t-1} \frac{\Pi_{F,t}}{\Pi_t} \tag{96}$$

$$\Gamma_{H,t} = \Gamma_{H,t-1} \frac{\Pi_{H,t}}{\Pi_t} \tag{97}$$

This is a system of 30 equations in 30 unknowns, $\tilde{\Lambda}_t$, $\tilde{C}_t$, $\tilde{C}_{H,t}$, R_t, $R_{F,t}$, Π_t, ΔS_t, Q_t, $\Pi_{F,t}$, $\Pi_{H,t}$, $\Pi_{X,t}$, $\Pi_{W,t}$, $H_{X1,t}$, $H_{X2,t}$, $H_{F1,t}$, $H_{F2,t}$, $H_{p1,t}$, $H_{p2,t}$, $H_{w1,t}$, $H_{w2,t}$, $\Gamma_{F,t}$, $\Gamma_{H,t}$, $\Gamma_{X,t}$, X_t, $\tilde{Y}_{F,t}$, MC_t, $\tilde{Y}_t$, N_t, $\tilde{W}_t$, $\tilde{B}_{F,t+1}$, plus shocks and foreign variables.

10.2 Steady State

The steady state relationships are:

Consumption decision:

$$\bar{\Lambda}\bar{C} = \frac{\mathcal{M} - \beta h}{\mathcal{M} - h} \tag{98}$$

Euler equation:

$$\bar{R} = \frac{\mathcal{M}\bar{\Pi}}{\beta} \tag{99}$$

UIP:

$$\bar{R} = \bar{R}_F \bar{\Delta S} \tag{100}$$

Real exchange rate:

$$\bar{\Delta S} = \frac{\bar{\Pi}}{\bar{\Pi}^*} \tag{101}$$

Interest rate on foreign borrowing:

$$\frac{\bar{B}_F}{\bar{Y}} = b_F \tag{102}$$

Imports demand:

$$\bar{Y}_F = \alpha(\bar{\Gamma}_F)^{-\tau}\bar{C} \tag{103}$$

Demand for home-produced goods:

$$\bar{C}_H = (1-\alpha)(\bar{\Gamma}_H)^{-\tau}\bar{C} \tag{104}$$

Consumer price index:

$$\bar{\Gamma}_F = \bar{\Gamma}_H \tag{105}$$

H_{X1}:

$$\bar{H}_{X1} = (1 - \beta\theta_X)\bar{\Lambda}\bar{\Gamma}_H\bar{X} \tag{106}$$

H_{X2}:

$$\bar{H}_{X2} = (1 - \beta\theta_X)\bar{\Lambda}\bar{\Gamma}_X\bar{X} \tag{107}$$

Exports inflation:

$$\bar{H}_{X1} = \bar{H}_{X2} \tag{108}$$

Exports demand:

$$\bar{X} = \alpha\left(\frac{\bar{\Gamma}_X}{\bar{Q}}\right)^{-\tau}\bar{Y}^* \tag{109}$$

H_{F1}:

$$\bar{H}_{F1} = (1-\theta_F)\bar{Q}\bar{\Lambda}\bar{Y}_F \tag{110}$$

H_{F2}:

$$\bar{H}_{F2} = (1-\theta_F)\bar{\Lambda}\bar{\Gamma}_F\bar{Y}_F \tag{111}$$

Imports inflation:

$$\bar{H}_{F1} = \bar{H}_{F2} \tag{112}$$

Production of domestically produced goods:

$$\bar{Y} = \bar{N} \tag{113}$$

Marginal costs

$$\bar{MC} = \frac{\bar{W}}{\bar{\Gamma}_H} \tag{114}$$

H_{p1}:

$$\bar{H}_{p1} = (1 - \beta\theta_p)\bar{\Lambda}\bar{MC}\bar{\Gamma}_H\bar{Y} \tag{115}$$

H_{p2}:

$$\bar{H}_{p2} = (1 - \beta\theta_p)\bar{\Lambda}\bar{\Gamma}_H\bar{Y} \tag{116}$$

