해석개론 및 연습 2 과제 #7

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1. We show that $\int_0^\infty s(x)\,dx < \infty$. Define $g(0) = 1$ and $g(x) = s(x)$ if $x \neq 0$. Then $\int_0^\infty s(x)\,dx = \int_0^\infty g(x)\,dx$, since

$$\int_0^\infty s(x)\,dx = \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \int_\epsilon^N s(x)\,dx = \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \int_\epsilon^N g(x)\,dx = \int_0^\infty g(x)\,dx.$$

$g(x)$ is bounded by 1, so $\int_0^A g(x)\,dx$ converges for $A > 0$. Now fix $A > 0$, and as for $\int_A^B g(x)\,dx$ $(B > A)$,

$$\int_A^B \frac{\sin x}{x}\,dx = \Big[-\frac{\cos x}{x}\Big]_A^B - \int_A^B \frac{\cos x}{x^2}\,dx.$$

So $\int_A^B \frac{\sin x}{x}\,dx < \infty$ if the right hand side converges as $B \to \infty$. Since

$$\lim_{B\to\infty}\left[\frac{\cos A}{A} - \frac{\cos B}{B}\right] = \frac{\cos A}{A}, \quad \int_A^\infty \frac{\cos x}{x^2}\,dx < \int_A^\infty \frac{1}{x^2}\,dx < \infty,$$

we have $\int_A^\infty \frac{\sin x}{x}\,dx < \infty$. Therefore $\int_0^\infty s(x)\,dx < \infty$. However, for $N \in \mathbb{N}$,

$$\begin{aligned}\int_0^\infty |s(x)|\,dx &\geq \int_0^{2\pi N} \left|\frac{\sin x}{x}\right|\,dx = \sum_{k=1}^N \int_{2(k-1)\pi}^{2k\pi} \left|\frac{\sin x}{x}\right|\,dx \\ &\geq \sum_{k=1}^N \frac{1}{2\pi k}\int_{2(k-1)\pi}^{2k\pi} |\sin x|\,dx = \sum_{k=1}^N \frac{1}{2\pi k}\int_0^{2\pi} |\sin x|\,dx = \sum_{k=1}^N \frac{2}{\pi k},\end{aligned}$$

which diverges to $+\infty$ as $N \to \infty$. Thus $s(x) \notin \mathcal{L}$ on $(0, \infty)$.

2. Rewrite $f(x)$ as

$$f(x) = \big(\log(m+1) - \log m\big)(x - m) + \log m$$

to see that $f(x)$ consists of line segments that connect $(m, \log m)$ and $(m+1, \log(m+1))$ for $m = 1, 2, \ldots$. Also, $g(x) = \frac{1}{m}(x - m) + \log m$ is the tangent line of $y = \log x$ at $(m, \log m)$, restricted to $\big[m - \frac{1}{2}, m + \frac{1}{2}\big)$. We additionally know that $\log x$ is concave, so $f(x) \leq \log x \leq g(x)$ for $x \geq 1$. Using the above results, the graphs of f and g should look like Figure 1.

Now by direct calculation,

$$\begin{aligned}\int_1^n f(x)\,dx &= \sum_{m=1}^{n-1} \int_m^{m+1} f(x)\,dx = \sum_{m=1}^{n-1} \left[-\frac{\log m}{2}(m+1-x)^2 + \frac{\log(m+1)}{2}(x-m)^2\right]_m^{m+1} \\ &= \sum_{m=1}^{n-1} \left[\frac{\log(m+1)}{2} + \frac{\log m}{2}\right] = \log n! - \frac{1}{2}\log n.\end{aligned}$$

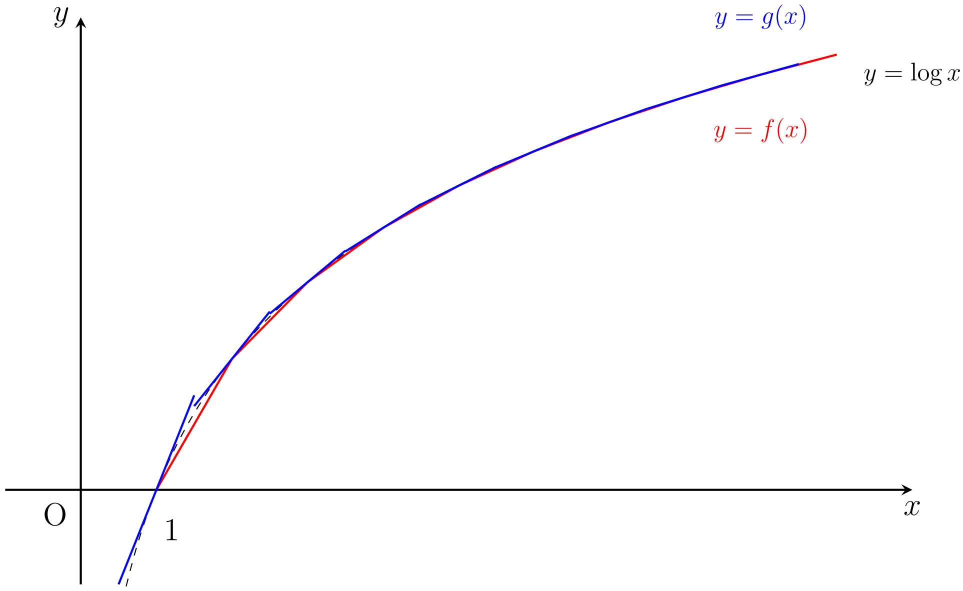

Figure 1: Graph of $\log x$ in dashed lines, $f(x)$ in red, $g(x)$ in blue.

