Introduction to Analysis II

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Introduction & Notice

- 7, 8장 나가고 중간고사, 11장 나가고 기말고사
- 연습 시간이 있는 수업 (목 6:30 ∼ 8:20)[1]
- 오늘 연습 시간: 지난학기 배운 내용 중 필요한 내용 복습

[1]가능하면 1시간 반 안에 끝내라고 하심 ㅋㅋ

Chapter 7

Sequences and Series of Functions

September 1st, 2022

기본적으로 수열에 관련된 내용, real/complex-valued 수열이 아니라 함수가 주어졌을 때. 함수들을 모은 '**sequence of functions**'의 극한을 생각하는 것.

Suppose E is a set[1], and let $f_n : E \to \mathbb{C}$. Then

$$(f_n)_{n=1}^{\infty}$$

is a sequence of (complex-valued) function.

Definition 7.1 (Pointwise Convergence) $(f_n)_{n=1}^{\infty}$ converges **pointwise** on E, if for each $x \in E$ the sequence $(f_n(x))_{n=1}^{\infty}$ converges in $\mathbb{C}$.

In other words, for each $x \in E$, there exists $a_x \in \mathbb{C}$ and

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |f_n(x) - a_x| < \epsilon.$$

Definition. If (f_n) converges pointwise, we can define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E)$$

We say that

- f is the *limit* or *limit function* of f_n.
- (f_n) to f pointwise on E.

[1] 사실은 *metric* space 이다.

Definition. If $\sum f_n(x)$ converges (pointwise) for every $x \in E$, we can define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E)$$

and the function f is called the *sum* of the series $\sum f_n$.

Recall. $f : (E, d) \to \mathbb{C}$ is continuous on $E \iff f$ is continuous at all $x \in E$.

Recall. (Theorem 4.6) If $p \in E$ and p is a limit point of E,

$$f \text{ is continuous at } p \iff \lim_{x \to p} f(x) = f(p)$$

Question. Suppose (f_n) is a sequence of functions. Does the limit function or the sum of the series preserve important properties?

(1) If f_n is continus, is f continuous?

(2) If f_n is differentiable/integrable, is f differentiable/integrable?

For (1), the question is equivalent to the following:

If p is a limit point, does the following hold?

$$\lim_{x \to p} \lim_{n \to \infty} f_n(x) \stackrel{?}{=} \lim_{n \to \infty} \lim_{x \to p} f_n(x)$$

And the answer is **No**.

Example 7.2 Suppose $a_{m,n} = \dfrac{m}{m+n}$ for $m, n \in \mathbb{N}$. We see that

$$\lim_{n \to \infty} \lim_{m \to \infty} a_{m,n} = 1 \neq 0 = \lim_{m \to \infty} \lim_{n \to \infty} a_{m,n}$$

Example. Define

$$f_n(x) = \begin{cases} 0 & \left(\frac{1}{n} \le x \le 1\right) \\ -nx + 1 & \left(0 \le x < \frac{1}{n}\right) \end{cases}$$

then we can easily see that

$$f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0 & (0 < x \le 1) \\ 1 & (x = 0) \end{cases}$$

Thus f is not continuous at $x = 0$.

Example. Define $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (n = 0, 1, 2, \dots)$$

by direct calculation,

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = 1 + x^2 \quad (x \neq 0)$$

since this is a geometric series when $x \neq 0$. If $x = 0$, $f(x) = 0$ and f is not continuous.

Does the limit function preserve Riemann integrability?

Example. For $m = 1, 2, \dots$, define

$$f_m(x) = \lim_{n\to\infty} (\cos m!\pi x)^{2n} = \begin{cases} 1 & (m!x \in \mathbb{Z}) \\ 0 & (m!x \notin \mathbb{Z}) \end{cases}$$

We see that $f_m(x)$ is Riemann integrable. However,

Claim.

$$f(x) = \lim_{m\to\infty} f_m(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

and $f(x)$ is nowhere continuous thus not Riemann integrable.

