해석개론 및 연습 1 과제 #5

2017-18570 컴퓨터공학부 이성찬

1. Fix $x < y$. Since f is a C^1-function on $[a, b]$, by Mean Value Theorem, there exists $c \in (a, b)$ s.t.

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

and since $f'(c) > 0$, $f(y) - f(x) > 0$. Thus f is increasing.

2. With the same conditions given in the problem, we prove the following.

- $\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)$

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \lim_{h\to 0} \left\{ g(x+h)\frac{f(x+h) - f(x)}{h} + f(x)\frac{g(x+h) - g(x)}{h} \right\} \\ &= f'(x)g(x) + f(x)g'(x)\end{aligned}$$

(Since $f(x), g(x)$ is differentiable and continuous)

- $\big(1/g(x)\big)' = -g'(x)/g(x)^2$

$$\begin{aligned}\lim_{h\to 0} \frac{1/g(x+h) - 1/g(x)}{h} &= \lim_{h\to 0} \frac{g(x) - g(x+h)}{hg(x)g(x+h)} \\ &= -\frac{1}{g(x)^2} \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = -\frac{g'(x)}{g(x)^2}\end{aligned}$$

Now combining these two result gives

$$\left(\frac{f(x)}{g(x)}\right)' = \left(f(x) \cdot \frac{1}{g(x)}\right)' = f'(x) \cdot \frac{1}{g(x)} + f(x) \cdot \frac{-g'(x)}{g(x)^2} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

3. **(1)** The following can be proved easily by induction.

$$f^{(i)}(x) = \sum_{k=i}^{n} \frac{k!}{(k-i)!} c_k x^{k-i} \quad (i = 0, \dots, n)$$

$(f^{(i+1)}(x) = \sum_{k=i+1}^{n} \frac{k!}{(k-i)!} c_k (k-i) x^{k-i-1} = \sum_{k=i+1}^{n} \frac{k!}{(k-i-1)!} c_k x^{k-i-1})$
And for $i > n$, $f^{(i)}(x) = 0$. Since $f^{(i)}(0) = i! \cdot c_i$ $(i = 0, \dots, n)$ we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{k! c_k}{k!} x^k + 0 = \sum_{k=0}^{n} c_k x^k$$

(2) By induction,

$$f^{(n)}(x) = 2^n e^{2x+1}$$

$((2^n e^{2x+1})' = 2^{n+1} e^{2x+1} = f^{(n+1)}(x))$
Thus $f^{(n)}(0) = e \cdot 2^n$, and

$$f(x) = \sum_{k=0}^{\infty} \frac{e \cdot 2^k}{k!} x^k$$

(3) Consider the $(2n+1)$-th degree Taylor expansion. By Taylor's Theorem, there exists x_* between 0 and x such that

$$\left|\cos x - \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}\right| = |\cos x_*| \frac{|x|^{2n+2}}{(2n+2)!} \leq \frac{|x|^{2n+2}}{(2n+2)!}$$

Now substitute x^2 in x. Since Taylor polynomials are unique (For two n-th degree polynomials, if their difference is in $o(x^n)$, they are equal) we have

$$\left|\cos(x^2) - \sum_{k=0}^{n} \frac{(-1)^k x^{4k}}{(2k)!}\right| \leq \frac{|x|^{4n+4}}{(2n+2)!}$$

and as $n \to \infty$, RHS $\to 0$.

$$\cos(x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k}}{(2k)!}$$

4. Use the Mean Value Theorem on (a, c) and (c, b). Then there exists $c_1 \in (a, c)$ and $c_2 \in (c, b)$ such that

$$\frac{f(c) - f(a)}{c - a} = f'(c_1) \qquad \frac{f(b) - f(c)}{b - c} = f'(c_2)$$

Since $(a, f(a)), (b, f(b)), (c, f(c))$ are on the same line, the slope is equal and $f'(c_1) = f'(c_2)$. By Rolle's Theorem, there exists $d \in (c_1, c_2) \subset [a, b]$ s.t. $f''(d) = 0$.

5. **(1)**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \geq \frac{x^n}{n!} \quad (x \geq 0)$$

(2) It is enough to check for $x = 0$. Check the left/right derivative.

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{e^{-1/h^2}}{h} \overset{(*)}{=} 0$$

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = 0$$

Thus f is differentiable on $\mathbb{R}$.

$(*)$: From (1),

$$0 \leq e^{-h} \leq \frac{n!}{h^n} \implies 0 \leq e^{-1/h^2} \leq n! \cdot h^{2n} \implies 0 \leq \frac{e^{-1/h^2}}{h} \leq n! \cdot h^{2n-1}$$

By Squeeze Theorem, the wanted limit approaches 0 as $h \to 0$.

