Introduction to Analysis II

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Introduction & Notice

- 7, 8장 나가고 중간고사, 11장 나가고 기말고사
- 연습 시간이 있는 수업 (목 6:30 ~ 8:20)[1]
- 오늘 연습 시간: 지난학기 배운 내용 중 필요한 내용 복습

[1] 가능하면 1시간 반 안에 끝내라고 하심 ㅋㅋ

Chapter 7

Sequences and Series of Functions

September 1st, 2022

함수들을 각각 보는 것보다 **함수들의 공간을 이해**하는 것이 해석학의 핵심이다! 공간을 이해해야 미분방정식도 풀고 실제 현실의 문제들을 풀 수 있는 것입니다.

$\mathbb{R}^n$을 단순히 좌표들의 모임으로 보는 것이 아니라, 거리 구조를 주고 열린/닫힌 집합과 같은 위상 구조를 줬었습니다. 이 전에 했던게 **수열의 수렴과 발산, 코시 수열**이죠. 이런 것들을 바탕으로 공간의 위상적 성질을 조금 더 효율적으로 공부할 수 있었습니다.

즉, 어떤 공간을 배우기 위해서는 수열의 수렴이나 발산을 배워야 합니다. **따라서 우리는 함수들의 공간을 공부하기 위해 우선 함수열을 공부합니다.**

Suppose E is a set[1], and let $f_n : E \to \mathbb{C}$ for all $n \in \mathbb{N}$. Then

$$(f_n)_{n=1}^{\infty}$$

is a sequence of (complex-valued) function.

수열을 공부했으니 수열의 **수렴**을 정의해야 할 것입니다.

Definition 7.1 (Pointwise Convergence) $(f_n)_{n=1}^{\infty}$ converges **pointwise** on E, if for each $x \in E$ the sequence $(f_n(x))_{n=1}^{\infty}$ converges in $\mathbb{C}$.

In other words, for each $x \in E$, there exists $a_x \in \mathbb{C}$ and

$$\forall \epsilon > 0, \exists N_x \in \mathbb{N} \text{ such that } n \geq N_x \implies |f_n(x) - a_x| < \epsilon.$$

[1] 사실은 *metric* space 이다.

Definition. If (f_n) converges pointwise, we can define a function f by

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E)$$

We say that

- f is the *limit* or *limit function* of f_n.
- (f_n) to f pointwise on E.

Definition. If $\sum f_n(x)$ converges (pointwise) for every $x \in E$, we can define

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E)$$

and the function f is called the *sum* of the series $\sum f_n$.

Recall. $f : (E, d) \to \mathbb{C}$ is continuous on $E \iff f$ is continuous at all $x \in E$.

Recall. (Theorem 4.6) If $p \in E$ and p is a limit point of E,

$$f \text{ is continuous at } p \iff \lim_{x\to p} f(x) = f(p)$$

Question. Suppose (f_n) is a sequence of functions. Does the limit function or the sum of the series preserve important properties?

(1) If f_n is continuous, is f continuous?

(2) If f_n is differentiable/integrable, is f differentiable/integrable?

For (1), the question is equivalent to the following:

If p is a limit point, does the following hold?

$$\lim_{x\to p}\lim_{n\to\infty} f_n(x) \stackrel{?}{=} \lim_{n\to\infty}\lim_{x\to p} f_n(x)$$

And the answer is **No**.

실수열이 주어졌을 때, 극한을 계산하여 극한값이 실수라는 것은 굉장히 중요한 것입니다. 우리가 **연속**함수열의 수렴을 정의할 때, 극한이 되는 함수 또한 **연속**이 되기를 기대하는 것은 굉장히 자연스러운 일입니다. 하지만 점별수렴하는 연속함수열의 극한은 연속이 아닐 수 있습니다. 즉, 점별수렴은 연속함수공간에서의 '수렴'으로 정의하기에는 부족합니다.

Example 7.2 Suppose $a_{m,n} = \dfrac{m}{m+n}$ for $m, n \in \mathbb{N}$. We see that

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} = 1 \neq 0 = \lim_{m\to\infty}\lim_{n\to\infty} a_{m,n}$$

Example. Define

$$f_n(x) = \begin{cases} 0 & \left(\frac{1}{n} \leq x \leq 1\right) \\ -nx+1 & \left(0 \leq x < \frac{1}{n}\right) \end{cases}$$

then we can easily see that

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & (0 < x \leq 1) \\ 1 & (x = 0) \end{cases}$$

Thus f is not continuous at $x = 0$.

Example. Define $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (n = 0, 1, 2, \dots)$$

by direct calculation,

$$f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = 1 + x^2 \quad (x \neq 0)$$

since this is a geometric series when $x \neq 0$. If $x = 0$, $f(x) = 0$ and f is not continuous.

Question. Does the limit function preserve Riemann integrability?

Example. For $m = 1, 2, \dots$, define

$$f_m(x) = \lim_{n\to\infty} (\cos m!\pi x)^{2n} = \begin{cases} 1 & (m!x \in \mathbb{Z}) \\ 0 & (m!x \notin \mathbb{Z}) \end{cases}$$

We see that $f_m(x)$ is Riemann integrable. However,

Claim.

$$f(x) = \lim_{m\to\infty} f_m(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \in \mathbb{R} \setminus \mathbb{Q}) \end{cases}$$

and $f(x)$ is nowhere continuous, also not Riemann integrable.

