Unanticipated Shocks and Forward Guidance at the Zero Lower Bound

Callum Jones*

New York University

Abstract

I study calendar based forward guidance when the nominal policy interest rate is at the zero lower bound (ZLB). To do this, I develop a very fast and accurate algorithm to approximate DSGE models with occasionally binding constraints. The method handles the ZLB but additionally accommodates credible announcements about the path of the policy rate intended to extend the duration of the ZLB regime. Using the algorithm, I develop a new decomposition of a ZLB duration into two components, one due to structural shocks and another to forward guidance. Using the Smets and Wouters (2007) model, I jointly estimate the model's parameters, expected ZLB durations, and a permanent decline in the trend rate of growth. With the estimated model, I identify forward guidance extensions of the ZLB of up to a year between 2011Q3 and 2013Q1. I also show: (i) unanticipated shocks which hit during a forward guidance duration can increase ex-post inflation and output volatility, (ii) a simple rule connecting the forward guidance duration to the output gap can mimic optimal policies, and (iii) government spending multipliers at the ZLB fall significantly when a forward guidance policy rule is in use.

*Department of Economics, New York University. Email: jonescallum@gmail.com. Many thanks to Jaroslav Borovicka, Tim Cogley, Mariano Kulish, Virgiliu Midrigan, Steven Pennings and seminar participants at the University of Western Australia, the University of New South Wales and the Reserve Bank of Australia for thoughtful comments and discussions. All remaining errors are mine. First version: December, 2014. Date: August, 2017.

1 Introduction

The policy interest rates of central banks around the world have been near the zero lower bound (ZLB) since 2009. In response, central banks have enacted unconventional policies, including forward guidance, in an attempt to guide expectations about the path of interest rates to stimulate consumption and employment.

To study forward guidance at the ZLB in large dynamic and stochastic general equilibrium (DSGE) models we need an efficient, accurate, and extremely fast way to compute the equilibrium path of the economy. In this paper I develop a piecewise-linear approximation of DSGE models with an occasionally binding ZLB in which monetary policy switches between a regime in which the ZLB binds and one in which it does not. In a rational expectations solution, the ZLB is a change in the structure of the economy which is anticipated by workers and firms. The algorithm iterates on the path of the nominal interest rate until it aligns with agents' expectations. The method is additionally used to impose calendar based forward guidance policies when the central bank announces and commits to an interest rate path.

I use the implementation to explore US calendar based forward guidance policy in an estimated New Keynesian model. First, I provide a solution to a key identification challenge that exists when the nominal interest rate is at the ZLB. The identification challenge arises because, in a rational expectations environment, to accurately describe the state of the economy it is crucial to know how long agents anticipate the ZLB to bind, and whether that anticipated duration is due to exogenous shocks or to forward guidance policy. With (i) information about the anticipated ZLB duration each period, and (ii) observable variables, the algorithm I develop in this paper delivers a decomposition that describes exactly how active monetary policy has been at the ZLB by identifying the fraction of an anticipated ZLB duration which is due to structural shocks and the fraction due to calendar based forward guidance. Characterizing the stance of monetary policy in this way is missing from existing implementations of structural models, but is necessary in light of evidence that the unconventional monetary policy used by the Federal Reserve at the ZLB has affected long-term interest rates (Swanson and Williams, 2014) and macroeconomic outcomes (Wu and Xia, 2016).

I illustrate the decomposition using the Smets and Wouters (2007) (SW) model, which I estimate from 1984Q1 to 2015Q3. The parameters and ZLB durations are jointly estimated, as in Kulish et al. (2017). Motivated by the evidence in Fernald (2015) and Gordon (2016), I also allow for per capita trend growth to fall in an unanticipated way, by estimating both the change and date of a permanent shock to trend growth. This is important because a decline in trend growth interacts in a highly nonlinear way with the ZLB, primarily because the decline in trend growth causes a fall in the steady-state nominal interest rate, bringing the nominal interest rate closer to the ZLB. I estimate that annual trend growth falls from about 2.15% to 1.35% in the early 2000s. Furthermore, I estimate the ZLB durations were between three quarters and two years from 2009Q1 to 2011Q2. Between 2011Q3 and 2013Q4, I find that these durations increase to between two years and three years, after which I estimate that they decline gradually to about one year. When the algorithm is used to decompose this path of ZLB durations, I find that the Federal Reserve has announced forward guidance extensions of the ZLB with durations between one quarter and one year, mainly between 2011Q3 and 2013Q1. This evidence lines up with the calendar-based guidance announced by the Federal Reserve Open Market Committee over this period. I use the ZLB algorithm to quantify the macroeconomic implications of these extensions, finding that, absent forward guidance, the cumulative drop in output and consumption would be about 3%, and about 8% for investment.

I study calendar based forward guidance in the estimated SW model by formulating a forward guidance rule under which the Federal Reserve announces a forward guidance duration which is increasing in the output gap that would have arisen in the absence of such an announcement, with that counterfactual output gap computed using this paper's ZLB algorithm. I show that calendar based forward guidance policies of the kind studied in theory by Eggertsson and Woodford (2003), Laseen and Svensson (2011) and Werning (2012), when formalized as the policy rule, can reduce inflation and output volatility, even if the economy is hit with additional stochastic shocks during the period that the Federal Reserve has committed to holding the policy interest rate constant. I also find that government spending multipliers fall when the Federal Reserve makes use of stimulatory forward guidance policy at the ZLB, with the size of fiscal policy multipliers at the ZLB falling to be around the size of the multiplier that arises when there is no ZLB constraint. This is because the stimulatory effect of an unexpected positive government spending shock at the ZLB relies on an inactive central bank (see Christiano et al., 2011), so that if monetary policy reacts through a forward guidance rule of the kind that I study, the central bank offsets the government spending shock. This conflicting policy behavior, which is typical in equilibrium models outside the ZLB, means that monetary policy can lower the fiscal multiplier at the ZLB.

Accounting for the ZLB in DSGE models is attracting more attention in the literature.[1] In work that is contemporaneous and independent to my analysis, Guerrieri and Iacoviello (2015) develop a procedure for handling occasionally binding constraints like the ZLB.[2] In their method, as here, linearized ZLB and non-ZLB regimes of a model are piecewise combined such that the periods in which those regimes apply accord with agents' expectations. This paper extends the techniques for handling occasionally binding

[1] Many papers have applied piecewise-linear techniques (see, for example Kulish et al., 2017; Del Negro et al., 2015).
[2] Wu (2015) develops an alternative algorithm based on anticipated monetary policy shocks.

constraints in two ways. First, my method accommodates forward guidance policies. When combined with techniques that estimate anticipated ZLB durations, my approach provides a new way to identify the fraction of an anticipated ZLB duration which can be assigned to forward guidance. Second, I expand the exposition to solution concepts which have clear predictions about the existence and uniqueness of the path of the economy during the ZLB period and show how they relate to techniques which approximate the path of the economy subject to anticipated structural changes (see Cagliarini and Kulish, 2013).

Other papers have studied forward guidance policies at the ZLB and have suggested that a commitment to calendar based forward guidance raises the risk of undesirable aggregate outcomes during an announcement period (Campbell et al., 2012; Coenen and Warne, 2014; Boneva et al., 2015). Boneva et al. (2015) show in a simple model that an alternative forward guidance policy, threshold-based guidance policy, can be an effective hedge against adverse outcomes during an announcement period. Under threshold-based guidance, a central bank announces it will hold its policy interest rate at zero so long as pre-announced thresholds (like numerical unemployment or inflation targets) are not breached.

Furthermore, the optimal calendar based forward guidance policies of Eggertsson and Woodford (2003) and Werning (2012) are derived in tractable models with few state variables. The piecewise linear algorithm I use instead allows me to analyze forward guidance policies in models with rich price ridigity and a number of sources of persistence, and with many shocks that hit every period, such as the SW model. I show that the optimal theoretical policies are largely replicated in the SW model, and that, because of the nonlinearities induced by the ZLB, the dynamic behavior of an economy under forward guidance is substantially different to the behavior of an economy that is not faced with the ZLB constraint.

The paper is organized as follows. Section 2 discusses the modified SW model used in the estimation and as a laboratory to illustrate the algorithm and the forward guidance decomposition. The algorithm and solution method is also discussed in section 2. Next, section 3 discusses the estimation results. Section 4 then examines (i) the forward guidance decomposition in the estimated model, (ii) how forward guidance as set by a simple rule linked to the output gap can mitigate the effect of shocks at the ZLB, and (iii) how government spending multiliers can change under such a rule. Section 5 concludes.

2 Model and solution method

I estimate the model of Smets and Wouters (2007) (SW) to illustrate the algorithm and to study a number of topics relating to calendar based forward guidance. The model is well-known, so the details of the linearized equations are left to the model appendix. I make two changes to the model.

The first, and main, difference between the SW model and the model used in this paper is the specification of monetary policy. Monetary policy operates in one of two possible regimes. In the first regime, the nominal interest rate can be set according to a standard Taylor rule, as in SW. In the second regime, the nominal interest rate is at zero and monetary policy enters the ZLB regime. It can be in the ZLB regime in two possible ways: first, if the Taylor rule calls for negative nominal interest rates, and second, if the Federal Reserve has announced contemporaneously or in an earlier period a commitment to a forward guidance duration.

To make this clear, let the variable FG_t indicate whether the Federal Reserve has made a forward guidance announcement and is holding the nominal interest rate r_t at zero.[3] If the Federal Reserve has made a forward guidance announcement, $\mathrm{FG}_t = 1$, and the nominal interest rate remains at its lower bound. When forward guidance is not in use, monetary policy follows a standard Taylor rule, with the nominal interest rate responding to deviations in inflation from a target rate π_t, deviations in output y_t from its potential level y_t^p, and the growth rate of potential output, and is subject to the ZLB:

$$r_t - r = \begin{cases} \max\left(0, \rho\left(r_{t-1} - r\right) + (1-\rho)\left\{r_\pi \pi_t + r_Y\left(y_t - y_t^p\right)\right\} + r_{\Delta y}\left[\Delta(y_t - y_t^p)\right]\right), & \text{if } \mathrm{FG}_t = 0 \\ 0, & \text{if } \mathrm{FG}_t > 0, \end{cases} \tag{1}$$

where r is the steady-state value of the Federal Funds rate. The periods that FG_t takes the value of zero or one will be estimated together with the parameters of the model. In section 4, I consider a simple rule that links the output gap to the announcement duration.

The second difference of the model I use to that of SW is that I allow for the trend growth parameter to change exogenously and unexpectedly. That is, there is a permanent, one-time, unanticipated shock to trend growth. The date and magnitude of that change will be estimated, as described below.

2.1 Solution method

In this section, I describe the algorithm that is developed to approximate the model subject to the ZLB.

2.1.1 Notation and initialization

Let x_t be the $n \times 1$ vector of state and jump variables of the model at t, one of which is the nominal interest rate r_t, and w_t be a vector of exogenous variables at t. The nonlinear rational expecations model is a system

[3] Unless otherwise stated, variables are expressed in log-deviations from their steady-state values.

of n equations $x_t = \Psi(x_{t-1}, \mathbb{E}_t x_{t+1}, w_t)$. The known variables at period t are the shock that hits at period t, w_t, and the initial vector of variables x_{t-1}.

Linearize the model around its non-stochastic steady-state and denote the resulting n equations as:

$$\mathbf{A}x_t = \mathbf{C} + \mathbf{B}x_{t-1} + \mathbf{D}\mathbb{E}_t x_{t+1} + \mathbf{F}w_t. \tag{2}$$

Provided the Blanchard-Kahn conditions are satisfied, standard methods are used to obtain the reduced form:

$$x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}w_t. \tag{3}$$

If the ZLB binds at t^*, denote the model system as:

$$\mathbf{A}^* x_{t^*} = \mathbf{C}^* + \mathbf{B}^* x_{t^*-1} + \mathbf{D}^* \mathbb{E}_{t^*} x_{t^*+1} + \mathbf{F}^* w_{t^*}. \tag{4}$$

The only difference between (2) and (4) is that the Taylor rule (1) in (2) is replaced with $r_t = 0$. In general, the system (4) will not have a solution in the form of (3) because the Blanchard-Kahn conditions are not satisfied if the nominal interest rate is fixed (see Davig and Leeper, 2007).

2.1.2 The ZLB algorithm

Given the exogenous variables w_t and a vector of state variables x_{t-1}, the first step of the algorithm is to forecast the path of all variables under a linear approximation when all future uncertainty is ignored and where the ZLB constraint is also ignored.[4] The second step is to check the computed path of the nominal interest rate, and in the time periods that the ZLB does bind, if at all, impose the ZLB so that it is anticipated by agents in the period the shock hits. An anticipated change in the future structure of the economy affects all endogenous variables before that change. In turn, changes to the endogenous variables affect the path of the nominal interest rate. To ensure the realized path is consistent with agents' expectations, an iterative step is needed to update the endogenous variables period-by-period.

Formally, the steps of the algorithm are:

0. Linearize the model around the non-stochastic steady state, ignoring the ZLB, to obtain (2) and (3).

1. For each period t:

[4] An implementation of the algorithm integrated with Dynare is available at `https://callumjones.github.io`

(a) Solve for the path $\{x_\tau\}_{\tau=t}^{T}$ with T large, using (3), given w_t and x_{t-1}, and assuming no future uncertainty $\{w_\tau\}_{\tau=t+1}^{T}=0$ (no future uncertainty), so that $x_t=\mathbf{J}+\mathbf{Q}x_{t-1}+\mathbf{G}w_t$, and $x_{t+1}=\mathbf{J}+\mathbf{Q}x_t$, up to $x_T=\mathbf{J}+\mathbf{Q}x_{T-1}$.

This step gives a path $\mathbf{i}_t^k=\{i_\tau\}_{\tau=t}^{T}$.

(b) Examine the path $\mathbf{i}_t^k=\{i_\tau\}_{\tau=t}^{T}$. If $\mathbf{i}_t^k\geq 0$, then the ZLB does not bind, and move to step (2). If $\mathbf{i}_t^k<0$, then move to step (1c). That is:

- If $i_\tau\geq 0$ for all $t\leq\tau<T$, accept $\{x_\tau\}_{\tau=t}^{T}$. The path of the nominal interest rate does not violate the ZLB today or in future.
- If $i_\tau<0$ for any $t\leq\tau<T$, move to step (1c).

(c) Update the path of $\{i_\tau\}_{\tau=t}^{T}$ for the ZLB. Denote by t^* the *first* time period where $\mathbf{i}_t^k<0$, and set the nominal interest rate in that period to zero. This changes the anticipated structure of the economy. Computing the new path $\{i_\tau\}_{\tau=t}^{T}$ under the structural change involves computing $\{x_\tau\}_{\tau=t}^{t^*}$ and $\{x_\tau\}_{\tau=t^*+1}^{T}$. At t^*, $\mathbb{E}_{t^*}x_{t^*+1}$ is computed using the the reduced form solution (3) and $w_{t^*+1}=0$. This expresses x_{t^*} as a function of x_{t^*-1}. Proceeding in this way with the correct structural matrices (either (4) or (2) at each time period), compute the path $\mathbf{i}_t^{k+1}=\{i_\tau\}_{\tau=t}^{T}$.

A convenient way to compute the new path $\mathbf{i}_t^{k+1}=\{i_\tau\}_{\tau=t}^{T}$ is to form the time varying matrices $\{\mathbf{J}_\tau,\mathbf{Q}_\tau,\mathbf{G}_\tau\}_{\tau=t}^{T}$ which satisfy the recursion:

$$
\begin{aligned}
\mathbf{Q}_t &= \left[\mathbf{A}_t-\mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}\mathbf{B}_t\\
\mathbf{J}_t &= \left[\mathbf{A}_t-\mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}\left(\mathbf{C}_t+\mathbf{D}_t\mathbf{J}_{t+1}\right)\\
\mathbf{G}_t &= \left[\mathbf{A}_t-\mathbf{D}_t\mathbf{Q}_{t+1}\right]^{-1}\mathbf{F}_t,
\end{aligned}
$$

with the final set of reduced form matrices for the recursion being the non-ZLB matrices $\mathbf{J}$, $\mathbf{Q}$, $\mathbf{G}$ from (3).

These time-varying matrices are then used to compute the path $\{x_\tau\}_{\tau=t}^{T}$ by calculating $x_\tau=\mathbf{J}_\tau+\mathbf{Q}_\tau x_{\tau-1}+\mathbf{G}_\tau w_\tau$. For more details on this particular recursion, see Kulish and Pagan (2017).

Iterate on steps 2 and 3 until convergence of $\mathbf{i}_t^{k+1}$ and $\mathbf{i}_t^k$.

2. Increment t. The initial vector of variables becomes x_t, which was solved for in step 1. Draw a new vector of unanticipated shocks w_{t+1} and return to step 1.

The algorithm yields a set of time-varying structural matrices:

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t w_t, \tag{5}$$

from which we get the path $\{x_\tau\}_{\tau=t}^{\infty}$ where the nominal interest rate is subject to the ZLB. Furthermore, if the ZLB does bind under the sequence of exogenous variables, the algorithm produces, at each period t, a sequence of expected durations of the ZLB.

2.1.3 Existence and uniqueness

In general, the path of the economy under this approximation will exist and be unique, as summarized in the following result.

Result 1. *If there are n distinct equations for n variables in the linearized model (2) under the non-ZLB regime and the Blanchard-Kahn conditions are satisfied for the linearized model under the non-ZLB regime, then the path during the ZLB period exists and is unique.*

The full reasoning is in the appendix. Intuitively, the result says that if the linearized model has a unique solution (3) when the ZLB does not bind, that solution pins down agents' expectations of when the ZLB regime ends. With those expectations pinned down, the path of the economy is uniquely described by the structural equations (4). For a discussion of issues of existence and uniqueness in the presence of foreseen structural changes, see Cagliarini and Kulish (2013).[5]

2.2 Forward guidance decomposition of a ZLB duration

Incorporating calendar based forward guidance involves setting, at step (1b) of the algorithm, the nominal interest rate regime to the ZLB regime in the periods that the central bank announces that it will hold interest rates at zero. The path $\{x_\tau\}_{\tau=t}^{\infty}$ under forward guidance announcements will differ from the endogenous ZLB path to the extent that credible announcements change agents' expectations of the periods that the ZLB will bind. As recognized by Eggertsson and Woodford (2003), Jung et al. (2005), and Werning (2012), this manipulation of agents' expectations can be stimulatory if it lowers the path of real rates.

The total ZLB duration can, in principle, be any length and can be decomposed into two components. The first component is one which is consistent with the structural shocks that prevail at each point in time.

[5] A worked example applying the Sims (2002) solution concept and setting up the structural matrices is given in the appendix. Sims (2002) proposal is to express the expectational variables as part of the linearized model's state vector, and append to the model equations defining expectations revisions $\eta_t = \mathbb{E}_t \mathbf{z}_t - \mathbb{E}_{t-1}\mathbf{z}_t$ with $\mathbb{E}_t \eta_{t+j} = 0$ for $j \geq 1$ which will be solved as part of the solution.

I call this duration the *endogenous duration*. The second component is the difference between the total expected duration and the endogenous duration. This second component is interpreted as a calendar based forward guidance duration. If the endogenous duration is expected by all agents in the economy, the central bank is acting passively in response to the structural shocks and expects to raise the nominal interest rate off the ZLB as soon as the policy rule prescribes. If, instead, the expected ZLB duration is longer than the endogenous duration, then agents in the economy believe the monetary authority is making a commitment to hold its policy rate at zero beyond the period implied by the endogenous duration.

The algorithm of this section can be used to compute this decomposition of the sequence of ZLB durations to uncover the fraction of the duration which can be assigned to a forward guidance announcement. To make this clear, suppose, at period t, the ZLB binds and we have in hand:

1. the duration T_t of the ZLB at t, so that the interest rate is expected to stay at zero until period $t+T_t$, and
2. the history of the states $\{x_\tau\}_{\tau=0}^{t-1}$ and the structural shocks $\{w_\tau\}_{\tau=1}^{t}$. Estimates of these can be obtained using the Kalman filter and smoother.

The ZLB durations and shocks recover the observed series. To decompose the estimated duration into a component due to the smoothed structural shocks and a forward guidance residual, at each point of time, we use the initial state x_0 and the sequence of exogenous shocks $\{w_\tau\}_{\tau=1}^{t}$ to compute, using the algorithm, a counterfactual series for the ZLB durations. This counterfactual sequence of ZLB durations defines the endogenous durations, with the difference between this series and the T_t series being forward guidance.

