Contents

1 Summary Paper

To better provide a narrative on the concept of closeness and how it has been developed throughout this course we will address its development as this class did. We will describe closeness as a pseudo-equality; that is to say that closeness is related to equality in that objects can only be truly "close" if they are "equal" to each other. However,as we once had gaps in our knowledge filled, we must make sure that that what we are looking at actually exists.

We begin our journey in a way contradicting this idea of closeness; to ensure that two numbers can get close to each other we need to ensure that those numbers actually exist and the numbers between exist. This leads us to the set of all real numbers $\mathbb{R}$. As Pythagoras demonstrated long ago there were gaps in the rational numbers with $\sqrt{2}$. To rectify this issue the set $\mathbb{R}$ was constructed: the collection of every rational and irrational number.

We can then transition into our first encounter with closeness: ϵ and its first use: supremums. The supremum s of a set is the least upper bound; that is s is an upper bound and $s \leq b$ where b is any upper bound for the set. Using Lemma 1.3.8 we can think about this differently. We consider any $\epsilon > 0$ then we can say that s is the supremum if and only if $s - \epsilon < a$ where a is in our set. Now using ϵ is such a manner is rather nondescript, but we assume that we want to look at particularly small numbers with ϵ, numbers that are infinitely small. The use of ϵ is the baseline for closeness. Let us refer back to lemma above, in plain english this is saying that no matter how small a number you produce, if you subtract that number from s then it is no longer an upper bound. Every rational person is going to consider s and $s - \epsilon$ to be the same number when considering a sufficiently small ϵ.

So then we need to answer the question of what makes a number equal another number. We say two numbers $a, b \in \mathbb{R}$ are equal if $|a - b| < \epsilon$. That is the distance between a and b is less than any number greater than 0– no matter how small a number is it will not be small enough. Back to the supremum, we have s and $s - \epsilon_0$, and using that formula we see $|s - (s - \epsilon_0)| = |s - s + \epsilon_0| = |\epsilon_0| < \epsilon$. Then depending on which values we pick for ϵ_0 and ϵ we see that s and $s - \epsilon_0$ can be equal by definition or they cannot. I've stated before that we imply that ϵ is only going to be extremely small, or more conceptually we imply that ϵ is going to constrain our argument such that we are forcing the numbers to be sufficiently close together.

A typical standard to ensure that ϵ is going to be extremely small is to say that $\epsilon < \frac{1}{n}$ where $n \in \mathbb{N}$. Or was it $\frac{1}{n} < \epsilon$? Here we encounter a situation similar to the supremum; that is if we specify a particular n, but consider we increment n. This problem then becomes even more complicated, but it does allow us to discuss sequences and convergence. This requires us to touch on what a sequence is, what convergence means, and what limits are. Informally, sequences are functions that takes natural numbers and turns them into real numbers; formally a sequence is a function whose domain is $\mathbb{N}$. The next preliminary is convergence. Here we can state Definition 2.2.3, and finally the limit of a sequence that converges is the convergence point– infinity otherwise.

In using Definition 2.2.3 we see our argument appear again. For a sequence (x_n) to converge we know that at some point in the sequence N, for each point n afterwards the distance between x_n and x is less than ϵ. Recall that this means $|x_n - x| < \epsilon$, so by definition this means that $x_n = x$. Then what about the sequence that began this $(\frac{1}{n})$. We know that this sequence can never equal 0, but we know that $(\frac{1}{n}) \to 0$. This is interesting, but it's addressing this pseudo-equality. Once we go far enough along the sequence we see that the terms are so close 0 that we can consider them equal without them actually being equal; it is the same reasoning behind $0.999\cdots = 1$. Technically they are not equal, but since the distance between the two numbers is sufficiently small– that is the distance between them is 0– we can call them equal and close.

