

**March 29th, 2019**

**Remark.**  $\limsup$  is the limit of  $\sup$ . If  $\sup$  is easy to calculate, find  $\sup$  and take the limit.

### Quiz 1 Solutions

#1. Given set  $A$ ,  $\text{int}(A)$ ,  $A'$ , determine whether the set is open or closed.

1.  $A = \mathbb{N} \subset \mathbb{R}$ .  $\text{int}(A) = \emptyset$ ,  $A' = \emptyset$ ,  $A$  is closed.
2.  $\mathbb{Q} \subset \mathbb{R}$ .  $\text{int}(\mathbb{Q}) = \emptyset$ ,  $\mathbb{Q}' = \mathbb{R}$ ,  $\mathbb{Q}$  is neither open nor closed.
3.  $C = [0, 1] \cup (2, 3) \cap \{4\} \subset \mathbb{R}$ .  $\text{int}(C) = (0, 1) \cup (2, 3)$ ,  $C' = [0, 1] \cup [2, 3]$ ,  $C$  is neither open nor closed.
4.  $D = \bigcup_{n=1}^{\infty} \{(\frac{1}{n}, y) : 0 \leq y \leq 1\} \subset \mathbb{R}^2$ .  $\text{int}(D) = \emptyset$ ,  $D' = D \cup \{(0, y) : 0 \leq y \leq 1\}$ ,  $D$  is neither open nor closed. ( $\because \text{int}D \neq D$ ,  $\overline{D} \neq D$ )

#2. Find a limit point of given set.

1.  $A = \mathbb{Q} \subset \mathbb{R}$ . 0 is a limit point. (Directly follows from Archimedes' principle)
2.  $B = \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ . 0 is a limit point of  $B$ . (Also directly follows from Archimedes')
3.  $C = \{2^{-n} + 3^{-m} : n, m \in \mathbb{N}\} \subset \mathbb{R}$ . 0 is a limit point of  $C$ . Given  $\epsilon > 0$ , exists  $N \in \mathbb{N}$  such that for  $n, m \geq N$ ,  $2^{-n} < \epsilon/2$ ,  $3^{-m} < \epsilon/2$ . Then  $0 \neq 2^{-n} + 3^{-m} < \epsilon$ .

#3. True or False? If false, find a counterexample.

1.  $\overline{A \cup B} = \overline{A} \cup \overline{B}$  **True**
2.  $\overline{A \cap B} = \overline{A} \cap \overline{B}$  **False**. Set  $A = (0, 1)$ ,  $B = (1, 2)$ .  
**Correct Statement:**  $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$
3.  $\text{int}(A \cup B) = \text{int}(A) \cup \text{int}(B)$  **False**. Set  $A = [0, 1]$ ,  $B = [1, 2]$ .  
**Correct Statement:**  $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$
4.  $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$  **True**

**Thm.**  $A \subset B \implies \overline{A} \subset \overline{B}, \text{int}(A) \subset \text{int}(B).$

**Proof.**

- We need to show  $A' \subset B'$ . Let  $x \in A'$ .  
 $\implies \forall \epsilon > 0, N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset.$   
 $\implies \forall \epsilon > 0, N(x, \epsilon) \cap (B \setminus \{x\}) \neq \emptyset$   
 $\implies x \in B'.$
- Let  $x \in \text{int}(A)$   
 $\implies \exists \epsilon > 0, N(x, \epsilon) \subset A \implies N(x, \epsilon) \subset B \implies x \in \text{int}(B).$

**Proof of (c).**  $A, B \subset A \cup B$

$\implies \text{int}(A), \text{int}(B) \subset \text{int}(A \cup B).$  Thus  $\text{int}(A) \cup \text{int}(B) \subset \text{int}(A \cup B)$

**Proof of (d).**  $A \cap B \subset A, B \implies \text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B).$  Thus  $\text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B)$   
Suppose  $x \in \text{int}(A) \cap \text{int}(B).$  Then  $\exists \epsilon_A, \epsilon_B > 0$  s.t.  $N(x, \epsilon_A) \subset A, N(x, \epsilon_B) \subset B.$  Take  $\epsilon = \min\{\epsilon_A, \epsilon_B\}/2.$  Then  $N(x, \epsilon) \subset A, B.$  Therefore  $N(x, \epsilon) \subset A \cap B, x \in \text{int}(A \cap B).$

**Example.**  $A = \{(x, y) : x^2 + 2y^2 < 1\}.$   $\text{int}(A) = A, A' = \{(x, y) : x^2 + 2y^2 \leq 1\}.$

Suppose  $(x_0, y_0) \in A.$   $x_0^2 + 2y_0^2 = 1 - \delta < 1$  for some  $\delta > 0.$  By symmetry, let  $x_0, y_0 > 0.$  From

$$(x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 = x_0^2 + 2y_0^2 + \epsilon(2x_0 + 4y_0 + 3\epsilon) < 1$$

