해석개론 및 연습 1 과제 #7

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1. (1) Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, $Q = \{a = y_0 < y_1 < \cdots < y_m = b\}$. If $P \subset Q$, there exists a sequence $\langle k(i) \rangle_{i=0}^n$ s.t. $k(0) = 0$ and $k(n) = m$, where $y_{k(i)} = x_i$. Let $X = \{0, 1, \ldots, n\}, Y = \{0, 1, \ldots, m\}$. Then

$$\begin{aligned} V(f, P) &= \sum_{i=1}^n |f(x_i) - f(x_{i-1})| = \sum_{i=1}^n \left| f(y_{k(i)}) - f(y_{k(i-1)}) \right| \\ &\le \sum_{i \in X} \left| f(y_{k(i)}) - f(y_{k(i-1)}) \right| + \sum_{j \in Y \setminus k(X)} |f(y_j) - f(y_{j-1})| \\ &= \sum_{i=1}^n |f(y_i) - f(y_{i-1})| = V(f, Q) \end{aligned}$$

(2) By definition of $V(f)$, for any $\epsilon > 0$, there exists $P_0 \in \mathcal{P}[a, b]$ s.t.

$$V(f) - \epsilon < V(f, P_0)$$

and by (1), if $P \supset P_0$, $V(f, P_0) \le V(f, P)$. Also, it is trivial that $V(f, P) \le V(f) < V(f) + \epsilon$. Thus we have the desired result,

$$|V(f, P) - V(f)| < \epsilon$$

2. (1) **False**. Consider

$$f(x) = \begin{cases} 0 & (a \le x \le 0) \\ 1 & (0 < x \le b) \end{cases}$$

Then for $a \le x \le 0$, $F(x) = 0$.
For $0 < x < \delta$, $P \in \mathcal{P}[a, x]$, define $P = \{a = x_0 < x_1 < \cdots < x_{l-1} \le 0 < x_l < \cdots < x_n = x\}$. Then $V_a^x(f, P) = |f(x_l) - f(0)| = 1$. Thus for $\epsilon = 1/2$, for any $\delta > 0$, there exists x s.t. $|x| < \delta$ and $|V_a^x(f, P)| \ge \epsilon$. $F(x)$ is discontinuous at $x = 0$.

(2) At $x_0 \in X = [a, b]$, for any $\epsilon > 0$, $x < x_0$, there exists $\delta > 0$ s.t. $|x - x_0| < \delta, x \in X \implies |f(x) - f(x_0)| < \epsilon/2$. $F(x_0) - F(x) = V_x^{x_0}(f)$ since f is a function of bounded variation. Take $y \in (x_0 - \delta, x_0)$ and consider a partition $P \in \mathcal{P}[y, x_0]$. For $y < x < x_0$,

$$V_y^{x_0}(f) < V_y^{x_0}(f, P) + \frac{\epsilon}{2}$$

$$V_y^x(f) \ge V_y^{x_0}(f, P) - |f(x_0) - f(x)| = V_y^x(f, P')$$

($V_y^{x_0}(f, P) - |f(x_0) - f(x)|$ is another variation on $[x, y]$ for a partition P'.) Combining these two gives

$$V_x^{x_0}(f) = V_y^{x_0}(f) - V_y^x(f) < V_y^{x_0}(f, P) + \frac{\epsilon}{2} - V_y^{x_0}(f, P) + |f(x) - f(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

We can use a similar argument for $x > x_0$, and this proves that $F(x)$ is continuous.

3. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

$$V_a^b(f, P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} \left|f'(t_i)\right| (x_i - x_{i-1}) \qquad t_i \in (x_{i-1}, x_i)$$

which is equal to $R(|f'|, P)$. By definition, $R(|f'|, P) = V_a^b(f, P) \le V_a^b(f)$. For all $\epsilon > 0$, there exists some partition P_1 s.t.

$$V_a^b(f) - \epsilon < V_a^b(f, P_1) = R(\left|f'\right|, P_1)$$

and if $P \supset P_1$, we have

$$\left|R(\left|f'\right|, P) - V_a^b(f)\right| < \epsilon$$

and since f' is integrable, $|f'|$ is integrable, and

$$V_a^b(f) = \int_a^b \left|f'(t)\right| \, dt$$

4. $a > b > 0$.

