HW Solution 7

1. (5 points) We first show that s(x) is improperly Riemann integrable on $(0, \infty)$. Since we observe that

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} s(x)$$

exists, it is enough to show that

$$\lim_{M \to \infty} S_M \equiv \lim_{M \to \infty} \int_{\pi}^{M\pi} s(x)\,dx \tag{1}$$

exists. To do this, we set

$$a_n = \int_{n\pi}^{(n+1)\pi} s(x) \quad \text{for any } n = 1, 2, \ldots.$$

Then we observe that

$$|a_n| = \int_{n\pi}^{(n+1)\pi} \left| \frac{sin(x)}{x} \right| dx > \int_{(n+1)\pi}^{(n+2)\pi} \left| \frac{sin(x)}{n+1} \right| dx > \int_{(n+1)\pi}^{(n+2)\pi} \left| \frac{sin(x)}{x} \right| dx = |a_{n+1}|,$$

$$\lim_{n \to \infty} |a_n| = 0$$

and a_n and a_{n+1} have different sign. Therefore, we have that

$$\sum_{n=1}^{\infty} a_n < \infty,$$

which implies that (1) is finite. We next prove that $s(x)$ is not in $\mathcal{L}$. This follows from the fact that

$$\int_{(0,\infty)} |s(x)|\,dx > \sum_{i=1}^{\infty} \int_{i\pi}^{(i+1)\pi} \left| \frac{sin(x)}{x} \right| dx > \sum_{i=1}^{\infty} \frac{\pi}{i+1} = \infty.$$

2. (5 points) Note that

$$\int_1^n \log x\,dx = n \log n - (n-1).$$

From the fact that

$$\int_1^n f(x)\,dx = \log(n!) - \frac{1}{2}\log n > -\frac{1}{8} + \int_1^n g(x)\,dx,$$

we have that

$$-\frac{1}{8} + n\log n - (n-1) < \log(n!) - \frac{1}{2}\log n \leq n\log n - (n-1).$$

After a few algebraic computations, we observe

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right)\log n + n < 1.$$

By taking exponential function for each number, we obtain the desired result.

3. (5 points) Note that Theorem 11.33 implies that f is continuous a.e. on $[a,b]$. Then it suffices to show that if f is continuous at $x \in [a,b]$, then $F'(x) = f(x)$. Suppose x is a such point. Then We observe that

$$\begin{aligned}\left|\left(\frac{F(x+h)-F(x)}{h} - f(x)\right)\right| &= \left|\int_x^{x+h}\left(\frac{f(y)-f(x)}{h}\right)\,dx\right| \\ &= \left|\int_0^h\left(\frac{f(x+s)-f(x)}{h}\right)\,ds\right| \\ &\leq \sup_{s\in[0,h]}\|f(x+s)-f(x)\|\end{aligned}$$

tends to 0 as $h \to 0$. This implies the desired result.

4. (5 points) Note that

$$|F(x+h)-F(x)| = \left|\int_x^{x+h} f(y)\,dy\right| \leq \int_a^b |f(y)|K_{[x,x+h]}\,dy$$

Since $|f(y)|K_{[x,x+h]} \leq |f(y)| \in \mathcal{L}$ and $|f(y)|K_{[x,x+h]} \to 0$ as $h \to 0$, LDCT yields that $|F(x+h)-F(x)| \to 0$ as $h \to 0$.

5. (5 points) Since proving that $\mathcal{L}(\mu)$ is a metric space follows from simple computations, we only show that it is a complete space. Suppose f_n is a Cauchy sequence in $\mathcal{L}(\mu)$. Then there is a subsequence f_{n_k} such that

$$\int_X \left|f_{n_k} - f_{n_{k+1}}\right|\,d\mu \leq \frac{1}{2^k}.$$

By following the same line as in the proof of Theorem 11.42, we have that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

exists a.e. on X. Using Fatou's lemma, we observe that

$$\begin{aligned}\int |f - f_n|\, d\mu &\le \liminf_{i\to\infty} \int |f_{n_i} - f_n|\, d\mu \\ &\le \liminf_{i\to\infty} \int |f_{n_i} - f_{n_k}|\, d\mu + \int |f_{n_k} - f_n|\, d\mu.\end{aligned}$$

By taking n and n_k sufficiently large, we see that

$$\int |f - f_n|\, d\mu \le \epsilon,$$

which gives that $f \in \mathcal{L}(\mu)$ and it is a limit of f_n in $\mathcal{L}(\mu)$.

6. Take $h_n = g_n - f_n$ to see that

$$h_n \ge 0.$$

Applying Fatou's lemma to h_n, we have that

$$\begin{aligned}&\int (g - f)\, d\mu = \int \liminf h_n\, d\mu \\ &\le \liminf \int h_n\, d\mu = \int g\, d\mu + \liminf \int -f_n\, d\mu = \int g\, d\mu - \limsup \int f_n\, d\mu.\end{aligned}$$

And this yields that

$$\int f\, d\mu \ge \limsup \int f_n\, d\mu.$$

Similarly, we deduce

$$\int f\, d\mu \le \liminf \int f_n\, d\mu$$

by taking $h_n = g_n + f_n$. Thus we conclude the proof.

7. (5 points) Using triangle inequality,

$$\int_X |f_n - f| \, d\mu \to 0$$

implies

$$\int_X |f_n| \to \int_X |f|.$$

On the other hand, suppose

$$\int_X |f_n| \to \int_X |f|.$$

Let us define

$$h_n = |f_n - f| \quad \text{and} \quad g_n = |f_n| + |f|.$$

to see that

$$|h_n| \le g_n, \lim_{n\to\infty} g_n = g = 2|f| \quad \text{and} \quad \int g_n \, d\mu \to \int g \, d\mu.$$

Since $\lim_{n\to\infty} h_n = 0$ a.e. on X, we have that

$$\int_X |f_n| \to \int_X |f|,$$

where we have used the result in the problem 6.