Introduction to Analysis II

Study Notes

Sungchan Yi

Fall 2021

Chapter 6

Sequence of Functions

6.1 Sequence of Continuous Functions

Definition. (Sequence of Functions) Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$. Given

$$f_n : X \to Y$$

for each $n \in \mathbb{N}$, we call $\langle f_n \rangle$ a **sequence of functions from X to Y**.

Definition. (Pointwise Convergence) The sequence $\langle f_n \rangle$ **converges pointwise** to the function $f : X \to Y$ if and only if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for each $x \in X$. In other words, given $\epsilon > 0$ and for all $x \in X$,

$$\exists N \in \mathbb{N} \quad \text{s.t.} \quad n \geq N \implies \|f_n(x) - f(x)\| < \epsilon.$$ [1]

Definition. (Sequence of Continuous Functions) $\langle f_n \rangle$ is a sequence of continuous functions if and only if f_n is continuous for all $n \in \mathbb{N}$.

Question. *Suppose $\langle f_n \rangle$ is a sequence of continuous functions that converges pointwise to f. Is f also continuous?*

Definition. (Uniform Convergence) Let $\langle f_n \rangle$ be a sequence of functions defined on $X \subseteq \mathbb{R}^n$ and let f be a function defined on X. We say that $\langle f_n \rangle$ is **uniformly convergent on** X if and only if for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N, \ x \in X \implies \|f_n(x) - f(x)\| < \epsilon$$

[1] 여기서 주의해야 할 점은 자연수 N 이 양수 $\epsilon > 0$ 뿐 아니라 정의역의 점 $x \in X$ 에도 의존한다는 점이다.

Problem 6.1.1. Following are equivalent.

(1) $\langle f_n \rangle$ is uniformly convergent on X.

(2) $\lim_{n\to\infty} \|f_n - f\|_{\sup} := \lim_{n\to\infty} \sup\{\|f_n - f\| : x \in X\} = 0$.

Proof. ($1 \implies 2$) Uniformly convergent on $X \implies \forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N, x \in X \implies \|f_n(x) - f(x)\| < \epsilon/2$. Then $0 \leq \sup\{\|f_n(x) - f(x)\| : x \in X\} < \epsilon/2 < \epsilon$, and we have the desired result. ($2 \implies 1$) If $\lim_{n\to\infty} \sup\{\|f_n - f\| : x \in X\} = 0$, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$, $\sup\{\|f_n(x) - f(x)\| : x \in X\} < \epsilon/2$. Then $\|f_n(x) - f(x)\|$ should be less than ϵ for all $x \in X$, and thus $\langle f_n \rangle$ is uniformly convergent.

Problem 6.1.2. $f_n(x) = \dfrac{1}{n}x$ is not uniformly convergent on $\mathbb{R}$.

Proof. Suppose $\langle f_n \rangle$ is converges uniformly on $\mathbb{R}$ to 0. Then for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N, x \in \mathbb{R} \implies \left|\frac{1}{n}x\right| < \epsilon$. But this can't be true, because for any ϵ, we can take x to be as large as we want. Take $x = 2\epsilon n$ for example, then $\left|\frac{1}{n}x\right| = 2\epsilon > \epsilon$. Contradiction.

Theorem 6.1.1. If a sequence $\langle f_n \rangle$ of continuous functions from X to Y converges uniformly to $f : X \to Y$, then f is a continuous function.

Proof. Given $\epsilon > 0$ and $x_0 \in X$, choose large enough $N \in \mathbb{N}$ such that

$$x \in X \implies \|f(x) - f_N(x)\| < \frac{\epsilon}{3}$$

Since f_N is continuous, there exists $\delta > 0$ such that

$$x \in X,\ \|x - x_0\| < \delta \implies \|f_N(x) - f_N(x_0)\| < \frac{\epsilon}{3}$$

If $x \in X$ and $\|x - x_0\| < \delta$, then we have

$$\begin{aligned} \|f(x) - f(x_0)\| &\leq \|f(x) - f_N(x)\| + \|f_N(x) - f_N(x_0)\| + \|f_N(x_0) - f(x_0)\| \\ &= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

So we can conclude that f is continuous at x_0. (Also note that uniform convergence implies pointwise convergence.)

