HW Set 1. (Due day: September 16, 23:59)

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded (See Definition 7.19).

2. If complex-valued (f_n) and (g_n) converge uniformly on a metric space E, prove that $(f_n + g_n)$ converges uniformly on E. If, in addition, (f_n) and (g_n) are sequences of bounded complex-valued functions, prove that $(f_n g_n)$ converges uniformly on E.

3. Let $f_n(x) = \begin{cases} 0 & \text{if } x < \frac{1}{n+1}, \\ \sin^2 \frac{\pi}{x} & \text{if } \frac{1}{n+1} \le x \le \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < x. \end{cases}$

   Show that (f_n) converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

4. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

   converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

5. For $n = 1, 2, 3, \cdots$, and any $x \in \mathbb{R}$, define

$$f_n(x) = \frac{x}{1 + nx^2}.$$

   Show that (f_n) converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n \to \infty} f'_n(x)$$

   is correct if $x \ne 0$, but false if $x = 0$.

6. Define $I(x) = \begin{cases} 0 & \text{if } x \le 0, \\ 1 & \text{if } x > 0. \end{cases}$ Suppose that (x_n) is a sequence of distinct points of (a, b) and that $\sum |c_n|$ converges. Prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \le x \le b)$$

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converges uniformly, and that f is continuous for every $x \neq x_n$.

7. Let (f_n) be a sequence of continuous functions which converges uniformly to a function f on a metric space E. Prove that

$$\lim_{n \to \infty} f_n(x_n) = f(x) \tag{1}$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$.
Is the converse of this true?(i.e. if a sequence of continuous functions (f_n) on a metric space E satisfies (1) for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$, then does (f_n) necessarily converge uniformly to f?)

8. Suppose (f_n) and (g_n) are real-valued functions defined on a metric space E, and

   - $\sum f_n$ has uniformly bounded partial sums;
   - $g_n \to 0$ uniformly on E;
   - $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

   Prove that $\sum f_n g_n$ converges uniformly on E. (*Hint*: Compare with Theorem 3.42.)