, we want  $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \delta.$  Set  $\epsilon < 1/10.$  Then  $\epsilon(2x_0 + 4y_0 + 3\epsilon) < \epsilon(2x_0 + 4y_0 + 3) < \delta.$

Now set  $\epsilon = \min \left\{ \frac{1}{2(2x_0 + 4y_0 + 3)}, \frac{1}{100} \right\} > 0.$

Then  $|x - x_0| < \epsilon, |y - y_0| < \epsilon.$   $x_0^2 + 2y_0^2 < (x_0 + \epsilon)^2 + 2(y_0 + \epsilon)^2 < 1.$   $N((x_0, y_0), \epsilon) \subset A.$

Interior points are limit points, and for the points  $(x_0, y_0)$  on the border, consider a sequence  $(x_0 - 1/n, y_0 - 1/n).$  Then the elements are in  $A$  and they converge to  $(x_0, y_0).$  Thus the border is also included in  $A'.$

April 1st, 2019

$\text{int}A : x \in A \text{ s.t. } N(x, \epsilon) \subset A \text{ for some } \epsilon > 0.$

$A' : x \in \mathbb{R}^d \text{ s.t. } N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset \text{ for } \forall \epsilon > 0$

$\overline{A} : x \in \mathbb{R}^d \text{ s.t. } N(x, \epsilon) \cap A \neq \emptyset, \forall \epsilon > 0, \overline{A} = A \cup A'$

**Example.**  $A = [0, 1) \cup \{2\}$ .  $1 \in A', 2 \notin A', 2 \in \overline{A}$

**Prop 2.3.3**  $x \in A' \implies N(x, \epsilon) \cap (A \setminus \{x\})$  는 무한집합이다.

**Proof.** 유한집합이라고 가정하자.  $N(x, \epsilon) \cap (A \setminus \{x\}) = \{x_1, \dots, x_n\}$  이라 할 수 있다. Set  $\delta = \min\{\|x - x_i\| : \forall i\}$ . Then  $N(x, \delta) \cap (A \setminus \{x\}) = \emptyset$ . 모순.

그래서 사실은 공집합이 아닌 것으로 정의했지만 **사실은** 무한집합이다.

**Remark.**  $A' \neq \emptyset \implies A$ 는 무한집합.

(대우)  $A$ 가 유한집합이면 극한점이 존재하지 않는다. (2.2 보기 4)

(역) **거짓.**  $A = \{1, 2, \dots\}$  이면  $A' = \emptyset$ .

그러면 역이 언제 성립하나요? 다음 단원 내용!

**Definition.** Convergence in  $\mathbb{R}^d$

Let  $\langle x_n \rangle$  be a sequence in  $\mathbb{R}^d$ .

$$\lim_{n \rightarrow \infty} x_n = x \iff \forall \epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies \|x_n - x\| < \epsilon)$$

**Exercise.**  $x_n = (x_n^{(1)}, \dots), x = (x^{(1)}, \dots)$  일 때,  $x_n \rightarrow x \iff \forall i, x_n^{(i)} \rightarrow x^{(i)}$

**Notation.**  $A \subset \mathbb{R}^d; \langle x_n \rangle$  is a sequence in  $A \iff \forall n, x_n \in A$

**Theorem 2.2.2**

1.  $x \in A' \iff \exists \langle x_n \rangle$  in  $A \setminus \{x\}$  such that  $x_n \rightarrow x$

2.  $x \in \overline{A} \iff \exists \langle x_n \rangle$  in  $A$  such that  $x_n \rightarrow x$

**Proof.**

1. ( $\implies$ )  $x_n \in N(x, \frac{1}{n}) \cap (A \setminus \{x\})$  이라 하자. (공집합이 아니므로 이러한 원소가 존재한다.)  
그러면  $\|x_n - x\| < 1/n$  이므로  $x_n$  은  $x$  로 수렴한다. 그리고  $x_n \in A \setminus \{x\}$  이므로 수열이  $A \setminus \{x\}$  에 있다.

2. Left as exercise. Replace  $A \setminus \{x\}$  with  $A$ .

**Theorem 2.2.3.** The following are equivalent.

1.  $F$  is closed.
2.  $F' \subset F$ .
3.  $F = \overline{F}$
4. For a sequence  $\langle x_n \rangle$  in  $F$ ,  $\lim_{n \rightarrow \infty} x_n = x \implies x \in F$ .

