해석개론 및 연습 2 과제 #3

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1. **(a)** Yes. Suppose the series $\sum c_n x^n$ converges for $|x| < R$, where

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \quad (R \in (0, \infty]).$$

Then for any $|y| < |x|$,

$$\limsup_{n\to\infty} \sqrt[n]{|c_n y^n|} = |y| \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \frac{|y|}{R} < \frac{|x|}{R} < 1.$$

Thus $\sum c_n y^n$ converges.

(b) To apply Theorem 7.17, we take

$$f_n(x) = \sum_{k=1}^{n} c_k x^k.$$

For x_0, we can take any $x_0 \in [-R+\epsilon, R-\epsilon]$ $(\epsilon > 0)$.

2. If either $\displaystyle\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$ or $\displaystyle\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$ is convergent, we can conclude that the given equality holds because we can use Theorem 8.3. So it is sufficient to check for the case where the sum diverges to infinity. For $N \in \mathbb{N}$,

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N}\sum_{i=1}^{N} a_{ij}.$$

Let $N \to \infty$. If either sum diverges to infinity, the other sum should diverge to infinity.

3. For $|x| < 1$, we can expand as the following.

$$\begin{aligned}
\log(1+x) &= \int_0^x \frac{1}{1+t}\,dt = \int_0^x \sum_{n=0}^{\infty} (-t)^n\,dt = \int_0^x \lim_{k\to\infty} \sum_{n=0}^{k} (-t)^n\,dt \\
&\overset{(*)}{=} \lim_{k\to\infty} \int_0^x \sum_{n=0}^{k} (-t)^n\,dt = \lim_{k\to\infty} \sum_{n=0}^{k} \int_0^x (-t)^n\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.
\end{aligned}$$

The equality in $(*)$ holds because $\sum_{n=0}^{\infty} (-t)^n$ converges uniformly on $t \in (-1, 1)$.

Let $\epsilon \in (0, a)$ be given and set $x = a + y$. Consider $\log x = \log(a+y)$, for small $|y| < \epsilon$.

$$\begin{aligned}
\log x &= \log a \left(1 + \frac{y}{a}\right) = \log a + \log\left(1 + \frac{y}{a}\right) \\
&= \log a + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(y/a)^n}{n} = \log a + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{na^n} y^n \\
&= \log a + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{na^n} (x-a)^n. \qquad (x \in (a-\epsilon, a+\epsilon))
\end{aligned}$$

This works because $|y/a| < 1$. $\log x$ is real-analytic on $(0, \infty)$.

4. Let $I = [-a, a] \setminus \{0\} \subseteq \mathbb{R}$ for $0 < a \ll 1$.

(a) The denominator and the numerator both converge to 0 as $x \to 0$, and both are differentiable on I. Let

$$f(x) = e - (1+x)^{1/x} \quad \text{and} \quad g(x) = x.$$

Since $g'(x) \neq 0$ on I, we can use L'Hôpital's Rule. The given limit is equal to $\lim_{x\to 0} f'(x)$, if this limit exists. Note that

$$f'(x) = \frac{(1+x)\log(1+x) - x}{x^2(x+1)} \cdot (1+x)^{1/x}.$$

Therefore, we restrict x so that $|x| < 1$ and

$$\begin{aligned}
\lim_{x\to 0} f'(x) &= \lim_{x\to 0} \frac{(1+x)\log(1+x) - x}{x^2(x+1)} \cdot (1+x)^{1/x} \\
&= e \lim_{x\to 0} \frac{(1+x)\sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n} - x}{x^2(x+1)} \\
&= e \lim_{x\to 0} \frac{(1+x)\sum_{n=2}^{\infty} \frac{(-1)^{n-1}x^n}{n} + x^2}{x^2(x+1)} \\
&= e \lim_{x\to 0} \left(\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} x^{n-2} + \frac{1}{x+1} \right) \\
&= e + e \lim_{x\to 0} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} x^{n-2} = e + \lim_{x\to 0} \lim_{k\to\infty} \sum_{n=2}^{k} \frac{(-1)^{n-1}}{n} x^{n-2} \\
&\overset{(*)}{=} e + \lim_{k\to\infty} \lim_{x\to 0} \sum_{n=2}^{k} \frac{(-1)^{n-1}}{n} x^{n-2} = e - \frac{e}{2} = \frac{e}{2}.
\end{aligned}$$

The equality in $(*)$ holds because $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n} x^{n-2}$ converges uniformly on $|x| < 1$. The limit exists, so $\frac{e}{2}$ is the desired value.