Just now I pointed out that the distance between 0 and $\frac{1}{n}$ is 0 even though $\frac{1}{n} \neq 0$ for all $n \in \mathbb{N}$. This sounds like a contradiction. Luckily, it isn't. Let me explain why; fix ϵ such that $\frac{1}{n} < \epsilon$. We know that $\mathbb{N}$ is countably infinite, but we also know that $\mathbb{Q}$ is dense. Here our dance begins. We fixed ϵ, but from density from that there is a number between 0 and $\frac{1}{n}$. Let us make ϵ equal to that number, well if we can make ϵ we can make n larger as well using the Archimedean Principle. It appears then that while we can make $\frac{1}{n}$ closer 0 we can then go ahead and make them not close anymore. What we are actually doing is unconditionally making $|\frac{1}{n} - 0|$ smaller each time we do this dance. Which means that as we make ϵ smaller and n larger we make 0 and $\frac{1}{n}$ more equal. We are then confident in our assessment that $(\frac{1}{n}) \to 0$, and as we alluded to earlier the limit of this sequence is then 0.

A sequence is defined as a function whose domain if $\mathbb{N}$, in particular let us formally notate this: $f\colon \mathbb{N} \to \mathbb{R}$. We know that if $\lim_{n\to\infty} f(n)$ exists then the sequence converges. This is just a reinterpretation of Definition

2.2.3 and what a limit is. To elaborate more, what I'm saying is that as $|n - \infty|$ gets smaller then $|f(n) - f(\infty)|$ gets smaller as well. Let us pause here since that feels extremely uncomfortable thinking of $|n - \infty|$ getting smaller. It is without a doubt that $1 < 2 < 3 \cdots < \infty$, and it appears that what I'm saying is that $\infty - 2 < \infty - 1$, but the properties of ∞ make that mute. Instead I'm saying that consider $\delta > 0$, then when $|n - \infty| < \delta$ we see that $|f(n) - f(\infty)| < \epsilon$ where $\epsilon > 0$. We can make δ greater than 0, which means that we can make δ infinity. Since $\mathbb{N}$ is countably infinite then n can get infinitely large. We already know that there are degrees of infinity and that the infinity of $\mathbb{N}$ is less than the infinity of $\mathbb{R}$. So then we will see that $|n - \infty| < \delta$ with δ being a higher level infinity which forces $f(n)$ and $f(\infty)$ to be close.

Now if I were you I would be thinking about how talking about infinities has nothing to do with closeness, and that my argument is at a standstill. In the former infinity is what allows us to get close to a number; since infinity exists we are able to get infinitely close to a number, and thus force them to be pseudo-equivalent. For the latter we see that we just defined a Functional Limit, the next piece to this puzzle. Functional Limits are an adjustment to the previous way we used ϵ. Instead of the inputs of the function going to infinity they go to a finite number. This is important, since as I just asserted above that infinity is vital when concerning closeness. To derive this new infinity we need only use that $\mathbb{R}$ is dense: there are an infinite number of numbers between x and c for any value x and c. This was proven when we created a bijection between $[a, b]$ with $a, b \in \mathbb{R}$ and $\mathbb{R}$. So just as we made n infinitely bigger to force (x_n) to be closer to it's limit we increase x infinitely to force $f(x)$ to be closer to $f(c)$. In a way we are doing the exact same process as what we were doing to for sequences. It's like taking a picture and running it through a filter.

There are indeed many other ways in which this idea of closeness has manifested in this class, and each of them adjust they way in which we talk about closeness. It is also true that each manifestation is deeply rooted in the ideas, concepts, and theorems that form their basis. Just like how depending on how we look at Functional Limits it can be built from the struggles of convergence or it can be exactly what we were doing before. So while we have only been discussing closeness in terms of numbers being pseudo-equal we can also think of each of these topics presented as being close. We did just show that functional limits are both exactly like sequential limits and still different; just as we have the dance of ϵ and $(\frac{1}{n})$. Ultimately, the biggest instance of closeness seen in this course is how closely connected each of the topics are.