3 Estimation

I estimate the expanded SW model on a sample from 1984Q1 to 2015Q3. The ZLB first binds in 2009Q1. The data series and construction are the same as in SW, and are plotted in Figure 1. The prior distributions and posterior estimates for each parameter are given in Table 1. The priors used for the structural parameters are the same as those used in SW except that I set a wider prior on the trend growth rate (Table 2). For the trend growth parameter, I set a normal prior, centered at 0.5% a quarter, with a standard deviation of 0.25%. For the estimated break date in the trend rate of growth, I use a uniform prior between the 20th and 80th percentiles of the sample, which restricts the trend growth break to occur between 1990Q1 and 2009Q2. I follow Kulish et al. (2017) in setting priors on the expected ZLB durations, which are based on the New York Federal Reserve's Survey of Primary Dealers. In that survey, primary dealers were asked when they expected the Federal Reserve would first lift the Federal Funds rate off the ZLB. The range of responses

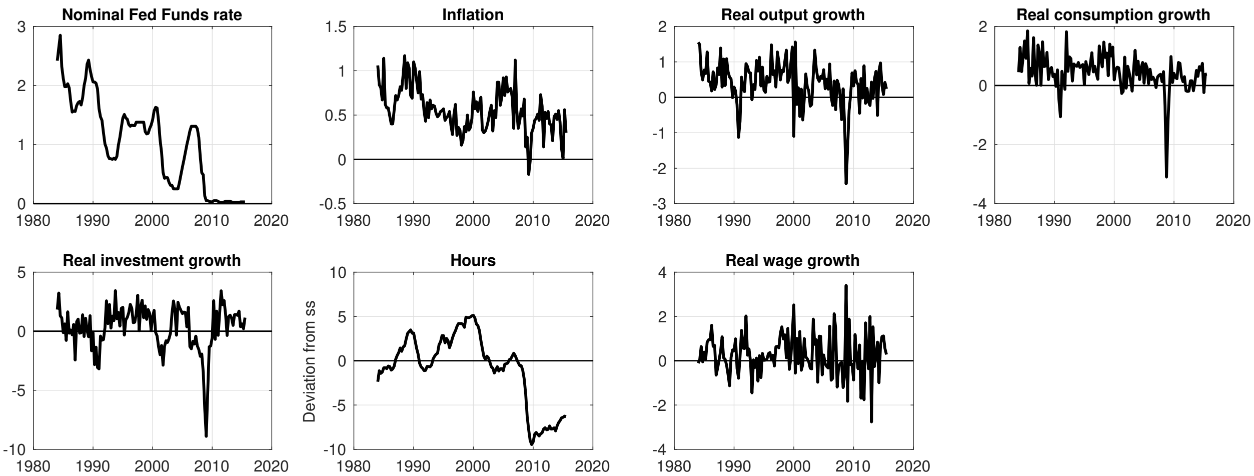


Figure 1: Quarterly data.

to those surveys are used to assign probabilities to liftoff dates. For the Bayesian Markov Chain Monte Carlo estimation, I construct four chains of half a million, and discard the first quarter million draws of each chain.

The posterior estimates of the structural parameters are broadly similar to the estimates of SW. To highlight some key differences, the persistence of the risk premia shock is substantially higher, with the posterior mode of ρ_b being 0.938 as compared to the SW value of 0.18. Offsetting this, the posterior mode of the variance of risk premia shocks σ_b is estimated here to be 0.062, as compared to a higher value of 0.2 in SW. Taken together, the unconditional variance of risk premia shocks in my estimates is around three times the size of the unconditional variance calculated from SW estimates. Accordingly, risk premia shocks play a much larger role in explaining the forecast error variance of output. These differences are most likely due to the large and persistent contraction in output, consumption, and hours associated with the financial crisis. The estimate of the annual inflation target is about 2.25%, which is lower than the estimate of SW. This is because SW use a longer sample, starting in 1948, covering the period of elevated inflation in the 1970s and early 1980s. Wages and prices are also much less flexible than the estimates of SW, with the average duration of price contracts lasting about 4 years, significantly longer than the 3Q estimate in SW. Estimating less flexible prices, particularly using data from more recent samples, is consistent with other studies documenting the flattening of the Philips curve (see, for example Del Negro et al., 2015).

The Taylor rule parameters are important because they determine the expected behavior of the Federal Funds rate outside of the ZLB, and therefore when nominal interest rates can be expected to be lifted off the ZLB. For example, observing long durations at the ZLB together with large output gaps is more

Table 1: Estimated parameters

Parameter	Prior			Posterior			
	Distribution	Mean	St Dev	Mode	Mean	10%	90%
σ_a	IG	0.100	2.000	0.417	0.426	0.390	0.464
σ_b	IG	0.100	2.000	0.062	0.063	0.053	0.073
σ_g	IG	0.100	2.000	0.380	0.381	0.350	0.413
σ_i	IG	0.100	2.000	0.300	0.315	0.255	0.376
σ_r	IG	0.100	2.000	0.116	0.118	0.105	0.132
σ_p	IG	0.100	2.000	0.112	0.115	0.100	0.131
σ_w	IG	0.100	2.000	0.441	0.445	0.397	0.495
ρ_a	B	0.500	0.200	0.964	0.955	0.931	0.976
ρ_b	B	0.500	0.200	0.938	0.938	0.920	0.955
ρ_g	B	0.500	0.200	0.978	0.973	0.958	0.987
ρ_i	B	0.500	0.200	0.678	0.685	0.591	0.784
ρ_r	B	0.500	0.200	0.500	0.485	0.390	0.580
ρ_p	B	0.500	0.200	0.712	0.672	0.531	0.810
ρ_w	B	0.500	0.200	0.931	0.792	0.585	0.935
ρ_{ga}	N	0.500	0.250	0.380	0.407	0.300	0.513
$\bar{l}$	N	0.000	1.500	1.373	1.394	1.215	1.635
$\bar{\pi}$	N	0.500	0.100	0.549	0.546	0.482	0.610
$100*(\beta^{-1}-1)$	G	0.250	0.100	0.161	0.168	0.091	0.255
μ_w	B	0.500	0.200	0.900	0.748	0.531	0.907
μ_p	B	0.500	0.200	0.624	0.577	0.397	0.751
α	N	0.300	0.050	0.148	0.152	0.128	0.176
ψ	B	0.500	0.150	0.801	0.784	0.671	0.884
φ	N	4.000	1.500	6.034	6.194	5.065	7.687
σ_c	N	1.500	0.375	0.918	0.983	0.850	1.135
h	B	0.700	0.100	0.603	0.590	0.519	0.659
Φ	N	1.250	0.125	1.375	1.399	1.299	1.500
ι_w	B	0.500	0.150	0.306	0.387	0.213	0.596
ξ_w	B	0.500	0.100	0.886	0.861	0.798	0.916
ι_p	B	0.500	0.150	0.217	0.255	0.148	0.372
ξ_p	B	0.500	0.100	0.939	0.940	0.927	0.953
σ_l	N	2.000	0.750	2.687	2.790	2.001	3.629
r_π	N	1.500	0.250	1.308	1.308	1.009	1.599
$r_{\Delta y}$	N	0.125	0.050	0.116	0.120	0.088	0.153
r_y	N	0.125	0.050	0.182	0.185	0.146	0.226
ρ	B	0.750	0.100	0.819	0.811	0.769	0.850

indicative of a strong response of the nominal interest rates to the output gap outside of the ZLB. The posterior estimates suggest a relatively weak long-run reaction r_π to deviations of inflation from target $\bar{\pi}$ (1.31) as compared to the estimate of SW (2.03). In contrast, I estimate the long-run response to output gap deviations to be 0.185, which is much higher than the SW estimate of 0.08, and which are consistent with the narrative that the Federal Reserve has been more focused on employment outcomes than deviations in inflation from a 2% target (see also Rudebusch and Williams, 2016). One way to shed light on how these parameters are identified is to compare (i) a counterfactual series for the Federal Funds rate computed using the parameter values of SW, and (ii) the observed Federal Funds rate. That comparison suggests that, under the SW estimates, the Federal Funds rate would have been lifted off the ZLB as early as 2010.

Table 2 summarizes the prior and posterior distributions for the trend rate of growth before and after its break on the 1984Q1 to 2015Q3 sample. The data prefer an estimate of trend growth before the structural break of around 0.53% a quarter, and around 0.34% a quarter after the break. The posterior distribution of the date that trend growth falls is centered around 2001Q2. The posterior distribution of the break date is shown in Figure 2, illustrating how the data are supportive of the change in trend growth occuring in the early 2000s. This is consistent with a number of studies of trend growth and with Federal Reserve / Congressional Budget Office projections of potential output (also see Luo and Startz, 2014; Fernald, 2015).

At the posterior mode, the implied annual steady-state real and nominal interest rates are 2.63% and 4.85% before 2001Q2, and fall to 1.91% and 4.11% after 2001Q2. This steady-state real rate is close to the values estimated by Johannsen and Mertens (2016), while the steady-state nominal rate is higher than the 3% median long-run Federal Funds rate projected by the Federal Reserve. The implied posterior distribution for the real interest rate places very little mass on the real interest rate being below 1%. This estimate is important, because of the potential for very low and negative real interest rates to give rise to a persistent output gap when the ZLB binds (Eggertsson and Mehrotra, 2014).

Table 3 presents the estimates of the posterior distributions of the ZLB durations each quarter from 2009Q1 to the end of the sample in 2015Q3. The data are quite informative about the estimates of the duration, with the posterior distributions having a much smaller variance than the prior distributions. For the periods between 2009 and 2010, and again between 2014 and 2015, the posterior distributions exhibit thick right tails, with some probability mass assigned to quite long durations. The mode of the ZLB durations suggests that agents expected a relatively rapid return to a positive Federal Funds rate between 2009 and 2010. The estimated durations then increase notably in 2011Q3, coinciding with Federal Reserve announcements about the future path of the Federal Funds rate. The estimated ZLB durations stay high at

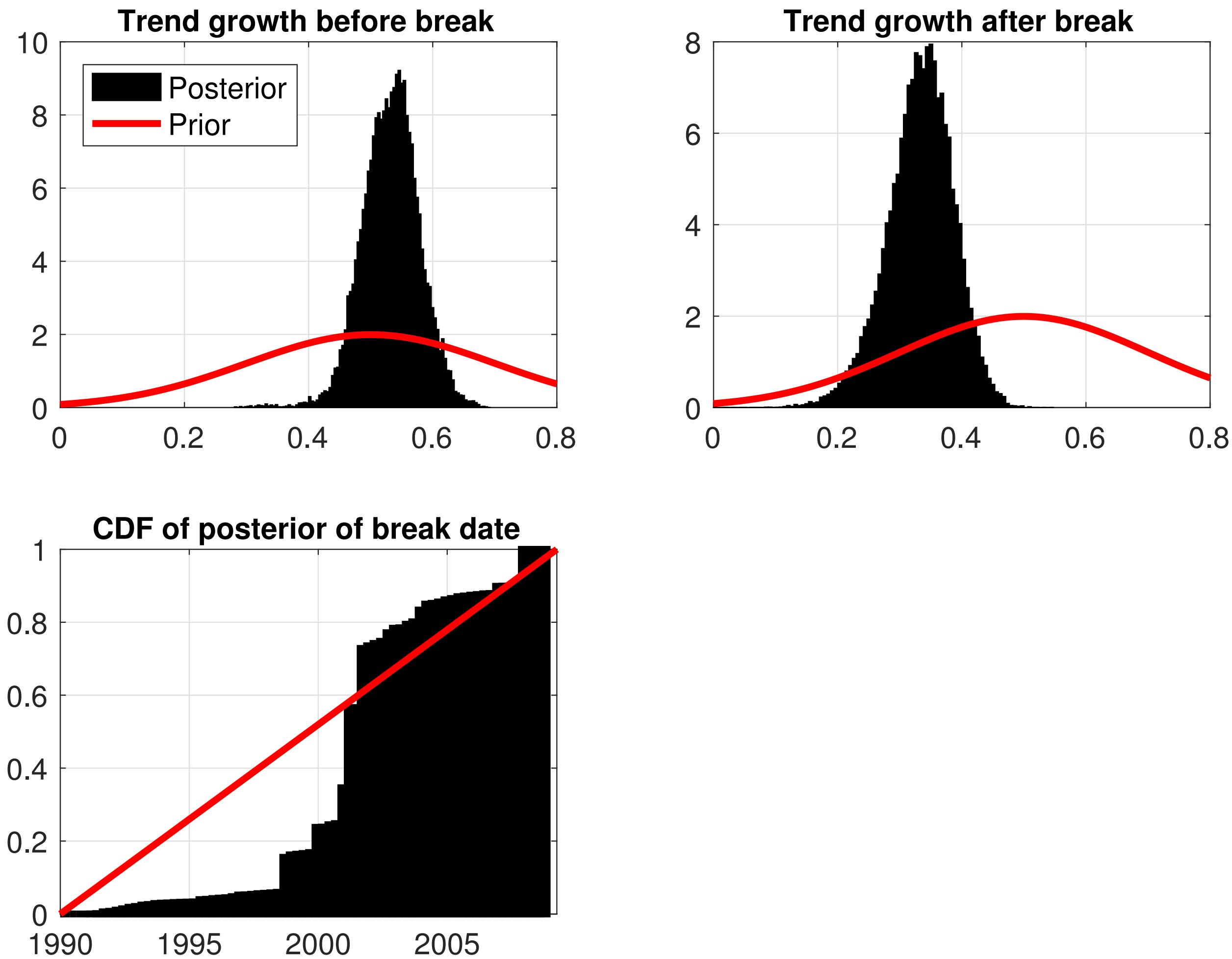


Figure 2: Trend growth posterior distributions.

Table 2: Trend growth estimates

	Prior				Posterior		
	Dist	Median	10%	90%	Median	10%	90%
γ after break	N	0.5	0.25	0.76	0.533	0.476	0.586
γ after break	N	0.5	0.25	0.76	0.336	0.262	0.398
Date of break*					2001Q2	1998Q4	2008Q1

*: Note, break date prior is uniform between 1990Q1 and 2009Q2.

Table 3: Estimated ZLB durations

	Prior			Posterior				
Date	Median	10%	90%	Mode	Median	10%	90%	Futures Data
2009Q1	5	1	18	4	7	4	13	2
2009Q2	5	1	18	4	8	4	14	1
2009Q3	3	1	18	4	9	3	16	1
2009Q4	3	1	18	5	8	3	14	1
2010Q1	3	1	18	3	9	3	14	2
2010Q2	4	1	18	4	10	3	16	2
2010Q3	4	2	18	5	7	4	12	4
2010Q4	5	2	18	5	8	4	15	3
2011Q1	6	2	18	7	7	4	12	2
2011Q2	5	2	18	6	7	4	13	3
2011Q3	9	3	18	9	9	6	14	6
2011Q4	9	3	18	9	10	6	14	6
2012Q1	10	3	19	10	10	6	15	5
2012Q2	10	3	19	9	10	6	15	7
2012Q3	11	4	18	10	10	7	14	8
2012Q4	11	3	19	12	11	7	15	7
2013Q1	10	3	18	10	10	6	14	6
2013Q2	7	2	18	8	9	5	14	4
2013Q3	8	3	18	9	9	6	13	5
2013Q4	8	3	18	8	9	6	14	4
2014Q1	7	3	18	7	8	5	13	4
2014Q2	6	2	18	6	8	4	14	3
2014Q3	4	1	18	4	7	4	14	2
2014Q4	3	1	18	4	8	3	15	2
2015Q1	3	1	18	4	8	3	14	2
2015Q2	3	1	18	4	8	3	14	2
2015Q3	2	1	18	9	9	3	15	–

around 2 to 2.5 years throughout 2012 and into 2013, before gradually declining to expected durations of one year in 2015. The last column of Table 3 shows a data series of the sequence of expected durations computed by Morgan Stanley using Federal Funds futures data. This series displays the same pattern as the mode of the posterior distribution, but is consistently shorter by about two to three quarters, most likely reflecting positive risk premia in Federal Funds futures prices.

Next, I extract the smooth estimates of the structural model by setting the parameters to the mode of their respective posterior kernels, by setting the sequence of ZLB durations to the mode of the distribution implied by the New York Survey of Primary Dealers, and by setting the date that trend growth changes to its modal estimate. The smoothed estimates of the structural shocks are presented in Figure 3. To fit the 2008-09 recession and the persistent deviation in employment from its steady-state, the model requires a large negative risk-premia shock. When the ZLB binds, the monetary policy shock is zero.

With the structural shocks, we can now examine a number of counterfactual paths of the economy

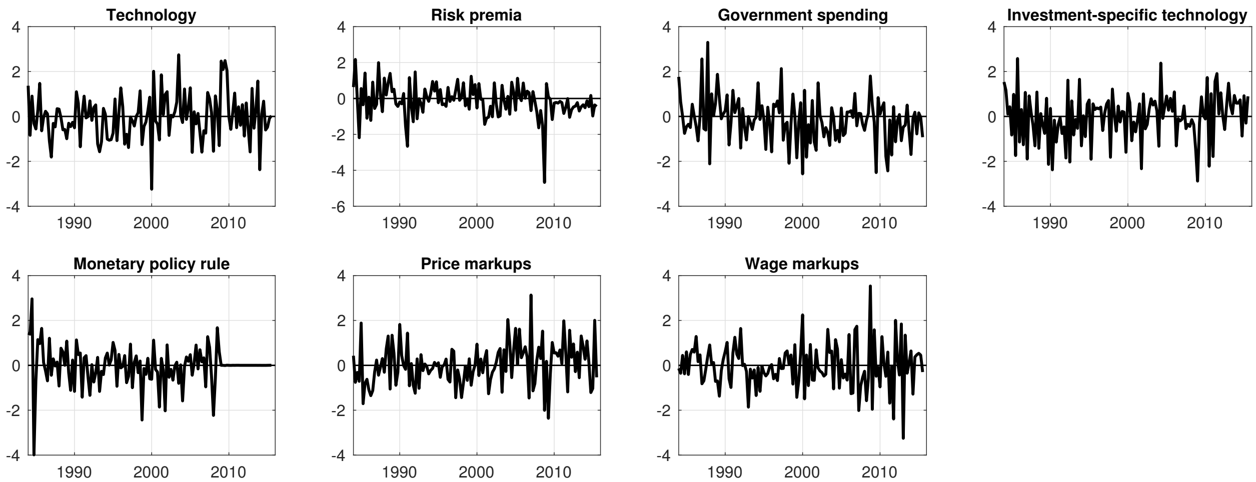


Figure 3: Structural shocks.

(Figure 4). We first look at what the model says would have occurred had trend per capita growth not fallen by an estimated 0.75% annual percentage points. Strikingly, log per capita output would be about 13 percentage points higher in 2015, while the recovery of the economy's pre-recession output level would have been swift, occurring by mid-2010. This compares to a similar period of time it took for output to recover from the early 1990s recession (two years). Furthermore, while the change in trend output growth also causes the steady-state nominal interest rate to rise, it was not enough for the economy to avoid the ZLB. On the other hand, the increase in the steady-state nominal interest rate results in shorter counterfactual ZLB durations–instead of being around 12 quarters in 2013, in the counterfactual with higher trend growth, they are, at most, 6 quarters in duration. The nonlinearities associated with the ZLB are also apparent when comparing the ZLB durations under the counterfactual with high trend growth to the ZLB durations used to compute the shocks. Focusing in particular on the period between 2011 and 2014 when the expected duration the ZLB binds doubles from about 6 quarters to 12 quarters in a year, in the counterfactual with higher trend growth, they increase much more modestly from around 4 to 6 quarters.

