

Introduction to Analysis II

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Introduction & Notice

- 7, 8장 나가고 중간고사, 11장 나가고 기말고사
- 연습 시간이 있는 수업 (목 6:30 ~ 8:20)¹
- 오늘 연습 시간: 지난학기 배운 내용 중 필요한 내용 복습

¹가능하면 1시간 반 안에 끝내라고 하심 ㅋㅋ

Chapter 7

Sequences and Series of Functions

September 1st, 2022

함수들을 각각 보는 것보다 **함수들의 공간을 이해**하는 것이 해석학의 핵심이다! 공간을 이해해야 미분방정식도 풀고 실제 현실의 문제들을 풀 수 있는 것입니다.

\mathbb{R}^n 을 단순히 좌표들의 모임으로 보는 것이 아니라, 거리 구조를 주고 열린/닫힌 집합과 같은 위상 구조를 줬었습니다. 이 전에 했던게 **수열의 수렴과 발산, 코시 수열**이죠. 이런 것들을 바탕으로 공간의 위상적 성질을 조금 더 효율적으로 공부할 수 있었습니다.

즉, 어떤 공간을 배우기 위해서는 수열의 수렴이나 발산을 배워야 합니다. **따라서 우리는 함수들의 공간을 공부하기 위해 우선 함수열을 공부합니다.**

Suppose E is a set¹, and let $f_n : E \rightarrow \mathbb{C}$ for all $n \in \mathbb{N}$. Then

$$(f_n)_{n=1}^{\infty}$$

is a sequence of (complex-valued) function.

수열을 공부했으니 수열의 **수렴**을 정의해야 할 것입니다.

Definition 7.1 (Pointwise Convergence) $(f_n)_{n=1}^{\infty}$ converges **pointwise** on E , if for each $x \in E$ the sequence $(f_n(x))_{n=1}^{\infty}$ converges in \mathbb{C} .

In other words, for each $x \in E$, there exists $a_x \in \mathbb{C}$ and

$$\forall \epsilon > 0, \exists N_x \in \mathbb{N} \text{ such that } n \geq N_x \implies |f_n(x) - a_x| < \epsilon.$$

¹사실은 *metric space* 이다.

Definition. If (f_n) converges pointwise, we can define a function f by

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad (x \in E)$$

We say that

- f is the *limit* or *limit function* of f_n .
- (f_n) to f pointwise on E .

Definition. If $\sum f_n(x)$ converges (pointwise) for every $x \in E$, we can define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E)$$

and the function f is called the *sum* of the series $\sum f_n$.

Recall. $f : (E, d) \rightarrow \mathbb{C}$ is continuous on $E \iff f$ is continuous at all $x \in E$.

Recall. (Theorem 4.6) If $p \in E$ and p is a limit point of E ,

$$f \text{ is continuous at } p \iff \lim_{x \rightarrow p} f(x) = f(p)$$

Question. Suppose (f_n) is a sequence of functions. Does the limit function or the sum of the series preserve important properties?

- (1) If f_n is continuous, is f continuous?
- (2) If f_n is differentiable/integrable, is f differentiable/integrable?

For (1), the question is equivalent to the following:

If p is a limit point, does the following hold?

$$\lim_{x \rightarrow p} \lim_{n \rightarrow \infty} f_n(x) \stackrel{?}{=} \lim_{n \rightarrow \infty} \lim_{x \rightarrow p} f_n(x)$$

And the answer is **No**.

실수열이 주어졌을 때, 극한을 계산하여 극한값이 실수라는 것은 굉장히 중요한 것입니다. 우리가 **연속함수열**의 수렴을 정의할 때, 극한이 되는 함수 또한 **연속**이 되기를 기대하는 것은 굉장히 자연스러운 일입니다. 하지만 점별수렴하는 연속함수열의 극한은 연속이 아닐 수 있습니다. 즉, 점별수렴은 연속함수공간에서의 '수렴'으로 정의하기에는 부족합니다.

