해석개론 및 연습 2 과제 #1

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1. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on X, and that $|f_n| \le M_n$ for all $n \in \mathbb{N}$.

By uniform convergence, we can choose $N \in \mathbb{N}$ such that for $n \ge N$,

$$|f_n(x) - f(x)| < 1, \quad \forall x \in X.$$

Thus, for $x \in X$ and $n \ge N$, we can write

$$|f_n(x)| \le |f_n(x) - f(x)| + |f(x) - f_N(x)| + |f_N(x)| < 2 + M_N.$$

Now set $M = \max\{M_1, M_2, \dots, M_{N-1}, 2 + M_N\}$. Then for all $n \in \mathbb{N}$,

$$|f_n(x)| \le M,$$

which shows that $\{f_n\}$ is uniformly bounded.

2. Suppose that $f_n \to f$, $g_n \to g$ uniformly on E, and let $\epsilon > 0$ be given. By uniform convergence of f_n, g_n, we can choose $N_1, N_2 \in \mathbb{N}$ such that

$$n \ge N_1 \implies |f_n(x) - f(x)| < \frac{\epsilon}{2} \text{ and } n \ge N_2 \implies |g_n(x) - g(x)| < \frac{\epsilon}{2}$$

for all $x \in E$. Set $N = \max\{N_1, N_2\}$, we find that for $n \ge N$,

$$\left|f_n(x) + g_n(x) - \big(f(x) + g(x)\big)\right| \le |f_n(x) - f(x)| + |g_n(x) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $x \in E$. Thus $f_n + g_n$ converges uniformly to $f + g$ on E.

If f_n, g_n are bounded, we know that they are both uniformly bounded by the first problem. Additionally, we know that f is bounded by Theorem 7.15. Thus there exists $F, G \in \mathbb{R}\backslash\{0\}$ such that $|f_n(x)| \le F$, $|f(x)| \le F$ and $|g_n(x)| \le G$, for all $x \in E$.

Let $\epsilon > 0$ be given. Using the uniform convergence of f_n and g_n, we can choose $M_1, M_2 \in \mathbb{N}$ such that

$$n \ge M_1 \implies |f_n(x) - f(x)| < \frac{\epsilon}{2G} \text{ and } n \ge M_2 \implies |g_n(x) - g(x)| < \frac{\epsilon}{2F}$$

for all $x \in E$. Set $M = \max\{M_1, M_2\}$, we find that for $n \ge M$,

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)g_n(x) - f(x)g_n(x) + f(x)g_n(x) - f(x)g(x)| \\ &\le |g_n(x)|\,|f_n(x) - f(x)| + |f(x)|\,|g_n(x) - g(x)| \\ &\le G \cdot \frac{\epsilon}{2G} + F \cdot \frac{\epsilon}{2F} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

for all $x \in E$. Thus $f_n g_n$ converges uniformly to fg on E.

3. It is easy to see that $f_n \to f \equiv 0$, making f a continuous function. But the convergence is not uniform. For instance, take $\epsilon = 1/2$. For all $n \in \mathbb{N}$, there exists some $x \in \mathbb{R}$ such that

$$|f_n(x) - f(x)| = |f_n(x)| \geq \frac{1}{2}.$$

$x_0 = \dfrac{1}{n+1/2}$ is such x, because

$$|f_n(x_0)| = \sin^2 \frac{\pi}{x_0} = \sin^2\left(n\pi + \frac{\pi}{2}\right) = 1 \geq \frac{1}{2}.$$

Now we calculate $\sum f_n(x)$. For $x \leq 0$, $x \geq 1$, $\sum f_n(x) = 0$.

For $x \in (0,1)$,

(a) If $x = \dfrac{1}{N}$ for some $N \in \mathbb{N}$, $f_n(x) = 0$ for all $n \in \mathbb{N}$.