The model is well suited to quantifying the extent that the ZLB is a binding constraint, by examining the path of the economy absent the ZLB. In this case the Federal Reserve acts according to its Taylor rule at its estimated parameter values, and when faced with the same exogenous shocks. Absent the ZLB, the annual Federal Funds rate would have declined to about -2 per cent in 2009 and stayed there through to 2015. Since inflation is large unchanged when the ZLB constraint is removed, output and consumption recover faster as a lower real interest rate follows from slacker monetary policy. The essentially unchanged path of inflation when the ZLB constraint is removed is consistent with the flat Philips curve in the estimated

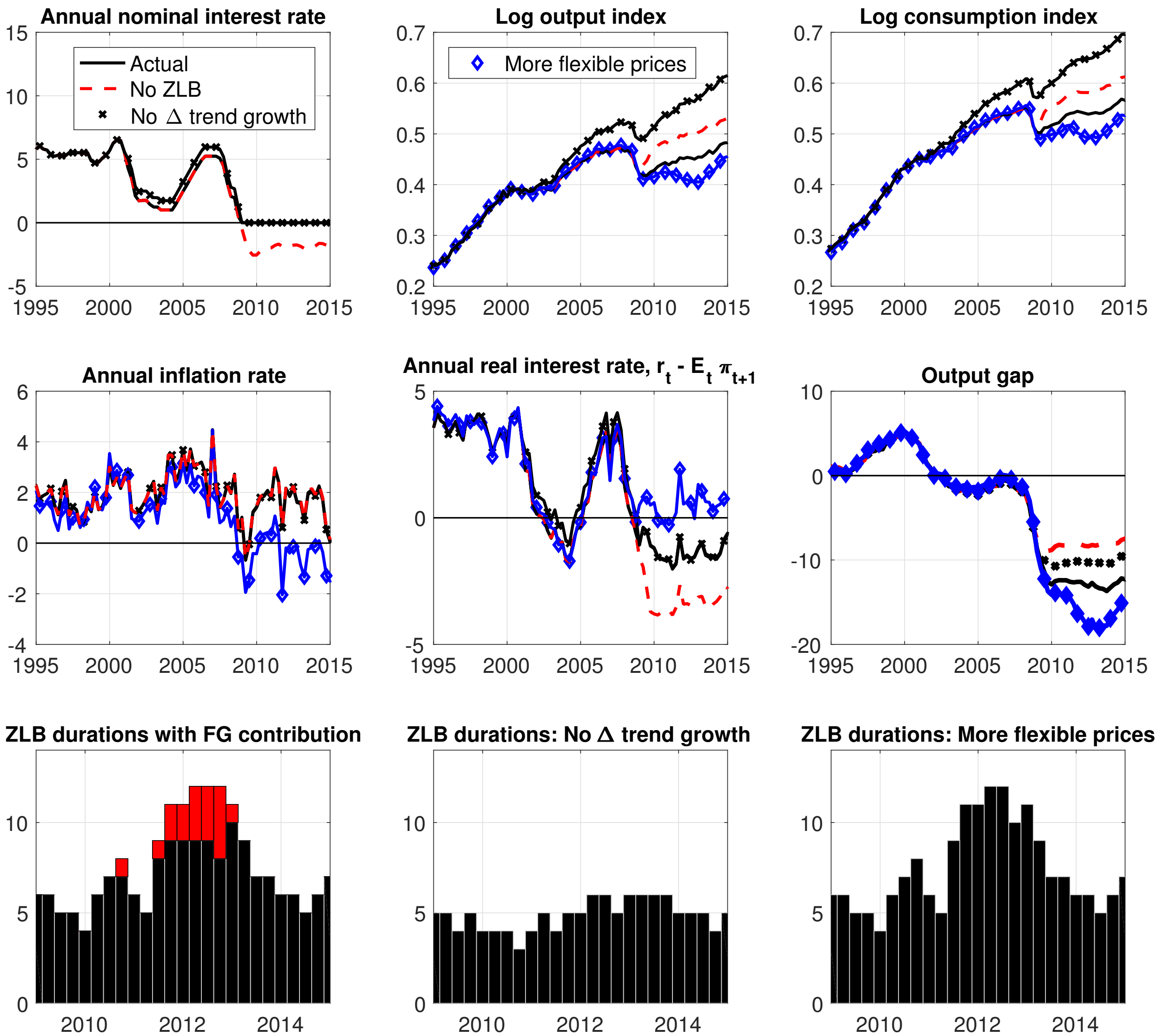


Figure 4: Counterfactual series. In computing the counterfactual with more flexible prices, the contribution of forward guidance is removed.

model, and with the results of Kulish et al. (2017) and the analysis of Del Negro et al. (2015). These papers discuss how, in New Keynesian models with a forward looking Philips curve, inflation is a function of the future path of marginal costs. Following risk premia shocks of the kind that drive the 2008-09 recession in the SW model, forecasted marginal costs recover sufficiently fast so that inflation does not fall much. This force is particularly strong when prices react less to current marginal costs so that the Philips curve is essentially horizontal. To make this point, I consider a counterfactual path of the economy where prices are more flexible. In particular, I compute, using the structural shocks, a counterfactual path of the economy where the average duration of a price contract is about half as long as in the baseline estimated results (8 quarters versus the estimated 15 quarters). In this counterfactual, I remove forward guidance stimulus. Inflation reacts more to low marginal costs caused by the recessionary shocks. At the ZLB, the response of the real interest is particularly acute, rising notably from the observed real interest rate. As a result, output contracts. An implication of this observation is that guidance made about the path of the Federal Funds rate will have more power when prices are more flexible. The flattening of the Philips curve therefore tempers the severity of the forward guidance puzzle highlighted by Del Negro et al. (2012).

4 Calendar based forward guidance at the ZLB

Calendar based forward guidance – or the explicit announcement of the length of time for which the policy rate is to be held at zero – is a tool that has been argued can provide monetary stimulus when the ZLB binds (Werning, 2012). By credibly lowering expectations about the path of interest rates, the central bank encourages households to intertemporally shift consumption to simulate output and inflation. In this section, I analyze the quantative implications of calendar based forward guidance policies with the estimated model.

4.1 Decomposition of the ZLB durations

I first decompose the estimated ZLB durations into the endogneous component due to the shocks interacting with the ZLB, and a component which cannot be explained by the shocks alone, and is therefore attributed to an announcement by the Federal Reserve to hold interest rates at zero. The decomposition follows the procedure laid out in section 2.2. The result of this decomposition of the ZLB durations is shown in the sixth panel of Figure 4. The durations expected by all agents in the economy are the same as those that are consistent with the structural shocks of the model from 2009Q1 to 2011Q3, after which forward guidance

Table 4: Percentage change in aggregates due to forward guidance

Date	Inflation	Output	Consumption	Investment
2011Q2	0.002	0.042	0.062	0.148
2011Q3	0.003	0.116	0.138	0.301
2012Q1	0.004	0.168	0.181	0.442
2012Q2	0.005	0.257	0.261	0.671
2012Q3	0.005	0.299	0.286	0.833
2012Q4	0.009	0.478	0.451	1.263
2013Q1	0.005	0.378	0.319	1.203
2013Q2	0.004	0.289	0.216	1.083
2013Q3	0.003	0.228	0.156	0.964
2013Q4	0.003	0.187	0.122	0.850
2014Q1	0.003	0.156	0.102	0.742
2014Q2	0.002	0.133	0.091	0.640

policy becomes more apparent. From 2011Q3 to 2013Q1, the decomposition reveals that the ZLB duration expected is between 1Q and 4Q longer than the ZLB duration implied by the structural shocks, with the largest forward guidance duration occurring in 2012Q4. These results are consistent with the Federal Reserve making explicit calendar-based announcements about when the Federal Funds rate would be raised. For example, in August 2011, the Federal Open Market Committee announced in its statement that the Federal Funds rate would be held at 'exceptionally low' levels 'at least through mid-2013' – consistent with the 9Q duration expected in 2011Q3. In October 2012, the FOMC further committed to hold the Federal Funds rate low until at least mid-2015, which is also consistent with the 12Q duration used to extract the model's shocks.

The decomposition suggests that announcements did affect expectations by lengthening the expected ZLB period by one year. Table 4 documents the difference in macroeconomic aggregates due to these announcements. The macroeconomic implications of calendar based forward guidance in the estimated model are fairly small because of the largely unchanged response of inflation. At its largest, an announced 4Q increase in the ZLB duration over the endogenous duration due to structural shocks causes inflation to increase by 0.01 log points, while output and consumption increase by about 0.5 log points. Investment increases by a little more–about 1.2 log points. Cumulatively, absent forward guidance the reduction in output was 2.9%, for consumption the reduction was 2.7%, and 8.0% for investment.

4.2 A forward guidance policy rule

In principle, the central bank can announce any sequence of ZLB durations. I consider a simple rule for a forward guidance commitment, where the ZLB duration announced is explicitly linked to the output gap that would prevail in the absence of active policy. The central bank, when the policy interest rate is unconstrained, follows a Taylor rule. However, when the ZLB binds or is expected to bind, I consider a rule where the central bank forecasts output under the prevailing shocks and forecasts the anticipated length of time the ZLB will bind. Under these forecasts, it announces a lengthening of the time period for which it expects to hold the nominal interest rate at zero. Explicitly, if the Federal Funds rate is expected to hit or go below zero, the rule takes the form:

$$T_t = \max\left\{T_t^* + \lfloor -\gamma_y \left(y_t - y_t^p\right)^* \rfloor, T_t^* + T_{t-1} - T_{t-1}^* - 1\right\}, \tag{6}$$

where T_t^* represents, at time t, the length of time the ZLB is expected to bind under the sequence of shocks which push the nominal rate towards zero before forward guidance is provided, and $\left(y_t - y_t^p\right)^*$ represents the output gap which would prevail under no forward guidance policy. The value $\lfloor -\gamma_y \left(y_t - y_t^p\right)^* \rfloor$ represents the floor of $-\gamma_y \left(y_t - y_t^p\right)^*$, so that, for example, an additional reaction of 2.5 quarters becomes 2 quarters.[6] The parameter γ_y is a positive number, so that the announced duration is increasing in the output gap. A rule such as this says that if the output gap is significantly negative, then the central bank will keep the interest rate at zero for longer than is expected under the sequence of shocks causing the ZLB to bind. This forward guidance policy is stimulatory at time t. To commit the central bank to a prior forward guidance announcement, the second element of the rule (6) says that the policy component of a ZLB duration can contract by, at most, one period $(T_{t-1} - T_{t-1}^* - 1)$. For example, suppose the central bank reacts to a bad shock that would ordinarily cause the ZLB to bind for 4Q by announcing an extension of the ZLB by an additional 2Q. Suppose that in the next period, positive shocks would call for an endogenous duration of 2Q. To satisfy the policy announcement, the shortest duration agents can expect in that period is 3Q, which is the endogenous duration plus one quarter less than the duration caused by a policy action.

To illustrate the operation of the rule, I parameterize $\gamma_y = 0.85$. The initial responses of inflation to risk premia shocks of different sizes in the estimated model are shown in Figure 5. The length of additional forward guidance is given by the gap between the two lines of the fourth panel of the figure. The stimulus of the forward guidance policy can be seen by comparing the initial reactions of inflation and output growth with and without the forward guidance rule. The additional duration under the rule can be substantial. For example, for a large risk premia shock that would cause the ZLB to bind for 20 quarters, the forward guidance rule calls for the interest rate to remain at zero for an additional 11 quarters. Under this policy,

[6] This is needed because of the discrete nature of an announcement period.

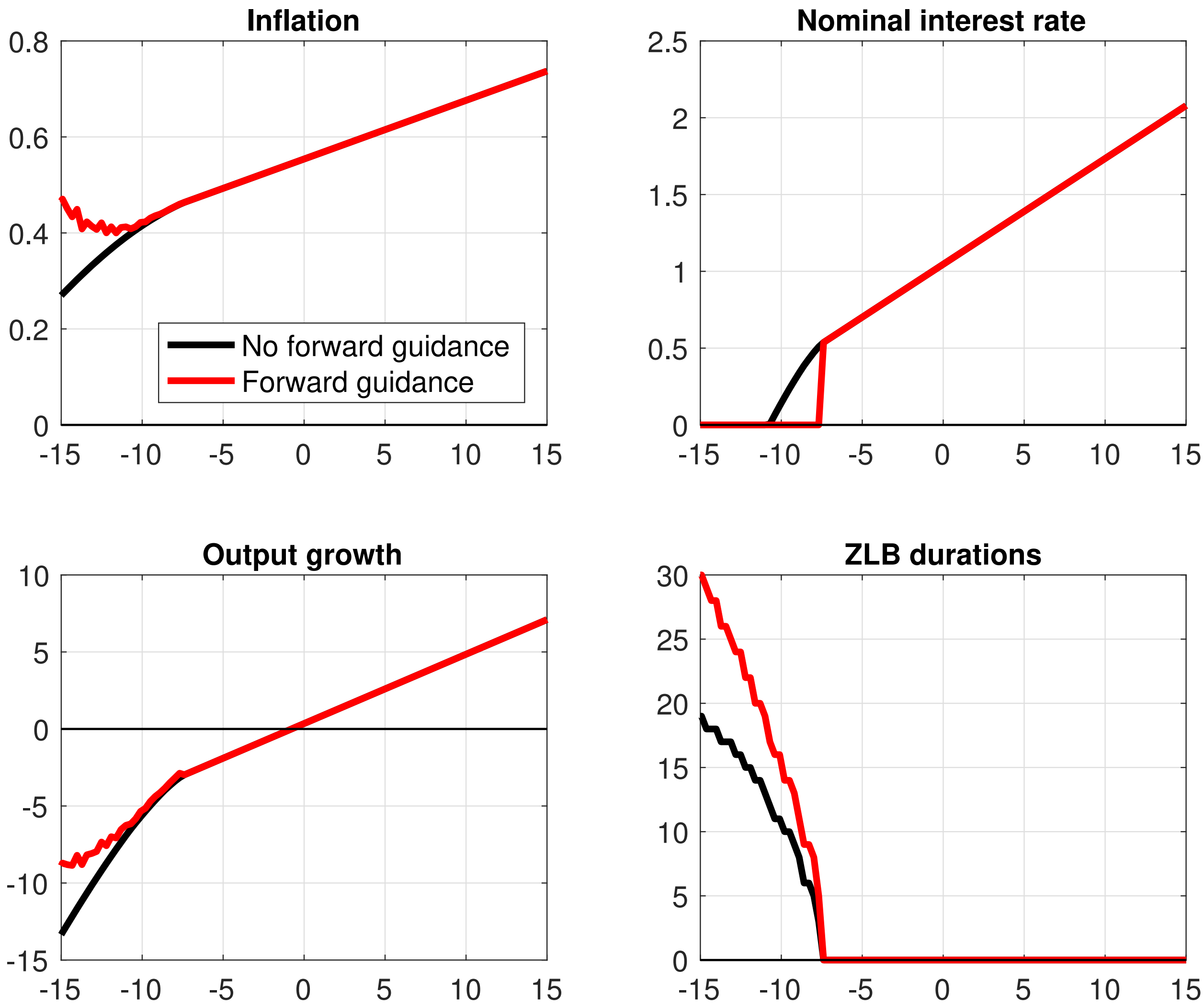


Figure 5: Initial response of variables to risk premia shocks with forward guidance.

output growth is about 4.5 percentage points higher, and the quarterly inflation rate is about 0.2 percentage points higher. Also, under the forward guidance rule, the initial reaction of the interest rate can be zero when the interest rate is positive without a rule. The reason for this is that, due to persistence in the policy rule, the interest rate is not forecasted to hit zero immediately but after a lag, and therefore attracts forward guidance immediately.

Under a calibration of $\gamma_y = 1.2$, Figure 6 plots the impulse response to a large negative risk premia shock when the central bank follows (6). The calendar-based forward guidance policy causes inflation to not fall as much, while output is subject to a less pronounced decline when the shock hits, followed by a relatively rapid recovery, eventually overshooting relative to steady-state. The overshooting of output mimics the optimal forward guidance policy discussed by Werning (2012). The impulse responses show that a rule of the form (6) generates paths for inflation and output growth that are substantially different to

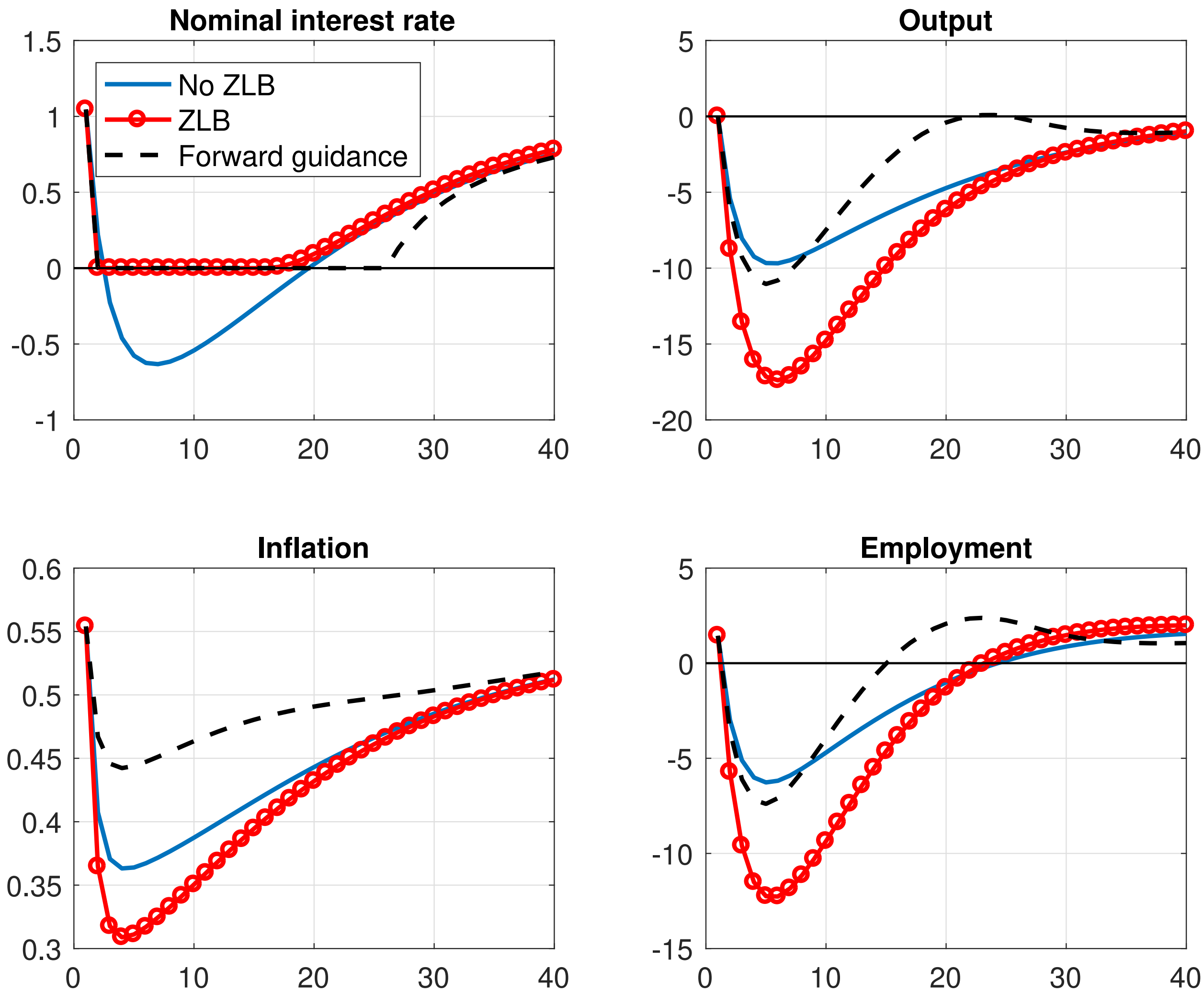


Figure 6: Impulse response to negative risk premia shock

the paths of the economy without the ZLB. In particular, the drop in output under the forward guidance rule is similar to the drop in output in the economy without the ZLB, although the drop in inflation under the forward guidance rule is about half the drop in inflation without the ZLB. This illustrates how nonlinear the dynamics can become when the economy is at the ZLB. The relatively strong response of inflation occurs because the forecasted path of marginal costs recovers relatively quickly along with the recovery in output and employment.

4.3 The calendar based forward guidance tradeoff

The previous section showed, in the estimated SW model featuring a range of pricing frictions, that the central bank can use credible announcements about the policy rate to react against shocks that might drive the Federal Funds rate to the ZLB, and mitigate potentially large fluctuations in inflation and output growth.

For forward guidance to be a time-consistent policy, a central bank must commit to holding the policy rate at zero for the duration of the announcement. A commitment to holding the interest rate constant for a period of time exposes the economy to volatility from unexpected shocks that would ordinarily call for an increase in the policy rate. There is, then, a trade-off between stabilizing contemporaneous shocks with an announcement rule and exposing the economy to excess volatility during the commitment period. This trade-off raises a number of questions including: how strong should a central bank following (6) tie its announcements to the realized shocks; do these policies improve overall macroeconomic stability; and how does a forward guidance rule affect the fiscal multiplier at the ZLB?

In this section, I examine the first two questions by simulating 100 paths of the estimated SW model for 1,000 periods under two policy regimes: when monetary policy is using the forward guidance rule (6) and when it is inactive at the ZLB. I compare the volatility of inflation and of output under the two policies for the same set of shocks but under different γ_y values.

For each simulation, I compute the following loss function, which is motivated by a central bank which maximizes households' utility function over time:

$$\text{Loss} = \text{var}(\pi_t) + \lambda \text{var}(y_t).$$

The value of λ determines the relative weight between the variance of inflation and the variance of output.

Figures 7 plots the mean, across simulated paths of the economy, of $\text{var}(\pi_t)$, $\text{var}(y_t)$ and the loss when $\lambda = 0.1$, normalized by their values for $\gamma_y = 0$. The forward guidance rule changes the dynamic behavior of the economy substantially. First, the mean length of the ZLB duration is increasing in γ_y, as expected under the policy rule (6). Second, for the estimated parameters, forward guidance reduces the variance of inflation and output, because it tempers large negative inflation and output responses when the ZLB binds. The loss function is minimized for values of γ_y between 1 and 1.5. For these values, the ZLB can be expected to be at zero twice as often as compared to an economy without forward guidance.