Proof. Suppose $x = p/q \in \mathbb{Q}$. $(p, q \in \mathbb{Z})$ If we take $m \geq q$, we see that $m!x \in \mathbb{Z}$. Thus $f_m(x) = 1$. If $x \notin \mathbb{Q}$, $m!x$ can never be in $\mathbb{Z}$ and $f_m(x) = 0$.

Question. Uniform continuity를 할 때 uniform이 어디서 나오죠? 해석학에서 그 점에서 뭐가 성립한다, 그러면 그 점과 그 근방에서만 확인하면 됐었죠. Continuity는 local property죠. 그런데 uniform continuity는 전체가 다 uniform하게 성립한다는 의미입니다.

Recall. $f : (X, d) \to (Y, d)$ is uniformly continuous on X[2] if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_X(p, q) < \delta \implies d_Y(f(p), f(q)) < \epsilon$$

즉, 모든 점에서 똑같이 잡을 수 있다!

Recall. (Theorem 4.19) If X is compact and f is continuous on X, then f is uniformly continuous on X.[3]

이제부터 나오는 uniform convergence는 sequence에 관한 것입니다!

[2]Subspace of metric space is also a metric space
[3]갑자기 왜 uniform continuity 얘기를 하냐, 헷갈리지 말고 기억하시라고!

Definition 7.7 (Uniform Convergence) Suppose $f_n : E \to \mathbb{C}$ is a sequence of functions. $(f_n)_{n=1}^{\infty}$ **converges uniformly** on E to a function f if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall x \in E, n \geq N \implies |f_n(x) - f(x)| \leq \epsilon.$$[4]

Also, we say that the series $\sum f_n(x)$ converges uniformly on E if the sequence of partial sums $(\sum_{k=1}^{n} f_k(x))$ converges uniformly on E.

Pointwise convergence의 경우 $N \in \mathbb{N}$ 이 $x \in E$ 에 의존하지만, uniform convergence의 경우 N 이 x와 무관하다!

[똑같은 ϵ-띠를 둘러서 $y = f(x)$ 의 근방 안에 $f_n(x)$ $(n \geq N)$ 가 모두 들어가 있어야 한다]는 의미에서 uniform 이다.

Theorem 7.9 Suppose

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E)$$

Then $f_n \to f$ converges uniformly on E if and only if

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$$

which can also be written as

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies \sup_{x\in E} |f_n(x) - f(x)| \leq \epsilon$$

Notation. $f_n \to f$ uniformly on $E \iff f_n \xrightarrow{u} f$ on E.[5]

Theorem 7.8 (Cauchy Criterion for Uniform Convergence) $f_n \xrightarrow{u} f$ on $E \iff$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n, m \geq N \implies \sup_{x\in E} |f_n(x) - f_m(x)| \leq \epsilon.$$[6]

Proof.

$(\implies)$ For given $\epsilon > 0$, fix $x \in E$. Since f_n converges uniformly on E, we can find $N \in \mathbb{N}$ such that for $n, m \geq N$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$(\impliedby)$ Uniform Cauchy property implies that (f_n) is a Cauchy sequence in $\mathbb{C}$. By the completeness of $\mathbb{C}$, the limit function $f(x)$ exists. Now we show that this convergence is uniform. For given $\epsilon > 0$ choose $N \in \mathbb{N}$ such that for all $n, m \geq N$,

[4] 등호를 붙이는 것이 극한 잡기 편하다???
[5] 교수님: 책에서는 나중에 $\|f_n(x) - f(x)\|_\infty \to 0$ 으로 적었던 것 같은데...
[6] Uniform Cauchy Property

$$\sup_{x \in E} |f_n(x) - f_m(x)| \leq \epsilon$$

Then

$$\begin{aligned} |f_n(x) - f(x)| = |f_n(x) - f_m(x) + f_m(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq |f_m(x) - f(x)| + \epsilon \end{aligned}$$

Fix $n \geq N$ and let $m \to \infty$. Observe that $|f_m(x) - f(x)| \to 0$ due to pointwise convergence. Therefore for every $x \in E$,

$$n \geq N \implies |f_n(x) - f(x)| \leq \epsilon$$

September 1st, 2022 (Practice)

해석개론 1 복습

1. Real Number System

Let $A \subseteq \mathbb{R}$.

