

# 해석개론 및 연습 2 과제 #1

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1. Suppose that  $\{f_n\}_{n=1}^{\infty}$  converges uniformly to  $f$  on  $X$ , and that  $|f_n| \leq M_n$  for all  $n \in \mathbb{N}$ .

By uniform convergence, we can choose  $N \in \mathbb{N}$  such that for  $n \geq N$ ,

$$|f_n(x) - f(x)| < 1, \quad \forall x \in X.$$

Thus, for  $x \in X$  and  $n \geq N$ , we can write

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x) - f_N(x)| + |f_N(x)| < 2 + M_N.$$

Now set  $M = \max\{M_1, M_2, \dots, M_{N-1}, 2 + M_N\}$ . Then for all  $n \in \mathbb{N}$ ,

$$|f_n(x)| \leq M,$$

which shows that  $\{f_n\}$  is uniformly bounded.

2. Suppose that  $f_n \rightarrow f$ ,  $g_n \rightarrow g$  uniformly on  $E$ , and let  $\epsilon > 0$  be given. By uniform convergence of  $f_n$ ,  $g_n$ , we can choose  $N_1, N_2 \in \mathbb{N}$  such that

$$n \geq N_1 \implies |f_n(x) - f(x)| < \frac{\epsilon}{2} \text{ and } n \geq N_2 \implies |g_n(x) - g(x)| < \frac{\epsilon}{2}$$

for all  $x \in E$ . Set  $N = \max\{N_1, N_2\}$ , we find that for  $n \geq N$ ,

$$|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all  $x \in E$ . Thus  $f_n + g_n$  converges uniformly to  $f + g$  on  $E$ .

If  $f_n, g_n$  are bounded, we know that they are both uniformly bounded by the first problem. Additionally, we know that  $f$  is bounded by Theorem 7.15. Thus there exists  $F, G \in \mathbb{R} \setminus \{0\}$  such that  $|f_n(x)| \leq F$ ,  $|f(x)| \leq F$  and  $|g_n(x)| \leq G$ , for all  $x \in E$ .

Let  $\epsilon > 0$  be given. Using the uniform convergence of  $f_n$  and  $g_n$ , we can choose  $M_1, M_2 \in \mathbb{N}$  such that

$$n \geq M_1 \implies |f_n(x) - f(x)| < \frac{\epsilon}{2G} \text{ and } n \geq M_2 \implies |g_n(x) - g(x)| < \frac{\epsilon}{2F}$$

for all  $x \in E$ . Set  $M = \max\{M_1, M_2\}$ , we find that for  $n \geq M$ ,

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)| \\ &\leq |g_n(x)| |f_n(x) - f(x)| + |f(x)| |g_n(x) - g(x)| \\ &\leq G \cdot \frac{\epsilon}{2G} + F \cdot \frac{\epsilon}{2F} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

for all  $x \in E$ . Thus  $f_n g_n$  converges uniformly to  $f g$  on  $E$ .

- 3.** It is easy to see that  $f_n \rightarrow f \equiv 0$ , making  $f$  a continuous function. But the convergence is not uniform. For instance, take  $\epsilon = 1/2$ . For all  $n \in \mathbb{N}$ , there exists some  $x \in [0, 1]$  such that

$$|f_n(x) - f(x)| = |f_n(x)| \geq \frac{1}{2}.$$

$x_0 = \frac{1}{n + 1/2}$  is such  $x$ , because

$$|f_n(x_0)| = \sin^2 \frac{\pi}{x_0} = \sin^2 \left( n\pi + \frac{\pi}{2} \right) = 1 \geq \frac{1}{2}.$$

Now we calculate  $\sum f_n(x)$ . For  $x \leq 0$ ,  $x \geq 1$ ,  $\sum f_n(x) = 0$ .

For  $x \in (0, 1)$ ,

(a) If  $x = \frac{1}{N}$  for some  $N \in \mathbb{N}$ ,  $f_n(x) = 0$  for all  $n \in \mathbb{N}$ .

(b) Otherwise,  $\exists N \in \mathbb{N}$  such that  $\frac{1}{N+1} < x < \frac{1}{N}$ . Then

$$f_n(x) = \begin{cases} \sin^2 \frac{\pi}{x} & (n = N) \\ 0 & (n \neq N) \end{cases}.$$

Thus,  $\sum f_n(x) = \sin^2 \frac{\pi}{x}$  for  $x \in (0, 1)$ . Overall,

$$f(x) = \sum f_n(x) = \begin{cases} \sin^2 \frac{\pi}{x} & (x \in (0, 1)) \\ 0 & (\text{otherwise}) \end{cases}.$$

Since all the terms are non-negative, the series converges absolutely.