Example 7.2 Suppose $a_{m,n} = \frac{m}{m+n}$ for $m, n \in \mathbb{N}$. We see that

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} a_{m,n} = 1 \neq 0 = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a_{m,n}$$

Example. Define

$$f_n(x) = \begin{cases} 0 & (\frac{1}{n} \leq x \leq 1) \\ -nx + 1 & (0 \leq x < \frac{1}{n}) \end{cases}$$

then we can easily see that

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & (0 < x \leq 1) \\ 1 & (x = 0) \end{cases}$$

Thus f is not continuous at $x = 0$.

Example. Define $f_n : \mathbb{R} \rightarrow \mathbb{R}$ as

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (n = 0, 1, 2, \dots)$$

by direct calculation,

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = 1 + x^2 \quad (x \neq 0)$$

since this is a geometric series when $x \neq 0$. If $x = 0$, $f(x) = 0$ and f is not continuous.

Does the limit function preserve Riemann integrability?

Example. For $m = 1, 2, \dots$, define

$$f_m(x) = \lim_{n \rightarrow \infty} (\cos m! \pi x)^{2n} = \begin{cases} 1 & (m!x \in \mathbb{Z}) \\ 0 & (m!x \notin \mathbb{Z}) \end{cases}$$

We see that $f_m(x)$ is Riemann integrable. However,

Claim.

$$f(x) = \lim_{m \rightarrow \infty} f_m(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

and $f(x)$ is nowhere continuous thus not Riemann integrable.

Proof. Suppose $x = p/q \in \mathbb{Q}$. ($p, q \in \mathbb{Z}$) If we take $m \geq q$, we see that $m!x \in \mathbb{Z}$. Thus $f_m(x) = 1$. If $x \notin \mathbb{Q}$, $m!x$ can never be in \mathbb{Z} and $f_m(x) = 0$.

Question. Uniform continuity를 할 때 uniform이 어디서 나오죠? 해석학에서 그 점에서 뭐가 성립한다, 그러면 그 점과 그 근방에서만 확인하면 됐었죠. Continuity는 local property죠. 그런데 uniform continuity는 전체가 다 uniform하게 성립한다는 의미입니다.

Recall. $f : (X, d) \rightarrow (Y, d)$ is uniformly continuous on X ² if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_X(p, q) < \delta \implies d_Y(f(p), f(q)) < \epsilon$$

즉, 모든 점에서 똑같이 잡을 수 있다!

Recall. (Theorem 4.19) If X is compact and f is continuous on X , then f is uniformly continuous on X .³

이제부터 나오는 uniform convergence는 sequence에 관한 것입니다!

Definition 7.7 (Uniform Convergence) Suppose $f_n : E \rightarrow \mathbb{C}$ is a sequence of functions. $(f_n)_{n=1}^\infty$ converges uniformly on E to a function f if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall x \in E, n \geq N \implies |f_n(x) - f(x)| \leq \epsilon.$$

Also, we say that the series $\sum f_n(x)$ converges uniformly on E if the sequence of partial sums $(\sum_{k=1}^n f_k(x))$ converges uniformly on E .

점별수렴의 경우 $N_x \in \mathbb{N}$ 이 $x \in E$ 에 의존하지만, 고른수렴의 경우 N 이 x 와 무관합니다!

[똑같은 ϵ -띠를 둘러서 $y = f(x)$ 의 근방 안에 $f_n(x)$ ($n \geq N$) 가 모두 들어가 있어야 한다]는 의미에서 uniform입니다. 한꺼번에 ϵ 으로 누를 수 있다는 것입니다.

Theorem 7.9 Suppose

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad (x \in E)$$

Then $f_n \rightarrow f$ converges uniformly on E if and only if

$$\lim_{n \rightarrow \infty} \sup_{x \in E} |f_n(x) - f(x)| = 0$$

which can also be written as

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies \sup_{x \in E} |f_n(x) - f(x)| \leq \epsilon$$

Notation. $f_n \rightarrow f$ uniformly on $E \iff f_n \xrightarrow{u} f$ on E .⁴

Theorem 7.8 (Cauchy Criterion for Uniform Convergence) $f_n \xrightarrow{u} f$ on $E \iff$

²Subspace of metric space is also a metric space.

³갑자기 왜 uniform continuity 얘기를 하나, 헛갈리지 말고 기억하시라고!

⁴교수님: 책에서는 나중에 $\|f_n(x) - f(x)\|_\infty \rightarrow 0$ 으로 적었던 것 같은데...

