해석개론 및 연습 2 과제 #6

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1. Let $A = I_1 \times I_2 \times \cdots \times I_p \subseteq \mathbb{R}^p$ where $I_i \subseteq \mathbb{R}$ and I_i has endpoints $a_i, b_i \in \mathbb{R}$, $a_i \leq b_i$ for $i = 1, 2, \ldots, p$. I_i can be any type of intervals - $[a, b], (a, b), (a, b], [a, b)$. Now let $\epsilon > 0$ be given.

Suppose that $m(A) < \epsilon$. We can take $F = \varnothing$, and then F is closed and satisfies $m(A) \leq m(F) + \epsilon$. Now assume that $m(A) \geq \epsilon$. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = m\left(\prod_{i=1}^{p} [a_i + x, b_i - x]\right) = \prod_{i=1}^{p} (b_i - a_i - 2x),$$

where $0 \leq x \leq M = \min_{1 \leq i \leq p} \frac{|b_i - a_i|}{2}$. Note that $a_i \neq b_i$ for all i since $m(A) \geq \epsilon \neq 0$. It is trivial that f is continuous and decreasing. Since $f(0) = m(A), f(M) = 0$, there exists $c \in (0, M)$ such that $f(c) = m(A) - \epsilon$. (Intermediate Value Theorem) With this c, construct a closed box $F = \prod_{i=1}^{p} [a_i + c, b_i - c]$. Then $F \subseteq A$ and $m(A) \leq m(F) + \epsilon$ holds.

To find an open set G, consider a function $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = m\left(\prod_{i=1}^{p} (a_i - x, b_i + x)\right) = \prod_{i=1}^{p} (b_i - a_i + 2x),$$

where $x \geq 0$. In a similar manner, g is continuous and increasing, diverges to ∞ as $x \to \infty$. Thus there exists $d \in (0, \infty)$ such that $g(d) = m(A) + \epsilon$. With this d, construct an open box $G = \prod_{i=1}^{p} (a_i - d, b_i + d)$. Then $A \subseteq G$ and $m(G) - \epsilon \leq m(A)$ holds.

Since any elementary set is a finite union of disjoint intervals, take open sets/closed sets for each interval as above, and union open sets/closed sets respectively. Since m is additive on Σ, it can be concluded that m is regular on Σ.

2. It is enough to show the following claim, since intervals are m-measurable.

Claim. $f^{-1}\big((a, \infty]\big) = \{x : f(x) > a\}$ is an interval.

Proof. Suppose that $t \in f^{-1}\big((a, \infty]\big)$. Then for all $u \geq t$, $a < f(t) \leq f(u) < \infty$. So $u \in f^{-1}\big((a, \infty]\big)$. Thus this interval is one of $(-\infty, \infty)$, (z, ∞), $[z, \infty)$. (z is some constant)

3. Define $N = \{x \in \mathbb{R} : f(x) \neq g(x)\}$. Then $m(N) = 0$, and by the completeness of the Lebesgue measure, any subset of N is measurable and has measure 0. Now we show that g is measurable. Take any $a \in \mathbb{R}$.

$$\begin{aligned}
\{x \in \mathbb{R} : g(x) > a\} &= \{x \in \mathbb{R} \setminus N : g(x) > a\} \cup \{x \in N : g(x) > a\} \\
&= \{x \in \mathbb{R} \setminus N : f(x) > a\} \cup \{x \in N : g(x) > a\}
\end{aligned}$$

Since $\{x \in N : g(x) > a\} \subseteq N$, the set $\{x \in N : g(x) > a\}$ is measurable and has measure zero. Also, $\{x \in \mathbb{R} \setminus N : f(x) > a\} = \{x \in \mathbb{R} : f(x) > a\} \setminus \{x \in N : f(x) > a\}$ is measurable. (σ-algebra) Thus $\{x \in \mathbb{R} : g(x) > a\}$ is measurable for all $a \in \mathbb{R}$.

