

HW 3 Solution

1. In the proof of Theorem 8.1, describe the following details:
 - (a) Does every power series converge absolutely in the interior of its interval of convergence? explain it by using root test.
 - (b) When we apply the theorem 7.17, what we choose for $(f_n)_{n=1}^{\infty}$ and x_0 ?

Solution. **(a)** Let R be the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$. If $|a| < R$, then there exists a $|a| < |b| < R$ s.t. $\sum_{n=0}^{\infty} c_n b^n$ converges. Since $\lim_{n \rightarrow \infty} c_n b^n = 0$, there exists a $N \in \mathbb{N}$ s.t. $n \geq N$ implies $|c_n b^n| < 1$, thus

$$\sum_{n=0}^{\infty} |c_n a^n| = \sum_{n=0}^N |c_n a^n| + \sum_{n=N+1}^{\infty} |c_n b^n| \left(\frac{|a|}{|b|} \right)^n \leq \sum_{n=0}^N |c_n a^n| + \sum_{n=N+1}^{\infty} \left(\frac{|a|}{|b|} \right)^n < \infty,$$

and $\sum_{n=0}^{\infty} |c_n a^n|$ converges.

(b) Let $f_n(x) := \sum_{k=0}^n c_k x^k$. then $f'_n(x) = \sum_{k=1}^n k c_k x^{k-1}$ uniformly converges on $[R - \varepsilon, R + \varepsilon]$ and $f_n(x_0)$ converges for any $x_0 \in [R - \varepsilon, R + \varepsilon]$, so we can apply theorem 7.17. \square

2. Prove that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

if $a_{ij} \geq 0$ for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$ (the case $+\infty = +\infty$ may occur).

Solution. First note that every addition is done in $[0, \infty]$ so there

is no confusion. Let $b_i := \sum_{j=1}^{\infty} |a_{ij}| = \sum_{j=1}^{\infty} a_{ij}$. If $\sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ converges, then $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ by theorem 8.3. Similarly if $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ converges then $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$ by theorem 8.3. Remaining case is $\infty = \infty$ \square

3. Show that $\log x$ is real-analytic on $(0, \infty)$, that is, for every $a \in (0, \infty)$ $\log x$ can be expressed $\log x = \sum_{n=0}^{\infty} a_n (x-a)^n$ in some interval $(a-\varepsilon, a+\varepsilon) \subset (0, \infty)$.

Solution.

$$\frac{1}{x} = \frac{1}{a + (x-a)} = \frac{1}{a} \cdot \frac{1}{1 + \frac{x-a}{a}} = \frac{1}{a} \left(\sum_{n=0}^{\infty} (-1)^n \left(\frac{x-a}{a} \right)^n \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} (x-a)^n$$

For $0 < x < 2a$. by Integrating both sides, we get

$$\log x = \log a + \int_a^x \frac{1}{t} dt = \log a + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{na^n} (x-a)^n$$

on $(0, 2a)$. Since $a \in (0, \infty)$ was arbitrary, $\log x$ is analytic.

4. Find the following limits:

(a)

$$\lim_{x \rightarrow 0} \frac{e - (1+x)^{1/x}}{x}.$$

(b)

$$\lim_{n \rightarrow \infty} \frac{n}{\log n} [n^{1/n} - 1].$$

(c)

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)}.$$

(d)

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x}.$$

Hint: You can use the power series of trigonometric functions, identity $f(x) = e^{\log f(x)}$, and L'Hôpital's rule. When you use L'Hôpital's rule, check the conditions necessary to apply it.

Solution. (a) Note that $(1+x)^{\frac{1}{x}} = e^{\frac{\log(1+x)}{x}}$ is differentiable on $(-1, 0) \cup (0, \infty)$. Since $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} e^{\frac{\log(1+x)}{x}} = e^{\frac{1}{x}|_{x=1}} = e$, given limit is equal to

$$\lim_{x \rightarrow 0} \frac{(e - (1+x)^{\frac{1}{x}})'}{(x)'} = \lim_{x \rightarrow 0} \left(e - e^{\frac{\log(1+x)}{x}} \right)' = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \left(-\frac{1}{x(x+1)} + \frac{1}{x^2} \log(1+x) \right)$$

if it exists. By the way,

$$-\frac{1}{x(x+1)} + \frac{1}{x^2} \log(1+x) = \frac{1}{x+1} \cdot \frac{-x + (1+x) \log(1+x)}{x^2}$$

and $-x + (1+x) \log(1+x)$ has power series expansion $-x + (1+x)(x - \frac{x^2}{2} + \dots) = \frac{x^2}{2} + \dots$ near $x = 0$. Thus (a) is equal to $\frac{e}{2}$.

(b)

$$\frac{n}{\log n} [n^{\frac{1}{n}} - 1] = \frac{e^{\frac{\log n}{n}} - 1}{\frac{\log n}{n}}$$

and $\lim_{n \rightarrow \infty} \frac{\log n}{n} = 0$, $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \lim_{x \rightarrow 0} \frac{e^x - e^0}{x} = (e^x)'|_{x=0} = 1$. Thus limit is 1.