2 Portfolio Items

2.1 A High Point

In the summary paper I mention the convergence of sequences, but I neglect a discussion on the Cauchy property. In particular I chose this proof since it continues the argument on closeness with sequences by adding another facet. With Cauchy we see that instead of the terms getting closer to a particular point, the terms get closer to each other. That is that once n has surpassed a particular threshold we see that each successive term is going to be pseudo-equal to every other successive term. Formally we say that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that with $n > m \geq N$ it follows that $|x_n - x_m| < \epsilon$ where (x_n) is the sequence. There are a few key points in this proof: Order Limit Theorem, Cauchy Criterion, and the Definition of Convergence of Series, and the Monotone Convergence Theorem. We need the Order Limit Theorem in order to show that (a_k) converges to 0, but it is also necessary to produce the ϵ necessary for showing that (s_{a_k}) is Cauchy. Reflecting on this proof I never actually use the fact that (a_k) is Cauchy. I state it in the proof, but it could be replaced with the definition of convergence instead. It is incredibly difficult to show that a series is convergent just using the series. In this class we spent so much time finding ways in which a sequence converges, since we are then able to transform the question of a series converging to a sequence converging the process becomes much simpler. This also lets substitute the problem: if the sequence of partial sums converges then the series converges. The Monotone Convergence Theorem is so powerful. It is difficult to summarize how useful this theorem is in part b of this exercise. This includes the Monotone Convergence Theorem we use in part b of the exercise. It is the foundation for the entire proof, but in particular with it we are able to make better use of (s_{b_k}). The sequence is bounded which means we have a bound for our other sequence. Though, in this proof I do use it unnecessarily to show that (a_k) converges. The same logic from part a works here as well. So for the latter it drives the progression from (s_{b_k}) being bound and convergent to (s_{a_k}) being convergent. I chose this proof in particular for this section because of the time I spent working on it. By having such an extensive relationship with this problem I was able to better understand the theorem I was trying to prove while also adjusting my confidence in the unit as a whole. Working on this problem for so long etched this unit into my brain.

The proof for Exercise 2.7.3: a)

Proof. First we will prove part i of the Comparison Test. Let (a_k) and (b_k) be sequences such that $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Let $\sum b_k$ converge. Using Theorem 2.7.3 we see that $\sum b_k$ converging implies that $(b_k) \to 0$. It then follows from the Order Limit Theorem that $\lim a_k \leq \lim b_k$ and $\lim a_k \geq 0$. The inequality $0 \leq \lim a_k \leq \lim b_k = 0$ forces $\lim a_k = 0$, so $(a_k) \to 0$. Now we need to show that (s_{a_k}) is Cauchy. Since $(a_k) \to 0$ we see that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ with $n \geq N$ it follows that $|a_n - 0| < \epsilon$. Then, let $\epsilon > 0$ where $N > \frac{\epsilon}{n-m}$ and $n > m \geq N$. It follows that $|s_{a_n} - s_{a_m}| = |a_{m+1} + a_{m+2} + \cdots + a_n| < |a_{m+1} + a_{m+1} + \cdots + a_{m+1}| = |(n-m)a_{m+1}| < |(n-m)\frac{\epsilon}{n-m}| = \epsilon$. Hence we see that (s_{a_k}) is Cauchy, and thus by the Cauchy Criterio we see that (s_{a_k}) converges. By the definition of convergence of series it then follows that $\sum a_k$ converges. Thus part i is proven.

We see that part ii is the contrapositive of part i, and the proof for that is above. $\square$

b)

Proof. First we will prove part i of the Comparison Test. Let (a_k) and (b_k) be sequences such that $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$. Let $\sum b_k$ converge. Using Theorem 2.7.3 we see that $\sum b_k$ converging implies that $(b_k) \to 0$. It then follows from the Order Limit Theorem that $\lim a_k \leq \lim b_k$ and $\lim a_k \geq 0$. The inequality $0 \leq \lim a_k \leq \lim b_k = 0$ forces $\lim a_k = 0$, so $(a_k) \to 0$. By the definition of convergence of series we know that (s_{b_k}) converges since $\sum b_k$ converges. We also know from our theorem that $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$, thus by algebra it follows that $0 \leq (s_{a_k}) \leq (s_{b_k})$. Thus by definition we see that (s_{a_k}) is bounded. In our theorem we established that $0 \leq a_k$, and a_k is always positive. From algebra we see that (s_{a_k}) is only summing positive numbers, and is thus increasing by definition. From the Monotone Convergence Theorem we then see that since (s_{a_k}) is monotone increasing and bounded then it must converge. Thus by the definition of convergence of series we see that $\sum a_k$ converges as well. Hence part i is proven.

