HW Set 4. (Due day: October 12, 23:59)

1. Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x + 2\pi) = f(x)$ for all x.

   (a) Compute the Fourier coefficients of f.

   (b) Conclude that

   $$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$

   (c) Deduce from Parseval's theorem that

   $$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

   (d) Let $\delta \to 0$ and prove that

   $$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

   (e) Put $\delta = \pi/2$ in (c). What do you get?

2. Prove that

   $$(\pi - |x|)^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx \qquad \text{for all } x \in [-\pi, \pi]$$

   and deduce that

   $$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \frac{1}{n^4} = \frac{\pi^4}{90}.$$

3. With $D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \frac{\sin(n+\frac{1}{2})x}{\sin(x/2)}$, put

   $$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x}$$

and that

(a) $K_N \geq 0$,

(b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\, dx = 1$,

(c) $K_N(x) \leq \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta}$ if $0 < \delta \leq |x| \leq \pi$.

If $s_N = s_N(f;x)$ is the Nth partial sum of the Fourier series of f, consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\, dt,$$

and hence prove Fejer's theorem: *If f is continuous, with period 2π, then $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi,\pi]$.*

Hint: Use properties (a), (b), (c) to proceed as in Theorem 7.26.

Note. σ_N defined above is the Cesàro mean. So if $s_N(f;x)$ converges, then $\sigma_N(f;x)$ also converges to the same value. The fact that there exists a continuous function whose fourier series doesn't converge to itself suggests that converse is not true.

4. In this problem we generalize the theorem 8.14. Let f be a Riemann-integrable function with period 2π. Define $f(a\pm) := \lim_{x\to a\pm} f(x)$ if it exists. Assume that both $f(a\pm)$ exist and there exists a positive number ε, δ, $M > 0$ s.t.

$$|t| < \delta \implies \left| \frac{f(a+t)+f(a-t)}{2} - \frac{f(a+)+f(a-)}{2} \right| \leq M|t|^{\varepsilon}.$$

In these conditions we will show that $s_N(f;a)$ converges to $\frac{f(a+)+f(a-)}{2}$.

(a) Show that $s_N(f;x)$ can be written as

$$\frac{1}{2\pi}\int_0^{\pi}\{f(x+t)+f(x-t)\}\frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

(b) Prove that

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_0^{\pi}\{f(x+t)+f(x-t)\}\left(\frac{1}{\sin\frac{t}{2}}-\frac{2}{t}\right)\sin\left(N+\frac{1}{2}\right)tdt=0.$$

(c) Now we only have to show that the below limit

$$\lim_{N\to\infty}\left(s_N(f;a)-\frac{f(a+)+f(a-)}{2}\right)$$

$$=\lim_{N\to\infty}\frac{1}{\pi}\int_0^{\pi}\left(\frac{f(a+t)+f(a-t)}{t}-\frac{f(a+)+f(a-)}{t}\right)\sin\left(N+\frac{1}{2}\right)tdt$$

converges to zero. However, this time we cannot do as we did in the proof of theorem 8.14, because $\frac{f(a+t)+f(a-t)-f(a+)-f(a-)}{t}$ is no longer Riemann-integrable on $[-\pi,\,\pi]$(don't confuse it with the integrability of whole integrand). Although we won't deal with improper integral, there is a breakthrough.
Define $f_n:\{\frac{1}{p}\,|\,p\in\mathbb{N}\}\to\mathbb{C}$ by

$$f_n\left(\frac{1}{m}\right)=\frac{1}{\pi}\int_{\frac{1}{n}}^{\pi}\frac{f(a+t)+f(a-t)-f(a+)-f(a-)}{t}\sin\left(m+\frac{1}{2}\right)tdt.$$

Prove that f_n uniformly converges.

(d) Use theorem 7.11(limit interchange theorem) to conclude that $s_N(f;a)$ converges to $\frac{f(a+)+f(a-)}{2}$.

Note. This theorem is a generalization of theorem 8.14 in two aspects. f can be a discontinuous function and ε can be less than 1.