Domestic price inflation:

$$\bar{H}_{p1} = \bar{H}_{p2} \tag{117}$$

H_{w1}:

$$\bar{H}_{w1} = (1 - \beta\theta_w)\nu\bar{N}^{1+\varphi} \tag{118}$$

H_{w2}:

$$\bar{H}_{w2} = (1 - \beta\theta_w)\bar{\Lambda}\bar{W}\bar{N} \tag{119}$$

Wage index:

$$\bar{H}_{w1} = \bar{H_{w2}} \tag{120}$$

Wage inflation:

$$\bar{\Pi}^w = \bar{\Pi}\mathcal{M} \tag{121}$$

Domestic goods market clearing:

$$\bar{Y} = \bar{C}_H + \bar{W} \tag{122}$$

Financial market clearing:

$$\bar{\Gamma}_x\bar{X} = \bar{Q}\bar{Y}_F \tag{123}$$

Monetary policy:

$$\bar{\Pi} = \bar{\Pi} \tag{124}$$

Relative prices exports:

$$\bar{\Pi}_W = \frac{\bar{\Pi}}{\Delta\bar{S}} \tag{125}$$

Relative price imports:

$$\bar{\Pi}_F = \bar{\Pi} \tag{126}$$

Relative price home goods:

$$\bar{\Pi}_H = \bar{\Pi} \tag{127}$$

10.3 Log-linearized Equilibrium Conditions

Consumption decision:

$$(\mathcal{M}^2 + \beta h^2)c_t = \beta h \mathcal{M} E_t\{c_{t+1}\} + h\mathcal{M}c_{t-1} - (\mathcal{M} - \beta h)(\mathcal{M} - h)\lambda_t \\ - h\mathcal{M}(\mu_t - \beta E_t\{\mu_{t+1}\}) + \mathcal{M}((\mathcal{M} - h)\xi_t - \beta h E_t\{\xi_{t+1}\}) \quad (128)$$

Discounted Euler equation:

$$\lambda_t = \kappa \mathbb{E}_t\left\{\lambda_{t+1}\right\} + \left(r_t - \mathbb{E}_t\left\{\pi_{t+1} + \mu_{t+1}\right\}\right) \quad (129)$$

UIP:

$$r_t = r_{F,t} + \mathbb{E}_t\left\{\Delta s_{t+1}\right\} \quad (130)$$

Real exchange rate:

$$q_t = q_{t-1} + \Delta s_t + \pi_t^* - \pi_t \quad (131)$$

Interest rate on foreign borrowing:

$$r_{F,t} = r_t^* - \psi_H \hat{b}_{F,t+1} + \psi_t \quad (132)$$

Imports demand

$$y_{F,t} = c_t - \tau\gamma_{F,t} \quad (133)$$

Demand for home-produced goods:

$$c_{H,t} = c_t - \tau\gamma_{H,t} \quad (134)$$

Consumer price index:

$$\pi_t = \alpha(\pi_{F,t} + \gamma_{F,t-1}) + (1-\alpha)(\pi_{H,t} + \gamma_{H,t-1}) \quad (135)$$

$H_{X1,t}$:

$$h_{X1,t} = (1 - \beta\theta_x)(\lambda_t + \gamma_{H,t} + x_t) + \beta\theta_X \mathbb{E}_t\left\{(h_{X1,t+1} + \epsilon_X(\pi_{X,t+1} + \chi_X\pi_{X,t}))\right\} \quad (136)$$

$H_{X2,t}$:

$$h_{X2,t} = (1 - \beta\theta_x)(\lambda_t + \gamma_{X,t} + x_t) + \beta\theta_X \mathbb{E}_t\left\{(h_{X2,t+1} + (\epsilon_X - 1)(\pi_{X,t+1} + \chi_X\pi_{X,t}))\right\} \quad (137)$$

Exports inflation:

$$\pi_{X,t} = \frac{1-\theta_X}{\theta_X}(h_{x1,t} - h_{x2,t} + \xi_{X,t}) + \chi_X\pi_{X,t-1} \quad (138)$$

Exports demand:

$$x_t = y_t^* - \tau(\gamma_{X,t} - q_t) \quad (139)$$

$H_{F1,t}$:

$$h_{F1,t} = (1 - \beta\theta_F)(q_t + \lambda_t + y_{F,t}) + \beta\theta_F \mathbb{E}_t\left\{(h_{F1,t+1} + \epsilon_F(\pi_{F,t+1} - \chi_F\pi_{F,t}))\right\} \quad (140)$$

$H_{F2,t}$:

$$h_{F2,t} = (1-\beta\theta_F)(\gamma_{F,t} + \lambda_t + y_{F,t}) + \beta\theta_F \mathbb{E}_t \left\{(h_{F2,t+1} + (\epsilon_F - 1)(\pi_{F,t+1} - \chi_F \pi_{F,t}))\right\} \tag{141}$$

Imports inflation:

$$\pi_{F,t} = \frac{1-\theta_F}{\theta_F}(h_{F1,t} - h_{F2,t} + \xi_{F,t}) + \chi_F \pi_{F,t-1} \tag{142}$$