(b) Otherwise, $\exists\, N \in \mathbb{N}$ such that $\dfrac{1}{N+1} < x < \dfrac{1}{N}$. Then

$$f_n(x) = \begin{cases} \sin^2 \dfrac{\pi}{x} & (n = N) \\ 0 & (n \neq N) \end{cases}.$$

Thus, $\sum f_n(x) = \sin^2 \dfrac{\pi}{x}$ for $x \in (0,1)$. Overall,

$$f(x) = \sum f_n(x) = \begin{cases} \sin^2 \dfrac{\pi}{x} & (x \in (0,1)) \\ 0 & (\text{otherwise}) \end{cases}.$$

Since all the terms are non-negative, the series converges absolutely.

If $\sum f_n(x)$ were to converge uniformly to f, f should have been continuous. But since f is not continuous at $x = 0$, $\sum f_n(x)$ cannot converge uniformly.

4. The given series can be considered as the sum of the two following series

$$A(x) = x^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \quad B = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

because both $A(x)$ and B converge, by the alternating series test.

Let $a > 0$ and set $X = [-a, a]$. It is sufficient to show uniform convergence for X, because a can be chosen arbitrarily large so that it would contain any bounded interval.

$A(x)$ converges uniformly on X by using the Weierstrass M-test with $M_n = \dfrac{a^2}{n^2}$. However, the given series does not converge absolutely by the comparison test since

$$\left|(-1)^n \frac{x^2+n}{n^2}\right| \geq \frac{n}{n^2} = \frac{1}{n}$$

and the harmonic series diverges.

5. Since $f_n(0) = 0$, $f(0) = 0$. Now consider the case $x \neq 0$. For $\epsilon > 0$, choose $N = \dfrac{1}{4\epsilon^2}$. Then for $n \geq N$,

$$|f_n(x) - 0| = \frac{1}{\left|nx + \dfrac{1}{x}\right|} \leq \frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{N}} = \epsilon, \quad (x \in \mathbb{R} \setminus \{0\})$$

(AM-GM inequality was used in the first inequality) Thus $f(x) = 0$ also for $x \neq 0$. f_n converges uniformly to $f(x) = 0$ for $x \in \mathbb{R}$.

We directly calculate $f_n'(x)$ and get

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}.$$

We see that

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 1 & (x = 0) \\ 0 & (x \neq 0) \end{cases},$$

whereas $f'(x) = 0$. The given equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is false only for $x = 0$.

6. We directly see that

$$|c_n I(x - x_n)| \leq |c_n|$$

and since $\sum |c_n| < \infty$, the given series converges uniformly on $[a, b]$ by the Weierstrass M-test.

Define the partial sum $s_n(x) = \displaystyle\sum_{k=1}^{n} c_k I(x - x_k)$. We know that $s_n(x)$ is already continuous at $x_0 \neq x_n$, since each term in the sum is continuous at x_0 and the sum is finite. Thus

$$\lim_{t\to x_0} s_n(t) = s_n(x_0).$$

Since $s_n(x)$ converges uniformly to $f(x)$, $\{s_n(x_0)\}$ converges to $f(x_0)$.

By Theorem 7.11, (the conditions for the theorem are indeed satisfied)

$$\lim_{t\to x_0} f(t) = \lim_{n\to\infty} \lim_{t\to x_0} s_n(t) = \lim_{n\to\infty} s_n(x_0) = f(x_0),$$

showing that f is continuous for every $x_0 \neq x_n$.

7. We are given that $\{f_n\}$ is a sequence of continuous functions converging uniformly to f. We know that f is continuous on E. Let $\epsilon > 0$ be given.

First, we choose $N_1 \in \mathbb{N}$ such that for $n \geq N_1$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}, \quad \forall x \in E.$$

Next, since f_n is continuous at $x \in E$, (the limit of x_n) there exists $\delta > 0$ such that

$$|y - x| < \delta \implies |f_n(y) - f_n(x)| < \frac{\epsilon}{2}, \quad \forall y \in E.$$

Lastly, since $x_n \to x$, we choose choose $N_2 \in \mathbb{N}$ such that for $n \geq N_2$,

$$|x_n - x| < \delta,$$

which implies that

$$|f_n(x_n) - f_n(x)| < \frac{\epsilon}{2}.$$

Therefore, for $n \geq \max\{N_1, N_2\}$,

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is equivalent to $\lim_{n\to\infty} f_n(x_n) = f(x)$.

