공학수학 2 Extra Homework

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(a)

$$f(x,y) = \frac{2}{\pi r^2}$$

(b)

$$f(y) = \frac{4}{\pi r^2}\sqrt{r^2 - y^2}, \quad \mathbf{E}[Y] = \frac{4r}{3\pi}$$

(c)

$$\mathbf{E}[Y] = \int_{-r}^{r}\int_{0}^{\sqrt{r^2-x^2}} y\frac{2}{\pi r^2}dydx = \frac{4r}{3\pi}$$

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(a)

$$\mathbf{E}[X] = \frac{13}{6}, \quad \Pr(A) = \frac{5}{8}, \quad f_{X|A}(x) = \frac{2}{5}x, \quad \mathbf{E}[X|A] = \frac{38}{15}$$

(b)

$$\mathbf{E}[Y] = 5, \quad \mathbf{Var}[Y] = \frac{16}{3}$$

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(a)

$$f(x,y) = \frac{1}{ly}$$

(b)

$$f_X(x) = \frac{\log(l/x)}{l}$$

(c), (d)

$$\mathbf{E}[X] = \frac{1}{4}l$$

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(a)

$$f(x,y) = 1$$

(b)

$$f_Y(y) = -\frac{1}{2}y + 1$$

(c)

$$f_{X|Y}(x,y) = \frac{2}{2-y}$$

(d)

$$\mathbf{E}[X|Y=y] = \frac{2-y}{4}, \quad \mathbf{E}[X] = \frac{1}{3} \quad \mathbf{E}[Y] = \frac{2}{3}$$

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자명. □

그래도 아쉬우니 몇 자 적어보자면...

$$f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z) \cdot f_{Y,Z}(y,z) = f_{X|Y,Z}(x|y,z) \cdot f_{Y|Z}(y|z) \cdot f_Z(z)$$

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(a)

$$\Pr(X = \mathrm{H}) = \int_0^1 \Pr(X = \mathrm{H}|P = p) \cdot f_P(p)dp = \int_0^1 p^2 e^p dp = e - 2$$

(b)

$$f_{P|X=\mathrm{H}}(p) = \frac{p^2 e^p}{e-2}$$

(c)

$$\begin{aligned}\Pr(X_2 = \mathrm{H}|X_1 = \mathrm{H}) &= \frac{\Pr(X_1 = \mathrm{H} \cap X_2 = \mathrm{H})}{e-2} \\ &= \frac{\int_0^1 \Pr(X_1 = \mathrm{H} \cap X_2 = \mathrm{H}|P = p) \cdot f_P(p)dp}{e-2} \\ &= \frac{\int_0^1 p^3 e^p dp}{e-2} = \frac{6-2e}{e-2}\end{aligned}$$

Note: X_1, X_2 는 dependent 하지만 $X_1|P, X_2|P$ 는 independent.

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(a) $X = x$ 일 때, $Y = z - x$ 이며 X, Y는 독립.

$$f_{Z|X}(z|x) = \frac{f_{Z,X}(z,x)}{f_X(x)} = \frac{f_Y(z-x)f_X(x)}{f_X(x)} = f_Y(z-x)$$

(b)

$$\begin{aligned}f_{X|Z=z}(x) &= \frac{f_{Z|X}(z|x) \cdot f_X(x)}{f_Z(z)} = \frac{f_Y(z-x) \cdot f_X(x)}{\int_0^\infty f_{Z|X}(z|x) \cdot f_X(x)dx} \\ &= \frac{f_Y(z-x) \cdot f_X(x)}{\int_0^z f_Y(z-x) \cdot f_X(x)dx} = \frac{\lambda^2 e^{-\lambda z}}{z\lambda^2 e^{-\lambda z}} = \frac{1}{z}\end{aligned}$$

Note: $Z = X + Y$ 이고 $Y \geq 0$ 이므로 $X \leq Z$.