Consider partition

$$P = \left\{ \left(\frac{2}{(2n+1)\pi} \right)^{1/b} \right\}_{n=0}^{\infty}$$

Then

$$V(f, P) = \left(\frac{2}{\pi}\right)^{a/b} + 2\sum_{i=1}^{\infty} \left(\frac{2}{(2i+1)\pi} \right)^{a/b} < \infty \iff a > b > 0$$

5. **(1 $\Longrightarrow$ 2)** $f \in \mathcal{R}(\alpha)$, $\int_a^b f \, d\alpha = A$. There exists P_1, P_2 s.t.

$$U(f, P_1, \alpha) < A + \epsilon \qquad L(f, P_2, \alpha) > A - \epsilon$$

Setting $P_0 = P_1 \cup P_2$, and if $P \supset P_0$,

$$A - \epsilon < L(f, P_0, \alpha) \le L(f, P, \alpha) \le S(f, P, \alpha) \le U(f, P, \alpha) \le U(f, P_0, \alpha) < A + \epsilon$$

Thus we have

$$|S(f, P, \alpha) - A| < \epsilon$$

(2 $\Longrightarrow$ 1) For all $\epsilon > 0$, there exists P_0 s.t for all $P \supset P_0$,

$$A - \frac{\epsilon}{3} < S(f, P, \alpha) < A + \frac{\epsilon}{3}$$

Take infimum on the left inequality, supremum on the right inequality to get

$$A - \frac{\epsilon}{3} \le L(f, P, \alpha) \qquad U(f, P, \alpha) \le A + \frac{\epsilon}{3}$$

Therefore

$$U(f, P, \alpha) - L(f, P, \alpha) < \frac{2\epsilon}{3} < \epsilon \implies f \in \mathcal{R}(\alpha)$$

Since

$$A - \frac{\epsilon}{3} < L(f, P, \alpha) < \underline{\int_a^b} f \, d\alpha \le \overline{\int_a^b} f \, d\alpha < U(f, P, \alpha) < A + \frac{\epsilon}{3}$$

setting $\epsilon \to 0$ will give $\int_a^b f \, d\alpha = A$ since $f \in \mathcal{R}(\alpha)$.

6. **(1)** α is monotone on $[0,1]$ and $[1,2]$. Therefore α is of bounded variation on each interval, thus α is of bounded variation on $[0,2]$.

(2) Set

$$\alpha_1(x) = \begin{cases} 0 & (x < 1) \\ 3 & (x \geq 1) \end{cases} \quad \alpha_2(x) = x^2 \quad \alpha_3(x) = \begin{cases} 0 & (x < 1) \\ 2x^2 & (x \geq 1) \end{cases}$$

so that α_i are increasing. (Also BV) Then

$$\int_0^2 f\, d\alpha = \int_0^2 f\, d(\alpha_1 + \alpha_2) - \int_0^2 f\, d\alpha_3 = \int_0^2 f\, d\alpha_1 + \int_0^2 f\, d\alpha_2 - \int_0^2 f\, d\alpha_3$$

where the last equality holds since f is Stieltjes integrable w.r.t. α_1, α_2. Evaluating each integral gives

$$\begin{aligned} \int_0^2 f\, d\alpha &= \int_0^2 x^3\, d\alpha_1 + \int_0^2 x^3\, d(x^2) - \int_0^2 x^3\, d(2x^2) \\ &= f(1) + \int_0^2 x^3 \cdot 2x dx - \int_1^2 x^3 \cdot 4x dx \\ &= 1 + \frac{64}{5} - \frac{124}{5} = -11 \end{aligned}$$