HW5 Solution

1. Show that $\Sigma \subset \mathcal{P}(S)$ is an algebra on S if and only if $\Sigma \subset \mathcal{P}(S)$ is a ring on S with $S \in \Sigma$. Show that $\Sigma \subset \mathcal{P}(S)$ is a σ-algebra on S if and only if $\Sigma \subset \mathcal{P}(S)$ is a σ-ring on S with $S \in \Sigma$.

   Solution. [$\Rightarrow$] First suppose that Σ is an algebra on S. By definition we have $S \in \Sigma$. For any $A, B \in \Sigma$, $A \cup B \in \Sigma$ follows directly from the definition of algebra. Finally, $A \setminus B = A \cap B^C = (A \cup B^C)^C \in \Sigma$ for any $A, B \in \Sigma$ since Σ is closed under complements and unions. This shows that Σ is a ring on S containing S. If, in addition, we assume that Σ is a σ-algebra, then we have the condition that $A_n (n \in \mathbb{N})$ implies $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, so Σ is a σ-ring.

   [$\Leftarrow$] Suppose that Σ is a ring on S and $S \in \Sigma$. Then for any $A, B \in \Sigma$, $A^C = S \setminus A \in \Sigma$ and $A \cup B \in \Sigma$ by properties of ring. This shows that Σ is an algebra on S. If, in addition, we assume that Σ is a σ-ring, then we have the condition that $A_n (n \in \mathbb{N})$ implies $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, showing that Σ is a σ-algebra. $\square$

2. E, S are sets and f is a map from E to S. Suppose Σ is a σ-algebra on S ($\Sigma \subset \mathcal{P}(S)$). Show that $f^{-1}(\Sigma) := \{f^{-1}(A) : A \in \Sigma\}$ is a σ-algebra on E.

   Solution. (1) $E = f^{-1}(S) \in f^{-1}(\Sigma)$ since $S \in \Sigma$.
   (2) Any element of $f^{-1}(\Sigma)$ is of the form $f^{-1}(A)$ with $A \in \Sigma$, and $E \setminus f^{-1}(A) = f^{-1}(S \setminus A) \in f^{-1}(\Sigma)$ since $S \setminus A \in \Sigma$. The set identity is true since $x \in E \setminus f^{-1}(A) \iff f(x) \notin A \iff f(x) \in S \setminus A$.
   (3) For any countable collection $\{B_n\}_n \subset f^{-1}(\Sigma)$, there are corresponding collection $\{A_n\}_n \subset \Sigma$ such that $B_n = f^{-1}(A_n)$ for each $n \in \mathbb{N}$. Then $\bigcup_n B_n = \bigcup_n f^{-1}(A_n) = f^{-1}\left(\bigcup_n A_n\right) \in f^{-1}(\Sigma)$ since $\bigcup_n A_n \in \Sigma$. The set identity is true since $x \in f^{-1}(\bigcup_n A_n) \iff f(x) \in \bigcup_n A_n \iff \exists n, f(x) \in A_n \iff \exists n, x \in f^{-1}(A_n) \iff x \in \bigcup_n f^{-1}(A_n)$. $\square$

3. Show that an arbitrary intersection of σ-algebras on S is a σ-algebra on S and show that union of two σ-algebras may not be a σ-algebra by a counterexample.

Solution. Let $\{\Sigma_\alpha\}_{\alpha \in I}$ be a collection of σ-algebras on S, indexed by the set I (note that this may be uncountably infinite).
(1) For all $\alpha \in I$, $S \in \Sigma_\alpha$. Hence $S \in \bigcap_{\alpha \in I} \Sigma_\alpha$.
(2) Let $A \in \bigcap_\alpha \Sigma_\alpha$. Then for all $\alpha \in I$, $A \in \Sigma_\alpha \implies S \setminus A \in \Sigma_\alpha$ since each Σ_α is a σ-algebra. Hence $S \setminus A \in \bigcap_\alpha \Sigma_\alpha$.
(3) Let $\{A_n\}_{n \in \mathbb{N}} \subset \bigcap_\alpha \Sigma_\alpha$. Then for all $\alpha \in I$, for all $n \in \mathbb{N}$, $A_n \in \Sigma_\alpha \implies \bigcup_{n=1}^\infty A_n \in \Sigma_\alpha$ since each Σ_α is a σ-algebra. Hence $\bigcup_n A_n \in \bigcap_\alpha \Sigma_\alpha$.
By (1), (2), (3), $\bigcap_\alpha \Sigma_\alpha$ is a σ-algebra.

