해석개론 및 연습 2 과제 #4

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1. **(a)** First of all,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-inx}\,dx.$$

If $n = 0$, $c_0 = \delta/\pi$. Now for $n \neq 0$,

$$c_n = \frac{1}{2\pi}\left[-\frac{1}{in}e^{-inx}\right]_{-\delta}^{\delta} = \frac{1}{2\pi}\frac{e^{in\delta} - e^{-in\delta}}{in} = \frac{\sin n\delta}{n\pi}.$$

(b) f satisfies the Lipschitz condition at $x = 0$, since for $|t| < \delta$,

$$|f(x+t) - f(x)| = |f(t) - 1| = 0.$$

By Theorem 8.14, the Fourier series of f converges at $x = 0$. Therefore

$$f(0) = \lim_{N\to\infty} s_N(f;0) = \lim_{N\to\infty}\sum_{n=-N}^{N} c_n = c_0 + 2\lim_{N\to\infty}\sum_{n=1}^{N} c_n = \frac{\delta}{\pi} + 2\sum_{n=1}^{\infty}\frac{\sin n\delta}{n\pi}.$$

Reordering terms give

$$\sum_{n=1}^{\infty}\frac{\sin n\delta}{n\pi} = \frac{f(0) - \frac{\delta}{\pi}}{2} \implies \sum_{n=1}^{\infty}\frac{\sin n\delta}{n} = \frac{\pi - \delta}{2}.$$

(c) We plan to use Parseval's identity.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\delta}^{\delta} dx = \frac{\delta}{\pi}.$$

$$\lim_{N\to\infty}\sum_{n=-N}^{N} |c_n|^2 = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty}\frac{\sin^2 n\delta}{n^2\pi^2}.$$

Therefore,

$$\frac{\delta}{\pi} = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty}\frac{\sin^2 n\delta}{n^2\pi^2} \implies \sum_{n=1}^{\infty}\frac{\sin^2 n\delta}{n^2} = \frac{\delta\pi - \delta^2}{2},$$

and we get the desired result,

$$\sum_{n=1}^{\infty}\frac{\sin^2 n\delta}{n^2\delta} = \frac{\pi - \delta}{2}. \qquad (\delta > 0)$$

(d) We first show that this integral converges. Since $\frac{\sin x}{x} \to 1$ as $x \to 0$, the integral is well-defined. For some $K > 0$,

$$\left|\int_K^{\infty}\left(\frac{\sin x}{x}\right)^2 dx\right| \le \int_K^{\infty}\left|\frac{\sin^2 x}{x^2}\right| dx \le \int_K^{\infty}\frac{1}{x^2}\,dx < \infty.$$

Therefore,

$$\int_0^K \left(\frac{\sin x}{x}\right)^2 dx + \int_K^\infty \left(\frac{\sin x}{x}\right)^2 dx = \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$$

also converges.

Let $\epsilon > 0$ be given. We will prove the equality by 4 steps. First, choose large enough $M_0 > 0$ such that for all $M \geq M_0$,

$$\left| \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx - \int_0^M \left(\frac{\sin x}{x}\right)^2 dx \right| < \frac{\epsilon}{4}.$$

Since $\left(\frac{\sin x}{x}\right)^2$ is continuous, we can write the integral as a Riemann sum as follows,

$$\int_0^M \left(\frac{\sin x}{x}\right)^2 dx = \lim_{N\to\infty} \sum_{k=1}^N \left(\frac{\sin \frac{M}{N}k}{\frac{M}{N}k}\right)^2 \cdot \frac{M}{N} = \lim_{N\to\infty} \sum_{k=1}^N \frac{\sin^2 k\delta_N}{k^2\delta_N}.$$

Here, $\delta_N = \frac{M}{N}$. Now choose $N_M \in \mathbb{N}$ such that for all $N \geq N_M$,

$$\left| \int_0^M \left(\frac{\sin x}{x}\right)^2 dx - \sum_{k=1}^N \frac{\sin^2 k\delta_N}{k^2\delta_N} \right| < \frac{\epsilon}{4}.$$

