HW Set 2 Solution

1. Suppose f is a real valued continuous function on $\mathbb{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \cdots$, and (f_n) is equicontinuous on $[0, 1]$. Show that f is a constant on $[0, \infty)$.

   Solution) If there exists a $t > 0$ s.t. $f(t) \neq f(0)$, then take $\varepsilon = |f(t) - f(0)|$. Since (f_n) is equicontinuous on $[0,\, 1]$, there exists a $\delta > 0$ s.t.

   $$x,\, y \in [0,\, 1],\, n \in \mathbb{N},\, |x - y| < \delta \Longrightarrow |f_n(x) - f_n(y)| < \varepsilon.$$

   Take a $m \in \mathbb{N}$ s.t. $0 < \frac{t}{m} < \min\{1,\, \delta\}$, then $|f_m(\frac{t}{m}) - f_m(0)|$
   $= |f(t) - t(0)| = \varepsilon < \varepsilon$ gives contradiction, so $f(t) = f(0)$ for all $t \in (0,\, \infty)$. □

2. Suppose (f_n) is an equicontinuous sequence of functions on a compact set K, and (f_n) converges pointwise on K. Prove that (f_n) converges uniformly on K. Take a counterexample(without proof) when K is not compact.
   Hint: Review the proof of theorem 7.25.

   Solution) For given $\varepsilon > 0$, since (f_n) is equicontinuous, there exists a $\delta > 0$ s.t.

   $$x,\, y \in K,\, n \in \mathbb{N},\, d(x,\, y) < \delta \Longrightarrow d(f_n(x),\, f_n(y)) < \frac{\varepsilon}{3}.$$

   Consider a open cover $\bigcup_{x \in K} B_\delta(x)$ of K. Then finite subcover $\bigcup_{k=1}^{l} B_\delta(x_k)$ contains K. Let N_k be natural numbers s.t. $n,\, m \geq N_k$ implies
   $d(f_n(x_k),\, f_m(x_k)) < \frac{\varepsilon}{3}$ ($\because$ pointwise convergence). Define
   $N := \max\{N_k \,|\, 1 \leq k \leq l\}$.
   Now if $m,\, n \geq N$, for any $x \in K$, there exists a $1 \leq i \leq n$ s.t. $x \in B_\delta(x_i)$. Then we have

   $$d(f_m(x),\, f_n(x)) \leq d(f_m(x),\, f_m(x_i)) + d(f_m(x_i),\, f_n(x_i)) + d(f_n(x_i),\, f_n(x))$$
   $$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

and (f_n) uniformly converges by theorem 7.8.

For a counterexample, take $f_n = \frac{x}{n}$ defined on $\mathbb{R}$. For any $\varepsilon > 0$ set $\delta = \varepsilon$ then $|x-y| < \varepsilon$ implies $f_n(x) - f_n(y)| = |\frac{x}{n} - \frac{y}{n}| < \frac{|x-y|}{n} \leq |x-y| < \varepsilon$ so (f_n) is equicontinuous. Surely $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = 0$ for each fixed x and (f_n) is pointwise converges. But (f_n) doesn't converge uniformly, because for any $\varepsilon > 0$, and for any $N \in \mathbb{N}$, there exists $x = (N+1)\varepsilon$ so that $|f_N(x)| = |\frac{N+1}{N}\varepsilon| > \varepsilon$. $\square$

3. If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n \, dx = 0 \quad \text{for all } n = 0, 1, 2, \cdots,$$

prove that $f(x) = 0$ on $[0,1]$.

Hint: The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem (Theorem 7.26) to show that $\int_0^1 f^2(x)\, dx = 0$.

