# Introduction to Analysis II

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## Introduction & Notice

- 7, 8장 나가고 중간고사, 11장 나가고 기말고사
- 연습 시간이 있는 수업 (목 6:30 ∼ 8:20)[1]
- 오늘 연습 시간: 지난학기 배운 내용 중 필요한 내용 복습

[1] 가능하면 1시간 반 안에 끝내라고 하심 ㅋㅋ

# Chapter 7

# Sequences and Series of Functions

## September 1st, 2022

기본적으로 수열에 관련된 내용, real/complex-valued 수열이 아니라 함수가 주어졌을 때. 함수들을 모은 '**sequence of functions**'의 극한을 생각하는 것.

Suppose $E$ is a set[1], and let $f_n : E \to \mathbb{C}$. Then

$$(f_n)_{n=1}^{\infty}$$

is a sequence of (complex-valued) function.

**Definition 7.1.** $(f_n)_{n=1}^{\infty}$ converges **pointwise** on $E$, if for each $x \in E$ the sequence $(f_n(x))_{n=1}^{\infty}$ converges in $\mathbb{C}$.

In other words, $\forall x \in E, \exists\, a_x \in \mathbb{C}$ and

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } n \geq N \implies |f_n(x) - a_x| < \epsilon.$$

**Definition.** If $(f_n)$ converges pointwise, we can define a function $f$ by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E)$$

We say that

- $f$ is the *limit* or *limit function* of $f_n$.
- $(f_n)$ to $f$ pointwise on $E$.

[1] 사실은 *metric* space 이다.

**Definition.** If $\sum f_n(x)$ converges (pointwise) for every $x \in E$, we define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E)$$

and the function $f$ is called the *sum* of the series $\sum f_n$.

**Recall.** $f : (E, d) \to \mathbb{C}$ is continuous on $E \iff f$ is continuous at all $x \in E$.

**Recall.** (Theorem 4.6) If $p \in E$ and $p$ is a limit point of $E$,

$$f \text{ is continuous at } p \iff \lim_{x \to p} f(x) = f(p)$$

**Question.** Suppose $(f_n)$ is a sequence of continous functions that converges pointwise on $E$. Is $f$ continuous? *No...*

If $p$ is a limit point, does the following hold?

$$\lim_{x \to p} \lim_{n \to \infty} f_n(x) \stackrel{?}{=} \lim_{n \to \infty} \lim_{x \to p} f_n(x)$$

**Example.** $a_{m,n} = \frac{m}{m+n}$, $a_{m,n} \to 1$ as $m \to \infty$ and then $n \to \infty$. But, as $n \to \infty$ $a_{m,n} = 0$

Example. Define

$$f_n(x) = \begin{cases} 0 & (1/n \leq x \leq 1) \\ -nx + 1 & (0 \leq x < 1/n) \end{cases}$$

then we can easily see that

$$f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & (0 < x \leq 1) \\ 1 & (x = 0) \end{cases}$$

, thus $f$ is not continuous at $x = 0$.

Example. Define $f_n : \mathbb{R} \to \mathbb{R}$.

$$f_n(x) = \frac{x^2}{(1 + x^2)^n} \qquad (n = 0, 1, \dots)$$

by direct calculation,

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1 + x^2)^n} = 1 + x^2 \qquad (x \neq 0)$$

If $x = 0$, $f(x) = 0$. Thus $f$ is not continuous.

Question. Suppose $(f_n)$ is a sequence of **Riemann integrable** functions that converges pointwise. Is $f$ **Riemann integrable**? Also **No...**

Example. For $m = 1, 2, \ldots$, define

$$\begin{aligned} f_m(x) &= \lim_{n\to\infty} (\cos m!\pi x)^{2n} \\ &= \begin{cases} 1 & (m!x \in \mathbb{Z}) \\ 0 & (m!x \notin \mathbb{Z}) \end{cases} \end{aligned}$$

Note that $f_m(x) \in \mathcal{R}[a, b]$

Claim.

$$f(x) = \lim_{m\to\infty} f_m(x) = \begin{cases} 1 & (x \in Q) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

and $f(x)$ is nowhere continuous thus not Riemann integrable.