Union of two σ-algebras may not be a σ-algebra (not even an algebra). Consider $S = \{1, 2, 3\}$, $\Sigma_1 = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\Sigma_2 = \{\emptyset, \{2\}, \{1, 3\}, \{1, 2, 3\}\}$. It is easy to check that Σ_1 and Σ_2 are both σ-algebras on S. However, $\Sigma_1 \cup \Sigma_2$ is not an algebra since $\{1\}, \{2\} \in \Sigma_1 \cup \Sigma_2$ but $\{1, 2\} \notin \Sigma_1 \cup \Sigma_2$. $\square$

4. For a finite set A, S is a set, $x \in S$ and $\Sigma = \mathcal{P}(S)$. Show that following are measures.

$$\mu_1(A) := \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A. \end{cases}$$

$$\mu_2(A) := \begin{cases} \text{Cardinality (number of elements) of } A & \text{if } A \text{ is finite} \\ \infty & \text{if } A \text{ is infinite .} \end{cases}$$

Solution. Clearly $\mathcal{P}(S)$ is a σ-algebra and μ_1, μ_2 are nonnegative set functions on $\mathcal{P}(S)$. We only need to show countable additivity over disjoint sets.
(1) Let $\{A_n\}_{n \in \mathbb{N}}$ be a collection of mutually disjoint subsets of S, and put $A = \bigcup_n A_n$. If $x \in A$, then $x \in A_m$ for some m, and $x \notin A_n$ for any $n \neq m$ since $A_m \cap A_n = \emptyset$. Thus $\mu_1(A_m) = 1$ while $\mu_1(A_n) = 0$ for all $n \neq m$. Hence

$$\mu_1(\bigcup_n A_n) = 1 = \mu_1(A_m) = \sum_{n=1}^\infty \mu_1(A_n).$$

Otherwise, $x \notin A_n$ for all $n \in \mathbb{N}$, so $\mu_1(A) = 0 = \sum_n \mu_1(A_n)$.

(2) Let $\{A_n\}_{n \in \mathbb{N}}$ be a collection of mutually disjoint subsets of S. We

divide into three cases.
(i) When all but finitely many A_n's are empty, and all A_n's are finite: Say $A_{n_1}, \dots, A_{n_r}$ are the nonempty ones among A_n's. Note that $\mu_2(A_n) = 0$ for all A_n that are not one of A_{n_i}'s. Then $A = \bigcup_{i=1}^r A_{n_i}$ and thus $\mu_2(A) = |A_{n_1} \cup \cdots \cup A_{n_r}| = |A_{n_1}| + \cdots + |A_{n_r}| = \sum_{i=1}^r \mu_2(A_{n_i}) = \sum_{n=1}^\infty \mu_2(A_n)$.
(ii) When all but finitely many A_n's are empty, but one of A_n's is infinite: Say A_m is the infinite one. Then $A_m \subset A$, so A is infinite and thus $\mu_2(A) = \infty$. This coincides with $\sum_n \mu_2(A_n)$ since $\sum_n \mu_2(A_n) \geq \mu_2(A_m) = \infty$.
(iii) When there are infinitely many nonempty A_n's: Let $A_{n_1}, A_{n_2}, \dots$ be any subsequence of nonempty elements of A_n. Choose $x_i \in A_{n_i}$, so that x_i are all distinct (by disjointness of A_n) and $\{x_1, x_2, \dots\} \subset A$. Hence A is infinite and $\mu_2(A) = \infty$. On the other hand, since $\mu_2(A_{n_i}) \geq 1$ for each i, we have $\sum_n \mu_2(A_n) \geq \sum_i \mu_2(A_{n_i}) = \infty$. □

5. Let μ be a measure on a σ-algebra $\mathcal{F}$. Suppose $\{A_i\}_{i=1}^\infty \subset \mathcal{F}$ and $A_{i+1} \subset A_i$. Show that $\mu(A_1) < \infty$, then

$$\lim_{i\to\infty} \mu(A_i) = \mu(\cap_{i=1}^\infty A_i).$$

Solution. For each $i \geq 1$, $B_i = A_1 \setminus A_i$. If $i < j$, then $B_i \subset B_j$ since $A_j \subset A_i$. That is, $\{B_i\}$ is an ascending sequence of sets and thus

$$\mu(\bigcup_{i=1}^\infty B_i) = \lim_{i\to\infty} \mu(B_i) = \lim_{i\to\infty} \mu(A_1 \setminus A_i) = \mu(A_1) - \lim_{i\to\infty} \mu(A_i). \quad (1)$$

Note that $\lim_{i\to\infty} \mu(A_i)$ exists since $\{\mu(A_i)\}$ is a decreasing sequence bounded below by 0. Now, observe that

$$\begin{aligned} x \in \bigcup_i B_i &\iff x \in A_1 \text{ and } \exists i, x \in B_i \\ &\iff x \in A_1 \text{ and } \exists i, x \notin A_i \\ &\iff x \in A_1 \setminus \bigcap_{i=1}^\infty A_i, \end{aligned}$$

so that $\mu(\bigcup_i B_i) = \mu(A_1 \setminus \bigcap_i A_i) = \mu(A_1) - \mu(\bigcap_i A_i)$. Comparing this with (1) we obtain the desired result. □

6. Let μ be a measure on a σ-algebra $\mathcal{F}$. Show that if $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F}$, then

$$\mu(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

Solution. Let $B_1 = A_1$ and $B_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$ for each $n > 1$. Then B_n's are mutually disjoint since $m < n$ implies $B_m \subset A_m \subset \bigcup_{k=1}^{n-1} A_k$ and $B_n \cap \bigcup_{k=1}^{n-1} A_k = \emptyset$. Next, it is clear that $\bigcup_n B_n \subset \bigcup_n A_n$, and the reverse inclusion holds since for any $x \in \bigcup_n A_n$ one can choose the smallest k for which $x \in A_k$ and then $x \in B_k \subset \bigcup_n B_n$. Thus we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

by countable additivity and monotonicity of measures. $\square$