Solution) From above condition, $\int_0^1 f(x)P(x)dx = 0$ for any polynomial function P. Since f is continuous, there exists a $M > 0$ s.t. $|f(x)| \leq M$. For any $\varepsilon > 0$, choose a polynomial function P s.t. $|f(x) - P(x)| < \frac{\varepsilon}{M}$ for any $x \in [0,\, 1]$ using theorem 7.26. Then

$$\left|\int_0^1 f(x)^2 dx\right| \leq \left|\int_0^1 f(x)\{f(x) - P(x)\}dx\right| + \left|\int_0^1 f(x)P(x)dx\right|$$

$$\leq \int_0^1 |f(x)|\,|f(x) - P(x)|\,dx + 0 \leq \varepsilon.$$

This means that $\int_0^1 f(x)^2 dx = 0$. If $f(a) \neq 0$ for some $a \in [0,\, 1]$, then by continuity of $f(x)^2$, there exists a closed interval $I \subsetneq [0,\, 1]$ that is not a one point set s.t. $f(x)^2 > \frac{1}{2}f(a)^2$ on I, and $\int_0^1 f(x)^2 dx \geq \int_I f(x)^2 dx \geq \frac{1}{2}(\text{length of } I)f(a)^2 > 0$ gives contradiction. Therefore $f(x) = 0$ for all $x \in [0,\, 1]$. $\square$

4. Assume that (f_n) is a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \le f_n(x) \le 1$ for all x and all n.

   (a) Prove that there is a function f and a sequence (n_k) such that
   $$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$
   for every $x \in \mathbb{R}$. (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem.*)

   (b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets.

   Hint: (i) Some subsequence (f_{n_l}) converges at all rational points r, say, to $f(r)$. (ii) Define $f(x)$, for any $x \in \mathbb{R}$, to be $\sup f(r)$, the sup being taken over all $r \le x$. (iii) Show that $f_{n_l}(x) \to f(x)$ at every x at which f is continuous. (This is where monotonicity is strongly used.) (iv) A subsequence of (f_{n_l}) converges at every point of discontinuity of f since there are at most countably many such points. This proves (a). To prove (b), modify your proof of (iii) appropriately.

Solution)

(a) Let $(q_n)_{n=1}^{\infty}$ be an enumeration of all (different) rational numbers. By theorem 7.23 there exists a subsequence (f_{n_l}) of (f_n) s.t. f_{n_l} converges pointwise on $\mathbb{Q}$. Define $g : \mathbb{Q} \to \mathbb{R}$, $g(q) := \lim_{l\to\infty} f_{n_l}(q)$. Since each f_{n_l} is increasing function, g is. Define
$h : \mathbb{R} \to \mathbb{R}$, $h(x) := \sup\{g(q) \,|\, q \le x,\, q \in \mathbb{Q}\}$. From definition $h(x)$ is increasing function. Note that $h(p) = g(p) = \lim_{l\to\infty} f_{n_l}(p)$ for $p \in \mathbb{Q}$; observe $g(q) \le g(p)$ for any rational number $q \le p$ and apply the definition of g.
If $h(x)$ is continuous at $x = a$, then for $\varepsilon > 0$, there exists a $\delta > 0$ s.t. $|x - a| < \delta$ implies $|h(x) - h(a)| < \frac{\varepsilon}{2}$. Choose two rational numbers $a - \delta < p < a < q < a + \delta$, and choose a natural number $N \in \mathbb{N}$ s.t. $l \ge N$ implies $|f_{n_l}(p) - g(p)|$, $|f_{n_l}(q) - g(q)| < \frac{\varepsilon}{2}$. Then $l \ge N$ implies
$$h(a) - \varepsilon < h(p) - \frac{\varepsilon}{2} = g(p) - \frac{\varepsilon}{2} < f_{n_l}(p) \le f_{n_l}(a),$$
$$f_{n_l}(a) \le f_{n_l}(q) < g(q) + \frac{\varepsilon}{2} = h(q) + \frac{\varepsilon}{2} < h(a) + \varepsilon$$

and $|f_{n_l}(a) - h(a)| < \varepsilon$. This shows $h(a) = \lim_{l\to\infty} f_{n_l}(a)$.
On the other hand, h has only (at most) countably many discontinuous points by theorem 4.30. Let $A \subset \mathbb{R}$ be the set of discontinuous points of h. Apply theorem 7.23 once again to (f_{n_l}), then there exists further subsequence (f_{n_m}) s.t. f_{n_m} also converges on A as well as on $\mathbb{R} \setminus A$.
Now define $f(x) := \lim_{m\to\infty} f_{n_m}(x)$.
Caution: $\lim_{m\to\infty} f_{n_m}(x)$ may be different from $h(x)$, for example, if
$f_n = \begin{cases} 0 & (x < \sqrt{2}) \\ 1 & (x \geq \sqrt{2}) \end{cases}$.