Proposition. If $\langle f_n \rangle$ converges uniformly to $f : X \to Y$ and if $\lim_{x\to x_0} f_n(x)$ exists for all n, the following holds.

$$\lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = \lim_{x\to x_0} \lim_{n\to\infty} f_n(x), \qquad x_0 \in X'$$

Proof. Let $\lim_{x\to x_0} f_n(x) = a_n$ for $n \in \mathbb{N}$. We want to show that $\lim_{n\to\infty} a_n = \lim_{x\to x_0} f(x)$.
First, we show that $\{a_n\}$ converges by proving that $\{a_n\}$ is a Cauchy sequence.

Take any $\epsilon > 0$. By uniform convergence, there exists $N \in \mathbb{N}$ such that

$$n \geq N,\ x \in X \implies \|f_n(x) - f(x)\| < \epsilon$$

Furthermore, because $\lim_{x \to x_0} f_n(x) = a_n$, there exists $\delta > 0$ such that

$$\|x - x_0\| < \delta \implies \|f_n(x) - a_n\| < \epsilon$$

Take $m, n \geq N$. Then we have

$$\|a_n - a_m\| \leq \|a_n - f_n(x)\| + \|f_n(x) - f(x)\| + \|f(x) - f_m(x)\| + \|f_m(x) - a_m\| < 4\epsilon$$

, since we can take x to be as close as we want to x_0. Therefore $\{a_n\}$ is a Cauchy sequence, let its limit be a.
Now it is enough to show that $\lim_{x \to x_0} f(x) = a$. For $\epsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}\|x - x_0\| < \delta \implies \|f(x) - a\| &\leq \|f(x) - f_n(x)\| + \|f_n(x) - a_n\| + \|a_n - a\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

The second inequality holds because each of the terms are smaller than $\epsilon/3$ due to uniform convergence, $\lim_{x \to x_0} f_n(x) = a_n$, $\lim_{n \to \infty} a_n = a$, respectively.[2]

Theorem 6.1.2. The following are equivalent for sequence of functions $\langle f_n \rangle$ from X to Y.

(1) $\langle f_n \rangle$ converges uniformly to f.

(2) For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$m, n \geq N,\ x \in X \implies \|f_m(x) - f_n(x)\| < \epsilon \tag{$*$}$$

Proof. ($1 \implies 2$) (Similar to the proof of Proposition 2.3.6).
($2 \implies 1$) Fix $x \in X$. Then we directly see that $\{f_n(x)\}$ is a Cauchy sequence. Suppose its limit is $f(x)$. For $\forall \epsilon > 0$, take N such that $(*)$ holds and set $m \to \infty$. Then for each $n \geq N$, $\|f_n - f\|_{\sup} \leq \epsilon$, thus $\langle f_n \rangle$ converges uniformly to f.

Theorem 6.1.3. (Weierstrass M-test) Suppose that $\langle f_n \rangle$ is a sequence of functions from a set X to $\mathbb{R}$, and that there is a sequence of non-negative numbers $\{M_n\}$ such that $|f_n| \leq M_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges (absolutely) and uniformly on X.[3]

[2] 조건 $x \in X'$ 은 어디서 이용된 것일까?
[3] 여기서 중요한 부분은 M_n 이 $x \in X$ 에 의존하지 않는 식이어야 한다는 점이다.

Proof. Since $\sum_{n=1}^{\infty} M_n$ converges and $M_n > 0$, given any $\epsilon > 0$, $\exists\, N \in \mathbb{N}$ such that

$$n > m \geq N \implies M_m + M_{m+1} + \cdots + M_n < \epsilon$$

by Corollary 2.3.9.

Meanwhile for all $x \in X$,

$$\begin{aligned} |f_m(x) + f_{m+1}(x) + \cdots + f_n(x)| &\leq |f_m(x)| + |f_{m+1}(x)| + \cdots + |f_n(x)| \\ &\leq M_m + M_{m+1} + \cdots + M_n < \epsilon \end{aligned}$$

Therefore by Theorem 6.1.2 (2), the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on X.

Absolute uniform convergence of $\sum_{n=1}^{\infty} |f_n(x)|$ can be shown analogously.

Problem 6.1.3.