4. $f_n(x)$ converges $\iff \forall M > 0, \exists N \in \mathbb{N}$ such that $m, n \geq N \implies |f_n(x) - f_m(x)| < \frac{1}{M}$. The set C which $f_n(x)$ converges can be written as

$$C = \bigcap_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcap_{m=N}^{\infty} \left\{x : |f_n - f_m| < \frac{1}{M}\right\}.$$

We take intersections for M, n, m since $|f_n - f_m| < \frac{1}{M}$ has to hold for all values of M, n, m. However, we take unions for N since we have to include all N if such N can make $|f_n - f_m| < \frac{1}{M}$ hold.

Since f_n are measurable, $f_n - f_m$ is measurable and $|f_n - f_m|$ is also measurable. Thus the set $\left\{x : |f_n - f_m| < \frac{1}{M}\right\}$ is measurable, and its countable union C is measurable.

5. Define $E_n = \{x \in E : f(x) > \frac{1}{n}\}$ for $n \in \mathbb{N}$, and let $A = \bigcup_{n=1}^{\infty} E_n$. Since $\mu(E_n) \leq \mu(A) \leq \sum_{n=1}^{\infty} \mu(E_n)$, we show that $\mu(A) = 0$ by showing that $\mu(E_n) = 0$ for all $n \in \mathbb{N}$. Suppose that $\mu(E_n) > 0$ for some $n \in \mathbb{N}$. Then

$$\int_E f\, d\mu = \int_{E \setminus E_n} f\, d\mu + \int_{E_n} f\, d\mu \geq \int_{E_n} f\, d\mu \geq \int_{E_n} \frac{1}{n}\, d\mu = \frac{\mu(E_n)}{n} > 0,$$

leading to a contradiction. Thus $\mu(E_n) = 0$ for all $n \in \mathbb{N}$, which leads to $\mu(A) = 0$. Therefore $\{x \in E : f(x) > 0\}$ is a measure zero set.

6. Since $\int_0^1 g(1-x)\, dx = \int_0^1 g(x)\, dx$, $\int_0^1 f_n(x)\, dx = \int_0^1 g(x)\, dx = \frac{1}{2}$ regardless of the parity of n. Check that

$$\{f_n(x)\} = \begin{cases} 1, 0, 1, 0, 1, \ldots & (x \in [0, 1/2]) \\ 0, 1, 0, 1, 0, \ldots & (x \in (1/2, 1]) \end{cases}$$

Therefore for $0 \leq x \leq 1$, $\inf_{k \geq n} f_k(x) = 0$ for any $n \in \mathbb{N}$. So we can conclude that $\liminf_{n \to \infty} f_n(x) = 0$.

7. We show that $N = \{x \in E : f(x) \neq 0\}$ has measure zero. Let $S_1 = \{x \in E : f(x) > 0\}$, then S_1 is measurable. So $\int_{S_1} f\, d\mu = 0$. Since $f \geq 0$ on S_1, we use Problem 5 and conclude that $\mu(S_1) = 0$. Similarly, define $S_2 = \{x \in E : f(x) < 0\}$. Then S_2 is measurable, so $\int_{S_2} f\, d\mu = 0$. Since $-f \geq 0$ on S_2, we use Probelm 5 again and conclude that $\mu(S_2) = 0$. Therefore, $\mu(N) = \mu(S_1) + \mu(S_2) = 0$, and $f(x) = 0$ μ-almost everywhere on E.

8. ($\Longrightarrow$) For any $A \in \mathfrak{M}$, set $N_A = \{x \in A : f(x) \neq g(x)\}$. Since $\mu(N_A) = 0$, $\int_{N_A} f\,d\mu = \int_{N_A} g\,d\mu = 0$. Now we have

$$\int_A f\,d\mu = \int_{A \setminus N_A} f\,d\mu + \int_{N_A} f\,d\mu = \int_{A \setminus N_A} g\,d\mu + \int_{N_A} g\,d\mu = \int_A g\,d\mu.$$

($\Longleftarrow$) Since $\int_A f\,d\mu$, $\int_B f\,d\mu$ are finite, ($f, g \in \mathcal{L}^1(X, \mu)$) we have $\int_A (f - g)\,d\mu = 0$ for any $A \in \mathfrak{M}$. By Problem 7, $f - g = 0$ μ-almost everywhere since A is any measurable subset of X. Therefore $f = g$ μ-almost everywhere.