- $b \in \mathbb{R}$ is an upper bound of A: $\forall a \in A \implies a \le b$.
- $b \in \mathbb{R}$ is a lower bound of A: $\forall a \in A \implies a \ge b$.
- Least uppoer bound is denoted as $\sup A$.
- Greatest lower bound is denoted as $\inf A$.
- Least upper bound property: If $A \neq \varnothing$, $\exists \sup A$.
- Extended Real Numbers: $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$
- Now, if $\varnothing = A \subseteq \overline{\mathbb{R}}$, $\sup A = -\infty$.

2. Metric Spaces

Metric space: (X, d_X) where $d_X : X \times X \to \mathbb{R}$. For all $x, y, z \in X$ the following must hold.

(1) $d_X(x, y) = 0 \iff x = y$.

(2) $d_X(x, y) = d_X(y, x)$ (Symmetric)

(3) $d_X(x, y) + d_X(y, z) \ge d_X(x, z)$

Notation. (Neighborhood) Ball of radius r, centered at p is denoted as

$$B_r(p) = \{x \in X \mid d_X(x, p) < r\}$$

- $U \subseteq X$ is open $\iff \forall p \in U, \exists\, r > 0$ such that $B_r(p) \subseteq U$.
- $C \subseteq X$ is closed $\iff C$ contains every limit point of C. Or alternatively, C^C is open.
- Union of open sets is open, <u>finite</u> intersection of open sets is open.
- $p \in B \subseteq X$ is a limit point of $B \iff \forall r \ge 0, (B_r(p) \setminus \{p\}) \cap B \neq \varnothing$.[7]
- A' is the set of limit points of A.

[7] 임의의 근방에서 자기자신을 제외하고 B의 점이 존재한다.

- $\overline{A} = A \cup A'$, which is the smallest closed set containing A.
- $A \subseteq X$ is dense in $X \iff \overline{A} = X$.
- $A \subseteq X$ is bounded $\iff \exists\, r > 0$ such that $A \subseteq B_r(p)$ for some $p \in X$.
- Sets A and B are separated $\iff \overline{A} \cap B = \varnothing = A \cap \overline{B}$.
- Set C is disconnected $\iff \exists$ non-empty separated sets A, B such that $C \subseteq A \cup B$.

Suppose $\{U_\alpha\}$ is a collection of open sets in X.

- $\{U_\alpha\}$ is an open cover of $A \iff A \subseteq \bigcup_\alpha U_\alpha$.
- $K \subseteq X$ is compact $\iff$ for every open cover of K, there exists a finite subcover of K.

$$\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \text{ such that } K \subseteq \bigcup_{k=1}^{n} U_{\alpha_k}$$

- (Heine-Borel) In $\mathbb{R}^n$, compact $\iff$ bounded and closed.
- If K is compact and $A \subseteq K$ is closed, then A is also compact.
- If $\{K_\alpha\}$ is a collection of compact sets and $\bigcap_\alpha K_\alpha = \varnothing$, then

$$\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \text{ such that } \bigcap_{k=1}^{n} K_{\alpha_k} = \varnothing.$$ [8]

3. Sequences

A sequence $a : \mathbb{N} \to A$, is a function. We write $a(i) = a_i$, and we usually consider sequences in metric spaces.