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n, m \geq N \implies \sup_{x \in E} |f_n(x) - f_m(x)| \leq \epsilon. \text{ }^5$$

Proof.

(\implies) For given $\epsilon > 0$, fix $x \in E$. Since f_n converges uniformly on E , we can find $N \in \mathbb{N}$ such that for $n, m \geq N$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(\impliedby) Uniform Cauchy property implies that (f_n) is a Cauchy sequence in \mathbb{C} . By the completeness of \mathbb{C} , the limit function $f(x)$ exists. Now we show that this convergence is uniform. For given $\epsilon > 0$ choose $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$\sup_{x \in E} |f_n(x) - f_m(x)| \leq \epsilon$$

Then

$$\begin{aligned} |f_n(x) - f(x)| &= |f_n(x) - f_m(x) + f_m(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq |f_m(x) - f(x)| + \epsilon \end{aligned}$$

Fix $n \geq N$ and let $m \rightarrow \infty$. Observe that $|f_m(x) - f(x)| \rightarrow 0$ due to pointwise convergence. Therefore for every $x \in E$,

$$n \geq N \implies |f_n(x) - f(x)| \leq \epsilon.$$

코시 수열이 중요한 이유는 completeness 뿐만 아니라, 극한값을 알지 못할 때 수열의 수렴성을 논할 수 있기 때문입니다. 특히 급수의 경우 그 극한값을 알 수 없기 때문에 급수의 수렴판정 등에서 유용하게 사용됩니다.

⁵Uniform Cauchy Property. 실수에서 알고있던 성질과 동일합니다.

September 6th, 2022

More examples.

Example 7.5 Consider $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ for $x \in \mathbb{R}$.

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \equiv 0$$

but,

$$f'_n(x) = \sqrt{n} \cos nx \implies f'_n(0) = \sqrt{n}.$$

As $n \rightarrow \infty$, $f'_n(0)$ does not converge to $f'(0)$.

Example 7.6 Consider $f_n(x) = nx(1 - x^2)^n$ for $x \leq 0 \leq 1$. Note that

$$f_n(0) = 0, f_n(1) = 0.$$

When $0 < x < 1$, $f_n \rightarrow f \equiv 0$. (Theorem 3.20 (d)) Thus $\lim_{n \rightarrow \infty} f_n(x) = 0$ for $0 \leq x \leq 1$.

But

$$\int_0^1 nx(1 - x^2)^n dx = \left[\frac{-n}{2n+2} (1 - x^2)^{n+1} \right]_0^1 = \frac{n}{2n+2},$$

and thus

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \frac{1}{2} \neq 0 = \int_0^1 f(x) dx.$$

Definition. $\sum_{n=1}^{\infty} f_n$ converges uniformly on $E \iff \left(\sum_{k=1}^n f_k \right)$ converges uniformly on E .

Theorem 7.10 (Weierstrass M -test) Suppose $f_n : E \rightarrow \mathbb{C}$ and that for every n , $\exists M_n \in \mathbb{R}$ such that

$$|f_n(x)| \leq M_n, \quad (x \in E)$$

and $\sum_{n=1}^{\infty} M_n < \infty$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on E .

Proof. We want to show that the series is Cauchy.

For $m > n$, we have

$$\left| \sum_{k=n}^m f_k(x) \right| \leq \sum_{k=n}^m |f_k(x)| \leq \sum_{k=n}^m M_k.$$

Given $\epsilon > 0$, choose $m, n \in \mathbb{N}$ such that for $m, n \geq N$, $\sum_{k=n}^m M_k < \epsilon$. Then we get

$$\left| \sum_{k=n}^m f_k(x) \right| < \epsilon, \text{ for all } m, n \geq N.$$

By Theorem 7.8, $\sum f_n$ converges uniformly.

