International Spillovers of Forward Guidance Shocks

Appendix

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Contents

1 Solution with Two Fixed-rate Regimes

In our case, policy interest rates in the US and Canada can be fixed at different periods, or at the same time. The economy can therefore be in one of the following four possible regimes at a given point in our sample: (i) interest rates follow feedback rules, (ii) only the interest rate of the large economy is fixed, (iii) only the interest rate of the small economy is fixed, and (iv) both interest rates are fixed. Figure 1 illustrates one possibility, in which in an initial sub-sample conventional policy applies to both economies, then there is a period of time for which the interest rate is fixed only in the large economy. After that, interest rates are fixed in both economies, and eventually there is a return to conventional policy which takes place out-of-sample.

We first linearize the model around the steady state for which policy rates follow feedback rules, and write the resulting system of equations in matrix form as:

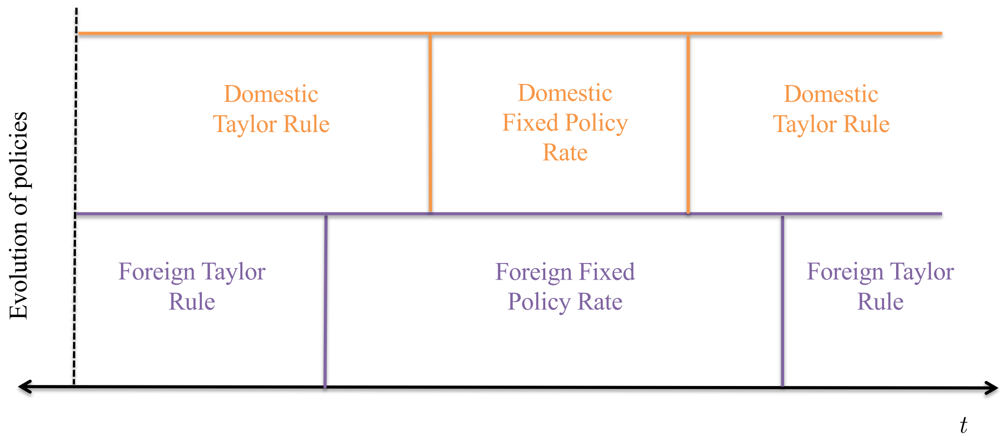
$$\mathbf{A}x_t = \mathbf{C} + \mathbf{B}x_{t-1} + \mathbf{D}\mathbb{E}_t x_{t+1} + \mathbf{F}\varepsilon_t. \tag{1}$$

where x_t is the state vector and ε_t is the vector of structural shocks, which we take to be i.i.d. without loss of generality. If it exists and is unique, the standard rational expectations solution to (1) is $x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}\varepsilon_t$.

When only the *foreign* interest rate is fixed the structural equations are given by:

$$\mathbf{A}^{\star}x_t = \mathbf{C}^{\star} + \mathbf{B}^{\star}x_{t-1} + \mathbf{D}^{\star}\mathbb{E}_t x_{t+1} + \mathbf{F}^{\star}\varepsilon_t. \tag{2}$$

Figure 1: Timing and Four Possible Regimes in the Model



Note: This figure shows the timing and 4 possible regimes that the model solution accounts for. During the sample period, the US interest rate can either be governed by a policy rule (foreign conventional policy) or fixed at its lower bound. For each of these two regimes, the Canadian interest rate can either be governed by a rule (domestic conventional policy) or at its lower bound.

where the only equation that has changed in the starred system relative to (1) is the equation defining the *foreign* policy interest rate rule, which is now specified such that the nominal interest rate is fixed.[1]

When only the *domestic* interest rate is fixed the structural equations are given by:

$$\bar{\mathbf{A}}x_t = \bar{\mathbf{C}} + \bar{\mathbf{B}}x_{t-1} + \bar{\mathbf{D}}\mathbb{E}_t x_{t+1} + \bar{\mathbf{F}}\varepsilon_t. \tag{3}$$

where the only equation that has changed relative to (1) is the equation defining the *domestic* policy interest rate rule, which is now specified such that the nominal interest rate is fixed.

And when both *foreign* and *domestic* interest rates are fixed the structural equations are given by:

$$\bar{\mathbf{A}}^\star x_t = \bar{\mathbf{C}}^\star + \bar{\mathbf{B}}^\star x_{t-1} + \bar{\mathbf{D}}^\star\mathbb{E}_t x_{t+1} + \bar{\mathbf{F}}^\star\varepsilon_t. \tag{4}$$

If interest rates follow feedback rules at time t, then $\mathbf{A}_t = \mathbf{A}$, $\mathbf{C}_t = \mathbf{C}$, $\mathbf{B}_t = \mathbf{B}$, and so. If both domestic and foreign policy interest rates are fixed at time t then $\mathbf{A}_t = \bar{\mathbf{A}}^\star$, $\mathbf{C}_t = \bar{\mathbf{C}}^\star$, $\mathbf{B}_t = \bar{\mathbf{B}}^\star$, and so on.

Assume then that at time $t = 1$ agents expect fixed interest rates in the large economy for $\mathbf{d}_1^*$ periods and fixed interest rates in the small economy for $\mathbf{d}_1$ periods. After $\max(\mathbf{d}_1^*, \mathbf{d}_1)$ both economies would have reverted back to (1) and the standard solution applies. From $t = 1$, $\mathbf{d}_1^*$ and $\mathbf{d}_1$ imply an expectation of which of the four possible regimes will be in place at each point in time. Let the expected regimes be summarised by the sequence

$$\{\mathbf{A}_t, \mathbf{C}_t, \mathbf{B}_t, \mathbf{D}_t, \mathbf{F}_t\}_{t=1}^{\max(\mathbf{d}_1^*, \mathbf{d}_1)}$$

Following Kulish and Pagan (2017), the solution is a time-varying coefficient VAR of the form

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t\varepsilon_t, \tag{5}$$

where the reduced form matrices solve the recursions below:

$$\begin{aligned}
\mathbf{J}_t &= \left[\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}(\mathbf{C}_t + \mathbf{D}_t\mathbf{J}_{t+1}) \\
\mathbf{Q}_t &= \left[\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}\mathbf{B}_t \\
\mathbf{G}_t &= \left[\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}\mathbf{F}_t.
\end{aligned} \tag{6}$$

With the solution in hand for a given foreign and domestic duration, the likelihood is constructed following Kulish, Morley and Robinson (2017). We denote the estimated durations of the small economy by $\{\mathbf{d}_t\}_{t=1}^T$ and those of the large economy by $\{\mathbf{d}_t^*\}_{t=1}^T$ and adopt the convention that in periods where the Taylor rule is in operation in the small economy, $\mathbf{d}_t = 0$, and in the large econonmy, $\mathbf{d}_t^* = 0$.

[1] The notation accommodates additional structural changes which have to be accounted for if the expansion point of the approximation changes. In our application we work around the intended steady state.

2 Lower Bound Implementation

There are two occasionally binding lower bound constraints to impose in this model, one to the US nominal interest rate, and one to the Canadian nominal interest rate. A flexible algorithm is developed that relies on constructing a perfect foresight path of the nominal interest rate in both countries, and piecing together linear systems in a step-by-step way. These methods are based on the solution concepts developed in Cagliarini and Kulish (2013); Kulish and Pagan (2017), Guerrieri and Iacoviello (2015) and Jones (2017). As shown in these papers, the approximation does a good job at capturing the non-linear effects induced by the occasionally binding constraints.

2.1 Notation

Denote by x_t^* the vector of endogenous variables for the large country at time period t, one of which is the nominal interest rate in the large country, and x_t the vector of endogenous variables for the small country at time period t, one of which is the nominal interest rate R_t. The initial conditions are $[{x_{t-1}^*}' \; {x_{t-1}}']'$ and the initial vector of unanticipated exogenous variables, denoted by ε_t. The model is a system of n equations.

2.2 Initialization at t

We know, at period t:

- The shock that hits at period t: ε_t.
- The initial vector of variables x_{t-1}.

2.3 The Algorithm

The steps of the algorithm are:

0. Linearize the model around the non-stochastic steady state, ignoring the lower bounds in both countries.
1. For each period t:

   For the large country:

   (a) Solve for the path $\{x_\tau\}_{\tau=t}^T$ with T large, using the solution of the linearized economy from step (0), given ε_t and the initial vector of variables x_{t-1}, and assuming no future uncertainty. This gives a path for the nominal interest rate, $\mathbf{i}_t^k = \{i_\tau^k\}_{\tau=t}^T$.

