HW Set 5. (Due day: November 6, 12pm)

Notation: For a set S, the power set of S is defined and denoted as $\mathcal{P}(S) := \{A : A \subset S\}$.

Definition: $\Sigma \subset \mathcal{P}(S)$ is an algebra on S if

(1) $S \in \Sigma$, (2) If $A \in \Sigma$ then $S \setminus A \in \Sigma$ and (3) If $A, B \in \Sigma$ then $A \cup B \in \Sigma$.

Definition: An algebra on S is a σ-algebra on S if the following holds: if $A_n \in \Sigma$, $n = 1, 2, \ldots$ then $\cup_{n=1}^{\infty} A_n \in \Sigma$.

1. Show that $\Sigma \subset \mathcal{P}(S)$ is an algebra on S if and only if $\Sigma \subset \mathcal{P}(S)$ is a ring on S with $S \in \Sigma$. Show that $\Sigma \subset \mathcal{P}(S)$ is a σ-algebra on S if and only if $\Sigma \subset \mathcal{P}(S)$ is a σ-ring on S with $S \in \Sigma$.

2. E, S are sets and f is a map from E to S. Suppose Σ is a σ-algebra on S ($\Sigma \subset \mathcal{P}(S)$). Show that $f^{-1}(\Sigma) := \{f^{-1}(A) : A \in \Sigma\}$ is a σ-algebra on E.

3. Show that an arbitrary intersection of σ-algebras on S is a σ-algebra on S and show that two union of σ-algebras may not be a σ-algebra by a counterexample.

4. S is a set, $x \in S$ and $\Sigma = \mathcal{P}(S)$. Show that following are measures on Σ.
$$\mu_1(A) := \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A. \end{cases}$$
$$\mu_2(A) := \begin{cases} \text{Cardinality (number of elements) of } A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite .} \end{cases}$$

5. Let μ be a measure on a σ-algebra $\mathcal{F}$. Suppose $A_i \in \mathcal{F}$ and $A_{i+1} \subset A_i$, $i \in \mathbb{N}$. Show that if $\mu(A_1) < \infty$, then
$$\lim_{i \to \infty} \mu(A_i) = \mu(\cap_{i=1}^{\infty} A_i).$$

6. Let μ is a measure on a σ-algebra $\mathcal{F}$. Show that if $A_i \in \mathcal{F}$, $i \in \mathbb{N}$. then
$$\mu(\cup_{n=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i).$$