해석개론 및 연습 2 과제 #5

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1. Show that $\Sigma \subseteq \mathcal{P}(S)$ is an algebra on S if and only if $\Sigma \subseteq \mathcal{P}(S)$ is a ring on S with $S \in \Sigma$.

($\Longrightarrow$) $S \in \Sigma$ is trivial. If $A, B \in \Sigma$, then $A \cup B \in \Sigma$. (algebra) Also, it follows that

$$A \setminus B = S \setminus \big((S \setminus A) \cup B\big) \in \Sigma,$$

since $S \setminus A \in \Sigma$ and $(S \setminus A) \cup B \in \Sigma$. Therefore Σ is a ring on S with $S \in \Sigma$.

($\Longleftarrow$) $S \in \Sigma$ is trivial. If $A, B \in \Sigma$, then $A \cup B \in \Sigma$. (ring) Since $S, A \in \Sigma$, $S \setminus A \in \Sigma$. (ring) Therefore Σ is an algebra on S.

Show that $\Sigma \subseteq \mathcal{P}(S)$ is a σ-algebra on S if and only if $\Sigma \subseteq \mathcal{P}(S)$ is a σ-ring on S with $S \in \Sigma$.

By the proof above, we only need to show that any countable union of elements of Σ is also an element of Σ. Let $A_n \in \Sigma$ $(n = 1, 2, \ldots)$. Then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ holds for both cases where Σ is a σ-algebra or a σ-ring. Therefore the statement is proven.

2. (i) $E = f^{-1}(S) \in f^{-1}(\Sigma)$, since $S \in \Sigma$. (algebra)

(ii) Take two elements $f^{-1}(A), f^{-1}(B) \in f^{-1}(\Sigma)$, where $A, B \in \Sigma$. Then

$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) \in f^{-1}(\Sigma)$$

since $A \cup B \in \Sigma$. (algebra)

(iii) Take an element $f^{-1}(A) \in f^{-1}(\Sigma)$, where $A \in \Sigma$. Then

$$f^{-1}(S) \setminus f^{-1}(A) \overset{(*)}{=} f^{-1}\left(S \setminus A\right) \in f^{-1}(\Sigma)$$

since $S \setminus A \in \Sigma$. (algebra)

(iv) For $f^{-1}(A_n) \in f^{-1}(\Sigma)$ where $A_n \in \Sigma$, $(n = 1, 2, \ldots)$ the following holds.

$$\bigcup_{n=1}^{\infty} f^{-1}(A_n) = f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) \in f^{-1}(\Sigma)$$

since $\bigcup_{n=1}^{\infty} A_n \in \Sigma$. ($\sigma$-algebra)

Therefore, $f^{-1}(\Sigma)$ is a σ-algebra on E.

Proof of $(*)$

$$\begin{aligned} x \in f^{-1}(S) \setminus f^{-1}(A) &\iff x \in f^{-1}(S) \wedge x \notin f^{-1}(A) \\ &\iff f(x) \in S \wedge f(x) \notin A \\ &\iff f(x) \in S \setminus A \iff x \in f^{-1}\left(S \setminus A\right). \end{aligned}$$

3. Let Σ_n $(n = 1, 2, \ldots)$ be σ-algebras on S, and define $\Sigma = \bigcap_n \Sigma_n$. It is trivial that $S \in \Sigma$.

If $A, B \in \Sigma$, then $A, B \in \Sigma_i$ for all $i \in \mathbb{N}$. Then $S \setminus A \in \Sigma_i$ and $A \cup B \in \Sigma_i$ for all $i \in \mathbb{N}$. Therefore $S \setminus A \in \Sigma$ and $A \cup B \in \Sigma$.

Finally, if $A_j \in \Sigma$ for $j = 1, 2, \ldots$, then $A_j \in \Sigma_i$ for all $i, j \in \mathbb{N}$. So $\bigcup_{j=1}^{\infty} A_j \in \Sigma_i$ for all $i \in \mathbb{N}$, since Σ_i are σ-algebras. Therefore $\bigcup_{j=1}^{\infty} A_j \in \Sigma$.