The converse is not true. Consider the function f_n given in Problem #3, with the domain restricted to $E = [0, 1]$. First of all, f_n does not converge uniformly to $f \equiv 0$ on E.

Now, consider any sequence $\{x_n\} \to x \in E$. For very small $\epsilon > 0$ such that $x - \epsilon > 0$, choose $N \in \mathbb{N}$ such that for $n \geq N_1$, $|x_n - x| < \epsilon$. Then, we see that for $n \geq N_1$, $0 < x - \epsilon < x_n$.

Therefore we can choose M large enough so that $x - \epsilon > \dfrac{1}{M}$. Then $f_M(x - \epsilon) = 0$. Setting $N = \max\{N_1, M\}$ will give $f_n(x_n) = 0$ for $n \geq N$.

Hence, $\lim_{n\to\infty} f_n(x_n) = f(x)$ holds, but f_n does not converge uniformly.

8. First we prove the following lemma.

Lemma. *Given two sequences $\{a_n\}, \{b_n\}$ and a partial sum $A_n = \sum_{k=1}^{n} a_k$, (define $A_0 = 0$) the following holds for $m < n$.*

$$\sum_{k=m+1}^{n} a_n b_n = A_n b_{n+1} - A_m b_{m+1} - \sum_{k=m+1}^{n} A_k \big(b_{k+1} - b_k\big) \qquad (*)$$

Proof of Lemma. Observe that

$$\begin{aligned}
\sum_{k=1}^{n} a_k b_k &= \sum_{k=1}^{n} (A_k - A_{k-1}) b_k \\
&= \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1} \\
&= A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k).
\end{aligned}$$

For $m < n$, we use the result above to get

$$\sum_{k=m+1}^{n} a_k b_k = \sum_{k=1}^{n} a_k b_k - \sum_{k=1}^{m} a_k b_k = A_n b_{n+1} - A_m b_{m+1} - \sum_{k=m+1}^{n} A_k (b_{k+1} - b_k),$$

which was what we wanted. $\square$

Suppose that $f_n, g_n : E \to \mathbb{R}$. Define a partial sum of f_k as $F_n(x) = \sum_{k=1}^{n} f_k(x)$. From the assumption,

- There exists $M > 0$ such that $|F_n(x)| < M$ for all $n \in \mathbb{N}$.
- For large enough $m \in \mathbb{N}$, we can make $|g_n(x)|$ arbitrarily small.

Now we show that the partial sums of $\sum f_n g_n$ is a Cauchy sequence. For $m < n$,

$$\begin{aligned}
\left| \sum_{k=m+1}^{n} f_n g_n \right| &\overset{(*)}{=} \left| F_n g_{n+1} - F_m g_{m+1} - \sum_{k=m+1}^{n} F_k \big(g_{k+1} - g_k \big) \right| \\
&\leq |F_n| \, |g_{n+1}| + |F_m| \, |g_{m+1}| + \sum_{k=m+1}^{n} |F_k| \, (g_k - g_{k+1}) \\
&\leq M \Big(|g_{n+1}| + |g_{m+1}| + \sum_{k=m+1}^{n} (g_k - g_{k+1}) \Big) \\
&= M \Big(|g_{n+1}| + |g_{m+1}| + g_{m+1} - g_{n+1} \Big) \leq 4M \, |g_{n+1}| \, .
\end{aligned}$$

The variable x was omitted for the sake of brevity, and the third assumption $g_k(x) - g_{k+1}(x) \geq 0$ was used in the second line.

Finally, for any $\epsilon > 0$, choose $N \in \mathbb{N}$ large enough so that $|g_{n+1}(x)| < \epsilon/4M$. Then

$$\left| \sum_{k=m+1}^{n} f_n(x) g_n(x) \right| \leq 4M \, |g_{n+1}(x)| < 4M \cdot \frac{\epsilon}{4M} = \epsilon,$$

which shows that $\sum f_n(x) g_n(x)$ converges uniformly on E.