

Introduction to Analysis II

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Introduction & Notice

- 7, 8장 나가고 중간고사, 11장 나가고 기말고사
- 연습 시간이 있는 수업 (목 6:30 ~ 8:20)¹
- 오늘 연습 시간: 지난학기 배운 내용 중 필요한 내용 복습

¹가능하면 1시간 반 안에 끝내라고 하심 ㅋㅋ

Chapter 7

Sequences and Series of Functions

September 1st, 2022

기본적으로 수열에 관련된 내용, real/complex-valued 수열이 아니라 함수가 주어졌을 때. 함수들을 모은 ‘sequence of functions’의 극한을 생각하는 것.

Suppose E is a set¹, and let $f_n : E \rightarrow \mathbb{C}$. Then

$$(f_n)_{n=1}^{\infty}$$

is a sequence of (complex-valued) function.

Definition 7.1 (Pointwise Convergence) $(f_n)_{n=1}^{\infty}$ converges **pointwise** on E , if for each $x \in E$ the sequence $(f_n(x))_{n=1}^{\infty}$ converges in \mathbb{C} .

In other words, for each $x \in E$, there exists $a_x \in \mathbb{C}$ and

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |f_n(x) - a_x| < \epsilon.$$

Definition. If (f_n) converges pointwise, we can define a function f by

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad (x \in E)$$

We say that

- f is the *limit* or *limit function* of f_n .
- (f_n) to f pointwise on E .

¹사실은 *metric space* 이다.

Definition. If $\sum f_n(x)$ converges (pointwise) for every $x \in E$, we can define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E)$$

and the function f is called the *sum* of the series $\sum f_n$.

Recall. $f : (E, d) \rightarrow \mathbb{C}$ is continuous on $E \iff f$ is continuous at all $x \in E$.

Recall. (Theorem 4.6) If $p \in E$ and p is a limit point of E ,

$$f \text{ is continuous at } p \iff \lim_{x \rightarrow p} f(x) = f(p)$$

Question. Suppose (f_n) is a sequence of functions. Does the limit function or the sum of the series preserve important properties?

(1) If f_n is continuous, is f continuous?

(2) If f_n is differentiable/integrable, is f differentiable/integrable?

For (1), the question is equivalent to the following:

If p is a limit point, does the following hold?

$$\lim_{x \rightarrow p} \lim_{n \rightarrow \infty} f_n(x) \stackrel{?}{=} \lim_{n \rightarrow \infty} \lim_{x \rightarrow p} f_n(x)$$

And the answer is **No**.

Example 7.2 Suppose $a_{m,n} = \frac{m}{m+n}$ for $m, n \in \mathbb{N}$. We see that

$$\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} a_{m,n} = 1 \neq 0 = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} a_{m,n}$$

Example. Define

$$f_n(x) = \begin{cases} 0 & (\frac{1}{n} \leq x \leq 1) \\ -nx + 1 & (0 \leq x < \frac{1}{n}) \end{cases}$$

then we can easily see that

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = \begin{cases} 0 & (0 < x \leq 1) \\ 1 & (x = 0) \end{cases}$$

Thus f is not continuous at $x = 0$.

Example. Define $f_n : \mathbb{R} \rightarrow \mathbb{R}$ as

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (n = 0, 1, 2, \dots)$$

by direct calculation,

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = 1+x^2 \quad (x \neq 0)$$

since this is a geometric series when $x \neq 0$. If $x = 0$, $f(x) = 0$ and f is not continuous.

Does the limit function preserve Riemann integrability?

Example. For $m = 1, 2, \dots$, define

$$f_m(x) = \lim_{n \rightarrow \infty} (\cos m! \pi x)^{2n} = \begin{cases} 1 & (m!x \in \mathbb{Z}) \\ 0 & (m!x \notin \mathbb{Z}) \end{cases}$$

We see that $f_m(x)$ is Riemann integrable. However,

Claim.

$$f(x) = \lim_{m \rightarrow \infty} f_m(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

and $f(x)$ is nowhere continuous thus not Riemann integrable.