**Proof.**

- (1)  $\iff$  (3) ( $\overline{F}$ : smallest closed set containing  $F$ .)  
 (2)  $\iff$  (3) 은 자명.  
 (1)  $\iff$  (4) by the above theorem. (Thm 2.2.2)

**Applications.**

1.  $A'$  is closed.

*Proof.* We want to show that  $(A')' \subset A'$ .

We want to show:  $x \in (A')' \implies x \in A'$ .

( $A'$  이 공집합이면 자명. 공집합이 아니라고 가정하고...)

Given  $\epsilon > 0$ ,  $N(x, \epsilon) \cap (A' \setminus \{x\}) \neq \emptyset$ . Take an element  $y \in A'$  from this set. Now set  $\delta = \min\{\|x - y\|, \epsilon - \|x - y\|\}$  then we have  $N(y, \delta) \cap (A \setminus \{y\}) \neq \emptyset$ . ( $\because y \in A'$ )  
 $z \in N(y, \delta) \cap (A \setminus \{y\})$  라 하자.

(a)  $z \in A \setminus \{y\} \subset A$ .

(b)  $\|x - z\| \leq \|x - y\| + \|y - z\| < \|x - y\| + \delta \leq \epsilon$  ( $z \in N(y, \delta)$ )

(c)  $\|x - z\| \geq \|x - y\| - \|y - z\| > \|x - y\| - \delta \geq 0$  (By the choice of  $\delta$ .) Thus  $x \neq z$ .

Therefore  $z \in N(x, \epsilon)$  (by (b)),  $z \in A \setminus \{x\}$  (by (a), (c)).

$x \in A'$  since  $N(x, \epsilon) \cap (A \setminus \{x\})$  is not empty.

2.  $A \subset \mathbb{R}$ : closed and bounded  $\implies \inf A = \min A$ ,  $\sup A = \max A$ . (Existence)

*Proof.* Let  $\sup A = x \notin A$ . ( $\sup A \in A$  이면 자명)

*Claim.*  $x \in A'$ .

*Proof of Claim.*  $\forall \epsilon > 0$ ,  $N(x, \epsilon) = (x - \epsilon, x + \epsilon)$

$x = \sup A$  이므로  $x - \epsilon$  is not an upper bound.

$\exists y$  such that  $y \in (x - \epsilon, x)$

$y \in N(x, \epsilon) \cap (A \setminus \{x\}) \neq \emptyset$  이므로  $x$  는 극한점.

따라서  $x \in A' \subset A$  (closed set 이므로 Thm 2.2.3 (2)) 모순.

$\sup A \in A$  이므로 이 값이 최댓값이다.

## 2.3 유계집합과 코시수열

핵심: Thm 2.3.4, Thm 2.3.7

**Definition.**  $\langle x_n \rangle$ : 유계수열(bounded sequence)  $\iff \exists M > 0$  s.t.  $\|x_n\| \leq M$  for all  $n \in \mathbb{N}$ .

**Definition.**  $n_1 < n_2 < \dots$  : sequence in  $\mathbb{N}$  이라 하자.  $\langle x_{n_k} \rangle_{k=1}^\infty = (x_{n_1}, x_{n_2}, \dots)$  를  $\langle x_n \rangle$ 의 부분수열(subsequence)이라 한다.

**Theorem 2.3.4** (Bolzano-Weierstrass Theorem)

If  $\langle x_n \rangle$  is bounded, there exists a convergent subsequence of  $\langle x_n \rangle$ .

**Idea of Proof.** Equivalent formulation for sets.

**Definition.** Set  $A$  is bounded  $\iff \exists M > 0$  such that  $\|x\| < M$  for all  $x \in A$ .

**Theorem 2.3.2** (Equivalent of 2.3.4)  $A$ 가 유계이고 무한집합이면,  $A' \neq \emptyset$ .

**Remark.**  $A' \neq \emptyset \implies A$ : 무한집합.

역이 성립하기 위해서는  $A$ 가 유계라는 조건이 필요하다.

극한점이 중요한 이유는 계속 수열과 관련이 있기 때문이다.

**Example.**  $A = \{1/n : n \in \mathbb{N}\}$  을 고려하는 것은 수열  $x_n = 1/n$  을 고려하는 것이나 마찬가지이다. 이 수열  $x_n$  이  $x$  로 수렴하는 것은  $A' = \{x\}$  와 동치이다. (Hence the name “limit point”) 이로부터  $x \in A' \iff$  Exists a subsequence of  $\langle x_n \rangle$  in  $A \setminus \{x\}$  converging to  $x$ .