- $\{a_n\}$ converges to $\alpha \iff \forall \epsilon > 0, \exists\, N \in \mathbb{N}$ such that $n \geq N \implies d_X(a_n, \alpha) < \epsilon$.
- (Cauchy Sequence) $\{a_n\}$ is Cauchy
$\iff \forall \epsilon > 0, \exists\, N \in \mathbb{N}$ such that $n, m \geq N \implies d_X(a_n, a_m) < \epsilon$.
- (X, d) is complete $\iff$ every Cauchy sequence converges.[9]
- $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}$.
- $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$.

[8] 정의로 쉽게 보일 수 있다?

[9] 수렴하면 코시 수열이지만, 모든 코시 수열이 수렴하지는 않는다. Consider any sequence of rational numbers converging to an irrational real number.

- $\lim a_n = \alpha \iff \limsup a_n = \liminf a_n = \alpha$ $(\alpha \in \mathbb{R})$.
- For power series $\sum a_n x^n$, the radius of convergence $R \in \overline{R}$ is calculated as

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

- Absolute convergence implies convergence.

4. Limit of Functions

Given metric spaces X, Y, define a function $f : E \subseteq X \to Y$.

- If $p \in E'$[10] then we can define $\lim_{x\to p} f(x) = \alpha$ as

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } 0 < d_X(x,p) < \delta \implies d_Y(f(x), \alpha) < \epsilon.$$

  Or equivalently, for any sequence $\{a_n\}$ in X with $a_n \neq p$,

$$\text{if } \lim_{n\to\infty} a_n = p \text{ then } \lim_{n\to\infty} f(a_n) = \alpha.$$

- f is continuous at $p \in E$[11] $\iff$

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } x \in E, d_X(x,p) < \delta \implies d_Y(f(x), f(p)) < \epsilon.$$

  Or equivalently, for any sequence $\{a_n\}$ in X,[12]

$$\text{if } \lim_{n\to\infty} a_n = p \text{ then } \lim_{n\to\infty} f(a_n) = f(p).$$

- f is continuous $\iff$ for any open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.
- Suppose that f is continuous.
  - If $K \subseteq E$ is compact, $f(K)$ is also compact.
  - If $C \subseteq E$ is connected, $f(C)$ is also connected.
- (Extreme Value Theorem) Suppose $K \subseteq E$ is compact and $f : K \to \mathbb{R}$ is continuous. Because $f(K)$ is a compact set in $\mathbb{R}$, it is a closed interval. Hence f has a maximum/minimum.
- (Uniform Continuity) f is uniformly continuous on $E \iff$

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } \forall x, \forall y \in E,\ d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

[10] 함수의 극한은 극한점에서 논한다! 다가갈 점들이 있어야 하지 않겠는가?
[11] Limit point가 아니어도 정의할 수 있으며, 고립점에서는 연속이다.
[12] 여기서는 $a_n \neq p$ 조건이 빠진다.

- If $f : E \subseteq X \to Y$ is continuous and E is compact, f is uniformly continuous.

5. Differentiation

Function $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b] \iff$

$$\text{the limit } f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \text{ exists.}$$

- If f is differentiable at $x = p$, then f is continuous at $x = p$.
- If f is differentiable at $x = p$ and $g : f([a, b]) \to \mathbb{R}$ is differentiable at $x = f(p)$ $\implies g \circ f$ is differentiable at $x = p$ and

$$(g \circ f)'(p) = g'\big(f(p)\big)f'(p).$$

- (Fermat) If f is differentiable and has a local extremum at $x = a$, then $f'(a) = 0$.
- (Mean Value Theorem) If f is continuous on $[a, b]$ and differentiable on (a, b), there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

6. Integration

Given a <u>bounded</u> function $f : [a, b] \to \mathbb{R}$, a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and a monotonically increasing function $\alpha : [a, b] \to \mathbb{R}$, define

$$U(P, f, \alpha) = \sum_{i=0}^{n-1} \sup_{x \in [x_i, x_{i+1}]} f(x)\big(\alpha(x_{i+1}) - \alpha(x_i)\big)$$

$$L(P, f, \alpha) = \sum_{i=0}^{n-1} \inf_{x \in [x_i, x_{i+1}]} f(x)\big(\alpha(x_{i+1}) - \alpha(x_i)\big)$$

