HW 3 Solution

1. In the proof of Theorem 8.1, describe the following details:

   (a) Does every power series converge absolutely in the interior of its interval of convergence? explain it by using root test.

   (b) When we apply the theorem 7.17, what we choose for $(f_n)_{n=1}^{\infty}$ and x_0?

Solution. **(a)** Let R be the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$. If $|a| < R$, then there exists a $|a| < |b| < R$ s.t. $\sum_{n=0}^{\infty} c_n b^n$ converges. Since $\lim_{n\to\infty} c_n b^n = 0$, there exists a $N \in \mathbb{N}$ s.t. $n \geq N$ implies $|c_n b^n| < 1$, thus

$$\sum_{n=0}^{\infty} |c_n a^n| = \sum_{n=0}^{N} |c_n a^n| + \sum_{n=N+1}^{\infty} |c_n b^n| \left(\frac{|a|}{|b|}\right)^n \leq \sum_{n=0}^{N} |c_n a^n| + \sum_{n=N+1}^{\infty} \left(\frac{|a|}{|b|}\right)^n < \infty,$$

and $\sum_{n=0}^{\infty} |c_n a^n|$ converges.

(b) Let $f_n(x) := \sum_{k=0}^{n} c_k x^k$. then $f_n'(x) = \sum_{k=1}^{n} k c_k x^{k-1}$ uniformly converges on $[R-\varepsilon,\, R+\varepsilon]$ and $f_n(x_0)$ converges for any $x_0 \in [R-\varepsilon,\, R+\varepsilon]$, so we can apply theorem 7.17. □

2. Prove that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

if $a_{ij} \geq 0$ for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$ (the case $+\infty = +\infty$ may occur).

Solution. First note that every addition is done in $[0, \infty]$ so there is no confusion. Let $b_i := \sum_{j=1}^{\infty}|a_{ij}| = \sum_{j=1}^{\infty}a_{ij}$. If $\sum_{i=1}^{\infty}b_i = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$ converges, then $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$ by theorem 8.3. Similarly if $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$ converges then $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}$ by theorem 8.3. Remaining case is $\infty = \infty$ $\square$

3. Show that $\log x$ is real-analytic on $(0,\infty)$, that is, for every $a \in (0, \infty)$ $\log x$ can be expressed $\log x = \sum_{n=0}^{\infty} a_n(x-a)^n$ in some interval $(a-\varepsilon,\, a+\varepsilon) \subset (0,\, \infty)$.

Solution.

$$\frac{1}{x} = \frac{1}{a+(x-a)} = \frac{1}{a}\cdot\frac{1}{1+\frac{x-a}{a}} = \frac{1}{a}\left(\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-a}{a}\right)^n\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}(x-a)^n$$

For $0 < x < 2a$. by Integrating both sides, we get

$$\log x = \log a + \int_a^x \frac{1}{t}dt = \log a + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{na^n}(x-a)^n$$

on $(0,\, 2a)$. Since $a \in (0,\, \infty)$ was arbitraty, $\log x$ is analytic.

4. Find the following limits:

(a)

$$\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x}.$$

(b)

$$\lim_{n\to\infty}\frac{n}{\log n}[n^{1/n}-1].$$

(c)

$$\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)}.$$

(d)

$$\lim_{x\to 0} \frac{x-\sin x}{\tan x - x}.$$

Hint: You can use the power series of trigonometric functions, identity $f(x) = e^{\log f(x)}$, and L'Hôpital's rule. When you use L'Hôpital's rule, check the conditions necessary to apply it.

Solution. **(a)** Note that $(1+x)^{\frac{1}{x}} = e^{\frac{\log(1+x)}{x}}$ is differentiable on $(-1,\,0)\cup(0,\,\infty)$. Since $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} e^{\frac{\log(1+x)}{x}} = e^{\frac{1}{x}|_{x=1}} = e$, given limit is equal to

$$\lim_{x\to 0}\frac{(e-(1+x)^{\frac{1}{x}})'}{(x)'} = \lim_{x\to 0}\left(e - e^{\frac{\log(1+x)}{x}}\right)' = \lim_{x\to 0}(1+x)^{\frac{1}{x}}\left(-\frac{1}{x(x+1)} + \frac{1}{x^2}\log(1+x)\right)$$

if it exists. By the way,

$$-\frac{1}{x(x+1)} + \frac{1}{x^2}\log(1+x) = \frac{1}{x+1}\cdot\frac{-x+(1+x)\log(1+x)}{x^2}$$

and $-x+(1+x)\log(1+x)$ has power series expension $-x+(1+x)(x-\frac{x^2}{2}+\cdots) = \frac{x^2}{2}+\cdots$ near $x=0$. Thus (a) is equal to $\frac{e}{2}$.

