HW 1

1) a) $\langle x|\hat{x}|x'\rangle = x\,\delta(x-x')$

$\langle k|\hat{x}|k'\rangle = i\hbar\,\partial/\partial p\,\delta(p-p')$

This is from Fourier transform of $p = -i\hbar\,\partial/\partial x$

b) $\langle x|\hat{k}|x'\rangle = -i\hbar\,\partial/\partial x\,\delta(x-x')$

$\langle k'|\hat{k}|k'\rangle = k\,\delta(k-k')$

c) $\langle x|\hat{E}_{kin}|x'\rangle = \langle x|\frac{k^2}{2m}|x'\rangle \rightarrow \frac{1}{2m}\left(-i\hbar\,\partial/\partial x\right)^2\delta(x-x')$

$\langle k|\hat{E}_{kin}|k'\rangle = \langle k|\frac{k^2}{2m}|k'\rangle \rightarrow \frac{p^2}{2m}\,\delta(k-k')$

d) $\langle x|\hat{V}|x'\rangle \rightarrow \langle x|\hat{V}(x)|x'\rangle \rightarrow \hat{V}(x)\,\delta(x-x')$

$\langle k|\hat{V}|k'\rangle \rightarrow \langle k|\hat{V}(x)|k'\rangle \rightarrow V(i\hbar\,\partial/\partial p)\,\delta(k-k')$

e) $L = \hat{x}\times\hat{k} \rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ k_x & k_y & k_z \end{vmatrix}$

$L_x = yk_z - zk_y \qquad L_y = xk_z - zk_x \qquad L_z = xk_y - yk_x$

$\langle x|\hat{L}|x'\rangle \longrightarrow L_x = -i\hbar\left(y\,\partial/\partial z - z\,\partial/\partial y\right)\delta(x-x')$

$L_y = -i\hbar\left(x\,\partial/\partial z - z\,\partial/\partial x\right)\delta(x-x')$

$L_z = -i\hbar\left(x\,\partial/\partial y - y\,\partial/\partial x\right)\delta(x-x')$

$\langle k|\hat{L}|k'\rangle \longrightarrow L_x = i\hbar\left(k_z\,\partial/\partial k_y - k_y\,\partial/\partial k_z\right)\delta(k-k')$

$L_y = i\hbar\left(k_z\,\partial/\partial k_x - k_x\,\partial/\partial k_z\right)\delta(k-k')$

$L_z = i\hbar\left(k_y\,\partial/\partial k_x - k_x\,\partial/\partial k_y\right)\delta(k-k')$

Both delta function and partial derivative b/c function of both $\hat{x}/\hat{k}$

f) $\hat{v} = \frac{dx}{dt} \to \frac{i}{\hbar}[\hat{H}, \hat{x}] \to \frac{i}{\hbar}\left[\frac{\hat{k}^2}{2m} + \hat{V}(x), \hat{x}\right]$

$[\hat{V}(x), \hat{x}] = 0$

a general formula is $\frac{dA}{dt} = \frac{i}{\hbar}[\hat{H}, \hat{A}]$ Heisenberg eqn of motion

$= \frac{i}{\hbar}\left[\frac{\hat{k}^2}{2m}, \hat{x}\right] = \frac{i}{\hbar}\frac{1}{2m}[\hat{k}^2, \hat{x}]$ $\quad [\hat{k}^2, \hat{x}] = \hat{k}[\hat{k}, \hat{x}] + [\hat{k}, \hat{x}]\hat{k}$

not sure if you're expecting me to show how this is true, I just looked up the properties for it

$[\hat{x}, \hat{k}] = i\hbar \to [\hat{k}, \hat{x}] = -i\hbar \qquad [\hat{k}^2, \hat{x}] = -\hat{k}i\hbar - \hat{k}i\hbar = -2i\hbar\hat{k}$

$= \frac{\cancel{i}}{\cancel{\hbar}}\frac{1}{\cancel{2}m}(\cancel{-2}\cancel{i}\cancel{\hbar}\hat{k}) \quad i^2 \to -1$

$\hat{V} = \frac{\hat{k}}{m}$, so it is momentum over mass I guess...