(1) $f_n(x) = nx(1-x^2)^n$

Pointwise convergence on $-\sqrt{2} < x < \sqrt{2}$, because $f_n(x) \leq \sqrt{2}n\alpha^n$ for some $\alpha \in [1 - x^2, 1)$, and $f(x) = \lim_{n\to\infty} f_n(x) = 0$. Now take the derivative.

$$f_n'(x) = n(1-x^2)^{n-1}\{1 - (2n+1)x^2\}$$

We see that f_n has a local maximum at $x_* = \dfrac{1}{\sqrt{2n+1}}$. Thus,

$$\lim_{n\to\infty} \|f_n - f\|_{\sup} \geq \lim_{n\to\infty} f_n(x_*) = \lim_{n\to\infty} \frac{1}{\left(1 + \dfrac{1}{2n}\right)^n} \cdot \frac{n}{\sqrt{2n+1}} = \infty$$

and the given sequence does not uniformly converge on $[-\sqrt{2}, \sqrt{2}]$.

(2) $f_n(x) = \sum_{k=1}^{n} \dfrac{x^k}{x^k+1}$

Pointwise convergence on $x \in [0, 1)$, by comparing with x^k. Let $f(x) = \sum_{k=1}^{\infty} \dfrac{x^k}{x^k+1}$.

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} \frac{x^k}{x^k+1} \geq \frac{x^{n+1}}{1+x^{n+1}}$$

As $x \to 1^-$, $\dfrac{x^{n+1}}{1+x^{n+1}} \to \dfrac{1^-}{2}$, thus $\lim_{n\to\infty} \|f_n - f\|_{\sup} \geq \dfrac{1}{2}$ and the given sequence does not uniformly converge on $[0, 1)$.

(3) $f_n(x) = \dfrac{1}{n} e^{-x^2/n}$

Pointwise convergence on $x \in \mathbb{R}$, converges to $f(x) = 0$. Also, $f_n(x)$ has a maximum at $x = 0$. And we have

$$f_n(x) - f(x) = \frac{1}{n} e^{-x^2/n} \leq \frac{1}{n}.$$

As $n \to \infty$, $\lim_{n\to\infty} \|f_n - f\|_{\text{sup}} \le \lim_{n\to\infty} \frac{1}{n} = 0$ and the given sequence converges uniformly on $\mathbb{R}$.

(4) $f_n(x) = \sum_{k=1}^{n} \frac{1}{x^2+k^2}$
By comparison with $\sum 1/k^2$, the sequence uniformly converges on $\mathbb{R}$ by Weierstrass M-test.

(5) $f_n(x) = \frac{nx}{1+nx^2}$
If $x = 0$, convergence is trivial. Let $x \neq 0$, then

$$\lim_{n\to\infty} \frac{nx}{1+nx^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n}+x^2} = \frac{1}{x}$$

thus pointwise convergence on $\mathbb{R}$. Take $x = 1/n$. Then

$$\|f_n(x) - f(x)\| = \left|\frac{1}{x(1+nx^2)}\right| = \frac{n^2}{n+1}.$$

As $n \to \infty$, $\|f_n(x) - f(x)\| \to \infty$. The given sequence does not uniformly converge on $\mathbb{R}$.

(6) $f_n(x) = \sum_{k=1}^{n} \frac{1}{1+k^2x}$
For $x = 0$, $f_n(x) \to \infty$. Now let $x \neq 0$, then $f_n(x) = \frac{1}{x}\sum_{k=1}^{n}\frac{1}{1/x+k^2}$

(a) If $x > 0$, $\frac{1}{1/x+k^2} < \frac{1}{k^2}$, thus uniformly converges on $x > 0$ by Weierstrass M-test.

(b) If $x < 0$ and $x \neq -\frac{1}{k^2}$ for $k \in \mathbb{N}$, take large enough $K \in \mathbb{N}$ such that for $k \ge K$, $k^2x < 1 + k^2x < 0$. Then we have

$$\sum_{k=K}^{\infty}\left|\frac{1}{1+k^2x}\right| \le \sum_{k=K}^{\infty}\frac{1}{k^2|x|} < \infty$$

and thus uniformly converges on the interval by Weierstrass M-test.