(b)

$$\frac{n}{\log n}[n^{\frac{1}{n}}-1] = \frac{e^{\frac{\log n}{n}}-1}{\frac{\log n}{n}}$$

and $\lim\limits_{n\to\infty}\frac{\log n}{n} = 0$, $\lim\limits_{x\to 0}\frac{e^x-1}{x} = \lim\limits_{x\to 0}\frac{e^x-e^0}{x} = (e^x)'|_{x=0} = 1$. Thus limit is 1.

(c) We know $\sin x = x - \frac{x^3}{3!}+\cdots$, $\cos x = 1-\frac{x^2}{2!}+\cdots$. Since $\cos 0 = 1 \neq 0$, power series expension of $\tan x$ at $x=0$ is given by

$$\frac{\sin x}{1+(\cos x - 1)} = \sin x\cdot\left\{1-(\cos x-1)+(\cos x-1)^2-\cdots\right\}$$

$$= \left(x-\frac{x^3}{3!}+\cdots\right)\left\{1-\left(-\frac{x^2}{2!}+\cdots\right)+\left(-\frac{x^2}{2!}+\cdots\right)^2+\cdots\right\} = x+\frac{1}{3}x^3+\cdots.$$

We also know that every converging power series is continuous, so

$$\lim_{x\to 0}\frac{\tan x - x}{x(1-\cos x)} = \lim_{x\to 0}\frac{\frac{1}{3}x^3+\cdots}{x(\frac{1}{2}x^2+\cdots)} = \lim_{x\to 0}\frac{\frac{1}{3}+\cdots}{\frac{1}{2}+\cdots} = \frac{2}{3}.$$

(d) Similarly,

$$\lim_{x\to 0}\frac{x-\sin x}{\tan x - x} = \lim_{x\to 0}\frac{\frac{x^3}{3!}+\cdots}{\frac{x^3}{3}+\cdots} = \lim_{x\to 0}\frac{\frac{1}{3!}+\cdots}{\frac{1}{3}+\cdots} = \frac{1}{2}.$$

5. Prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1 \quad \text{for all } 0 < x < \frac{\pi}{2}.$$

Solution. $\frac{\sin x}{x} = \frac{\sin x - \sin 0}{x-0} = \cos x^* < 1$ by mean value theorem, so $\sin x < x$ for $0 < x < \frac{\pi}{2}$. Also note that $(\sin x)'' = -\sin x \le 0$ and the equation of the line passing $(0, \sin 0)$, $(\frac{\pi}{2}, \sin\frac{\pi}{2})$ is $y = \frac{2}{\pi}x$. So it is enough to prove the following theorem:
Theorem. Let f be a twice differentiable function on $[a, b]$ s.t. $f''(x) < 0$ on (a, b). then $f(x) > \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$ on (a, b)
Proof of theorem: Let $g(x) := f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$, then $g(a) = g(b) = 0$, $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. By mean value theorem, there exists a $c \in (a, b)$ s.t. $\frac{f(b)-f(a)}{b-a} = f'(c)$. Since $g''(x) = f''(x) < 0$, $g'(x)$ strictly decreases on (a, b), Thus $g'(x) > 0$ on (a, c), $g'(x) < 0$ on (c, b). This means that g strictly increases on $[a, c]$ and strictly decreases on $[c, b]$, and this proves $g(x) > 0$ on (a, b).

6. Prove that

$$|\sin nx| \le n|\sin x| \quad \text{for all } n = 0, 1, 2, \cdots, \text{ and } x \in \mathbb{R}$$

Note that this inequality may be false for other values of n. For instance,

$$\left|\sin\frac{1}{2}\pi\right| > \frac{1}{2}|\sin\pi|.$$

Solutioin. Use induction on n, with $n = 0$, 1 trivial. If given inequality holds when $n = k$, then $|\sin(k+1)x| = |\sin kx \cos x + \cos kx \sin x|$ $\leq |\sin kx||\cos x| + |\cos kx||\sin x| \leq |\sin kx| + |\sin x|$ $\leq k|\sin x| + |\sin x| = (k+1)|\sin x|$ and the inequality also holds when $n = k+1$.

Alternative Solution 1. Note that the period of $|\sin x|$ is π, and $|\sin(\pi - x)| = |\sin x|$, $|\sin n(\pi - x)| = |\sin nx|$. So it suffices to assume that $0 \leq x \leq \frac{\pi}{2}$. On $[0, \frac{\pi}{2n}]$, $\sin x$, $\sin nx \geq 0$. Let $g_n(x) := n\sin x - \sin nx$. $g'(x) = n\cos x - n\cos nx = n(\cos x - \cos nx)$. Since $0 \leq x \leq nx \leq \frac{\pi}{2}$, $g'(x) \geq 0$ and this proves $g(x) \geq 0$, or equivalently $n|\sin x| \geq |\sin nx|$ on $[0, \frac{\pi}{2n}]$. If $x \in [\frac{\pi}{2n}, \frac{\pi}{2}]$, from $\sin > \frac{2}{\pi}x$, $n\sin x \geq \frac{2n}{\pi}x \geq 1 \geq |\sin nx|$.