(b) Since every compact set is bounded, it suffices to consider when $K = [-M,\, M]$ for some $M > 0$. For $\varepsilon > 0$, take $\delta > 0$ s.t. $x,\, y \in K$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{2}$($\because$ f is continuous on compact set). Choose $a \in \mathbb{N}$ s.t. $\frac{2M}{a} < \delta$, and let $x_k := -M + \frac{2M}{a}k$, $0 \leq k \leq a$.
Let N_k be natural numbers s.t. $m \geq N_k$ implies $|f_{n_m}(x_k) - f(x_k)| < \frac{\varepsilon}{2}$.
Let $N := \max\{N_k \,|\, 0 \leq k \leq a\}$.
Then if $m \geq N$ and $x \in [-M,\, M]$, there exists i s.t. $x_i \leq x \leq x_{i+1}$, and we have

$$f_{n_m}(x) \leq f_{n_m}(x_{i+1}) < f(x_{i+1}) + \frac{\varepsilon}{2} < f(x) + \varepsilon,$$

$$f(x) - \varepsilon < f(x_i) - \frac{\varepsilon}{2} < f_{n_m}(x_i) \leq f_{n_m}(x),$$

and $|f_{n_m}(x) - f(x)| < \varepsilon$ holds. That is, $(f_{n_m})_{m=1}^{\infty}$ uniformly converges to f. □

5. Recall that $\mathcal{R}(\alpha)$ denotes the family of Riemann-Stieltjes integrable functions with respect to α over $[a, b]$.

   Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 \, d\alpha \right\}^{1/2}.$$

   Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality (as in the proof of Theorem 1.37).

Solution) First we prove the Schwarz inequality. For any $f,\, g \in \mathcal{R}(\alpha)$, $|tf-g|^2 = (tf-g)(\bar{t}\bar{f}-\bar{g}) = |t|^2|f|^2 - 2\mathfrak{Re}(tf\bar{g}) + |g|^2 \geq 0$. for any complex number t(note that all above functions are in $\mathcal{R}(\alpha)$ by theorem 6.11, 6.13). Let $t = re^{i\theta}$, then $|f|^2r^2 - 2r\mathfrak{Re}(e^{i\theta}f\bar{g}) + |g|^2 \geq 0$. By integrating, we have

$$\left(\int_a^b |f|^2 d\alpha\right) r^2 - 2r\mathfrak{Re}\left(e^{i\theta}\int_a^b f\bar{g}d\alpha\right) + \int_a^b |g|^2 d\alpha \geq 0.$$

From middle school math, we already know if $a \geq 0$ then

$$ax^2 + bx + c \geq 0\,(\forall x \in \mathbb{R}) \implies b^2 - 4ac \leq 0.$$

So we have

$$\left(\mathfrak{Re}\left(e^{i\theta}\int_a^b f\bar{g}d\alpha\right)\right)^2 \leq \left(\int_a^b |f|^2 d\alpha\right)\left(\int_a^b |g|^2 d\alpha\right),$$

or set $\theta = -\mathrm{Arg}\left(\int_a^b f\bar{g}d\alpha\right)$ and we get

$$\left|\int_a^b f\bar{g}d\alpha\right|^2 \leq \left(\int_a^b |f|^2 d\alpha\right)\left(\int_a^b |g|^2 d\alpha\right).$$

If equality holds, then we have two cases.
(1) If $\int_a^b |f|^2 d\alpha = 0$, then Schwarz inequality gives
$|\int_a^b |f|\cdot 1 d\alpha| \leq \left(\int_a^b |f|^2 d\alpha\right)^{\frac{1}{2}}\left(\int_a^b 1 d\alpha\right)^{\frac{1}{2}} = 0$ so $\int_a^b |1\cdot f + 0\cdot g| d\alpha = 0$.
(2) If $\int_a^b |f|^2 d\alpha \neq 0$, then from the form $ax^2+bx+c \geq 0$ with $b^2-4ac = 0$, $\int_a^b |re^{i\theta}f - g|^2 d\alpha = 0$, and similar process gives
$\int_a^b |re^{i\theta}\cdot f + (-1)\cdot g| d\alpha = 0$.
Conclusion: there exists $a,\, b \in \mathbb{C}$, not both zero, s.t. $\int_a^b |af+bg| d\alpha = 0$. You can also check that this condition forces the equlity to hold.

