Paper Aeroplanes: A Dynamical Systems Perspective

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Adapted from Lab 5.4, p. 562 of "Differential Equations" by Blanchard, Devaney & Hall (2011).

Consider a glider flying in the xy-plane (see Figure 1 below). Let $s(t)$ be the speed of the glider along its path at time t and $\theta(t)$ be the angle of the velocity vector of the path with the x-direction at time t. Note that, since the body of the glider points in the direction of motion, θ is also the angle between the glider and the x-direction.

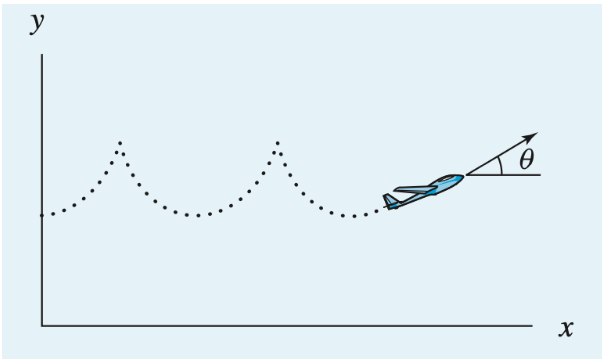

Figure 1: The angle θ for the motion of a glider.

The forces involved are gravity, lift provided by the wings (a force perpendicular to the velocity vector), and drag (a force parallel but in the opposite direction to the velocity vector). Using $F = ma$, we can obtain equations for d^2x/dt^2 and d^2y/dt^2 and then derive the system

$$\frac{d\theta}{dt} = \frac{s^2 - \cos\theta}{s} \tag{1a}$$

$$\frac{ds}{dt} = -\sin\theta - Ds^2 \tag{1b}$$

This model assumes that both the lift and drag are proportional to the square of the velocity.[1] The most remarkable thing about system (1) is that there is only one parameter, D. This parameter is the drag force caused by air resistance divided by the lift, the "drag-to-lift ratio." The designer of a glider tries to maximize lift while minimizing drag, so

[1]This model appears in the book *Theory of Oscillators* by A. A. Andronov, A. A. Vitt, and S. E. Khaikin, Dover Publishing, 1987.

the parameter D can be viewed as a measure of the quality of the design (a small value for D is preferred).

Your task is to prepare a comprehensive and well-illustrated report relating the flight and performance of the glider to the system of equations given by (1).

Your **group report** (5 pages max) should address the following questions:

1. Study the solutions of the system above with $D = 0$ (that is, for the perfect glider with zero drag). Are there equilibrium points? What is the physical interpretation of these?

2. Repeat your analysis for values of D between 0 and 4 (that is, for increasing drag-to-lift ratio). How do the phase portraits change as D changes? How do the possible glider paths change as D increases? Write some code to produce figures to illustrate your phase plane analysis and enhance your explanations.

3. Apply the theory of linearization[2] to classify all equilibria for values of D in the interval $0 \leq D \leq 4$. Determine the bifurcation values of D and write some code to plot a bifurcation diagram with D as the bifurcation parameter (indicate stability/instability visually).

4. Given a value of D and an initial condition (θ_0, s_0), the motion of the glider is determined by the equations (1). Write some code to solve the system (1) using an appropriate numerical method (justify your choice). Use this code to produce some figures illustrating the different types of solutions/flight conditions determined in your analysis in Parts 1, 2, and 3.

5. Why is it more natural to think of the "phase cylinder" for this system rather than the phase plane? What changes if you analyze the system using a phase cylinder in place of a phase plane? Modify your code from the previous questions to illustrate trajectories in the phase cylinder.

6. Construct a paper glider and relate test flights to your answers in Parts 1, 2 and 3. *Note:* Gliders made from higher-quality paper demonstrate the dynamics much better.[3]

Building on your **group report**, you should prepare an **individual extension** component of up to three pages (including figures). Marks will be awarded for creativity, originality, and quality and clarity of presentation. Some possible ideas to explore for your extension include:

- Incorporating factors such as wind, turbulence, or (random) noise into the model,
- Record more detailed data from your test flights and try to compare them to the model quantitatively, e.g., can you estimate D for a given glider design?
- The current model assumes that drag and lift are quadratic in speed. Investigate what happens if these forces depend on s differently (e.g., linear or with a saturation effect). How does this affect the dynamics?
- Suppose the glider can make slight adjustments to its angle during flight (like a

[2]You could consult Ch. 5 of *Differential Equations* by Blanchard, Devaney & Hall, 2011, or another appropriate reference on qualitative analysis of nonlinear differential equations.

[3]Depending on your experience in paper aeroplane design, you may wish to consult the appropriate literature, e.g., J. M. Collins, *The Gliding Flight*, Ten Speed Press, Berkeley, 1989.

bird or drone). Propose a simple feedback rule for $\theta(t)$ that might improve glide performance and simulate the trajectories numerically.

You are not limited to the ideas above. You are strongly encouraged to pursue your own interests and ideas — originality will be rewarded!