Also,

$$\int_{m-1/2}^{m} g(x)\,dx = \frac{1}{2}\log m - \frac{1}{8m}, \qquad \int_{m}^{m+1/2} g(x)\,dx = \frac{1}{2}\log m + \frac{1}{8m},$$

so $\displaystyle\int_{m-1/2}^{m+1/2} g(x)\,dx = \log m$. Therefore,

$$\begin{aligned}\int_1^n g(x)\,dx &= \int_1^{3/2} g(x)\,dx + \sum_{m=2}^{n-1}\int_{m-1/2}^{m+1/2} g(x)\,dx + \int_{n-1/2}^{n} g(x)\,dx \\ &= \frac{1}{8} + \log(n-1)! + \frac{1}{2}\log n - \frac{1}{8n} = \log n! - \frac{1}{2}\log n + \frac{1}{8} - \frac{1}{8n} \\ &= \int_1^n f(x)\,dx + \frac{1}{8} - \frac{1}{8n}. \qquad (*)\end{aligned}$$

By $(*)$,

$$\int_1^n f(x)\,dx = -\frac{1}{8} + \frac{1}{8n} + \int_1^n g(x)\,dx > -\frac{1}{8} + \int_1^n g(x)\,dx.$$

For $n \geq 2$, integrating $f(x) \leq \log x \leq g(x)$ over $[1, n]$ gives

$$\log n! - \frac{1}{2}\log n = \int_1^n f(x)\,dx < n\log n - n + 1 < \int_1^n g(x)\,dx < \frac{1}{8} + \log n! - \frac{1}{2}\log n.$$

Subtracting $n\log n - n$ from all sides and a bit of reordering terms will give

$$\log n! - \left(n + \frac{1}{2}\right)\log n + n < 1, \quad 1 - \frac{1}{8} < \log n! - \left(n + \frac{1}{2}\right)\log n + n$$

which is the desired inequality. (The equalities were dropped because for $n \geq 2$, it is evident that the areas under the curve $f(x), \log x, g(x)$ from $x = 1$ to $x = n$ are different)

Finally, it suffices to show that $\exp\left(\log n! - \left(n + \frac{1}{2}\right)\log n + n\right) = \frac{n!}{(n/e)^n\sqrt{n}}$.

$$\exp\left(\log n! - \left(n + \frac{1}{2}\right)\log n + n\right) = \frac{\exp(\log n!)\cdot\exp(n)}{\exp(n\log n)\cdot\exp(\log\sqrt{n})} = \frac{n!\cdot e^n}{n^n\sqrt{n}} = \frac{n!}{(n/e)^n\sqrt{n}}.$$

3. By Theorem 6.20, if $f(x)$ is continuous at x_0, then $F(x)$ is differentiable at x_0 and $F'(x_0) = f(x_0)$. Also by Theorem 11.33 (b), $f \in \mathcal{R}$ so f is continuous almost everywhere. Let $N = \{x \in [a,b] : f(x) \text{ is discontinuous at } x\}$ then $m(N) = 0$. On $[a,b] \setminus N$, $F(x)$ is differentiable and $F'(x) = f(x)$. Thus $F'(x) = f(x)$ almost everywhere.

4. Take any sequence $\{x_n\}$ in $[a,b]$, that converges to $x \in [a,b]$. Define $f_n = \chi_{[a,x_n]} f$ then f_n is a sequence of measurable functions (χ, f are measurable), dominated by $|f| \in \mathcal{L}$. We can see that $f_n \to \chi_{[a,x]} f$ as $n \to \infty$ almost everywhere. (Possibly except for x, but a point has measure 0) Upon direct calculation,

$$\lim_{n\to\infty} \int_a^b f_n(t)\,dt = \lim_{n\to\infty} \int_a^{x_n} f(t)\,dt = \lim_{n\to\infty} F(x_n), \qquad \int_a^b \chi_{[a,x]} f\,dt = \int_a^x f(t)\,dt = F(x).$$

By Lebesgue's dominated convergence theorem, $\lim\limits_{n\to\infty} F(x_n) = F(x)$. Since x_n was arbitrary, we can conclude that F is continuous at $x \in [a,b]$.