If  $\sum f_n(x)$  were to converge uniformly to  $f$ ,  $f$  should have been continuous. But since  $f$  is not continuous at  $x = 0$ ,  $\sum f_n(x)$  cannot converge uniformly.

- 4.** The given series can be considered as the sum of the two following series

$$A(x) = x^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \quad B = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

because both  $A(x)$  and  $B$  converge, by the alternating series test.

Let  $a > 0$  and set  $X = [-a, a]$ . It is sufficient to show uniform convergence for  $X$ , because  $a$  can be chosen arbitrarily large so that it would contain any bounded interval.

$A(x)$  converges uniformly on  $X$  by using the Weierstrass  $M$ -test with  $M_n = \frac{a^2}{n^2}$ . However, the given series does not converge absolutely by the comparison test since

$$\left| (-1)^n \frac{x^2 + n}{n^2} \right| \geq \frac{n}{n^2} = \frac{1}{n}$$

and the harmonic series diverges.

- 5.** Since  $f_n(0) = 0$ ,  $f(0) = 0$ . Now consider the case  $x \neq 0$ . For  $\epsilon > 0$ , choose  $N = \frac{1}{4\epsilon^2}$ . Then for  $n \geq N$ ,

$$|f_n(x) - 0| = \frac{1}{\left|nx + \frac{1}{x}\right|} \leq \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} = \epsilon, \quad (x \in \mathbb{R} \setminus \{0\})$$

(AM-GM inequality was used in the first inequality) Thus  $f(x) = 0$  also for  $x \neq 0$ .  $f_n$  converges uniformly to  $f(x) = 0$  for  $x \in \mathbb{R}$ .

We directly calculate  $f'_n(x)$  and get

$$f'_n(x) = \frac{1 - nx^2}{(1 + nx^2)^2}.$$

We see that

$$\lim_{n \rightarrow \infty} f'_n(x) = \begin{cases} 1 & (x = 0) \\ 0 & (x \neq 0) \end{cases},$$

whereas  $f'(x) = 0$ . The given equation  $f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$  is false only for  $x = 0$ .

- 6.** We directly see that

$$|c_n I(x - x_n)| \leq |c_n|$$

and since  $\sum |c_n| < \infty$ , the given series converges uniformly on  $[a, b]$  by the Weierstrass  $M$ -test.

Define the partial sum  $s_n(x) = \sum_{k=1}^n c_k I(x - x_k)$ . We know that  $s_n(x)$  is already continuous at  $x_0 \neq x_n$ , since each term in the sum is continuous at  $x_0$  and the sum is finite. Thus

$$\lim_{t \rightarrow x_0} s_n(t) = s_n(x_0).$$

Since  $s_n(x)$  converges uniformly to  $f(x)$ ,  $\{s_n(x_0)\}$  converges to  $f(x_0)$ .

By Theorem 7.11, (the conditions for the theorem are indeed satisfied)

$$\lim_{t \rightarrow x_0} f(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x_0} s_n(t) = \lim_{n \rightarrow \infty} s_n(x_0) = f(x_0),$$

showing that  $f$  is continuous for every  $x_0 \neq x_n$ .

- 7.** We are given that  $\{f_n\}$  is a sequence of continuous functions converging uniformly to  $f$ .

We know that  $f$  is continuous on  $E$ . Let  $\epsilon > 0$  be given.

First, we choose  $N_1 \in \mathbb{N}$  such that for  $n \geq N_1$ ,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}, \quad \forall x \in E.$$

Next, since  $f_n$  is continuous at  $x \in E$ , (the limit of  $x_n$ ) there exists  $\delta > 0$  such that

$$|y - x| < \delta \implies |f_n(y) - f_n(x)| < \frac{\epsilon}{2}, \quad \forall y \in E.$$

Lastly, since  $x_n \rightarrow x$ , we choose choose  $N_2 \in \mathbb{N}$  such that for  $n \geq N_2$ ,

$$|x_n - x| < \delta,$$

which implies that

$$|f_n(x_n) - f_n(x)| < \frac{\epsilon}{2}.$$

Therefore, for  $n \geq \max\{N_1, N_2\}$ ,

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is equivalent to  $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$ .