### Proof of 2.3.2

#### 1. Lemma 2.3.1 축소구간정리 in $\mathbb{R}^d$ .

$B$  is a closed box in  $\mathbb{R}^d \iff B = I_1 \times I_2 \times \dots \times I_d$ , where  $I_i = [a_i, b_i]$  for  $i = 1, \dots, d$ . ( $I_i$  is a closed and bounded interval.)

$$B_1 \supset B_2 \supset \dots \implies \bigcap_{n=1}^\infty B_n \neq \emptyset$$

**Proof.** 각 ‘좌표’  $I_i$  별로 1차원 축소구간정리를 적용하면 된다.

#### 2. Divide and Conquer Strategy

$B$ : Box 일 때,  $\text{diam}(B) = \sup\{\|x - y\| : x, y \in B\} = \sqrt{(a_1 - b_1)^2 + \dots + (a_d - b_d)^2}$

**Claim.** There exists closed boxes  $B_1, B_2, \dots$  s.t.

(a)  $B_1 \supset B_2 \supset \dots$

(b)  $\text{diam} B_n = \frac{1}{2^n} \text{diam} B_1$

(c)  $B_n \cap A$ : 무한집합

**Proof.** (Induction)  $n = 1$ ;  $B_1$ : 충분히 커서  $A \subset B_1$  인 box 를 잡으면 된다.

Suppose we have  $B_1, \dots, B_n$ ;  $B_n$ 을  $2^d$  등분하면 적어도 하나는  $A$ 의 원소를 무한개 포함하고 있다. 그 집합을  $B_{n+1}$  으로 잡는다. (비둘기집의 원리)

이제  $x \in \bigcap_{n=1}^{\infty} B_n$  으로 잡으면 (축소구간정리에 의해 잡을 수 있다)  $x \in A'$ . ( $A' \neq \emptyset$ )

$\because \forall \epsilon > 0$ ,  $\text{diam} B_n < \epsilon$  인  $N \in \mathbb{N}$  을 찾아  $n \geq N$  일 때 부등식이 성립하도록 할 수 있다.

이러한  $n$  들에 대하여  $B_n \subset N(x, \epsilon)$ . 그러면  $N(x, \epsilon) \cap (A \setminus \{x\}) \supset B_n \cap (A \setminus \{x\})$ .

April 3rd, 2019

우리가 지금 2.3 을 하고 있는데, 2 가지 중요한 결과가 있어요.

**Theorem 2.3.4**  $\langle x_n \rangle$  이 bounded 이면 수렴하는 부분수열을 갖는다.<sup>1</sup>

**Theorem 2.3.2**  $A$ 가 유계인 집합이고 무한집합이면 극한점을 가진다.  $A' \neq \emptyset$   
증명은 축소구간정리를 박스로 확장해가지고 분할 정복하면 된다.

**Recall 2.3.3**  $x \in A' \implies N(x, \epsilon) \cap (A \setminus \{x\})$  는 무한집합이다.

**Proof of 2.3.4.**  $A = \{x_1, x_2, \dots, x_n\}$  라고 하면 이 집합은 유계이다. (수열이 유계이므로)

1.  $A$ 가 유한집합: 자명.

$\exists x$  such that  $x$  appears infinitely many times in  $\langle x_n \rangle$ . (PHP) 이 경우에는 부분수열을  $x, x, \dots$  로 잡으면 된다. 이는 수렴하는 부분수열이다.

2.  $A$ 가 무한집합<sup>2</sup>

$A' \neq \emptyset$  이므로  $\alpha \in A'$  이라 하자.

**Claim.**  $\exists n_1 < n_2 < \dots$  such that  $\|x_{n_k} - \alpha\| < 1/k$ .

**Proof.** (첨자들이 증가하면서 가까워져야 한다는 것이 유일하게 tricky 한 부분이다. 귀납법을 사용하자.)  $k = 1$ :  $x_{n_1} \in N(\alpha, 1) \cap (A \setminus \{\alpha\})$  로 잡으면 된다.

$x_{n_1}, \dots, x_{n_k}$  를 잡았다고 가정:  $N(\alpha, \frac{1}{k+1}) \cap (A \setminus \{\alpha\})$  에서  $x_{n_{k+1}}$  를 잡아야 하는데 이 집합은 무한집합이다. (Recall 2.3.3) 이 집합에서 첨자가  $n_k$ 보다 큰 항이 반드시 존재하므로 그 중 하나를  $x_{n_{k+1}}$  이라 잡으면 된다.

따라서  $\lim_{k \rightarrow \infty} x_{n_k} = \alpha$  (Check as exercise)

**Application.** (Characterization of  $\limsup$  and  $\liminf$ )

$x_n$  이 bounded 이면,  $A = \{x : \exists \text{ subsequence of } x_n \text{ converging to } x\}$ . 이 때 Theorem 2.3.4에 의해  $A \neq \emptyset$  임을 증명하였다.

