해석개론 및 연습 2 과제 #8

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1. Set $u_n = |f - f_n|$, $v_n = |f - f_n|^{1/2}$. Then u_n is measurable, and v_n is also measurable since for $a \in \mathbb{R}$,

$$\{x \in \mathbb{R} : v_n > a\} = \{x \in \mathbb{R} : u_n > a^2\} \in \mathfrak{M}.$$

Next, since $\lim_{n\to\infty} \int_{\mathbb{R}} |f - f_n|\, dx = 0$ and $\lim_{n\to\infty} \int_{\mathbb{R}} |f - f_n|^2\, dx = 0$, both $\int_{\mathbb{R}} |f - f_n|\, dx$ and $\int_{\mathbb{R}} |f - f_n|^2\, dx$ should be finite for large enough $n \in \mathbb{N}$. Therefore, there exists $N \in \mathbb{N}$ such that $n \geq N \implies u_n, v_n \in \mathcal{L}^2$. Now, from Schwarz inequality,

$$0 \leq \int_{\mathbb{R}} |f - f_n|^{3/2}\, dx \leq \left(\int_{\mathbb{R}} |f - f_n|^2\, dx \right)^{1/2} \left(\int_{\mathbb{R}} |f - f_n|\, dx \right)^{1/2}.$$

Taking limit $n \to \infty$ on both sides will give $\lim_{n\to\infty} \int_{\mathbb{R}} |f - f_n|^{3/2}\, dx = 0$.

2. Let $\phi_n = e^{inx}$ for $n \in \mathbb{N}$. We use the fact that $\{\phi_n\}_{n\in\mathbb{N}}$ is a complete orthonormal set. Let $c_n = \frac{1}{n}$ and $s_m = \sum_{n=1}^{m} c_n \phi_n$. Since $\sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, s_m converges to some function $f \in \mathcal{L}^2[-\pi, \pi]$ by Riesz-Fischer theorem. Therefore $\sum_{n=1}^{\infty} \frac{1}{n} e^{inx} \in \mathcal{L}^2[-\pi, \pi]$.

Now suppose that $g = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e^{inx} \in \mathcal{L}^2[-\pi, \pi]$. Then by Theorem 11.45, we have that $\sum_{n=1}^{\infty} \left| \frac{1}{\sqrt{n}} \right|^2 = \int_{-\pi}^{\pi} |g|^2\, dx$. However, the left hand side diverges to ∞ (harmonic series), contradicting that $g \in \mathcal{L}^2[-\pi, \pi]$ (the integral should be finite).

3. First, boundedness can be checked easily by calculation.

$$\|f_n\|_2^2 = \int_{-\pi}^{\pi} \sin^2 nx\, dx = \int_{-\pi}^{\pi} \frac{1 - \cos 2nx}{2}\, dx = \pi. \quad (n \in \mathbb{N})$$

Next, to see that $A = \{f_n : n \in \mathbb{N}\} \subseteq \mathcal{L}^2[-\pi, \pi]$ is closed, we show that all $f_i \in A$ are isolated points. Direct calculation yields

$$\begin{aligned} \|f_n - f_m\|_2^2 &= \int_{-\pi}^{\pi} |\sin nx - \sin mx|^2\, dx \\ &= \int_{-\pi}^{\pi} \sin^2 nx\, dx + \int_{-\pi}^{\pi} \sin^2 mx\, dx - 2\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = 2\pi. \end{aligned}$$

Thus, each f_i are $\sqrt{2\pi}$ distance apart. So if we define

$$B_{\frac{\sqrt{2\pi}}{2}}(f_i) = \left\{ f \in \mathcal{L}^2[-\pi, \pi] : \|f - f_i\|_2 < \frac{\sqrt{2\pi}}{2} \right\},$$

then $B_{\sqrt{2\pi}/2}(f_i) \setminus \{f_i\} \cap A = \varnothing$. So A has no limit point, and $A' = \varnothing$. Since $A' \subseteq A$, A is closed.

Finally, suppose that A is compact. Since A is infinite, A has a limit point in A by Theorem 2.37. But A has no limit point, which is a contradiction. Thus A cannot be compact.

