HW Set 7 (Due day: Nov 25 , 12pm)

1. Show that $s(x) = \frac{\sin x}{x}$ is improperly Riemann integrable on $(0, \infty)$ and show that it is not in $\mathcal{L}$ on $(0, \infty)$.

2. The following simple computation yields a good approximation to Stirling's formula. For $m = 1, 2, 3, \cdots$, define

$$f(x) = (m + 1 - x) \log m + (x - m) \log(m + 1)$$

if $m \leq x \leq m + 1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - \frac{1}{2} \leq x < m + \frac{1}{2}$. Draw the graphs of f and g. Note that $f(x) \leq \log x \leq g(x)$ if $x \geq 1$ and that

$$\int_1^n f(x)\ dx = \log(n!) - \frac{1}{2} \log n > -\frac{1}{8} + \int_1^n g(x)\ dx.$$

Integrate $\log x$ over $[1, n]$. Conclude that

$$\frac{7}{8} < \log(n!) - \big(n + \frac{1}{2}\big) \log n + n < 1$$

for $n = 2, 3, 4, \cdots$ (Note: $\log \sqrt{2\pi} \sim 0.918...$). Thus

$$e^{7/8} < \frac{n!}{(n/e)^n \sqrt{n}} < e.$$

3. If $f \in \mathcal{R}$ on $[a, b]$ and if $F(x) = \int_a^x f(t)\ dt$, prove that $F'(x) = f(x)$ almost everywhere on $[a, b]$.

4. Let $f \in \mathcal{L}$ on $[a, b]$. Prove that the function $F(x) = \int_a^x f(t)\ dt$ is continuous on $[a, b]$.

5. If $f, g \in \mathcal{L}(\mu)$ on X, define the distance between f and g by

$$\int_X |f - g|\ d\mu.$$

Prove that $\mathcal{L}(\mu)$ is a complete metric space. (Note that $f = g$ in $\mathcal{L}(\mu)$ means that $f = g$ a.e. in X with respect to the measure μ.)

6. Let (X, F, μ) be a measure space and $f_n, g_n : X \to \mathbb{R}$ be measurable functions. Suppose that $|f_n| \leq g_n$, $\lim_{n\to\infty} g_n = g$ a.e. in X, $g_n, g \in \mathcal{L}$, and $\lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu$. Prove that $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$ provided that $\lim_{n\to\infty} f_n = f$ a.e. in X with respect to the measure μ.

7. Let (X, F, μ) be a measure space and $f_n, f : X \to \mathbb{R}$ be measurable functions with $f_n, f \in \mathcal{L}(\mu)$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} f_n = f$ a.e in X with respect to the measure μ. Prove that

$$\int_X |f_n - f| \, d\mu \to 0 \quad \text{if and only if} \quad \int_X |f_n| \, d\mu \to \int_X |f| \, d\mu.$$

(Hint: Use the result in the problem 6.)