1.  $A$ : closed and bounded  $\implies \max(A), \min(A)$  가 존재한다.

**Proof.**  $B = \{x_1, x_2, \dots\}$ ,  $C = \{\langle x_n \rangle \text{ 에 무한 번 나타나는 수} \}$  로 잡자.  $A = B' \cup C$ ,  $C \subset B$ ,  $C' \subset B'$  임을 확인해보라! 이를 이용하면  $B' \cup C = (B' \cup C') \cup C = B' \cup (C' \cup C) = B' \cup \overline{C}$  가 되어 닫힌집합의 합집합은 닫힌 집합이다.  $A$ 는 closed and bounded 이다.

2.  $\limsup x_n = \max(A)$ ,  $\liminf x_n = \min(A)$

(부분수열이 가질 수 있는 극한값들 중 가장 큰 값이  $\limsup$ , 가장 작은 값이  $\liminf$ )

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<sup>1</sup>증명이 가장 테크니컬 해요!

<sup>2</sup>이제 Thm 2.3.2 를 사용할 수 있다. 사실 경우를 나눈 것은 예외적인 case 를 처리하기 위한 것이었다.

**Proof.** Recall

$$\limsup x_n = \alpha \iff \begin{cases} \text{(i) } \forall \epsilon > 0, \exists N \text{ s.t. } (n \geq N \implies x_n < \alpha + \epsilon) \\ \text{(ii) } \forall \epsilon > 0, x_n > \alpha - \epsilon \text{ for infinitely many } n \end{cases}$$

(a) 부분수열  $\langle x_{n_k} \rangle \rightarrow \beta$  이면 (i)에 의해  $k \geq N \implies x_{n_k} < \alpha + \epsilon$  이 되어  $\beta \leq \alpha + \epsilon$ .  $\beta \leq \alpha$ . 그러므로  $\max(A) \leq \alpha$  이다.

(b)  $\forall \epsilon > 0$ , (i), (ii)에 의해  $x_n \in (\alpha - \epsilon, \alpha + \epsilon)$  인  $n$  이 무한히 많다. 이 유계인 구간에 속하는 수열의 항들에 대해 부분수열을 잡아 (further subsequence)  $\gamma$  로 수렴하도록 할 수 있다. (Theorem 2.3.4) 그러면  $\langle x_{m_k} \rangle \rightarrow \gamma \in [\alpha - \epsilon, \alpha + \epsilon]$ . 따라서  $\alpha - \epsilon \leq \gamma \leq \max(A)$  가 되어  $\alpha \leq \max(A)$ .

따라서  $\max(A) = \alpha$ .

**Definition.**  $\langle x_n \rangle$ : Cauchy Sequence  $\iff \forall \epsilon > 0, \exists N \text{ s.t. } [m, n \geq N \implies \|x_m - x_n\| < \epsilon]$

**Prop 2.3.6, Thm 2.3.8**  $\langle x_n \rangle$ : convergent  $\iff \langle x_n \rangle$ : Cauchy sequence<sup>3</sup>

**Proof.** ( $\implies$ ) 자명.  $\|x_m - x_n\| \leq \|x_m - \alpha\| + \|x_n - \alpha\| < \epsilon/2 + \epsilon/2 = \epsilon$  인  $m, n \geq N$  존재.

( $\impliedby$ ) 수렴 값이 없는 상태에서 증명해야 한다. 먼저 수렴 값을 찾아보자.

1.  $\langle x_n \rangle$  is bounded.

**Proof.**  $\exists N$  s.t.  $\|x_m - x_n\| < 1$  for all  $m, n \geq N$ .

Set  $M = \max\{\|x_1\|, \dots, \|x_{N-1}\|, \|x_N\| + 1\}$ . ( $\|x_m\| < \|x_N\| + 1$ )

따라서  $\|x_n\| \leq M$  for all  $n \in \mathbb{N}$ .

2. There exists a subsequence  $\langle x_{n_k} \rangle$  converging to some  $\alpha$ . (Thm 2.3.4)

3.  $\langle x_n \rangle$  converges to  $\alpha$ .

**Proof.**  $\epsilon > 0$  에 대해,

(a) 코시 수열의 성질에 의해  $\exists N_1$  s.t.  $\|x_m - x_n\| < \epsilon/2$  for all  $m, n \geq N_1$ .

(b) 부분수열이  $\alpha$ 로 수렴하므로  $\exists N_2$  s.t.  $\|x_{n_k} - \alpha\| < \epsilon/2$  for all  $k \geq N_2$ .

