1. $A \in M_{m\times n}(F)$, $B \in M_{n\times m}(F) \Rightarrow tr(AB) = tr(BA)$

⌜ $tr(AB) = \sum_{i=1}^{m} (AB\text{의 } ii\text{성분}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{m} b_{ji}a_{ij} = \sum_{j=1}^{n} (BA\text{의 } jj\text{성분}) = tr(BA)$ ⌟

2. $V\times W$는 V와 isomorphic한 subspace를 갖는다.

⌜ Let $V' = \{(v,0) \in V\times W \mid v \in V\}$

Then, V' is a subspace of $V\times W$: $a(v,0) + b(w,0) = (av+bw,0) \in V'$ for $a,b \in F$, $v,w \in V$

Define a map $\varphi : V \to V'$ by $v \mapsto (v,0)$.

Then, · φ is linear: $\varphi(av+bw,0) = a(v,0) + b(w,0) = a\varphi(v) + b\varphi(w)$.

· φ is bijective: $(v,0) \longmapsto v$ is the inverse of φ.

Hence, V is isomorphic to V' ⌟

3. $U = \{(x,y,z)^t \in F^3 \mid 2x-y+4z=0\}$
$W = \{(x,y,z)^t \in F^3 \mid 5y-3z=0\}$ $\Rightarrow \dim(U+W) = ?$

⌜ we use the formula: $\dim(U+W) = \dim U + \dim W - \dim U\cap W$

① $\dim U = 2$: $\{(1,2,0), (0,4,1)\}$ is a basis of U

② $\dim W = 2$: $\{(1,0,0), (0,3,5)\}$ is a basis of W

③ $\dim U\cap W = 1$: $U\cap W = \{(x,y,z)^t \in F^3 \mid 2x-y+4z=0,\ 5y-3z=0\}$

$\Rightarrow \{(-17, 6, 1)\}$ is a basis of $U\cap W$.

By ①~③, we conclude $\dim(V+W) = 2+2-1 = 3$ ⌟

4. $L \in \mathcal{L}(F^n, F^n)$, $x_1, \dots, x_n \in F^n$

$\{L(x_1), \cdots, L(x_n)\}$: linearly independent $\Rightarrow$ L is an isomorphism.

$\ulcorner$

Since the cardinality of linearly independent subset $\{L(x_1), \cdots, L(x_n)\}$ is n and $\dim F^n = n$,

we have $\{L(x_1), \cdots, L(x_n)\}$ is a basis of F^n.

This implies that L is surjective.

Since F^n is a finite dimensional vector space, we conclude that L is bijective,

hence an isomorphism. $\lrcorner$

5. prove "Rank theorem". : row rank = column rank.

Dimension theorem : $n = \dim \ker L_A + \dim \operatorname{im} L_A$ 와 $\operatorname{im} L_A$ = column space of A

로부터 $n = \dim \ker L_A$ + row rank of A 만 보이면 충분하다.

A의 row-reduced echelon form R을 생각하면 A의 row rank는 R의 [최초의 1]의 개수와

같음을 알 수 있다.

한편 $AX = 0$ 의 solution space 의 dimension은 R의 [최초의 1]을 <u>갖지 않는</u> column의 개수와

같음을 알 수 있다.

그러므로 $n = \dim \ker L_A$ + row rank of A.

6. $L \in \mathcal{L}(F^n, F^m) \Rightarrow L_{[L]} = L$.

⌜ Since linear map is completely determined by its value on a basis,

it is enough to show $L_{[L]}(e_i) = L(e_i)$ for each i, where $\{e_i\}_{i=1}^{n}$ is

a canonical basis of F^n. Indeed, $L_{[L]}(e_i) = [L] \cdot e_i = L(e_i)$.

We conclude $L_{[L]} = L$ ⌟

7. $A \in M_{m \times n}(F)$, $B = M_{n \times r}(F)$, $AB = 0 \Rightarrow \operatorname{rank} A + \operatorname{rank} B \leq n$.

⌜ We consider linear maps L_A, L_B. By assumption, we have $L_A \cdot L_B = 0$.

This implies $\operatorname{im} L_{~~A~~B} \leq \ker L_A$, hence $\dim \operatorname{im} L_B \leq \dim ~~\ker~~ L_A$. (*)

On the ~~other~~ hand, we have $n \overset{(**)}{=} \dim \ker L_A + \dim \operatorname{im} L_A$ by dimension theorem.

By (*) & (**), we conclude $\operatorname{rank} A + \operatorname{rank} B = \dim \operatorname{im} L_A + \dim \operatorname{im} L_B$

$$\overset{(*)}{\leq} \dim \operatorname{im} L_A + \dim \ker L_A \overset{(**)}{=} n$$

⌟

8. $A \in M_{n \times n}(F)$, $AX = 0$ 이 trivial sol만 가진다. $\Rightarrow$ A : invertible.

⌜ $AX = 0$ 이 trivial sol만 가진다.

$\Rightarrow$ $\ker L_A = 0$

$\Rightarrow$ L_A is a monomorphism.

$\Rightarrow$ ~~By Since dim~~ By dimension theorem (~~or domain of L_A is finite dimensional~~ ~~and range~~),

L_A is an isomorphism.

$\Rightarrow$ A is invertible

⌟