Theorem 7.11 Given metric space (Y, d) and $E \subseteq Y$, suppose that $f_n \xrightarrow{u} f$ on E and $x \in E'$.
If

$$\lim_{t \rightarrow x} f_n(t) = A_n \in \mathbb{C}, \quad (\text{limit exists})$$

then the sequence (A_n) converges, and

$$\lim_{n \rightarrow \infty} A_n = \lim_{t \rightarrow x} f(t).$$

In conclusion,

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t) = \lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t).$$

Proof. ⁶

$((A_n)$ converges in \mathbb{C}) Since \mathbb{C} is complete, we will show that (A_n) is a Cauchy sequence. Let $\epsilon > 0$. Since $f_n \xrightarrow{u} f$ on E ,

$$\exists N \in \mathbb{N} \text{ such that } n, m \geq N \implies |f_n(t) - f_m(t)| \leq \epsilon. \quad (\forall t \in E)$$

From $\lim_{t \rightarrow x} f_n(t) = A_n$, we can choose t arbitrarily close to x , such that for $n, m \geq N$,

$$|f_n(t) - A_n| < \epsilon \text{ and } |f_m(t) - A_m| < \epsilon.$$

Therefore for all $n, m \geq N$,

$$\begin{aligned} |A_n - A_m| &= |A_n - f_n(t) + f_n(t) - A_m + f_m(t) - f_m(t)| \\ &\leq |f_n(t) - A_n| + |f_m(t) - A_m| + |f_n(t) - f_m(t)| < 3\epsilon, \end{aligned}$$

and thus (A_n) is a Cauchy Sequence.

$(\lim_{n \rightarrow \infty} A_n = \lim_{t \rightarrow x} f(t))$ Let $A = \lim_{n \rightarrow \infty} A_n$. We want to show that for all $\epsilon > 0$,

$$\exists \delta > 0 \text{ such that } 0 < d(t, x) < \delta \implies |f(t) - A| < \epsilon.$$

Now,

$$|f(t) - A| \leq \sup_{s \in E} |f(s) - f_n(s)| + |f_n(t) - A_n| + |A_n - A|.$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that for $n \geq N$,

$$\sup_{s \in E} |f(s) - f_n(s)| < \frac{\epsilon}{3} \text{ and } |A_n - A| < \frac{\epsilon}{3}.$$

Fix such N and choose δ such that for $0 < d(x, t) < \delta$ and $t \in E$,

$$|f_N(t) - A_N| \leq \frac{\epsilon}{3}.$$

⁶고른수렴이 어디에서 쓰였는지 확인하는 것이 중요하다.

Thus for $t \in E$ and $0 < d(x, t) < \delta$,

$$|f(t) - A| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Theorem 7.12 Suppose $f: E \rightarrow \mathbb{C}$ is continuous on E and $f_n \xrightarrow{u} f$ on E . Then f is continuous on E .

Proof. Let $x \in E$. If $x \in E'$, f is continuous at x by Theorem 7.11. If x is an isolated point (not a limit point), f is continuous at x by definition of continuity.

고른수렴이 연속함수열의 올바른 수렴입니다.

앞으로는 E 를 전부 metric space라고 가정할게요.

이 정리는 언제 uniformly converge 하는지 알려줍니다.

Theorem 7.13 Given a compact metric space K , suppose that

(1) f_n and $f : K \rightarrow \mathbb{C}$ are continuous on K .

(2) $f_n \rightarrow f$ pointwise.

(3) $f_n(x) \geq f_{n+1}(x)$ for $x \in K$.⁷

Then $f_n \xrightarrow{u} f$ on K .

Proof. Let $g_n(x) = f_n(x) - f(x)$. Then $g_n(x)$ is continuous, decreasing and $g_n \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in K$. Let $\epsilon > 0$ be given.

Claim. There exists $N \in \mathbb{N}$ such that $0 \leq g_n(x) < \epsilon$ for all $x \in K$.

Proof. Let $K_n = \{x \in K : g_n(x) \geq \epsilon\}$. Then $K_n = K \cap g_n^{-1}([\epsilon, \infty))$.⁸ Since g_n is decreasing, $K_{n+1} \subseteq K_n$, but because $g_n \rightarrow 0$, $\bigcap_{n=1}^{\infty} K_n = \emptyset$. By Theorem 2.36, there exists $N \in \mathbb{N}$ such that $K_N = \emptyset$, and then $K_n = \emptyset$ for $n \geq N$. Thus, $0 \leq g_n(x) < \epsilon$ for $\forall x \in K, \forall n \geq N$.

