HW Solution 8

1. (~~5~~ 6 points) Since we have

$$|f-f_n|^{\frac{3}{2}} = |f-f_n|^{\frac{1}{2}}|f-f_n|,$$

   Hölder's inequality yields that

$$\left(\int_{\mathbb{R}} |f-f_n|^{\frac{3}{2}}\,dx\right) \leq \left(\int_{\mathbb{R}} |f-f_n|\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f-f_n|^2\,dx\right)^{\frac{1}{2}}.$$

   By the assumption, we have the desired result.

~~2.~~

3. (6 points) Since $\|f_n\|_{L^2} \leq 2\pi$, $\{f_n\}$ is bounded set. Moreover, we observe that

$$\|f_n - f_m\|_{L^2} > \frac{1}{10000} \quad \text{if } n \neq m.$$

   It implies that any ball centered at f_n with radius $\frac{1}{10000}$ contains only f_n. Therefore, $\{f_n\}$ is closed and compact set.

2. (1)

: Let $f_k(x) = \sum_{n=1}^{k} \frac{1}{n} e^{inx}$. $\Rightarrow f_k \in \mathcal{L}^2$

Since $\| f_k - f_m \|_{\mathcal{L}^2}^2 \leq \sum_{n \geq \max\{k,m\}} \frac{2\pi}{n^2}$,

f_k is a Cauchy sequence in $\mathcal{L}^2$

Let f be a limit of f_k in $\mathcal{L}^2$.

We see that $\int f e^{-inx} = \lim_{k\to\infty} \int f_k e^{-inx}$.

$$= \frac{2\pi}{n},$$

where we have used $f_k \to f$ in $\mathcal{L}^2$ and Hölder's inequality.

Thus $f = \sum_{n=1}^{\infty} \frac{2\pi}{n} e^{inx} \in \mathcal{L}^2$.

(2) Suppose $\sum_{n=1}^{\infty} \frac{e^{inx}}{\sqrt{n}} \in \mathcal{L}^2$.

By Riesz thm, $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}}\right)^2 < \infty$. which is contradiction

$\square$

4. (6 points)

Let $E_{n,k} = \bigcup_{i \geq n} \{x \in [a,b] ; |f_i(x) - f(x)| \geq \frac{1}{k}\}$.

Since $f_n \to f$ a.e, $\mu(\bigcap_{n=1}^{\infty} E_{n,k}) = 0$. $\forall k \in \mathbb{N}$.

Since $E_{n+1,k} \subset E_{n,k}$ and $\mu(E_{1,k}) < \infty$,

$$\lim_{n \to \infty} \mu(E_{n,k}) = \mu\left(\bigcap_{n=1}^{\infty} E_{n,k}\right) = 0.$$

Given δ and $\varepsilon > 0$, $\exists\, n \in \mathbb{N}$ s.t $\mu(E_{n,n}) < \delta$.

and $\frac{1}{n} < \varepsilon$.

It implies that $\forall k > n$, $x \in [a,b] \setminus E_{n,k}$,

$$|f_k(x) - f(x)| < \varepsilon.$$

$\square$

5. (6 points)

Let $E = \{ x \in (-\pi, \pi) \; ; \; \lim f_k(x) = f(x) \}$.

Since $f(x)$ is bounded, $f \in L^2(E)$.

① $\forall A \subseteq E$: $\int_A \sin n_k x \, dx \to 0$ as $n_k \to \infty$

$\because$ Note that $\forall m \in \mathbb{Z}$,

$$0 = \lim_{n_k \to \infty} \int \sin n_k x \, e^{imx} dx = \int f(x) e^{imx} dx,$$

where we have used $n_k \gg m$ for the first equality and LDCT for the second inequality $\left(\begin{array}{l} |\sin n_k x| \le 1 \\ \sin n_k x \to f(x) \end{array} \right.$

This implies that $\|f\|_{L^2(A)} = 0 \cdots (*)$.

Thus $\int_A \sin n_k x \, dx \to \int_A f \, dx = 0$ by Hölder ineq.

② $\forall A \subseteq E$, $2 \int_A (\sin n_k x)^2 \to m(A)$ as $n_k \to \infty$.

$\because$ Note that

$$2 \int_A (\sin n_k x)^2 = \int_A (1 - \cos 2n_k x).$$

As in ①, we deduce that

$$\int_A \cos 2n_k x \to 0 \quad \text{as } n_k \to \infty.$$

Thus, ② is true.

Suppose $m(E) \neq 0$. Then by ②,

$$2\int_E (\sin n_k x)^2 \to 2\int_E (f(x))^2 = m(E).$$

But it contradicts (*). Thus $m(E) = 0$.

$\square$

6. (6 points)

Suppose there is a countable set $\{n_k\}$ s.t $n_k \in \mathbb{N}$ and $\sin n_k x \geq \delta$ $\forall x \in E$.

(1) Since $\chi_E \in \mathcal{L}^2$, and $\|\chi_E\|_{\mathcal{L}^2} \neq 0$.

$$\chi_E = \sum_{n \in \mathbb{Z}} c_n e^{inx}$$

Note that $c_{n_k} = \frac{1}{2\pi} \int \chi_E e^{-in_k x} dx$

$$= \frac{1}{2\pi} \int_E \cos n_k x + i \sin n_k x \quad dx$$

$$\geq \frac{1}{2\pi} (a + i\delta m(E)).$$

It gives $|c_{n_k}| \geq \frac{\delta}{2\pi} m(E)$. which contradicts the fact that $\chi_E \in \mathcal{L}^2$.

$\square$.

7.

$\Rightarrow$. It is easy consequence

$\Leftarrow$.

WLOG WMA $\|f\| \neq 0$. ($\| \ \| = \| \ \|_{L^2}$).

select $c \in \mathbb{C}$ s.t $|c| = \frac{\|g\|}{\|f\|}$ and $\bar{c} \int g \bar{f} = |c| \left| \int g \bar{f} \right|$.

① ②

Then we observe that

$$\|g - cf\|^2 + \|g + cf\|^2 = 2 \left(\|g\|^2 + \|cf\|^2 \right).$$

Thus

$$\|g - cf\|^2 = 2 \left(\|g\|^2 + \|cf\|^2 \right) - \|g + cf\|^2$$

$$= \|g\|^2 + \|cf\|^2 - \left(\int g \, \overline{cf} + \int \bar{g} \, cf \right)$$

$$= \|g\|^2 + \|cf\|^2 - 2|c| \left| \int g \bar{f} \right| \quad ②$$

$$= \|g\|^2 + \|cf\|^2 - 2 \|g\| \|cf\|$$

$$= \|g\|^2 - \|g\| \|cf\| + \|cf\|^2 - \|g\| \|cf\| \quad ①$$

$$= \|g\|^2 - \|g\|^2 + \|cf\|^2 - \|cf\|^2 = 0$$

It implies that $g = Cf$ a.e.

$\square$