Proof. Suppose $x = p/q \in \mathbb{Q}$. ($p, q \in \mathbb{Z}$) If we take $m \geq q$, we see that $m!x \in \mathbb{Z}$. Thus $f_m(x) = 1$. If $x \notin \mathbb{Q}$, $m!x$ can never be in \mathbb{Z} and $f_m(x) = 0$.

Question. Uniform continuity를 할 때 uniform이 어디서 나오죠? 해석학에서 그 점에서 뭐가 성립한다, 그러면 그 점과 그 근방에서만 확인하면 됐었죠. Continuity는 local property죠. 그런데 uniform continuity는 전체가 다 uniform하게 성립한다는 의미입니다.

Recall. $f : (X, d) \rightarrow (Y, d)$ is uniformly continuous on X ² if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_X(p, q) < \delta \implies d_Y(f(p), f(q)) < \epsilon$$

즉, 모든 점에서 똑같이 잡을 수 있다!

Recall. (Theorem 4.19) If X is compact and f is continuous on X , then f is uniformly continuous on X .³

이제부터 나오는 uniform convergence는 sequence에 관한 것입니다!

²Subspace of metric space is also a metric space

³갑자기 왜 uniform continuity 얘기를 하나, 헛갈리지 말고 기억하시라고!

Definition 7.7 (Uniform Convergence) Suppose $f_n : E \rightarrow \mathbb{C}$ is a sequence of functions. $(f_n)_{n=1}^\infty$ **converges uniformly** on E to a function f if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall x \in E, n \geq N \implies |f_n(x) - f(x)| \leq \epsilon.^4$$

Also, we say that the series $\sum f_n(x)$ converges uniformly on E if the sequence of partial sums $(\sum_{k=1}^n f_k(x))$ converges uniformly on E .

Pointwise convergence의 경우 $N \in \mathbb{N}$ 이 $x \in E$ 에 의존하지만, uniform convergence의 경우 N 이 x 와 무관하다!

[똑같은 ϵ -띠를 둘러서 $y = f(x)$ 의 근방 안에 $f_n(x)$ ($n \geq N$) 가 모두 들어가 있어야 한다]는 의미에서 uniform 이다.

Theorem 7.9 Suppose

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad (x \in E)$$

Then $f_n \rightarrow f$ converges uniformly on E if and only if

$$\lim_{n \rightarrow \infty} \sup_{x \in E} |f_n(x) - f(x)| = 0$$

which can also be written as

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \implies \sup_{x \in E} |f_n(x) - f(x)| \leq \epsilon$$

Notation. $f_n \rightarrow f$ uniformly on $E \iff f_n \xrightarrow{u} f$ on E .⁵

Theorem 7.8 (Cauchy Criterion for Uniform Convergence) $f_n \xrightarrow{u} f$ on $E \iff$

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n, m \geq N \implies \sup_{x \in E} |f_n(x) - f_m(x)| \leq \epsilon.^6$$

Proof.

(\implies) For given $\epsilon > 0$, fix $x \in E$. Since f_n converges uniformly on E , we can find $N \in \mathbb{N}$ such that for $n, m \geq N$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(\impliedby) Uniform Cauchy property implies that (f_n) is a Cauchy sequence in \mathbb{C} . By the completeness of \mathbb{C} , the limit function $f(x)$ exists. Now we show that this convergence is uniform. For given $\epsilon > 0$ choose $N \in \mathbb{N}$ such that for all $n, m \geq N$,

⁴등호를 붙이는 것이 극한 잡기 편하다???

⁵교수님: 책에서는 나중에 $\|f_n(x) - f(x)\|_\infty \rightarrow 0$ 으로 적었던 것 같은데...