4. Let μ denote the Lebesgue measure for this problem. Let

$$C_{m,N} = \bigcup_{n=N}^{\infty} \left\{ x \in [a,b] : |f_n(x) - f(x)| \geq \frac{1}{m} \right\}, \quad (m, N \in \mathbb{N}).$$

Note that $C_{m,N}$ is indeed measurable (because f_n, f are) and $C_{m,N+1} \subseteq C_{m,N}$ for fixed m. Also let $C_m = \bigcap_{N=1}^{\infty} C_{m,N}$. For $x \in [a,b]$, if $f_n(x)$ converges to $f(x)$, x cannot be in $C_{m,N}$ for all N, since for large enough N_0, $|f_{N_0}(x) - f(x)| < \frac{1}{m}$ for fixed m. So if C is the set of $x \in [a,b]$ where $f_n(x)$ does not converge to $f(x)$, then $\mu(C) = 0$ by the given condition. Also, we can write $C = \bigcup_{m=1}^{\infty} C_m$. So using the completeness of Lebesgue measure, $\mu(C_m) = 0$. ($C_m \subseteq C$) Since $\mu([a,b]) < \infty$ and $C_{m,N} \searrow C_m$, continuity of the measure from above gives $\mu(C_{m,N}) \to 0$ as $N \to \infty$.

To construct a measurable set $A \subseteq [a,b]$ with $\mu(A) < \delta$, we can choose $N_m \in \mathbb{N}$ for each $m \in \mathbb{N}$ to construct a sequence $\{N_m\}_{m=1}^{\infty}$ such that $\mu(C_{m,N_m}) < \frac{\delta}{2^m}$. If we set $A = \bigcup_{m=1}^{\infty} C_{m,N_m}$, then

$$\mu(A) \leq \sum_{m=1}^{\infty} \mu(C_{m,N_m}) < \sum_{m=1}^{\infty} \frac{\delta}{2^m} = \delta.$$

Now we show that $f_n(x)$ converges uniformly on $[a,b] \setminus A$. We need to choose N large enough so that for $n \geq N$, $|f_n(x) - f(x)| < \epsilon$ on $[a,b] \setminus A$. For given $\epsilon > 0$ choose m such that $\frac{1}{m} < \epsilon$ and with this m, choose N_* such that $N_* > N_m$. Then if we consider

$$\sup_{n \geq N_*,\, x \in [a,b] \setminus A} |f_n(x) - f(x)|,$$

we see that all $x \in [a,b]$ such that $|f_n(x) - f(x)| \geq \frac{1}{m}$ for $n \geq N_*$, $m \in \mathbb{N}$ has been taken out by A.[1] Therefore,

$$\sup_{n \geq N_*,\, x \in [a,b] \setminus A} |f_n(x) - f(x)| < \frac{1}{m} < \epsilon.$$

5. First we prove the following claim.

Claim. Let A be a measurable subset of $[-\pi, \pi]$. Then

$$\int_A \sin nx \, dx, \int_A \cos nx \, dx \to 0 \text{ as } n \to \infty.$$

[1] By the choice of $A = \bigcup_{m=1}^{\infty} C_{m,N_m}$, for each m, all $[x \in [a,b]$ such that $|f_n(x) - f(x)| \geq \frac{1}{m}$ for $n \geq N_m]$ has been taken out. But by the choice of N_m, there may still be $x \in [a,b] \setminus A$ such that $n < N_m$ and $|f_n(x) - f(x)| \geq \frac{1}{m}$. Thus, we have to set N_m large enough so that $|f_n(x) - f(x)| < \frac{1}{m}$ for $n \geq N_m$.

Proof. On $\mathcal{L}^2[-\pi,\pi]$, we consider the Fourier coefficients of χ_A. Then by Bessel's inequality,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi_A e^{inx}\,dx = \frac{1}{2\pi}\int_A e^{inx}\,dx \to 0$$

as $n \to \infty$. Therefore $\int_A \sin nx\,dx, \int_A \cos nx\,dx \to 0$.

For measurable $A \subseteq E$, we have $\int_A \sin n_k x\,dx \to 0$ and

$$2\int_A (\sin n_k x)^2\,dx = \int_A (1-\cos 2n_k x)\,dx = m(A) - \int_A \cos 2n_k x\,dx \to m(A)$$

as $k \to \infty$ by the above claim and the half-angle formula.

