Introduction to Analysis II

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Introduction & Notice

- 7, 8장 나가고 중간고사, 11장 나가고 기말고사
- 연습 시간이 있는 수업 (목 6:30 ∼ 8:20)[1]
- 오늘 연습 시간: 지난학기 배운 내용 중 필요한 내용 복습

[1]가능하면 1시간 반 안에 끝내라고 하심 ㅋㅋ

Chapter 7

Sequences and Series of Functions

September 1st, 2022

기본적으로 수열에 관련된 내용, real/complex-valued 수열이 아니라 함수가 주어졌을 때. 함수들을 모은 '**sequence of functions**'의 극한을 생각하는 것.

Suppose E is a set[1], and let $f_n : E \to \mathbb{C}$. Then

$$(f_n)_{n=1}^{\infty}$$

is a sequence of (complex-valued) function.

Definition 7.1 (Pointwise Convergence) $(f_n)_{n=1}^{\infty}$ converges **pointwise** on E, if for each $x \in E$ the sequence $(f_n(x))_{n=1}^{\infty}$ converges in $\mathbb{C}$.

In other words, for each $x \in E$, there exists $a_x \in \mathbb{C}$ and

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } n \geq N \implies |f_n(x) - a_x| < \epsilon.$$

Definition. If (f_n) converges pointwise, we can define a function f by

$$f(x) = \lim_{n \to \infty} f_n(x) \quad (x \in E)$$

We say that

- f is the *limit* or *limit function* of f_n.
- (f_n) to f pointwise on E.

[1] 사실은 *metric* space 이다.

Definition. If $\sum f_n(x)$ converges (pointwise) for every $x \in E$, we can define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E)$$

and the function f is called the *sum* of the series $\sum f_n$.

Recall. $f : (E, d) \to \mathbb{C}$ is continuous on $E \iff f$ is continuous at all $x \in E$.

Recall. (Theorem 4.6) If $p \in E$ and p is a limit point of E,

$$f \text{ is continuous at } p \iff \lim_{x \to p} f(x) = f(p)$$

Question. Suppose (f_n) is a sequence of functions. Does the limit function or the sum of the series preserve important properties?

(1) If f_n is continus, is f continuous?

(2) If f_n is differentiable/integrable, is f differentiable/integrable?

For (1), the question is equivalent to the following:

If p is a limit point, does the following hold?

$$\lim_{x \to p} \lim_{n \to \infty} f_n(x) \stackrel{?}{=} \lim_{n \to \infty} \lim_{x \to p} f_n(x)$$

And the answer is **No**.

Example 7.2 Suppose $a_{m,n} = \dfrac{m}{m+n}$ for $m, n \in \mathbb{N}$. We see that

$$\lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = 1 \neq 0 = \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n}$$

Example. Define

$$f_n(x) = \begin{cases} 0 & \left(\frac{1}{n} \leq x \leq 1\right) \\ -nx + 1 & \left(0 \leq x < \frac{1}{n}\right) \end{cases}$$

then we can easily see that

$$f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & (0 < x \leq 1) \\ 1 & (x = 0) \end{cases}$$

Thus f is not continuous at $x = 0$.