HW Solution 6

1. (3 points) Let A be an elementary set. Then it is a finite union of intervals, $A = \sum_{i=1}^{k} I_i$. Let us assume that $I_1 = (a_1, b_1)$ for some $a_1 < b_1 \in \mathbb{R}$. We first show the following claim.

   claim: For any ϵ, there are closed set F and open set O such that

   $$F \subset I_1 \subset O \quad \text{and} \quad \mu(O) - \epsilon \leq \mu(I_1) \leq \mu(F) + \epsilon. \tag{1}$$

   It follows from taking $F = \left[a_1 + \frac{\delta}{4}, b_1 - \delta 4\right]$ and $) = \left(a_1 - \frac{\delta}{4}, b_1 + \delta 4\right)$ where $\delta = \min\{\epsilon, b_1 - a_1\}$. Similarly, we prove (1) if I_1 is of the form $(a_1, b_1], [a_1, b_1)$ and $[a_1, b_1]$. Since intersection of closed sets and union of open sets are closed set and open set, respectively, we also prove the claim for the case A.

2. (3 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Let $a \in \mathbb{R}$. It suffices to show that

   $$f_a = \{x \in \mathbb{R}\,;\, f(x) > a\}$$

   is a measurable set. We may assume that f_a is nonempty set.

   (a) Suppose that $n \in f_a$ for any $n \in \mathbb{Z}$. Then this implies that $f_a = \mathbb{R}$.

   (b) Suppose that there is a $M \in \mathbb{Z}$ such that $M \notin f_a$. Then f_a is bounded below because f is non-decreasing. Thus there is a real number $M_a = \inf\{x \in \mathbb{R}\,;\, x \in f_a\}$. Since f is non-decreasing, f_a is either $(-\infty, M_a]$ or $(-\infty, M_a)$.

   In any case, we have shown that f_a is measurable set.

3. (5 points) Let a be a real number. Then it suffices to show that

   $$g_a = \{x \in \mathbb{R}\,;\, g(x) > a\}$$

   is a Lebesgue measurable set. Let us define

   $$B = \{x \in \mathbb{R}\,;\, f(x) \notin g(x)\}.$$

   Then we have that

   $$g_a = A \cup \{x \in \mathbb{R}\,;\, f(x) > a\}$$

   for some subset $A \subset B$. By completeness of Lebesgue measure, A is Lebesgue measurable, so is g_a.

4. (5 points) It suffices to show that

$$S \equiv \left\{ x\,;\ \liminf_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x) \quad \text{and} \quad |\limsup_{n\to\infty} f_n(x)| < \infty \right\}$$

is measurable set. Let us define

$$B = \left\{ x\,;\ |\liminf_{n\to\infty} f(x)| = \infty \quad \text{or} \quad |\limsup_{n\to\infty} f(x)| = \infty \right\}.$$

and

$$A = X \setminus B.$$

Then we observe that

$$S = \left\{ x\,;\ \liminf_{n\to\infty} f_n|_A(x) = \limsup_{n\to\infty} f_n|_A(x) \right\},$$

where $f_n|_A$ is the restriction of f_n to the set A. Since $\liminf_{n\to\infty} f_n|_A - \limsup_{n\to\infty} f_n|_A$ is a well defined real-valued measurable function, S is the measurable set.

Another proof: Use the following relation

$$S = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \left\{ x\,;\ |f_m(x) - f_l(x)| < \frac{1}{n} \quad \text{and} \quad m, l \geq i \right\}.$$

5. (3 points) Define

$$E_n = \left\{ x\,;\ f(x) \geq \frac{1}{n} \right\}.$$

By the assumption,

$$\frac{1}{n}\mu(E_n) \leq \int_{E_n} f\, d\mu = 0.$$

Therefore, we have that

$$\mu(\{x\,;\ f(x) > 0\}) \leq \sum_{n=1}^{\infty} \mu(E_n) = 0$$

which implies that $f = 0$ a.e. on E.

6. (3 points) Let $x \in [0,1]$. Then $f_{2k}(x) = 0$ or $f_{2k+1}(x) = 0$ for any $k \in \mathbb{N}$. Thus we observe that

$$\liminf_{n\to\infty} f_n(x) = 0.$$

On the other hand,

$$\int_0^1 f_n(x)\,dx = \frac{1}{2}.$$

7. (3 points) Let

$$C = \{x\,;\, f(x) \leq 0\} \quad \text{and} \quad D = \{x\,;\, f(x) \geq 0\}.$$

Then $f|_C$ and $-f|_D$ satisfy the assumption given in the problem 5. Thus we have that $f = 0$ a.e. on C and D. Since $E = C \cup D$, we obtain the desired result.

8. (5 points) Let us define

$$S = \{x\,;\, g(x) = f(x) = \infty \quad \text{or} \quad f(x) = g(x) = -\infty\} \quad \text{and} \quad B = X\backslash S.$$

Then we observe that $h = f|_B - g|_B$ is a well defined measurable function. In addition, since f and g are in $\mathcal{L}^1$, $\mu(S) = 0$. We now prove the equivalent relation given in the problem 8.

(a) Suppose that $f = g$ a.e. on X with respect to the measure μ. Note that by the fact that $\mu(S) = 0$,

$$\int_S g\,d\mu = \int_S f\,d\mu = 0.$$

Then for any $\mathcal{A} \in \mathcal{M}$,

$$\int_A f\,d\mu = \int_{A\backslash S} f\,d\mu = \int_{A\backslash S} g\,d\mu = \int_A g\,d\mu.$$

(b) Suppose that for any $A \in \mathcal{M}$,

$$\int_A f\,d\mu = \int_A g\,d\mu.$$

Then we have that

$$0 = \int_{A\cap B} f - g, d\mu = \int_A h\,d\mu = 0$$

for any $A \in \mathcal{M}$. By the problem 7, we have that $h = 0$ a.e. on X. Since $\mu(S) = 0$, $f = g$ a.e. on X.

Another proof: If $f, g \in L^1$, then $f - g \in L^1$.(Theorem 11.29) Then the desired result follows from the problem 5.