We see that part ii is the contrapositive of part i, and the proof for that is above. $\square$

2.2 Most Improved

I would like to first reflect on this proof. My biggest weakness involving this proof was my overconfidence. I knew that I felt good about the question despite being told by the professor how difficult it was. In my overconfidence I made some hasty generalizations regarding the ideas I used. I went to office hours to discuss the issue with Dr. Downes, and during the visit I felt confident that I would be able to replicate what I was hearing. I couldn't fully. The problem baffles me. I asked someone else to explain their proof to try and kickstart some ideas; that made the situation worse. Here I am with time winding down thinking about what exactly I'm missing. This section asks for the proof I improved the most on, but overall this portfolio is for my benefit so I am going to rebel a little. The proof below is the exact proof in all its infamy. I admit that currently I am not capable of solving this problem. Thus, I'm going to make my Most Improved proof a work in progress– a personal goal so to say. A reminder. Hence while I may not understand the correct answer I can explain why my answer is incorrect. This is something that I definitely took with me out of office hours.

This proof but another simultaneous example on how intricately laced series and sequences are together and how convergence represents closeness. One of the ways in which this problem is so difficult for me is how close these ideas are to each other. I do not understand when I need to use series or sequences for this problem– among several other things of course. In my head I am forcibly blending the two together creating this idea of psuedo-equality. They are the same, but they are incredibly different. On the case of convergence representing closeness, this is just another example we can demonstrate. From little I do know about the solution to the problem, I know that we need to utilize $|c_n - c|\epsilon$. We know that (c_n) converges, and so it gets close to c, with this closeness we can show that other sequences or series are also close.

The "Proof" for Exam 3 Question 4: a)

Proof. Suppose that the sequence of partial sums (s_n) of $\sum a_n$ is bounded, and $(b_n) \to 0$ and is decreasing. By supposition we know that (s_n) is bounded and $\lim b_n = 0$, then by Exercise 2.3.9 we know that $(s_n b_n) \to 0$. Using some algebra know that $(s_n b_n) = \sum a_n b_n$. Applying the definition of convergence of series we know that when the sequence of partial sums converges then the series converges. Then since $(s_n b_n)$ converges we see that $\sum a_n b_n$ converges as well. □

b)

Proof. Suppose that $\sum a_n$ converges and (c_n) is monotonic and bounded. We are going to show that $\sum a_n c_n$ converges. Since (c_n) is bounded we know that by definition there exits $M \in \mathbb{R}$ such that $c_n \leq M$ for all $n \in \mathbb{N}$. By our supposition we know that $\sum a_n$ converges, and using Theorem 2.7.1 we know that $\sum M a_n$ converges as well. We established that $c_n \leq M$, and so it follows that $a_n c_n \leq M a_n$ for all $n \in \mathbb{N}$. Then we can apply the Comparison Test to $\sum a_n c_n$, so it must converge as well. Hence the theorem is proven. □

2.3 A Touchstone

The first key point with closeness I refer to is supremums, and in particular we reference Lemma 1.3.8. This proof uses a slight variation of Lemma 1.3.8 in it's theorem, so it allows us to continue the discussion. We get an intimate view to what I was referring to. We see that the supremum is not maleable; we cannot deviate from that position. No matter how close we to the supremum– including them being equal by definition– any value less than or greater than is not going to be the supremum. For this proof the biggest ideas we use are the Density of $\mathbb{Q}$ and Lemma 1.3.8. With it we are able to show that $s + \frac{1}{n}$ can not be the supremum since we can always a number between s and $s + \frac{1}{n}$. The lemma also is what explains why $s - \frac{1}{n}$ is not going to be an upper bound. Instead of directly influecing the proof, it helps more in rationalizing the theorem. I know that this section asked for something from class or the book, honestly this proof was incredibly important to me. It is the first proof I can remember doing in this class that made me feel like I can do this. I mentioned something similar in my coversheet, but with my mindset being rooted in grades taking two C's back to back hurt my ego. Knowing that there was at least something I did well and felt good about helped me to get to my feet and finish this class.

