Assignment 7

Due Date: None

1. Suppose that $f : [a, b] \to \mathbb{R}$ is a function of bounded variation.

(1) If $P, Q \in \mathcal{P}[a, b]$ satisfy $P \subset Q$, then prove that $V(f; P) \le V(f; Q)$.

(2) For any $\epsilon > 0$, prove that there exists P_0 such that for any $P \supset P_0$ satisfies

$$|V(f, P) - V(f)| < \epsilon .$$

2. Suppose that $f : [a, b] \to \mathbb{R}$ is a function of bounded variation, and define $F : [a, b] \to \mathbb{R}$ as

$$F(x) = V_a^x(f)$$

(1) Prove or disprove: F is a continuous function.

(2) Prove or disprove: If f is continuous, then F is continuous as well.

3. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function and let f' be a Riemann integrable function. Prove that

$$V(f) = \int_a^b |f'(x)| dx .$$

4. For $a, b > 0$, define a function $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^a \sin \frac{1}{x^b} & \text{if } x \in (0, 1] \\ 0 & \text{if } x = 0 \end{cases} .$$

Find all (a, b) such that f is a function of bounded variation.

5. Let $\alpha : [a, b] \to \mathbb{R}$ be a monotonically increasing function, and let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that two statements below are equivalent:
[1] $f \in \mathcal{R}(\alpha)$ and $\int_a^b f d\alpha = A$

[2] For all $\epsilon > 0$, there exists $P_0 \in \mathcal{P}[a, b]$ such that $|S(f, P, \alpha) - A| < \epsilon$ for all $P \supset P_0$.

6. Define $\alpha : [0, 2] \to \mathbb{R}$ as

$$\alpha(x) = \begin{cases} x^2 & \text{if } x \in [0, 1) \\ 3 - x^2 & \text{if } x \in [1, 2] \,. \end{cases}$$

(1) Prove that α is a function of bounded variation.

(2) For $f(x) = x^3$, compute $\int_0^2 f d\alpha$.
Hint. There are (at least) two nice ways; first, decompose $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ so that α_i's are simpleenough. Then, you can compute $\sum_{i=1}^{3} \int_0^2 f d\alpha_i$. Second, use $\int_0^2 \alpha df$.