HW Set 3. (Due day: October 5, 23:59)

1. In the proof of Theorem 8.1, describe the following details:

   (a) Does every power series converge absolutely in the interior of its interval of convergence? explain it by using root test.

   (b) When we apply the theorem 7.17, what we choose for $(f_n)_{n=1}^{\infty}$ and x_0?

2. Prove that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$

   if $a_{ij} \geq 0$ for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$ (the case $+\infty = +\infty$ may occur).

3. Show that $\log x$ is real-analytic on $(0, \infty)$, that is, for every $a \in (0, \infty)$ $\log x$ can be expressed $\log x = \sum\limits_{n=1}^{\infty} a_n (x-a)^n$ in some interval $(a-\varepsilon,\, a+\varepsilon) \subset (0,\, \infty)$.

4. Find the following limits:

   (a)
$$\lim_{x\to 0} \frac{e - (1+x)^{1/x}}{x}.$$

   (b)
$$\lim_{n\to\infty} \frac{n}{\log n}[n^{1/n} - 1].$$

   (c)
$$\lim_{x\to 0} \frac{\tan x - x}{x(1-\cos x)}.$$

   (d)
$$\lim_{x\to 0} \frac{x - \sin x}{\tan x - x}.$$

Hint: You can use the power series of trigonometric functions, identity $f(x) = e^{\log f(x)}$, and L'Hôpital's rule. When you use L'Hôpital's rule, check the conditions necessary to apply it.

5. Prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1 \quad \text{for all } 0 < x < \frac{\pi}{2}.$$

6. Prove that

$$|\sin nx| \leq n|\sin x| \quad \text{for all } n = 0, 1, 2, \cdots, \text{ and } x \in \mathbb{R}$$

Note that this inequality may be false for other values of n. For instance,

$$\left|\sin \frac{1}{2}\pi\right| > \frac{1}{2}|\sin \pi|.$$

7. (a) Put $s_N = 1 + (\frac{1}{2}) + \cdots + (\frac{1}{N})$. Prove that

$$\lim_{N\to\infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ, is called Euler's constant.)

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?

8. Suppose that f is Riemann integrable on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to +\infty$. Prove that

$$\lim_{t\downarrow 0} t \int_0^\infty e^{-tx} f(x)\, dx = 1.$$