(b) Let $x = \frac{\log n}{n}$ then $n \to \infty$ as $x \to 0^+$.

$$\lim_{n\to\infty} \frac{n}{\log n}(n^{1/n} - 1) = \lim_{n\to\infty} \frac{e^{\log n/n} - 1}{\log n/n} = \lim_{x\to 0^+} \frac{e^x - 1}{x} = \frac{d}{dx}e^x \bigg|_{x=0} = 1.$$

(c) We first calculate the following limit.

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3}.$$

L'Hôpital's Rule was used sequentially, since all the terms in the denominator and the numerator approach 0 as $x \to 0$. Also they are differentiable and derivative of the numerator is not zero on I.

Now we compute the following limit.

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \lim_{x\to 0} \frac{\sin^2 x}{x^2(1+\cos x)} = \frac{1}{2}.$$

Therefore,

$$\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)} = \frac{\displaystyle\lim_{x\to 0}\frac{\tan x - x}{x^3}}{\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}} = \frac{1/3}{1/2} = \frac{2}{3}.$$

(d) We first calculate the following limit.

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2} = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}.$$

L'Hôpital's Rule was used, since both of the terms in the denominator and the numerator approach 0 as $x \to 0$. Also they are differentiable and derivative of the numerator is not zero on I. The last limit was computed using the result from **(c)**.

Therefore, using the results from **(c)**, we get

$$\lim_{x\to 0}\frac{x-\sin x}{\tan x - x} = \frac{\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x^3}}{\displaystyle\lim_{x\to 0}\frac{\tan x - x}{x^3}} = \frac{1/6}{1/3} = \frac{1}{2}.$$

5. Let $I = (0, \frac{\pi}{2})$.

(i) Let $f(x) = x - \sin x$. $f'(x) = 1 - \cos x > 0$ in I. Thus f is increasing. Therefore $f(x) > f(0) = 0$, which proves that $\sin x < x$.

(ii) Let $g(x) = \sin x - \frac{2}{\pi}x$. $g(x)$ is continuous on $[0, \frac{\pi}{2}]$ and differentiable on I. By Rolle's Theorem,

$$\exists\, \alpha \in I \text{ such that } g'(\alpha) = \cos\alpha - \frac{2}{\pi} = 0.$$

Since $g''(x) = -\sin x < 0$ on I, $g'(x)$ is decreasing, which makes the above α a unique solution to $g'(x) = 0$. For $(0, \alpha)$, we see that $g'(x) > 0$, and $g'(x) < 0$ for $(\alpha, \frac{\pi}{2})$. Thus, $g(x)$ increases then decreases on I. Therefore, $g(x) > \min\{g(0), g\left(\frac{\pi}{2}\right)\} = 0$, which proves that $\frac{2}{\pi}x < \sin x$.

Overall, we have $\frac{2}{\pi}x < \sin x < x$ for $x \in I$. We divide by x (> 0) to get

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1, \quad (x \in I).$$

6. For $n = 0, 1$, the given inequality is trivial. For $n = 2$,

$$|\sin 2x| = 2\,|\sin x\cos x| \le 2\,|\sin x|. \quad (\because |\cos x| \le 1)$$

Now we prove the inequality by induction. Suppose that

$$|\sin nx| \le n\,|\sin x|$$

for $n \geq 2$. Then,

$$\begin{aligned}|\sin(n+1)x| &= |\sin nx \cos x + \cos nx \sin x| \\ &\leq |\sin nx|\,|\cos x| + |\cos nx|\,|\sin x| \\ &\leq n\,|\sin x|\,|\cos x| + |\sin x| \\ &\leq n\,|\sin x| + |\sin x| = (n+1)\,|\sin x|\,.\end{aligned}$$

Thus the given inequality holds for all $n = 0, 1, 2, \ldots$ and equality holds when $x = k\pi$ for $k \in \mathbb{Z}$.

7. (a) Let $\gamma_n = s_n - \log n$. We first show that γ_n is decreasing. For $n \leq x \leq n+1$, $(n \in \mathbb{N})$

$$\frac{1}{n+1} \leq \frac{1}{x}.$$

Using this fact,

$$0 \leq \int_n^{n+1} \left(\frac{1}{x} - \frac{1}{n+1}\right) dx = \log\left(\frac{n+1}{n}\right) - \frac{1}{n+1} = \gamma_n - \gamma_{n+1}.$$

Therefore $\gamma_n \geq \gamma_{n+1}$.