Return to the original problem, then we have

$$(||F||_2 + ||G||_2)^2 = ||F||_2^2 + 2||F||_2||G||_2 + ||G||_2^2$$

$$
\begin{aligned}
&= ||F||_2^2 + ||G||_2^2 + 2\left(\int_a^b |F|^2 d\alpha\right)^{\frac{1}{2}}\left(\int_a^b |G|^2 d\alpha\right)^{\frac{1}{2}} \\
&\geq ||F||_2^2 + ||G||_2^2 + 2\left|\int_a^b F\bar{G} d\alpha\right| \\
&= ||F||_2^2 + ||G||_2^2 + \left|\int_a^b F\bar{G} d\alpha\right| + \left|\int_a^b \bar{F}G d\alpha\right| \\
&\geq \left|\int_a^b |F|^2 d\alpha + \int_a^b |G|^2 d\alpha + \int_a^b F\bar{G} d\alpha + \int_a^b \bar{F}G d\alpha\right| \\
&= \int_a^b |F+G|^2 d\alpha = |F+G|_2^2
\end{aligned}
$$

and we get $||F||_2 + ||G||_2 \geq ||F+G||_2$. Finally, let $F = f - g$, $G = g - h$ □

6. With the notations of 5, suppose $f \in \mathcal{R}(\alpha)$ and $\epsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \epsilon$.
   Hint: Let $P = \{x_0, \cdots, x_n\}$ be a suitable partition of $[a, b]$, define

$$
g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)
$$

   if $x_{i-1} \leq t \leq x_i$.

   Solution) First assume f is real valued. Since $f \in \mathcal{R}(\alpha)$, there exists $M > 0$ s.t. $|f| \leq M$. Choose a partition P s.t.
   $U(f,\, P,\, \alpha) - L(f,\, P,\, \alpha) < \frac{\varepsilon^2}{3M}$. define g as above. Indeed, g on $[x_{i-1},\, x_i]$ is just a line segment which passes $(x_{i-1},\, f(x_{i-1}))$ and $(x_i,\, f(x_i))$, hence continuous. Let
   $M_{i,\,f} := \sup\{f(x) \,|\, x_{i-1} \leq x \leq x_i\}$ and
   $m_{i,\,f} := \inf\{f(x) \,|\, x_{i-1} \leq x \leq x_i\}$. From elementary geometry we know $m_{i,\,f} \leq \min\{f(x_{i-1}),\, f(x_i)\} \leq g(x) \leq \max\{f(x_{i-1}),\, f(x_i)\} \leq M_{i,\,f}$ on $[x_{i-1},\, x_i]$ Therefore we have

$$
||f - g||_2^2 = \int_a^b |f - g|^2 d\alpha \leq 2M \int_a^b |f - g| d\alpha \leq 2M \overline{\int_a^b} |f - g| d\alpha
$$

$$\leq 2M\sum_{i=1}^{n}M_{i,\,|f-g|}(\alpha(x_i)-\alpha(x_{i-1})) \leq 2M\sum_{i=1}^{n}(M_{i,\,f}-m_{i,\,f})(\alpha(x_i)-\alpha(x_{i-1}))$$

$$\leq 2M\cdot\frac{\varepsilon^2}{3M} < \varepsilon^2.$$

Now if $f = f_1 + if_2$, then pick g_1, g_2 s.t. $||f_1 - g_1||_2, ||f_2 - g_2||_2 < \frac{\varepsilon}{2}$ then $||f - (g_1 + ig_2)||_2 \leq ||f_1 - g_1||_2 + |i| \cdot ||f_2 - g_2||_2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$