Proof. Suppose $x = p/q \in \mathbb{Q}$. $(p, q \in \mathbb{Z})$ If we take $m \geq q$, we see that $m!x \in \mathbb{Z}$. Thus $f_m(x) = 1$. If $x \notin \mathbb{Q}$, $m!x$ can never be in $\mathbb{Z}$ and $f_m(x) = 0$.

Uniform continuity를 할 때 uniform이 어디서 나오죠? 해석학에서 그 점에서 뭐가 성립한다, 그러면 그 점과 그 근방에서만 확인하면 됐었죠. Continuity는 local property죠. 그런데 uniform continuity는 전체가 다 uniform하게 성립한다는 의미입니다.

Recall. $f:(X,d) \to (Y,d)$ is **uniformly continuous** on X[2] if

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } d_X(p,q) < \delta \implies d_Y(f(p), f(q)) < \epsilon$$

즉, 모든 점에서 똑같이 잡을 수 있다!

Recall. (Theorem 4.19, Heine-Cantor) If X is compact and f is continuous on X, then f is uniformly continuous on X.[3]

이제부터 나오는 uniform convergence는 sequence에 관한 것입니다!

Definition 7.7 (Uniform Convergence) Suppose $f_n : E \to \mathbb{C}$ is a sequence of functions. $(f_n)_{n=1}^{\infty}$ **converges uniformly** on E to a function f if

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } \forall x \in E, n \geq N \implies |f_n(x) - f(x)| \leq \epsilon.$$

Also, we say that the series $\sum f_n(x)$ converges uniformly on E if the sequence of partial sums $(\sum_{k=1}^{n} f_k(x))$ converges uniformly on E.

점별수렴의 경우 $N_x \in \mathbb{N}$ 이 $x \in E$ 에 의존하지만, 고른수렴의 경우 N 이 x와 무관합니다!

[똑같은 ϵ-띠를 둘러서 $y = f(x)$ 의 근방 안에 $f_n(x)$ $(n \geq N)$ 가 모두 들어가 있어야 한다]는 의미에서 *uniform*입니다. 한꺼번에 ϵ으로 누를 수 있다는 것입니다.

Theorem 7.9 Suppose

$$f(x) = \lim_{n\to\infty} f_n(x) \quad (x \in E)$$

Then $f_n \to f$ converges uniformly on E if and only if

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$$

which can also be written as

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } n \geq N \implies \sup_{x\in E} |f_n(x) - f(x)| \leq \epsilon$$

Notation. $f_n \to f$ uniformly on $E \iff f_n \xrightarrow{u} f$ on E.[4]

Theorem 7.8 (Cauchy Criterion for Uniform Convergence) $f_n \xrightarrow{u} f$ on $E \iff$

[2]Subspace of metric space is also a metric space.
[3]갑자기 왜 uniform continuity 얘기를 하냐, 헷갈리지 말고 기억하시라고!
[4]교수님: 책에서는 나중에 $\|f_n(x) - f(x)\|_\infty \to 0$ 으로 적었던 것 같은데...

$$\forall \epsilon > 0, \exists\, N \in \mathbb{N} \text{ such that } n, m \geq N \implies \sup_{x \in E} |f_n(x) - f_m(x)| \leq \epsilon.$$[5]

Proof.

($\Longrightarrow$) For given $\epsilon > 0$, fix $x \in E$. Since f_n converges uniformly on E, we can find $N \in \mathbb{N}$ such that for $n, m \geq N$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

($\Longleftarrow$) Uniform Cauchy property implies that (f_n) is a Cauchy sequence in $\mathbb{C}$. By the completeness of $\mathbb{C}$, the limit function $f(x)$ exists. Now we show that this convergence is uniform. For given $\epsilon > 0$ choose $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$\sup_{x \in E} |f_n(x) - f_m(x)| \leq \epsilon$$

Then

$$\begin{aligned} |f_n(x) - f(x)| = |f_n(x) - f_m(x) + f_m(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \\ &\leq |f_m(x) - f(x)| + \epsilon \end{aligned}$$

Fix $n \geq N$ and let $m \to \infty$. Observe that $|f_m(x) - f(x)| \to 0$ due to pointwise convergence. Therefore for every $x \in E$,

$$n \geq N \implies |f_n(x) - f(x)| \leq \epsilon.$$

코시 수열이 중요한 이유는 completeness 뿐만 아니라, 극한값을 알지 못할 때 수열의 수렴성을 논할 수 있기 때문입니다. 특히 급수의 경우 그 극한값을 알 수 없기 때문에 급수의 수렴판정 등에서 유용하게 사용됩니다.

[5] Uniform Cauchy Property. 실수에서 알고있던 성질과 동일합니다.

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More examples.

Example 7.5 Consider $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$ for $x \in \mathbb{R}$.

$$f(x) = \lim_{n\to\infty} f_n(x) \equiv 0$$

but,

$$f_n'(x) = \sqrt{n}\cos nx \implies f_n'(0) = \sqrt{n}.$$

As $n \to \infty$, $f_n'(0)$ does not converge to $f'(0)$.

Example 7.6 Consider $f_n(x) = nx(1-x^2)^n$ for $x \in [0,1]$. Note that

$$f_n(0) = 0,\ f_n(1) = 0.$$

When $0 < x < 1$, $f_n \to f \equiv 0$. (Theorem 3.20 (d)) Thus $\lim_{n\to\infty} f_n(x) = 0$ for $0 \le x \le 1$.