⁶Uniform Cauchy Property

$$\sup_{x \in E} |f_n(x) - f_m(x)| \leq \epsilon$$

Then

$$\begin{aligned} |f_n(x) - f(x)| &= |f_n(x) - f_m(x) + f_m(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq |f_m(x) - f(x)| + \epsilon \end{aligned}$$

Fix $n \geq N$ and let $m \rightarrow \infty$. Observe that $|f_m(x) - f(x)| \rightarrow 0$ due to pointwise convergence.

Therefore for every $x \in E$,

$$n \geq N \implies |f_n(x) - f(x)| \leq \epsilon$$

September 1st, 2022 (Practice)

해석개론 1 복습

1. Real Number System

Let $A \subseteq \mathbb{R}$.

- $b \in \mathbb{R}$ is an upper bound of A : $\forall a \in A \implies a \leq b$.
- $b \in \mathbb{R}$ is a lower bound of A : $\forall a \in A \implies a \geq b$.
- Least upper bound is denoted as $\sup A$.
- Greatest lower bound is denoted as $\inf A$.
- Least upper bound property: If $A \neq \emptyset$, $\exists \sup A$.
- Extended Real Numbers: $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$
- Now, if $\emptyset = A \subseteq \overline{\mathbb{R}}$, $\sup A = -\infty$.

2. Metric Spaces

Metric space: (X, d_X) where $d_X : X \times X \rightarrow \mathbb{R}$. For all $x, y, z \in X$ the following must hold.

- (1) $d_X(x, y) = 0 \iff x = y$.
- (2) $d_X(x, y) = d_X(y, x)$ (Symmetric)
- (3) $d_X(x, y) + d_X(y, z) \geq d_X(x, z)$

Notation. (Neighborhood) Ball of radius r , centered at p is denoted as

$$B_r(p) = \{x \in X \mid d_X(x, p) < r\}$$

- $U \subseteq X$ is open $\iff \forall p \in U, \exists r > 0$ such that $B_r(p) \subseteq U$.
- $C \subseteq X$ is closed $\iff C$ contains every limit point of C . Or alternatively, C^C is open.
- Union of open sets is open, finite intersection of open sets is open.
- $p \in B \subseteq X$ is a limit point of $B \iff \forall r \geq 0, (B_r(p) \setminus \{p\}) \cap B \neq \emptyset$.⁷
- A' is the set of limit points of A .

⁷임의의 근방에서 자기자신을 제외하고 B 의 점이 존재한다.

- $\overline{A} = A \cup A'$, which is the smallest closed set containing A .
- $A \subseteq X$ is dense in $X \iff \overline{A} = X$.
- $A \subseteq X$ is bounded $\iff \exists r > 0$ such that $A \subseteq B_r(p)$ for some $p \in X$.
- Sets A and B are separated $\iff \overline{A} \cap B = \emptyset = A \cap \overline{B}$.
- Set C is disconnected $\iff \exists$ non-empty separated sets A, B such that $C \subseteq A \cup B$.

Suppose $\{U_\alpha\}$ is a collection of open sets in X .

- $\{U_\alpha\}$ is an open cover of $A \iff A \subseteq \bigcup_{\alpha} U_\alpha$.
- $K \subseteq X$ is compact \iff for every open cover of K , there exists a finite subcover of K .

$$\exists \alpha_1, \alpha_2, \dots, \alpha_n \text{ such that } K \subseteq \bigcup_{k=1}^n U_{\alpha_k}$$

- (Heine-Borel) In \mathbb{R}^n , compact \iff bounded and closed.
- If K is compact and $A \subseteq K$ is closed, then A is also compact.
- If $\{K_\alpha\}$ is a collection of compact sets and $\bigcap_{\alpha} K_\alpha = \emptyset$, then

$$\exists \alpha_1, \alpha_2, \dots, \alpha_n \text{ such that } \bigcap_{k=1}^n K_{\alpha_k} = \emptyset.^8$$

3. Sequences

A sequence $a : \mathbb{N} \rightarrow A$, is a function. We write $a(i) = a_i$, and we usually consider sequences in metric spaces.