Let  $N = \max\{N_1, N_2\}$ .  $n \geq N, n_N \geq n_{N_1} \geq N_1$  이므로,

$$n > N \implies \|x_n - \alpha\| \leq \|x_n - x_{n_N}\| + \|x_{n_N} - \alpha\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

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<sup>3</sup>중간고사 전 까지 가장 중요한 정리.



**Remark.** 우리의 여정을 돌아보자.

1. Archimedes' Principle 을 가정하면

Completeness Axiom  $\implies$  Monotone Convergence Theorem  $\implies$  축소구간정리  $\implies$   
Bolzano-Weierstrass Theorem  $\implies$  **Cauchy Convergent Theorem**<sup>4</sup>  
(Exercise)  $\implies$  Completeness Axiom

2. **Example.**  $X = C([0, 1])$ . (Set of functions that are continuous in  $[0, 1]$ ) How would we define  $\|f - g\|$ ?  $\int_0^1 |f(x) - g(x)| dx$  ?  $\max\{|f(x) - g(x)| : x \in [0, 1]\}$  ? Only the second choice gives completeness for  $X$ .

3. **Convergence Test** without limit value. (**Theorem 2.3.9**)

$\sum_{n=1}^{\infty} a_n$  is convergent  $\iff \forall \epsilon > 0, \exists N$  s.t.  $(n > m \geq N \implies |a_{m+1} + \cdots + a_n| < \epsilon)$

**Proof.** Trivial.

**Definition.**  $\sum a_n$  is **absolutely convergent**  $\iff \sum |a_n|$  is convergent

**Theorem.** An absolutely convergent series converges.

**Proof.** Suppose  $\sum |a_n|$  converges. For  $\forall \epsilon > 0$ , there exists  $N$  such that  $|a_{m+1}| + \cdots + |a_n| < \epsilon$  for all  $m, n \geq N$ . Therefore, for  $m, n \geq N$ ,

$$|a_{m+1} + \cdots + a_n| \leq |a_{m+1}| + \cdots + |a_n| < \epsilon$$

and  $\sum a_n$  converges.

---

<sup>4</sup>In any metric spaces, this is the condition for completeness.

**April 5th, 2019**

**Theorem.**  $\overline{A \cup B} = \overline{A} \cup \overline{B}$

**Proof.** ( $\subset$ ) Trivial.

( $\supset$ )  $A \subset \overline{A}, B \subset \overline{B} \implies A \cup B \subset \overline{A} \cup \overline{B} \implies \overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$ . The closure of a closed set is itself.

**6. (2)**  $a_n = \cos \sqrt{2019 + n^2 \pi^2}$

Consider  $\delta > 0$ , such that

$$(n\pi - \delta)^2 < 2019 + n^2 \pi^2 < (n\pi + \delta)^2$$

$$-2n\pi < \frac{2019}{\delta} \pm \delta < 2n\pi$$

We can find large enough  $N$  such that the above inequality holds for  $n \geq N$ .

Now we want  $b_n = \sqrt{2019 + n^2 \pi^2}$  bounded by  $n\pi \pm \delta$ .

$n \geq N, n \text{ even} \implies n\pi - \delta < b_n < n\pi + \delta$

$\implies 1 \geq a_n > 1 - \epsilon$

$n \geq N, n \text{ odd} \implies -1 \leq a_n < -1 + \epsilon$

**Problem 2.3.5**

$$(1) \ x_{n+2} = \frac{x_n + x_{n+1}}{2}$$

$$(2) \ x_{n+1} = x_n + x(-1)^n \frac{1}{3n+1}$$

**Solution.**

(1) Write  $x_{n+2} - x_{n+1} = a(x_{n+1} - x_n)$  and observe that  $a = -1/2$ . Write as

$$x_n = x_{n-1} + \left(-\frac{1}{2}\right)^{n-2} (x_2 - x_1)$$

Then we have

$$x_n = x_2 + \sum_{k=1}^{n-2} \left(-\frac{1}{2}\right)^k (x_2 - x_1)$$

This series converges to  $\frac{2x_2 + x_1}{3}$ .

(2) This is an alternating series. Write as

$$x_n = x_1 + \sum_{k=1}^{n-1} (-1)^k \frac{x}{3n+1}$$

By alternating series test, the second summation term converges, and the series converges to  $x_1$ .

Since a converging sequence is a Cauchy sequence,  $x_1, x_2$  can be any real number.

April 8th, 2019

Section 2.3: Bolzano-Weierstrass Theorem, Cauchy Convergent Theorem

In section 2.4, we will be studying about Convergence Tests.

정

## 2.4 급수의 수렴판정

**Cor 2.3.9.**  $\sum_{n=1}^{\infty} a_n$  is convergent  $\iff s_n = \sum_{k=1}^n a_k, \langle s_n \rangle$  is convergent  $\iff \langle s_n \rangle$  is Cauchy.