The Proof for Exercise 1.4.2:

Proof. By the Axiom of Completeness we know that there exists a supremum for A. From the Theorem we know that any value less that s is not an upper bound. We know that $s + \frac{1}{n}$ is an upper bound. From the density of $\mathbb{Q}$ we know that there will always a number r such that $s < r < s + \frac{1}{n}$. Thus $s + \frac{1}{n}$ cannot be the least upper bound since I can always make n larger and thus

$\frac{1}{n}$ smaller. Suppose then that s is not an upper bound, then it follows that A has no supremum since no number less than s is an upper bound and no number greater than s is a least upper bound. Thus it follows that s is an upper bound for A, and by Lemma 1.3.8 we know that s must then be the least upper bound for A. □

2.4 Compare and Contrast: Supremum

In the context of describing closeness I referenced two vital topics covered in Analysis: supremums and Density. However I used them merely as a stepping stone unable to give them the proper attention they deserved. I understand I dedicated an entire argument to supremums, but I still failed to present a sufficient number of examples. By using these problems I am able to provide some examples to give justice to supremums and Density. Then technically I am admitting that I'm not necessarily developing the exact concept of closeness, but I am instead having them provide a better platform for which I base my argument on. Im the beginning I would definitely say that these problems would have been troublesome considering the situation at the beginning of the semester; I cannot say for certain given the sheer amount of time between solving each question. This is a postive though since I can not only quantify my improvement with the grade performance on each question as well the time needed to solve each question.

Now just as we have been having our little dialogue, I'm going to have these proofs converse. The first exercise requires more steps since there is less information we can work with. That or experience has allowed me to draw more information out of the question. We see that they both use the Axiom of Completeness to show existence, but one uses the definition of least upper bound and the other uses Lemma 1.3.8. It is plain to see proving using definition isn't always the most efficient. Though the existence of c is the first question complicates using just the definition; it has complications that the second question doesn't have. Upon further inspection it becomes evident that Exam 1 Question 1 has another complication: equality. We need to have two numbers equal each other. Then, Exam 1 Question 1 does appear to be more difficult since it involves more showing more details. With the second question we can rely heavily on Density and Lemma 1.3.8.

The Proof for Exam 1 Question 1:

Proof. Let $A \subset \mathbb{R}$ such that A is both bounded above and nonempty. Let

$c \geq 0$, and $cA = \{ca : a \in A\}$. We know that since A is nonempty and bound by the Axiom of Completeness it must have a supremum, and since $c \in \mathbb{R}$ by construction we then know that cA is also bounded and nonempty which implies that it has a supremum as well. Now we need to show that $\sup cA = c\sup A$. We are presented with two distinct cases: $c = 0$ and $c > 0$. Let us begin with the first one. If $c = 0$, then we see that $cA = \{0\}$, and thus this case is trivial. $\sup cA = 0$, $c\sup A = 0 * c = 0$, where c is the value of $\sup A$. Let us now look at the remaining case. To prove that $c\sup A$ is the supremum for cA we need to show that it is an upper bound, and that it is less than or equal to every upper bound. Then, let $s = \sup A$; we see that $a \leq s$ for all $a \in A$. From this we see that $ca \leq cs$ for all $a \in A$. Hence the first part of the definition is true. Now, let b be an upper bound for cA which is to say that $ca \leq b$ for all $a \in A$. From this we can conclude that $a \leq \frac{b}{c}$ since we stated that $c > 0$. Since we established that s is the supremum for A, then $s \leq \frac{b}{c}$. Using a little algebra we know that $cs \leq b$, and thus we see that the second part of the definition is true. Hence the theorem is proven. $\square$

The Proof for Exam 3 Question 1:

Proof. Let $x \in \mathbb{R}$. Consider the set $A = \{q \in \mathbb{Q} \colon q < x\}$. We are going to show that $x = \sup A$. By construction we know that x is going to ber an upper bound for A. Since A is bounded above by construction and nonempty by construction we see that it must have a least upper bound. Now we must show that it is the least upper bound for A. Let $\epsilon > 0$. By the Density of $\mathbb{Q}$ we know that there exists $y \in \mathbb{Q}$ such that $x - \epsilon < y < x$. Then by Lemma 1.3.8 we know that $x = \sup A$. Hence the theorem is proven. $\square$