Now consider $f(x) = \lim_{k\to\infty} \sin n_k x$ for $x \in E$. f is measurable, and since $|2\sin n_k x| \le 2$ for all $x \in E$, we can use LDCT and conclude that

$$m(A) = \lim_{k\to\infty}\int_A 2\sin^2 n_k x\,dx = \int_A 2f^2\,dx.$$

Therefore $\int_A [2f^2 - 1]\,dx = 0$ for all measurable $A \subseteq E$. By Problem 7 of Homework 6, $f^2(x) = \frac{1}{2}$ a.e. on E. Now consider $f^{-1}(1/\sqrt{2})$ and $f^{-1}(-1/\sqrt{2})$. $f^{-1}(1/\sqrt{2}) \subseteq E$ trivially, and

$$0 = \int_{f^{-1}(1/\sqrt{2})} f(x)\,dx = \int_{f^{-1}(1/\sqrt{2})} \frac{1}{\sqrt{2}}\,dx = \frac{1}{\sqrt{2}} m(f^{-1}(1/\sqrt{2})).$$

Thus $m(f^{-1}(1/\sqrt{2})) = 0$, and similarly $m(f^{-1}(-1/\sqrt{2})) = 0$. Therefore,

$$m(E) = m(f^{-1}(1/\sqrt{2})) + m(f^{-1}(-1/\sqrt{2})) = 0.$$

6. Suppose that $\sin nx \ge \delta$ for all $x \in E$ holds for infinitely many $n \in \mathbb{N}$. Then from

$$\int_E \sin nx\,dx \ge \int_E \delta\,dx = \delta m(E) > 0.$$

But the left hand side should approach 0 as $n \to \infty$. ($\because$ Claim in Problem 5) This is impossible, so $\sin nx \ge \delta$ can only hold for finitely many $n \in \mathbb{N}$.

7. If $f \sim 0$ or $g \sim 0$ in $\mathcal{L}^2(\mu)$, the statement is trivial, so we assume that this is not the case. (In fact, the statement is false when $f \sim 0$ but $g \not\sim 0$.)

($\Longleftarrow$) Suppose that $\exists c \in \mathbb{C}$ such that $g(x) = cf(x)$ μ-a.e.. Then $f\overline{g} = \overline{c}\,|f|^2$, $|g|^2 = |c|^2\,|f|^2$ μ-a.e.. Therefore,

$$\text{LHS} = \left|\int f\overline{g}\,d\mu\right|^2 = \left|\int \overline{c}\,|f|^2\,d\mu\right|^2 = |c|^2\,\|f\|_2^4,$$

$$\text{RHS} = \left(\int |f|^2\,d\mu\right)\left(\int |c|^2\,|f|^2\,d\mu\right) = |c|^2\,\|f\|_2^4.$$

($\Longrightarrow$) Denote $\langle f, g\rangle = \int f\overline{g}\,d\mu$. The assumption can be written as

$$|\langle f, g\rangle|^2 = \|f\|_2^2\,\|g\|_2^2.$$

Let $u = f - \frac{\langle f,g \rangle}{\|g\|_2^2} g$. Then u is indeed measurable. Now by calculation,

$$\begin{aligned}
\int |u|^2 \, d\mu &= \int \left(f - \frac{\langle f,g \rangle}{\|g\|_2^2} g \right) \overline{\left(f - \frac{\langle f,g \rangle}{\|g\|_2^2} g \right)} \, d\mu \\
&= \int |f|^2 \, d\mu - \int \frac{\langle f,g \rangle}{\|g\|_2^2} \overline{f} g \, d\mu - \int \frac{\overline{\langle f,g \rangle}}{\|g\|_2^2} f \overline{g} \, d\mu + \int \frac{|\langle f,g \rangle|^2}{\|g\|_2^4} |g|^2 \, d\mu \\
&= \|f\|_2^2 - \frac{\langle f,g \rangle}{\|g\|_2^2} \langle g,f \rangle - \frac{\overline{\langle f,g \rangle}}{\|g\|_2^2} \langle f,g \rangle + \frac{|\langle f,g \rangle|^2}{\|g\|_2^4} \cdot \|g\|_2^2 \\
&= \|f\|_2^2 - \frac{\langle f,g \rangle}{\|g\|_2^2} \overline{\langle f,g \rangle} - \frac{\overline{\langle f,g \rangle}}{\|g\|_2^2} \langle f,g \rangle + \|f\|_2^2 \\
&= \|f\|_2^2 - \|f\|_2^2 - \|f\|_2^2 + \|f\|_2^2 = 0.
\end{aligned}$$

$\langle f,g \rangle = \overline{\langle g,f \rangle}$ and the given assumption was used. Since $|u|^2 \geq 0$, $u = 0$ μ-a.e. by Problem 5 of Homework 6. Therefore, $g = cf$ for some constant $c \in \mathbb{C}$ μ-a.e..