Alternative Solution 2. Assume $x \neq 2n\pi$. Recall $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$. Let $a := e^{ix}$, $b := e^{-ix}$.

$$\frac{|\sin nx|}{|\sin x|} \leq \frac{|e^{inx} - e^{-inx}|}{|e^{ix} - e^{-ix}|} \leq \left|\frac{a^n - b^n}{b - a}\right| = |a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}|$$

$$\leq |a|^{n-1} + |a|^{n-2}|b| + \cdots + |a||b|^{n-2} + |b|^{n-1} \leq n.$$

□

7. (a) Put $s_N = 1 + (\frac{1}{2}) + \cdots + (\frac{1}{N})$. Prove that

$$\lim_{N\to\infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ, is called Euler's constant.)

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?

Solution. **(a)** From

$$\sum_{k=1}^{N} \int_k^{k+1} \frac{1}{x} dx \leq s_N = \sum_{k=1}^{N} \frac{1}{k} \leq 1 + \sum_{k=2}^{N} \int_{k-1}^{k} \frac{1}{x} dx,$$

wa have

$$\log(N+1) \le s_N \le 1 + \log N, \ \ \log\left(1 + \frac{1}{N}\right) \le s_N - \log N \le 1$$

so $s_N - \log N$ is bounded. On the other hand,

$$\{s_{N+1} - \log(N+1)\} - \{s_N - \log N\} = \frac{1}{N+1} - \int_N^{N+1} \frac{1}{x} dx \le 0.$$

Therefore, $(s_N - \log N)_{N=1}^{\infty}$ is decreasing sequence and has a lower bound, hence converges.

(b) From $\log(N+1) \le s_N \le 1+\log N$, We know that $\log(N+1) > 100$ is enough to have $s_N > 100$, and $1 + \log N > 100$ is necessary to have $s_N > 100$. Thus the threshold value for s_N is between e^{99} and e^{100}. In the form of $N = 10^m$, range of the threshold value for m is

$$\frac{99}{\log 10} < m_0 < \frac{100}{\log 10}.$$

From the definition of e, $2 = 1 + \frac{1}{1!} < e = \sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \frac{1}{1!} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 3$ and we get $2 < \log x < 4$, because $3^2 < 10 < 2^4$. So $24.75 < m_0 < 50.\square$

8. Suppose that f is Riemann integrable on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to +\infty$. Prove that

$$\lim_{t \downarrow 0} t \int_0^{\infty} e^{-tx} f(x) \ dx = 1.$$

Solution. First we show that given improper integral exists if $t > 0$. For any $\varepsilon > 0$, there exists a $B \in \mathbb{R}^+$ s.t. $x > B$ implies $|f(x) - 1| < \varepsilon$. If $A \ge B$, then

$$\int_0^A |e^{-tx} f(x)| dx = \int_0^B |e^{-tx} f(x)| dx + \int_B^A |e^{-tx} f(x)| dx$$

is increasing function of A and bounded above because

$$\int_B^A |e^{-tx} f(x)| dx \leq (1+\varepsilon) \int_B^A e^{-tx} dx = (1+\varepsilon) \frac{e^{-Bt} - e^{-At}}{t} < (1+\varepsilon) \frac{e^{-Bt}}{t}.$$

Therefore $\int_0^A |e^{-tx} f(x)| dx$ converges. Moreover, since
$0 \leq \frac{|x|+x}{2},\ \frac{|x|-x}{2} \leq |x|$,

$$\int_0^A \frac{|e^{-tx} f(x)| + e^{-tx} f(x)}{2} dx,\ \int_0^A \frac{|e^{-tx} f(x)| - e^{-tx} f(x)}{2} dx$$

are both increasing function of A and bounded above, so they converges.
Hence

$$\int_0^A e^{-tx} f(x) dx = \int_0^A \frac{|e^{-tx} f(x)| + e^{-tx} f(x)}{2} dx - \int_0^A \frac{|e^{-tx} f(x)| - e^{-tx} f(x)}{2} dx$$

also converges.
Now for any $\varepsilon > 0$, Let B like before, then

$$\left| t \int_0^\infty e^{-tx} f(x) dx - 1 \right| = \left| t \int_0^\infty e^{-tx} \left\{ f(x) - 1 \right\} dx \right|$$

$$\leq t \int_0^B e^{-tx} |f(x) - 1| dx + t \int_B^\infty e^{-tx} |f(x) - 1| dx.$$

($t \int_0^\infty e^{-tx} dx = 1$ was used.) Since f is integrable $[0,\ B]$, so f is bounded on $[0,\ B]$ by, say M. Thus we have

$$\left| t \int_0^\infty e^{-tx} f(x) dx - 1 \right| \leq t(M+1)B + \varepsilon e^{-Bt},$$

$$\limsup_{t \to 0+} \left| t \int_0^\infty e^{-tx} f(x) dx - 1 \right| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this means

$$\lim_{t \to 0+} t \int_0^\infty e^{-tx} f(x) dx = 1.$$