(c) (b) 에서와 마찬가지로 한다.

$$f_{X|Z=z}(x) = \frac{\frac{1}{2\pi\sigma_1\sigma_2}\exp(-\frac{x^2}{2\sigma_1^2} - \frac{(z-x)^2}{2\sigma_2^2})}{\int_{-\infty}^\infty \frac{1}{2\pi\sigma_1\sigma_2}\exp(-\frac{x^2}{2\sigma_1^2} - \frac{(z-x)^2}{2\sigma_2^2})dx}$$

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(a)

$$\frac{1/2}{1/2 + 1/3 \cdot 1/2} = \frac{3}{4}$$

(b) Define X: 맞은 문제 개수, Θ: Soo 가 아는 문제 개수.

$$\begin{aligned}\Pr(\Theta = k | X = 6) &= \frac{\Pr(X = 6 | \Theta = k) \cdot \Pr(\Theta = k)}{\Pr(X = 6)} \\ &= \frac{\binom{10-k}{6-k}(1/3)^{6-k}(2/3)^4\binom{10}{k}(1/2)^k}{\sum_{k=0}^{6} \Pr(X = 6|\Theta = k)\Pr(\Theta = k)} \\ &= \frac{\binom{10-k}{6-k}(1/3)^{6-k}(2/3)^4\binom{10}{k}(1/2)^k}{\sum_{k=0}^{6} \binom{10-k}{6-k}(1/3)^{6-k}(2/3)^4\binom{10}{k}(1/2)^k}\end{aligned}$$

Note: $k \leq 6$.

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(a) Define J: 아는 문제의 개수.

$$\Pr(\Theta = \theta_i | k) = \frac{\Pr(k|\Theta = \theta_i)\Pr(\Theta = \theta_i)}{\Pr(k)}$$

$$\begin{aligned}\Pr(k|\Theta = \theta_i) &= \sum_{j=0}^{k} \Pr(k, J = j | \Theta = \theta_i) \\ &= \sum_{j=0}^{k} \binom{10}{j} (\theta_i)^j (1 - \theta_i)^{10-j}) \cdot \binom{10-j}{k-j} (1/3)^{k-j} (2/3)^{10-k}\end{aligned}$$

(b)

$$\begin{aligned}\Pr(M = m | X = 5) &= \frac{\Pr(M = m \cap X = 5)}{\Pr(X = 5)} \\ &= \sum_{i=1}^{3} \Pr(M = m | X = 5, \Theta = \theta_i)\Pr(\Theta = \theta_i)\end{aligned}$$

$$\Pr(M = m | X = 5, \Theta = \theta_i) = \sum_{k=0}^{m} \binom{10}{k} (\theta_i)^k (1 - \theta_i)^{10-k} \cdot \binom{10-k}{5-k} \left(\frac{1}{3}\right)^{5-k} \left(\frac{2}{3}\right)^{10-k}$$

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$\Pr(\Theta = 1) = 0.3, \Pr(\Theta = 2) = 0.7$ 임이 주어져 있다. 그리고 $\int_5^{60} f_{X|\Theta}(x|\Theta = i)dx = 1$ 로부터 c_i 의 값을 찾는다.

(a) $X = 20$.

$$\begin{aligned}
\Pr(\Theta = 1|X = 20) &= \frac{\Pr(\Theta = 1, X = 20)}{\Pr(X = 20)} \\
&= \frac{f_{X|\Theta=1}(20) \cdot \Pr(\Theta = 1)}{f_{X|\Theta=1}(20) \cdot \Pr(\Theta = 1) + f_{X|\Theta=2}(20) \cdot \Pr(\Theta = 2)} \\
&= \frac{0.3c_1 e^{-0.04 \cdot 20}}{0.3c_1 e^{-0.04 \cdot 20} + 0.7c_2 e^{-0.16 \cdot 20}}
\end{aligned}$$

$$\begin{aligned}
\Pr(\Theta = 2|X = 20) &= \frac{\Pr(\Theta = 2, X = 20)}{\Pr(X = 20)} \\
&= \frac{f_{X|\Theta=2}(20) \cdot \Pr(\Theta = 2)}{f_{X|\Theta=1}(20) \cdot \Pr(\Theta = 1) + f_{X|\Theta=2}(20) \cdot \Pr(\Theta = 2)} \\
&= \frac{0.7c_2 e^{-0.16 \cdot 20}}{0.3c_1 e^{-0.04 \cdot 20} + 0.7c_2 e^{-0.16 \cdot 20}}
\end{aligned}$$

MAP 는 두 확률 중 큰 쪽을 택한다. 작은 쪽이 error probability.

