Paper Aeroplanes and Dynamical Systems

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Computational Mathematics II Individual Project Report

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Plagiarism declaration

> This piece of work is a result of my own work and I have complied with the Department's guidance on multiple submission and on the use of AI tools. Material from the work of others not involved in the project has been acknowledged, quotations and paraphrases suitably indicated, and all uses of AI tools have been declared.

I, **Callum Wright_Parish**, confirm that I have read, understood, and have adhered to the plagiarism declaration above.

AI Usage Declaration

- Google Gemini was used to debug code.
- Grammarly was used to enhance the writing style, grammar, and spelling of the report throughout.

Link to GitHub Codebase

All codes used to produce the analysis and results in this report can be found at the GitHub repository: https://github.com/callumpw06/Individual_Project.

In the group project, we focused on the flight of paper aeroplane, using a simple model. However, this has limited real-world applications, which is why I want to focus on making this model more widely applicable to learning about the characteristics that determine the flight of an aeroplane. It is important to note that this model will ignore many of the complex intricacies of aerodynamics, as we are more interested in the behaviour of the plane under different conditions.

1 Additional Force: Thrust

Our model can be extended to account for an additional force, thrust, to affect the aeroplane's trajectory. This changes our model to be defined by the following:

$$\frac{d\theta}{dt} = \frac{s^2 - \cos\theta}{s} \quad (1) \qquad \frac{ds}{dt} = -\sin\theta - Ds^2 + T_{thrust} \quad (2)$$

where $s(t)$, $\theta(t)$ are defined as previously, while T_{thrust} is the additional force, thrust. This more accurately models a plane capable of sustained flight, such as a commercial airliner or fighter jet. We also previously studied the equilibrium points in our (θ, s)-space. Therefore, I would like to investigate how these equilibrium points change with the introduction of this additional force. Again, we let $\frac{d\theta}{dt} = \frac{ds}{dt} = 0$ when $(\theta, s) = (\theta^*, s^*)$:

$$0 = \frac{(s^*)^2 - \cos\theta^*}{s^*} \implies (s^*)^2 = \cos\theta^* \quad (3)$$

$$0 = -\sin\theta^* - D(s^*)^2 + T_{thrust} \implies T_{thrust} = \sin\theta^* + Dcos\theta^* \quad (4)$$

If we want our plane to ascend at an angle θ, for example, it would require a thrust of at least $\sin\theta + D\cos\theta$. This naturally raises questions about what happens if our plane is capable of producing thrust greater than this amount. In theory, our plane could ascend indefinitely, which is not realistic because lift and thrust decrease with altitude as air density decreases, according to [1]. For simplicity, we will model air density and thrust exponentially decaying with altitude, leading to the following system:

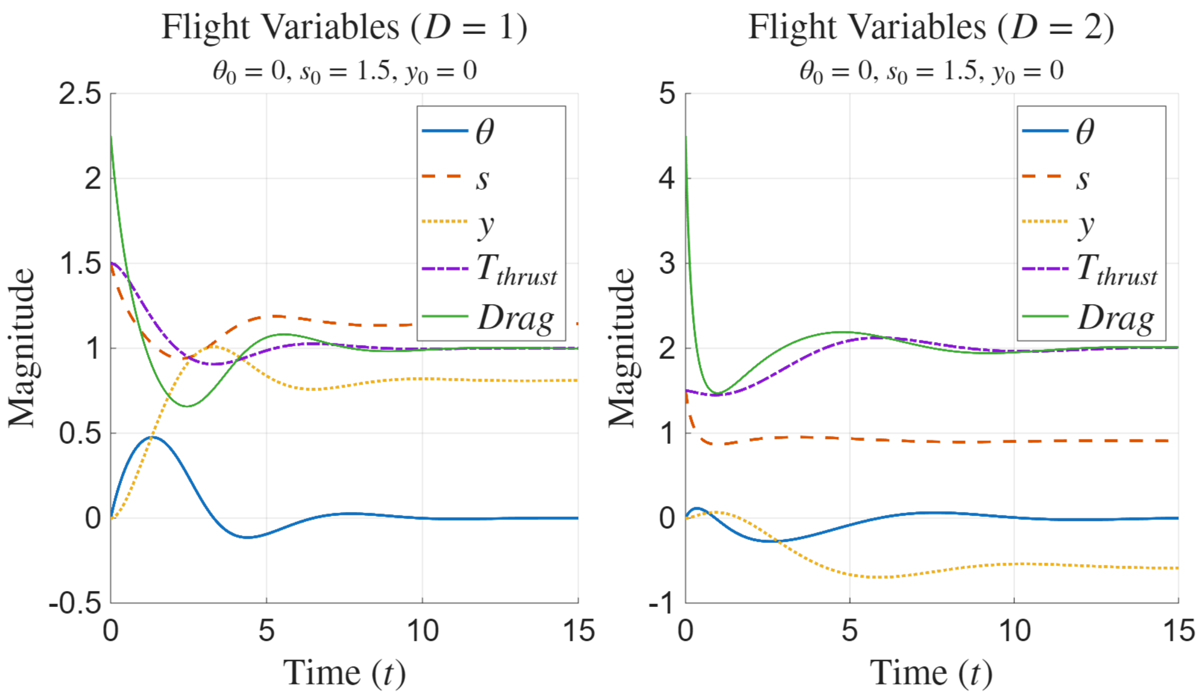
$$\frac{d\theta}{dt} = \frac{s^2 e^{-y/H} - \cos\theta}{s} \quad (5)$$