   (b) Examine the path $\mathbf{i}_t^k$. If $\mathbf{i}_t^k \geq 0$, then the lower bound does not bind, so move onto step (2). If $\mathbf{i}_t^k < 0$, then move onto step (1c).

   (c) For the *first* time period where $\mathbf{i}_t^k < 0$, set the nominal interest rate in that period to zero. This changes the anticipated structure of the economy. Under this new structure, calculate the path of all variables, including the new path for the nominal interest rate $\mathbf{i}_t^{k+1} = \{i_\tau^{k+1}\}_{\tau=t}^T$.

   Iterate on steps 1a and 1c until convergence of $\mathbf{i}_t^*$.

   Repeat steps 1a to 1c for the small country.

2. Increment t by one. The initial vector of variables now becomes x_t, which was solved for in step 1. Draw a new vector of unanticipated shocks ε_{t+1} and return to step 1.

To compute the path $\{x_\tau\}_{\tau=t}^T$ under forward guidance, compute step (1c) first, imposing the sequence of structural matrices corresponding to the lower bound and non-lower bound periods. Then examine the path $\{i_\tau\}_{\tau=t}^T$ for subsequent violations of the lower bound.

2.4 Details of Each Step

At the following steps:

0. Write the n equations of the linearized structural model at t as:

$$\mathbf{A}x_t = \mathbf{C} + \mathbf{B}x_{t-1} + \mathbf{D}\mathbb{E}_t x_{t+1} + \mathbf{F}\varepsilon_t, \tag{SM}$$

where x_t is a $n \times 1$ vector of state and jump variables and ε_t is a $l \times 1$ vector of exogenous variables. Use standard methods to obtain the reduced form:

$$x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}\varepsilon_t. \tag{RF}$$

1. For each period t:

(a) Using (RF), obtain the path $\{x_\tau\}_{\tau=t}^{T}$ given ε_t. Set T to be large. Assume $\{w_\tau\}_{\tau=t+1}^{T} = 0$ (no future uncertainty), so that $x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}\varepsilon_t$, and $x_{t+1} = \mathbf{J} + \mathbf{Q}x_t$, up to $x_T = \mathbf{J} + \mathbf{Q}x_{T-1}$.
This step gives a path $\mathbf{i}_t = \{i_\tau\}_{\tau=t}^{T}$.

(b) Examine the path $\{i_\tau\}_{\tau=t}^{T}$.

- If $i_\tau \geq 0$ for all $t \leq \tau < T$, accept $\{x_\tau\}_{\tau=t}^{T}$. The i_t path does not violate lower bound today or in future.
- If $i_\tau < 0$ for any $t \leq \tau < T$, move to step (1c).

(c) Update the path of $\{i_\tau\}_{\tau=t}^{T}$ for the lower bound. For the first time period t^* where $i_{t^*} < 0$, set $i_{t^*} = 0$. The model system at t^* therefore becomes:

$$\mathbf{A}^* x_{t^*} = \mathbf{C}^* + \mathbf{B}^* x_{t^*-1} + \mathbf{D}^*\mathbb{E}_{t^*} x_{t^*+1} + \mathbf{F}^* w_{t^*}, \tag{7}$$

Compute the new path $\{i_\tau\}_{\tau=t}^{T}$. This involves computing $\{x_\tau\}_{\tau=t}^{t^*}$ and $\{x_\tau\}_{\tau=t^*+1}^{T}$. At t^*, $\mathbb{E}_{t^*} x_{t^*+1}$ is computed using the the reduced form solution (RF) and $w_{t^*+1} = 0$. This expresses x_{t^*} as a function of x_{t^*-1}. Proceeding in this way with the correct structural matrices (either lower bound $*$ or no lower bound at each time period), compute the path $\{i_\tau\}_{\tau=t}^{T}$.
A convenient way to compute the new path $\{i_\tau\}_{\tau=t}^{T}$ is to form the time varying matrices $\{\mathbf{J}_\tau, \mathbf{Q}_\tau, \mathbf{G}_\tau\}_{\tau=t}^{T}$ which satisfy the recursion:

$$\mathbf{Q}_t = \left[\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}\mathbf{B}_t \tag{8}$$

$$\mathbf{J}_t = \left[\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}\left(\mathbf{C}_t + \mathbf{D}_t\mathbf{J}_{t+1}\right) \tag{9}$$

$$\mathbf{G}_t = \left[\mathbf{A}_t - \mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}\mathbf{F}_t, \tag{10}$$

with the final set of reduced form matrices for the recursion being the non-lower bound matrices $\mathbf{J}$, $\mathbf{Q}$, $\mathbf{G}$ from (RF).
These time-varying matrices are then used to compute the path $\{x_\tau\}_{\tau=t}^{T}$ by calculating $x_\tau = \mathbf{J}_\tau + \mathbf{Q}_\tau x_{\tau-1} + \mathbf{G}_\tau w_\tau$.

2.5 Output of the Algorithm

The algorithm yields a set of time-varying structural matrices:

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t\varepsilon_t, \tag{11}$$

from which we get the path of $\{x_\tau\}_{\tau=t}^{\infty}$ where the nominal interest rate is subject to the lower bound. Both the current value of the nominal interest rate, and expectations of the lower bound binding, affect current values of state variables.

2.6 Identifying Forward Guidance

Here, we explain how to use the algorithm in Section 2.3 to decompose an anticipated duration of the lower bound into a component due to structural shocks, and a component due to forward guidance. Assume that at period t, the lower bound binds and we have used procedures to estimate the model parameters and the anticipated length of the lower bound at period t. We have in hand at period t:

1. An estimated duration $\tilde{T}$ of the lower bound at t, so that the interest rate is expected to stay at zero until time period $t + \tilde{T}$.
2. An estimate of the history of the states $\{x_\tau\}_{\tau=0}^{t-1}$ and an estimate of the structural shocks $\{w_\tau\}_{\tau=1}^{t}$, computed using the Kalman smoother.

The estimated parameters, durations and shocks recover the observed series and give an estimate of the model's state variables x_t. To decompose the proportion of the estimated duration due to structural shocks, so that the remainder is due to forward guidance policies, at each point of time:

1. Use the state x_{t-1} and the structural shock ε_t to compute, using the lower bound algorithm of Section 2.3, the endogenous duration of the lower bound.
2. If the computed endogenous duration is less than the estimated duration, then the additional time is assigned to commitment forward guidance policy.

The endogenous duration is the duration that would have occured had the central bank simply set the nominal interest rate to zero in periods where the policy rule would have specified that it be negative, and set the interest rate to its positive value when the policy rule specifies that it be positive.

2.7 Kalman Filter

The model in state space representation is:

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t \varepsilon_t \tag{State Eqn}$$
$$z_t = \mathbf{H}_t x_t. \tag{Obs Eqn}$$

The structural shocks are Gaussian, so that $\varepsilon_t \sim N(0, \mathbf{Q})$, where $\mathbf{Q}$ is the covariance matrix of ε_t. There is no observation error by assumption. The Kalman filter recursion is given by the following equations. The state of the system is the state vector and its covariance matrix $(\hat{x}_t, P_{t-1})$. The predict step involves using the structural matrices $\mathbf{J}_t$, $\mathbf{Q}_t$ and $\mathbf{G}_t$:

$$\hat{x}_{t|t-1} = \mathbf{J}_t + \mathbf{Q}_t \hat{x}_t \tag{12}$$
$$\mathbf{P}_{t|t-1} = \mathbf{Q}_t \mathbf{P}_{t-1} \mathbf{Q}_{t|t-1}^\top + \mathbf{G}_t \mathbf{Q} \mathbf{G}_t^\top. \tag{13}$$

This formulation differs from the time-invariant Kalman filter step because in the forecast stage the structural matrices $\mathbf{J}_t$, $\mathbf{Q}_t$ and $\mathbf{G}_t$ can vary over time. We update these forecasts with imperfect observations of the state vector. Also note that $\mathbf{H}_t$ is time-varying, reflecting that when the nominal interest rate is at its lower bound, we lose it as an observable variable. The update step involves computing forecast errors $\tilde{y}_t$ and its associated covariance matrix $\mathbf{S}_t$:

$$\tilde{y}_t = z_t - \mathbf{H}_t \hat{x}_{t|t-1} \tag{14}$$
$$\mathbf{S}_t = \mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^\top. \tag{15}$$

The Kalman gain matrix is given by:

$$\mathbf{K}_t = \mathbf{P}_{t|t-1} \mathbf{H}_t^\top \mathbf{S}_t^{-1}. \tag{16}$$

With $\tilde{y}_t$, $\mathbf{S}_t$ and $\mathbf{K}_t$ in hand, the optimal update of the state x_t and its associated covariance matrix is:

$$\hat{x}_t = \hat{x}_{t|t-1} + \mathbf{K}_t \tilde{y}_t \tag{17}$$
$$\mathbf{P}_t = (I - \mathbf{K}_t \mathbf{H}_t) \mathbf{P}_{t|t-1}. \tag{18}$$

The Kalman filter is initialized with x_0 and $\mathbf{P}_0$ computed from their unconditional moments. The recursion is computed until the final time period T of data.