(b) Define $X = (X_1, \cdots, X_5)$, 각각 TA 들이 푸는데 걸린 시간.

$$\begin{aligned}
\Pr(\Theta = 1|X = (20, 10, 25, 15, 35)) &= \frac{\Pr(\Theta = 1, X = (20, 10, 25, 15, 35))}{\Pr(X = (20, 10, 25, 15, 35))} \\
&= \frac{f_{X|\Theta=1}((20, 10, 25, 15, 35)) \cdot \Pr(\Theta = 1)}{\Pr(X = (20, 10, 25, 15, 35))}
\end{aligned}$$

$$f_{X|\Theta=1}(X) = \prod_{i=1}^{5} f_{X_i|\Theta=1}(x) = c_1^5 e^{-0.04(x_1 + \cdots + x_5)}$$

비슷한 방법으로 $\Theta = 2$ 일 때도 할 수 있다.

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(a)

$$\Pr(\text{box 1}|\text{black}) = \frac{p \cdot 1/3}{p \cdot 1/3 + (1-p) \cdot 2/3} = \frac{p}{2-p}$$

$$\Pr(\text{box 2}|\text{black}) = \frac{(1-p) \cdot 2/3}{p \cdot 1/3 + (1-p) \cdot 2/3} = \frac{2-2p}{2-p}$$

$$\Pr(\text{box 1}|\text{white}) = \frac{p \cdot 2/3}{p \cdot 2/3 + (1-p) \cdot 1/3} = \frac{2p}{p+1}$$

$$\Pr(\text{box 2}|\text{white}) = \frac{(1-p) \cdot 1/3}{p \cdot 2/3 + (1-p) \cdot 1/3} = \frac{1-p}{p+1}$$

(b) $p = 1/2$ 일 때, 각각의 확률은 차례대로 $1/3, 2/3, 2/3, 1/3$. 검은 공을 뽑았을 때 box 1 에서 나왔다면 error probability 는 $2/3$. 나머지 경우에도 비슷하게 한다.

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실제 차의 속도 $\Theta \sim \mathrm{U}[55, 75]$, 측정값 $X = \Theta + W$, $W \sim \mathrm{U}[0, 5]$.

$$f_{X|\Theta} = \frac{1}{5}, \quad \theta \le x \le \theta + 5$$
$$f_{\Theta,X}(\theta, x) = f_\Theta(\theta) \cdot f_{X|\Theta}(x|\theta) = \frac{1}{20} \cdot \frac{1}{5} = \frac{1}{100}$$

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$$\begin{aligned}\Pr(\Theta = i|X = j) &= \frac{\Pr(X = j|\Theta = i) \cdot \Pr(\Theta = i)}{\Pr(X = j)} \\ &= \frac{(1/i) \cdot (1/100)}{\sum_{i=j}^{100} \Pr(X = j|\Theta = i) \cdot \Pr(\Theta = i)} \\ &= \frac{1/i}{\sum_{i=j}^{100} 1/i}\end{aligned}$$

조화급수... (답이 없다)

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(a) Independent, identically distributed 이므로 $\mathbf{E}[Y_i|Y] = \mathbf{E}[Y_j|Y]$.
By linearity of expectation,

$$Y = \mathbf{E}[Y|Y] = \sum_{i=1}^{n} \mathbf{E}[Y_i|Y] = n\mathbf{E}[Y_i|Y] \qquad \Rightarrow \qquad \mathbf{E}[Y_i|Y] = \frac{Y}{n}$$

(b) $X \sim \mathcal{N}(\mu_X, \sigma_X^2), \quad Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ 이고 X, Y 가 독립이면

$$X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$$

이제 Θ, W 를 표준정규분포들의 합으로 생각하면,

$$\mathbf{E}[\Theta|\Theta + W] = \frac{k}{k + m}(\Theta + W)$$

(c) Poisson random variable 의 합은 poisson random variable. 따라서 $X \sim \mathrm{Poi}(1)$ 들의 합으로 생각하면,

$$\mathbf{E}[\Theta|\Theta + W] = \frac{\lambda}{\lambda + \mu}(\Theta + W)$$