Assignment 4

Due Date: 2019/05/16, 1:30 PM

1. Prove that the following functions $f : X \to \mathbb{R}$ are continuous. Determine whether each function is uniformly continuous or not, and verify your answer.

(1) $f(x) = \frac{1}{(x+1)^3}$ with $X = (-1, 1)$

(2) $f(x) = \frac{1}{x+3}$ with $X = (0, \infty)$

(3) $f(x) = \frac{1}{x^2+1}$ with $X = \mathbb{R}$.

(4) $f(x) = \sqrt{x^2 + 1}$ with $X = (0, \infty)$.

2. Answer the following questions.

(1) Suppose that $f : X \to Y$ is a continuous function on X, and that the set X is bounded (in some Euclidean space). Then, prove that f is uniformly continuous on X if and only if f has a continuous extension to $\overline{X}$.

(2) Use part (1), show that

$$f(x, y) = \sqrt{x^{2020} + y^{2020} + x^2 + 1}$$

is a uniformly continuous function on $X = \{(x, y) : x^2 + y^2 < 3\}$.

(3) For $a < b$, suppose that $f : (a, b) \to \mathbb{R}$ is a continuous function. Then, prove that f is uniformly continuous on (a, b) if and only if $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist.

3. Compute the following values and verify your answer rigorously.

(1) $\lim_{x\to 0^-} \frac{\max\{x,0\}}{x} - \lim_{x\to 0^+} \frac{\max\{x,0\}}{x}$

(2) $\lim_{x\to 0^+} \frac{x^3}{|x|}$

4. For uniformly continuous functions $f,\, g : X \to \mathbb{R}$, prove or disprove the following statements:

(1) $f + g$ is uniformly continuous on X.

(2) fg is uniformly continuous on X.

(3) $\max\{f,\, g\}$ is uniformly continuous on X.

5. Let $f : [0,\, 1] \to \mathbb{R}$ be a continuous function on $[0,\, 1]$ and suppose that $f(x) \in [0,\, 1]$ for all $x \in [0,\, 1]$. Prove that there exists $x_0 \in [0,\, 1]$ such that $f(x_0) = x_0^{2016}$.

6. A function $f : X \to Y$ is called a Hölder continuous function on X if there exists $\alpha > 0$ and $M > 0$ such that

$$\|f(x) - f(y)\| \le M\|x - y\|^{\alpha} \text{ for all } x,\, y \in X\ .$$

(1) Prove that a Hölder continuous f on X is uniformly continuous (and hence continuous) on X.

(2) Suppose that $X = \mathbb{R}^d$ for some $d \in \mathbb{N}$, and that f is Hölder continuous on $\mathbb{R}^d$ with some $\alpha > 1$. Prove that f is merely a constant function.

7. Suppose that $f : \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function.

(1) Prove that $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset \mathbb{R}^m$.

(2) Prove or disprove that $f(\overline{A}) = \overline{f(A)}$ for all $A \subset \mathbb{R}^m$.

8. Suppose that $A,\, B$ are disjoint closed subsets of $\mathbb{R}^d$, i.e., $A \cap B = \emptyset$. Prove that there exists a continuous function $f : \mathbb{R}^d \to [0,\, 1]$ such that $f(x) = 1$ for all $x \in A$ and $f(x) = 0$ for all $x \in B$. (Hint. Use $g(x) = \mathrm{dist}(\{x\},\, A)$)

9. Suppose that $X \subset \mathbb{R}^m$ is a compact set and that $f : X \to Y$ is a function for some $Y \subset \mathbb{R}^n$. Define a set $E \subset \mathbb{R}^{m+n}$ by $E = \{(x,\, f(x)) : x \in X\}$. Prove that f is continuous on X if and only if E is a compact set in $\mathbb{R}^{m+n}$.

10. A function $f : (a,\, b) \to \mathbb{R}$ is called convex on $(a,\, b)$ if and only if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \text{ for all } x,\, y \in (a,\, b) \text{ and } \lambda \in [0,\, 1]\ .$$

Prove that, all convex function f on $(a,\, b)$ is continuous on $(a,\, b)$.

11. Suppose that $f : [a,\, b] \to \mathbb{R}$ is a continuous function on $[a,\, b]$. Prove that

$$f^*(x) = \sup\{f(y) : y \in [a,\, x]\}$$

is an increasing continuous function on $[a,\, b]$.

12. Does there exists a continuous function on $(0,\, 1]$, which does not have maximum and minimum? Prove your answer.