2.8 Kalman Smoother

With the estimates of the parameters and durations in hand at time period T, the Kalman smoother gives an estimate of $x_{t|T}$, or an estimate of the state vector at each point in time given all available information. With $\hat{x}_{t|t-1}$, $\mathbf{P}_{t|t-1}$, $\mathbf{K}_t$ and $\mathbf{S}_t$ in hand from the filter, the vector $x_{t|T}$ is computed by:

$$x_{t|T} = \hat{x}_{t|t-1} + \mathbf{P}_{t|t-1} r_{t|T}, \tag{19}$$

where the vector $r_{T+1|T} = 0$ and is updated with the recursion:

$$r_{t|T} = \mathbf{H}_t^\top \mathbf{S}_t^{-1} \left(z_t - \mathbf{H}_t \hat{x}_{t|t-1} \right) + \left(I - \mathbf{K}_t \mathbf{H}_t \right)^\top \mathbf{P}_{t|t-1}^\top r_{t+1|T}. \tag{20}$$

Finally, to get an estimate of the shocks to each state variable, denoted by e_t, we compute:

$$e_t = \mathbf{G}_t \varepsilon_t = \mathbf{G}_t r_{t|T}. \tag{21}$$

2.9 Sampler

This section describes the sampler used to obtain the posterior distribution of interest. Denote by ϑ the vector of parameters to be estimated and $\mathbf{T}$ the vector of durations to be estimated. Contained in $\mathbf{T}$ are a set of durations for both the foreign and domestic countries. Denote by $Z = \{z_\tau\}_{\tau=1}^T$ the sequence of observable vectors. The posterior $\mathcal{P}(\vartheta, \mathbf{T} \mid Z)$ satisfies:

$$\mathcal{P}(\vartheta, \mathbf{T} \mid Z) \propto \mathcal{L}(Z \mid \vartheta, \mathbf{T}) \times \mathcal{P}(\vartheta, \mathbf{T}). \tag{22}$$

With Gaussian errors, the likelihood function $\mathcal{L}(Z \mid \vartheta, \mathbf{T})$ is computed using the appropriate sequence of structural matrices and the Kalman filter:

$$\log \mathcal{L}(Z \mid \vartheta, \mathbf{T}) = -\left(\frac{N_z T}{2} \right) \log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log \det \mathbf{H}_t \mathbf{S}_t \mathbf{H}_t^\top - \frac{1}{2} \sum_{t=1}^{T} \tilde{y}_t^\top \left(\mathbf{H}_t \mathbf{S}_t \mathbf{H}_t^\top \right)^{-1} \tilde{y}_t.$$

The prior is simply computed using priors over ϑ which are consistent with the literature, and with flat priors over $\mathbf{T}$.[2]

The Markov Chain Monte Carlo posterior sampler has two blocks, corresponding to ϑ and $\mathbf{T}$. Initialize the sampler at step j with the last accepted draw of the structural parameters, the period of the breaking parameters and durations, denoted by ϑ_{j-1} and $\mathbf{T}_{j-1}$ respectively. The blocks are, in order of computation:

1. In the first block, randomly choose up to $\bar{T}$ durations to test in each country, corresponding to up to $\bar{T}$ time periods that each economy is at the lower bound. For each of those time periods, randomly choose a duration in the interval $[1, T^*]$ for each country to generate a new $\mathbf{T}_j$ proposal. Recompute the sequence of structural matrices associated with $(\vartheta_{j-1}, \mathbf{T}_j)$, compute the posterior $\mathcal{P}(\vartheta_{j-1}, \mathbf{T}_{j-1} \mid Z)$, and accept the proposal $(\vartheta_{j-1}, \mathbf{T}_j)$ with probability $\frac{\mathcal{P}(\vartheta_{j-1}, \mathbf{T}_j | Z)}{\mathcal{P}(\vartheta_{j-1}, \mathbf{T}_{j-1} | Z)}$. If $(\vartheta_{j-1}, \mathbf{T}_j)$ is accepted, then set $\mathbf{T}_{j-1} = \mathbf{T}_j$.

2. The second block is a more standard Metropolis-Hastings random walk step. Start by selecting which structural parameters to propose a new value for. For those parameters, draw a new proposal ϑ_j from a proposal density centered at ϑ_{j-1} and with thick tails to ensure sufficient coverage of the parameter space and an acceptance rate of roughly 20%. The proposal ϑ_j is accepted with probability $\frac{\mathcal{P}(\vartheta_j, \mathbf{T}_{j-1} | Z)}{\mathcal{P}(\vartheta_{j-1}, \mathbf{T}_{j-1} | Z)}$. If $(\vartheta_j, \mathbf{T}_{j-1})$ is accepted, then set $\vartheta_{j-1} = \vartheta_j$.

2.10 A Worked Example of the Algorithm

Consider the simple example, log-linearized around steady-state where y_t is output and the nominal interest rate i_t ignores the ZLB:

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - \bar{i}) + \varepsilon_t \\ i_t - \bar{i} &= \rho \left(i_{t-1} - \bar{i} \right) + \gamma y_t. \end{aligned}$$

[2] For practical convenience, we require that each estimated duration lies below some maximum value T^* which, in practice, is rarely visited by the sampler.

Putting this model in the form of (SM) requires $x_t = [\ i_t \quad y_t\ ]'$ and:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -\gamma \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ \rho & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \bar{i} \\ \bar{i}(1-\rho) \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The routines of Sims (2002) are used to obtain the linear system:

$$x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}\varepsilon_t. \tag{23}$$

The algorithm proceeds as follows. Given a shock ε_t:

1. Using the reduced form system without the ZLB (23), obtain the path $\mathbf{y}_t$ up to some large T. Assume no future shocks:

$$\begin{aligned} x_t &= \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}\varepsilon_t \\ x_{t+1} &= \mathbf{J} + \mathbf{Q}x_t \\ &\vdots \\ x_T &= \mathbf{J} + \mathbf{Q}x_{T-1}. \end{aligned}$$

2. Examine $\{i_\tau\}_{\tau=t}^{T}$. If $i_\tau > 0 \ \forall\ \tau$, then stop the algorithm. Otherwise, move to the next step.

3. Find the first time period where $i_\tau < 0$. Suppose $i_{t+1} < 0$ under the shock ε_t. Then, we want the following system to apply at time period $t+1$:

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - \bar{i}) + \varepsilon_t \\ i_t &= 0, \end{aligned}$$

   and the non-ZLB system to apply for t and time periods $\tau > t+1$. The system at $t+1$ translates into the following structural matrices:

$$\mathbf{A}^*_{t+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B}^*_{t+1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C}^*_{t+1} = \begin{bmatrix} \bar{i} \\ 0 \end{bmatrix}, \quad \mathbf{D}^*_{t+1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{F}^*_{t+1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

   while $\Pi^*_{t+1} = \Pi$. Then use the solution to the non-ZLB system to obtain x_{t+j} for $j > t+1$ again assuming no future shocks.

   Return to step 2 with the the new path of $\{i_\tau\}_{\tau=t}^{T}$.

4. Examine the new path of $\{i_\tau\}_{\tau=t}^{T}$. If $i_\tau > 0 \ \forall\ \tau$, then stop the algorithm: the ZLB applies only for time period $t+1$. Otherwise, move to the next step having already imposed the ZLB at time period $t+1$.