$$\frac{ds}{dt} = -\sin\theta - Ds^2 e^{-y/H} + T_{y=0} e^{-ky/H} \quad (6)$$

$$\frac{dy}{dt} = s\sin\theta \quad (7)$$

where $y(t)$ is the plane's altitude. $H > 0$ is a constant determining how quickly the air density decreases, and $k > 0$ is a constant determining how quickly thrust decreases with altitude. Now, we can determine the equilibrium conditions (θ^*, s^*, y^*) leading to a straight flight, using the `ode45` solver.

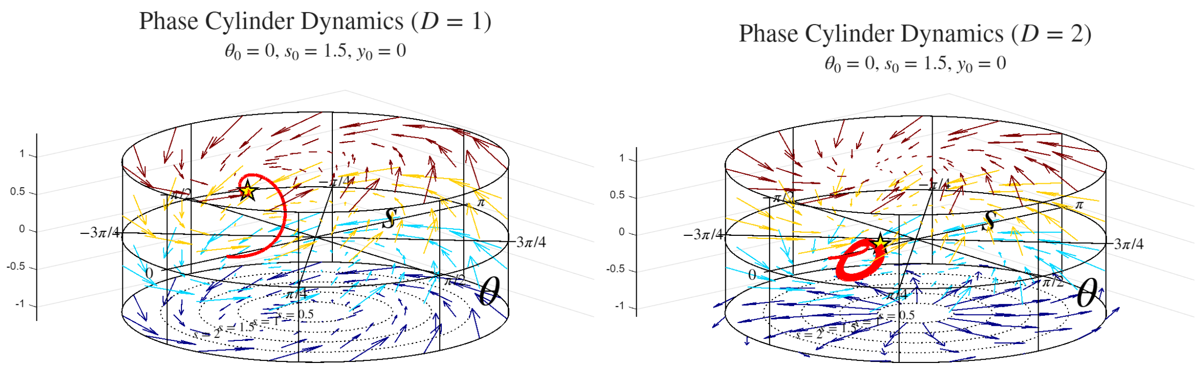
Firstly, we will study the plane's behaviour in flight for varying values of D, reflecting different aeroplane designs in the real world. This will enable us to interpret how the aeroplane trajectory is affected by both our drag-to-lift ratio and thrust. For ease of interpretation, we will study the values of the system variable for $D = \{1, 2\}$, $k = 1.5$, $H = 3$, $T_{y=0} = 1.5$ and initial conditions $(\theta_0, s_0, y_0) = (0, 1.5, 0)$ - the same figures for other values of D can be found on the GitHub.



(a) System Variables for $D = 1$. (b) System Variables for $D = 2$.

Figure 1: System variables over time for $D = \{1, 2\}$.

Each variable settles into an equilibrium over time, which is expected for an aircraft producing constant thrust. This equilibrium point, (θ^*, s^*, y^*), marks the conditions that achieve the most efficient flight as it maximises speed whilst minimising drag. The plane with $D = 1$ is the only plane to gain altitude, as the other planes are not producing enough thrust to get up to speeds to generate enough lift to gain altitude. It can be more natural to consider our parameters in a cylindrical coordinate basis, as it allows for θ to be periodic, and restricts $s > 0$. Hence, we can plot our system as a phase cylinder describing how these parameters change throughout the flight.



(a) Phase Cylinder for $D = 1$. (b) Phase Cylinder for $D = 2$.

Figure 2: Phase Cylinders describing our system for $D = \{1, 2\}$.

A plane with smaller D-value quickly tends towards its equilibrium position (the star), whilst larger values of D oscillate more, with increasing amplitude (see GitHub), and take longer to decay to their equilibrium points in the (θ, s, y)-space; reflected in Figure 1 by the lengthened period in oscillations of each variable for larger D. Hence, introducing these additional factors still leads to equilibrium points, just at different altitudes.

