Introduction to Analysis II

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Introduction & Notice

- 7, 8장 나가고 중간고사, 11장 나가고 기말고사
- 연습 시간이 있는 수업 (목 6:30 ∼ 8:20)[1]
- 오늘 연습 시간: 지난학기 배운 내용 중 필요한 내용 복습

[1]가능하면 1시간 반 안에 끝내라고 하심 ㅋㅋ

Chapter 7

Sequences and Series of Functions

September 1st, 2022

기본적으로 수열에 관련된 내용, real/complex-valued 수열이 아니라 함수가 주어졌을 때. 함수들을 모은 '**sequence of functions**'의 극한을 생각하는 것.

Suppose E is a set[1], and let $f_n : E \to \mathbb{C}$. Then

$$(f_n)_{n=1}^{\infty}$$

is a sequence of (complex-valued) function.

Definition 7.1 (Pointwise Convergence) $(f_n)_{n=1}^{\infty}$ converges **pointwise** on E, if for each $x \in E$ the sequence $(f_n(x))_{n=1}^{\infty}$ converges in $\mathbb{C}$.

In other words, for each $x \in E$, there exists $a_x \in \mathbb{C}$ and

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } n \geq N \implies |f_n(x) - a_x| < \epsilon.$$

Definition. If (f_n) converges pointwise, we can define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E)$$

We say that

- f is the *limit* or *limit function* of f_n.
- (f_n) to f pointwise on E.

[1] 사실은 *metric* space 이다.

Definition. If $\sum f_n(x)$ converges (pointwise) for every $x \in E$, we can define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E)$$

and the function f is called the *sum* of the series $\sum f_n$.

Recall. $f : (E, d) \to \mathbb{C}$ is continuous on $E \iff f$ is continuous at all $x \in E$.

Recall. (Theorem 4.6) If $p \in E$ and p is a limit point of E,

$$f \text{ is continuous at } p \iff \lim_{x \to p} f(x) = f(p)$$

Question. Suppose (f_n) is a sequence of functions. Does the limit function or the sum of the series preserve important properties?

(1) If f_n is continus, is f continuous?

(2) If f_n is differentiable/integrable, is f differentiable/integrable?

For (1), the question is equivalent to the following:

If p is a limit point, does the following hold?

$$\lim_{x \to p} \lim_{n \to \infty} f_n(x) \stackrel{?}{=} \lim_{n \to \infty} \lim_{x \to p} f_n(x)$$

And the answer is **No**.

Example 7.2 Suppose $a_{m,n} = \dfrac{m}{m+n}$ for $m, n \in \mathbb{N}$. We see that

$$\lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = 1 \neq 0 = \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n}$$

Example. Define

$$f_n(x) = \begin{cases} 0 & \left(\frac{1}{n} \leq x \leq 1\right) \\ -nx + 1 & \left(0 \leq x < \frac{1}{n}\right) \end{cases}$$

then we can easily see that

$$f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & (0 < x \leq 1) \\ 1 & (x = 0) \end{cases}$$

Thus f is not continuous at $x = 0$.

Example. Define $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (n = 0, 1, 2, \dots)$$

by direct calculation,

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = 1 + x^2 \quad (x \neq 0)$$

since this is a geometric series when $x \neq 0$. If $x = 0$, $f(x) = 0$ and f is not continuous.

Does the limit function preserve Riemann integrability?

Example. For $m = 1, 2, \dots$, define

$$f_m(x) = \lim_{n \to \infty} (\cos m!\pi x)^{2n} = \begin{cases} 1 & (m!x \in \mathbb{Z}) \\ 0 & (m!x \notin \mathbb{Z}) \end{cases}$$

We see that $f_m(x)$ is Riemann integrable. However,

Claim.

$$f(x) = \lim_{m \to \infty} f_m(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

and $f(x)$ is nowhere continuous thus not Riemann integrable.

