이산수학 HW3

2017-18570 이성찬

3.1 #28

Given a list of n increasing integers $A[1, 2, \cdots, n]$ split the list into 4 sublists. Suppose we want to find an integer k in the list. The following algorithm returns the index of k. If the list doesn't contain k, it will return -1.

4-ary Search(A, i, j, k)

- $i = 1$, $j = n$ // consider list from $A[i, \cdots, j]$
- **while** $i < j - 2$
    - $q_1 = \lfloor (i+j)/4 \rfloor$
    - $q_2 = \lfloor (i+j)/2 \rfloor$
    - $q_3 = \lfloor 3(i+j)/4 \rfloor$
    - **if** $k > A[q_2]$
        - **if** $k \leq A[q_3]$
            - $i = q_2 + 1$, $j = q_3$
        - **else** $i = q_3 + 1$
    - **else if** $k > A[q_1]$
        - $i = q_1 + 1$, $j = q_2$
    - **else** $j = q_1$
- **if** $k == A[i]$
    - $location = i$
- **else if** $k == A[j]$
    - $location = j$
- **else if** $k == A[\lfloor (i+j)/2 \rfloor]$
    - $location = \lfloor (i+j)/2 \rfloor$
- **else** $location = -1$
- **return** $location$

3.1 #44

Suppose we are given a sorted list $A[1, 2, \cdots, n]$ and we want to insert an element k. We suppose the elements are comparable. The following algorithm will return the index of k.

insert(A, i, j, k)
 $i = 1$, $j = n$
 while $i < j$
  $m = \lfloor (i+j)/2 \rfloor$
  if $k > A[m]$
   $i = m+1$
  else $j = m$
 if $k < A[i]$
  $location = i$
 else $location = i+1$
 return $location$

3.1 #56

We give a counterexample for this problem.

Suppose we want to pay 16 cents.

Using the greedy algorithm, we first pay 12 cents with 12-cent coin, the remaining 4 cents with pennies. Thus we need 5 coins in total.

But $16 = 10 + 5 + 1$, so we only need 3 coins.

In case of paying 12 cents, the greedy algorithm gives the fewest coins. (1 coin)

Thus the greedy algorithm with 12-cent coin does not always give the fewest coins possible.

3.2 #26

a) The given form of function is $n^p(\log n)^q$ thus we find a smallest $p, q \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{n\log(n^2+1) + n^2\log n}{n^p(\log n)^q} < \infty$$

Simplify to get

$$\lim_{n\to\infty}\left(\frac{n\log(n^2+1)}{n^p(\log n)^q} + \frac{n^2\log n}{n^p(\log n)^q}\right) = \lim_{n\to\infty}\left(\frac{n^{1-p}\log(n^2+1)}{\log(n^2)}\frac{\log(n^2)}{(\log n)^q} + n^{2-p}(\log n)^{1-q}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\log(n^2+1)}{\log(n^2)}2n^{1-p}(\log n)^{1-q} + n^{2-p}(\log n)^{1-q}\right)$$

Since $\lim_{n\to\infty}\log(n^2+1)/\log(n^2) = 1$, this limit will converge when $2-p \le 0$, $1-q \le 0$.

Set $p = 2, q = 1$, thus $n\log(n^2+1) + n^2\log n \in O(n^2\log n)$

b) The form of the function suggests, that we should compare with $n^p(\log n)^q$. We now find the smallest possible values of $p, q \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{(n\log n+1)^2+(\log n+1)(n^2+1)}{n^p(\log n)^q} < \infty$$

We only need to compare the highest order of n^p and $(\log n)^q$.

The numerator contains a term with n^2 and $(\log n)^2$ thus this suggests that the function is $O(n^2(\log n)^2)$.

We compute the limit,

$$\lim_{n\to\infty} \frac{(n\log n+1)^2+(\log n+1)(n^2+1)}{n^2(\log n)^2} = \lim_{n\to\infty} \frac{n^2(\log n)^2+n^2\log n+2n\log n+n^2+\log n+2}{n^2(\log n)^2}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{\log n}+\frac{2}{n\log n}+\frac{1}{(\log n)^2}+\frac{1}{n^2\log n}+\frac{2}{n^2(\log n)^2}\right) = 1 < \infty$$

Thus $(n\log n+1)^2+(\log n+1)(n^2+1) \in O(n^2(\log n)^2)$.

c) The given function $n^{2^n}+n^{n^2}$ is either $O\left(n^{2^n}\right)$ or $O\left(n^{n^2}\right)$ (They are the only terms in the function.) But 2^n grows faster than n^2 it must be $O\left(n^{2^n}\right)$.

$$\lim_{n\to\infty} \frac{n^{2^n}+n^{n^2}}{n^{2^n}} = \lim_{n\to\infty}\left(1+n^{n^2-2^n}\right) = 1 < \infty$$

cf. $\lim_{n\to\infty} n^{n^2-2^n} = \exp\left(\log\left(\lim_{n\to\infty} n^{n^2-2^n}\right)\right) = \exp\left(\lim_{n\to\infty}\log\left(n^{n^2-2^n}\right)\right) = \exp\left(\lim_{n\to\infty}(n^2-2^n)\log n\right) = 0$

Thus $n^{2^n}+n^{n^2} \in O\left(n^{2^n}\right)$

3.2 #32

$f(x) \in O(g(x))$

$\Leftrightarrow$ $\exists\, C>0, \exists\, k>0$ s.t. for all $x>k$, $|f(x)| \le C|g(x)|$

$\Leftrightarrow$ $\exists\, D=1/C>0, \exists\, k>0$ s.t. for all $x>k$, $|g(x)| \ge D|f(x)|$

$\Leftrightarrow$ $g(x) \in \Omega(f(x))$

3.2 #46

$f_1(x) \in \Theta(g_1(x))$, $f_2(x) \in \Theta(g_2(x))$ then $\exists\, C_1, C_2, D_1, D_2 > 0$ s.t. for all $x>k$,

$$C_1|g_1(x)| \le f_1(x) \le C_2|g_1(x)|,\ D_1|g_2(x)| \le f_2(x) \le D_2|g_2(x)|.$$

Thus there exists $C_1D_1, C_2D_2 > 0$ s.t. for all $x>k$,

$$\therefore\ C_1D_1|g_1(x)g_2(x)| \le f_1(x)f_2(x) \le C_2D_2|g_1(x)g_2(x)|$$

Therefore, $f_1(x)f_2(x) \in \Theta(g_1(x)g_2(x))$