For large values of γ_y, however, the variance of inflation across the forecast horizon rises significantly. This reflects two factors: first, the central bank may be holding the interest rate at zero for much longer than is optimal, which significantly raises initial inflation and output, and second, the volatility caused by shocks while constrained at the ZLB. The first force is related to the forward guidance puzzle (see Del Negro et al., 2012; McKay et al., 2015), whereby announcements about the future path of the Federal Funds can have large aggregate consequences in the period of the announcement.

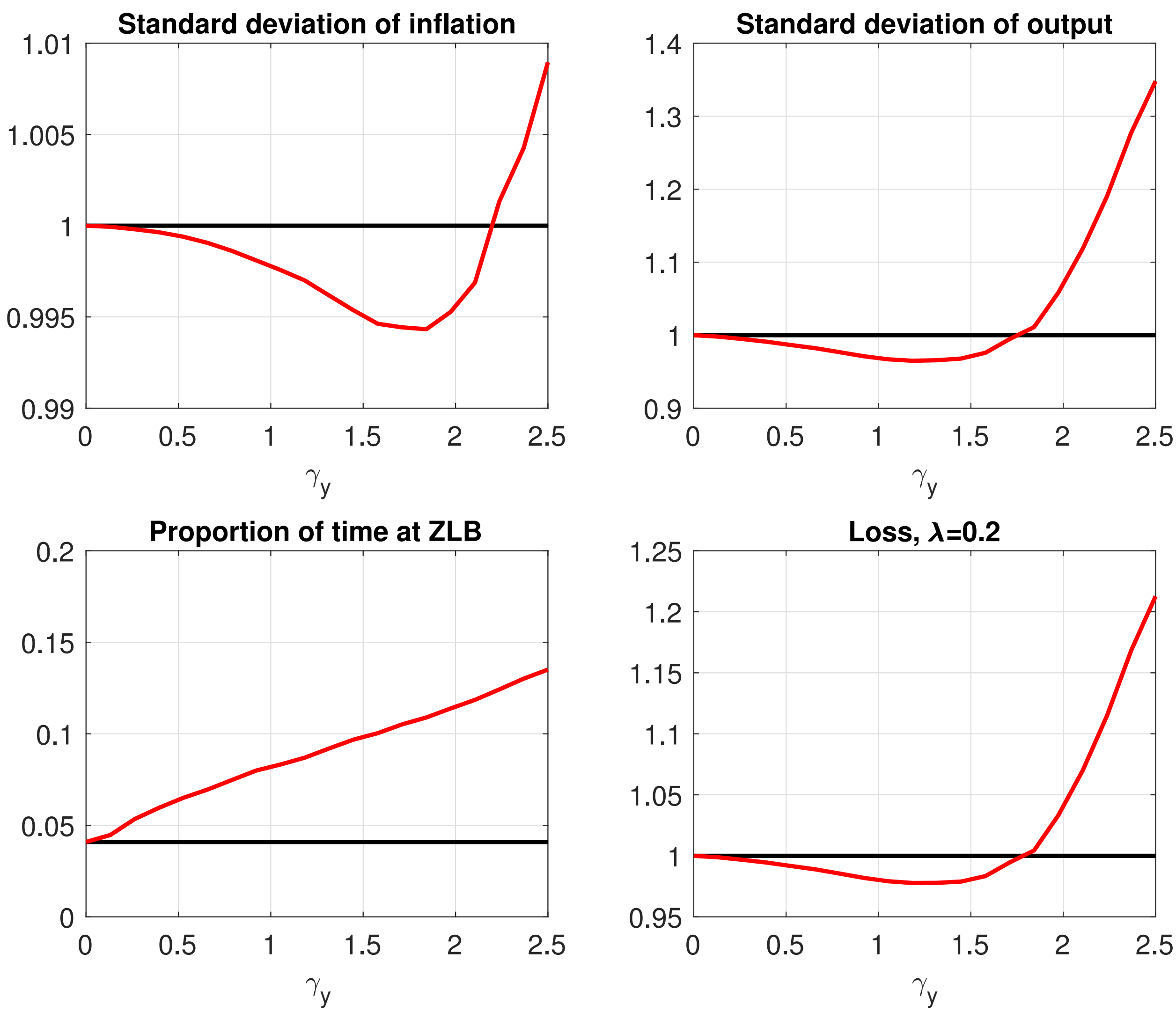


Figure 7: Volatility of inflation and output. The standard deivation for each series is normalized to the corresponding standard deviation when no forward guidance rule is in use.

4.4 The government spending multiplier under a forward guidance rule

The previous section showed that at the ZLB, a forward guidance rule like (6) can be a stimulatory. A key prediction of the New Keynesian model when monetary policy is inactive is that the returns to an increase in government spending are larger at the ZLB, driven by a decrease in the real interest rate when the nominal interest rate is constant (Christiano et al., 2011). These multipliers are derived when the central bank has no policy options at the ZLB. In this section, I investigate how an endogenous rule of the kind (6) can change the magnitude of the fiscal multiplier at the ZLB.

To illustrate this, I use the ZLB algorithm and the forward guidance rule calibrated at $\gamma_y = 0.85$ and compute the path of the economy under a large negative risk premia shock sufficient to cause the Federal Funds rate to hit the ZLB, both with and without a government spending shock which arrives at the same time as the risk premia shock. With these paths in hand, I compute the multiplier:

$$\frac{y_t^g - y_t}{g_1},$$

as in Fernández-Villaverde et al. (2012) where y_t^g is the path of output under the risk premia and government spending shock, y_t is the path of output under the risk premia shock only, and g_1 is the size of the government spending shock in the first period.

Figure 8 plots the path of the economy both with and without the government spending shock. First, consider how the economy reacts when the ZLB does not bind. The government spending shock causes output and inflation to rise. The Federal Reserve responds by raising the Federal Funds rate, increasing the real interest rate and dampening the rise in output. The government spending multiplier is well below 1 both when the government spending shock hits and in subsequent periods. Next, consider how the economy behaves when the ZLB binds. For large negative risk premia shocks, the real interest rate rises significantly when the ZLB binds, causing a substantial decline in output. The shock to government spending leads to a decline in the real interest rate, driven by an increase in the rate of inflation under the government spending shock. This generates a relatively larger government spending multiplier, consistent with the theory, of above 0.8 and remaining quite high across the forecast horizon. The increase in government spending at the ZLB also affects the date when the ZLB is expected to rise. This endogenous response of the ZLB reduces the government spending multiplier and explains why the multiplier remains below 1, as compared to the multipliers above one documented in Christiano et al. (2011).

Under the forward guidance rule, the forward guidance channel lowers real interest rates compared to

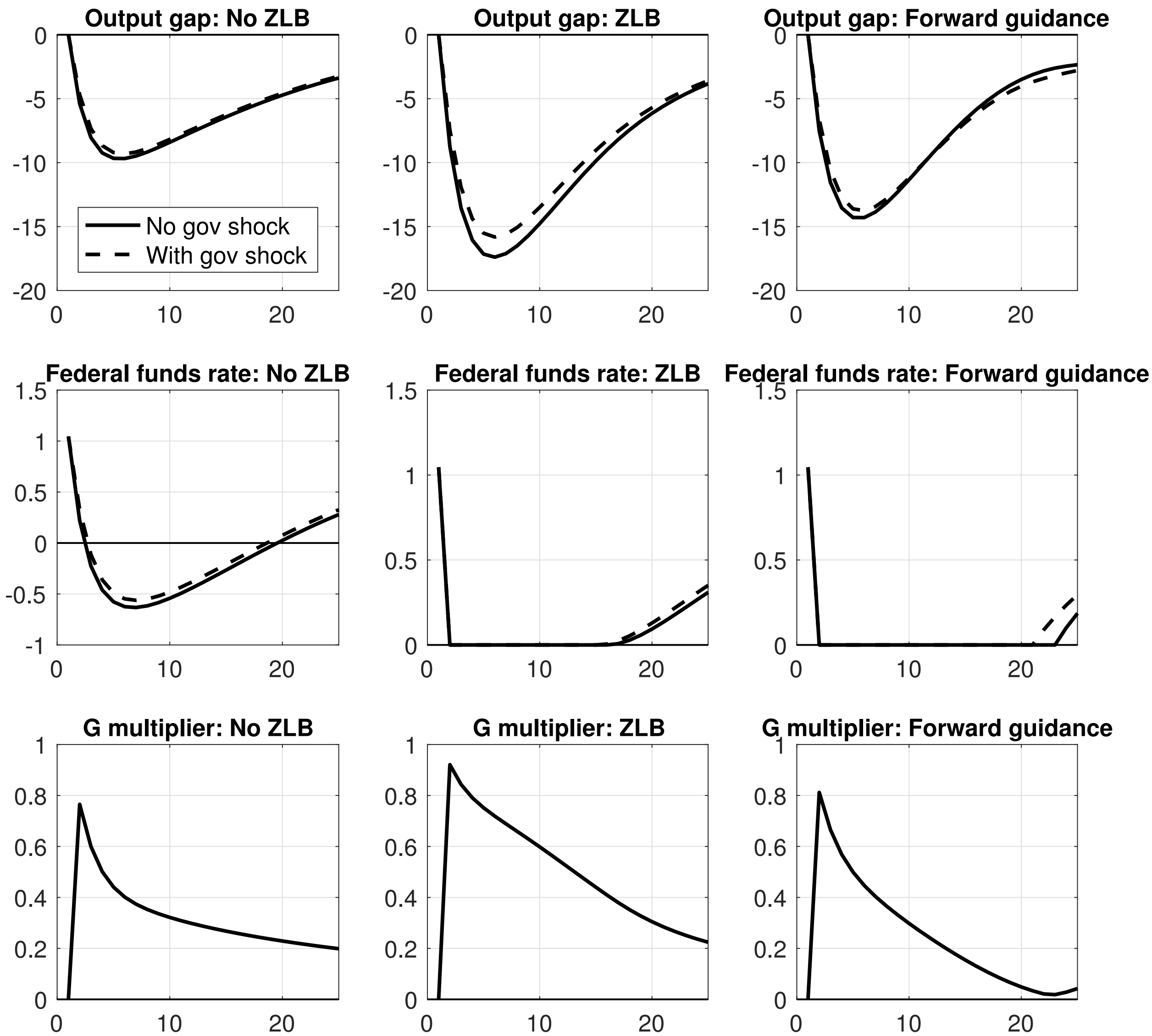


Figure 8: Negative risk premia shock and government spending

the endogenous durations case, and narrows the output gap. Adding the government spending shock under the forward guidance rule, does not change the the output gap much. This is because the government spending shock causes the output gap to narrow. As a result, the central bank refrains from announcing a longer ZLB duration. This can be seen by observing the Federal Funds rate lifts off the ZLB two quarters earlier under the government spending shock. The resulting multiplier is comparable to the multiplier that arises without the ZLB, reinforcing the observation that in New Keynesian models, higher government spending multipliers rely on the Federal Funds rate remaining fixed, at the same time as the government spending shock, and in the date the ZLB is expected to stop binding. Under a forward guidance rule, the exit date from the ZLB is endogenous and affected by government spending. This interaction has been discussed by policymakers following the 2016 election, after which expectations for stronger fiscal policy went hand in hand with a steepening yield curve.[7]

5 Conclusion

This paper introduced an efficient algorithm for implementing the ZLB and calendar based forward guidance for DSGE models typically used in policy analysis. The algorithm piecewise combines linear approximations of a model economy in a recursive way so that the periods that the ZLB is expected to bind aligns with the expectations of agents in the model. I show how a simple extension of the algorithm easily accommodates calendar based forward guidance policies and admits a very useful decomposition of a ZLB duration into a component due to structural shocks and a residual which can be interpreted as calendar based forward guidance. The decomposition provides a solution to an identification problem that arises at the ZLB, answering how stimulatory monetary policy is at the ZLB.

I estimated a version of the Smets and Wouters (2007) model, modified to accommodate forward guidance policy and a one-time but permanent change in trend growth. The results showed there was a substantial decline in trend growth occurring around the start of the 2000s, and that the ZLB was a substantial constraint on monetary policy. Applying the decomposition to the estimated model, I find evidence for forward guidance policy between 2011Q3 and 2013Q1, which in turn stimulated the economy.

With the estimated model and the ZLB algorithm, I studied a simple inflation-linked forward guidance policy rule, and showed that it replicated the features of the optimal forward guidance policies studied by, among others, Werning (2012). Exploiting the computational efficiency of the algorithm, the paper then asked how well the policy rule performs when the economy is buffeted by stochastic shocks and found that

[7] For a discussion of these observations, see Bernanke (2017).

a forward guidance rule does reduce the ex-post volatility of inflation and output growth when monetary policy uses forward guidance to stabilize large shocks, and exposes itself to the risk of undesirable volatility from shocks that hit during the commitment period. Furthermore, I showed that because a forward guidance rule formalizes a way for monetary policy to stimulate the economy at the ZLB, the stimulatory effect of government spending which occurs at the ZLB can be undone by the response of monetary policy to those shocks via shortened forward guidance.

Appendix

A Smets and Wouters (2007) model

The linearized equations of the Smets and Wouters (2007) model are the following equations, plus the monetary policy rule (1). Expectations are dropped for brevity.

A.1 Sticky price economy

Factor prices:

$$mc_t = \alpha r_t + (1-\alpha) w_t - \varepsilon_{a,t} \tag{A.1}$$

$$r_t = w_t + l_t - k_t^s \tag{A.2}$$

$$z_t = \tfrac{1-\psi}{\psi} r_t \tag{A.3}$$

Investment equations:

$$i_t = \tfrac{1}{1+\bar{\beta}\gamma}\left(i_{t-1} + \bar{\beta}\gamma i_{t+1} + \tfrac{1}{\gamma^2\phi} q_t\right) + \varepsilon_{i,t} \tag{A.4}$$

$$q_t = \tfrac{\sigma_c(1+\lambda/\gamma)}{1-\lambda/\gamma}\varepsilon_{b,t} + \tfrac{1-\delta}{1-\delta+R^k} q_{t+1} + \tfrac{R^k}{1-\delta+R^k} r_{t+1} - r_t + \pi_{t+1} \tag{A.5}$$

Consumption decision:

$$c_t = \varepsilon_{b,t} + \tfrac{\lambda/\gamma}{1+\lambda/\gamma} c_{t-1} + \tfrac{1}{1+\lambda/\gamma} c_{t+1} + \tfrac{(\sigma_c - 1) W^* L^*/C^*}{\sigma_c,(1+\lambda/\gamma)} (l_t - l_{t+1}) - \tfrac{1-\lambda/\gamma}{\sigma_c,(1+\lambda/\gamma)} (r_t - \pi_{t+1}) \tag{A.6}$$

Resource constraint:

$$y_t = c_t c_y + i_t i_y + \varepsilon_{g,t} + z_t z_y \tag{A.7}$$

Production function:

$$y_t = \phi_p \left(\varepsilon_{a,t} + \alpha k_t^s + (1-\alpha)\, l_t\right) \tag{A.8}$$

$$k_t^s = z_t + k_{t-1} \tag{A.9}$$

Evolution of capital:

$$k_t = (1-i_k)\, k_{t-1} + i_k i_t + \varepsilon_{i,t}\, \phi\, \gamma^2\, i_k \tag{A.10}$$

Price and wage Philips curves:

$$\pi_t = \tfrac{1}{1+\bar{\beta}\gamma\iota_p}\left(\bar{\beta}\gamma\pi_{t+1} + \iota_p \pi_{t-1} + mc_t \tfrac{\frac{(1-\xi_p)(1-\bar{\beta}\gamma\xi_p)}{\xi_p}}{1+(\phi_p-1)\varepsilon_p}\right) + \varepsilon_{p,t} \tag{A.11}$$

$$w_t = w_1 w_{t-1} + w_2 w_{t+1} + w_3 \pi_{t-1} - w_4 \pi_t + w_2 \pi_{t+1} + w_5 \left(\sigma_l\, l_t + \tfrac{1}{1-\lambda/\gamma} c_t - \tfrac{\lambda/\gamma}{1-\lambda/\gamma} c_{t-1} - w_t\right) + \varepsilon_{w,t} \tag{A.12}$$

where $w_1 = \frac{1}{1+\bar{\beta}\gamma}$, $w_2 = \frac{\bar{\beta}\gamma}{1+\bar{\beta}\gamma}$, $w_3 = \frac{\iota_w}{1+\bar{\beta}\gamma}$, $w_4 = \frac{1+\bar{\beta}\gamma\iota_w}{1+\bar{\beta}\gamma}$, and $w_5 = \frac{(1-\xi_w)(1-\bar{\beta}\gamma\xi_w)}{(1+\bar{\beta}\gamma)\xi_w}\frac{1}{1+(\phi_w-1)\varepsilon_w}$.

A.2 Flexible price economy

The corresponding equations defining the flexible price economy are:

$$\varepsilon_{a,t} = \alpha r_t^f + (1-\alpha) w_t^f \tag{A.13}$$

$$r_t^f = w_t^f + l_t^f - k_t^f \tag{A.14}$$

$$z_t^f = \tfrac{1-\psi}{\psi} r_t^f \tag{A.15}$$

$$k_t^f = z_t^f + kp_{t-1}^f \tag{A.16}$$

$$i_t^f = \frac{1}{1+\bar{\beta}\gamma}\left(i_{t-1}^f + \bar{\beta}\gamma i_{t+1}^f + \frac{1}{\gamma^2\phi}q_t^f\right) + \varepsilon_{i,t} \tag{A.17}$$

$$q_t^f = \frac{1-\delta}{1-\delta+Rk}q_{t+1}^f + \frac{R^k}{1-\delta+R^k}rk_{t+1}^f - rr_t^f + \frac{\sigma_c,(1+\lambda/\gamma)}{1-\lambda/\gamma}\varepsilon_{b,t} \tag{A.18}$$

$$c_t^f = \varepsilon_{b,t} + \frac{\lambda/\gamma}{1+\lambda/\gamma}c_{t-1}^f + \frac{1}{1+\lambda/\gamma}c_{t+1}^f + \frac{(\sigma_c-1)W^*L^*/C^*}{\sigma_c,(1+\lambda/\gamma)}\left(l_t^f - l_{t+1}^f\right) - \frac{1-\lambda/\gamma}{\sigma_c,(1+\lambda/\gamma)}rr_t^f \tag{A.19}$$

$$y_t^f = c_t^f c_y + i_t^f i_y + \varepsilon_{g,t} + z_t^f z_y \tag{A.20}$$

$$y_t^f = \phi_p\left(\varepsilon_{a,t} + \alpha k_t^f + (1-\alpha)\, l_t^f\right) \tag{A.21}$$

$$k_t^{p,f} = k_{t-1}^{p,f}\,(1-i_k) + i_t^f\, i_k + \varepsilon_{i,t}\,\gamma^2\,\phi\, i_k \tag{A.22}$$

$$w_t^f = \sigma_l l_t^f + \frac{1}{1-\lambda/\gamma}c_t^f - \frac{\lambda/\gamma}{1-\lambda/\gamma}c_{t-1}^f \tag{A.23}$$

A.3 Shocks

$$\varepsilon_{a,t} = \rho_a\,\varepsilon_{a,t-1} + \sigma_a\,\eta_{a,t} \tag{A.24}$$

$$\varepsilon_{b,t} = \rho_b\,\varepsilon_{b,t-1} + \sigma_b\,\eta_{b,t} \tag{A.25}$$

$$\varepsilon_{g,t} = \rho_g\,\varepsilon_{g,t-1} + \sigma_g\,\eta_{g,t} + \eta_{a,t}\,\sigma_a\,\rho_{ga} \tag{A.26}$$

$$\varepsilon_{i,t} = \rho_i\,\varepsilon_{i,t-1} + \sigma_i\,\eta_{i,t} \tag{A.27}$$

$$\varepsilon_{r,t} = \rho_r\,\varepsilon_{r,t-1} + \sigma_r\,\eta_{r,t} \tag{A.28}$$

$$\varepsilon_{p,t} = \rho_p\,\varepsilon_{p,t-1} + \eta_{p,ma,t} - \mu_p\,\eta_{p,ma,t-1} \tag{A.29}$$

$$\eta_{p,ma,t} = \sigma_p\,\eta_{p,t} \tag{A.30}$$

$$\varepsilon_{w,t} = \rho_w\,\varepsilon_{w,t-1} + \eta_{w,ma,t} - \mu_w\,\eta_{w,ma,t-1} \tag{A.31}$$

$$\eta_{w,ma,t} = \sigma_w\,\eta_{w,t} \tag{A.32}$$

A.4 Measurement equations

$$dy_t = \bar{\gamma} + y_t - y_{t-1} \tag{A.33}$$

$$dc_t = \bar{\gamma} + c_t - c_{t-1} \tag{A.34}$$

$$di_t = \bar{\gamma} + i_t - i_{t-1} \tag{A.35}$$

$$dw_t = \bar{\gamma} + w_t - w_{t-1} \tag{A.36}$$

$$\pi_t^{obs} = \bar{\pi} + \pi_t \tag{A.37}$$

$$r_t^{obs} = \bar{r} + r_t \tag{A.38}$$

$$l_t^{obs} = \bar{l} + l_t \tag{A.39}$$

B Solution methods

B.1 Binder and Pesaran (1995)