Next, there exists large enough $N_1 \in \mathbb{N}, N_1 \geq N_M$, such that for all $N \geq N_1$,

$$\left| \sum_{k=1}^N \frac{\sin^2 k\delta_N}{k^2\delta_N} - \frac{\pi - \delta_N}{2} \right| < \frac{\epsilon}{4}.$$

Finally, take even larger $N_2 \in \mathbb{N}, N_2 \geq N_1$, such that for all $N \geq N_2$,

$$\left| \frac{\pi - \delta_N}{2} - \frac{\pi}{2} \right| < \frac{\epsilon}{4}.$$

Using the results from above, for large enough $M \geq M_0$ and $N \geq N_2$,

$$\left| \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx - \frac{\pi}{2} \right| < \epsilon.$$

(e) Set $\delta = \pi/2$. For $n \in \mathbb{N}$,

$$\sin^2 \frac{n\pi}{2} = \begin{cases} 0 & (n \text{ is even}) \\ 1 & (n \text{ is odd}) \end{cases}.$$

Therefore,

$$\sum_{n=1}^\infty \frac{\sin^2 \frac{n\pi}{2}}{n^2 \cdot \frac{\pi}{2}} = \sum_{n=1}^\infty \frac{1}{(2n-1)^2 \frac{\pi}{2}} = \frac{\pi}{4} \implies \sum_{n=1}^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

2. Let $f(x) = (\pi - |x|)^2$.

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - |x|)^2(\cos nx - i\sin nx)\,dx \\ &= \frac{1}{\pi}\int_0^{\pi}(\pi - x)^2\cos nx\,dx. \end{aligned}$$

The last equality comes from the fact that $\sin nx$ is odd and $\cos nx$ is even. Now using integration by parts, we get

$$c_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)^2\cos nx\,dx = \begin{cases} \dfrac{\pi^2}{3} & (n = 0) \\ \dfrac{2}{n^2} & (n \neq 0) \end{cases}.$$

We can directly see that $c_n = c_{-n}$. For $n \neq 0$,

$$c_n e^{inx} + c_{-n}e^{-inx} = 2c_n(e^{inx} + e^{-inx}) = \frac{4}{n^2}\cos nx,$$

so

$$s_N(f; x) = \frac{\pi^2}{3} + \sum_{n=1}^{N}\frac{4}{n^2}\cos nx.$$

We want to show that the partial sum converges for $x \in [-\pi, \pi]$. To use Theorem 8.14, we try to prove the Lipschitz condition. Take some small $\delta > 0$. For $|t| < \delta$,

$$\begin{aligned} |f(x+t) - f(x)| &= \left|(\pi - |x+t|)^2 - (\pi - |x|)^2\right| = \left|2xt + t^2 - 2\pi\,|x+t| + 2\pi\,|x|\right| \\ &= |t(2x+t) - 2\pi(|x+t| - |x|)| \le |t|\,|2x+t| + 2\pi\,||x+t| - |x|| \\ &\le |t|\,M + 2\pi\,|t| = (M + 2\pi)\,|t|, \end{aligned}$$

since $|2x + t|$ can be bounded by some $M > 0$. Therefore by Theorem 8.14,

$$(\pi - |x|)^2 = f(x) = \lim_{N\to\infty} s_N(f; x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx, \quad (-\pi \le x \le \pi).$$

Take $x = 0$ to get

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2} \implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

The last equation will come from Parseval's identity.

$$\lim_{N\to\infty}\sum_{n=-N}^{N}|c_n|^2 = c_0^2 + 2\sum_{n=1}^{\infty}|c_n|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty}\frac{4}{n^4}.$$

Also,

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi - |x|)^4\,dx &= \frac{1}{\pi}\int_0^{\pi}(\pi - x)^4\,dx \\ &= \frac{1}{\pi}\left[-\frac{1}{5}(\pi - x)^5\right]_0^{\pi} = \frac{\pi^4}{5}. \end{aligned}$$