- $\{a_n\}$ converges to $\alpha \iff \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N \implies d_X(a_n, \alpha) < \epsilon$.
- (Cauchy Sequence) $\{a_n\}$ is Cauchy
 $\iff \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n, m \geq N \implies d_X(a_n, a_m) < \epsilon$.
- (X, d) is complete \iff every Cauchy sequence converges.⁹
- $\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \sup\{a_k : k \geq n\}$.
- $\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \inf\{a_k : k \geq n\}$.

⁸정의로 쉽게 보일 수 있다?

⁹수렴하면 코시 수열이지만, 모든 코시 수열이 수렴하지는 않는다. Consider any sequence of rational numbers converging to an irrational real number.

- $\lim a_n = \alpha \iff \limsup a_n = \liminf a_n = \alpha$ ($\alpha \in \mathbb{R}$).
- For power series $\sum a_n x^n$, the radius of convergence $R \in \overline{\mathbb{R}}$ is calculated as

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}.$$

- Absolute convergence implies convergence.

4. Limit of Functions

Given metric spaces X, Y , define a function $f : E \subseteq X \rightarrow Y$.

- If $p \in E$ ¹⁰ then we can define $\lim_{x \rightarrow p} f(x) = \alpha$ as

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < d_X(x, p) < \delta \implies d_Y(f(x), \alpha) < \epsilon.$$

Or equivalently, for any sequence $\{a_n\}$ in X with $a_n \neq p$,

$$\text{if } \lim_{n \rightarrow \infty} a_n = p \text{ then } \lim_{n \rightarrow \infty} f(a_n) = \alpha.$$

- f is continuous at $p \in E$ ¹¹ \iff

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } x \in E, d_X(x, p) < \delta \implies d_Y(f(x), f(p)) < \epsilon.$$

Or equivalently, for any sequence $\{a_n\}$ in X ,¹²

$$\text{if } \lim_{n \rightarrow \infty} a_n = p \text{ then } \lim_{n \rightarrow \infty} f(a_n) = f(p).$$

- f is continuous \iff for any open set $V \subseteq Y$, $f^{-1}(V)$ is open in X .
- Suppose that f is continuous.
 - If $K \subseteq E$ is compact, $f(K)$ is also compact.
 - If $C \subseteq E$ is connected, $f(C)$ is also connected.
- (Extreme Value Theorem) Suppose $K \subseteq E$ is compact and $f : K \rightarrow \mathbb{R}$ is continuous. Because $f(K)$ is a compact set in \mathbb{R} , it is a closed interval. Hence f has a maximum/minimum.
- (Uniform Continuity) f is uniformly continuous on $E \iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall x, \forall y \in E, d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

¹⁰함수의 극한은 극한점에서 논한다! 다가갈 점들이 있어야 하지 않겠는가?

¹¹Limit point가 아니어도 정의할 수 있으며, 고립점에서는 연속이다.

¹²여기서는 $a_n \neq p$ 조건이 빠진다.

- If $f : E \subseteq X \rightarrow Y$ is continuous and E is compact, f is uniformly continuous.

5. Differentiation

Function $f : [a, b] \rightarrow \mathbb{R}$ is differentiable at $x \in [a, b] \iff$

$$\text{the limit } f'(x) = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x} \text{ exists.}$$

- If f is differentiable at $x = p$, then f is continuous at $x = p$.
- If f is differentiable at $x = p$ and $g : f([a, b]) \rightarrow \mathbb{R}$ is differentiable at $x = f(p)$
 $\implies g \circ f$ is differentiable at $x = p$ and

$$(g \circ f)'(p) = g'(f(p))f'(p).$$

- (Fermat) If f is differentiable and has a local extremum at $x = a$, then $f'(a) = 0$.
- (Mean Value Theorem) If f is continuous on $[a, b]$ and differentiable on (a, b) , there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

