Introduction to Analysis II

Study Notes

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Chapter 6

Sequence of Functions

6.1 Sequence of Continuous Functions

Definition. (Sequence of Functions) Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$. Given

$$f_n : X \to Y$$

for each $n \in \mathbb{N}$, we call $\langle f_n \rangle$ a **sequence of functions from X to Y**.

Definition. (Pointwise Convergence) The sequence $\langle f_n \rangle$ **converges pointwise** to the function $f : X \to Y$ if and only if

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for each $x \in X$. In other words, given $\epsilon > 0$ and for all $x \in X$,

$$\exists N \in \mathbb{N} \quad \text{s.t.} \quad n \geq N \implies \|f_n(x) - f(x)\| < \epsilon.$$ [1]

Definition. (Sequence of Continuous Functions) $\langle f_n \rangle$ is a sequence of continuous functions if and only if f_n is continuous for all $n \in \mathbb{N}$.

Question. *Suppose $\langle f_n \rangle$ is a sequence of continuous functions that converges pointwise to f. Is f also continuous?*

Definition. (Uniform Convergence) Let $\langle f_n \rangle$ be a sequence of functions defined on $X \subseteq \mathbb{R}^n$ and let f be a function defined on X. We say that $\langle f_n \rangle$ is **uniformly convergent on X** if and only if for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N,\ x \in X \implies \|f_n(x) - f(x)\| < \epsilon$$

[1] 여기서 주의해야 할 점은 자연수 N 이 양수 $\epsilon > 0$ 뿐 아니라 정의역의 점 $x \in X$ 에도 의존한다는 점이다.

Problem 6.1.1. Following are equivalent.

(1) $\langle f_n \rangle$ is uniformly convergent on X.

(2) $\lim_{n\to\infty} \sup \{\|f_n - f\| : x \in X\} = 0$.

Proof. ($1 \implies 2$) Uniformly convergent on $X \implies \forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N, x \in X \implies \|f_n(x) - f(x)\| < \epsilon/2$. Then $0 \leq \sup \{\|f_n(x) - f(x)\| : x \in X\} < \epsilon/2 < \epsilon$, and we have the desired result.
($2 \implies 1$) If $\lim_{n\to\infty} \sup \{\|f_n - f\| : x \in X\} = 0$, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$, $\sup \{\|f_n(x) - f(x)\| : x \in X\} < \epsilon/2$. Then $\|f_n(x) - f(x)\|$ should be less than ϵ for all $x \in X$, and thus $\langle f_n \rangle$ is uniformly convergent.

Problem 6.1.2. $f_n(x) = \dfrac{1}{n}x$ is not uniformly convergent on $\mathbb{R}$.

Proof. Suppose $\langle f_n \rangle$ is converges uniformly on $\mathbb{R}$ to 0. Then for any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N, x \in \mathbb{R} \implies \left|\frac{1}{n}x\right| < \epsilon$. But this can't be true, because for any ϵ, we can take x to be as large as we want. Take $x = 2\epsilon n$ for example, then $\left|\frac{1}{n}x\right| = 2\epsilon > \epsilon$. Contradiction.