HW Set 4. (Due day: October 12, 23:59)

1. Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \le \delta$, $f(x) = 0$ if $\delta < |x| \le \pi$, and $f(x + 2\pi) = f(x)$ for all x.

   (a) Compute the Fourier coefficients of f.

   (b) Conclude that
   $$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$

   (c) Deduce from Parseval's theorem that
   $$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

   (d) Let $\delta \to 0$ and prove that
   $$\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}.$$

   (e) Put $\delta = \pi/2$ in (c). What do you get?

Solution. **(a)** $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}dx = \frac{1}{2\pi}[\frac{1}{-in}e^{-inx}]_{-\delta}^{\delta}$
$= \frac{1}{\pi n}\frac{e^{in\delta}-e^{-in\delta}}{2i} = \frac{1}{\pi n}\sin n\delta$ for $n \neq 0$, and $c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$
$= \frac{1}{2\pi}\int_{-\delta}^{\delta} 1dx = \frac{\delta}{\pi}$.

(b) Since $f(x)$ is constant on the neighbor of $x = 0$, by theorem 8.14, Fourier series of f converges at $x = 0$ to $f(0)$. Thus
$$\lim_{N\to\infty} \sum_{k=-N}^{N} c_k = f(0) = 1.$$

Note that $c_{-n} = c_n$ for $n \neq 0$, so we have
$$\frac{\delta}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = 1$$

and this gives

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$

(c) f is Riemann-integrable because it is not continuous only at two points $t = \pm\delta$. $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\delta}^{\delta} 1 dx = \frac{\delta}{\pi}$, and Parseval's theorem gives

$$\lim_{N\to\infty} \sum_{k=-N}^{N} |c_k|^2 = \frac{\delta}{\pi}, \quad \frac{\delta^2}{\pi^2} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2} = \frac{\delta}{\pi}, \quad \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) Define $g(x) = \begin{cases} \left(\frac{\sin x}{x}\right)^2 & (x \neq 0) \\ 1 & (x = 0) \end{cases}$. $g(x)$ is continuous on $\mathbb{R}$, and

$$\int_0^A |g(x)|dx \leq \int_0^1 |g(x)|dx + \int_1^A \frac{1}{x^2}dx \leq \int_0^1 |g(x)|dx + 1 - \frac{1}{A}$$
$$\leq \int_0^1 |g(x)|dx + 1$$

so $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$ exists.
For $\varepsilon > 0$, there exists $A \in \mathbb{R}^+$ s.t. $B \geq A$ implies

$$\left| \int_0^B \left(\frac{\sin x}{x}\right)^2 dx - \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx \right| < \frac{\varepsilon}{5}.$$

Let B be s.t. $B > \max\{A,\ \frac{10}{\varepsilon}\}$.
Since $g(x)$ is Riemann-integrable on $[0,\ B]$, there exists δ_0 s.t. $U(g,\ P) - L(g,\ P) < \frac{\varepsilon}{5}$ for any partition $P = \{0 = t_0 < t_1 < \cdots < t_n = B\}$ which satiefies $\sup\limits_{1\leq i\leq n} |t_i - t_{i-1}| < \delta_0$.
Let $M_\delta := \lfloor \frac{B}{\delta} \rfloor$ and observe that

$$\sum_{M_\delta+1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} \leq \sum_{M_\delta+1}^{\infty} \frac{1}{n^2\delta} \leq \frac{1}{\delta}\int_{M_\delta}^{\infty} \frac{1}{x^2}dx = \frac{1}{\delta M_\delta} \leq \frac{1}{B - \delta}.$$

Let M be s.t. $|g(x)| \leq M$ for all $x \in \mathbb{R}$(such M exists because $\lim_{x\to\pm\infty} g(x) = 0$).