But

$$\int_0^1 nx(1-x^2)^n\,dx = \left[\frac{-n}{2n+2}(1-x^2)^{n+1}\right]_0^1 = \frac{n}{2n+2},$$

and thus

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \frac{1}{2} \neq 0 = \int_0^1 f(x)\,dx.$$

Definition. $\displaystyle\sum_{n=1}^{\infty} f_n$ converges uniformly on $E \iff \left(\sum_{k=1}^{n} f_k\right)$ converges uniformly on E.

Theorem 7.10 (Weierstrass M-test) Suppose $f_n : E \to \mathbb{C}$ and that for every n, $\exists\, M_n \in \mathbb{R}$ such that

$$|f_n(x)| \le M_n, \quad (x \in E)$$

and $\sum_{n=1}^{\infty} M_n < \infty$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

Proof. We want to show that the series is Cauchy.

For $m > n$, we have

$$\left|\sum_{k=n}^{m} f_k(x)\right| \le \sum_{k=n}^{m} |f_k(x)| \le \sum_{k=n}^{m} M_k.$$

Given $\epsilon > 0$, choose $m, n \in \mathbb{N}$ such that for $m, n \ge N$, $\sum_{k=n}^{m} M_k < \epsilon$. Then we get

$$\left|\sum_{k=n}^{m} f_k(x)\right| < \epsilon, \text{ for all } m, n \ge N.$$

By Theorem 7.8, $\sum f_n$ converges uniformly.

Theorem 7.11 Given metric space (Y, d) and $E \subseteq Y$, suppose that $f_n \xrightarrow{u} f$ on E and $x \in E'$. If

$$\lim_{t \to x} f_n(t) = A_n \in \mathbb{C}, \quad \text{(limit exists)}$$

then the sequence (A_n) converges, and

$$\lim_{n \to \infty} A_n = \lim_{t \to x} f(t).$$

In conclusion,

$$\lim_{n \to \infty} \lim_{t \to x} f_n(t) = \lim_{t \to x} \lim_{n \to \infty} f_n(t).$$

Proof. [6]

((A_n) converges in $\mathbb{C}$) Since $\mathbb{C}$ is complete, we will show that (A_n) is a Cauchy sequence. Let $\epsilon > 0$. Since $f_n \xrightarrow{u} f$ on E,

$$\exists\, N \in \mathbb{N} \text{ such that } n, m \geq N \implies |f_n(t) - f_m(t)| \leq \epsilon. \ (\forall t \in E)$$

From $\lim\limits_{t \to x} f_n(t) = A_n$, we can choose t arbitrarily close to x, such that for $n, m \geq N$,

$$|f_n(t) - A_n| < \epsilon \text{ and } |f_m(t) - A_m| < \epsilon.$$

Therefore for all $n, m \geq N$,

$$\begin{aligned} |A_n - A_m| &= |A_n - f_n(t) + f_n(t) - A_m + f_m(t) - f_m(t)| \\ &\leq |f_n(t) - A_n| + |f_m(t) - A_m| + |f_n(t) - f_m(t)| < 3\epsilon, \end{aligned}$$

and thus (A_n) is a Cauchy Sequence.

($\lim\limits_{n \to \infty} A_n = \lim\limits_{t \to x} f(t)$) Let $A = \lim\limits_{n \to \infty} A_n$. We want to show that for all $\epsilon > 0$,

$$\exists\, \delta > 0 \text{ such that } 0 < d(t, x) < \delta \implies |f(t) - A| < \epsilon.$$

Now,

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|.$$

Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that for $n \geq N$,

$$|f(t) - f_n(t)| < \frac{\epsilon}{3} \text{ for all } t \in E \text{ and } |A_n - A| < \frac{\epsilon}{3}.$$

Fix such N and choose δ such that for $0 < d(x, t) < \delta$ and $t \in E$,

$$|f_N(t) - A_N| \leq \frac{\epsilon}{3}.$$

[6] 고른수렴이 어디에서 쓰였는지 확인하는 것이 중요하다.

Thus for $t \in E$ and $0 < d(x,t) < \delta$,

$$|f(t) - A| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Theorem 7.12 Suppose $f : E \to \mathbb{C}$ is continuous on E and $f_n \xrightarrow{u} f$ on E. Then f is continuous on E.

Proof. Let $x \in E$. If $x \in E'$, f is continuous at x by Theorem 7.11. If x is an isolated point (not a limit point), f is continuous at x by definition of continuity.

고른수렴이 연속함수열의 올바른 수렴입니다.

앞으로는 E를 전부 metric space라고 가정할게요.

이 정리는 언제 uniformly converge 하는지 알려줍니다.

Theorem 7.13 (Dini) Given a compact metric space K, suppose that

(1) f_n and $f : K \to \mathbb{C}$ are continuous on K.

(2) $f_n \to f$ pointwise.

(3) $f_n(x) \geq f_{n+1}(x)$ for $x \in K$.[7]

Then $f_n \xrightarrow{u} f$ on K.

Proof. Let $g_n(x) = f_n(x) - f(x)$. Then $g_n(x)$ is continuous, decreasing and $g_n \to 0$ as $n \to \infty$ for all $x \in K$. Let $\epsilon > 0$ be given.

Claim. There exists $N \in \mathbb{N}$ such that $0 \leq g_n(x) < \epsilon$ for all $x \in K$.

Proof. Let $K_n = \{x \in K : g_n(x) \geq \epsilon\}$. Then $K_n = K \cap g_n^{-1}([\epsilon, \infty))$.[8] Since g_n is decreasing, $K_{n+1} \subseteq K_n$, but because $g_n \to 0$, $\bigcap_{n=1}^{\infty} K_n = \varnothing$. By Theorem 2.36, there exists $N \in \mathbb{N}$ such that $K_N = \varnothing$, and then $K_n = \varnothing$ for $n \geq N$. Thus, $0 \leq g_n(x) < \epsilon$ for $\forall x \in K$, $\forall n \geq N$.