Consider a rational expectations model $x_t = \Psi(x_{t-1}, \mathbb{E}_t x_{t+1}, w_t)$. Linearize the model around a non-stochastic steady state to get:

$$\mathbf{A}x_t = \mathbf{C} + \mathbf{B}x_{t-1} + \mathbf{D}\mathbb{E}_t x_{t+1} + \mathbf{F}w_t.$$

In an economy where all agents know the regime and expectations are formed under that regime, the solution is a reduced-form VAR:

$$x_t = \mathbf{J} + \mathbf{Q}x_{t-1} + \mathbf{G}w_t.$$

where $\mathbf{J}$, $\mathbf{Q}$ and $\mathbf{G}$ are conformable matrices which are functions of the structural matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{F}$. As in Binder and Pesaran (1995) and Kulish and Pagan (2017), $\mathbf{Q}$ is solved by iterating on the quadratic expression:

$$\mathbf{Q} = [\mathbf{A} - \mathbf{D}\mathbf{Q}]^{-1}\mathbf{B}.$$

With $\mathbf{Q}$ in hand, we compute $\mathbf{J}$ and $\mathbf{G}$ with:

$$\mathbf{J} = [\mathbf{A} - \mathbf{D}\mathbf{Q}]^{-1}(\mathbf{C} + \mathbf{D}\mathbf{J})$$
$$\mathbf{G} = [\mathbf{A} - \mathbf{D}\mathbf{Q}]^{-1}\mathbf{F}.$$

B.2 Sims (2002)

The following is adapted from Cagliarini and Kulish (2013). Write a model in matrix form as:

$$\widetilde{\Gamma}_0\mathbf{y}_t = \widetilde{\Gamma}_1\mathbf{y}_{t-1} + \widetilde{C} + \widetilde{\Psi}\varepsilon_t, \tag{B.1}$$

where the state vector is defined by and ordered according to:

$$\mathbf{y}_t = \begin{bmatrix} \mathbf{y}_{1,t} \\ \mathbf{y}_{2,t} \\ \mathbb{E}_t\mathbf{z}_{t+1} \end{bmatrix},$$

and where $\mathbf{y}_{1,t}$ is an $(n_1 \times 1)$ vector of exogenous and some endogenous variables, and $\mathbf{y}_{2,t}$ is an $(n_2 \times 1)$ vector with those endogenous variables for which conditional expectations appear; $\mathbf{z}_{t+1}$, $(k \times 1)$, where $k = n_2 \times s$ contains s leads of $\mathbf{y}_{2,t}$; in most models like Ireland (2004) and Smets and Wouters (2007), however, $\mathbf{z}_{t+1} = \mathbf{y}_{2,t+1}$ so that $s = 1$ and $k = n_2$. The dimension of $\mathbf{y}_t$ is $n \times 1$, where $n = n_1 + n_2 + k$. Also, we assume ε_t to be an $l \times 1$ vector of serially uncorrelated processes, $\widetilde{\Gamma}_0$ and $\widetilde{\Gamma}_1$ are $(n_1 + n_2) \times n$ matrices, $\widetilde{C}$ is $(n_1 + n_2) \times 1$ and $\widetilde{\Psi}$ is $(n_1 + n_2) \times l$.

Because of the presence of expectations, we cannot invert $\widetilde{\Gamma}$ and estimate a reduced form version of (B.1). Sims's (2002) proposal is to append to (B.1) expectations revisions which will be solved as part of the solution. Let η_t be the vector of expectations revisions:

$$\eta_t = \mathbb{E}_t\mathbf{z}_t - \mathbb{E}_{t-1}\mathbf{z}_t, \tag{B.2}$$

where $\mathbb{E}_t\eta_{t+j} = 0$ for $j \geq 1$. For example, if $z_t = y_{2,t}$, then η_t are forecast revisions.

Augment the system defined by Equation (B.1) with the k equations from Equation (B.2) to obtain:

$$\Gamma_0\mathbf{y}_t = C + \Gamma_1\mathbf{y}_{t-1} + \Psi\varepsilon_t + \Pi\eta_t. \tag{B.3}$$

where the matrices Γ_0, Γ_1,C,Ψ, and Π are of conformable dimensions. Γ_0 is now an $n \times n$ matrix, which we will invert with a Schur (QZ) decomposition, and impose conditions such that we can remove the η_t from the system.

To solve (B.3) as Sims (2002), take a Schur (QZ) decomposition of (Γ_0, Γ_1) to get:

$$Q'\Lambda Z' = \Gamma_0 \quad \text{and} \quad Q'\Omega Z' = \Gamma_1,$$

where Λ and Ω are both upper triangular. The matrices Q and Z are unitary, so that $QQ' = I$ and $ZZ' = I$.

Pre-multiply model equation by Q and define $w_t = Z' y_t$ to rewrite the system as:

$$\Lambda w_t = \Omega w_{t-1} + Q(C + \Psi \varepsilon_t + \Pi \eta_t).$$

Define $w_{1,t} = Z_1' y_t$ and $w_{2,t} = Z_2' y_t$. Λ and Ω are upper triangular and have the property that the generalized eigenvalues of (Γ_0, Γ_1) are ratios of diagonal elements of Ω and Λ. Rearrange the system so that the explosive eigenvalues correspond to the lower right blocks of Λ and Σ, partitioning w_t and rewriting the system as:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & \Lambda_{22} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ 0 & \Omega_{22} \end{bmatrix} \begin{bmatrix} w_{1,t-1} \\ w_{2,t-1} \end{bmatrix} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} (C + \Psi \varepsilon_t + \Pi \eta_t).$$

The lower block of the system are those equations which correspond to the m explosive generalised eigenvalues of (Γ_0, Γ_1). The lower set of equations are not affected by $w_{1,t}$. Isolate these:

$$\Lambda_{22} w_{2,t} = \Omega_{22} w_{2,t-1} + Q_2 \left(C + \Psi \varepsilon_t + \Pi \eta_t\right).$$

For stability of the system, we need η_t to offset the effect of ε_t on $w_{2,t}$. To see this, first solve $w_{2,t}$ forward to get:

$$w_{2,t} = \left(\Lambda_{22} - \Omega_{22}\right)^{-1} Q_2 C - \sum_{j=1}^{\infty} \left(\Omega_{22}^{-1} \Lambda_{22}\right)^{j-1} \Omega_{22}^{-1} Q_2 (\Psi \varepsilon_{t+j} + \Pi \eta_{t+j}).$$

This says that $w_{2,t}$ requires having in hand all future values of ε_t and η_t at time t. Take expectations of this expression at time t to get:

$$w_{2,t} = \left(\Lambda_{22} - \Omega_{22}\right)^{-1} Q_2 C - \mathbb{E}_t \sum_{j=1}^{\infty} \left(\Omega_{22}^{-1} \Lambda_{22}\right)^{j-1} \Omega_{22}^{-1} Q_2 \Psi \varepsilon_{t+j}.$$

Also take expectations at time $t+1$ to get:

$$w_{2,t} = \left(\Lambda_{22} - \Omega_{22}\right)^{-1} Q_2 C - \mathbb{E}_{t+1} \sum_{j=1}^{\infty} \left(\Omega_{22}^{-1} \Lambda_{22}\right)^{j-1} \Omega_{22}^{-1} Q_2 \Psi \varepsilon_{t+j} - \Omega_{22}^{-1} Q_2 \Pi \eta_{t+1}.$$

Note the left hand side has not changed, so equating these two expressions implies:

$$Q_2 \Pi \eta_{t+1} = \Omega_{22} \sum_{j=1}^{\infty} \left(\Omega_{22}^{-1} \Lambda_{22}\right)^{j-1} \Omega_{22}^{-1} Q_2 \left(\mathbb{E}_t \varepsilon_{t+j} - \mathbb{E}_{t+1} \varepsilon_{t+j}\right).$$

This says that for the system to be stable, the expectations revisions must offset the effect that shocks ε_t have on the explosive component of the system, $w_{2,t}$. Expectations revisions ensure that the system is placed on the saddle path to stability. For this to be true, Sims (2002) shows that what is required for a unique solution is that the number of explosive eigenvalues of (Γ_0, Γ_1), m equals the number of variables which appear as expectations in the system, k. Under this condition, the system is on a saddle path to a steady-state from any initial condition. (Note, there are weaker conditions just for stability.) If this is true, and if the solution is stable then there is a matrix Φ such that:

$$Q_1 \Pi = \Phi Q_2 \Pi.$$

By premultiplying the system by $[I_{n-p}, -\Phi]$, the coefficient on η_t is:

$$Q_1 \Pi - \Phi Q_2 \Pi.$$

Since existence of a solution requires $Q_1\Pi = \Phi Q_2\Pi$, the η_t drop out of (B.1), so that a solution to the model can be written as:

$$\mathbf{y}_t = S_0 + S_1\mathbf{y}_{t-1} + S_2\varepsilon_t + S_y\mathbb{E}_t\sum_{j=1}^{\infty} M^{j-1}\Omega_{22}^{-1}Q_2\Psi\varepsilon_{t+j}, \tag{B.4}$$

where:

$$H = Z\begin{bmatrix} \Sigma_{11}^{-1} & -\Sigma_{11}^{-1}(\Sigma_{12}-\Phi\Sigma_{22}) \\ 0 & I \end{bmatrix}, \qquad S_0 = H\begin{bmatrix} Q_1-\Phi Q_2 \\ (\Sigma_{22}-\Omega_{22})^{-1}Q_2 \end{bmatrix}C,$$

$$S_1 = H\begin{bmatrix} \Omega_{11} & \Omega_{12}-\Phi\Omega_{22} \\ 0 & 0 \end{bmatrix}, \qquad S_2 = H\begin{bmatrix} Q_1-\Phi Q_2 \\ 0 \end{bmatrix}\Psi,$$

$$S_y = -H\begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

The solution (B.4) is in the desired VAR(1) form.

B.3 Foreseen structural changes

Suppose the structural parameters of the economy are known to change into the future. In particular, suppose the economy is expected to evolve with the following structure: in time period $t = 1$, the economy starts with the following structure:

$$\widetilde{\Gamma}_{0,1}\mathbf{y}_t = \widetilde{\Gamma}_{1,1}\mathbf{y}_0 + \widetilde{C}_1 + \widetilde{\Psi}_1\varepsilon_1,$$

and in time periods $2 \le t \le T$, the structural parameters of the economy evolve according to:

$$\Gamma_{0,t}\mathbf{y}_t = \Gamma_{1,t}\mathbf{y}_{t-1} + C_t + \Pi\eta_t + \Psi_t\left(\varepsilon_t^u + \varepsilon_t^a\right),$$

where ε_t^u are shocks which are unanticipated at time period $t = 1$, ε_t^a are shocks which are anticipated at $t = 1$. In this implementation, expectations revisions are included as the system evolves. Unanticipated shocks are added to show that it is possible to solve the model subject to foreseen structural changes and unanticipated shocks, though the solution would need to be computed each time period. Also notice that the matrices specifying the structural parameters are time-varying. After time period $T+1$, the structural parameters of the economy are fixed, so that the system becomes:

$$\bar{\Gamma}_0\mathbf{y}_t = \bar{\Gamma}_1\mathbf{y}_{t-1} + \bar{C} + \bar{\Pi}\eta_t + \bar{\Psi}\varepsilon_t.$$

Stacking $T \times (n_1 + n_2 + k) + \widetilde{m} - k$ equations and imposing $\mathbb{E}_1\eta_2 = \mathbb{E}_1\eta_3 = \ldots = \mathbb{E}_1\eta_T = 0$ (rational expectations) yields:

$$\begin{bmatrix} \widetilde{\Gamma}_{0,1} & 0 & \ldots & \ldots & 0 \\ -\Gamma_{1,2} & \Gamma_{0,2} & \ddots & & \vdots \\ 0 & -\Gamma_{1,3} & \Gamma_{0,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & -\Gamma_{1,T} & \Gamma_{0,T} \\ 0 & \ldots & \ldots & 0 & \bar{Z}_2' \end{bmatrix}\begin{bmatrix} \mathbf{y}_1 \\ \mathbb{E}_1\mathbf{y}_2 \\ \vdots \\ \mathbb{E}_1\mathbf{y}_T \end{bmatrix} = \begin{bmatrix} \widetilde{C}_1 + \widetilde{\Gamma}_{1,1}\mathbf{y}_0 + \widetilde{\Psi}_1\varepsilon_1 \\ C_2 + \Psi_2\varepsilon_2^a \\ \vdots \\ C_T + \Psi_T\varepsilon_T^a \\ \widetilde{w}_{2,T} \end{bmatrix}. \tag{B.5}$$

More concisely:

$$A\mathscr{Y} = \mathbf{b}.$$

The necessary condition to invert A is that the final (bar) structure of the economy has a solution, which ensures the economy reaches its saddle path. In particular, Cagliarini and Kulish (2013) show that uniqueness of final system is necessary for the intermediate path of the economy to be unique.

Practically, $\mathscr{Y} = (\mathbf{y}_1', \ldots, \mathbf{y}_T')$ is a $T \times (n_1 + n_2 + k)$ vector. As discussed in Sims (2002) and in Section B.2, a solution to the final system implies $\bar{m} = k$. This condition implies A is a square matrix. Given uniqueness of the solution to the final system, it is necessary for A be full rank for the path $\mathscr{Y}$ to be unique. This is generally the case unless perverse parameters are used. The system can largely be unconstrained in the intermediate stage. If the system is on a saddle path eventually, there is usually a unique path.

C Estimation details

C.1 Kalman filter

The model in state space representation is:

$$x_t = \mathbf{J}_t + \mathbf{Q}_t x_{t-1} + \mathbf{G}_t w_t \tag{C.1}$$
$$z_t = \mathbf{H}_t x_t. \tag{C.2}$$

The error is distributed $w_t \sim N(0, \mathbf{Q})$ where $\mathbf{Q}$ is the covariance matrix of w_t. By assumption, there is no observation error of the data given in the vector z_t. The Kalman filter recursion is given by the following equations, conceptualized as the predict and update steps. The state of the system is $(\hat{x}_t, \mathbf{P}_{t-1})$. In the predict step, the structural matrices $\mathbf{J}_t$, $\mathbf{Q}_t$ and $\mathbf{G}_t$ are used to compute a forecast of the state $\hat{x}_{t|t-1}$ and the forecast covariance matrix $\mathbf{P}_{t|t-1}$ as:

$$\begin{aligned}\hat{x}_{t|t-1} &= \mathbf{J}_t + \mathbf{Q}_t \hat{x}_t \\ \mathbf{P}_{t|t-1} &= \mathbf{Q}_t \mathbf{P}_{t-1} \mathbf{Q}_{t|t-1}^\top + \mathbf{G}_t \mathbf{Q} \mathbf{G}_t^\top.\end{aligned}$$

This formulation differs from the time-invariant Kalman filter because in the forecast stage the structural matrices $\mathbf{J}_t$, $\mathbf{Q}_t$ and $\mathbf{G}_t$ can vary over time. We update these forecasts with imperfect observations of the state vector. This update step involves computing forecast errors $\widetilde{y}_t$ and its associated covariance matrix $\mathbf{S}_t$ as:

$$\begin{aligned}\widetilde{y}_t &= z_t - \mathbf{H}_t \hat{x}_{t|t-1} \\ \mathbf{S}_t &= \mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^\top.\end{aligned}$$

The Kalman gain matrix is given by:

$$\mathbf{K}_t = \mathbf{P}_{t|t-1} \mathbf{H}_t^\top \mathbf{S}_t^{-1}.$$

With $\widetilde{y}_t$, $\mathbf{S}_t$ and $\mathbf{K}_t$ in hand, the optimal filtered update of the state x_t is

$$\hat{x}_t = \hat{x}_{t|t-1} + \mathbf{K}_t \widetilde{y}_t,$$

and for its associated covariance matrix:

$$\mathbf{P}_t = (I - \mathbf{K}_t \mathbf{H}_t) \mathbf{P}_{t|t-1}.$$

The Kalman filter is initialized with x_0 and $\mathbf{P}_0$ determined from their unconditional moments, and is computed until the final time period T of data.

C.2 Kalman smoother

With the estimates of the parameters and durations in hand at time period T, the Kalman smoother gives an estimate of $x_{t|T}$, or an estimate of the state vector at each point in time given all available information (see **?**). With $\hat{x}_{t|t-1}$, $\mathbf{P}_{t|t-1}$, $\mathbf{K}_t$ and $\mathbf{S}_t$ in hand from the Kalman filter, the vector $x_{t|T}$ is computed by:

$$x_{t|T} = \hat{x}_{t|t-1} + \mathbf{P}_{t|t-1} r_{t|T},$$

where the vector $r_{T+1|T} = 0$ and is updated with the recursion:

$$r_{t|T} = \mathbf{H}_t^\top \mathbf{S}_t^{-1} \left(z_t - \mathbf{H}_t \hat{x}_{t|t-1} \right) + \left(I - \mathbf{K}_t \mathbf{H}_t \right)^\top \mathbf{P}_{t|t-1}^\top r_{t+1|T}.$$

Finally, to get an estimate of the shocks to each state variable, denoted by e_t, we compute:

$$e_t = \mathbf{G}_t w_t = \mathbf{G}_t r_{t|T}.$$

From these, we get an estimate of the structural shocks used to compute counterfactuals with the model.

C.3 Sampler

This section describes the sampler used to construct an estimate of the posterior distributions. Denote by ϑ the vector of parameters to be estimated, $\mathbf{U}$ the vector of breaking structural parameters (trend growth), and $\mathbf{T}$ the vector of durations to be estimated. Denote by $\mathbf{z} = \{z_\tau\}_{\tau=1}^{T}$ the sequence of observable vectors. The posterior $\mathscr{P}(\vartheta, \mathbf{U}, \mathbf{T} \mid \mathbf{z})$ satisfies:

$$\mathscr{P}(\vartheta, \mathbf{U}, \mathbf{T} \mid \mathbf{z}) \propto \mathscr{L}(\mathbf{z} \mid \vartheta, \mathbf{U}, \mathbf{T}) \times \mathscr{P}(\vartheta, \mathbf{U}, \mathbf{T}).$$

With Gaussian errors, the likelihood function $\mathscr{L}(\mathbf{z} \mid \vartheta, \mathbf{U}, \mathbf{T})$ is computed using the appropriate sequence of structural matrices and the Kalman filter:

$$\log \mathscr{L}(\mathbf{z} \mid \vartheta, \mathbf{U}, \mathbf{T}) = -\left(\frac{N_z T}{2} \right) \log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log \det \mathbf{H}_t \mathbf{S}_t \mathbf{H}_t^\top - \frac{1}{2} \sum_{t=1}^{T} \widetilde{y}_t^\top \left(\mathbf{H}_t \mathbf{S}_t \mathbf{H}_t^\top \right)^{-1} \widetilde{y}_t.$$

The prior is simply computed using priors over ϑ which are consistent with the literature, and with flat priors on $\mathbf{U}$ and $\mathbf{T}$.[8]

The Markov Chain Monte Carlo posterior sampler has three blocks, corresponding to ϑ, $\mathbf{U}$ and $\mathbf{T}$. Initialize the sampler at step j with the last accepted draw of the structural parameters, the period of the breaking parameters and durations, denoted by ϑ_{j-1}, $\mathbf{U}_{j-1}$ and $\mathbf{T}_{j-1}$ respectively. The three blocks are, in order of computation:

1. In the first block, propose a new $\mathbf{U}_j$ by randomly choosing a date between $[T_1, T_2]$, where T_1 and T_2 are the bounds of the interquartile range of the sample. With $\mathbf{U}_j$, recompute the sequence of structural matrices associated with $(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1})$, compute the posterior $\mathscr{P}(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1} \mid \mathbf{z})$, and accept $(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1})$ with probability $\frac{\mathscr{P}(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1} \mid \mathbf{z})}{\mathscr{P}(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_{j-1} \mid \mathbf{z})}$. If $(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1})$ is accepted, then set $\mathbf{U}_{j-1} = \mathbf{U}_j$.