5. Let $d(f,g) = \int_X |f-g|\,d\mu$ for $f, g \in \mathcal{L}(\mu)$. We first show that $d(\cdot,\cdot)$ is a metric on $\mathcal{L}(\mu)$. For $f, g, h \in \mathcal{L}(\mu)$,

- If $f \sim g$, $d(f,g) = 0$. Otherwise, $d(f,g) = \displaystyle\int_X |f-g|\,d\mu > 0$.
- $d(f,g) = \displaystyle\int_X |f-g|\,d\mu = \int_X |g-f|\,d\mu = d(g,f)$.
- $d(f,g) = \displaystyle\int_X |f-g|\,d\mu \le \int_X \big(|f-h| + |h-g|\big)\,d\mu = d(f,h) + d(h,g)$.

Now we show that $\big(\mathcal{L}(\mu), d\big)$ is complete. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}(\mu)$. Take a sequence $\{n_k\}$ such that $d(f_{n_k}, f_{n_{k+1}}) < \frac{1}{2^k}$ for $k = 1, 2, \ldots$. Then

$$\sum_{k=1}^{\infty} d(f_{n_k}, f_{n_{k+1}}) = \sum_{k=1}^{\infty} \int_X \left|f_{n_k} - f_{n_{k+1}}\right|\,d\mu = \int_X \sum_{k=1}^{\infty} \left|f_{n_k} - f_{n_{k+1}}\right|\,d\mu \le 1,$$

since the partial sums of the series on the left hand side is non-negative and increasing. Monotone convergence theorem was applied to switch the order of summation and integration. Using the lemma covered in class, we can conlude that $\sum_{k=1}^{\infty} \left|f_{n_k} - f_{n_{k+1}}\right| < \infty$ μ-a.e. on X. Therefore $(*) = \sum_{k=1}^{\infty} \big(f_{n_k} - f_{n_{k+1}}\big)$ converges μ-a.e. on X. Let

$$f = \sum_{k=1}^{\infty} \big(f_{n_k} - f_{n_{k+1}}\big) + f_1 = \lim_{k\to\infty} f_{n_k},$$

except for points that $(*)$ does not converge. On the points that $(*)$ does not converge, set $f(x) = 0$. Now we show that $f_n \to f$ as $n \to \infty$. Let $\epsilon > 0$ be given. Take large enough $M \in \mathbb{N}$, so that if $n_t, n_k > M$ then $d(f_{n_t}, f_{n_k}) < \epsilon$. By Fatou's lemma,

$$d(f, f_{n_k}) = \int_X |f - f_{n_k}|\,d\mu = \int_X \liminf_{t\to\infty} |f_{n_t} - f_{n_k}|\,d\mu \le \liminf_{t\to\infty} \int_X |f_{n_t} - f_{n_k}|\,d\mu < \epsilon.$$

Therefore we see that $f - f_{n_k} \in \mathcal{L}(\mu)$, which implies $f \in \mathcal{L}(\mu)$. Also, for large enough k, $d(f, f_{n_k}) < \epsilon$, so the right hand side of

$$d(f, f_n) \leq d(f, f_{n_k}) + d(f_{n_k}, f_n)$$

can be made arbitrarily small by choosing n, n_k large enough. Therefore any Cauchy sequence in $\mathcal{L}(\mu)$ converges, and $\big(\mathcal{L}(\mu), d\big)$ is complete.

6. We show that $\int_X |f_n - f| \, d\mu \to 0$ as $n \to \infty$. Since

$$|f_n - f| \leq |f_n| + |f| \leq |g_n| + |g|$$

μ-a.e. and $|f_n - f|, |g_n|, |g| \in \mathcal{L}$, Define $h = |g| + |g_n| - |f_n - f|$ then $h \in \mathcal{L}$. Note that $\liminf_{n\to\infty} h = 2|g|$. By Fatou's lemma,

$$\int_X 2|g| \, d\mu = \int_X \liminf_{n\to\infty} h \, d\mu \leq \liminf_{n\to\infty} \int_X h \, d\mu = \int_X 2|g| \, d\mu - \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu.$$

Therefore $0 \leq \liminf_{n\to\infty} \int_X |f_n - f| \, d\mu \leq \limsup_{n\to\infty} \int_X |f_n - f| \, d\mu \leq 0$, and thus $\int_X |f_n - f| \, d\mu \to 0$ as $n \to \infty$.

7. $(\implies)$ Observe that

$$\left| \int_X |f_n| \, d\mu - \int_X |f| \, d\mu \right| = \left| \int_X (|f_n| - |f|) \, d\mu \right| \leq \int_X \big||f_n| - |f|\big| \, d\mu \leq \int_X |f_n - f| \, d\mu \to 0$$

as $n \to \infty$. So $\int_X |f_n| \, d\mu \to \int_X |f| \, d\mu$ as $n \to \infty$.

$(\impliedby)$ Set $g_n = |f_n|$ in Problem 6. Then $|f_n| \leq |g_n|$, $\lim_{n\to\infty} g_n = \lim_{n\to\infty} |f_n| = |f| = |g|$ μ-a.e. and $\int_X g_n \, d\mu \to \int_X g \, d\mu$ as $n \to \infty$. All assumptions hold, so we can use the result of Problem 6 to conclude that $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$, which is equivalent to $\int_X |f_n - f| \, d\mu \to 0$ as $n \to \infty$.