5. Continue iterating until the nominal interest rate satisfies the ZLB across the forecast horizon.

3 Data Sources

The model is estimated using 15 macroeconomic time series.

- **US real output growth**: The quarterly log change in US real GDP per capita. We construct the latter series by dividing US real GDP, seasonally adjusted (FRED GDPC1) by the US civilian population aged over 16 years (FRED CNP16OV).

- **US real consumption growth**: The quarterly log change in real US personal consumption expenditures (Fred code PCE) divided by the population (FRED CNP16OV).

- **US inflation**: The quarterly log change in the US core PCE price index, seasonally adjusted (FRED PCEPILFE).

- **US policy rate**: The quarterly average of the target US Federal Funds rate (FRED DFF).

- **US 2 year bond yield**: The quarterly average of the US 2-year constant maturity treasury bond yield (FRED GS2).
- **US Nominal Wages**: The quarterly log change in average hourly earnings of private sector production and non-supervisory employees, seasonally adjusted, (FRED code AHETPI).
- **Canada real GDP growth**: The quarterly log change in Canadian real GDP per capita. We construct the latter series by dividing Canadian real GDP, seasonally adjusted (CANSIM 380-0064) by the Canadian Working Age Population (CANSIM 051-0001).
- **Canada real consumption growth**: Quarterly log change in household consumption per capita (Statcan Table 37-10-0107-01).
- **Canada inflation**: The quarterly log change in the Canadian CPI excluding food, energy and indirect taxes, seasonally adjusted (CANSIM 326-0022).
- **Canada policy rate**: The quarterly average of the Bank of Canada target rate rate (CANSIM v122530).
- **Canada nominal wages**: The quarterly log change in total compensation of employees divided by number of workers (Fred Codes, CANCOMPQDSNAQ / LFEMTTTTCAQ647S).
- **Canada 2 year bond yield**: The quarterly average of the Canadian 2-year constant maturity treasury bond yield (CANSIM v122538).
- **Canada-US exchange rate**: The log change in the quarterly average level of the Canada-US exchange rate (CANSIM 176-0064).
- **Canada import volumes per capita, growth**: The log change in quarterly import volumes per capita (CANSIM 380-0064).
- **Canada export growth**: The log change in quarterly export volumes per capita (CANSIM 380-0064).

Our economic model implies some relatively strong restrictions on the joint behaviour of the observed variables. For example, balanced growth requires that wages grow at the product of labour productivity growth and inflation, while exports, imports, consumption and GDP should all grow at the same rate. In practice, the relative growth rates of many of the observed variables differs from that implied by the model. Because many of these differences reflect economic forces, including changes in worker bargaining power, or in tariff barriers, that are outside the scope of our empirical exercise, we accommodate them by transforming the data. The specific transformations we make are:

1. We calculate the average growth rate of US and Canadian GDP per capita prior to 2009. Define this value as $\bar{Y}$.
2. We adjust the means of US and Canadian GDP, consumption and trade data so that they equal $\bar{Y}$ prior to 2009.
3. We adjust the mean of wages growth so that it equals $\bar{Y}$ plus the country-specific average inflation rate prior to 2009.
4. We normalize the change in the Canadian nominal exchange rate so that it equals 0 prior to 2009.

Figure 2 plots the data we use in the estimation.

4 Robustness

4.1 With Discounting in Euler Equation

We estimate the model with discounting in the Euler Equation, with $\gamma = 0.99$. Tables 1 and 2 contain the estimates of the structural parameters and the shock processes. Figure 4 shows the mean of the estimated durations and mean of the decomposed lower bound durations for the US and Canada under these estimates. Compared to our baseline estimates the estimated US interest rate durations are longer by, on average, one quarter, while the decomposition is largely the same.

Figures 5 and 6 plot contemporanous, and expected 1Y and 2Y-ahead monetary policy shocks, as in Del Negro et al. (2012). The impulse responses compared to our baseline specification and with discounting in the Euler equation ($\gamma = 0.97$) generates similar impulse responses for the expected monetary policy shocks, with the paths under discounting slightly damped compared to our baseline estimates.

4.2 GIRFs to a Forward Guidance Shock

A forward guidance shock is an unanticipated change in forward guidance component of the expected duration. This gives rise to a change in $\mathbf{d}_t$ orthogonal to the structural shocks. A foreign forward guidance shock at t changes the reduced form matrices that prevail at t as well as those that agent expected to prevail in the forecast horizon according to the solution given in (5) and (6)

We select a base duration, $\mathbf{d}^{\text{base}}$, a quarter of the fixed interest rate regime, t and compute generalized impulse responses conditional on the history of the observed variables. We take draws from the posterior and keep draws for which the foreign duration at t corresponds to our base duration, that is $\mathbf{d}_t^* = \mathbf{d}^{\text{base}}$. For admissible draws from the posterior, the Kalman smoother gives estimates of the state, $\hat{x}_{t-1|T}$ and structural shocks, $\hat{\varepsilon}_{t|T}$. We then assume no future structural shocks, $\varepsilon_{t+j} = 0$, and no future forward guidance shocks, to obtain forecasts for the state

$$\mathbb{E}_t(x_{t+n}|\mathbf{d}_t^* = \mathbf{d}^{\text{base}}, \hat{x}_{t-1|T}, \hat{\varepsilon}_{t|T}) \quad \text{for} \quad n = 0, 1, 2, \ldots$$

where for $n = 0$ we recover the smooth estimate $\hat{x}_{t|T}$. We then consider a foreign forward guidance shock, $\varepsilon_t^{\text{fg}^*}$, which changes the duration from $\mathbf{d}_t^*$ to $\mathbf{d}_t^* + \varepsilon_t^{\text{fg}^*}$. This changes the reduced-form matrices at t and in the forecast horizon as well. We forecast x_t under the forward guidance shock to obtain

$$\mathbb{E}_t(x_{t+n}|\mathbf{d}_t^* = \mathbf{d}^{\text{base}} + \varepsilon_t^{\text{fg}^*}, \hat{x}_{t-1|T}, \hat{\varepsilon}_{t|T}) \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Generalized impulse response are given by

$$\text{GIRF}(x_{t+n}) = \mathbb{E}_t(x_{t+n}|\mathbf{d}_t^* = \mathbf{d}^{\text{base}} + \varepsilon_t^{\text{fg}^*}, \hat{x}_{t-1|T}, \hat{\varepsilon}_{t|T}) - \mathbb{E}_t(x_{t+n}|\mathbf{d}_t^* = \mathbf{d}^{\text{base}}, \hat{x}_{t-1|T}, \hat{\varepsilon}_{t|T})$$

Figures 7 to 10 plots four generalized impulse responses to a two quarter US forward guidance shock in 2010Q3 and 2013Q3, for base US lower bound durations of two quarters and 8 quarters.