2 Even Faster: Supersonic Flight

We can take this model one step further to investigate the behaviour of our plane at supersonic speeds. Once an aircraft breaks the speed of sound, drag increases, and lift efficiency decreases. This means our drag-to-lift ratio is now a function of speed, $D(s)$, and lift is no longer proportional to s^2. Using Ackeret's theory [4][5], this leads to the following updated model at supersonic speeds:

$$\frac{d\theta}{dt} = \frac{4(\alpha-\theta)se^{-y/H}}{\sqrt{(s/s_v)^2-1}} - \frac{\cos\theta}{s} \tag{8}$$

$$\frac{ds}{dt} = -\sin\theta - D(s)s^2e^{-y/H} + T_{y=0}e^{-ky/H}, \quad D(s) = D_{sub} + \frac{D_{super}-D_{sub}}{1+e^{-b(s-s_v)}} \tag{9}$$

$$\frac{dy}{dt} = s\sin\theta \tag{10}$$

where b is another decay constant, D_{sub} and D_{super} are drag-to-lift ratios at sub- and supersonic speeds, s_v is the speed of sound, and α is a pilot-inputed trim. This system is valid for $s > s_v$, and models the drag-to-lift ratio increasing and approaching a larger value, $D_{super} > D_{sub}$.

The most important factor when travelling at supersonic speeds is whether an aircraft is stable - does it return to equilibrium with small perturbations? We can study this by applying the theory of linearisation[2] and finding the Jacobian matrix:

$$J = \begin{pmatrix} \frac{-4se^{-y/H}}{\sqrt{(s/s_v)^2-1}} + \frac{\sin\theta}{s} & -\frac{4(\alpha-\theta)se^{-y/H}}{((s/s_v)^2-1)^{3/2}} + \frac{\cos\theta}{s^2} & \frac{4(\alpha-\theta)se^{-y/H}}{H\sqrt{(s/s_v)^2-1}} \\ -\cos\theta & -(2sD(s)+s^2D'(s))e^{-y/H} & \frac{1}{H}(D(s)s^2e^{-y/H} - kT_{y=0}e^{-y/H}) \\ s\cos\theta & \sin\theta & 0 \end{pmatrix}$$

with eigenvalues λ_i, $i = 1,2,3$. If $\max(\mathrm{Re}(\lambda_i)) < 0$, then the plane is stable[3].

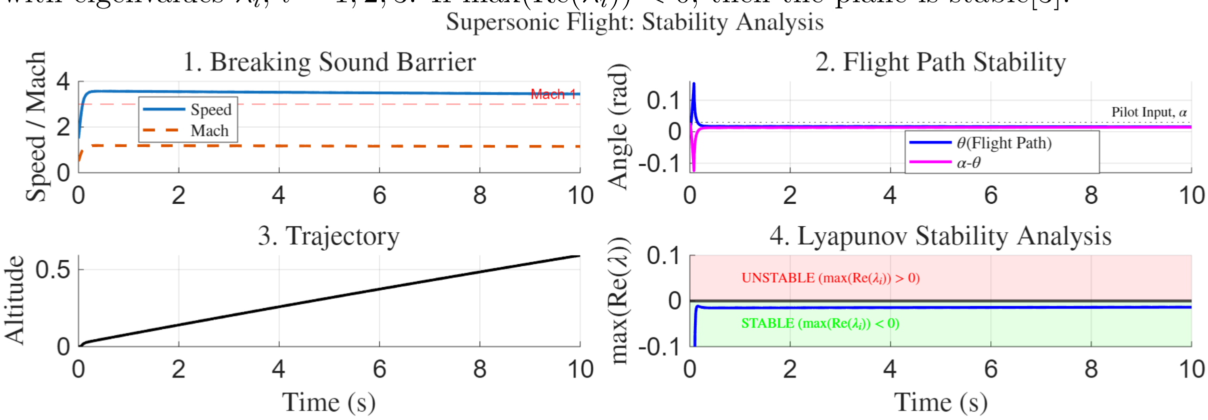


Figure 3: Supersonic flight: $T_{y=0} = 25; \alpha = 0.03; s_v, H = 3; D_{sub} = 1; D_{super} = 2; b = 5; k = 1.5$.

To reach these much higher speeds, our aircraft has to produce much more thrust than at subsonic speeds. When we compare this to Figure 1, we can see that it becomes a lot more difficult to gain altitude at these speeds, reflecting the reduced lift efficiency. However, through plots 3.2 and 3.4, we can conclude that we have a stable flight using this model. Using other aircraft designs (different values of D) may not lead to stable flight.

We have effectively analysed the behaviour of an aircraft in a more developed model, concluding that this leads to more complex flight dynamics. However, this report abstracts some physics principles - a more in-depth model is available in the GitHub repository.

References

[1] How altitude affects aircraft performance – what every aviation enthusiast should know. Aeropeep, 2025.

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[4] NACA. Naca report 851. Technical report, NACA, 1958.

[5] S. University. Compressible Thin Airfoil Theory, chapter 13. Stanford University, 2013.