Now If $\delta < \min\{\delta_0,\ \frac{\varepsilon}{5M},\ \frac{B}{2}\}$, then

$$\left|\int_0^\infty g(x)dx - \sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta}\right| \leq \left|\int_0^\infty g(x)dx - \int_0^B g(x)dx\right|$$

$$+\left|\int_0^B g(x)dx - \sum_{n=1}^{M_\delta} g(n\delta)\delta - g(B)(B - M_\delta\delta)\right| + |g(B)(B - M_\delta\delta)|$$

$$+\left|\sum_{M_\delta+1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta}\right| \leq \frac{\varepsilon}{5} + \leq \frac{\varepsilon}{5} + M\delta + \frac{1}{B-\delta} \leq \frac{4\varepsilon}{5} < \varepsilon.$$

(e) $\sin^2\frac{n\pi}{2} = \begin{cases} 0 & (n \text{ is even}) \\ 1 & (n \text{ is odd}) \end{cases}$. so we get

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi}{4},\ \ \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

□

2. Prove that

$$(\pi - |x|)^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx \qquad \text{for all } x \in [-\pi, \pi]$$

and deduce that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution. Let $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$ and $f(x + 2\pi) = f(x)$. Then f is well-defined continuous periodic function. f is differentiable at $x \neq 2n\pi$, and $\lim\limits_{x\to 2n\pi\pm} \frac{f(2n\pi+x)-f(2n\pi)}{x} = \mp 2\pi$ exist. Therefore, The Fourier series of f converges at every point to f by theorem 8.14.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$$

since f is even, and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{0}^{\pi} (\pi - x)^2 \cos nx dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n}(\pi-x)^2\sin nx - \frac{2}{n^2}(\pi - x)\cos nx - \frac{2}{n^3}\sin nx\right]_0^{\pi} = \frac{2}{n^2}.$$

Also $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}(\pi-x)^2 dx = \frac{\pi^2}{3}$. Thus

$$f(x) = (\pi - |x|)^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\{c_n e^{inx} + c_{-n}e^{-inx}\}$$

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left\{\left(\frac{2}{n^2} - 0i\right)e^{inx} + \left(\frac{2}{n^2} + 0i\right)e^{-inx}\right\} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$$

on $[-\pi, \pi]$. Let $x = 0$ then we have

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}, \quad \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)^4 dx = \frac{1}{\pi}\left[-\frac{1}{5}(\pi - x)^5\right]_0^{\pi} = \frac{\pi^4}{5},$$

Parseval's theorem gives

$$\frac{\pi^4}{9} + 2\sum_{n=1}^{\infty}\frac{4}{n^4} = \frac{\pi^4}{5}, \quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

$\square$

3. With $D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \frac{\sin(n+\frac{1}{2})x}{\sin(x/2)}$, put

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x}$$

and that

(a) $K_N \geq 0$,

(b) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\, dx = 1$,

(c) $K_N(x) \leq \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta}$ if $0 < \delta \leq |x| \leq \pi$.

If $s_N = s_N(f;x)$ is the Nth partial sum of the Fourier series of f, consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\, dt,$$

and hence prove Fejer's theorem: *If f is continuous, with period 2π, then $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi, \pi]$.*

Hint: Use properties (a), (b), (c) to proceed as in Theorem 7.26.

Note. σ_N defined above is the Cesàro mean. So if $s_N(f;x)$ converges, then $\sigma_N(f;x)$ also converges to the same value. The fact that there exists a continuous function whose fourier series doesn't converge to itself suggests that converse is not true.

