선형대수학 2 숙제 #3

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13.3.11 Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\begin{aligned} A \in \mathbf{SL}_2(F) &\iff \det A = ad - bc = 1 \\ &\iff A^{\mathbf{t}} \cdot JA = \begin{pmatrix} 0 & ad-bc \\ bc-ad & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = J \\ &\iff A \in \mathbf{Sp}_2^J(F) \end{aligned}$$

$\therefore \mathbf{SL}_2(F) = \mathbf{Sp}_2^J(F)$.

13.4.4 $\mathbf{SO}^\circ(1,1)$ is a normal subgroup of $\mathbf{O}(1,1)$. For any $A \in \mathbf{O}(1,1)$, $S \in \mathbf{SO}^\circ(1,1)$, we know that $A^{-1}SA \in \mathbf{SO}(1,1)$. Since all elements of $\mathbf{SO}^\circ(1,1)$ have positive trace, and $\operatorname{tr}(A^{-1}SA) = \operatorname{tr} S > 0$, $A^{-1}SA \in \mathbf{SO}^\circ(1,1)$. $\therefore \mathbf{SO}^\circ(1,1) \lhd \mathbf{O}(1,1)$.

13.4.6 Suppose there existed an isomorphism $\varphi : \mathbf{O}(2) \to \mathbf{O}(1,1)$. Take $R = R_{2\pi/3} \in \mathbf{O}(2)$. Then $R^3 = I$, so $I = \varphi(I) = \varphi(R^3) = (\varphi(R))^3$, and $\varphi(R) \in \mathbf{O}(1,1)$. Since φ is an isomorphism, there exists $I \neq \varphi(R) \in \mathbf{O}(1,1)$. Now we show that this $S = \varphi(R) \neq I$ doesn't exist.
First, we see that $S^3 = I$, and since S is a real matrix, its determinant must be 1. Using the coset decomposition of $\mathbf{O}(1,1)$,
Case 1. $S \in \mathbf{SO}^\circ(1,1)$. Let $S = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$, then $S^3 = \begin{pmatrix} \cosh 3t & \sinh 3t \\ \sinh 3t & \cosh 3t \end{pmatrix} = I$ then $t = 0$, so $S = I$, contradiction.
Case 2. $S \in -I \cdot \mathbf{SO}^\circ(1,1)$. Let $S = \begin{pmatrix} -\cosh t & -\sinh t \\ -\sinh t & -\cosh t \end{pmatrix}$, then $S^3 = \begin{pmatrix} -\cosh 3t & -\sinh 3t \\ -\sinh 3t & -\cosh 3t \end{pmatrix} = I$ then t does not exist since $\cosh 3t \geq 1$. Such S doesn't exist.
Case 3. $S \in \operatorname{diag}(1,-1) \cdot \mathbf{SO}^\circ(1,1)$ or $S \in \operatorname{diag}(-1,1) \cdot \mathbf{SO}^\circ(1,1)$. In both cases, since all matrices in $\mathbf{SO}^\circ(1,1)$ have determinant 1, $\det S = -1$, contradiction.
Thus, such $I \neq S \in \mathbf{O}(1,1)$ does not exist, and φ cannot be an isomorphism.

$\therefore \mathbf{O}(2) \not\approx \mathbf{O}(1,1)$.

13.5.7 Define a linear map $\varphi_w : V \to \mathbb{R}$ as

$$\varphi_w(x) = B(w,x) \qquad (x \in V)$$

Since $\varphi_w(w) \neq 0$, φ_w is surjective. (You can scale w by any scalar c then $\varphi_w(cw)$ will span $\mathbb{R}$.) Now, by Dimension Theorem, $\dim \langle w \rangle^\perp = \dim \ker \varphi_w = \dim V - \dim \operatorname{im} \varphi_w = \dim V - 1 = \dim V - \dim \langle w \rangle$ $\therefore \dim \langle w \rangle^\perp = \dim V - \dim \langle w \rangle$.