We define upper integral and lower integral as follows:

$$\overline{\int_a^b} f \, d\alpha = \inf_{P \in \mathcal{P}[a,b]} U(P, f, \alpha) \qquad \underline{\int_a^b} f \, d\alpha = \sup_{P \in \mathcal{P}[a,b]} L(P, f, \alpha).$$

f is Stieltjes integrable with respect to $\alpha \iff$

$$\forall \epsilon > 0, \exists\, P \in \mathcal{P}[a, b] \text{ such that } U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Or equivalently, $\overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha$. We write $f \in \mathcal{R}(\alpha)$.

Supplementary Material

F is a field for this section.

Definition. (Vector Space) A set V with addition $+ : V \times V \to V$ and scalar multiplication $\cdot : F \times V \to V$ is a vector space over F if the following properties hold.

(1) (Associativity of $+$) $u + (v + w) = (u + v) + w$ for all $v, w, u \in V$.

(2) (Commutativity of $+$) $v + w = w + v$ for all $v, w \in V$.

(3) (Identity of $+$) $\exists\, 0_V \in V$ such that $v + 0 = 0 + v = v$ for all $v \in V$.

(4) (Inverse of $+$) For each $v \in V$, $\exists\, x \in V$ such that $v + x = x + v = 0_V$.

(5) (Identity of $\cdot$) $1v = v$ for $v \in V$, where $1 \in F$ is the multiplicative identity in F.

(6) (Distributive Property of $\cdot$ w.r.t. Vector $+$) For $a \in F$ and $v, w \in V$, $a(v + w) = av + aw$.

(7) (Distributive Property of $\cdot$ w.r.t. Field $+$) For $a, b \in F$ and $v \in V$, $(a + b)v = av + bv$.

(8) (Compatibility of $\cdot$ w.r.t. $+$) $a(bv) = (ab)v$ for $a, b \in F$, $v \in V$.

We write $V = (V, +, \cdot)$.

Definition. (Normed Vector Space) A vector space V with a norm $\|\cdot\| : V \to \mathbb{R}$ if a normed vector space if the following properties hold.

(1) $\|v\| \geq 0$ for all $v \in V$.

(2) $\|v\| = 0 \iff v = 0$.

(3) For all $\alpha \in F$ and $v \in V$, $\|\alpha v\| = |\alpha|\,\|v\|$.

(4) (Triangle Inequality) For all $v, w \in V$, $\|v + w\| \leq \|v\| + \|w\|$.

For inner product spaces, $F = \mathbb{C}$ or $F = \mathbb{R}$.

Definition. (Inner Product Space) A vector space V with an inner product $\langle \cdot, \cdot \rangle : V \times V \to F$ is an inner product space if the following properties hold.

(1) (Linearity in the first argument) For $x, y, z \in V$ and $a, b \in F$, $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$.

(2) (Conjugate Symmetry) For $x, y \in V$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

(3) (Positive Definiteness) If $0 \neq x \in V$, $\langle x, x \rangle > 0$.

Remark. An inner product can induce a norm by $\|v\| = \sqrt{\langle v, v\rangle}$. With norm as the distance metric, the following holds.

$$\text{Inner Product Space} \implies \text{Normed Vector Space} \implies \text{Metric Space}$$

If the inner product space is complete with respect to the distance metric, it is said to be a Hibert space.

September 6th, 2022

More examples.

Example 7.5 Consider $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ for $x \in \mathbb{R}$.

$$f(x) = \lim_{n\to\infty} f_n(x) \equiv 0$$

but,

$$f_n'(x) = \sqrt{n}\cos nx \implies f_n'(0) = \sqrt{n}.$$

As $n \to \infty$, $f_n'(0)$ does not converge to $f'(0)$.