2. In the second block, randomly choose up to $\bar{T}$ durations to test, corresponding to up to $\bar{T}$ time periods that the economy is at the ZLB. For each of those time periods, randomly choose a duration in the interval $[1, T^*]$ and mix that value with the previously accepted draw to generate a new $\mathbf{T}_j$ proposal. As with the first block, recompute the sequence of structural matrices associated with $(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_j)$,

[8] I require the structural break date to lie in the interquartile range of the sample to avoid issues of erroneous errors found in short samples, and I require that each estimated duration lies below some maximum value T^* which, in practice, is rarely visited by the sampler.

compute the posterior $\mathscr{P}(\vartheta_{j-1}, \mathbf{U}_j, \mathbf{T}_{j-1} \mid \mathbf{z})$, and accept the proposal $(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_j)$ with probability $\frac{\mathscr{P}(\vartheta_{j-1},\mathbf{U}_{j-1},\mathbf{T}_j|\mathbf{z})}{\mathscr{P}(\vartheta_{j-1},\mathbf{U}_{j-1},\mathbf{T}_{j-1}|\mathbf{z})}$. If $(\vartheta_{j-1}, \mathbf{U}_{j-1}, \mathbf{T}_j)$ is accepted, then set $\mathbf{T}_{j-1} = \mathbf{T}_j$.

3. The third block is a more standard Metropolis-Hastings random walk step. First start by selecting which structural parameters to propose a new value for. For those parameters, draw a new proposal ϑ_j from a proposal density centered at ϑ_{j-1} and with thick tails to ensure sufficient coverage of the parameter space and an acceptance rate of roughly 20%. The proposal ϑ_j is accepted with probability $\frac{\mathscr{P}(\vartheta_j,\mathbf{U}_{j-1},\mathbf{T}_{j-1}|\mathbf{z})}{\mathscr{P}(\vartheta_{j-1},\mathbf{U}_{j-1},\mathbf{T}_{j-1}|\mathbf{z})}$. If $(\vartheta_j, \mathbf{U}_{j-1}, \mathbf{T}_{j-1})$ is accepted, then set $\vartheta_{j-1} = \vartheta_j$.

D A worked example of the algorithm

Consider the simple example, log-linearized around steady-state where y_t is output and the nominal interest rate i_t ignores the ZLB:

$$y_t = \mathbb{E}_t y_{t+1} - (i_t - \bar{i}) + \varepsilon_t$$
$$i_t - \bar{i} = \rho\,(i_{t-1} - \bar{i}) + \gamma y_t.$$

Putting this model in the form of (B.1) requires $\mathbf{y}_t = [\ i_t \quad y_t \quad \mathbb{E}_t y_{t+1}\ ]'$ and:

$$\widetilde{\Gamma}_0 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -\gamma & 0 \end{bmatrix}, \quad \widetilde{\Gamma}_1 = \begin{bmatrix} 0 & 0 & 0 \\ \rho & 0 & 0 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} \bar{i} \\ \bar{i}(1-\rho) \end{bmatrix}, \quad \widetilde{\Psi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Adding expectations revisions requires appending a single equation:

$$\eta_t = y_t - \mathbb{E}_{t-1} y_t.$$

The matrices become:

$$\Gamma_0 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -\gamma & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} 0 & 0 & 0 \\ \rho & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} \bar{i} \\ \bar{i}(1-\rho) \\ 0 \end{bmatrix}, \quad \widetilde{\Psi} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and $\Pi = [\ 0 \quad 0 \quad 1\ ]'$. The routines of Sims (2002) are used to obtain the linear system:

$$\mathbf{y}_t = S_0 + S_1 \mathbf{y}_{t-1} + S_2 \varepsilon_t. \tag{D.1}$$

The algorithm proceeds as follows. Given a shock ε_t:

1. Using the reduced form system without the ZLB (D.1), obtain the path $\mathbf{y}_t$ up to some large T. Assume no future shocks:

$$\begin{aligned} \mathbf{y}_t &= S_0 + S_1 \mathbf{y}_{t-1} + S_2 \varepsilon_t \\ \mathbf{y}_{t+1} &= S_0 + S_1 \mathbf{y}_t \\ &\vdots \\ \mathbf{y}_T &= S_0 + S_1 \mathbf{y}_{T-1}. \end{aligned}$$

2. Examine $\{i_\tau\}_{\tau=t}^T$. If $i_\tau > 0\ \forall\ \tau$, then stop the algorithm. Otherwise, move to the next step.

3. Find the first time period where $i_\tau < 0$. Suppose $i_{t+1} < 0$ under the shock ε_t. Then, we want the following

system to apply at time period $t+1$:

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - \bar{i}) + \varepsilon_t \\ i_t &= 0, \end{aligned}$$

and the non-ZLB system to apply for t and time periods $\tau > t+1$. The system at $t+1$ translates into the following structural matrices:

$$\Gamma^*_{0,t+1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Gamma^*_{1,t+1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C^*_{t+1} = \begin{bmatrix} \bar{i} \\ 0 \\ 0 \end{bmatrix}, \quad \Psi^*_{t+1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

while $\Pi^*_{t+1} = \Pi$. We now put these structural matrices in place in the format of (B.5):

$$\begin{bmatrix} \widetilde{\Gamma}_0 & 0 \\ -\Gamma^*_{1,t+1} & \Gamma^*_{0,t+1} \\ 0 & \widetilde{Z}'_2 \end{bmatrix} \begin{bmatrix} \mathbf{y}_t \\ \mathbb{E}_t \mathbf{y}_{t+1} \end{bmatrix} = \begin{bmatrix} \widetilde{C} + \widetilde{\Gamma}_1 \mathbf{y}_{t-1} + \widetilde{\Psi}\varepsilon_t \\ C^*_{t+1} \\ \widetilde{w}_2 \end{bmatrix}, \tag{D.2}$$

where $\widetilde{Z}'_2$ and $\widetilde{w}_2$ are the matrices defined in section B.2 associated with the solution to the non-ZLB system. Invert (D.2) to obtain $\mathbf{y}_t$ and $\mathbf{y}_{t+1}$. Then use the solution to the non-ZLB system to obtain $\mathbf{y}_{t+j}$ for $j > t+1$ again assuming no future shocks.

Return to step 2 with the the new path of $\{i_\tau\}_{\tau=t}^T$.

4. Examine the new path of $\{i_\tau\}_{\tau=t}^T$. If $i_\tau > 0 \ \forall\ \tau$, then stop the algorithm: the ZLB applies only for time period $t+1$. Otherwise, move to the next step having already imposed the ZLB at time period $t+1$.

5. Find the new first time period where $i_\tau < 0$. Suppose $i_t < 0$ under the shock ε_t with $i_{t+1} = 0$. This could happen as the imposition of $i_{t+1} = 0$ is a contactionary monetary policy relative to $i_{t+1} < 0$. Then, we want the following system to apply at t:

$$\begin{aligned} y_t &= \mathbb{E}_t y_{t+1} - (i_t - \bar{i}) + \varepsilon_t \\ i_t &= 0, \end{aligned}$$

and the non-ZLB system to apply for time periods $\tau > t+1$. The system at t translates into the following structural matrices:

$$\widetilde{\Gamma}^*_0 = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \widetilde{\Gamma}^*_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \widetilde{C}^* = \begin{bmatrix} \bar{i} \\ 0 \end{bmatrix}, \quad \widetilde{\Psi}^*_t = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

Again, putting these structural matrices in place in the format of (B.5):

$$\begin{bmatrix} \widetilde{\Gamma}^*_0 & 0 \\ -\Gamma^*_{1,t+1} & \Gamma^*_{0,t+1} \\ 0 & \widetilde{Z}'_2 \end{bmatrix} \begin{bmatrix} \mathbf{y}_t \\ \mathbb{E}_t \mathbf{y}_{t+1} \end{bmatrix} = \begin{bmatrix} \widetilde{C}^* + \widetilde{\Gamma}^*_1 \mathbf{y}_{t-1} + \widetilde{\Psi}\varepsilon_t \\ C^*_{t+1} \\ \widetilde{w}_2 \end{bmatrix}.$$

And so, invert the LHS matrix to obtain the path of the model variables during the ZLB episode including the path of the nominal interest rate. Now the ZLB applies for two periods.

6. Continue iterating until the nominal interest rate satisfies the ZLB across the forecast horizon.

E Result 1

Result 1. *If there are n distinct equations for n variables in the linearized model (2) under the non-ZLB regime and the Blanchard-Kahn conditions are satisfied for the linearized model under the non-ZLB regime, then the path during the ZLB period exists and is unique.*

Given the discussion in section B.3, Result 2 involves checking the rank of the A matrix of equation (B.5). In particular, we need that the rank of A is $n \times T$. Without loss of generality, take the case where $T = 3$:

$$A = \begin{bmatrix} \widetilde{\Gamma}_{0,1} & 0 & 0 \\ -\Gamma_{1,2} & \Gamma_{0,2} & 0 \\ 0 & -\Gamma_{1,3} & \Gamma_{0,3} \\ 0 & 0 & \bar{Z}_2' \end{bmatrix}.$$

Uniqueness of the final solution implies the rank of $\bar{Z}_2' = k$. From our assumption that there are $n_1 + n_2$ unique equations of the original system, appending k expectations revisions to the system ensures there are $n = n_1 + n_2 + k$ unique equations to the non-ZLB system. And so, if all that has changed is that the interest rate rule is now specified to force the nominal interest rate to the zero, then we have:

$$\text{rank}\left(\begin{bmatrix} -\Gamma_{1,2} & \Gamma_{0,2} & 0 \\ 0 & -\Gamma_{1,3} & \Gamma_{0,3} \end{bmatrix} \right) = 2n.$$

It remains to argue that the final set of equations $\begin{bmatrix} 0 & 0 & \bar{Z}_2' \end{bmatrix}$ cannot contain an equation which is a linear combination of the equations in $\begin{bmatrix} 0 & -\Gamma_{1,3} & \Gamma_{0,3} \end{bmatrix}$. To see that this is the case, suppose that indeed there is an equation in $\begin{bmatrix} 0 & 0 & \bar{Z}_2' \end{bmatrix}$ that is a linear combination of the equations in $\begin{bmatrix} 0 & -\Gamma_{1,3} & \Gamma_{0,3} \end{bmatrix}$. Then we have that $\mathbb{E}_3\mathbf{y}_{2,4}$ is defined in terms of $\mathbf{y}_{1,3}$ and $\mathbf{y}_{2,3}$ in the same way as implied by the system under the ZLB regime. In our application of the ZLB, this implies that there are some expectations that behave the same way as the structure when the interest rate is held constant. But this is not consistent with the final system implying a unique and determinate solution. Therefore, $\text{rank}(A) = 3 \times T$, as needed for A to be invertible.

F Illustration of algorithm with the three equation New Keynesian model

The example in this section illustrates the algorithm and the identification method in a simulation of a simple model where known forward guidance policies are specified.

F.1 The model

The log-linearized economy is summarized by the following three equations, where i_t is the nominal interest rate, y_t is detrended output and π_t is the rate of inflation. There are three autoregressive shocks, to permanent technology z_t, to demand ξ_t and to the pricing equation a_t. There is a monetary policy shock $\varepsilon_{i,t}$ when the interest rate is positive. All variables are expressed as deviations from their steady-state values. The equations are, first, the Euler equation:

$$y_t = \mathbb{E}_t[y_{t+1}] - (i_t - \mathbb{E}_t[\pi_{t+1}]) + (1 - \rho_\xi)\xi_t, \tag{F.1}$$

second, the pricing equation:

$$\pi_t = \beta\mathbb{E}_t[\pi_{t+1}] + \kappa[y_t - a_t], \tag{F.2}$$

and third, the policy rule, subject to the ZLB:

$$i_t = \max\left\{-i_{ss},\ \rho_i i_{t-1} + \phi_\pi \pi_t + \phi_g(y_t - y_{t-1} + z_t) + \varepsilon_{i,t}\right\}. \tag{F.3}$$

Since i_t is expressed as a deviation from steady-state, the ZLB binds when $i_t = -i_{ss}$ where $i_{ss} = \frac{\pi z}{\beta}$, with π^* and z^* being the inflation target and steady-state permanent rate of technology growth. The markup shock is autoregressive:

$$a_t = \rho_a a_{t-1} + \varepsilon_{a,t},$$

as is the shock to technology:

$$z_t = \rho_z z_{t-1} + \varepsilon_{z,t},$$

and the demand shock:

$$\xi_t = \rho_\xi \xi_{t-1} + \varepsilon_{\xi,t}.$$

F.2 Simulation

The model is calibrated to values that reflect the estimated parameters in analogous New Keynesian models (as in, for example, Ireland, 2004). The following table gives the baseline calibration.

β	z	π	κ	ρ_i	ϕ_π	ϕ_g	ρ_a	ρ_z	ρ_ξ	σ_ξ	σ_a	σ_i	σ_z
.99	1.0025	$1.01^{1/4}$	.2	.8	1.7	.1	.8	.2	.8	.04	.01	.003	.01

The inflation target is set to 1 per cent to lower the steady-state nominal interest rate and make the ZLB more likely to be visited. Demand and pricing shocks are relatively more persistent than technology shocks and demand shocks are large compared to the other disturbances. Monetary policy reacts stronger to deviations in inflation from target as compared to the growth rate of output.

Figure 9 plots a simulated series of the interest rate, inflation and output generated by the model and a random set of shocks. The figure plots three series. The series labelled 'endogenous durations' is the series that transpires under the ZLB algorithm if the nominal interest rate was simply constrained by the ZLB, and becomes positive as soon as the policy rule requires it. The series labelled 'no ZLB' is the simulated series where the ZLB is unconstrained. I plot the unconstrained path as a comparison to the third series labelled 'forward guidance', which is the simulated path where the central bank announces and commits to a path for the interest rate at zero for a period of time. The path the central bank commits to, as summarized by a sequence of anticipated durations, is illustrated in Figure 10 and discussed below.

This particular set of shocks drives the nominal interest rate to the ZLB in period 13 and keeps it there until period 16. In period 17, the shocks lead the central bank to raise the interest rate, but it hovers around zero until period 22. Under the forward guidance path, the interest rate stays at zero until period 19 and returns to the ZLB for a single period at period 21. The decline in output and inflation is much more pronounced in the endogenous durations path as compared to the path under active forward guidance.

The durations that agents in the model expect the ZLB to bind at each period are plotted in Figure 10. The first panel shows the anticipated durations for the endogenous durations case. The shocks keep the anticipated duration at 2 periods in both periods 14 and 15. This stability in the anticipated duration corresponds to a decline in inflation and output from period 14 to period 15, so that this particular sequence of shocks pushes out the ex-post ZLB exit date by one period.

The second panel of Figure 10 shows the anticipated durations under the forward guidance policy. The central bank announces that it will abandon its policy rule for the specified duration and credibly commit to holding the nominal interest rate at zero for that duration. This announcement could, in practice, take any value. However, if the announced path is less than or the same as the endogenous duration, the announcement has no aggregate effects because the announcement confirms agents' expectations of the ZLB duration. Furthermore, to ensure an announcement policy is time-consistent, if the central bank commits to a particular duration T, then in the next time period I require it to commit to a duration which is at least as high as $T-1$. While in some sense arbitrary, the forward guidance values chosen for this exercise can be rationalized with a policy rule which activates when the interest rate becomes zero and which is

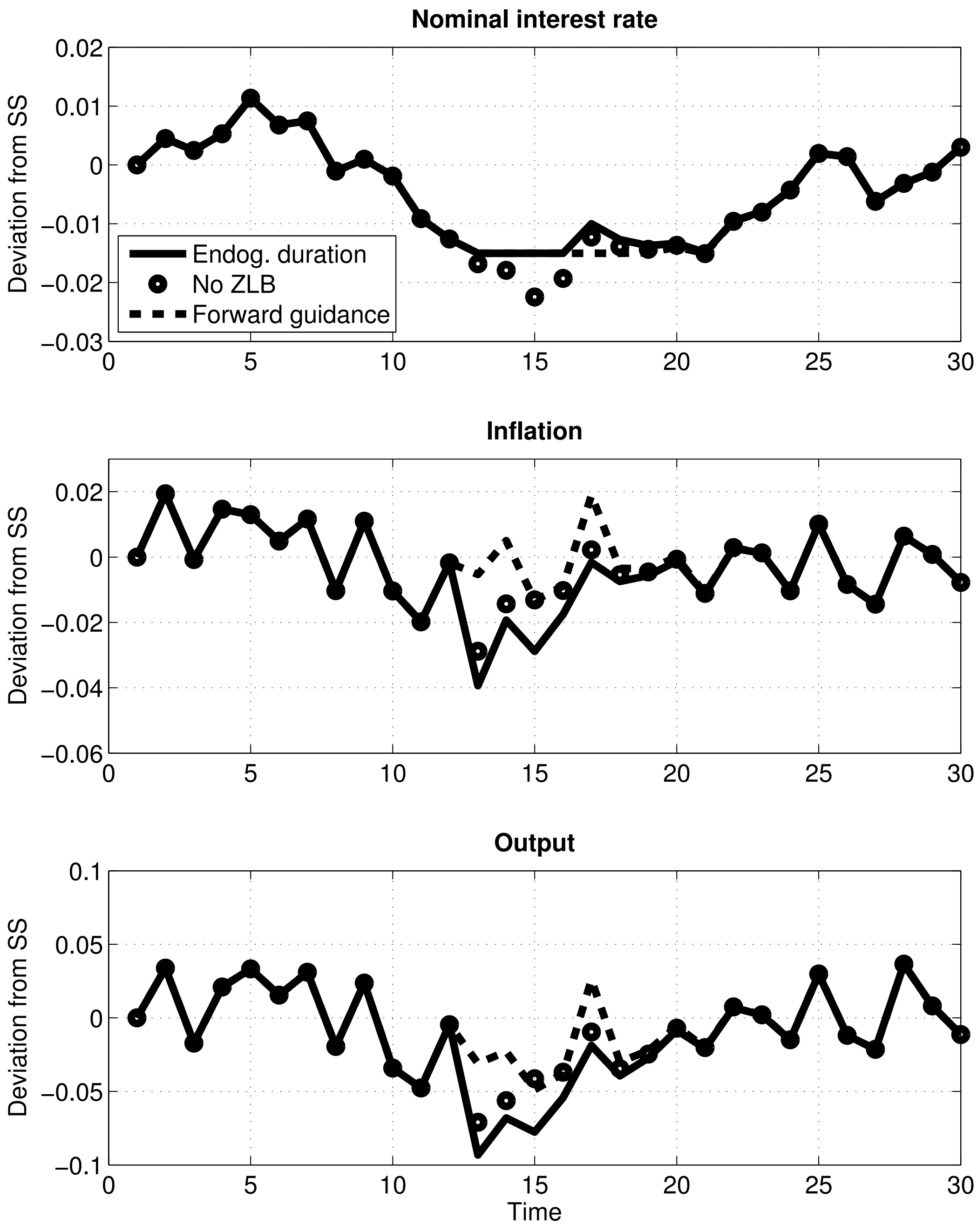


Figure 9: Simulation. This figure shows the interest rate, inflation, and output under a set of random shocks, when the interest rate is subject to the ZLB (without forward guidance), when there is no ZLB imposed, and when the central bank announces a sequence of interest rate ZLB durations above those implied endogenously by the shocks.

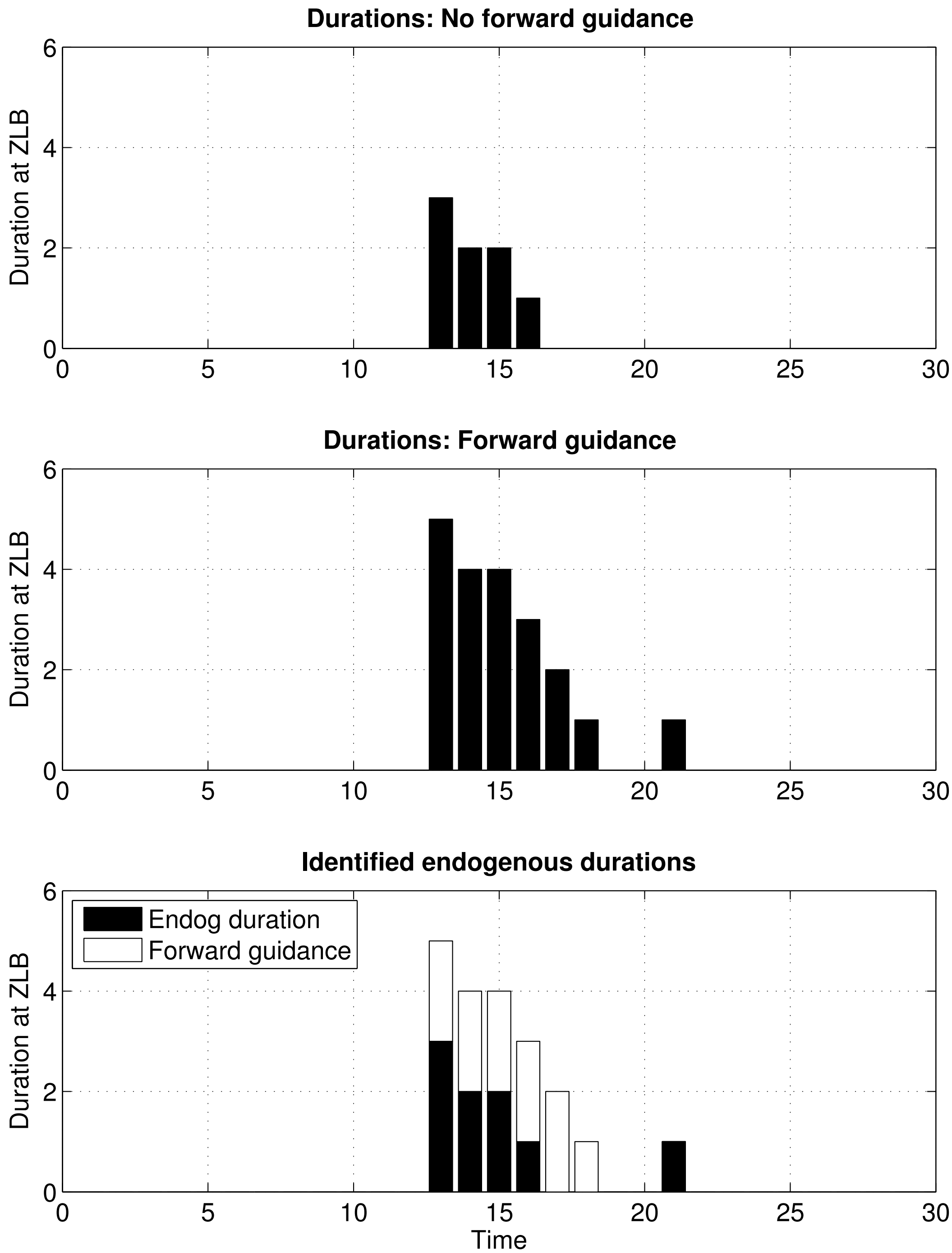


Figure 10: Estimated and decomposed durations. The top two panels of this figure show the anticipated durations of the ZLB in the 'endogenous durations' (no forward guidance) and 'forward guidance' simulations. The third panel shows the outcome of the identification procedure using the forward guidance durations.

tied to the rate of inflation that would arise in the absence of forward guidance policies. An example rule with this feature is outlined and studied in section 4.

