Discrete Mathematics Homework #5

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* Let $\mathrm{P}(E)$: probability of event E and let $\mathbf{E}(X)$: expectaion of a random variable X.

7.1.38 Let H, T denote heads and tails, respectively. Also note that for the sample space S, $|S| = 8$, and each element of the sample space have equally likely outcomes.

(a) $E_1 = \{\text{THH, THT, TTH, TTT}\}$, and $E_2 = \{\text{HHH, HHT, THH, THT}\}$. Thus $\mathrm{P}(E_1) = \mathrm{P}(E_2) = 4/8 = 1/2$. And since $E_1 \cap E_2 = \{\text{THH, THT}\}$, $\mathrm{P}(E_1 \cap E_2) = 2/8 = 1/4$, which is equal to $\mathrm{P}(E_1)\mathrm{P}(E_2) = 1/4$. E_1, E_2 are independent.

(b) $E_1 = \{\text{THH, THT, TTH, TTT}\}$, and $E_2 = \{\text{HHT, THH}\}$. Thus $\mathrm{P}(E_1) = 1/2$, $\mathrm{P}(E_2) = 1/4$. Since $E_1 \cap E_2 = \{\text{THH}\}$, $\mathrm{P}(E_1 \cap E_2) = 1/8$, which is equal to $\mathrm{P}(E_1)\mathrm{P}(E_2) = 1/8$. E_1, E_2 are independent.

(c) $E_1 = \{\text{HTH, HTT, TTH, TTT}\}$, and $E_2 = \{\text{HHT, THH}\}$. Thus $\mathrm{P}(E_1) = 1/2$, $\mathrm{P}(E_2) = 1/4$. Since $E_1 \cap E_2 = \varnothing$, $\mathrm{P}(E_1 \cap E_2) \neq \mathrm{P}(E_1)\mathrm{P}(E_2)$. Thus E_1, E_2 are dependent.

□

7.2.24 Let

A : First flip comes up tails when a fair coin is flipped 5 times.

B : Exactly 4 heads appear when a fair coin is flipped 5 times.

We want to calculate $\mathrm{P}(B \mid A)$.
$\mathrm{P}(A) = 1/2$ since we only care about the outputs of the first flip, and $\mathrm{P}(A \cap B) = 1/32$ since THHHH is the only possible case. From the definition, $\mathrm{P}(B \mid A) = \mathrm{P}(A \cap B)/\mathrm{P}(A) = 1/16$. □

7.2.34 The probability of success is p, and failure is $1 - p$.

(a) Fails for n times. Thus $(1-p)^n$.

(b) Since $1 = \mathrm{P}(\text{No successes}) + \mathrm{P}(\text{At least 1 success})$, the wanted probability is $1 - (1-p)^n$.

(c) For the case $n = 0$, it is trivial that the answer is 1. For $n > 0$, the only possible cases are: No successes, and only 1 success. The probability of 1 success is $np(1-p)^{n-1}$ since we fail for $n - 1$ times and succeed once, and we can succeed on any n trials, out of n trials. Thus the wanted probability is: $(1-p)^n + np(1-p)^{n-1}$.

(d) For the case $n = 0$, it is trivial that the answer is 0. For $n > 0$, since $1 = \mathrm{P}(\text{At most 1 success}) + \mathrm{P}(\text{At least 2 successes})$, the wanted probability is equal to $1-$ (answer of **(c)**). Thus the wanted probability is $1 - \left[(1-p)^n + np(1-p)^{n-1}\right]$.

□

7.3.14 From Bayes' theorem,

$$
\begin{aligned}
\mathrm{P}(F_2|E) = \frac{\mathrm{P}(E\cap F_2)}{\mathrm{P}(E)} &= \frac{\mathrm{P}(E|F_2)\mathrm{P}(F_2)}{\mathrm{P}(E\cap F_1)+\mathrm{P}(E\cap F_2)+\mathrm{P}(E\cap F_3)} \\
&= \frac{\mathrm{P}(E|F_2)\mathrm{P}(F_2)}{\mathrm{P}(E|F_1)\mathrm{P}(F_1)+\mathrm{P}(E|F_2)\mathrm{P}(F_2)+\mathrm{P}(E|F_3)\mathrm{P}(F_3)} \\
&= \frac{3/8\cdot 1/2}{2/7\cdot 1/6+3/8\cdot 1/2+1/2\cdot 1/3} = \frac{7}{15}
\end{aligned}
$$

□

7.3.22 **(a)** We can estimate that $\mathrm{P}(S) = s/(s+h)$, and $\mathrm{P}(\bar{S}) = h/(s+h)$.

(b) Let W be the event that an incoming message contains the word w. We want to calculate $\mathrm{P}(S \mid W)$. By Bayes' theorem,

$$
\begin{aligned}
\mathrm{P}(S\mid W) = \frac{\mathrm{P}(S\cap W)}{\mathrm{P}(W)} &= \frac{\mathrm{P}(W\mid S)\mathrm{P}(S)}{\mathrm{P}(W\cap S)+\mathrm{P}(W\cap \bar{S})} = \frac{\mathrm{P}(W\mid S)\mathrm{P}(S)}{\mathrm{P}(W\mid S)\mathrm{P}(S)+\mathrm{P}(W\mid \bar{S})\mathrm{P}(\bar{S})} \\
&= \frac{p(w)\cdot \dfrac{s}{s+h}}{p(w)\cdot \dfrac{s}{s+h}+q(w)\cdot \dfrac{h}{s+h}} = \frac{sp(w)}{sp(w)+hq(w)}
\end{aligned}
$$

□

7.4.18 Using that X, Y are non-negative and by the definition of Z, $Z(s) \le X(s)+Y(s)$. Thus,

$$
\mathbf{E}(Z) = \sum_{s\in S} Z(s)\mathrm{P}(s) \le \sum_{s\in S}(X(s)+Y(s))\mathrm{P}(s) = \sum_{s\in S} X(s)\mathrm{P}(s) + \sum_{s\in S} Y(s)\mathrm{P}(s) = \mathbf{E}(X)+\mathbf{E}(Y)
$$

□

7.4.38 **(a)** Let X be the number of cans filled on a day. By Markov's inequality,

$$
\mathrm{P}(X > 11000) < \frac{\mathbf{E}(X)}{11000} = \frac{10000}{11000} = \frac{10}{11}
$$

(b) By Chebyshev's inequality,

$$
\begin{aligned}
\mathrm{P}(9000 < X < 11000) &= \mathrm{P}(|X-\mathbf{E}(X)| < 1000) = 1-\mathrm{P}(|X-\mathbf{E}(X)| \ge 1000) \\
&\ge 1-\frac{\mathbf{V}(X)}{1000^2} = \frac{999}{1000}
\end{aligned}
$$

□

7.4.48 Define a random variable X_i as follows.

$$
X_i = \begin{cases} 1 & \text{if } i\text{-th ball is in the first bin} \\ 0 & \text{otherwise} \end{cases}
$$

And define $X = \sum_{i=1}^{m} X_i$. Then the expected number of balls that fall into the first bin is $\mathbf{E}(X)$.
By the linearity of expectation, $\mathbf{E}(X) = \sum_{i=1}^{m} \mathbf{E}(X_i)$. Now we calculate $\mathbf{E}(X_i)$.
By definition, $\mathbf{E}(X_i) = 1\cdot \mathrm{P}(X_i = 1) + 0\cdot \mathrm{P}(X_i = 0) = 1/n$. (Clearly $\mathrm{P}(X_i = 1) = 1/n$, and this situation is symmetric for all i). Thus the expected number of balls in the first bin is m/n. □