# Assignment 3.5

Due Date: None

**1.** Let $X \subset \mathbb{R}^d$. Answer the following questions.

**(1)** For $x \in X$ and $r > 0$, verify that $N_X(x, r) = N(x, r) \cap X$.

**(2)** Prove that $\emptyset$ and $X$ are both open and closed in $X$.

**(3)** Suppose that $\{U_i : i \in \mathcal{I}\}$ is a family of open sets in $X$. Prove that $\bigcup_{i \in \mathcal{I}} U_i$ is an open set in $X$.

**(4)** Suppose that $U_1, \ldots, U_n$ are open sets in $X$. Prove that $\bigcap_{i=1}^n U_i$ is an open set in $X$.

**(5)** Suppose that $\{F_i : i \in \mathcal{I}\}$ is a family of closed sets in $X$. Prove that $\bigcap_{i \in \mathcal{I}} F_i$ is a closed set in $X$.

**(6)** Suppose that $F_1, \ldots, F_n$ are closed sets in $X$. Prove that $\bigcup_{i=1}^n F_i$ is a closed set in $X$.

**2.** Let $X \subset \mathbb{R}^d$, and let $\{U_i : i \in \mathcal{I}\}$ be a family of open sets in $X$. Suppose that $\{U_i : i \in \mathcal{I}\}$ is a cover of $X$. If $X$ is a compact set, prove that $\{U_i : i \in \mathcal{I}\}$ has a finite subcover of $X$. (In this sense, we can also say that $\{U_i : i \in \mathcal{I}\}$ is an *open cover of* $X$)

**3.** Let $f : X \to Y$ where $X$ and $Y$ are subsets of Euclidean spaces. Suppose that $\{A_i : i \in \mathcal{I}\}$ and $\{B_i : i \in \mathcal{I}\}$ are family of subsets of $X$ and $Y$, respectively. Then, prove the following properties.

$$
\begin{aligned}
f(\cup_{i \in \mathcal{I}} A_i) &= \bigcup_{i \in \mathcal{I}} f(A_i) \\
f(\cap_{i \in \mathcal{I}} A_i) &\subset \bigcap_{i \in \mathcal{I}} f(A_i) \\
f^{-1}(\cup_{i \in \mathcal{I}} B_i) &= \bigcup_{i \in \mathcal{I}} f^{-1}(B_i) \\
f^{-1}(\cap_{i \in \mathcal{I}} B_i) &= \bigcap_{i \in \mathcal{I}} f^{-1}(B_i)
\end{aligned}
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