Remark. Compactness is necessary here. Consider $f_n(x) = \frac{1}{nx+1}$ on $x \in E = (0, 1)$. f_n does not converge to 0 uniformly.

Proof. Suppose $f_n \xrightarrow{u} 0$, and take $\epsilon = 1/2$. Then,

$$\exists N \in \mathbb{N} \text{ such that } x \in (0, 1) \implies \frac{1}{Nx+1} < \frac{1}{2}.$$

⁷ f_n only needs to be monotone. See Dini's Theorem.

⁸Closed subset of a compact set is also compact, and the inverse image of closed set is closed if the function is continuous

This gives a contradiction because the equation above gives $Nx > 1$, but we can choose x arbitrarily close to 0.

Definition. Let (X, d) be a metric space. Define

$$C(X, \mathbb{C}) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is continuous and bounded}\}.$$

If there is no ambiguity, we write $C(X) = C(X, \mathbb{C})$.

Let $\|f\| = \sup_{x \in X} |f(x)|$. Then $\|\cdot\|$ is a norm on $C(X)$.

$$(1) \quad \|f\| = 0 \iff f \equiv 0.$$

$$(2) \quad \|f\| < \infty.$$

$$(3) \quad \|f + g\| \leq \|f\| + \|g\|.$$

Define $d(f, g) = \|f - g\|$, then $(C(X), d)$ is a metric space.

Therefore, $f_n \xrightarrow{u} f \iff f_n \rightarrow f$ in $(C(X), d)$.

September 8th, 2022

September 1st, 2022 (Practice)

해석개론 1 복습

1. Real Number System

Let $A \subseteq \mathbb{R}$.

- $b \in \mathbb{R}$ is an upper bound of A : $\forall a \in A \implies a \leq b$.
- $b \in \mathbb{R}$ is a lower bound of A : $\forall a \in A \implies a \geq b$.
- Least upper bound is denoted as $\sup A$.
- Greatest lower bound is denoted as $\inf A$.
- Least upper bound property: If $A \neq \emptyset$, $\exists \sup A$.
- Extended Real Numbers: $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$
- Now, if $\emptyset = A \subseteq \overline{\mathbb{R}}$, $\sup A = -\infty$.

2. Metric Spaces

Metric space: (X, d_X) where $d_X : X \times X \rightarrow \mathbb{R}$. For all $x, y, z \in X$ the following must hold.

- (1) $d_X(x, y) = 0 \iff x = y$.
- (2) $d_X(x, y) = d_X(y, x)$ (Symmetric)
- (3) $d_X(x, y) + d_X(y, z) \geq d_X(x, z)$

Notation. (Neighborhood) Ball of radius r , centered at p is denoted as

$$B_r(p) = \{x \in X \mid d_X(x, p) < r\}$$

- $U \subseteq X$ is open $\iff \forall p \in U, \exists r > 0$ such that $B_r(p) \subseteq U$.
- $C \subseteq X$ is closed $\iff C$ contains every limit point of C . Or alternatively, C^C is open.
- Union of open sets is open, finite intersection of open sets is open.
- $p \in B \subseteq X$ is a limit point of $B \iff \forall r \geq 0, (B_r(p) \setminus \{p\}) \cap B \neq \emptyset$.⁹
- A' is the set of limit points of A .

⁹임의의 근방에서 자기자신을 제외하고 B 의 점이 존재한다.

- $\overline{A} = A \cup A'$, which is the smallest closed set containing A .
- $A \subseteq X$ is dense in $X \iff \overline{A} = X$.
- $A \subseteq X$ is bounded $\iff \exists r > 0$ such that $A \subseteq B_r(p)$ for some $p \in X$.
- Sets A and B are separated $\iff \overline{A} \cap B = \emptyset = A \cap \overline{B}$.
- Set C is disconnected $\iff \exists$ non-empty separated sets A, B such that $C \subseteq A \cup B$.

Suppose $\{U_\alpha\}$ is a collection of open sets in X .

