Assignment 1

Due Date: 2019/03/20, 1:30 PM

Remark. In this assignment, all the sequences are sequence of real numbers, and lim is a shorthand of $\lim_{n\to\infty}$.

1. Suppose that a subset S of $\mathbb{R}$ contains a minimum $m \in \mathbb{R}$, namely $\min(S) = m$. Prove that $\inf(S) = m$.

2. Find infimum and supremum of the following sets, and prove your answer.
(1) $(1, 2) = \{x \in \mathbb{R} : 1 < x < 2\}$

(2) $\left\{\frac{1}{1+n^2} : n \in \mathbb{N}\right\}$

(3) $\{(-1)^n + (-1/2)^m : n,\, m \in \mathbb{N}\}$

3. Define $X - Y = \{x - y : x \in X,\, y \in Y\}$. Let A and B be bounded and non-empty subsets of $\mathbb{R}$.
(1) Prove that $A - B$ is also bounded.

(2) Express $\sup(A - B)$ in terms of $\sup A$, $\sup B$, $\inf A$, and $\inf B$.

4. Suppose that a sequence $\langle a_n \rangle$ satisfies $\lim a_n = 0$, and that another sequence $\langle s_n \rangle$ satisfies $|s_n - s| < a_n$ for all $n \in \mathbb{N}$. Prove that $\lim s_n = s$.

5. Find the following limits and verify your answer rigorously.
(1) $\lim \frac{\sqrt{n}}{2\sqrt{n}+7}$

(2) $\lim \frac{2n^5+\cos(n^8+1)}{n^5+1}$

(3) $\lim \frac{3n^2+n(-1)^n}{n^2+2}$

6. Prove that the sequence $\langle a_n \rangle$ defined by $a_n = \frac{(-2)^n + n}{2^n}$ is divergent.

7. Let $\langle s_n \rangle$ be a convergent sequence and let $\lim s_n = s$. Prove that $\lim f(s_n) = f(s)$ where f is given by:

(1) $f(x) = x^2 + 4x + 5$

(2) $f(x) = \sqrt{\frac{1}{1+x^2}}$

(3) $f(x) = x^{2019}$

8. Let $\langle a_n \rangle$ be a divergent sequence. Define another sequence $\langle b_n \rangle$ as

$$b_n = \frac{1}{n} \sum_{k=1}^{n} a_k \ ; \ n \in \mathbb{N} \, .$$

Prove, or disprove by a counter example that the following statement: *the sequence* $\langle b_n \rangle$ *is divergent.*