Remark. Compactness is necessary here. Consider $f_n(x) = \dfrac{1}{nx+1}$ on $x \in E = (0,1)$. f_n does not converge to 0 uniformly.[9]

Proof. Suppose $f_n \xrightarrow{u} 0$, and take $\epsilon = 1/2$. Then,

$$\exists\, N \in \mathbb{N} \text{ such that } x \in (0,1) \implies \frac{1}{Nx+1} < \frac{1}{2}.$$

[7] f_n only needs to be monotone.

[8] Closed subset of a compact set is also compact, and the inverse image of closed set is closed if the function is continuous.

[9] 정의역을 $[0,1]$로 바꾼다면 f의 연속성이 필요한 이유를 알게 된다.

This gives a contradiction because the equation above gives $Nx > 1$, but we can choose x arbitrarily close to 0.

Definition. Let (X, d) be a metric space. Define

$$C(X, \mathbb{C}) = \{f : X \to \mathbb{C} \mid f \text{ is continuous and bounded}\}.$$

If there is no ambiguity, we write $C(X) = C(X, \mathbb{C})$.

Let $\|f\| = \sup_{x \in X} |f(x)|$. Then $\|\cdot\|$ is a norm on $C(X)$, and we call it **sup norm**.

(1) $\|f\| = 0 \iff f \equiv 0$.

(2) $\|f\| < \infty$, $\|\alpha f\| = |\alpha| \, \|f\|$ for $\alpha \in \mathbb{C}$.

(3) $\|f + g\| \leq \|f\| + \|g\|$.

Define $d(f, g) = \|f - g\|$, then $(C(X), d)$ is a metric space.

Therefore, $f_n \overset{u}{\to} f \iff f_n \to f$ in $(C(X), d)$.

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Theorem 7.15 $(C(X), \|\cdot\|)$ is a complete metric space.

Proof. [10] Suppose (f_n) is a Cauchy sequence in $C(X)$. By Theorem 7.8, $\exists\, f : X \to \mathbb{C}$ such that $f_n \xrightarrow{u} f$ on X. Also by Theorem 7.12, f is continuous on X.

Choose $N \in \mathbb{N}$ such that $\|f_N - f\| \le 1$. Then

$$\|f\| \le \|f_N - f\| + \|f_N\| \le 1 + \|f_N\| < \infty,$$

and f is bounded. Therefore $f \in C(X)$, and $C(X)$ is complete.

다시 돌아와서 고른수렴이 리만적분가능성과 미분가능성을 보존하는지 확인합니다.

Theorem 7.16 Let $\alpha : [a,b] \to \mathbb{R}$ be a monotonically increasing function. Suppose that

(1) $f_n \in \mathcal{R}(\alpha)$,

(2) $f_n \xrightarrow{u} f$ on $[a,b]$.

Then $f \in \mathcal{R}(\alpha)$ and $\displaystyle\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha$.

Proof. WLOG, let $f_n : [a,b] \to \mathbb{R}$. We know that f_n, f are bounded.[11]

Let $\epsilon_n = \displaystyle\sup_{x\in[a,b]} |f_n(x) - f(x)| < \infty$, and because $f_n \xrightarrow{u} f$, $\epsilon_n \to 0$ as $n \to \infty$. Therefore

$$f_n - \epsilon_n \le f \le f_n + \epsilon_n$$

since upper/lower integrals preserve monotonicity,

$$\int_a^b (f_n - \epsilon_n)\, d\alpha = \underline{\int_a^b} (f_n - \epsilon_n)\, d\alpha \le \underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} (f_n + \epsilon_n)\, d\alpha = \int_a^b (f_n + \epsilon_n)\, d\alpha.$$

The equality at each end is given by $f_n \pm \epsilon_n \in \mathcal{R}(\alpha)$. Thus,

$$0 \le \overline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f\, d\alpha \le 2\int_a^b \epsilon_n\, d\alpha = 2\epsilon_n\big(\alpha(b) - \alpha(a)\big),$$

and $f \in \mathbb{R}(\alpha)$. Moreover,

$$\left|\int_a^b f_n\, d\alpha - \int_a^b f\, d\alpha\right| \le \epsilon_n\big(\alpha(b) - \alpha(a)\big) \to 0$$

as $n \to \infty$ gives the existence of the limit.

[10] 코시 수열이 수렴하며 수렴값이 $C(X)$에 있는지 보이면 된다.
[11] 리만적분가능하면 유계이다.

Corollary. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and $\sum f_n(x)$ converges uniformly. Then

$$\int_a^b \sum_{n=1}^{\infty} f_n \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha.$$

Theorem 7.17 For sequence of functions f_n defined on $[a,b]$, suppose the following.

(1) f_n is differentiable on $[a,b]$.

(2) For some point $x_0 \in [a,b]$, $\big(f_n(x_0)\big)_{n=1}^{\infty}$ converges.

(3) f_n' converges uniformly on $[a,b]$.

Then the following holds.

(R1) $f_n \xrightarrow{u} f$ on $[a,b]$.

(R2) $f'(x) = \lim_{n\to\infty} f_n'(x)$.