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Figure 2: Quarterly Data

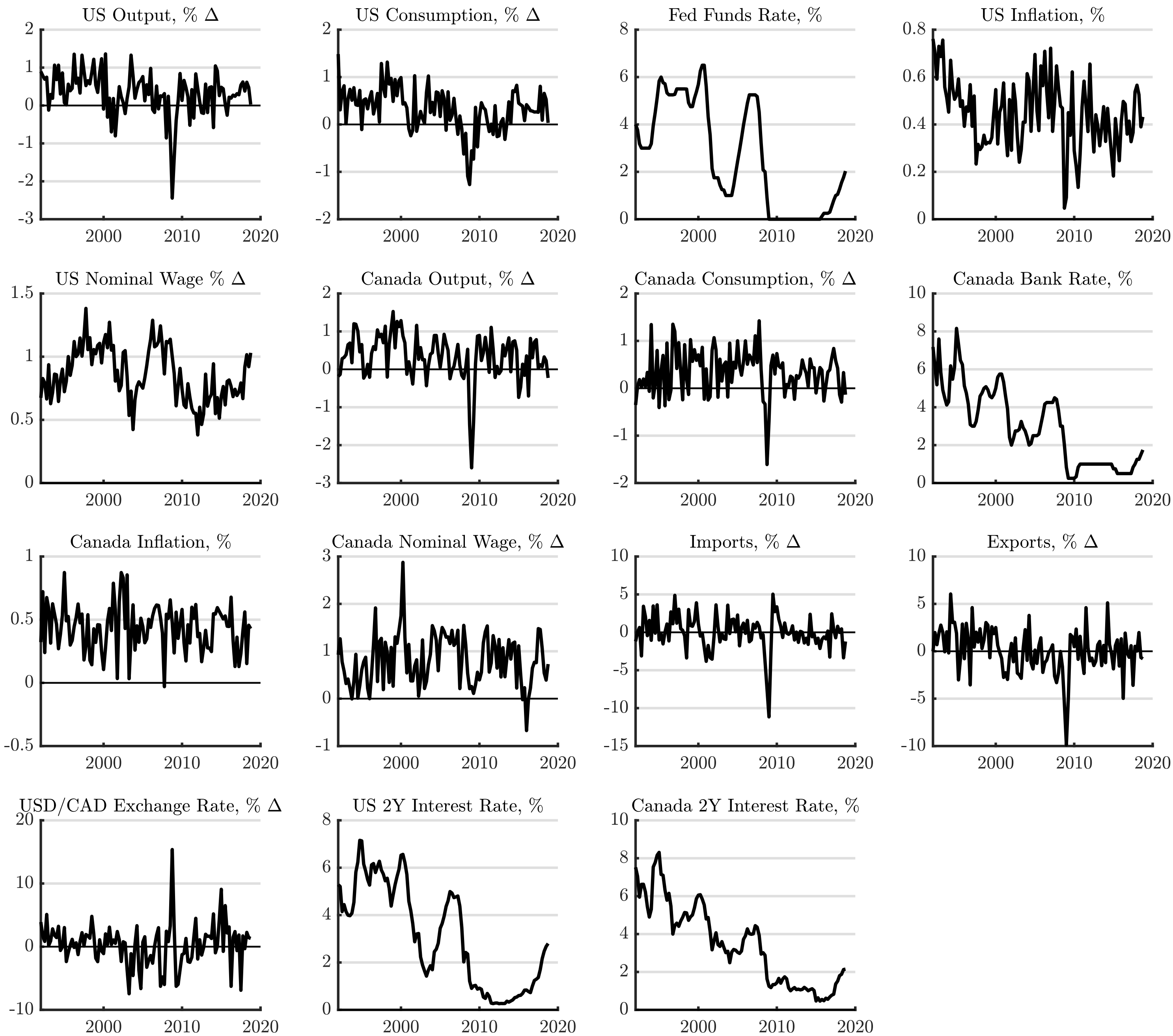

Figure 3: Gelman Chain Diagnostics

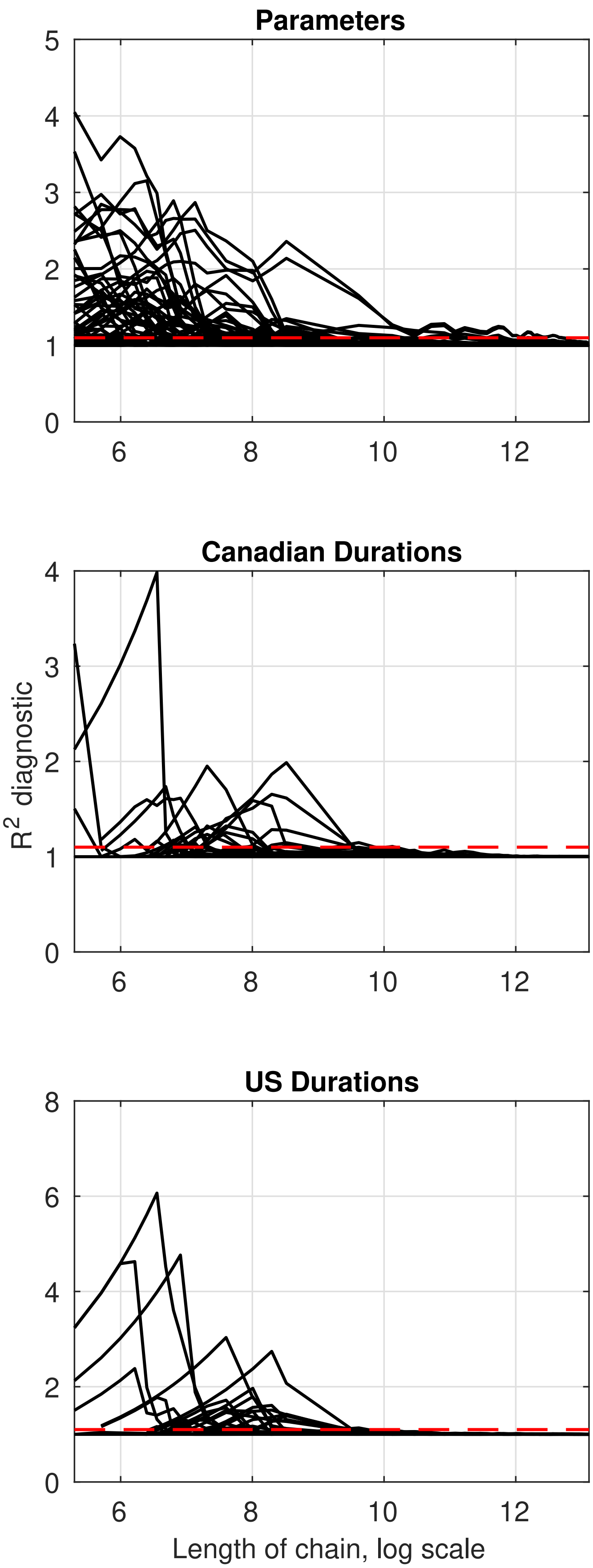

Figure 4: Fixed Interest Rate Duration and Forward Guidance, Mean Across Draws, $\gamma = 0.99$