- $\{U_\alpha\}$ is an open cover of $A \iff A \subseteq \bigcup_{\alpha} U_\alpha$.
- $K \subseteq X$ is compact \iff for every open cover of K , there exists a finite subcover of K .

$$\exists \alpha_1, \alpha_2, \dots, \alpha_n \text{ such that } K \subseteq \bigcup_{k=1}^n U_{\alpha_k}$$

- (Heine-Borel) In \mathbb{R}^n , compact \iff bounded and closed.
- If K is compact and $A \subseteq K$ is closed, then A is also compact.
- If $\{K_\alpha\}$ is a collection of compact sets and $\bigcap_{\alpha} K_\alpha = \emptyset$, then

$$\exists \alpha_1, \alpha_2, \dots, \alpha_n \text{ such that } \bigcap_{k=1}^n K_{\alpha_k} = \emptyset. \text{ }^{10}$$

3. Sequences

A sequence $a : \mathbb{N} \rightarrow A$, is a function. We write $a(i) = a_i$, and we usually consider sequences in metric spaces.

- $\{a_n\}$ converges to $\alpha \iff \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N \implies d_X(a_n, \alpha) < \epsilon$.
- (Cauchy Sequence) $\{a_n\}$ is Cauchy
 $\iff \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n, m \geq N \implies d_X(a_n, a_m) < \epsilon$.
- (X, d) is complete \iff every Cauchy sequence converges.¹¹
- $\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\}$.
- $\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\}$.

¹⁰정의로 쉽게 보일 수 있다?

¹¹수렴하면 코시 수열이지만, 모든 코시 수열이 수렴하지는 않는다. Consider any sequence of rational numbers converging to an irrational real number.

- $\lim a_n = \alpha \iff \limsup a_n = \liminf a_n = \alpha$ ($\alpha \in \mathbb{R}$).
- For power series $\sum a_n x^n$, the radius of convergence $R \in \overline{\mathbb{R}}$ is calculated as

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

- Absolute convergence implies convergence.

4. Limit of Functions

Given metric spaces X, Y , define a function $f : E \subseteq X \rightarrow Y$.

- If $p \in E$ ¹² then we can define $\lim_{x \rightarrow p} f(x) = \alpha$ as

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < d_X(x, p) < \delta \implies d_Y(f(x), \alpha) < \epsilon.$$

Or equivalently, for any sequence $\{a_n\}$ in X with $a_n \neq p$,

$$\text{if } \lim_{n \rightarrow \infty} a_n = p \text{ then } \lim_{n \rightarrow \infty} f(a_n) = \alpha.$$

- f is continuous at $p \in E$ ¹³ \iff

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } x \in E, d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \epsilon.$$

Or equivalently, for any sequence $\{a_n\}$ in X ,¹⁴

$$\text{if } \lim_{n \rightarrow \infty} a_n = p \text{ then } \lim_{n \rightarrow \infty} f(a_n) = f(p).$$

- f is continuous \iff for any open set $V \subseteq Y$, $f^{-1}(V)$ is open in X .
- Suppose that f is continuous.
 - If $K \subseteq E$ is compact, $f(K)$ is also compact.
 - If $C \subseteq E$ is connected, $f(C)$ is also connected.
- (Extreme Value Theorem) Suppose $K \subseteq E$ is compact and $f : K \rightarrow \mathbb{R}$ is continuous. Because $f(K)$ is a compact set in \mathbb{R} , it is a closed interval. Hence f has a maximum/minimum.
- (Uniform Continuity) f is uniformly continuous on $E \iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, \forall y \in E, d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

¹²함수의 극한은 극한점에서 논한다! 다가갈 점들이 있어야 하지 않겠는가?

¹³Limit point가 아니어도 정의할 수 있으며, 고립점에서는 연속이다.

¹⁴여기서는 $a_n \neq p$ 조건이 빠진다.

- If $f : E \subseteq X \rightarrow Y$ is continuous and E is compact, f is uniformly continuous.

