Assignment 6

Due Date: 2019/06/07, 9:00 AM

1. The function $f : [0,\ 1] \to \mathbb{R}$ is defined by $f(x) = x^2$. For $n \in \mathbb{N}$, let

$$P_n = \left\{0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1\right\} \in \mathcal{P}[a,\ b]$$

be a (equi) partition of the interval $[0,\ 1]$.

(1) Compute $U(f,\ P_n)$ and $L(f,\ P_n)$.

(2) Prove that

$$\lim_{n\to\infty} U(f,\ P_n) = \lim_{n\to\infty} L(f,\ P_n) = \frac{1}{3}\ .$$

(3) (This problem will not be graded, but work for yourself) Deduce $\int_0^1 f = \frac{1}{3}$ from (2) without recalling any high technology such as the fundamental theorem of calculus.

2. Determine whether the following function $f : [0,\ 1] \to \mathbb{R}$ is Riemann integrable or not, and demonstrate your answer.

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap [0,\ 1] \\ 0 & \text{if } x \in \mathbb{Q}^c \cap [0,\ 1] \end{cases}$$

3. Answer the following questions.

(1) Let $f : [a,\ b] \to \mathbb{R}$ be a constant function such that $f(x) = c$ for all $x \in [a,\ b]$. Prove that f is Riemann integrable and that

$$\int_a^b f = c(b-a)\ .$$

(2) Let $f : [a,\ b] \to \mathbb{R}$ be a function such that $f(x) = c$ for all $x \in [a,\ b]$, except for **finitely** many points. Prove that f is Riemann integrable and that

$$\int_a^b f = c(b-a)\ .$$

4. Suppose that two functions $f,\ g : [a,\ b] \to \mathbb{R}$ are Riemann integrable and satisfy

$$f(x) \geq g(x) \text{ for all } x \in [a,\ b]\ .$$

Prove that $\int_a^b f \geq \int_a^b g$.

5. Suppose that $f : [a,\ b] \to \mathbb{R}$ is a Riemann integrable function and satisfies $f(x) \geq 0$ for all $x \in [a,\ b]$. By problem **3-(1)** and problem **4** with $g \equiv 0$, we know that $\int_a^b f \geq 0$. In this problem, we shall assume in addition that, there exists $x_0 \in [a,\ b]$ such that $f(x_0) > 0$.

(1) Explain why we cannot assert that $\int_a^b f > 0$.

(2) Suppose moreover that f is a continuous function. Prove that $\int_a^b f > 0$.

6. Let $f : [a,\ b] \to \mathbb{R}$ be a bounded function, and define

$$D_f = \{x : x \in [a,\ b] \text{ and } f \text{ is not continuous at } x\}\ .$$

(1) Suppose that D_f is finite set; in other words, f is discontinuous only at finite points. Prove that f is a Riemann integrable function.

(2) Suppose that D_f is countable set. Prove that f is a Riemann integrable function.

7. Suppose that two functions $f,\ g : [a,\ b] \to \mathbb{R}$ are Riemann integrable. Prove or disprove that $fg : [a,\ b] \to \mathbb{R}$ is a Riemann integrable function.