Proof. Let $\epsilon > 0$ be given. We choose $N \in \mathbb{N}$ such that for $n, m \geq N$ and $t \in [a,b]$,

$$|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \text{ and } |f_n'(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)}.$$

Then for $n, m \geq N$, by using the Mean Value Theorem on $f_n - f_m$,

$$|f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq |x - t| \cdot \frac{\epsilon}{2(b-a)} \leq \frac{\epsilon}{2}. \tag{$*$}$$

Thus, set $t = x_0$ to get

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

and (R1) is proven. Let $f(x) = \lim_{n\to\infty} f_n(x)$.

For (R2), fix $x \in [a,b]$. Define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \qquad \phi(t) = \frac{f(t) - f(x)}{t - x},$$

for $t \in [a,b] \setminus \{x\}$. Then for each n,

$$\lim_{t\to x} \phi_n(t) = f_n'(x).$$

We want to show that $\phi_n \xrightarrow{u} \phi$ (on $t \neq x$), so that

$$f'(x) = \lim_{t\to x} \phi(t) = \lim_{t\to x} \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \lim_{t\to x} \phi_n(t) = \lim_{n\to\infty} f_n'(x).$$

By $(*)$, we see that for $n, m \geq N$,

$$|\phi_n(t) - \phi_m(t)| \leq \frac{\epsilon}{2(b-a)},$$

which directly shows that ϕ_n converges uniformly on $t \neq x$.

Remark. f_n'의 연속성을 가정하면 적분을 이용해서 훨씬 간편하게 증명할 수 있습니다.

Remark. (2)번 조건이 조금 부자연스럽게 느껴질 수 있으나, 최소한의 조건으로 최대의 결과를 얻고 싶었던 것입니다. 사실 (R1) $f_n \xrightarrow{u} f$ 임을 가정에 포함시켜 버린다면, (R2)를 얻을 수 있습니다.

Corollary. For sequence of functions f_n defined on $[a, b]$, suppose the following.

(1) f_n is differentiable on $[a, b]$.

(2) $f_n \xrightarrow{u} f$ on $[a, b]$.

(3) f_n' converges uniformly on $[a, b]$.

Then $f'(x) = \lim_{n\to\infty} f_n'(x)$.

Theorem 7.18 There exists $f \in C(\mathbb{R}, \mathbb{R})$ such that f' does not exist for all $x \in \mathbb{R}$.[12]

Proof. Define $\varphi(x) = |x|$ for $x \in [-1, 1]$, and $\varphi(x+2) = \varphi(x)$. Then for all $x, y \in \mathbb{R}$,

(1) $0 \leq \varphi \leq 1$,

(2) $|\varphi(x) - \varphi(y)| \leq |x - y|$.

Now define

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x).$$

By (1), $0 \leq f(x) \leq \sum (3/4)^n$ and $f(x)$ converges uniformly by M-Test, hence continuous on $\mathbb{R}$. Now we show that f is nowhere differentiable.

Fix $x \in \mathbb{R}$, let $m \in \mathbb{N}$. Choose $\delta_m = \pm\frac{1}{2} \cdot 4^{-m}$ so that (3) there are no integers between $4^m x$ and $4^m(x + \delta_m)$. If $n > m$, then $4^n \delta_m = \pm\frac{1}{2} \cdot 4^{n-m}$ is even. Then by periodicity,

$$a_n = \frac{\varphi(4^n(x + \delta_m)) - \varphi(4^n x)}{\delta_m} = 0.$$

[12] 브라운 운동도 또 하나의 예.

If $0 \le n \le m$, by (2),

$$|a_n| = \left| \frac{\varphi(4^n(x+\delta_m)) - \varphi(4^n x)}{\delta_m} \right| \le \frac{|4^n \delta_m|}{|\delta_m|} = 4^n.$$

If $n = m$, by (3),

$$|a_m| = \left| \frac{4^m(x+\delta_m) - 4^m x}{\delta_m} \right| = 4^m.$$

Therefore,

$$\begin{aligned}
\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n a_n \right| &\ge \overbrace{0}^{n>m} + \overbrace{\left(\frac{3}{4}\right)^m |a_m|}^{n=m} - \left| \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n a_n \right| \\
&\ge 3^m - \sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n |a_n| \\
&\ge 3^m - \sum_{n=0}^{m-1} 3^n = \frac{3^m + 1}{2}.
\end{aligned}$$

As $m \to \infty$, $\delta_m \to 0$, but the difference diverges. Thus f is not differentiable anywhere.

September 13th, 2022

Equicontinuous 하기 전에 7.23을 먼저 할게요. 7.23을 하기 위한 motivation이 필요합니다.

Recall. (Bolzano-Weierstrass) Suppose (a_n) is a sequence in $\mathbb{C}$. If (a_n) is bounded, then there exists a convergent subsequence of (a_n).

Recall. A sequence (a_n) in a metric space (E, d) is bounded if

$$\exists\, x_0 \in E \text{ and } \exists\, r > 0 \text{ such that } a_n \in B_r(x_0) \text{ for all } n \in \mathbb{N}.$$

Definition. (Pointwise Bounded) Suppose $f_n : E \to \mathbb{C}$. We say that (f_n) is **pointwise bounded** on E if the sequence $(f_n(x))_{n=1}^{\infty}$ is bounded for all $x \in E$. Or equivalently,

$$\sup_{n\in\mathbb{N}} |f_n(x)| < \infty \text{ for each } x \in E.$$

Theorem 7.23 Suppose E is at most countable.[13] Suppose $f_n : E \to \mathbb{C}$ and (f_n) is pointwise bounded. Then there exists a subsequence (f_{n_k}) of (f_n) such that $(f_{n_k}(x))$ converges for all $x \in E$.