1.  $\sum_{n=1}^{\infty} a_n$  is convergent  $\implies \lim_{n \rightarrow \infty} a_n = 0$ .
2.  $\sum_{n=1}^{\infty} |a_n|$  is convergent  $\implies \sum_{n=1}^{\infty} a_n$  convergent.

**Theorem 2.4.3** (Comparison Test) Suppose  $\sum b_n$  converges. If  $0 \leq a_n \leq b_n$  for all  $n \in \mathbb{N}$ ,  $\sum a_n$  converges.

**Proof** Let  $M = \sum b_n$ ,  $s_n = \sum_{k=1}^n a_k$ .  $s_n$  is increasing and  $s_n$  is bounded by  $M$ .  $s_n$  is convergent by Monotone Convergence Theorem.

**Theorem.** Suppose sequences  $a_n, b_n$  satisfy  $0 \leq |a_n| \leq b_n$ <sup>5</sup> and  $\sum b_n$  converges. Then  $\sum a_n$  is convergent.<sup>6</sup>

**Proof.** By comparison test and absolute convergence.

**Prop 2.4.4** (Root Test) Suppose  $\alpha = \limsup_{n \rightarrow \infty} |a_n|^{1/n}$ .

If  $\alpha < 1$ ,  $\sum a_n$  converges. If  $\alpha > 1$ ,  $\sum a_n$  diverges.

**Proof.**

1.  $\alpha < 1$ . Take  $\epsilon > 0$  such that  $\alpha < \alpha + \epsilon < 1$ . Then there exists  $N$  such that  $|a_n|^{1/n} < \alpha + \epsilon$  for all  $n \geq N$ . Therefore  $|a_n| < (\alpha + \epsilon)^n$ . Since  $\alpha + \epsilon < 1$ ,  $\sum (\alpha + \epsilon)^n$  converges. Apply the comparison test to see that  $\sum a_n < \infty$ .
2.  $\alpha > 1$ . Take  $\epsilon > 0$  such that  $\alpha > \alpha - \epsilon > 1$ . Then  $|a_n|^{1/n} > \alpha - \epsilon$  for infinitely many  $n$ . Then  $|a_n| > (\alpha - \epsilon)^n > 1$ . Therefore  $\lim a_n \neq 0$ .  $\sum a_n$  diverges.

**Prop 2.4.5** (Ratio Test) Suppose  $a_n \neq 0$ . Let  $\beta = \limsup |a_{n+1}/a_n|$ ,  $\gamma = \liminf |a_{n+1}/a_n|$ .

If  $\beta < 1$ ,  $\sum a_n$  converges. If  $\beta > 1$ ,  $\sum a_n$  diverges.

**Proof.**

1.  $\beta < 1$ . Take  $\epsilon > 0$  such that  $\beta < \beta + \epsilon < 1$ . Then  $\exists N$  s.t.  $|a_{n+1}/a_n| < \beta + \epsilon$  for  $n \geq N$ .  
 $\implies |a_n| = |a_N| |a_{N+1}/a_N| \cdots |a_n/a_{n-1}| < |a_N| (\beta + \epsilon)^{n-N}$ .  
Set  $b_n = |a_N| (\beta + \epsilon)^{n-N}$  and apply comparison test to see that  $\sum a_n < \infty$ .

---

<sup>5</sup>Note that this condition can fail for finitely many  $n$ .

<sup>6</sup> $a_n$  may be a very complex expression, but we want  $b_n$  to be simple, an expression we know that it is convergent.

2.  $\gamma > 1$ . Take  $\epsilon > 0$  such that  $\gamma > \gamma - \epsilon > 1$ . Then  $\exists N$  s.t.  $|a_{n+1}/a_n| > \gamma - \epsilon$  for  $n \geq N$ .

Then we see that  $|a_n|$  is increasing for  $n \geq N$ . Thus  $a_n$  cannot converge to 0.  $\sum a_n$  is divergent.

**Remark.** If the above limits (ratio, root) exist, elementary tests can be applied. But if the limits turn out to be 1, the test fails. (ND: Non-Deterministic) Check it for  $\sum 1/n, \sum 1/n^2$ .

Also, these are *weak tests*. For most of the series, the limit is 1. Moreover...