13.5.13 (나) Suppose $x \in (U+W)^\perp$. $\forall u \in U, \forall w \in W$, $B(u+w,x) = 0$. Now if we set $u = 0$, then $B(w,x) = 0$. If we set $w = 0$, we have $B(u,x) = 0$. We conclude that $x \in U^\perp$ and $x \in W^\perp$. Therefore $x \in U^\perp \cap W^\perp$. $\therefore (U+W)^\perp \subseteq U^\perp \cap W^\perp$

Now suppose $x \in U^\perp \cap W^\perp$. Then $B(w,x) = 0$ and $B(u,x) = 0$. Since B is bilinear, $B(u+w,x) = 0$. Thus $x \in (U+W)^\perp$. $\therefore U^\perp \cap W^\perp \subseteq (U+W)^\perp$

$$(U+W)^\perp = U^\perp \cap W^\perp$$

(다) From the result above, replace U with $U^\perp$, W with $W^\perp$ and take $\perp$ on both sides. Then we have

$$((U^\perp + W^\perp)^\perp)^\perp = ((U^\perp)^\perp \cap (W^\perp)^\perp)^\perp$$

Using the fact that $(W^\perp)^\perp = W$ ($\forall\, W \leq V$),

$$U^\perp + W^\perp = (U \cap W)^\perp$$

13.5.20 (나) $B^J_{\mathcal{E}}(Y,Y) = \begin{pmatrix}\cosh\frac{t}{2} & -\sinh\frac{t}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cosh\frac{t}{2}\\ -\sinh\frac{t}{2}\end{pmatrix} = \cosh^2\frac{t}{2} - \sinh^2\frac{t}{2} = 1 \neq 0.$

For the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$,

$$S_Y(\mathbf{e}_1) = \mathbf{e}_1 - 2\frac{B^J_{\mathcal{E}}(\mathbf{e}_1, Y)}{B^J_{\mathcal{E}}(Y,Y)}Y = \begin{pmatrix}1 - 2\cosh^2\frac{t}{2}\\ 2\cosh t\sinh t\end{pmatrix} = \begin{pmatrix}-\cosh t\\ \sinh t\end{pmatrix}$$

$$S_Y(\mathbf{e}_2) = \mathbf{e}_2 - 2\frac{B^J_{\mathcal{E}}(\mathbf{e}_2, Y)}{B^J_{\mathcal{E}}(Y,Y)}Y = \begin{pmatrix}-2\cosh t\sinh t\\ 1 + 2\sinh^2\frac{t}{2}\end{pmatrix} = \begin{pmatrix}-\sinh t\\ \cosh t\end{pmatrix}$$

Therefore,

$$[S_Y]^{\mathcal{E}}_{\mathcal{E}} = \begin{pmatrix}-\cosh t & -\sinh t\\ \sinh t & \cosh t\end{pmatrix} \in \mathbf{O}(1,1)$$

Since $S_Y(Y) = -Y$ and for $v = \left(\sinh\frac{t}{2}, \cosh\frac{t}{2}\right)^{\mathsf{t}}$, $S_Y(v) = v$. (This v was found by solving $B^J_{\mathcal{E}}(Y,v) = 0$.) $\therefore S_Y$ is a reflection of $\mathbf{O}(1,1)$.

13.7.5 (가) $\forall w \in W$, if $f \in W^{\mathsf{perp}}$, $\varepsilon(w,f) = 0$. $w \in (W^{\mathsf{perp}})^{\mathsf{perp}}$. $\therefore W \leq (W^{\mathsf{perp}})^{\mathsf{perp}}$.
$\dim W = \dim V - \dim W^{\mathsf{perp}} = \dim (W^{\mathsf{perp}})^{\mathsf{perp}}$, and now the result directly follows by dimension argument. $\therefore W = (W^{\mathsf{perp}})^{\mathsf{perp}}$.

13.7.7 (가) Use definitions.

$$\begin{aligned} g \in \ker L^* &\iff L^*(g) = g \circ L = 0\\ &\iff 0 = (g \circ L)(v) = g(Lv) = \varepsilon(Lv, g) \quad (v \in V)\\ &\iff g \in (\mathrm{im}\, L)^{\mathsf{perp}} \quad (Lv \in \mathrm{im}\, L)\end{aligned}$$

$\therefore \ker L^* = (\mathrm{im}\, L)^{\mathsf{perp}}$.

(나) If $f \in \mathrm{im}\, L^*$, $\exists g \in W^*$ s.t. $L^*(g) = g \circ L = f$. Now we have $0 = g(Lv) = (g \circ L)(v) = f(v) = \varepsilon(v, f)$ $(v \in \ker L)$. $f \in (\ker L)^{\mathsf{perp}}$, $\therefore \mathrm{im}\, L^* \leq (\ker L)^{\mathsf{perp}}$.
Now we show $(\mathrm{im}\, L^*)^{\mathsf{perp}} \leq \ker L$ instead. If $v \in (\mathrm{im}\, L^*)^{\mathsf{perp}}$, $\varepsilon(v, L^*(g)) = 0$ for any $L^*(g) \in \mathrm{im}\, L^*$, where $g \in W^*$. Since $0 = \varepsilon(v, g \circ L) = (g \circ L)(v) = g(Lv)$ for any g, Lv must be 0. $v \in \ker L$. Take perp on both sides, $\therefore (\ker L)^{\mathsf{perp}} \leq \mathrm{im}\, L^*$.
$\therefore \mathrm{im}\, L^* = (\ker L)^{\mathsf{perp}}$.