Example 7.6 Consider $f_n(x) = nx(1-x^2)^n$ for $x \le 0 \le 1$. Note that

$$f_n(0) = 0,\ f_n(1) = 0.$$

When $0 < x < 1$, $f_n \to f \equiv 0$. (Theorem 3.20 (d)) Thus $\lim\limits_{n\to\infty} f_n(x) = 0$ for $0 \le x \le 1$.

But

$$\int_0^1 nx(1-x^2)^n\,dx = \left[\frac{-n}{2n+2}(1-x^2)^{n+1}\right]_0^1 = \frac{n}{2n+2},$$

and thus

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \frac{1}{2} \ne 0 = \int_0^1 f(x)\,dx.$$

Definition. $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly on $E \iff \left(\sum_{k=1}^{n} f_k\right)$ converges uniformly on E.

Theorem 7.10 (Weierstrass M-test) Suppose $f_n : E \to \mathbb{C}$ and that for every n, $\exists\, M_n \in \mathbb{R}$ such that

$$|f_n(x)| \le M_n, \quad (x \in E)$$

and $\sum_{n=1}^{\infty} M_n < \infty$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

Proof. We want to show that the series is Cauchy.

For $m > n$, we have

$$\left|\sum_{k=n}^{m} f_k(x)\right| \le \sum_{k=n}^{m} |f_k(x)| \le \sum_{k=n}^{m} M_k.$$

Given $\epsilon > 0$, choose $m, n \in \mathbb{N}$ such that for $m, n \ge N$, $\sum_{k=n}^{m} M_k < \epsilon$. Then we get

$$\left|\sum_{k=n}^{m} f_k(x)\right| < \epsilon, \text{ for all } m, n \ge N.$$

By Theorem 7.8, $\sum f_n$ converges uniformly.

Theorem 7.11 Given metric space (Y, d) and $E \subseteq Y$, suppose that $f_n \xrightarrow{u} f$ on E and $x \in E'$. If

$$\lim_{t\to x} f_n(t) = A_n \in \mathbb{C}, \quad \text{(limit exists)}$$

then the sequence (A_n) converges, and

$$\lim_{n\to\infty} A_n = \lim_{t\to x} f(t).$$

In conclusion,

$$\lim_{n\to\infty} \lim_{t\to x} f_n(t) = \lim_{t\to x} \lim_{n\to\infty} f_n(t).$$

Proof.

((A_n) converges in $\mathbb{C}$) Since $\mathbb{C}$ is complete, we will show that (A_n) is a Cauchy sequence. Let $\epsilon > 0$. Since $f_n \xrightarrow{u} f$ on E,

$$\exists\, N \in \mathbb{N} \text{ such that } n, m \geq N \implies |f_n(t) - f_m(t)| \leq \epsilon. \ (\forall t \in E)$$

From $\lim_{t\to x} f_n(t) = A_n$, we can choose t arbitrarily close to x, such that for $n, m \geq N$,

$$|f_n(t) - A_n| < \epsilon \text{ and } |f_m(t) - A_m| < \epsilon.$$

Therefore for all $n, m \geq N$,

$$\begin{aligned} |A_n - A_m| &= |A_n - f_n(t) + f_n(t) - A_m + f_m(t) - f_m(t)| \\ &\leq |f_n(t) - A_n| + |f_m(t) - A_m| + |f_n(t) - f_m(t)| < 3\epsilon, \end{aligned}$$

and thus (A_n) is a Cauchy Sequence.