F.3 Illustrating the decomposition

The aggregate consequences of forward guidance announcements, as plotted in Figure 9, illustrates how the anticipated duration series is a crucial variable when the interest rate is at zero. Under the forward guidance path, both output and inflation lie well above the paths that arise when no forward guidance is used. Information on the ZLB durations together with the observable variables can be submitted to the ZLB algorithm to identify how stimulatory those announcements are in practice.

To illustrate this, assume we observe the interest rate, inflation and output series under the 'forward guidance' series, and we estimate the anticipated ZLB durations under these series. Suppose the procedure perfectly estimates the parameters and the ZLB durations, so that they exactly equal the durations shown in the second panel of Figure 10. The smoother provides an estimate of the structural shocks $\{w_\tau\}_{\tau=1}^{T}$. Using the estimated structural shocks, the model and the ZLB algorithm, the third panel of Figure 10 illustrates the estimated ZLB durations as decomposed into the component due to structural shocks and the component due to active forward guidance policy. The identification procedure finds that forward guidance contributed an additional two periods to each duration from time periods 13 to 17, and an additional period in time period 18.

Figure 11 illustrates the identification procedure period-by-period. The first time period the ZLB binds is in period 13, plotted in the first panel. At this period, the estimated duration is five periods. Also plotted is the forecast of the interest rate that is implied by the state x_{t-1} and structural shock w_t at time period 13, labelled the 'shadow rate'. The duration is at zero for two periods longer than the shadow rate is at zero, so that the identified forward guidance component is two periods. Most strikingly, in time period 17, the shadow rate is positive for the two periods that the ZLB is estimated to bind.

The example simulation makes it clear that the identified endogenous duration is not simply the difference between durations that arise when there is no forward guidance and the durations that arise under forward guidance (or the first two panels of Figure 10). This is because each announced duration under forward guidance changes the state vector at that period, and the different path of the state vector could change the endogenous binding of the ZLB. This is why, in the simulation, the ZLB also binds for one period endogenously in period 21–because the interest rate was kept at zero for longer under the forward guidance simulation, the desire of the central bank to smooth interest rates keeps it closer to zero relative to the no forward guidance simulation, so that a deflationary shock is more likely to make the ZLB bind after $t = 18$.

G Comparison to non-linear approximation

To show that the method provides a good approximation, here I compare the output of the algorithm to a non-linear approximation of a simplified version of the Ireland (2004) economy. The non-linear model solved consists of an equation relating consumption c_t to output y_t and includes a term for price adjustments costs:

$$y_t = c_t + \frac{\phi}{2}\left(\frac{\pi_t}{\pi} - 1\right)^2 y_t,$$

where π_t is the inflation rate. The Euler equation is:

$$\frac{1}{\beta}\frac{1}{i_t}\frac{a_t}{c_t} = \mathbb{E}_t\left[\frac{a_{t+1}}{c_{t+1}}\frac{1}{\pi_{t+1}}\right],$$

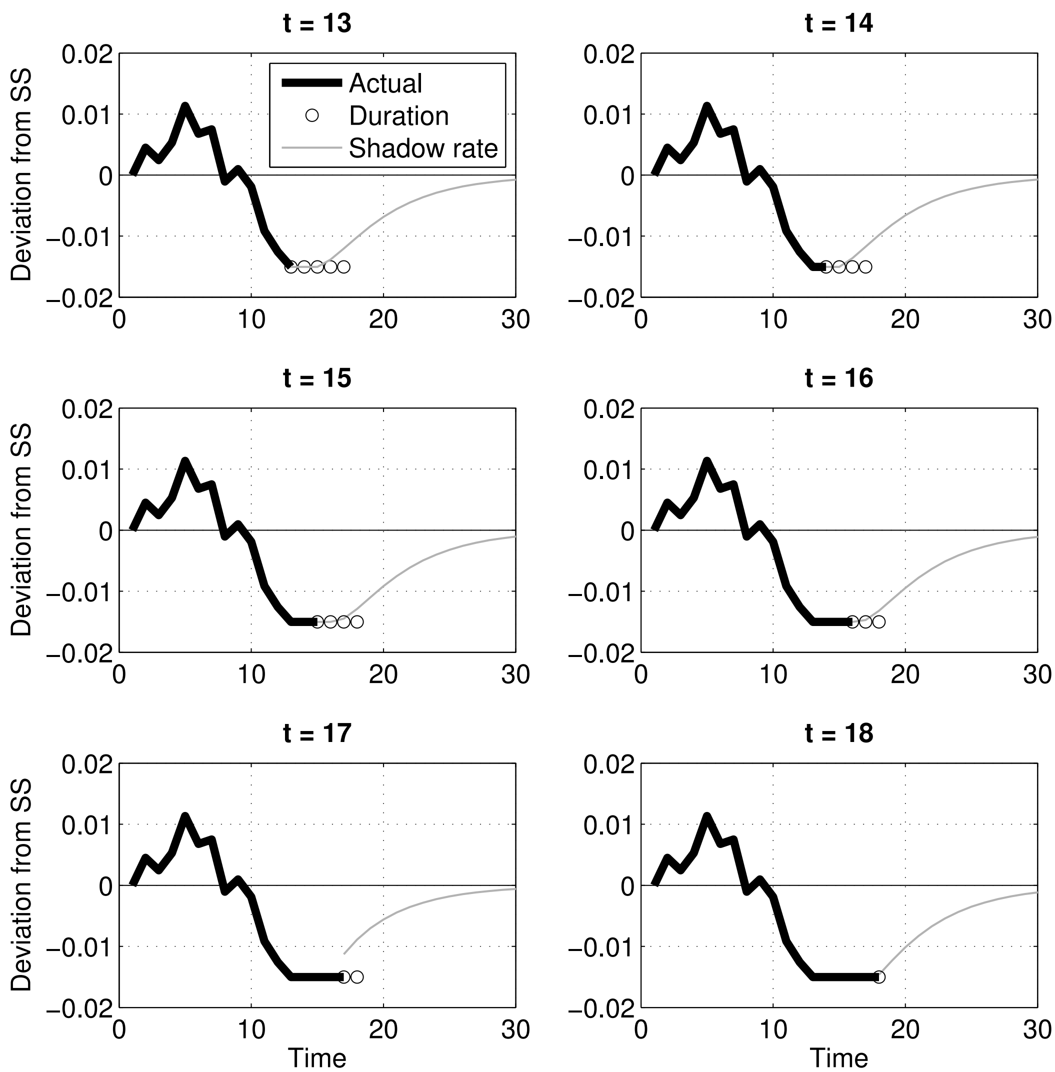


Figure 11: Interest rates during the ZLB episode. The figure shows, from time periods 13 to 18, forecasts of the interest rate under the estimated duration, and forecasts of the shadow interest rate. Forward guidance is active when the shadow rate forecast lies above the path under the estimated duration.

where i_t is the gross nominal interest rate, a_t is a demand shock, and z_t is a permanent productivity shock, and an equation derived from intermediate goods producing firms optimal price adjustment:

$$\mathbb{E}_t\left[\frac{a_{t+1}}{a_t}\frac{y_{t+1}}{c_{t+1}}\left(\frac{\pi_{t+1}}{\pi}-1\right)\frac{\pi_{t+1}}{\pi}\right]=\frac{1}{\beta\phi}\frac{y_t}{c_t}\left[\theta_t-1-\theta_t\left(\frac{c_t}{a_t}\right)y_t^{\eta-1}+\phi\left(\frac{\pi_t}{\pi}-1\right)\frac{\pi_t}{\pi}\right],$$

where θ_t is a shock to intermediate goods producing firms' desired markup. The central bank follows a Taylor rule. The ZLB on the gross nominal interest rate requires $i_t > 1$, so the rule becomes:

$$i_t = \max\left[1, \pi_t^{\rho_\pi} g_t^{\rho_g} x_t^{\rho_x}\right],$$

where g_t is the growth rate of output from $t-1$ to t:

$$g_t = \frac{y_t}{y_{t-1}} z_t,$$

and x_t is the efficient level of output:

$$x_t = \frac{y_t}{a_t^{1/\eta}}.$$

The demand shocks follows an autoregressive process:

$$\ln(a_t) = (1-\rho_a)\ln(a) + \rho_a \ln(a_{t-1}) + \varepsilon_{a,t}.$$

The two state variables of the model are a_{t-1} and y_{t-1}. To approximate the solution, I follow the exposition of Fernández-Villaverde et al. (2012). I first discretize a_t with a Tauchen approximation. The solution will be in terms of two functions $f_1(y_{t-1}, a_t)$ and $f_2(y_{t-1}, a_t)$ which approximate the expectations given by $\frac{1}{\beta}\frac{1}{i_t}\frac{a_t}{c_t}$ and $\frac{a_t}{\beta\phi}\frac{y_t}{c_t}\left[\theta-1-\theta\left(\frac{c_t}{a_t}\right)y_t^{\eta-1}+\phi\left(\frac{\pi_t}{\pi}-1\right)\frac{\pi_t}{\pi}\right]$ respectively. First, I constrain y_{t-1} to lie between 0.95 and 1.05 times the steady-state value of y. Together with the discretized a_t, this gives a grid $\{\mathbf{y}, \mathbf{a}\}$. I use the guess-and-verify method to approximate the policy functions across that grid. The algorithm is:

1. Guess the values of $f_1(s^i)$ and $f_2(s^i)$ at each point $s^i \in \{\mathbf{y}, \mathbf{a}\}$. Call these guesses $\hat{f}_1(s^i)$ and $\hat{f}_2(s^i)$.

2. Using the guesses $\hat{f}_1(s^i)$ and $\hat{f}_2(s^i)$:

   for each i (for each state):

   Guess π_t . Using the guess for π_t:

   (a) Determine the ratio $\frac{c_t}{y_t}$.
   (b) Using the ratio $\frac{c_t}{y_t}$ and the guess $\hat{f}_2(s^i)$, obtain y_t.
   (c) Using y_t, obtain c_t, a_t, g_t and then i_t, where i_t is calculated subject to $i_t = 1$.
   (d) Using $\hat{f}_1(s^i)$ and i_t, compute the implied consumption and call it $\widetilde{c}_t$.

   Check the computed $\widetilde{c}_t$ against c_t. Update guess of π_t until $|\widetilde{c}_t - c_t|$ converges.

3. With the equilibrium policy functions at time t, compute the expectations:

$$\mathbb{E}_t\left[\frac{a_{t+1}}{c_{t+1}}\frac{1}{\pi_{t+1}}\right] \quad \text{and} \quad \mathbb{E}_t\left[a_{t+1}\frac{y_{t+1}}{c_{t+1}}\left(\frac{\pi_{t+1}}{\pi}-1\right)\frac{\pi_{t+1}}{\pi}\right],$$

   using the transition matrix for the discretized θ_t. Adjust the guesses $\hat{f}_1(s^i)$ and $\hat{f}_2(s^i)$ until they converge with the computed $f_1(s^i)$ and $f_2(s^i)$.

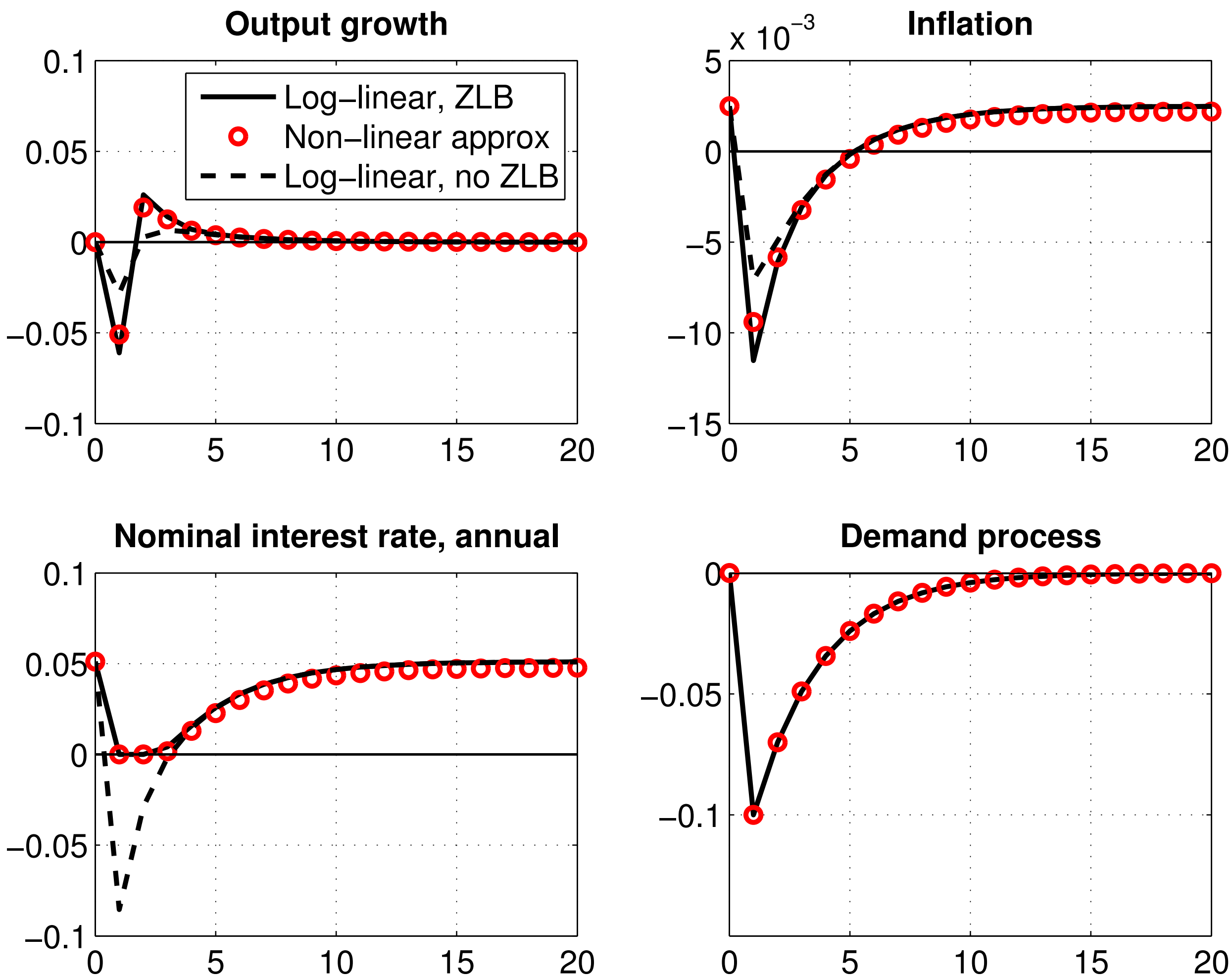


Figure 12: Comparison of ZLB algorithm and non-linear approximation. This figure plots the impulse response to a negative five standard deviation demand shock.

The algorithm approximates policy functions for all the endogenous variables.

I use the following calibration, which mirrors the calibrated and estimated results of Ireland (2004). The inflation target is set to 1 per cent per year so that the ZLB is more likely to bind following a negative demand shock.

β	ϕ	η	ρ_π	ρ_g	ρ_x	ρ_a	σ_a	θ	π
0.99	200	1/0.06	2.5	0.3	0.1	0.7	0.02	$0.1\frac{\phi}{\eta}+1$	$1.01^{1/4}$

Figure 12 compares the impulse response to a large negative five standard deviation demand shock under the non-linear approximation and the ZLB algorithm. Both methods have similar profiles for output growth and inflation, and show that the ZLB binds for two periods following the shock.

Figure 13 shows the two approximations have similar predictions for the stochastic paths of endogenous variables under 1000 simulations using the same set of shocks for both approximations.

G.1 Severity of shocks at the ZLB

I use the implementation to analyse the severity of shocks when the nominal interest rate is at the ZLB. Figure 9 showed that, when the central bank cannot stabilize a large negative demand shock pushing the

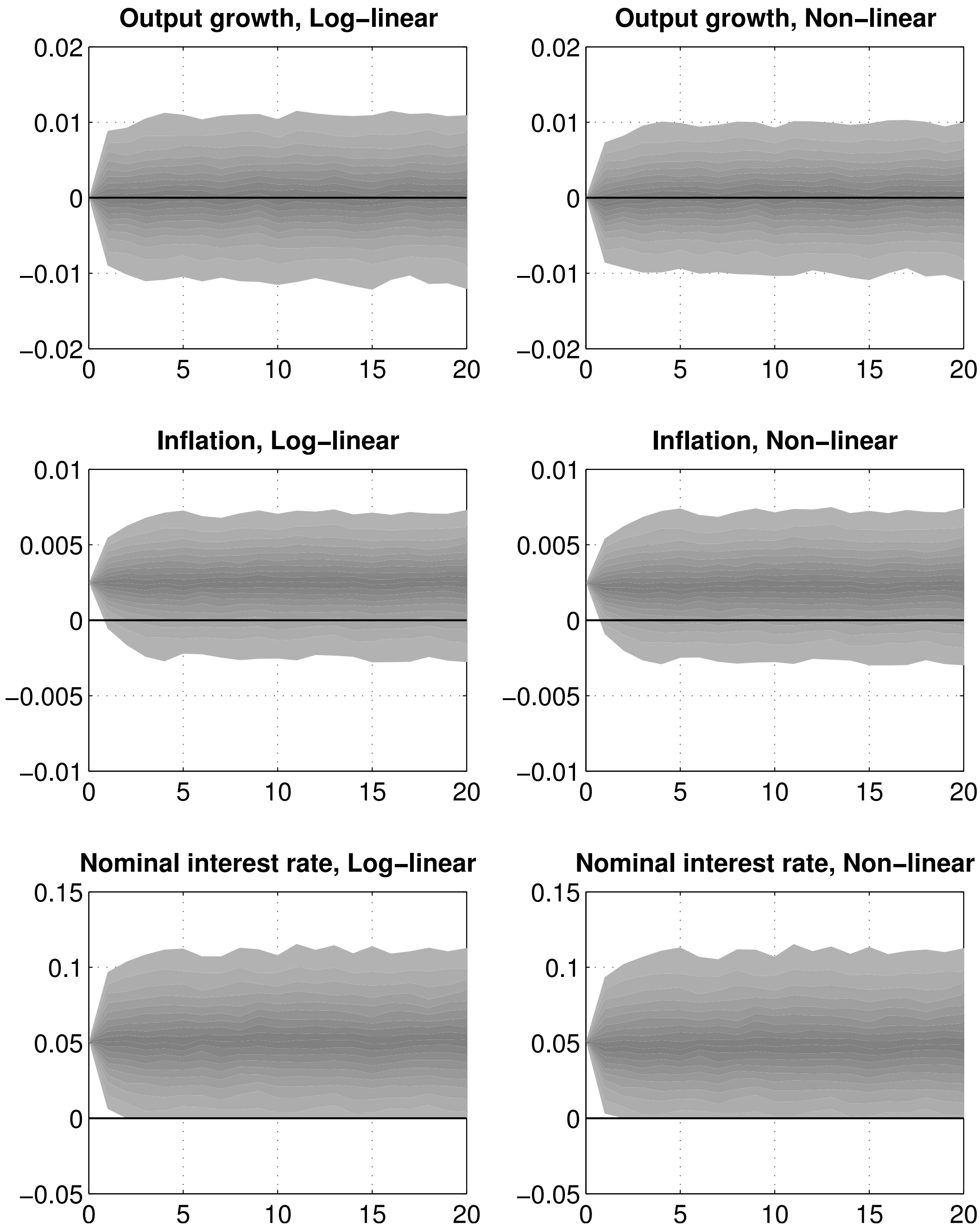


Figure 13: Comparison of ZLB algorithm and non-linear approximation over time. This figure plots fancharts for 1000 simulations of the log-linear and non-linear approximations under the same set of demand shock.