2.5 Compare and Contrast: Limits and Continuity

By including the first proof I am to build a solid foundation on the rather vague argument I produced about δ and ϵ representing closeness. By including the second I'm able to address continuity. Continuity is almost verbatum the same definition as Functional Limits, so just claiming that they are intimately related is justified. By putting them both in a conversation with each other I'm able to better represent the closeness of their relationship. Since they both use disturbingly similar definitions then together they make my vague argument concrete since it provides more examples. The first exercise uses exclusively the definition of Functional Limits in order to show the limits of the specified functions; whereas we see that we take alternative routes in the second exercise– other than the third part of Exercise 4.4.1. Where we use Algebraic theorems and $\epsilon - \delta$ proofs as well. Further examination of the exercises shows that we can substitute then process used in Exercise 4.4.1a and Exercise 4.2.5. This shows that the fundamental problem they are asking is the same, and so the problems and solutions are then closely related. However, there are some very distinct differences that we need to address. For instance Exercise 4.4.1 is working with the entire domain of the function whereas Exercise 4.2.5 uses a single point. I said that we can substitute the process to solve the problems with each other, but we will still need to adjust the new process for Exercise 4.2.5. We need to make sure that we are still looking at a single point instead of the entire function.

These examples are fairly straightforward and direct which makes them understandable, and it immediately gave me confidence when tackling each homework. I was able to root myself on a problem I understood. These units were some of the most confusing for me despite my actual performance. I'm sure it has more to do with it being different but the same, and I kept confusing this new idea with things that applied to other concepts. It was then nice that when I kept confusing things I had certainty to rely on to guide me through the fog.

The Proof for Exercise 4.2.5: a)

Proof. Let $\epsilon > 0$ and $\delta < \frac{\epsilon}{3}$. Then we see that $|f(x) - L| = |(3x+4) - 10| = |3x+4-10| = |3x-6| = |3||x-2| < |3|\delta < \epsilon$. Hence we see that by definition that $\lim_{x\to 2}(3x+4) = 10$. ☐

b)

Proof. Let $\epsilon > 0$ and $\delta < \sqrt[3]{\epsilon}$. Then we see that $|f(x) - L| = |x^3 - 0| = |x^3| = |x-0||x-0||x-0| < \delta * \delta * \delta < \frac{\epsilon}{3} * \frac{\epsilon}{3} * \frac{\epsilon}{3} = \epsilon$. Hence we see that by definition that $\lim_{x\to 0} x^3 = 0$. ☐

d)

Proof. Let $\epsilon > 0$ and we can consider δ such that $2 \leq x \leq 4$ and $\delta < \epsilon$. Then we see that $|f(x) - L| = |\frac{1}{x} - \frac{1}{3}| = |\frac{3-x}{3x}| = |\frac{x-3}{3x}| \leq |\frac{x-3}{6}| < \frac{\epsilon}{6} < \epsilon$. Hence we see that by definition that $\lim_{x\to 3}\frac{1}{x} = \frac{1}{3}$. ☐

The Proof for Exercise 4.4.1:

Proof. Let $f(x) = x^3$ and $c \in \mathbb{R}$. Using a little algebra we see that $f(x) = x{*}x{*}x$, and we have previously proven that x is continuous on $\mathbb{R}$. This means that x is continuous at c. Using the Algebra Continuity Theorem we see that $x * x * x$ is going to be continuous at c, and so it must be that $f(x) = x^3$ is continuous at c. Hence we have shown that $f(x) = x^3$ is continuous on $\mathbb{R}$. □

b)