13.8.7 **(가)** Let $v = (v_1, v_2)^{\mathbf{t}} \in \mathbb{R}^2$. Since $\varphi_B^V(v) = f$, we evaluate at $w = (x, y) \in \mathbb{R}^2$.

$$B_{\mathcal{E}}^J(w, v) = \varphi_B^V(v)(w) = f(w) = x + y$$

and

$$B_{\mathcal{E}}^J(w, v) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_1 x - v_2 y = x + y$$

This must hold for all x, y, thus $v_1 = 1, v_2 = -1$. $\therefore v = (1, -1)^{\mathbf{t}}$.

(나) In a similar fashion, we now easily see that

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = (2v_1 - v_2)x + (-v_1 + 2v_2)y = x + y \quad (x, y \in \mathbb{R})$$

Thus $2v_1 - v_2 = 1$ and $-v_1 + 2v_2 = 1$. $\Rightarrow v_1 = 1, v_2 = 1$. $\therefore v = (1, 1)^{\mathbf{t}}$.

13.8.8 Let $w_j = (\varphi_B^V)^{-1}(v_j^*)$. ($(\varphi_B^V)^{-1}$ exists since B is non-degenerate) Then

$$B(v_i, w_j) = \varphi_B^V(w_j)(v_i) = v_j^*(v_i) = \delta_{ij} \qquad (1 \le i, j \le n = \dim V)$$

Because φ_B^V is an isomorphism, and $\{v_i^*\}$ is a basis, $\{w_j\}$ is also a basis (of V).
Now we check if $[I]_{\mathfrak{C}}^{\mathfrak{B}} = [B]_{\mathfrak{B}}$. This is equivalent to

$$v_i = \sum_k B(v_k, v_i) w_k, \quad (i = 1, \ldots, n)$$

Suppose we let

$$v_i = \sum_k b_{ki} w_k, \quad (i = 1, \ldots, n)$$

We will show that $b_{ki} = B(v_k, v_i)$. With a bit of calculation, for all i, j,

$$\begin{aligned} B(v_j, v_i) &= \varphi_B^V(v_i)(v_j) = \varphi_B^V\left(\sum_k b_{ki} w_k\right)(v_j) \\ &= \left(\sum_k b_{ki} \varphi_B^V(w_k)\right)(v_j) = \left(\sum_k b_{ki} v_k^*\right)(v_j) \\ &= b_{ji} \end{aligned}$$

Which was what we wanted.

13.9.6 **(가)** Consider the standard basis $\mathcal{E}$ of $\mathbb{R}^4$. $x = (x_1, x_2, x_3, x_4)^{\mathbf{t}} \in \mathbb{R}^4$.

$$[L^{\mathbf{t}}(x)]_{\mathcal{E}}^{\mathcal{E}} = [L^{\mathbf{t}}]_{\mathcal{E}}^{\mathcal{E}} \cdot [x]_{\mathcal{E}} = \left([L]_{\mathcal{E}}^{\mathcal{E}}\right)^{\mathbf{t}} \cdot [x]_{\mathcal{E}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^{\mathbf{t}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{pmatrix}$$

$$\therefore L^{\mathbf{t}}\left((x_1, x_2, x_3, x_4)^{\mathbf{t}}\right) = (x_2, x_3, x_4, x_1)^{\mathbf{t}}.$$

(나) In a similar fashion, $x = (x, y)^{\mathbf{t}} \in \mathbb{R}^2$

$$\begin{aligned}[L^{\mathbf{t}}(x)]_{\mathcal{E}}^{\mathcal{E}} &= [L^{\mathbf{t}}]_{\mathcal{E}}^{\mathcal{E}} \cdot [x]_{\mathcal{E}} = J^{-1} \left([L]_{\mathcal{E}}^{\mathcal{E}}\right)^{\mathbf{t}} J \cdot [x]_{\mathcal{E}} \\ &= \operatorname{diag}(1, -1) \cdot \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}^{\mathbf{t}} \cdot \operatorname{diag}(1, -1) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ -x + 3y \end{pmatrix}\end{aligned}$$

$$\therefore L^{\mathbf{t}}\left((x_1, x_2)^{\mathbf{t}}\right) = (x + y, -x + 3y)^{\mathbf{t}}.$$