By Parseval's identity,

$$\frac{\pi^4}{9} + 2\sum_{n=1}^{\infty}\frac{4}{n^4} = \frac{\pi^4}{5} \implies \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$

3. We restrict the domain to $[-\pi, \pi]$, since e^{inx} is periodic with period 2π. Note that

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \begin{cases} \dfrac{\sin(n+1/2)x}{\sin(x/2)} & (x \in [-\pi, \pi]) \\ 2n+1 & (x = 0) \end{cases}. \tag{$\star$}$$

For $x = 0$,

$$K_N(0) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(0) = \frac{1}{N+1}\sum_{n=0}^{N}(2n+1) = \frac{(N+1)^2}{N+1} = N+1.$$

For $x \neq 0$,

$$K_N(x) = \sum_{n=0}^{N} \frac{\sin(n+1/2)x}{\sin(x/2)} = \frac{1}{N+1} \cdot \frac{1}{\sin(x/2)} \sum_{n=0}^{N} \sin\left(n + \frac{1}{2}\right)x.$$

We try to simplify the last sum. ($\Im$ denotes the imaginary part)

$$\begin{aligned}
\sum_{n=0}^{N} \sin\left(n+\frac{1}{2}\right)x &= \sum_{n=0}^{N} \Im\left(e^{i(n+1/2)x}\right) = \Im\left(\sum_{n=0}^{N} e^{i(n+1/2)x}\right) \\
&= \Im\left(\frac{e^{ix/2}\left(e^{i(N+1)x}-1\right)}{e^{ix}-1}\right) = \Im\left(\frac{e^{i(N+1)x}-1}{e^{ix/2}-e^{-ix/2}}\right) \\
&= \Im\left(\frac{\cos(N+1)x + i\sin(N+1)x - 1}{2i\sin(x/2)}\right) \\
&= \Im\left(\frac{\sin(N+1)x + i(1-\cos(N+1)x)}{2\sin(x/2)}\right) = \frac{1-\cos(N+1)x}{2\sin(x/2)}.
\end{aligned}$$

Therefore,

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1}{\sin(x/2)} \frac{1-\cos(N+1)x}{2\sin(x/2)} = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x}.$$

by the half-angle formula.

(a) For $x = 0$, $K_N(x) > 0$, and if $x \neq 0$,

$$1-\cos(N+1)x \geq 0, \quad 1-\cos x \geq 0.$$

Therefore $K_N(x) \geq 0$.

(b) We see this by direct calculation. Using $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} D_n(x)\,dx = 1$,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\,dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{N+1}\sum_{n=0}^{N} D_n(x)\,dx \\
&= \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\,dx = \frac{1}{N+1}\sum_{n=0}^{N} 1 = 1.
\end{aligned}$$

(c) Since $1 - \cos x$ is increasing on $\delta \le |x| \le \pi$,

$$0 < 1 - \cos\delta \le 1 - \cos x.$$

Thus,

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x} \le \frac{1}{N+1} \cdot \frac{2}{1-\cos x} \le \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta}.$$

For the last part, we use the fact that

$$s_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,dt.$$

Now manipulate the expression.

$$\begin{aligned}
\sigma_N(f;x) &= \frac{1}{N+1}\sum_{n=0}^{N} s_n(f;x) = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\cdot\frac{1}{N+1}\sum_{n=0}^{N} D_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt.
\end{aligned}$$

To prove Fejer's Theorem, suppose that f is continuous with period 2π on $[-\pi,\pi]$. Now we show that $\sigma_N(f;x)$ converges uniformly on $[-\pi,\pi]$.

$$\begin{aligned}
|\sigma_N(f;x) - f(x)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(x-t)K_N(t)\,dt - \int_{-\pi}^{\pi} f(x)K_N(t)\,dt\right| && (\text{from } \textsf{(b)}) \\
&= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} \big(f(x-t)-f(x)\big)K_N(t)\,dt\right| \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t)-f(x)|\,K_N(t)\,dt. && (K_N(t)\ge 0)
\end{aligned}$$

Now, split the integral into three parts, as $[-\pi,\pi] = [-\pi,-\delta]\cup[-\delta,\delta]\cup[\delta,\pi]$.