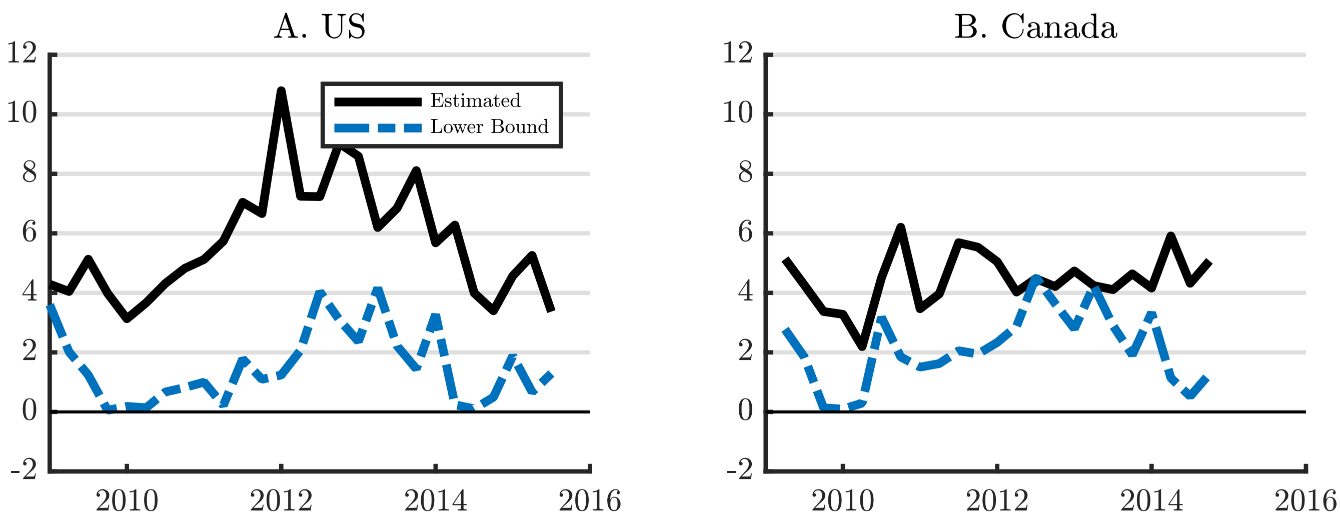

Figure 5: Contemporanous and Anticipated Monetary Policy Shocks

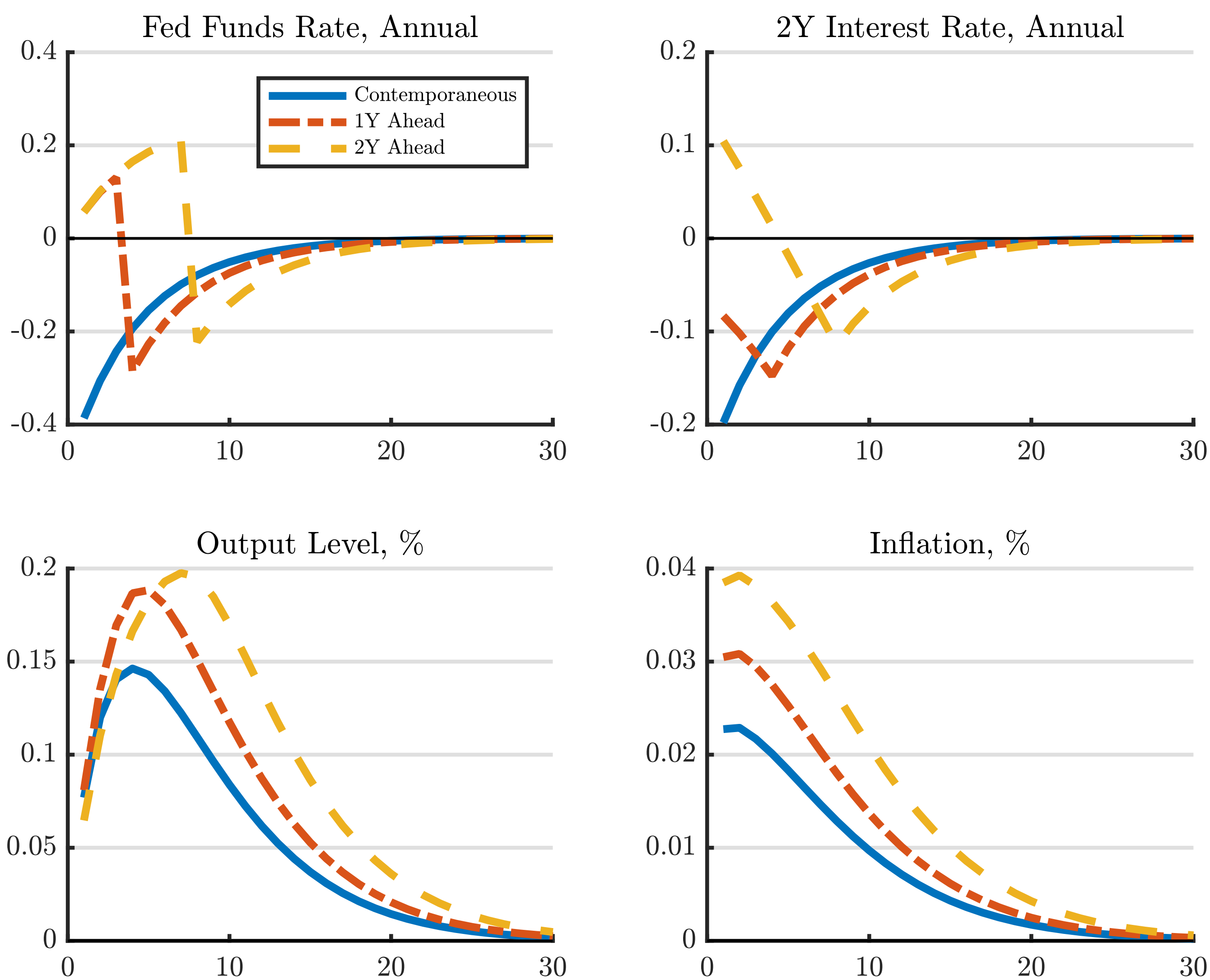

Figure 6: Contemporanous and Anticipated Monetary Policy Shocks, $\gamma = 0.97$

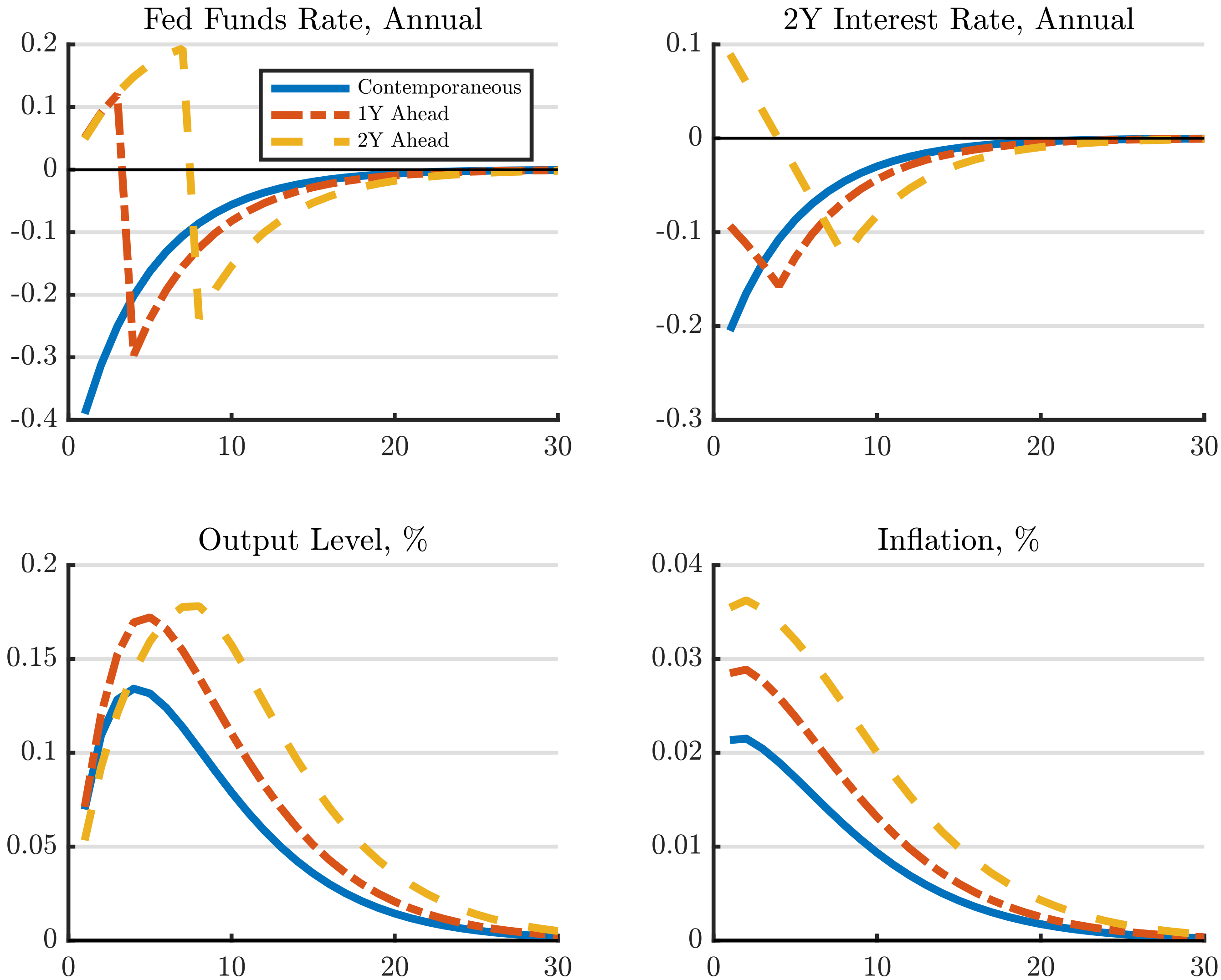

Table 1: Estimated Structural Parameters, $\gamma = 0.99$

	Prior				Posterior			
Parameter	Dist	Median	10%	90%	Mode	Median	10%	90%
				US				
h^*	B	0.7	0.6	0.8	0.70	0.70	0.64	0.76
θ_p^*	B	0.7	0.7	0.8	0.76	0.75	0.73	0.78
θ_w^*	B	0.7	0.7	0.8	0.79	0.79	0.76	0.81
ρ_r^*	B	0.5	0.2	0.8	0.89	0.89	0.87	0.91
ϕ_π^*	N	2.0	1.7	2.3	1.65	1.72	1.46	2.02
ϕ_g^*	G	0.5	0.3	0.7	0.09	0.09	0.07	0.12
ϕ_y^*	G	0.5	0.3	0.7	0.09	0.09	0.07	0.11
c_8^*	N	0.3	0.1	0.8	0.10	0.10	0.06	0.15
				Canada				
h	B	0.7	0.6	0.8	0.81	0.81	0.75	0.84
τ	N	1.0	0.4	1.6	3.01	3.02	2.73	3.38
θ_p	B	0.7	0.7	0.8	0.79	0.79	0.76	0.83
θ_w	B	0.7	0.7	0.8	0.78	0.77	0.74	0.80
θ_x	B	0.7	0.7	0.8	0.78	0.78	0.75	0.80
θ_F	B	0.7	0.7	0.8	0.76	0.76	0.72	0.80
ρ_r	B	0.5	0.2	0.8	0.91	0.91	0.87	0.93
ϕ_π	N	2.0	1.7	2.3	2.16	2.15	1.85	2.45
ϕ_g	G	0.5	0.3	0.7	0.11	0.11	0.09	0.14
ϕ_y	G	0.5	0.3	0.7	0.18	0.17	0.09	0.24
c_8	N	0.3	0.1	0.9	0.12	0.13	0.06	0.19