Proof. Let $f(x) = x^3$. Let $\epsilon_o = 3$. Consider the sequences $(x_n) = n + \frac{1}{n}$ and $(y_n) = n$. We see that $|x_n - y_n| = |n + \frac{1}{n} - n| = \frac{1}{n}$. We have proven that $\frac{1}{n}$ is Cauchy before. We are now going to show that $|f(x_n) - f(y_n)| \geq \epsilon_o$. $|f(x_n) - f(y_n)| = |(n^3 + 3n + \frac{3}{n} + \frac{1}{n^3}) - n^3| = |3n + \frac{3}{n} + \frac{1}{n^3}| > |3n| \geq 3 = \epsilon_o$. So by Theorem 4.4.5 we see that $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$. □

c)

Proof. Let $f(x) = x^3$ and consider that we are looking at a bounded subset A of $\mathbb{R}$ where $x, c \in A$. By Definition 3.3.3 we know then that there exists $M \in \mathbb{R}$ with $|a \in A| \leq M$. Let $\epsilon > 0$ and $\delta < \frac{\epsilon}{3M^2}$ we then see that $|f(x) - f(c)| = |x^3 - c^3| = |(x^2 + xc + c^2)(x - c)| \leq |x^2 + xc + c^2||x - c| < |M^2 + M^2 + M^2||x - c| < |3M^2|\delta < \epsilon$. Hence by the definition of uniform continuity we see that $f(x) = x^3$ is uniformly continuous on a bounded set. □

2.6 Additional Proof 1

In this proof we see a conversation between sequences and series which is something that I didn't fully address earlier. I made references to how interconnected the concepts in Analysis are to each other, and this is a perfect example. This proof is entirely reliant on our ability to weave between series and sequences. For instance the definition of convergence of series producing the sequence of partial sums. Since we are able to convert the series into a sequence we can take advantage of the Monotone Convergence Theorem to prove the reverse direction of the proof since each part necessary for the Monotone Convergence Theorem to work is given in the theorem. I chose this proof for its simplicity. I might be rather verbose in my explanation, but each direction of the proof is incredibly simple. As I was tackling this problem it felt it was going to be intricate and complex, but I find that in this simplicity we see how close series and sequences are connected. I was able to look at myself when working on this question. Despite the stress of this being an exam question, it is a textbook example of me judging a book by its cover. Working through this problem really emphasized that I judge a question harshly before I begin the proof. That being said, just because I recognize the problem does not mean I was able to resolve my prejudice against problems.

The Proof for Exam 2 Question 3:

Proof. Let (x_n) be a sequence with $x_n \geq 0$ for all $n \in \mathbb{N}$. We are going to first show that if $\sum x_n \to L$ then the sequence of partial sums is bounded. Consider $\sum x_n \to L$, then by the definition of convergence of series we know that the sequence of partial sums $(s_n) \to L$. By Theorem 2.3.2 we know that every convergent sequence is bounded; thus (s_n) is bounded, and the forward direction is proven.

Now we will show that if the sequence of partial sums for $\sum x_n$ (s_n) is bounded, then $\sum x_n \to L$. Suppose that (s_n) is bounded; then we know that $\sum x_n$ is bounded as well. We are going to use the Monotone Convergence Theorem to show that (s_n) converges. Now we need to show that (s_n) is monotone. By the construction of (s_n) we know that it is summing terms of (x_n), and by our supposition we know that $x_n \geq 0$ for all $n \in \mathbb{N}$. This means that (s_n) is summing only positive numbers, so (s_n) must be increasing. Then by the Monotone Convergence Theorem (s_n) converges. It follows that by the definition of convergence of series that since $(s_n) \to L$ then $\sum x_n \to L$. Hence we see that the reverse direction is true, and the theorem is proven. $\square$

2.7 Additional Proof 2

Part of my argument is hammering just how often things are equal without actually being equal: like how we can have two numbers be defined as equal without them actually being equal to each other. This proof is an argument against that proposal. Here we see strict equality with the limits, and equality with numbers is how get closesness. To be precise for this proof we force a number to close to two other numbers by making the other two numbers close to each other. This is the Squeeze Theorem and it is aptly named. Because of how incredibly short this proof the main idea behind it is abusing the Order Limit Theorem. With this theorem we force two inequalities that have one case where both are true, and the construction of construction of the terms requires that both inequalities be true. In many ways this question is similar in it's simplicity to the previous section. This time we see that the proof is actually short, but it also a change of pace. I do not speak much on limits by themselves barring Functional Limits. This proof allows me to elaborate and the limits I so neglected. I cannot say that this proof was formative in my experience with this class or my understanding. Honestly, the theorem and succeeding proof intrigue because the answer seems obvious. It reminds me of the transitive property.