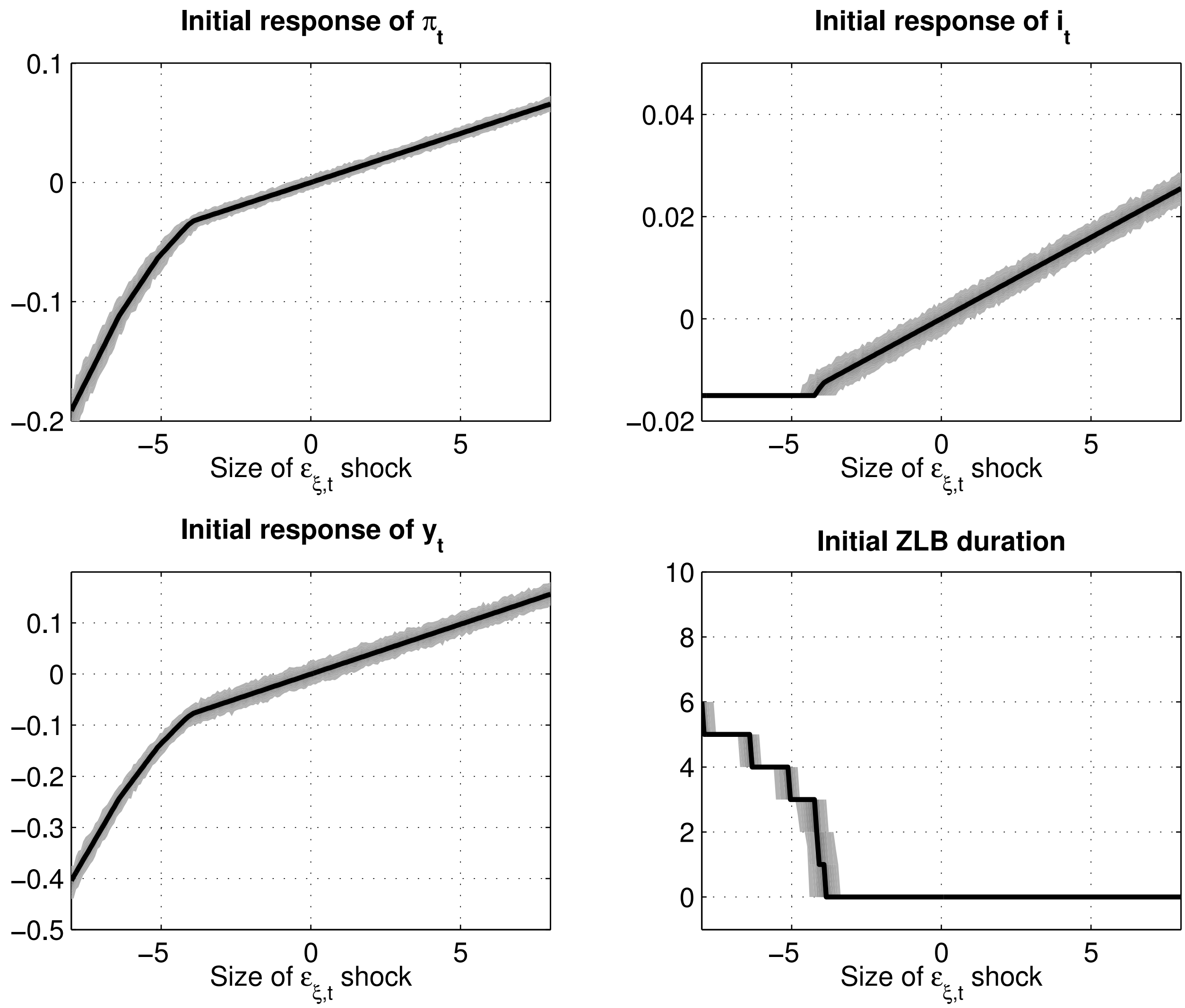


Figure 14: Initial response of variables to $\varepsilon_{\xi,t}$ shocks. This figure shows the initial responses of inflation, the nominal interest rate and output in response to demand shocks of varying size. The fanchart illustrates the band of the initial response when the economy is hit by other stochastic shocks.

nominal interest rate to its lower bound, the inflation and output decline is pronounced. I investigate this further in Figure 14 by plotting the initial response of inflation following demand (ξ_t) shocks of varying size. The non-stochastic response is the bold black line. Also plotted are fancharts for the range of the initial response when the economy is subject to random shocks to the other stochastic variables.

For the region in which the ZLB does not bind, the initial response is linear. Beyond large, negative demand shocks, the ZLB binds and the decline in inflation and output is more severe with more negative shocks. The width of the fanchart also widens for both inflation and output when the ZLB binds. To make this clear, Figure 15 plots a normalized measure of the width of the bands around the initial response in Figure 14. This exercise illustrates how, when the central bank is constrained and unable to act against the shock with the nominal interest rate, the effect of unanticipated stochastic shocks is large. This is intuitive: policy functions for inflation and output are steeper at the ZLB relative to when the central bank can stabilize contractionary demand shocks, and so further shocks that impact when the nominal rate is at that bound moves the economy along those steeper functions.

To illustrate the severity of shocks which impact the economy at the ZLB over time, Figure 16 plots

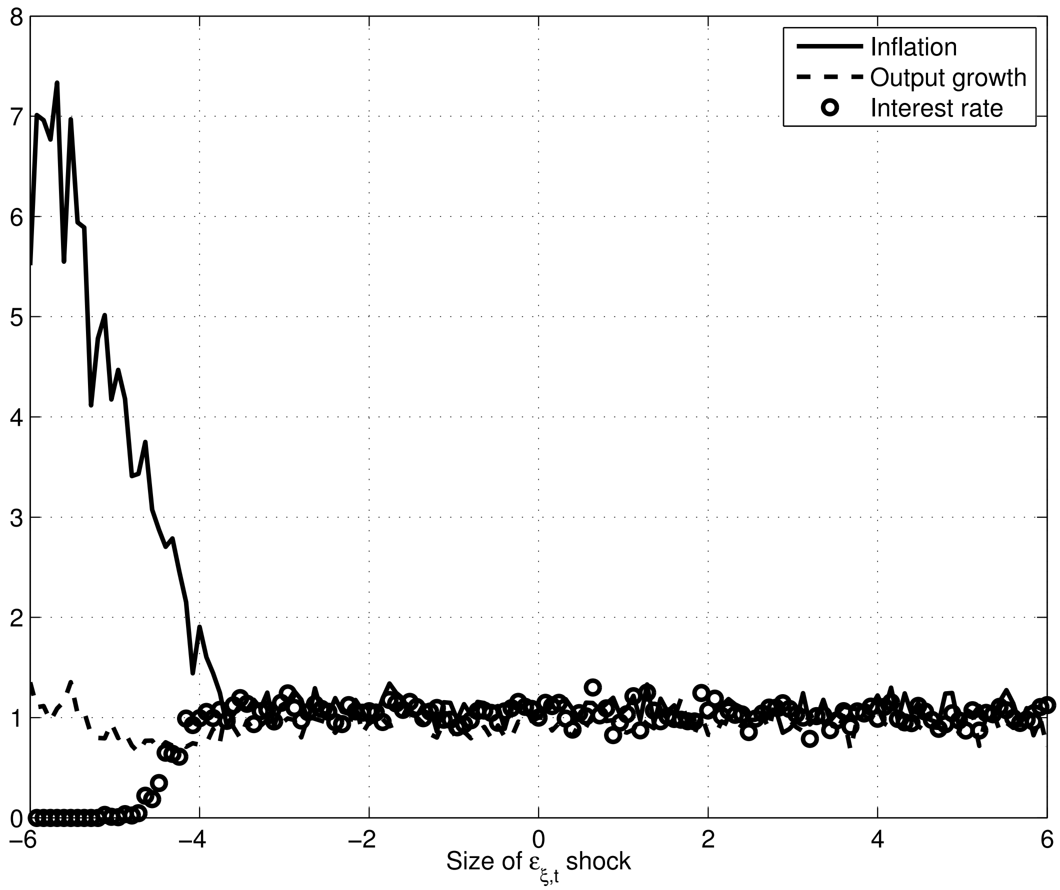
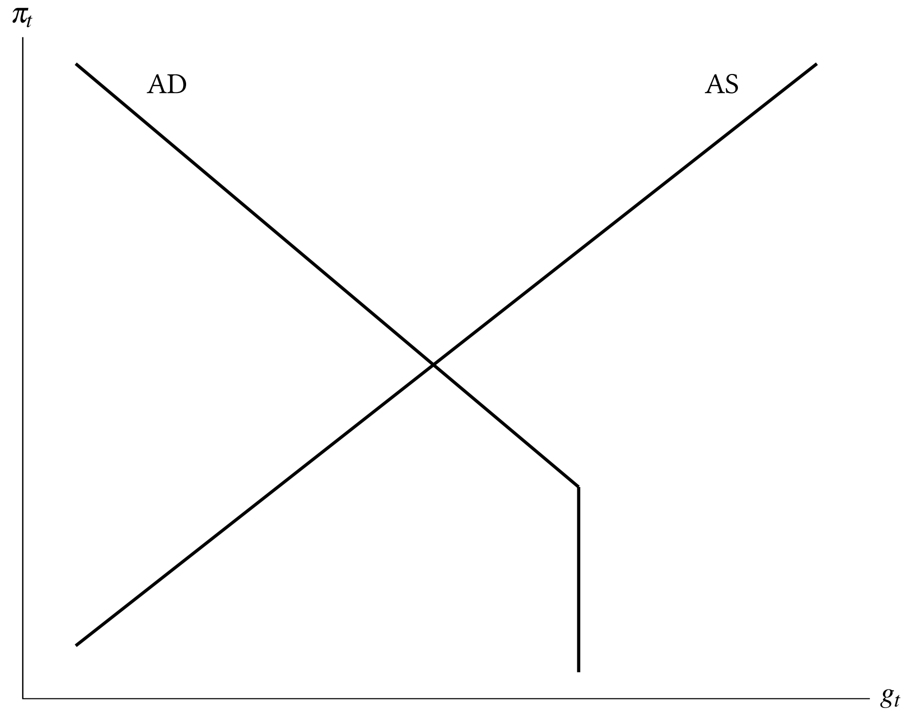


Figure 15: Standard Deviation of initial response of interest rate, inflation and output growth. The standard deviations of the interest rate, inflation and output growth are normalized by their respective standard deviations for $\varepsilon_\xi = 0$. The figure plots a normalized measure of the width of the fancharts around the initial responses of inflation, output growth and the interest rate, plotted in Figure 14.

AD/AS under ZLB



fancharts summarizing the range of the interest rate, inflation and output growth across 100 simulations of the model when the interest rate is subject to the ZLB, and for when it is not subject to the ZLB. The shocks are such that the economy is subject to three consecutive quarters of unanticipated negative two standard deviation risk premia shocks, in addition to random shocks. This pushes the nominal interest rate to its ZLB for an extended period. The black line gives the economy's path under three consecutive quarters of negative two standard deviation risk premia shocks and absent further stochastic shocks. The fancharts illustrate how, when the ZLB binds, some paths are particularly variable, so that the overall variance of inflation and output growth rises over the simulation period when compared to the variance in the no-ZLB simulation.

The figure below illustrates the aggregate demand (AD) and aggregate supply (AS) curves in the inflation-output growth space under the ZLB,[9] and motivates why shocks can increase the volatility of inflation and output growth, as a given shock affects inflation and output growth differently depending on whether the equilibrium lies on the sloped or vertical segment of the AD curve. The vertical component of the AD curve arises because monetary policy cannot manipulate the real interest rate when it is constrained by the ZLB. A given shift in the AS curve along the vertical component of AD can cause inflation to be more volatile, while a given shift in the AD curve when the equilibrium lies in the vertical component of the AD curve can cause both inflation and output growth to be more volatile.

H Aggregate demand and supply under ZLB

In this section, I derive the three equation New Keynesian model in the AD and AS framework as in Jones and Kulish (2016) but with the ZLB. Define output growth:

$$g_t = y_t - y_{t-1} + z_t. \tag{H.1}$$

[9] See the next section for the derivation of these curves in the three-equation New Keynesian model. If derived in the space of expected inflation $\mathbb{E}_t \pi_{t+1}$ and output as in Wieland (2014) the AD curve would be upward sloping in the ZLB region.

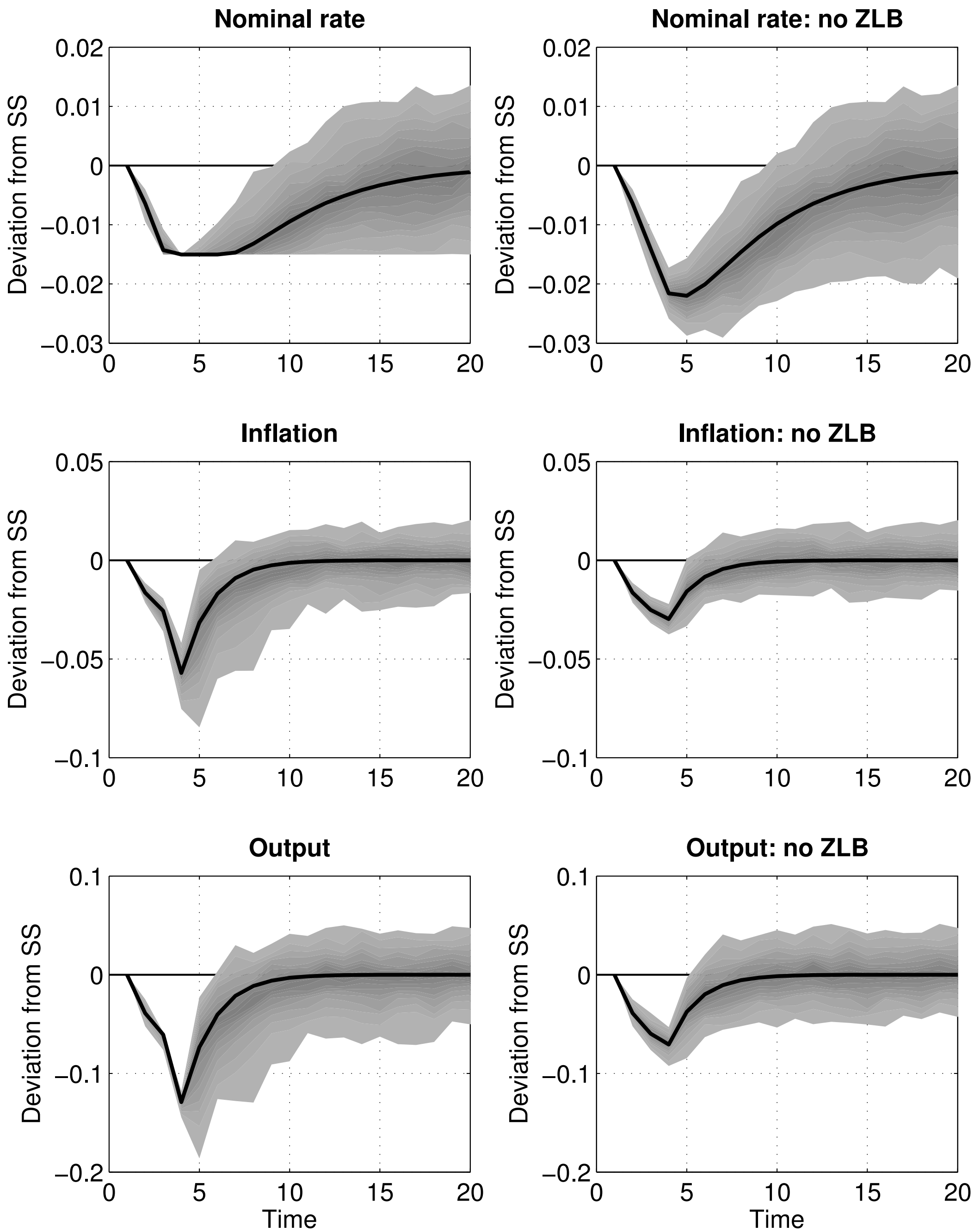


Figure 16: Unanticipated shocks each period. This figure plots fancharts of 100 simulations of the model economy when the ZLB binds and when it does not. Under the ZLB, the fancharts are wider, illustrating how shocks which hit at the ZLB can generate excess volatility.

To find the AS schedule, substitute equation (H.1) into (F.2) to get:

$$\pi_t = \kappa g_t + s_t + (\pi - \kappa g), \tag{H.2}$$

where $s_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa y_{t-1} - \kappa z_t - \kappa a_t$. In the space of contemporanous output growth and inflation (g_t, π_t), equation (H.2) expresses inflation as a linear function of output growth, with slope κ and intercept $s_t + (\pi - \kappa g)$. Note that the time-varying component of the intercept s_t is zero when the economy is on its growth path. Also note that the slope of the curve depends on the degree of nominal price rigidities. That is, as $\kappa \to \infty$ prices become fully flexible implying a vertical AS. Conversely as the cost of price adjustment rises, $\kappa \to 0$, so that AS flattens.

To obtain the AD schedule when the ZLB does not bind, substitute equations (F.3) and (H.1) into (F.1) to get:

$$\pi_t = -\left(\frac{1+\psi_g}{\psi_\pi}\right) g_t + d_t + \left(\pi + \left(\frac{1+\psi_g}{\psi_\pi}\right) g\right), \tag{H.3}$$

where $\psi_\pi d_t = -\rho_i i_{t-1} + \mathbb{E}_t y_{t+1} + \mathbb{E}_t \pi_{t+1} - y_{t-1} + z_t + (1-\rho_\xi)\xi_t - \varepsilon_{i,t}$. Note that, as for the time-varying intercept in the AS curve, when the economy is on its balanced growth path, d_t is zero. The slope of the curve (H.3) depends on the parameters of the policy rule. A greater response to deviations of inflation from target, ψ_π, flattens the curve. Vice versa, stronger responses to output growth, ψ_g, steepen the AD curve. The central bank uses changes in the nominal interest rate to affect growth and stabilize the inflation rate around the target.

The AD and AS curves reveal that shocks move both schedules simultaneously and so, to determine the overall effect of the shocks on π_t and g_t, we find their intersection. With the values of s_t and d_t in hand, the AS curve (H.2) and the AD curve (H.3) can be written as a system of two equations in two variables g_t and π_t. Inverting these equations:

$$\begin{bmatrix} \pi_t \\ g_t \end{bmatrix} = \begin{bmatrix} \pi + \frac{\kappa \psi_\pi}{1+\psi_g+\kappa\psi_\pi} d_t + \frac{1+\psi_g}{1+\psi_g+\kappa\psi_\pi} s_t \\ g + \frac{\psi_\pi}{1+\psi_g+\kappa\psi_\pi} d_t - \frac{\psi_\pi}{1+\psi_g+\kappa\psi_\pi} s_t \end{bmatrix}. \tag{H.4}$$

Now suppose the ZLB binds. The AD curve changes, while the AS curve is unchanged. To derive the new AD curve, notice that the rule (F.3) says that, at the ZLB, the parameters of the rule are equal to zero. This says the AD curve becomes vertical at a level of g_t determined by substituting (F.3) with the constraint binding, and (H.1) into (F.1) to get:

$$g_t = \widetilde{d_t} + g,$$

where $\widetilde{d_t} = -i_{ss} + \mathbb{E}_t y_{t+1} + \mathbb{E}_t \pi_{t+1} - y_{t-1} + z_t + (1-\rho_\xi)\xi_t$. The vertical component of the AD curve says that when the ZLB binds, the central bank cannot engineer an expansion of output by lowering the nominal interest rate.

To determine inflation and output growth at an equilibrium where the ZLB binds, write the AD and AS equations in inflation and output growth space, and solve:

$$\begin{bmatrix} \pi_t \\ g_t \end{bmatrix} = \begin{bmatrix} \pi + s_t + \kappa \widetilde{d_t} \\ g + \widetilde{d_t} \end{bmatrix}. \tag{H.5}$$

Comparing (H.4) with (H.5) reveals that shocks to e_t, a_t and z_t move both AD and AS simultaneously, directly through the shock and indirectly through expected inflation and output.

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