Solution.

$$K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}} = \frac{1}{(N+1)\sin\frac{x}{2}}\sum_{n=0}^{N}\frac{e^{i(n+\frac{1}{2})x} - e^{-i(n+\frac{1}{2})x}}{2i}$$

$$= \frac{1}{2i(N+1)\sin\frac{x}{2}}\left(e^{\frac{ix}{2}}\frac{e^{i(N+1)x}-1}{e^{ix}-1} - e^{-\frac{ix}{2}}\frac{e^{-i(N+1)x}-1}{e^{-ix}-1}\right)$$

$$\frac{1}{2i(N+1)\sin\frac{x}{2}}\left(\frac{e^{i(N+1)x}-1}{e^{\frac{ix}{2}}-e^{-\frac{ix}{2}}} + \frac{e^{-i(N+1)x}-1}{e^{\frac{ix}{2}}-e^{-\frac{ix}{2}}}\right) = \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{2\sin^2\frac{x}{2}}$$

$$= \frac{1}{N+1}\cdot\frac{1-\cos(N+1)x}{1-\cos x}$$

for $x \neq 2n\pi$, and $K_N(2n\pi) = N+1$. (*Note*: $\lim_{x\to 2n\pi} K_N(x) = (N+1)$.)

For $x \neq 2n\pi$, $\cos x$, $\cos(N+1)x \leq 1$ so $K_N(x) \geq 0$. Also $K_N(2n\pi) = N+1 > 0$. The mean value of K_N over $[-\pi,\, \pi]$ is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)dx = \frac{1}{2\pi}\frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=-n}^{n}\int_{-\pi}^{\pi} e^{ikx}dx = \frac{1}{2\pi}\frac{1}{N+1}\sum_{n=0}^{N}2\pi = 1.$$

If $0 < \delta \leq |x| \leq \pi$, then $K_N(x) \leq \frac{1}{N+1}\frac{1+1}{1-\cos x} = \frac{1}{N+1}\frac{2}{1-\cos x}$.

For $\sigma(f;\, x)$,

$$\sigma(f;\, x) = \frac{1}{N+1}\sum_{n=0}^{N} s_n(f;\, x) = \frac{1}{N+1}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\sum_{n=0}^{N} D_n(t)dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt.$$

For uniform convergence, Note the expression

$$|\sigma(f;\, x) - f(x)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|K_N(t)dt.$$

For $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|x-y| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{2}$ because f is periodic continuous function, and there exists $M > 0$ s.t. $|f(x)| \leq M$ for all $x \in \mathbb{R}$. Then if $n \geq \lfloor \frac{16\pi M}{\varepsilon(1-\cos\frac{\delta}{2})} \rfloor$ then

$$|\sigma(f;\, x) - f(x)|$$

$$\leq \frac{1}{2\pi}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} |f(x-t)-f(x)|K_n(t)dt + \frac{1}{2\pi}\int_{-\pi}^{-\frac{\delta}{2}} |f(x-t)-f(x)|K_n(t)dt$$

$$+\frac{1}{2\pi}\int_{\frac{\delta}{2}}^{\pi} |f(x-t)-f(x)|K_n(t)dt$$

$$\leq \frac{\varepsilon}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(t)dt + \frac{1}{2\pi}\int_{-\pi}^{-\frac{\delta}{2}} \frac{4M}{(n+1)(1-\cos\frac{\delta}{2})}dt + \frac{1}{2\pi}\int_{\frac{\delta}{2}}^{\pi} \frac{4M}{(n+1)(1-\cos\frac{\delta}{2})}dt$$

$$\leq \frac{\varepsilon}{2} + \frac{4M}{1-\cos\frac{\delta}{2}}\cdot\frac{1}{n+1} < \frac{\varepsilon}{2}+\frac{\varepsilon}{4} < \varepsilon.$$

$\square$

4. In this problem we generalize the theorem 8.14. Let f be a Riemann-integrable function with period 2π. Define $f(a\pm) := \lim_{x\to a\pm} f(x)$ if it exists. Assume that both $f(a\pm)$ exist and there exists a positive number $\varepsilon,\ \delta,\ M > 0$ s.t.

$$|t| < \delta \implies \left|\frac{f(a+t)+f(a-t)}{2} - \frac{f(a+)+f(a-)}{2}\right| \leq M|t|^{\varepsilon}.$$

In these conditions we will show that $s_N(f;a)$ converges to $\frac{f(a+)+f(a-)}{2}$.