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t)-f(x)|\,K_N(t)\,dt &= \frac{1}{2\pi}\int_{-\pi}^{-\delta} |f(x-t)-f(x)|\,K_N(t)\,dt \quad (\clubsuit) \\
&\quad + \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-t)-f(x)|\,K_N(t)\,dt \quad (\spadesuit) \\
&\quad + \frac{1}{2\pi}\int_{\delta}^{\pi} |f(x-t)-f(x)|\,K_N(t)\,dt \quad (\heartsuit)
\end{aligned}$$

Since the domain is compact, we know that f is uniformly continuous and bounded. Let $\epsilon > 0$ be given. There exists $\delta > 0$ such that

$$|t| = |(x-t)-x| < \delta \implies |f(x-t)-f(x)| < \pi\epsilon. \qquad (*)$$

Also, $|f| < M$ for some $M > 0$. We bound each integral like the following.

$$(\spadesuit) \le \frac{1}{2\pi}\int_{-\delta}^{\delta} \pi\epsilon K_N(t)\,dt < \frac{\epsilon}{2}. \qquad (\text{by } \textsf{(b)},\ (*))$$

$$
\begin{aligned}
(\clubsuit) + (\heartsuit) &\le \frac{1}{2\pi}\int_{-\pi}^{-\delta} 2MK_N(t)\,dt + \frac{1}{2\pi}\int_{\delta}^{\pi} 2MK_N(t)\,dt \\
&\le \frac{M}{\pi}\int_{-\pi}^{-\delta} \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}\,dt + \frac{M}{\pi}\int_{\delta}^{\pi} \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}\,dt \quad (\text{by } \textsf{(c)}) \\
&= \frac{4M}{\pi}\cdot\frac{1}{N+1}\cdot\frac{\pi-\delta}{1-\cos\delta}. \quad (\diamondsuit)
\end{aligned}
$$

We can set N large enough that $(\diamondsuit) < \frac{\epsilon}{2}$. Therefore,

$$
\begin{aligned}
|\sigma_N(f;x) - f(x)| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x-t) - f(x)|\, K_N(t)\,dt \\
&= (\spadesuit) + (\clubsuit) + (\heartsuit) < \frac{\epsilon}{2} + (\diamondsuit) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}
$$

and $\sigma_N(f;x)$ converges uniformly to $f(x)$ on $[-\pi,\pi]$.

4. **(a)** Since $D_N(t) = D_N(-t)$ almost directly from definition,

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{0} f(x-t)D_N(t)\,dt &= \frac{1}{2\pi}\int_{\pi}^{0} f(x+u)D_N(-u)(-du) \quad (u=-t) \\
&= \frac{1}{2\pi}\int_{0}^{\pi} f(x+u)D_N(u)\,du.
\end{aligned}
$$

Now we rewrite $s_N(f;x)$ as the following,

$$
\begin{aligned}
s_N(f;x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} f(x-t)D_N(t)\,dt + \frac{1}{2\pi}\int_{0}^{\pi} f(x-t)D_N(t)\,dt \\
&= \frac{1}{2\pi}\int_{0}^{\pi} f(x+t)D_N(t)\,dt + \frac{1}{2\pi}\int_{0}^{\pi} f(x-t)D_N(t)\,dt \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \big(f(x+t) + f(x-t)\big)D_N(t)\,dt \\
&= \frac{1}{2\pi}\int_{0}^{\pi} \big(f(x+t) + f(x-t)\big)\frac{\sin(N+1/2)t}{\sin(t/2)}\,dt,
\end{aligned}
$$

by $(\star)$ from Problem 3. (A difference at a single point $x = 0$ does not change the value of the integral.)