(c) We know $\sin x = x - \frac{x^3}{3!} + \dots$, $\cos x = 1 - \frac{x^2}{2!} + \dots$. Since $\cos 0 = 1 \neq 0$, power series expansion of $\tan x$ at $x = 0$ is given by

$$\begin{aligned} \frac{\sin x}{1 + (\cos x - 1)} &= \sin x \cdot \{1 - (\cos x - 1) + (\cos x - 1)^2 - \dots\} \\ &= \left(x - \frac{x^3}{3!} + \dots \right) \left\{ 1 - \left(-\frac{x^2}{2!} + \dots \right) + \left(-\frac{x^2}{2!} + \dots \right)^2 + \dots \right\} = x + \frac{1}{3}x^3 + \dots \end{aligned}$$

We also know that every converging power series is continuous, so

$$\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)} = \lim_{x \rightarrow 0} \frac{\frac{1}{3}x^3 + \dots}{x(\frac{1}{2}x^2 + \dots)} = \lim_{x \rightarrow 0} \frac{\frac{1}{3} + \dots}{\frac{1}{2} + \dots} = \frac{2}{3}.$$

(d) Similarly,

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x} = \lim_{x \rightarrow 0} \frac{\frac{x^3}{3!} + \dots}{\frac{x^3}{3} + \dots} = \lim_{x \rightarrow 0} \frac{\frac{1}{3!} + \dots}{\frac{1}{3} + \dots} = \frac{1}{2}.$$

5. Prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1 \quad \text{for all } 0 < x < \frac{\pi}{2}.$$

Solution. $\frac{\sin x}{x} = \frac{\sin x - \sin 0}{x - 0} = \cos x^*$ by mean value theorem, so $\sin x < x$ for $0 < x < \frac{\pi}{2}$. Also note that $(\sin x)'' = -\sin x \leq 0$ and the equation of the line passing $(0, \sin 0)$, $(\frac{\pi}{2}, \sin \frac{\pi}{2})$ is $y = \frac{2}{\pi}x$. So it is enough to prove the following theorem:

Theorem. Let f be a twice differentiable function on $[a, b]$ s.t.

$f''(x) < 0$ on (a, b) . then $f(x) > \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$ on (a, b)

Proof of theorem: Let $g(x) := f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$, then $g(a) = g(b) = 0$, $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. By mean value theorem, there exists a $c \in (a, b)$ s.t. $\frac{f(b)-f(a)}{b-a} = f'(c)$. Since $g''(x) = f''(x) < 0$, $g'(x)$ strictly decreases on (a, b) , Thus $g'(x) > 0$ on (a, c) , $g'(x) < 0$ on (c, b) . This means that g strictly increases on $[a, c]$ and strictly decreases on $[c, b]$, and this proves $g(x) > 0$ on (a, b) .

6. Prove that

$$|\sin nx| \leq n|\sin x| \quad \text{for all } n = 0, 1, 2, \dots, \text{ and } x \in \mathbb{R}$$

Note that this inequality may be false for other values of n . For instance,

$$\left| \sin \frac{1}{2}\pi \right| > \frac{1}{2} |\sin \pi|.$$

Solutioin. Use induction on n , with $n = 0, 1$ trivial. If given inequality holds when $n = k$, then $|\sin(k+1)x| = |\sin kx \cos x + \cos kx \sin x|$
 $\leq |\sin kx| |\cos x| + |\cos kx| |\sin x| \leq |\sin kx| + |\sin x|$
 $\leq k|\sin x| + |\sin x| = (k+1)|\sin x|$ and the inequality also holds when $n = k+1$.

Alternative Solution 1. Note that the period of $|\sin x|$ is π , and $|\sin(\pi - x)| = |\sin x|$, $|\sin n(\pi - x)| = |\sin nx|$. So it suffices to assume that $0 \leq x \leq \frac{\pi}{2}$. On $[0, \frac{\pi}{2n}]$, $\sin x, \sin nx \geq 0$. Let $g_n(x) := n \sin x - \sin nx$. $g'(x) = n \cos x - n \cos nx = n(\cos x - \cos nx)$. Since $0 \leq x \leq nx \leq \frac{\pi}{2}$, $g'(x) \geq 0$ and this proves $g(x) \geq 0$, or equivalently $n|\sin x| \geq |\sin nx|$ on $[0, \frac{\pi}{2n}]$. If $x \in [\frac{\pi}{2n}, \frac{\pi}{2}]$, from $\sin > \frac{2}{\pi}x$, $n \sin x \geq \frac{2n}{\pi}x \geq 1 \geq |\sin nx|$.