6. Integration

Given a bounded function $f : [a, b] \rightarrow \mathbb{R}$, a partition $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ and a monotonically increasing function $\alpha : [a, b] \rightarrow \mathbb{R}$, define

$$U(P, f, \alpha) = \sum_{i=0}^{n-1} \sup_{x \in [x_i, x_{i+1}]} f(x) (\alpha(x_{i+1}) - \alpha(x_i))$$

$$L(P, f, \alpha) = \sum_{i=0}^{n-1} \inf_{x \in [x_i, x_{i+1}]} f(x) (\alpha(x_{i+1}) - \alpha(x_i))$$

We define upper integral and lower integral as follows:

$$\overline{\int_a^b} f d\alpha = \inf_{P \in \mathcal{P}[a, b]} U(P, f, \alpha) \quad \underline{\int_a^b} f d\alpha = \sup_{P \in \mathcal{P}[a, b]} L(P, f, \alpha).$$

f is Stieltjes integrable with respect to $\alpha \iff$

$$\forall \epsilon > 0, \exists P \in \mathcal{P}[a, b] \text{ such that } U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Or equivalently, $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$. We write $f \in \mathcal{R}(\alpha)$.

Supplementary Material

F is a field for this section.

Definition. (Vector Space) A set V with addition $+: V \times V \rightarrow V$ and scalar multiplication $\cdot: F \times V \rightarrow V$ is a vector space over F if the following properties hold.

- (1) (Associativity of $+$) $u + (v + w) = (u + v) + w$ for all $v, w, u \in V$.
- (2) (Commutativity of $+$) $v + w = w + v$ for all $v, w \in V$.
- (3) (Identity of $+$) $\exists 0_V \in V$ such that $v + 0 = 0 + v = v$ for all $v \in V$.
- (4) (Inverse of $+$) For each $v \in V$, $\exists x \in V$ such that $v + x = x + v = 0_V$.
- (5) (Identity of \cdot) $1v = v$ for $v \in V$, where $1 \in F$ is the multiplicative identity in F .
- (6) (Distributive Property of \cdot w.r.t. Vector $+$) For $a \in F$ and $v, w \in V$, $a(v + w) = av + aw$.
- (7) (Distributive Property of \cdot w.r.t. Field $+$) For $a, b \in F$ and $v \in V$, $(a + b)v = av + bv$.
- (8) (Compatibility of \cdot w.r.t. $+$) $a(bv) = (ab)v$ for $a, b \in F$, $v \in V$.

We write $V = (V, +, \cdot)$.

Definition. (Normed Vector Space) A vector space V with a norm $\|\cdot\|: V \rightarrow \mathbb{R}$ is a normed vector space if the following properties hold.

- (1) $\|v\| \geq 0$ for all $v \in V$.
- (2) $\|v\| = 0 \iff v = 0$.
- (3) For all $\alpha \in F$ and $v \in V$, $\|\alpha v\| = |\alpha| \|v\|$.
- (4) (Triangle Inequality) For all $v, w \in V$, $\|v + w\| \leq \|v\| + \|w\|$.

For inner product spaces, $F = \mathbb{C}$ or $F = \mathbb{R}$.

Definition. (Inner Product Space) A vector space V with an inner product $\langle \cdot, \cdot \rangle: V \times V \rightarrow F$ is an inner product space if the following properties hold.

- (1) (Linearity in the first argument) For $x, y, z \in V$ and $a, b \in F$, $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$.
- (2) (Conjugate Symmetry) For $x, y \in V$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
- (3) (Positive Definiteness) If $0 \neq x \in V$, $\langle x, x \rangle > 0$.

Remark. An inner product can induce a norm by $\|v\| = \sqrt{\langle v, v \rangle}$. With norm as the distance metric, the following holds.

$$\text{Inner Product Space} \implies \text{Normed Vector Space} \implies \text{Metric Space}$$

If the inner product space is complete with respect to the distance metric, it is said to be a Hilbert space.