HW Set 8. (Due day:Dec. 4, 12pm)

1. Let $f, f_n : \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable functions. Suppose that
$$\lim_{n\to\infty} \int_{\mathbb{R}} |f - f_n|\, dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}} |f - f_n|^2\, dx = 0.$$
Prove that
$$\lim_{n\to\infty} \int_{\mathbb{R}} |f - f_n|^{\frac{3}{2}}\, dx = 0.$$

2. Prove that
$$\sum_{n=1}^{\infty} \frac{1}{n} e^{inx} \in \mathcal{L}^2 \text{ on } [-\pi, \pi] \quad \text{but} \quad \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} e^{inx} \notin \mathcal{L}^2 \text{ on } [-\pi, \pi].$$

3. Consider the functions $f_n(x) = \sin nx$ ($n = 1, 2, 3, \cdots$ and $-\pi \le x \le \pi$) as points of $\mathcal{L}^2$. Prove that the set of these points is closed and bounded, but not compact.

4. Suppose a sequence of Lebesgue measurable functions $f_n : [a, b] \to \mathbb{R}$ converges to $f : [a, b] \to \mathbb{R}$ a.e.. Then for every positive ε and δ, there exist $N > 1$ and a measurable set $A \subset [a, b]$ with $m(A) < \delta$ such that $\sup_{n\ge N, x\in[a,b]\setminus A} |f_n(x) - f(x)| < \varepsilon$.

5. Suppose $\{n_k\}$ is an increasing sequence of positive integers and E is the set of all $x \in (-\pi, \pi)$ at which $\{\sin n_k x\}$ converges. Prove that $m(E) = 0$. *Hint*: For every $A \subset E$, $\int_A \sin n_k x\ dx \to 0$, and
$$2 \int_A (\sin n_k x)^2\ dx = \int_A (1 - \cos 2n_k x)\ dx \to m(A) \text{ as } k \to \infty.$$

6. Suppose $E \subset (-\pi, \pi)$, $m(E) > 0$, $\delta > 0$. Use the Bessel inequality to prove that there are at most finitely many integers n such that $\sin nx \ge \delta$ for all $x \in E$.

7. Suppose $f, g \in \mathcal{L}^2(\mu)$. Prove that
$$\left| \int f\bar{g}\ d\mu \right|^2 = \left(|f|^2\ d\mu \right) \left(|g|^2\ d\mu \right)$$
if and only if there is a constant c such that $g(x) = cf(x)$ almost everywhere.