(b) First we prove a lemma.

Lemma.[1] Let f be Riemann integrable on an interval I. Then,

$$
\lim_{N\to\infty}\int_I f(t)\sin Nt\,dt = 0.
$$

Proof. We prove for the case $I = [-\pi,\pi]$. We use the fact that for

$$
\phi_0(t) = \frac{1}{\sqrt{2\pi}}, \quad \phi_{2n-1}(t) = \frac{\cos nt}{\sqrt{\pi}}, \quad \phi_{2n}(t) = \frac{\sin nt}{\sqrt{\pi}},
$$

[1] 구간 I 위에서의 특이적분이 수렴하면 된다. 일반적인 형태는 Riemann-Lebesgue Lemma이지만...

$\{\phi_n\}_{n=1}^N$ form an orthonormal system of functions. Therefore,

$$\int_I f(t)\sin Nt\,dt = \int_I \sqrt{\pi} f(t)\frac{\sin Nt}{\sqrt{\pi}}\,dt$$

can be viewed as the $2N$-th Fourier coefficient of $\sqrt{\pi} f$ relative to $\{\phi_n\}$. By Theorem 8.12, Fourier coefficients approach 0 as $N \to \infty$, so we have the desired result. (There is another proof that uses the denseness of step functions in $\mathcal{R}^1(I)$...)

Also we calculate the following limit by applying L'Hôpital's rule, since all the terms in the denominator and the numerator approach 0 as $t \to 0$. Also they are differentiable and derivative of the denominator is not zero in the neighborhood of 0, except for at 0 itself.

$$\begin{aligned}
\lim_{t\to 0}\left(\frac{1}{\sin(t/2)} - \frac{2}{t}\right) &= \lim_{t\to 0}\frac{t - 2\sin(t/2)}{t\sin(t/2)} = \lim_{t\to 0}\frac{1-\cos(t/2)}{\sin(t/2) + (t/2)\cos(t/2)} \\
&= \lim_{t\to 0}\frac{(1/2)\sin(t/2)}{(1/2)\cos(t/2) + (1/2)\cos(t/2) - (t/4)\sin(t/2)} \\
&= \lim_{t\to 0}\frac{2\sin(t/2)}{4\cos(t/2) - t\sin(t/2)} = 0
\end{aligned}$$

Therefore $\dfrac{1}{\sin(t/2)} - \dfrac{2}{t}$ is bounded on $[-\pi, \pi]$, thus

$$(f(x+t) + f(x-t))\left(\frac{1}{\sin(t/2)} - \frac{2}{t}\right)$$

is integrable, and by the lemma,

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_0^\pi (f(x+t) + f(x-t))\left(\frac{1}{\sin(t/2)} - \frac{2}{t}\right)\sin\left(N + \frac{1}{2}\right)t\,dt = 0.$$

(The lemma was proven for $[-\pi, \pi]$, but the integrand here is an even function, so we can only consider $[0, \pi]$.)

(c) From (b),

$$\lim_{N\to\infty} s_N(f;x) = \lim_{N\to\infty}\frac{1}{\pi}\int_0^\pi \frac{f(x+t) + f(x-t)}{t}\sin\left(N + \frac{1}{2}\right)t\,dt.$$

We want to show

$$\begin{aligned}
0 &= \lim_{N\to\infty}\left(s_N(f;a) - \frac{f(a+) + f(a-)}{2}\right) \\
&= \lim_{N\to\infty}\frac{1}{\pi}\int_0^\pi \frac{f(a+t) + f(a-t)}{t}\sin\left(N + \frac{1}{2}\right)t\,dt \\
&\qquad - \frac{f(a+) + f(a-)}{2}\lim_{N\to\infty}\frac{2}{\pi}\int_0^\pi \frac{\sin\left(N + 1/2\right)t}{t}\,dt \\
&= \lim_{N\to\infty}\frac{1}{\pi}\int_0^\pi \left(\frac{f(a+t) + f(a-t)}{t} - \frac{f(a+) + f(a-)}{t}\right)\sin\left(N + \frac{1}{2}\right)t\,dt.
\end{aligned}$$