$$\exists\, (n_k) \text{ such that } f(x) = \lim_{k\to\infty} f_{n_k}(x) \text{ for each } x \in E.$$

Proof. *Diagonalization!* Let $E = (x_i)_{i=1}^{\infty}$. (countable) Since $(f_n(x_1))$ is bounded, there exists a convergent subsequence by Bolzano-Weierstrass Theorem. Let the subsequence be $(f_{1,k})_{k=1}^{\infty}$. Similarly, since $(f_{1,k}(x_2))$ is bounded, there exists a convergent subsequence $(f_{2,k})_{k=1}^{\infty}$ of $(f_{1,k})_{k=1}^{\infty}$. Since $(f_{n,k})$ is pointwise bounded, we can repeat this procedure to get the next subsequence $(f_{n+1,k})_{k=1}^{\infty}$.

Now we arrange the sequence,

$$\begin{matrix} f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} & \cdots \\ f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \cdots \\ f_{4,1} & f_{4,2} & f_{4,3} & f_{4,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

and consider the diagonal elements. Then $\lim_{n\to\infty} f_{n,n}(x)$ converges for all $x \in E$. This is because $(f_{n+1,k})$ is always a subsequence of $(f_{n,k})$, and $(f_{n,k})$ was chosen to converge on $\{x_1, x_2, \ldots, x_n\}$.

Definition 7.19 Suppose $f_n : E \to \mathbb{C}$.

(1) (Pointwise Bounded) (f_n) is **pointwise bounded** on E if there exists a finite valued function $\varphi : E \to \mathbb{R}^+$ such that

[13] Countable 이거나 그거보다 작아서 finite 이거나. Countable이 아니면 안돼요.

$$|f_n(x)| \le \varphi(x) < \infty \text{ for all } x \in E.$$

(2) (Uniformly Bounded) (f_n) is **uniformly bounded** on E if there exists $M > 0$ such that

$$|f_n(x)| < M, \forall x \in E \text{ and } n \ge 1.$$

Or equivalently,

$$\sup_{n\in\mathbb{N}} \sup_{x\in E} |f_n(x)| < \infty$$

Remark.

(1) Uniform boundedness implies pointwise boundedness, but the converse is false.

(2) Every uniformly convergent sequence of bounded functions is uniformly bounded.

Question. Let E be a compact metric space.

(1) *Suppose that $(f_n)_{n=1}^{\infty}$ is uniformly bounded or continuous. Is there a pointwise convergent subsequence (f_{n_k}) of (f_n)?* **No.**

(2) *Suppose that (f_n) converges pointwise and is uniformly bounded. Is there a uniformly convergent subsequence (f_{n_k}) of (f_n)?* **No.**

아래 2개의 예시는 위 질문에 대한 대답이 '아니오'임을 알려줍니다.

Example 7.20 Let

$$f_n(x) = \sin nx, \quad (x \in [0, 2\pi]).$$

f_n is uniformly bounded ($|\sin nx| \le 1$). Suppose there exists a subsequence f_{n_k} such that $(f_{n_k}(x))$ converges for all $x \in [0, 2\pi]$. Then as $k \to \infty$, $f_{n_k}(x) - f_{n_{k+1}}(x) \to 0$ for all $x \in [0, 2\pi]$.

However, as $k \to \infty$,

$$\int_0^{2\pi} (f_{n_k} - f_{n_{k+1}})^2 \, dx \to \int_0^{2\pi} 0^2 \, dx = 0$$

by Theorem 11.32.[14] But simple calculation shows that

$$\int_0^{2\pi} (f_{n_k} - f_{n_{k+1}})^2 \, dx = 2\pi,$$

which leads to a contradiction. Thus such subsequence cannot exist.

Example 7.21 Let

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}, \quad (x \in [0, 1]).$$

[14] Lebesgue's theorem concerning integration of boundedly convergent sequences.

f_n is uniformly bounded (by 1), and $f_n \to 0$ pointwise. Also, note that $f_n(1/n) = 1$.

Suppose there exists a subsequence (f_{n_k}) such that $f_{n_k} \xrightarrow{u} 0$. Then, $\exists\, k_0 \in \mathbb{N}$ such that

$$\sup_{x\in[0,1]} \left| f_{n_{k_0}}(x) \right| < \frac{1}{2}. \tag{$*$}$$

However,

$$1 = \left| f_{n_{k_0}} \left(\frac{1}{n_{k_0}} \right) \right| \leq \sup_{x\in[0,1]} \left| f_{n_{k_0}}(x) \right| < \frac{1}{2},$$

leading to a contradiction. Thus such subsequence cannot exist.

Definition 7.22 (Equicontinuity) Let (X, d) be a metric space, and $E \subseteq X$. Let $\mathscr{F}$ be a family of complex-valued functions on E. We say that $\mathscr{F}$ is **equicontinuous** on E if

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } d(x,y) < \delta \implies |f(x) - f(y)| < \epsilon \text{ for all } x, y \in E, f \in \mathscr{F}.$$

That is, all $f \in \mathscr{F}$ are uniformly continuous on E with the same (ϵ, δ) in the definition.

f가 하나라면 그냥 uniform continuity의 정의입니다. 그런데 f가 family of functions이고, 이러한 ϵ을 모든 f에 대해서 잡을 수 있다는 의미에서 'equi' 입니다.

Remark. (Example 7.21) Let $\mathscr{F} = (f_n)$. Then $\mathscr{F}$ is not equicontinuous on E.