**Proof.** Suppose $x = p/q \in \mathbb{Q}$. $(p, q \in \mathbb{Z})$ If we take $m \geq q$, we see that $m!x \in \mathbb{Z}$. Thus $f_m(x) = 1$.

If $x \notin \mathbb{Q}$, $m!x$ can never be in $\mathbb{Z}$ and $f_m(x) = 0$. □

계속 예제...

Question. Uniform continuity를 할 때 uniform이 어디서 나오죠? 해석학에서 그 점에서 뭐가 성립한다, 그러면 그 점과 그 근방에서만 확인하면 됐었죠. Continuity는 local property죠. 그런데 uniform continuity는 전체가 다 uniform하게 성립한다.

Recall. $f : (X, d) \to (Y, d)$ is uniformly continuous on $X$[2] if

$$\forall \epsilon > 0, \exists\, \delta > 0, d_X(q, p) < \delta \implies d_Y(f(p), f(q)) < \epsilon$$

모든 점에서 똑같이 잡을 수 있다!

Fact. If $X$ is compact and $f$ is continuous on $X$, then $f$ is uniformly continuous on $X$. (Theorem 4.19)[3]

지금 나오는 uniform convergence는 sequence에 관한 것입니다!

**Definition Uniform Convergence.** Suppose $f_n : E \to \mathbb{C}$ is a sequence of functions.

$(f_n)_{n=1}^{\infty}$ converges uniformly on $E$ to a function $f$ if

$$\forall \epsilon > 0, \exists\, N : \forall x \in E, n \geq N, |(|\, f_n(x) - f(x)) \leq \epsilon$$

[2] Subspace of metric space is also a metric space

[3] 갑자기 왜 uniform continuity 얘기를 하냐, 헷갈리지 말고 기억하시라고!

[a]

$$\forall \epsilon > 0, \sup_{x\in E} |f_n(x) - f(x)| \le \epsilon, \forall n \ge N$$

$$\sum_{n=1}^{\infty} f_n$$

converges uniformly on $E$: $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges pointwise and $\sum_{k=1}^{n} f_k(x)$ converges uniformly to $f$.

[a] 등호를 붙이는 것이 극한 잡기 편하다???

[똑같은 $\epsilon$-띠를 둘러서 $y = f(x)$ 의 근방 안에 $f_n(x)$ $(n \ge N)$ 가 모두 들어가 있어야 한다]는 의미에서 uniform 이다.

Notation. $f_n \to f$ uniformly on $E$ $\iff$ $f_n \xrightarrow{u} f$ on $E$.[4]

7.8에 나와있는 내용이 Cauchy sequence...

Recall. Cauchy sequence converges!

**Theorem.** (7.8) $f_n \xrightarrow{u} f$ on $E$ $\iff$

$$\forall \epsilon > 0, \exists N : \sup_{x\in E} |f_n(x) - f_m(x)| \le \epsilon, \forall n, \forall m \ge N$$

[a]

[a] Uniform Cauchy

**Proof.** ( $\implies$ ) For given $\epsilon > 0$...

Fix $x \in E$,

$$|f_n(x) - f_m(x)| \le |f_n(x) - f(x)| + |f(x) - f_m(x)| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for $n, m \ge N$.

( $\impliedby$ ) Uniform Cauchy property implies that $(f_n)$ is a Cauchy sequence in $\mathbb{C}$. By the completeness of $C$, the limit function $f(x)$ exists. Now we show that this convergence is uniform.

For given $\epsilon > 0$ choose $N$ such that

$$\sup_{x\in E} |f_n(x) - f_m(x)| \le \epsilon$$

, for all $n, m \ge N$. Then

$$|f_n(x) - f(x)| \le ||f_n(x) - f_m(x)| - |f_n(x) - f(x)|| + |f_n(x) - f_m(x)| \le |f_m(x) - f(x)| + \epsilon$$

[4] 책에서는 나중에 $\|f_n(x) - f(x)\|_\infty \to 0$ 으로 적었던 것 같은데...

Fix $n \geq N$ and let $m \to \infty$. Observe that the first term converges to 0 due to pointwise convergence.

$\therefore |f_n(x) - f(x)| \leq \epsilon, \forall n \geq N, \forall x \in E$ $\square$