Table 2: Estimated Parameters, Exogenous Processes, $\gamma = 0.99$

Parameter	Prior				Posterior			
	Dist	Median	10%	90%	Mode	Median	10%	90%
				US				
ρ^*_ξ	B	0.5	0.2	0.8	0.95	0.95	0.93	0.96
ρ^*_g	B	0.5	0.2	0.8	0.95	0.95	0.93	0.97
$\rho^*_{\xi_p}$	B	0.5	0.2	0.8	0.99	0.98	0.97	0.99
$\rho^*_{\xi_w}$	B	0.5	0.2	0.8	0.80	0.78	0.67	0.86
ρ^*_{tp}	B	0.5	0.2	0.8	0.73	0.73	0.64	0.82
$100 \times \sigma_z$	IG	0.3	0.1	2.6	0.11	0.11	0.09	0.14
$100 \times \sigma^*_r$	IG	0.3	0.1	0.6	0.11	0.11	0.10	0.12
$100 \times \sigma^*_\xi$	IG	0.3	0.1	0.7	0.25	0.28	0.21	0.41
$10 \times \sigma^*_g$	IG	0.3	0.1	0.7	0.13	0.13	0.12	0.14
$100 \times \sigma^*_{\xi_p}$	IG	0.1	0.1	0.3	0.15	0.15	0.13	0.17
$100 \times \sigma^*_{\xi_w}$	IG	0.3	0.1	0.9	0.11	0.11	0.08	0.13
$100 \times \sigma^*_{r,8}$	IG	0.3	0.1	0.9	0.09	0.09	0.09	0.10
				Canada				
ρ_{rp}	B	0.5	0.2	0.8	0.96	0.95	0.79	0.97
ρ_ξ	B	0.5	0.2	0.8	0.69	0.72	0.58	0.93
ρ_g	B	0.5	0.2	0.8	0.94	0.93	0.90	0.96
ρ_{ξ_H}	B	0.5	0.2	0.8	0.86	0.83	0.71	0.91
ρ_{ξ_w}	B	0.5	0.2	0.8	0.24	0.26	0.16	0.37
ρ_{ξ_X}	B	0.5	0.2	0.8	0.91	0.91	0.87	0.95
ρ_{ξ_F}	B	0.5	0.2	0.8	0.99	0.96	0.92	0.99
ρ_{tp}	B	0.5	0.2	0.8	0.75	0.74	0.64	0.83
$100 \times \sigma_r$	IG	0.3	0.1	0.7	0.17	0.17	0.15	0.19
$100 \times \sigma_{rp}$	IG	0.1	0.1	0.2	0.29	0.34	0.27	0.58
$10 \times \sigma_g$	IG	0.3	0.1	0.9	0.14	0.14	0.13	0.15
$100 \times \sigma_\xi$	IG	0.3	0.1	0.9	0.23	0.24	0.21	0.29
$100 \times \sigma_{\xi_H}$	IG	0.2	0.1	0.3	0.36	0.37	0.33	0.41
$100 \times \sigma_{\xi_w}$	IG	0.3	0.1	0.8	0.49	0.50	0.45	0.55
$100 \times \sigma_{\xi_X}$	IG	0.3	0.1	0.9	1.31	1.34	1.15	1.57
$100 \times \sigma_{\xi_F}$	IG	0.1	0.1	0.2	1.03	1.08	0.89	1.34
$100 \times \sigma_{r,8}$	IG	0.3	0.1	0.7	0.12	0.12	0.11	0.13