$\langle x | \hat{V} | x' \rangle \to \langle x | \frac{\hat{k}}{m}, x' \rangle = -i\hbar/m\, \partial/\partial x\, \delta(x - x')$

$\langle k | \hat{V} | k' \rangle \to \langle k | \frac{k}{m} | k' \rangle = \frac{k}{m}\delta(k - k')$

2) a) (3D) $g_{dim}(\varepsilon) = \lim_{\Delta\varepsilon \to 0} \frac{\#\text{states within } [\varepsilon, \varepsilon + \Delta\varepsilon]}{\Delta\varepsilon \cdot V_{dim}} \left[\frac{1}{eV \cdot m^{dim}}\right]$

Set numerator to integral/V

$\Delta^{dim}k = \frac{(2\pi)^{dim}}{V}$

$\frac{\int_{\varepsilon_k}^{\varepsilon_k+\Delta\varepsilon} d^3k \,/\, \frac{(2\pi)^3}{\cancel{V}}}{\Delta\varepsilon \cdot \cancel{V}} = \frac{\int_{\varepsilon_k}^{\varepsilon_k+\Delta\varepsilon} d^3k}{(2\pi)^3} \cdot \frac{1}{\Delta\varepsilon}$ cartesian $\to$ spherical

$= \frac{4\pi}{(2\pi)^3}\int_{\varepsilon_k}^{\varepsilon_k+\Delta\varepsilon} k^2 dk \cdot \frac{1}{\Delta\varepsilon}$

$d^3k = dk_x dk_y dk_z$
$= k^2 \sin\theta\, dk\, d\theta\, d\phi$

convert back to limit

$= \lim_{\Delta\varepsilon \to 0} \frac{4\pi}{(2\pi)^3} k^2(\varepsilon)\frac{dk}{d\varepsilon}$ $\quad \varepsilon = \frac{\hbar^2 k^2}{2m}$ $\quad d\varepsilon = \frac{2\hbar^2 k}{2m} dk$

$= \lim_{\Delta\varepsilon \to 0} \frac{4\pi}{(2\pi)^3} k^2(\varepsilon)\frac{m}{\hbar^2 k} = \frac{4\pi}{(2\pi)^3}\frac{km}{\hbar^2}$ $\quad d\varepsilon = \frac{\hbar^2 k}{m} dk$

$= \frac{4\pi}{8\pi^3}\frac{m}{\hbar^2}\sqrt{\frac{2m\varepsilon}{\hbar^2}} = \boxed{\frac{1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon}}$ $\quad d\varepsilon/dk = \frac{\hbar^2 k}{m} \quad \frac{dk}{d\varepsilon} = \frac{m}{\hbar^2 k}$

$k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}$

b) 2D $\lim_{\Delta\varepsilon\to 0} \frac{\#\text{ states}}{\Delta\varepsilon \cdot V}$ $\frac{\int_{\varepsilon_k}^{\varepsilon_k+\Delta\varepsilon} d^2k}{(2\pi)^2} \cdot \frac{1}{\Delta\varepsilon \cdot \cancel{SA}}$

$d^2k = dk_x dk_y \to$ polar

$k\,dk\,d\theta$ $\quad \frac{\cancel{SA}}{(2\pi)^2} \int_0^{2\pi}\int_{\varepsilon_k}^{\varepsilon_k+\Delta\varepsilon} k\,dk\,d\theta \cdot \frac{1}{\Delta\varepsilon}$

$= \frac{2\pi}{(2\pi)^2} \int_{\varepsilon_k}^{\varepsilon_k+\Delta\varepsilon} k\,dk \cdot \frac{1}{\Delta\varepsilon}$

back to limit

$\lim_{\Delta\varepsilon\to 0} \frac{1}{2\pi} k(\varepsilon) \frac{dk}{d\varepsilon}$ $\quad dk/d\varepsilon = \frac{m}{\hbar^2 k}$

$\frac{1}{2\pi} k \frac{m}{\hbar^2 k} = \boxed{\frac{1}{2\pi} \frac{m}{\hbar^2}}$

c) 1D $\quad \frac{\int_{\varepsilon_k}^{\varepsilon_k+\Delta\varepsilon} dk}{\frac{2\pi}{\cancel{V}}} \cdot \frac{1}{\Delta\varepsilon \cdot \cancel{V}} \to \frac{1}{2\pi} \int_{\varepsilon_k}^{\varepsilon_k+\Delta\varepsilon} dk \cdot \frac{1}{\Delta\varepsilon}$ $\quad 2$