Proof. Suppose $\mathscr{F}$ is equicontinuous on E. For $\epsilon = 1$, there should exist $\delta > 0$ such that

$$\forall x, y \in [0,1] \text{ and } |x - y| < \delta \implies |f_n(x) - f_n(y)| < 1.$$

In particular, $|f_n(0) - f_n(1/n)| < 1$ if $1/n < \delta$, but $|f_n(0) - f_n(1/n)| = 1$. Contradiction.

Theorem 7.24 Let K be a compact set. Suppose $f_n \in C(K)$. If f_n converges uniformly on K, $\mathscr{F} = (f_n)$ is equicontinuous on K.

Proof. Let $\epsilon > 0$ be given. Since (f_n) is uniformly Cauchy, $\exists\, N \in \mathbb{N}$ such that for all $x \in E$,

$$n \geq N \implies |f_n(x) - f_N(x)| < \frac{\epsilon}{3}.$$

Since continuous function defined on a compact set is uniformly continuous,[15] f_n is uniformly continuous. Therefore $\exists\, \delta > 0$ such that

$$d(x,y) < \delta \implies |f_N(x) - f_N(y)| < \frac{\epsilon}{3} \text{ for } x, y \in K.$$

For $x, y \in K$, if $n \geq N$ and $d(x, y) < \delta$,

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

[15] Heine-Cantor Theorem.

Therefore $(f_n)_{n=N}^{\infty}$ is equicontinuous. Additionally, $(f_n)_{n=1}^{N}$, which is a finite union of uniformly continuous functions, is equicontinuous. Thus (f_n) is equicontinuous because it is a union of two equicontinuous family of functions.[16]

[16]유한개는 equicontinuous 로 누르고, 꼬리는 uniform convergence로 누르고.

September 15th, 2022

증명을 들여다 보면, 7.24의 조건이 optimal 조건은 아니에요.

Theorem 7.25 (Arzela-Ascoli) Let K be a compact set. Suppose $f_n \in C(K)$. (Bounded and continuous) Suppose that $(f_n)_{n=1}^{\infty}$ is pointwise bounded and equicontinuous on K. Then

(1) $(f_n)_{n=1}^{\infty}$ is uniformly bounded on K.

(2) There exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ which converge uniformly on K.

Proof. Since (f_n) is equicontinuous on K, choose $\epsilon = 1$. Then $\exists\, \delta > 0$ such that for all $n \in \mathbb{N}$,

$$\forall x, y \in K, d(x,y) < \delta \implies |f_n(x) - f_n(y)| < 1. \tag{$*$}$$

Since K is compact and $K \subseteq \displaystyle\bigcup_{x \in K} B_\delta(x)$,

$$\exists\, x_1, \dots, x_r \in K \text{ such that } K \subseteq \bigcup_{j=1}^{r} B_\delta(x_j).$$

By $(*)$, for all $n \in \mathbb{N}$ and $j = 1, \dots, r$,

$$|f_n(x)| \le |f_n(x_j)| + 1, \quad \forall x \in B_\delta(x_j).$$

Then we see that $\|f_n\| \le \displaystyle\max_{1 \le j \le r} |f_n(x_j)| + 1$, for all $n \in \mathbb{N}$. Now since (f_n) is pointwise bounded,

$$\sup_{n \in \mathbb{N}} \|f_n\| \le \sup_{n \in \mathbb{N}} \max_{1 \le j \le r} |f_n(x_j)| + 1 < \infty,$$

showing that (f_n) is uniformly bounded.

From Problem 2.25, a compact set has a countable dense subset. Let $E \subseteq K$ be such subset. (극한점이거나, E의 원소이거나) By Theorem 7.23, there exists a subsequence $g_i = f_{n_i}$ such that (g_i) converges pointwise in E.

Let $\epsilon > 0$ be given. We know that (g_i) is equicontinuous, $\exists\, \delta > 0$ such that for all $i \in \mathbb{N}$

$$\forall x, y \in K, d(x,y) < \delta \implies |g_i(x) - g_i(y)| < \frac{\epsilon}{3}. \tag{$**$}$$

Because E is dense, $K \subseteq \displaystyle\bigcup_{x \in E} B_\delta(x)$. Compactness of K gives

$$\exists\, x_1, \dots, x_m \in E \text{ such that } K \subseteq \bigcup_{s=1}^{m} B_\delta(x_s).$$

Then for any $x \in K$, there exists $s \le m$ such that $x \in B_\delta(x_s)$. By (**), for all $i \in \mathbb{N}$,

$$|g_i(x) - g_i(x_s)| < \frac{\epsilon}{3}.$$

Since g_i converges pointwise in E, $\exists\, N \in \mathbb{N}$ such that for $i, j \ge N$,

$$|g_i(x_s) - g_j(x_s)| < \frac{\epsilon}{3}. \quad (s = 1, \ldots, m)$$

Therefore, for all $i, j \ge N$ and $x \in K$,

$$\begin{aligned}|g_i(x) - g_j(x)| &\le |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

언제 uniformly converge 하는 subsequence가 존재하는지에 대한 조건들을 배운 것입니다.

Weierstrass Theorem. 어떤 관점으로 보면 좋냐면, 실제로 $\sin, \cos$ 전개도 안하고 쭉 얘기를 하고 있어요. 거리 공간에서 정의된 연속함수들의 공간을 생각하는데, 구체적인 함수 얘기가 없어요. 그렇다고 모든 연속함수를 구체적으로 적을 수는 없죠. (구체적이라는 것의 정의도 애매하지만) 다 정리를 못한다면, 어느 정도 근사하는 방법으로 정리할 수 있는가?