**Theorem 2.4.6** Suppose  $a_n \neq 0$ .

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq \liminf |a_n|^{\frac{1}{n}} \leq \limsup |a_n|^{\frac{1}{n}} \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|$$

Thus if the root test works, ratio test also works.<sup>7</sup>

**Proof.** We only need to prove the last inequality.

Let  $\beta = \limsup |a_{n+1}/a_n|$ ,  $\forall \epsilon > 0$ .  $\implies \exists N$  s.t.  $|a_{n+1}/a_n| \leq \beta + \epsilon$  for  $n \geq N$ . Then if  $n \geq N$ ,  $|a_n| \leq |a_N|(\beta + \epsilon)^{n-N}$ . (Similar to proof of 2.4.5) Then

$$|a_n|^{1/n} \leq (\beta + \epsilon) \left( \frac{|a_n|}{(\beta + \epsilon)^N} \right)^{1/n}$$

and take  $\limsup$  on both sides, then  $\limsup |a_n|^{1/n} \leq \beta + \epsilon$ .

**Example.**  $\langle a_n \rangle = \begin{cases} 1/2^n & n \text{ odd} \\ 1/2^{n-2} & n \text{ even} \end{cases}$

Check that  $\limsup |a_n|^{1/n} = 1/2 < 1$ , and the series  $\sum a_n$  converges by the root test.

But if we use the ratio test here,  $\limsup$  value is 2 and  $\liminf$  value is  $1/8$ .<sup>8</sup> The ratio test does not tell us anything about the convergence. Also note that the series converges to 2.

**Prop 2.4.1** (Rearrangement)  $a_n \geq 0$ .<sup>9</sup> Suppose a bijection  $r : \mathbb{N} \rightarrow \mathbb{N}$  exists.

$$1. \sum_{n=1}^{\infty} a_n = s \iff \sum_{n=1}^{\infty} a_{r(n)} = s$$

$$2. \sum_{n=1}^{\infty} a_n = \infty \iff \sum_{n=1}^{\infty} a_{r(n)} = \infty$$

**Proof.**

1. ( $\implies$ ) Let  $t_n = \sum_{k=1}^n a_{r(k)}$ . Then  $t_n$  is increasing and bounded by  $s$ . Thus  $t_n$  converges by MCT, and  $\lim t_n \leq s$ .

$$s_n = \sum_{k=1}^n a_k \leq \sum_{n=1}^{\infty} a_{r(n)} = t = \lim t_n. \quad (a_n \geq 0 \text{ was used here.})$$

( $\impliedby$ ) Use  $r^{-1}(n)$ .

---

<sup>7</sup>The limit for the ratio test is much easier to calculate than the root test. That's why we use the ratio test.

<sup>8</sup>The ratios are: 2,  $1/8$ , 2,  $1/8$  ...

<sup>9</sup>This is the important condition.

## 2. Contraposition of (1).

**Prop 2.4.2** (Alternating Series Test) For a given sequence  $x_n$ , suppose the following holds.

- $x_n$  is decreasing.
- $\lim x_n = 0$ .

Then the series  $\sum_{k=1}^{\infty} (-1)^{k-1} x_k$  is convergent.

**Proof.** Let  $s_n = \sum_{k=1}^n (-1)^{k-1} x_k$ . For  $m < n$ ,

$$|s_n - s_m| = |(-1)^m x_{m+1} + \cdots + (-1)^{n-1} x_n| = |x_{m+1} - x_{m+2} + \cdots \pm x_n| \stackrel{(*)}{\in} [0, x_{m+1}]$$

$$(*) : x_{m+1} - x_{m+2} + \cdots + x_n = (x_{m+1} - x_{m+2}) + \cdots + (x_{n-2} - x_{n-1}) + x_n \geq 0$$

$$= x_{m+1} - (x_{m+2} - x_{m+3}) - \cdots - (x_{n-1} - x_n) \leq x_{m+1}$$

Check for the case with last term  $-$ .

Now,  $\forall \epsilon > 0$ , find  $N$  such that  $|x_n| < \epsilon$  for  $n \geq N$ . Then for  $n > m \geq N$ ,  $|s_n - s_m| \leq x_{m+1} < \epsilon$ .

Thus  $\langle s_n \rangle$  is a Cauchy sequence and the given series converges.

**Example.**  $a_n = (-1)^{n-1}/n$ .  $\sum a_n$  converges by alternating series test and converges to  $\log 2$ .

**Remark.** The rearrangement of the above example may not converge, or converge to a different value than  $\log 2$ .

Exam: 1.1 - 2.6

After the midterms we will be covering functions and continuity.

Chapter 1 has been about  $\mathbb{R}$ , and in Chapter 2, we have talked about subsets of  $\mathbb{R}^n$ .

2.1: What is  $\mathbb{R}^n$  ? Vector Space, IPS, Metric Space, Normed Space...

2.2: Open, closed sets

2.3: Bounded sets and Cauchy sequences

(2.4: Convergence Tests)

2.5: Compact Sets

2.6: Connected Sets