HW Set 6 (Due day: Nov 18, 12pm)

1. Show that the set function defined at Page 303 is regular on $\mathcal{E}$.

2. Show that every non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

3. Show that $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $f = g$ a.e. with respect to Lebesgue measure, then $g : \mathbb{R} \to \mathbb{R}$ is also Lebesgue measurable.

4. If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.

5. If $f \geq 0$ and $\int_E f \, d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E. *Hint*: Let E_n be the subset of E on which $f(x) > 1/n$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n.

6. Put
$$g(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$
and
$$\begin{aligned} f_{2k}(x) &= g(x) & &\text{if } 0 \leq x \leq 1, \\ f_{2k+1}(x) &= g(1-x) & &\text{if } 0 \leq x \leq 1. \end{aligned}$$
Show that
$$\liminf_{n\to\infty} f_n(x) = 0 \qquad \text{for all } 0 \leq x \leq 1,$$
but
$$\int_0^1 f_n(x) \, dx = \frac{1}{2}.$$

7. If $\int_A f \, d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) = 0$ almost everywhere on E.

8. Suppose f and g are extended real-valued measurable functions in $\mathcal{L}^1(X, \mathcal{M}, \mu)$. Show that $f = g$ a.e. with respect to μ if and only if
$$\int_A f \, d\mu = \int_A g \, d\mu, \quad \forall A \in \mathcal{M}.$$