Production of domestic goods:

$$y_t = a_t + n_t \tag{143}$$

Marginal costs:

$$mc_t = w_t - \gamma_{H,t} - a_t \tag{144}$$

$H_{p1,t}$:

$$h_{p1,t} = (1-\beta\theta_p)(\lambda_t + mc_t + \gamma_{H,t} + y_t) + \beta\theta \mathbb{E}_t \left\{(h_{p1,t+1} + \epsilon_p(\pi_{H,t+1} - \chi_p \pi_{H,t}))\right\} \tag{145}$$

$H_{p2,t}$:

$$h_{p2,t} = (1-\beta\theta_p)(\lambda_t + \gamma_{H,t} + y_t) + \beta\theta \mathbb{E}_t \left\{(h_{p2,t+1} + (\epsilon_p - 1)(\pi_{H,t+1} - \chi_p \pi_{H,t}))\right\} \tag{146}$$

Domestic inflation:

$$\pi_{H,t} = \frac{1-\theta_p}{\theta_p}(h_{p1,t} - h_{p2,t} + \xi_{H,t}) + \chi_p \pi_{H,t-1} \tag{147}$$

$H_{w1,t}$:

$$h_{w1,t} = (1-\beta\theta_w)(1+\varphi)n_t + \beta\theta_w \mathbb{E}_t \left\{(h_{w1,t+1} + \epsilon_w(1+\varphi)(\pi^w_{t+1} - \chi_w \pi_{w,t}))\right\} \tag{148}$$

$H_{w2,t}$:

$$h_{w2,t} = (1-\beta\theta_w)(\lambda_t + w_t + n_t) + \beta\theta_w \mathbb{E}_t \left\{(h_{w2,t+1} + (\epsilon_w - 1)(\pi^w_{t+1} - \chi_w \pi_{w,t}))\right\} \tag{149}$$

Wage index:

$$\pi^w_t = \frac{1-\theta_w}{\theta_w(1+\epsilon_w\varphi)}(h_{w1,t} - h_{w2,t} + \xi_{w,t}) + \chi_w \pi^2_{t-1} \tag{150}$$

Wage inflation:

$$\pi^w_t = w_t - w_{t-1} + \pi_t + \mu_t \tag{151}$$

Domestic market clearing:

$$y_t = \frac{C_H}{Y} c_{H,t} + \frac{X}{Y} x_t \tag{152}$$

Financial market clearing:

$$\hat{b}_{F,t+1} = \hat{b}_{F,t} + \gamma_{X,t} + x_t - q_t - y_{F,t} \tag{153}$$

Monetary policy:

$$r_t = \rho_R r_{t-1} + (1-\rho_R)(\phi_\pi \pi_t + \phi_y y_t) + \varepsilon_{R,t} \tag{154}$$

Relative prices:

$$\gamma_{X,t} = \gamma_{X,t-1} + \Delta s_t + \pi_{X,t} - \pi_t \tag{155}$$

$$\gamma_{F,t} = \gamma_{F,t-1} + \pi_{F,t} - \pi_t \tag{156}$$

$$\gamma_{H,t} = \gamma_{H,t-1} + \pi_{H,t} - \pi_t \tag{157}$$

References

Cagliarini, Adam and Mariano Kulish, "Solving Linear Rational Expectations Models with Predictable Structural Changes," *Review of Economics and Statistics*, March 2013, *95* (1), 328–336.

Del Negro, Marco, Marc Giannoni, and Christina Patterson, "The Forward Guidance Puzzle," Staff Reports 574, Federal Reserve Bank of New York 2012.

Erceg, Christopher, Dale W. Henderson, and Andrew T. Levin, "Optimal monetary policy with staggered wage and price contracts," *Journal of Monetary Economics*, 2000, *46* (2), 281–313.

Guerrieri, Luca and Mateo Iacoviello, "OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily," *Journal of Monetary Economics*, 2015, *70*, 22–38.

Jones, Callum, "Unanticipated Shocks and Forward Guidance at the Zero Lower Bound," 2017.

Kulish, Mariano and Adrian Pagan, "Estimation and solution of models with expectations and structural changes," *Journal of Applied Econometrics*, 2017, *32* (2), 255–274.

_ , James Morley, and Tim Robinson, "Estimating DSGE models with zero interest rate policy," *Journal of Monetary Economics*, 2017, *88*, 35–49.

Sims, Christopher, "Solving Linear Rational Expectations Models," *Computational Economics*, 2002, *20*, 1–20.