Alternative Solution 2. Assume $x \neq 2n\pi$. Recall $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$. Let $a := e^{ix}$, $b := e^{-ix}$.

$$\begin{aligned} \frac{|\sin nx|}{|\sin x|} &\leq \frac{|e^{inx} - e^{-inx}|}{|e^{ix} - e^{-ix}|} \leq \left| \frac{a^n - b^n}{b - a} \right| = |a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1}| \\ &\leq |a|^{n-1} + |a|^{n-2}|b| + \dots + |a||b|^{n-2} + |b|^{n-1} \leq n. \end{aligned}$$

□

7. (a) Put $s_N = 1 + (\frac{1}{2}) + \dots + (\frac{1}{N})$. Prove that

$$\lim_{N \rightarrow \infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ , is called Euler's constant.)

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?

Solution. (a) From

$$\sum_{k=1}^N \int_k^{k+1} \frac{1}{x} dx \leq s_N = \sum_{k=1}^N \frac{1}{k} \leq 1 + \sum_{k=2}^N \int_{k-1}^k \frac{1}{x} dx,$$

we have

$$\log(N+1) \leq s_N \leq 1 + \log N, \quad \log\left(1 + \frac{1}{N}\right) \leq s_N - \log N \leq 1$$

so $s_N - \log N$ is bounded. On the other hand,

$$\{s_{N+1} - \log(N+1)\} - \{s_N - \log N\} = \frac{1}{N+1} - \int_N^{N+1} \frac{1}{x} dx \leq 0.$$

Therefore, $(s_N - \log N)_{N=1}^\infty$ is decreasing sequence and has a lower bound, hence converges.

(b) From $\log(N+1) \leq s_N \leq 1 + \log N$, We know that $\log(N+1) > 100$ is enough to have $s_N > 100$, and $1 + \log N > 100$ is necessary to have $s_N > 100$. Thus the threshold value for s_N is between e^{99} and e^{100} . In the form of $N = 10^m$, range of the threshold value for m is

$$\frac{99}{\log 10} < m_0 < \frac{100}{\log 10}.$$

From the definition of e , $2 = 1 + \frac{1}{1!} < e = \sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \frac{1}{1!} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 3$ and we get $2 < \log x < 4$, because $3^2 < 10 < 2^4$. So $24.75 < m_0 < 50$. \square

8. Suppose that f is Riemann integrable on $[0, A]$ for all $A < \infty$, and $f(x) \rightarrow 1$ as $x \rightarrow +\infty$. Prove that

$$\lim_{t \downarrow 0} t \int_0^\infty e^{-tx} f(x) dx = 1.$$

Solution. First we show that given improper integral exists if $t > 0$. For any $\varepsilon > 0$, there exists a $B \in \mathbb{R}^+$ s.t. $x > B$ implies $|f(x) - 1| < \varepsilon$. If $A \geq B$, then

$$\int_0^A |e^{-tx} f(x)| dx = \int_0^B |e^{-tx} f(x)| dx + \int_B^A |e^{-tx} f(x)| dx$$

is increasing function of A and bounded above because

$$\int_B^A |e^{-tx} f(x)| dx \leq (1+\varepsilon) \int_B^A e^{-tx} dx = (1+\varepsilon) \frac{e^{-Bt} - e^{-At}}{t} < (1+\varepsilon) \frac{e^{-Bt}}{t}.$$

Therefore $\int_0^A |e^{-tx} f(x)| dx$ converges. Moreover, since $0 \leq \frac{|x|+x}{2}, \frac{|x|-x}{2} \leq |x|$,

$$\int_0^A \frac{|e^{-tx} f(x)| + e^{-tx} f(x)}{2} dx, \int_0^A \frac{|e^{-tx} f(x)| - e^{-tx} f(x)}{2} dx$$

are both increasing function of A and bounded above, so they converges. Hence

$$\int_0^A e^{-tx} f(x) dx = \int_0^A \frac{|e^{-tx} f(x)| + e^{-tx} f(x)}{2} dx - \int_0^A \frac{|e^{-tx} f(x)| - e^{-tx} f(x)}{2} dx$$

also converges.

Now for any $\varepsilon > 0$, Let B like before, then

$$\begin{aligned} \left| t \int_0^\infty e^{-tx} f(x) dx - 1 \right| &= \left| t \int_0^\infty e^{-tx} \{f(x) - 1\} dx \right| \\ &\leq t \int_0^B e^{-tx} |f(x) - 1| dx + t \int_B^\infty e^{-tx} |f(x) - 1| dx. \end{aligned}$$

($t \int_0^\infty e^{-tx} dx = 1$ was used.) Since f is integrable $[0, B]$, so f is bounded on $[0, B]$ by, say M . Thus we have

$$\left| t \int_0^\infty e^{-tx} f(x) dx - 1 \right| \leq t(M+1)B + \varepsilon e^{-Bt},$$

$$\limsup_{t \rightarrow 0+} \left| t \int_0^\infty e^{-tx} f(x) dx - 1 \right| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this means

$$\lim_{t \rightarrow 0+} t \int_0^\infty e^{-tx} f(x) dx = 1.$$