Thus an arbitrary intersection of σ-algebras on S is a σ-algebra on S. For the case of unions, consider

$$\Sigma_1 = \{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}, \quad \Sigma_2 = \{\varnothing, \{2\}, \{1, 3\}, \{1, 2, 3\}\}.$$

Then $\{1\}, \{2\} \in \Sigma_1 \cup \Sigma_2$, but $\{1, 2\} \notin \Sigma_1 \cup \Sigma_2$.

4. We first check that Σ is a σ-ring. For $A, B \in \Sigma$, it is easy to see that $A \cup B, A \setminus B$ are both in Σ. Also if $A_i \in \Sigma$ $(i = 1, 2, \ldots)$, then $\bigcup_{i=1}^{\infty} A_i \subseteq S \in \mathcal{P}(\mathcal{S}) = \Sigma$.

It is easily seen that μ_1, μ_2 are non-negative. Now let $A_i \in \Sigma$ $(i = 1, 2, \ldots)$ be pairwise disjoint sets, and let $A = \bigcup_{i=1}^{\infty} A_i$.

(1) Suppose $x \in A$. Then because A_i are disjoint, $\exists\, N \in \mathbb{N}$ such that $x \in A_N$. So we can see that $\mu_1(A_i) = 1$ if $i = N$ and 0 otherwise. If $x \notin A$, $\mu_1(A_i) = 0$ for all $i \in \mathbb{N}$. Therefore, for both cases,

$$\mu_1\left(\bigcup_{i=1}^{\infty} A_i\right) = \begin{cases} 1 = \sum_{i=1}^{\infty} \mu_1(A_i) & (x \in A) \\ 0 = \sum_{i=1}^{\infty} \mu_1(A_i) & (x \notin A) \end{cases}$$

and μ_1 is a measure on Σ.

(2) If $|A_N| = \infty$ for some $N \in \mathbb{N}$, then $\mu_2\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_2(A_i) = \infty$. Now suppose that $|A_i| < \infty$ for all $i \in \mathbb{N}$. If $\sum_{i=1}^{\infty} \mu_2(A_i) = \sum_{i=1}^{\infty} |A_i| < \infty$, then

$$\mu_2\left(\bigcup_{i=1}^{\infty} A_i\right) = \left|\bigcup_{i=1}^{\infty} A_i\right| = \sum_{i=1}^{\infty} |A_i| < \infty.$$

If $\sum_{i=1}^{\infty} \mu_2(A_i) = \infty$, we can take any $K > 0$ and find $M \in \mathbb{N}$ such that $\sum_{i=1}^{M} \mu_2(A_i) > K$. Then

$$\mu_2\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \mu_2\left(\bigcup_{i=1}^{M} A_i\right) = \sum_{i=1}^{M} \mu_2(A_i) > K$$

for all $K > 0$. Thus $\mu_2(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu_2(A_i) = \infty$. Overall, μ_2 is a measure on Σ.

5. From the assumptions, $\mu(A_i) \leq \mu(A_1) < \infty$ for all $i \in \mathbb{N}$. Define $B_n = A_n \setminus A_{n+1}$ for $n = 1, 2, \ldots$. Then we directly see that B_n are pairwise disjoint. Note that

$$\sum_{i=1}^{n} \mu(B_i) = \sum_{i=1}^{n} [\mu(A_i) - \mu(A_{i+1})] = \mu(A_1) - \mu(A_{n+1}). \tag{$*$}$$

Therefore,

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \mu\left(A_1 \setminus \bigcup_{i=1}^{\infty} B_i\right) = \mu(A_1) - \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \mu(A_1) - \sum_{i=1}^{\infty} \mu(B_i)$$
$$= \lim_{n\to\infty}\left[\mu(A_1) - \sum_{i=1}^{n} \mu(B_i)\right] = \lim_{n\to\infty} \mu(A_{n+1}) = \lim_{n\to\infty} \mu(A_n)$$

by $(*)$.

6. Define $B_1 = A_1$, $B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$ for $n \geq 2$. Then B_n are pairwise disjoint. Since $B_n \subseteq A_n$ for all $n \in \mathbb{N}$, we have that $\mu(B_n) \leq \mu(A_n)$. Also note that

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

since $\mathcal{F}$ is a σ-algebra. Then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i).$$