Next, we show that $\gamma_n > 0$. For $k \leq x \leq k+1$, $(k \in \mathbb{N})$

$$\frac{1}{x} \leq \frac{1}{k}.$$

Therefore,

$$\begin{aligned}\gamma_n = s_n - \log n &= \frac{1}{n} + \sum_{k=1}^{n-1} \frac{1}{k} - \int_1^n \frac{1}{x}\,dx \\ &= \frac{1}{n} + \sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x}\,dx \\ &= \frac{1}{n} + \sum_{k=1}^{n-1} \int_k^{k+1} \left(\frac{1}{k} - \frac{1}{x}\right) dx > 0.\end{aligned}$$

$\gamma_n > 0$. Thus γ_n is decreasing and bounded below. γ_n converges by the monotone convergence theorem.

(b) From (a), we know that $\gamma_n > 0$, so

$$\log n < s_n.$$

Let $N = 10^m$ for $m \in \mathbb{N}$. Then

$$m \log 10 = \log N < s_N.$$

Therefore setting $m \log 10 \geq 100$ will work, which implies

$$m \geq \frac{100}{\log 10} \approx 43.42,$$

so m should be at least 44.

8. Let $\epsilon > 0$ be given. Since $f(x) \to 1$ as $x \to \infty$, we can choose some $K > 0$ such that

$$x \geq K \implies 1 - \frac{\epsilon}{2} < f(x) < 1 + \frac{\epsilon}{2}.$$

For $x \geq 0$, $|e^{-tx}| \leq 1$. Also f is bounded on $[0, A]$ for all $A < \infty$ because f is Riemann integrable. Let $|f| \leq M$ for $M > 0$.

$$\begin{aligned} t\int_0^\infty e^{-tx} f(x)\,dx &= t\int_0^K e^{-tx} f(x)\,dx + \lim_{b\to\infty} t\int_K^b e^{-tx} f(x)\,dx \qquad (*) \\ &\leq t\int_0^K M\,dx + \lim_{b\to\infty}\int_K^b \left(1 + \frac{\epsilon}{2}\right) te^{-tx}\,dx \\ &= KMt + \left(1 + \frac{\epsilon}{2}\right) e^{-tK}. \end{aligned}$$

The last expression approaches $1 + \epsilon/2$ as $t \to 0$. So we can choose $\delta_1 > 0$ such that

$$0 < t < \delta_1 \implies \left|KMt + \left(1 + \frac{\epsilon}{2}\right)(e^{-tK} - 1)\right| < \frac{\epsilon}{2}.$$

Therefore, if $t \in (0, \delta_1)$,

$$t\int_0^\infty e^{-tx} f(x)\,dx < KMt + \left(1 + \frac{\epsilon}{2}\right) e^{-tK} < 1 + \epsilon.$$

In a similar manner, we can start from $(*)$ and get a lower bound.

$$\begin{aligned} (*) &\geq t\int_0^K (-M)\,dx + \lim_{b\to\infty}\int_K^b \left(1 - \frac{\epsilon}{2}\right) te^{-tx}\,dx \\ &= -KMt + \left(1 - \frac{\epsilon}{2}\right) e^{-tK} \end{aligned}$$

The last expression approaches $1 - \epsilon/2$ as $t \to 0$. So we can choose $\delta_2 > 0$ such that

$$0 < t < \delta_2 \implies \left|-KMt + \left(1 - \frac{\epsilon}{2}\right)(e^{-tK} - 1)\right| < \frac{\epsilon}{2}.$$

Therefore, if $t \in (0, \delta_2)$,

$$t\int_0^\infty e^{-tx} f(x)\,dx > -KMt + \left(1 - \frac{\epsilon}{2}\right) e^{-tK} > 1 - \epsilon.$$

So we have proved that for any given $\epsilon > 0$, we can choose $\delta = \min\{\delta_1, \delta_2\}$ such that

$$0 < t < \delta \implies \left|t\int_0^\infty e^{-tx} f(x)\,dx - 1\right| < \epsilon.$$

Thus

$$\lim_{t\to 0^+} t\int_0^\infty e^{-tx} f(x)\,dx = 1.$$