5. Differentiation

Function $f : [a, b] \rightarrow \mathbb{R}$ is differentiable at $x \in [a, b] \iff$

$$\text{the limit } f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \text{ exists.}$$

- If f is differentiable at $x = p$, then f is continuous at $x = p$.
- If f is differentiable at $x = p$ and $g : f([a, b]) \rightarrow \mathbb{R}$ is differentiable at $x = f(p)$
 $\implies g \circ f$ is differentiable at $x = p$ and

$$(g \circ f)'(p) = g'(f(p))f'(p).$$

- (Fermat) If f is differentiable and has a local extremum at $x = a$, then $f'(a) = 0$.
- (Mean Value Theorem) If f is continuous on $[a, b]$ and differentiable on (a, b) , there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

6. Integration

Given a bounded function $f : [a, b] \rightarrow \mathbb{R}$, a partition $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ and a monotonically increasing function $\alpha : [a, b] \rightarrow \mathbb{R}$, define

$$U(P, f, \alpha) = \sum_{i=0}^{n-1} \sup_{x \in [x_i, x_{i+1}]} f(x) (\alpha(x_{i+1}) - \alpha(x_i))$$

$$L(P, f, \alpha) = \sum_{i=0}^{n-1} \inf_{x \in [x_i, x_{i+1}]} f(x) (\alpha(x_{i+1}) - \alpha(x_i))$$

We define upper integral and lower integral as follows:

$$\overline{\int_a^b} f d\alpha = \inf_{P \in \mathcal{P}[a, b]} U(P, f, \alpha) \quad \underline{\int_a^b} f d\alpha = \sup_{P \in \mathcal{P}[a, b]} L(P, f, \alpha).$$

f is Stieltjes integrable with respect to $\alpha \iff$

$$\forall \epsilon > 0, \exists P \in \mathcal{P}[a, b] \text{ such that } U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Or equivalently, $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$. We write $f \in \mathcal{R}(\alpha)$.

Supplementary Material

F is a field for this section.

Definition. (Vector Space) A set V with addition $+: V \times V \rightarrow V$ and scalar multiplication $\cdot: F \times V \rightarrow V$ is a vector space over F if the following properties hold.

- (1) (Associativity of $+$) $u + (v + w) = (u + v) + w$ for all $v, w, u \in V$.
- (2) (Commutativity of $+$) $v + w = w + v$ for all $v, w \in V$.
- (3) (Identity of $+$) $\exists 0_V \in V$ such that $v + 0 = 0 + v = v$ for all $v \in V$.
- (4) (Inverse of $+$) For each $v \in V$, $\exists x \in V$ such that $v + x = x + v = 0_V$.
- (5) (Identity of \cdot) $1v = v$ for $v \in V$, where $1 \in F$ is the multiplicative identity in F .
- (6) (Distributive Property of \cdot w.r.t. Vector $+$) For $a \in F$ and $v, w \in V$, $a(v + w) = av + aw$.
- (7) (Distributive Property of \cdot w.r.t. Field $+$) For $a, b \in F$ and $v \in V$, $(a + b)v = av + bv$.
- (8) (Compatibility of \cdot w.r.t. $+$) $a(bv) = (ab)v$ for $a, b \in F$, $v \in V$.

We write $V = (V, +, \cdot)$.

Definition. (Normed Vector Space) A vector space V with a norm $\|\cdot\|: V \rightarrow \mathbb{R}$ is a normed vector space if the following properties hold.

- (1) $\|v\| \geq 0$ for all $v \in V$.
- (2) $\|v\| = 0 \iff v = 0$.
- (3) For all $\alpha \in F$ and $v \in V$, $\|\alpha v\| = |\alpha| \|v\|$.
- (4) (Triangle Inequality) For all $v, w \in V$, $\|v + w\| \leq \|v\| + \|w\|$.

For inner product spaces, $F = \mathbb{C}$ or $F = \mathbb{R}$.

Definition. (Inner Product Space) A vector space V with an inner product $\langle \cdot, \cdot \rangle: V \times V \rightarrow F$ is an inner product space if the following properties hold.

- (1) (Linearity in the first argument) For $x, y, z \in V$ and $a, b \in F$, $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$.
- (2) (Conjugate Symmetry) For $x, y \in V$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
- (3) (Positive Definiteness) If $0 \neq x \in V$, $\langle x, x \rangle > 0$.

Remark. An inner product can induce a norm by $\|v\| = \sqrt{\langle v, v \rangle}$. With norm as the distance metric, the following holds.

$$\text{Inner Product Space} \implies \text{Normed Vector Space} \implies \text{Metric Space}$$

If the inner product space is complete with respect to the distance metric, it is said to be a Hilbert space.

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