(a) Show that $s_N(f;x)$ can be written as

$$\frac{1}{2\pi}\int_0^{\pi}\{f(x+t)+f(x-t)\}\frac{\sin(N+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

(b) Prove that

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_0^{\pi}\{f(x+t)+f(x-t)\}\left(\frac{1}{\sin\frac{t}{2}}-\frac{2}{t}\right)\sin\left(N+\frac{1}{2}\right)tdt = 0.$$

(c) Now we only have to show that the below limit

$$\lim_{N\to\infty}\left(s_N(f;a) - \frac{f(a+)+f(a-)}{2}\right)$$

$$= \lim_{N\to\infty} \frac{1}{\pi} \int_0^{\pi} \left(\frac{f(a+t)+f(a-t)}{t} - \frac{f(a+)+f(a-)}{t} \right) \sin\left(N+\frac{1}{2}\right) t dt$$

converges to zero. However, this time we cannot do as we did in the proof of theorem 8.14, because $\frac{f(a+t)+f(a-t)-f(a+)-f(a-)}{t}$ is no longer Riemann-integrable on $[-\pi, \pi]$(don't confuse it with the integrability of whole integrand). Although we won't deal with improper integral, there is a breakthrough.

Define $f_n : \{\frac{1}{p} \,|\, p \in \mathbb{N}\} \to \mathbb{C}$ by

$$f_n\left(\frac{1}{m}\right) = \frac{1}{\pi} \int_{\frac{1}{n}}^{\pi} \frac{f(a+t)+f(a-t)-f(a+)-f(a-)}{t} \sin\left(m+\frac{1}{2}\right) t dt.$$

Prove that f_n uniformly converges.

(d) Use theorem 7.11(limit interchange theorem) to conclude that $s_N(f;a)$ converges to $\frac{f(a+)+f(a-)}{2}$.

Note. This theorem is a generalization of theorem 8.14 in two aspects. f can be a discontinuous function and ε can be less than 1.

Solution. **(a)**

$$\begin{aligned} s_N(f;\, x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x+t) D_N(-t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2} D_N(t) dt \\ &= \frac{1}{2\pi} \int_0^{\pi} \{f(x+t)+f(x-t)\} D_N(t) dt. \end{aligned}$$

(b) Given limit is equal to

$$\lim_{N\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2} \left(\frac{1}{\sin\frac{t}{2}} - \frac{2}{t} \right) \sin\left(N+\frac{1}{2}\right) t dt.$$

Note that $\frac{1}{\sin\frac{t}{2}} - \frac{2}{t}$ is continuous on $\mathbb{R}$; $\lim_{t\to 0} \left(\frac{1}{\sin\frac{t}{2}} - \frac{2}{t} \right) = \lim_{t\to 0} \frac{t-2\sin\frac{t}{2}}{t\sin\frac{t}{2}}$ $= \lim_{t\to 0} \frac{\frac{t^3}{24}+\cdots}{t\sin\frac{t}{2}} = 0$. Thus $\frac{f(x+t)+f(x-t)}{2} \left(\frac{1}{\sin\frac{t}{2}} - \frac{2}{t} \right)$ is Riemann-integrable function on $[-\pi, \pi]$ and given limit is 0 by the argument in the proof of theorem 8.14.