The Proof for Exercise 2.3.3:

Proof. Let $x_n \leq y_n \leq z_n$, and $\lim x_n = \lim z_n = l$. We can apply the Order Limit Theorem to show that $\lim x_n \leq \lim y_n \leq \lim z_n$, and using substitution it follows that $l \leq \lim y_n \leq l$. The only way $\lim y_n$ can fulfill both inequalities is if $\lim y_n = l$. Hence the theorem is proven. □

2.8 Additional Proof 3

Not once do I mention Differentiability in the summary paper. That wasn't to say that Differentiability isn't related to closeness; this is in fact wrong since it defined using limits and relies on continuity. It was just the case that using Differentiability in my argument had its flaws. Still, it is a vital topic in the course since it is the culmination of everything we have learned. It felt improper to have a discussion on topic in Analysis without discussing Differentiability. This proof in particular brings closer to understanding Differentiability by actively trying to break it. In my conclusion I address how an aspect of closesness we learn about in this class is how the concepts themselves are close; this proof is an extension of that. In this instance what is being brought closer to the concept is ourselves. Seeing as I just described this proof as trying to break Differentiability, then it is only right that the biggest idea present is Definition 5.2.1. The definition isn't explicitly used throughout the exercise, but in solving each part of the exercise it is the most valuable resource. For example, twice we use constant functions, but it is the simplest examples that display understanding. Then take part d, here we actually use the definition. Since we everything we are working with is arbitrary it becomes important since we just need to demonstrate existence. I valued these questions the most after turning the homework in.

They facilitated a more intimate relationship with the concept involved in the homework. It cultivated thinking creatively and a better understanding of the material. Though, I will say that working on them was not as amusing as reflecting on them.

The Proof for Exercise 5.2.2: a) Consider the functions $f(x) = 1$ with $x < 0$ and $f(x) = -1$ with $x >= 0$ and $g(x) = -1$ with $x < 0$ and $g(x) = 1$ with $x >= 0$. We see that neither function is continuous at $x = 0$ by the Sequential Criterion, and thus they are not differentiable at $x = 0$ by Theorem 5.2.3. Then $(fg)(x) = -1$, $\forall x$. Since this is a constant function then we see that it is differentiable at $x = 0$.

b) Consider $f(x) = |x|$ and $g(x) = 0$. We have shown before that $f(x)$ is not differentiable at $x = 0$, and since $g(x)$ is a constant function we see that it is trivially differentiable. Then we see that $f(x)g(x) = 0$, $\forall x$, and such $f(x)g(x)$ is going to be differentiable.

c)

Proof. This request is impossible. Consider $f(x)$, $g(x)$, and $(f + g)(x)$ such that $g(x)$ and $(f+g)(x)$ are differentiable at 0 and $f(x)$ is not. Suppose to the contrary that the request is possible then. We see that $f(x) = (f + g)(x) - g(x)$. From our supposition know that $(f + g)(x)$ and $g(x)$ are differentiable at $x = 0$, and by the Algebraic Differentiability Theorem then it must be that $f(x)$ is differentiable at $x = 0$. This contradicts our supposition that $f(x)$ is not differentiable at $x = 0$, and thus the request is impossible. □

d) Consider $f(x) = 0$ with $x \notin \mathbb{Q}$ and $f(x) = x^2$ with $x \in \mathbb{Q}$. We see that at $x = 0$ $f(x)$ is going to be differentiable since

$$\lim_{x\to 0} f(x),$$
$$\lim_{x\to 0} f(c),$$
$$\lim_{x\to 0} x,$$
$$\lim_{x\to 0} 0$$

are all defined, so by the Algebra Limit Theorem we then know that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ is defined with $x \neq 0$. We also see that the function is not going to be continuous anywhere else by the Criterion for Discontinuity.