Proof. Suppose $x = p/q \in \mathbb{Q}$. $(p, q \in \mathbb{Z})$ If we take $m \geq q$, we see that $m!x \in \mathbb{Z}$. Thus $f_m(x) = 1$. If $x \notin \mathbb{Q}$, $m!x$ can never be in $\mathbb{Z}$ and $f_m(x) = 0$.

Question. Uniform continuity를 할 때 uniform이 어디서 나오죠? 해석학에서 그 점에서 뭐가 성립한다, 그러면 그 점과 그 근방에서만 확인하면 됐었죠. Continuity는 local property죠. 그런데 uniform continuity는 전체가 다 uniform하게 성립한다는 의미입니다.

Recall. $f : (X, d) \to (Y, d)$ is uniformly continuous on X[2] if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_X(p, q) < \delta \implies d_Y(f(p), f(q)) < \epsilon$$

즉, 모든 점에서 똑같이 잡을 수 있다!

Recall. (Theorem 4.19) If X is compact and f is continuous on X, then f is uniformly continuous on X.[3]

이제부터 나오는 uniform convergence는 sequence에 관한 것입니다!

[2] Subspace of metric space is also a metric space
[3] 갑자기 왜 uniform continuity 얘기를 하냐, 헷갈리지 말고 기억하시라고!

Definition 7.7 (Uniform Convergence) Suppose $f_n : E \to \mathbb{C}$ is a sequence of functions. $(f_n)_{n=1}^{\infty}$ **converges uniformly** on E to a function f if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall x \in E, n \geq N \implies |f_n(x) - f(x)| \leq \epsilon.$$[4]

Also, we say that the series $\sum f_n(x)$ converges uniformly on E if the sequence of partial sums $(\sum_{k=1}^{n} f_k(x))$ converges uniformly on E.

Pointwise convergence의 경우 $N \in \mathbb{N}$ 이 $x \in E$ 에 의존하지만, uniform convergence의 경우 N 이 x와 무관하다!

[똑같은 ϵ-띠를 둘러서 $y = f(x)$ 의 근방 안에 $f_n(x)$ $(n \geq N)$ 가 모두 들어가 있어야 한다]는 의미에서 uniform 이다.

Theorem 7.9 Suppose

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E)$$

Then $f_n \to f$ converges uniformly on E if and only if

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$$

which can also be written as

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies \sup_{x\in E} |f_n(x) - f(x)| \leq \epsilon$$

Notation. $f_n \to f$ uniformly on $E \iff f_n \xrightarrow{u} f$ on E.[5]

Theorem 7.8 (Cauchy Criterion for Uniform Convergence) $f_n \xrightarrow{u} f$ on $E \iff$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n, m \geq N \implies \sup_{x\in E} |f_n(x) - f_m(x)| \leq \epsilon.$$[6]

Proof.

$(\implies)$ For given $\epsilon > 0$, fix $x \in E$. Since f_n converges uniformly on E, we can find $N \in \mathbb{N}$ such that for $n, m \geq N$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$(\impliedby)$ Uniform Cauchy property implies that (f_n) is a Cauchy sequence in $\mathbb{C}$. By the completeness of $\mathbb{C}$, the limit function $f(x)$ exists. Now we show that this convergence is uniform. For given $\epsilon > 0$ choose $N \in \mathbb{N}$ such that for all $n, m \geq N$,

[4] 등호를 붙이는 것이 극한 잡기 편하다???
[5] 교수님: 책에서는 나중에 $\|f_n(x) - f(x)\|_\infty \to 0$ 으로 적었던 것 같은데...
[6] Uniform Cauchy Property

$$\sup_{x\in E} |f_n(x) - f_m(x)| \leq \epsilon$$

Then

$$\begin{aligned}|f_n(x) - f(x)| &= |f_n(x) - f_m(x) + f_m(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq |f_m(x) - f(x)| + \epsilon\end{aligned}$$

Fix $n \geq N$ and let $m \to \infty$. Observe that $|f_m(x) - f(x)| \to 0$ due to pointwise convergence. Therefore for every $x \in E$,

$$n \geq N \implies |f_n(x) - f(x)| \leq \epsilon$$