(3) (Induction) For $n = 1$, $f'(x) = \frac{2}{x^3} e^{-1/x^2}$, thus $Q_1(t) = 2t^3$, leading coefficient is positive, $\deg Q_1 = 3$. Suppose for n (≥ 1), $f^{(n)}(x) = Q_n(1/x) e^{-1/x^2}$, leading coefficient is positive, and $\deg Q_n = 3n$.

$$f^{(n+1)}(x) = \left(-\frac{1}{x^2} Q'_n\left(\frac{1}{x}\right) + \frac{2}{x^3} Q_n\left(\frac{1}{x}\right)\right) e^{-1/x^2} \quad (x > 0)$$

Let

$$P\left(\frac{1}{x}\right) = -\frac{1}{x^2} Q'_n\left(\frac{1}{x}\right) + \frac{2}{x^3} Q_n\left(\frac{1}{x}\right)$$

Then $P(t) = -t^2 Q'_n(t) + 2t^3 Q_n(t)$. $\deg -t^2 Q'_n(t) = 3n + 1$ and $\deg 2t^3 Q_n(t) = 3n + 3$. Therefore $P(t) = Q_{n+1}(t)$, with positive leading coefficient and degree $3n + 3$.

(4) For any n, we show that $f^{(n)}(x)$ is differentiable. From (3), we have

$$f^{(n)}(x) = \begin{cases} Q_n(1/x)e^{-1/x^2} & (x > 0) \\ 0 & (x < 0) \end{cases}$$

We will show that $f^{(n)}(0) = 0$ by induction to complete the proof. (2) handles the case for $n = 1$, and suppose $f^{(n)}(0) = 0$ for $n \geq 1$. The left hand derivative is obviously 0, and for the right hand derivative,

$$f_+^{(n+1)}(0) = \lim_{h \to 0^+} \frac{f^{(n)}(0) - f^{(n)}(0)}{h} = \lim_{h \to 0^+} \frac{Q_n(1/h)}{h} e^{-1/h^2}$$

Let $Q(t) = \sum_{i=0}^{3n} q_i t^i$. Then $Q(1/h) = \sum_{i=0}^{3n} q_i/t^i$ and

$$e^x \geq \frac{x^{2n}}{(2n)!} \implies (2n)! \cdot x^{4n} \geq e^{-1/x^2} \implies 0 \leq \frac{Q_n(1/h)e^{-1/h^2}}{h} \leq (2n)! \sum_{i=0}^{3n} q_i h^{4n-i-1}$$

Applying the Squeeze Theorem here gives us $f_+^{(n+1)}(0) = 0$. Therefore $f(x) \in C^\infty$.

(5) Define $g(x) = f(1+x)f(1-x)$. Then we immediately have $g(x) = 0$ for $|x| \geq 1$. Since $f(x) > 0$, we also have $f(1+x)f(1-x) > 0$ for $|x| < 1$. Finally, since $f(x) \in C^\infty$, its product $g(x)$ is also in C^∞.

6. **(1)** As $h \to 0$, denominator/numerator both approach 0. And we have

$$\begin{aligned} \lim_{h \to 0} \frac{\big(f(x+h) + f(x-h) - 2f(x)\big)'}{(h^2)'} &= \lim_{h \to 0} \frac{f'(x+h) + f'(x-h)}{2h} \\ &= \lim_{h \to 0} \frac{f'(x+h) - f'(x) + f'(x) + f'(x-h)}{2h} \\ &= \frac{1}{2}f''(x) + \frac{1}{2}f''(x) = f''(x) \end{aligned}$$

By L'Hospital's Rule, the original limit is equal to $f''(x)$.

(2) As $h \to 0$, denominator/numerator both approach 0. Thus we would like to calculate

$$\lim_{h \to 0} \frac{2f'(x+2h) - 3f'(x+h) + f'(x-h)}{3h^2}$$

For this limit, denominator/numerator also approach 0 as $h \to 0$. So instead we calculate

$$\lim_{h \to 0} \frac{4f''(x+2h) - 3f''(x+h) - f''(x-h)}{6h}$$

, hoping to use L'Hospital's Rule. The actual value is

$$\begin{aligned} &= \lim_{h \to 0} 4 \cdot \frac{f''(x+2h) - f''(x)}{6h} - \lim_{h \to 0} 3 \cdot \frac{f''(x+h) - f''(x)}{6h} + \lim_{h \to 0} \frac{f''(x) - f''(x-h)}{6h} \\ &= \frac{4}{3}f'''(x) - \frac{1}{2}f'''(x) + \frac{1}{6}f'''(x) = f'''(x) \end{aligned}$$

The original limit is equal to $f^{(3)}(x)$ by L'Hospital's Theorem.