Using the workaround, we show that f_n converges uniformly. For any $\epsilon > 0$, we show that there exists $N \in \mathbb{N}$ such that for all $m \in \mathbb{N}$,

$$n \geq N \implies \left| g\left(\frac{1}{m}\right) - f_n\left(\frac{1}{m}\right) \right| < \epsilon,$$

where

$$g\left(\frac{1}{m}\right) = \frac{1}{\pi} \int_0^{\pi} \left(\frac{f(a+t) + f(a-t)}{t} - \frac{f(a+) + f(a-)}{t} \right) \sin\left(m + \frac{1}{2}\right) t \, dt$$

for $m \in \mathbb{N}$. From the assumption, we can choose positive p, δ, M such that

$$\begin{aligned} 0 < |t| < \delta &\implies \left| \frac{f(a+t) + f(a-t)}{2} - \frac{f(a+) + f(a-)}{2} \right| \leq M |t|^p \\ &\implies \left| \frac{f(a+t) + f(a-t)}{t} - \frac{f(a+) + f(a-)}{t} \right| \leq 2M |t|^{p-1}. \end{aligned}$$

Therefore, for large enough N such that $N > \delta^{-1}$, $1/n \leq 1/N < \delta$.

$$\begin{aligned} &\left| g\left(\frac{1}{m}\right) - f_n\left(\frac{1}{m}\right) \right| \\ &\quad = \frac{1}{\pi} \left| \int_0^{1/n} \left(\frac{f(a+t) + f(a-t)}{t} - \frac{f(a+) + f(a-)}{t} \right) \sin\left(m + \frac{1}{2}\right) t \, dt \right| \\ &\quad \leq \frac{1}{\pi} \int_0^{1/n} \left| \frac{f(a+t) + f(a-t)}{t} - \frac{f(a+) + f(a-)}{t} \right| \left| \sin\left(m + \frac{1}{2}\right) t \right| \, dt \\ &\quad \leq \frac{1}{\pi} \int_0^{1/n} 2M |t|^{p-1} \, dt = \frac{2M}{\pi p} \cdot \frac{1}{n^p} < \epsilon, \end{aligned}$$

since the last term can be made arbitrarily small. Thus f_n converges uniformly on $\left\{ \frac{1}{m} : m \in \mathbb{N} \right\}$.

(d) Using the definition of f_n in (c), we can rephrase our objective as

$$\lim_{m\to\infty} \lim_{n\to\infty} \frac{1}{\pi} \int_{1/n}^{\pi} \left(\frac{f(a+t) + f(a-t)}{t} - \frac{f(a+) + f(a-)}{t} \right) \sin\left(m + \frac{1}{2}\right) t \, dt = 0.$$

But since we have uniform convergence, we can change the order of limits. So we can instead show that

$$\lim_{n\to\infty} \lim_{m\to\infty} \frac{1}{\pi} \int_{1/n}^{\pi} \left(\frac{f(a+t) + f(a-t)}{t} - \frac{f(a+) + f(a-)}{t} \right) \sin\left(m + \frac{1}{2}\right) t \, dt = 0.$$

Now we directly see that the integrand is well-defined and bounded in $[1/n, \pi]$. Thus by the lemma in (b),

$$\lim_{m\to\infty} \frac{1}{\pi} \int_{1/n}^{\pi} \left(\frac{f(a+t) + f(a-t)}{t} - \frac{f(a+) + f(a-)}{t} \right) \sin\left(m + \frac{1}{2}\right) t \, dt = 0.$$

And thus the result is proven.

$$\lim_{N\to\infty} s_N(f; a) = \frac{f(a+) + f(a-)}{2}.$$