($\lim_{n\to\infty} A_n = \lim_{n\to x} f(t)$) Let $A = \lim_{n\to\infty} A_n$. We want to show that for all $\epsilon > 0$,

$$\exists\, \delta > 0 \text{ such that } 0 < d(t, x) < \delta \implies |f(t) - A| < \epsilon.$$

Now,

$$|f(t) - A| \leq \sup_{s\in E} |f(s) - f_n(s)| + |f_n(t) - A_n| + |A_n - A|.$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that for $n \geq N$,

$$\sup_{s\in E} |f(s) - f_n(s)| < \frac{\epsilon}{3} \text{ and } |A_n - A| < \frac{\epsilon}{3}.$$

Fix such N and choose δ such that for $0 < d(x, t) < \delta$ and $t \in E$,

$$|f_N(t) - A_N| \leq \frac{\epsilon}{3}.$$

Thus for $t \in E$ and $0 < d(x,t) < \delta$,

$$|f(t) - A| \le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Theorem 7.12 Suppose $f_n E \to \mathbb{C}$ is continuous on E and $f_n \xrightarrow{u} f$ on E. Then f is continuous on E.

Proof. Let $x \in E$. If $x \in E'$, f is continuous at x by Theorem 7.11. If x is an isolated point (not a limit point), f is continuous at x by definition of continuity.

앞으로는 E를 전부 metric space라고 가정할게요.

이 정리는 언제 uniformly converge 하는지 알려줍니다.

Theorem 7.13 Given a compact metric space K, suppose that

(1) f_n and $f : K \to \mathbb{C}$ are continuous on K.

(2) $f_n \to f$ pointwise.

(3) $f_n(x) \ge f_{n+1}(x)$ for $x \in K$.[13]

Then $f_n \xrightarrow{u} f$ on K.

Proof. Let $g_n(x) = f_n(x) - f(x)$. Then $g_n(x)$ is continuous, decreasing and $g_n \to 0$ as $n \to \infty$ for all $x \in K$. Let $\epsilon > 0$ be given.

Claim. There exists $N \in \mathbb{N}$ such that $0 \le g_n(x) < \epsilon$ for all $x \in K$.

Proof. Let $K_n = \{x \in K : g_n(x) \ge \epsilon\}$. Then $K_n = K \cap g_n^{-1}([\epsilon, \infty))$.[14] Since g_n is decreasing, $K_{n+1} \subseteq K_n$, but because $g_n \to 0$, $\bigcap_{n=1}^{\infty} K_n = \varnothing$. By Theorem 2.36, there exists $N \in \mathbb{N}$ such that $K_N = \varnothing$, and then $K_n = \varnothing$ for $n \ge N$. Thus, $0 \le g_n(x) < \epsilon$ for $\forall x \in K$, $\forall n \ge N$.

Remark. Compactness is necessary here. Consider $f_n(x) = \dfrac{1}{nx+1}$ on $x \in E = (0,1)$. f_n does not converge to 0 uniformly.

Proof. Suppose $f_n \xrightarrow{u} 0$, and take $\epsilon = 1/2$. Then,

$$\exists\, N \in \mathbb{N} \text{ such that } x \in (0,1) \implies \frac{1}{Nx+1} < \frac{1}{2}.$$

This gives a contradiction because the equation above gives $Nx > 1$, but we can choose x arbitrarily close to 0.

[13] f_n only needs to be monotone. See Dini's Theorem.

[14] Closed subset of a compact set is also compact, and the inverse image of closed set is closed if the function is continuous

Definition. Let (X, d) be a metric space. Define

$$C(X, \mathbb{C}) = \{f : X \to \mathbb{C} \mid f \text{ is continuous and bounded}\}.$$

If there is no ambiguity, we write $C(X) = C(X, \mathbb{C})$.

Let $\|f\| = \sup\limits_{x \in X} |f(x)|$. Then $\|\cdot\|$ is a norm on $C(X)$.

(1) $\|f\| = 0 \iff f \equiv 0$.

(2) $\|f\| < \infty$.

(3) $\|f + g\| \leq \|f\| + \|g\|$.

Define $d(f, g) = \|f - g\|$, then $(C(X), d)$ is a metric space.

Therefore, $f_n \overset{u}{\to} f \iff f_n \to f$ in $(C(X), d)$.