(c) Now

$$\lim_{N\to\infty}\left(s_N(f;\, a) - \frac{f(a+)+f(a-)}{2}\right)$$
$$= \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(f(a-t) - \frac{f(a+)+f(a-)}{2}\right)D_N(t)dt$$
$$= \lim_{N\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f(a+t)+f(a-t)}{2} - \frac{f(a+)+f(a-)}{2}\right)D_N(t)dt$$
$$= \lim_{N\to\infty}\frac{1}{\pi}\int_{0}^{\pi}\left(\frac{f(a+t)+f(a-t)}{2} - \frac{f(a+)+f(a-)}{2}\right)D_N(t)dt$$
$$= \lim_{N\to\infty}\frac{1}{\pi}\int_{0}^{\pi}\left(\frac{f(a+t)+f(a-t)}{2} - \frac{f(a+)+f(a-)}{2}\right)\left(\frac{1}{\sin\frac{t}{2}} - \frac{2}{t}\right)\sin\left(N+\frac{1}{2}\right)tdt$$
$$+\lim_{N\to\infty}\frac{1}{\pi}\int_{0}^{\pi}\left(\frac{f(a+t)+f(a-t)}{t} - \frac{f(a+)+f(a-)}{t}\right)\sin\left(N+\frac{1}{2}\right)tdt$$
$$= \lim_{N\to\infty}\frac{1}{\pi}\int_{0}^{\pi}\left(\frac{f(a+t)+f(a-t)}{t} - \frac{f(a+)+f(a-)}{t}\right)\sin\left(N+\frac{1}{2}\right)tdt$$

because

$$\lim_{N\to\infty}\frac{1}{\pi}\int_{0}^{\pi}\left(\frac{f(a+t)+f(a-t)}{2} - \frac{f(a+)+f(a-)}{2}\right)\left(\frac{1}{\sin\frac{t}{2}} - \frac{2}{t}\right)\sin\left(N+\frac{1}{2}\right)tdt$$

is zero by (b)(substitute $f(x) - \frac{f(a+)+f(a-)}{2}$ instead of $f(x)$.)
For $k > 0$, since $\lim\limits_{t\to 0+} t^\varepsilon = 0$, there exists $N \in \mathbb{N}$ s.t. $n \geq N$ implies $\left(\frac{1}{n}\right)^\varepsilon < \min\{\frac{k\varepsilon\pi}{2M},\, \delta\}$. Then for any $l \geq n \geq N$ we have

$$\left|f_n\left(\frac{1}{m}\right) - f_l\left(\frac{1}{m}\right)\right| \leq \frac{1}{\pi}\int_{\frac{1}{l}}^{\frac{1}{n}} 2Mt^{\varepsilon-1}dt = \frac{2M}{\pi}\left[\frac{1}{\varepsilon}t^\varepsilon\right]_{\frac{1}{l}}^{\frac{1}{n}} \leq \frac{2M}{\pi\varepsilon}\left(\frac{1}{n}\right)^\varepsilon < k$$

and f_n uniformly converges.

(d) Let $g_n(t) := \begin{cases} \frac{f(a+t)+f(a-t)-f(a+)-f(a-)}{t} & (\frac{1}{n} \le t \le \pi) \\ 0 & (-\pi \le t < \frac{1}{n}) \end{cases}$. Surely g_n is Riemann-integrable on $[-\pi,\, \pi]$ and

$$f_n\left(\frac{1}{m}\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} g_n(t)\sin\left(m+\frac{1}{2}\right)tdt.$$

However, the right side goes 0 when $m \to 0$, again by the argument of the proof of theorem 8.14. Thus we have

$$\lim_{N\to\infty}\frac{1}{\pi}\int_0^{\pi}\left(\frac{f(a+t)+f(a-t)}{t} - \frac{f(a+)+f(a-)}{t}\right)\sin\left(N+\frac{1}{2}\right)tdt$$

$$= \lim_{m\to\infty}\lim_{n\to\infty} f_n\left(\frac{1}{m}\right) = \lim_{n\to\infty}\lim_{m\to\infty} f_n\left(\frac{1}{m}\right) = \lim_{n\to\infty} 0 = 0,$$

because

$$\lim_{t\to a+}\int_t^b f(x)dx = \int_a^b f(x)dx$$

for any Riemann-integrable function on $[a,\, b]$. $\square$