Figure 7: GIRF of 2 Quarter Forward Guidance Shock in 2010Q3, 2Q Base Duration

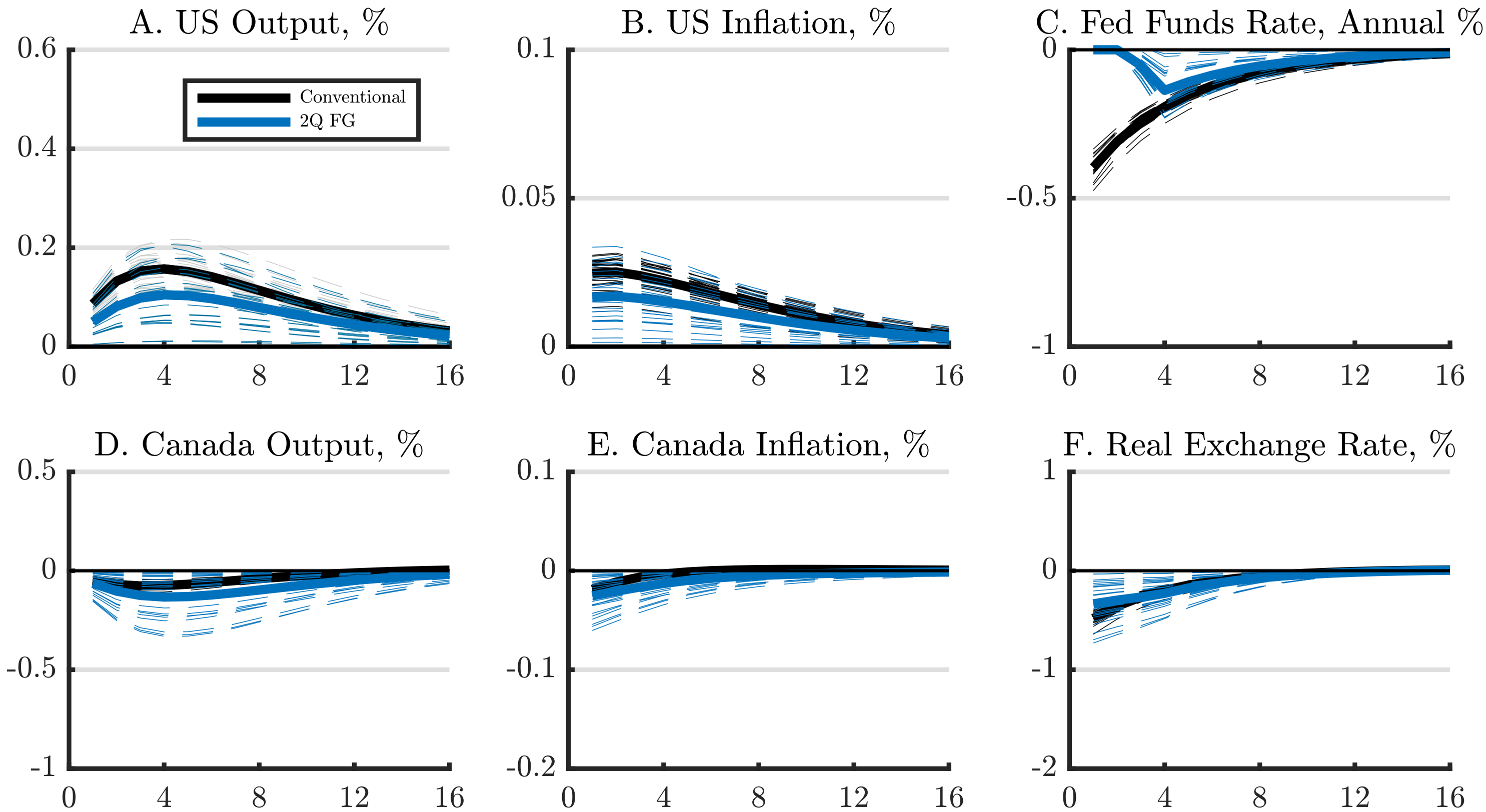

Figure 8: GIRF of 2 Quarter Forward Guidance Shock in 2010Q3, 6Q Base Duration

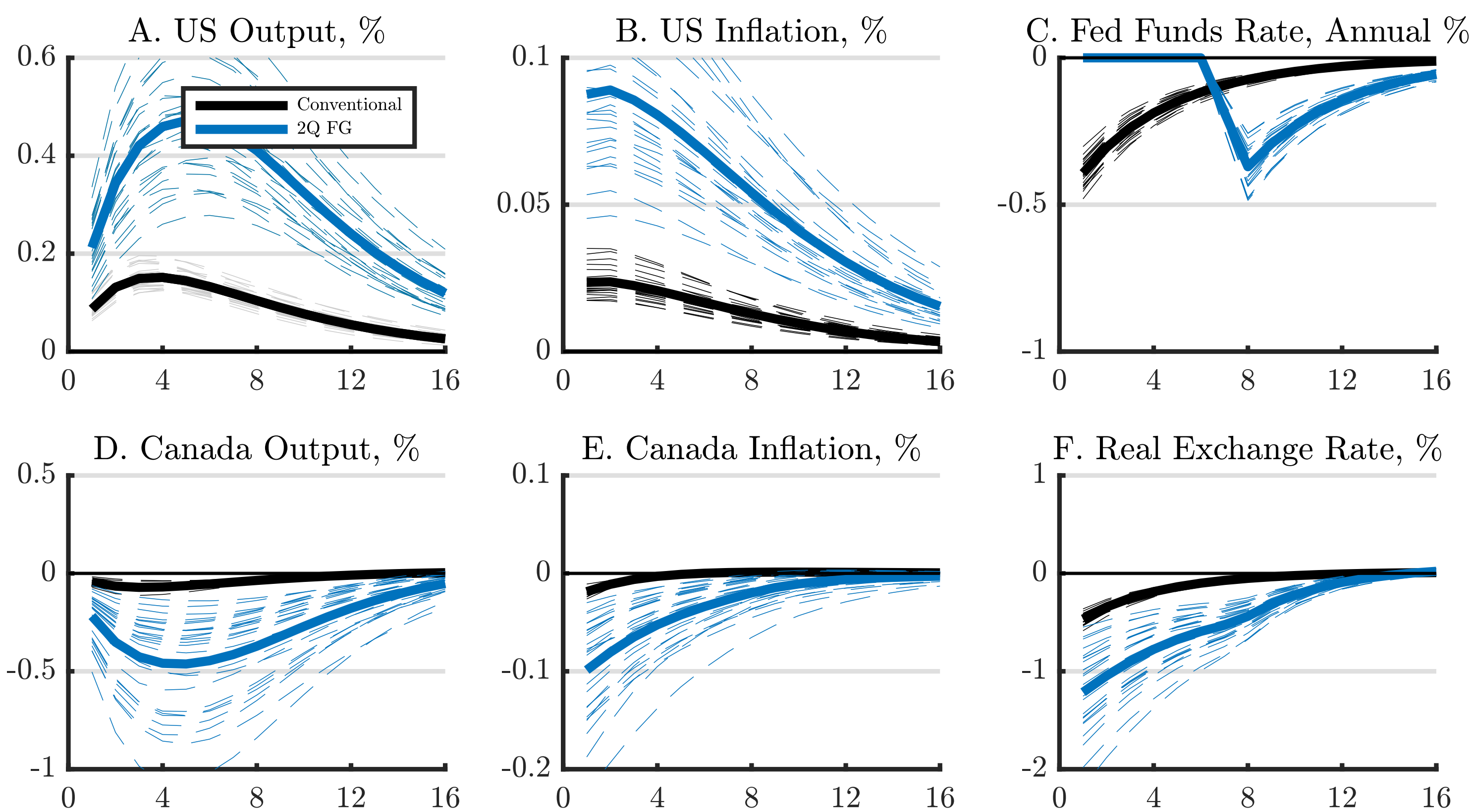

Figure 9: GIRF of 2 Quarter Forward Guidance Shock in 2013Q3, 2Q Base Duration

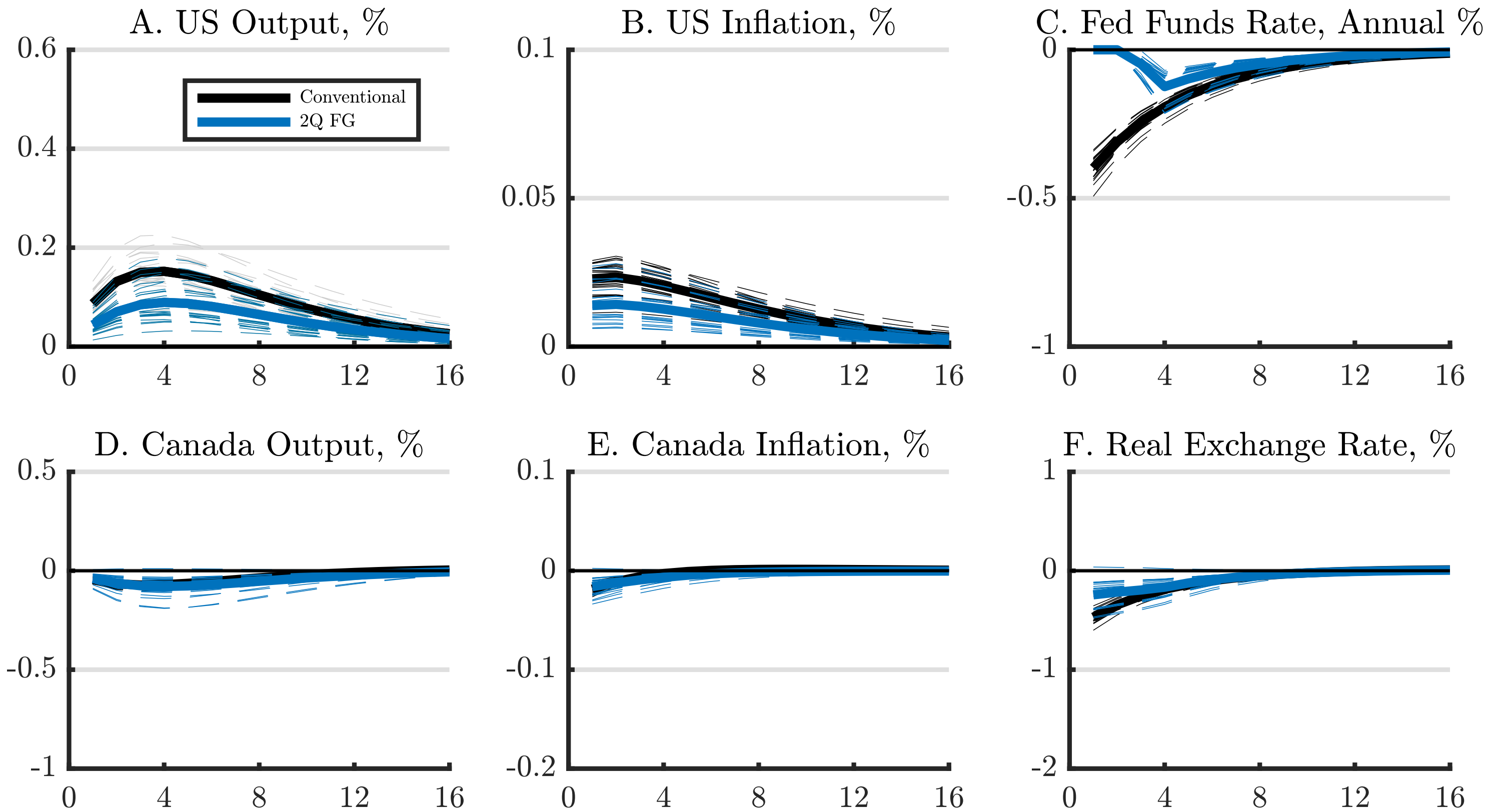

Figure 10: GIRF of 2 Quarter Forward Guidance Shock in 2013Q3, 6Q Base Duration