$= \frac{1}{\pi} \frac{dk}{d\varepsilon} = \frac{1}{\pi} \frac{m}{\hbar^2 k}$

$= \frac{m}{\pi\hbar^2 k}$ $\quad k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}$

$= \frac{m}{\pi\hbar^2}\sqrt{\frac{\hbar^2}{2m\varepsilon}} = \boxed{\frac{1}{\pi}\sqrt{\frac{m}{2\hbar^2\varepsilon}}}$

d) dim > 3 $\quad g_{dim}(\varepsilon) = \lim_{\Delta\varepsilon\to 0} \frac{\#\text{ states within } [\varepsilon, \varepsilon+\Delta\varepsilon]}{\Delta\varepsilon \cdot V_{dim}}$

$g_{dim}(\varepsilon) = \frac{\int_{\varepsilon_k}^{\varepsilon_k+\Delta\varepsilon} d^{dim}k}{(2\pi)^{dim}} \cdot \frac{1}{\Delta\varepsilon \cdot \cancel{V_{dim}}}$

$d^{dim}k \to r^{dim-1}\left(\frac{2\pi^{n/2}}{\Gamma(n/2)}\right)$

formula for total solid angle in ↑ space

$\frac{1}{(2\pi)^{dim}}\left(\frac{2\pi^{dim/2}}{\Gamma(dim/2)}\right)\int_{\varepsilon_k}^{\varepsilon_k+\Delta\varepsilon} k^{dim} \cdot \frac{1}{\Delta\varepsilon}$

to limit

$= \lim_{\Delta\varepsilon\to 0} \frac{1}{(2\pi)^{dim}} \cdot \left(\frac{2\pi^{dim/2}}{\Gamma(dim/2)}\right) k^{dim}(\varepsilon) \frac{dk}{d\varepsilon}\Big|_\varepsilon$ $\quad k = \sqrt{\frac{2m\varepsilon}{\hbar^2}}$ $\quad \frac{dk}{d\varepsilon}\Big|_\varepsilon = \frac{m}{\hbar^2 k}$

$= \frac{1}{(2\pi)^{dim}}\left(\frac{2\pi^{dim/2}}{\Gamma(dim/2)}\right)\left(\frac{2m\varepsilon}{\hbar^2}\right)^{\frac{dim-1}{2}} \frac{m}{\hbar^2} \to \boxed{\frac{1}{(2\pi)^{dim}}\left(\frac{2\pi^{dim/2}}{\Gamma(dim/2)}\right)\left(\frac{2m^2\varepsilon}{\hbar^4}\right)^{\frac{dim-1}{2}}}$

e) 0D) $g_{0D}(\varepsilon) = \sum_n \delta(\varepsilon - \varepsilon_n)$

$$g_{dim}(\varepsilon) = \frac{\int d^{dim}k \cdot \frac{1}{\Delta\varepsilon \cdot V_{dim}}}{\cancel{(2\pi)^{dim}}} \rightarrow \frac{1}{(2\pi)^{dim}} \int d^{dim} k \cdot / \Delta\varepsilon \rightarrow \frac{1}{(2\pi)^{dim}} \int d^{dim} \vec{k}\, \delta(\varepsilon - \varepsilon_{\vec{k}})$$

$$\frac{1}{(2\pi)^{dim}} \int d^{dim} \vec{k}\, \delta(\varepsilon - \varepsilon_k) \longrightarrow \frac{1}{(2\pi)^{dim}} \sum_n \delta(\varepsilon - \varepsilon_n),$$ or call it $\frac{1}{V_{dim}} \sum_n \delta(\varepsilon - \varepsilon_n)$

for 0D, $(2\pi)^0 \rightarrow 1$, so $g_{0D} = \sum_n \delta(\varepsilon - \varepsilon_n)$

3) $E_{r,n,k} = \frac{\hbar^2 \pi^2}{2mW^2} n^2 + \frac{\hbar^2 k^2}{2m}$ $\quad m = .067 m_0 \quad m_e = 9.1 \times 10^{-31} kg$