The converse is not true. Consider the function  $f_n$  given in **Problem #3**, with the domain restricted to  $E = [0, 1]$ . First of all,  $f_n$  does not converge uniformly to  $f \equiv 0$  on  $E$ .

Now, consider any sequence  $\{x_n\} \rightarrow x \in E$ . For very small  $\epsilon > 0$  such that  $x - \epsilon > 0$ , choose  $N \in \mathbb{N}$  such that for  $n \geq N_1$ ,  $|x_n - x| < \epsilon$ . Then, we see that for  $n \geq N_1$ ,  $0 < x - \epsilon < x_n$ .

Therefore we can choose  $M$  large enough so that  $x - \epsilon > \frac{1}{M}$ . Then  $f_M(x - \epsilon) = 0$ . Setting  $N = \max\{N_1, M\}$  will give  $f_n(x_n) = 0$  for  $n \geq N$ .

Hence,  $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$  holds, but  $f_n$  does not converge uniformly.

## 8. First we prove the following lemma.

**Lemma.** Given two sequences  $\{a_n\}, \{b_n\}$  and a partial sum  $A_n = \sum_{k=1}^n a_k$ , (define  $A_0 = 0$ ) the following holds for  $m < n$ .

$$\sum_{k=m+1}^n a_k b_k = A_n b_{n+1} - A_m b_{m+1} - \sum_{k=m+1}^n A_k (b_{k+1} - b_k) \quad (*)$$

**Proof of Lemma.** Observe that

$$\begin{aligned} \sum_{k=1}^n a_k b_k &= \sum_{k=1}^n (A_k - A_{k-1}) b_k \\ &= \sum_{k=1}^n A_k b_k - \sum_{k=1}^n A_k b_{k+1} + A_n b_{n+1} \\ &= A_n b_{n+1} - \sum_{k=1}^n A_k (b_{k+1} - b_k). \end{aligned}$$

For  $m < n$ , we use the result above to get

$$\sum_{k=m+1}^n a_k b_k = \sum_{k=1}^n a_k b_k - \sum_{k=1}^m a_k b_k = A_n b_{n+1} - A_m b_{m+1} - \sum_{k=m+1}^n A_k (b_{k+1} - b_k),$$

which was what we wanted.  $\square$

Suppose that  $f_n, g_n : E \rightarrow \mathbb{R}$ . Define a partial sum of  $f_k$  as  $F_n(x) = \sum_{k=1}^n f_k(x)$ . From the assumption,

- There exists  $M > 0$  such that  $|F_n(x)| < M$  for all  $n \in \mathbb{N}$ .
- For large enough  $m \in \mathbb{N}$ , we can make  $|g_n(x)|$  arbitrarily small.

Now we show that the partial sums of  $\sum f_n g_n$  is a Cauchy sequence. For  $m < n$ ,

$$\begin{aligned} \left| \sum_{k=m+1}^n f_n g_n \right| &\stackrel{(*)}{=} \left| F_n g_{n+1} - F_m g_{m+1} - \sum_{k=m+1}^n F_k (g_{k+1} - g_k) \right| \\ &\leq |F_n| |g_{n+1}| + |F_m| |g_{m+1}| + \sum_{k=m+1}^n |F_k| (g_k - g_{k+1}) \\ &\leq M(|g_{n+1}| + |g_{m+1}| + \sum_{k=m+1}^n (g_k - g_{k+1})) \\ &= M(|g_{n+1}| + |g_{m+1}| + g_{m+1} - g_{n+1}) \leq 4M |g_{n+1}|. \end{aligned}$$

The variable  $x$  was omitted for the sake of brevity, and the third assumption  $g_k(x) - g_{k+1}(x) \geq 0$  was used in the second line.

Finally, for any  $\epsilon > 0$ , choose  $N \in \mathbb{N}$  large enough so that  $|g_{n+1}(x)| < \epsilon/4M$ . Then

$$\left| \sum_{k=m+1}^n f_n(x) g_n(x) \right| \leq 4M |g_{n+1}(x)| < 4M \cdot \frac{\epsilon}{4M} = \epsilon,$$

which shows that  $\sum f_n(x) g_n(x)$  converges uniformly on  $E$ .