모든 연속함수는 다항식으로 근사 가능하다! Compact 하면 uniformly 근사가 된다.

Theorem 7.26 (Weierstrass) Suppose $f \in C([a,b], \mathbb{C})$. $(a, b \in \mathbb{R})$ Then there exists a sequence of polynomials $P_n \in C([a,b], \mathbb{C})$ such that

$$\lim_{n\to\infty} P_n(x) = f(x)$$

uniformly on $[a,b]$. If f is real, P_n may be taken real.

여러 가지 증명 방법이 있는데, 책의 증명이 굉장히 교육적입니다. 몇 가지 중요한 테크닉, 배울 점이 많은 증명이에요.

Proof. WLOG, we prove the theorem on $[0,1]$ and we extend to $\mathbb{R}$ by letting $f \equiv 0$ on $\mathbb{R} \setminus [0,1]$. We first assume that $f(0) = f(1) = 0$.

Then f is uniformly continuous on $\mathbb{R}$. We will find polynomials Q_n and let[17]

$$P_n(x) = \int_{-1}^{1} f(x+t) Q_n(t)\, dt.$$

[17] Idea. P_n의 모양은 간단해요. 이런걸 convolution 이라고 부르는데... P_n이 정말로 근사가 된다는걸 보이기 위해 $|P_n - f|$ 같은걸 해야하죠. Q_n은 n이 커짐에 따라 점점 몰아주고, $f(x+t) - f(x)$ 를 작게 만들고 싶습니다.

Let
$$Q_n(x) = c_n(1-x^2)^n, \quad c_n = \left(\int_{-1}^{1}(1-x^2)^n\,dx\right)^{-1},$$
where c_n is chosen to satisfy $\displaystyle\int_{-1}^{1} Q_n(x)\,dx = 1$. We need to control the size of c_n, so define
$$g(x) = (1-x^2)^n - (1-nx^2).$$
Then $g(0) = 0$, $g'(x) = 2nx(1-(1-x^2)^{n-1}) \geq 0$ for $x \in [0,1]$. ($g(x) \geq 0$) Thus,
$$\int_{-1}^{1}(1-x^2)^n\,dx = 2\int_{0}^{1}(1-x^2)^n\,dx \geq 2\int_{0}^{1/\sqrt{n}}(1-nx^2)\,dx = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}},$$
therefore $c_n < \sqrt{n}$.

By change of variables $s = x+t$ and because $f \equiv 0$ on $\mathbb{R} \setminus [0,1]$,
$$P_n(x) = \int_0^1 f(s)Q_n(s-x)\,ds = c_n\int_0^1 f(s)(1-s^2+2sx-x^2)^n\,ds.$$
The integral above is clearly a polynomial in x. Thus (P_n) is a sequence of polynomials.

Given $\epsilon > 0$, choose $\delta \in (0,1)$ such that
$$a, b \in [0,1], |a-b| < \delta \implies |f(a)-f(b)| < \frac{\epsilon}{2}.$$
Now,
$$|P_n(x) - f(x)| \leq \int_{-1}^{1} |f(x+t)-f(x)|\,Q_n(t)\,dt.$$
Split the last integral to $[-1,1] = [-1,-\delta] \cup [-\delta,\delta] \cup [\delta,1]$. Then we see that
$$\int_{-\delta}^{\delta} |f(x+t)-f(x)|\,Q_n(t)\,dt < \frac{\epsilon}{2}\int_{-\delta}^{\delta} Q_n(t)\,dt < \frac{\epsilon}{2},$$
$$\int_{-1}^{-\delta} |f(x+t)-f(x)|\,Q_n(t)\,dt + \int_{\delta}^{1} |f(x+t)-f(x)|\,Q_n(t)\,dt < 4c_n\,\|f\|\,(1-\delta^2)^n,$$
since $Q_n(t) \leq c_n(1-\delta^2)^n$ on $\delta \leq |t| \leq 1$.

Let $N \in \mathbb{N}$ such that for $n \geq N$,
$$4\,\|f\|\,\sqrt{n}(1-\delta^2)^n < \frac{\epsilon}{2}.$$
Then we finally have
$$|P_n(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
for $n \geq N$.

In general, let

$$g(x) = f(x) - f(0) - x\,(f(1) - f(0))$$

so that $g(1) = g(0) = 0$. By the previous case, there exists $P_n^* \xrightarrow{u} g$ on $[0, 1]$. Then

$$P_n(x) = P_n^*(x) + f(0) + x\,(f(1) - f(0)) \xrightarrow{u} f$$

on $[0, 1]$.

September 20, 2022

Weierstrass: 닫힌 구간에서 정의된 연속함수는 다항식으로 uniformly 근사할 수 있다!

Corollary. For $a > 0$, $\exists\, P_n \in C([-a,a],\mathbb{R})$ such that

$$P_n(0) = 0 \text{ and } P_n(x) \xrightarrow{u} |x| \text{ on } [-a,a].$$

Proof. By Weierstrass Theorem, there exists $P_n^*(x) \in C([-a,a],\mathbb{R})$ such that $P_n^*(x) \xrightarrow{u} |x|$ on $[-a,a]$. Let $P_n(x) = P_n^*(x) - P_n^*(0)$ so that $P_n(0) = 0$. Then $P_n(x)$ will have the desired property.

Definition 7.28 Given a metric space (E,d), let $\mathcal{A}$ be a collection of complex-valued functions on E.

(1) $\mathcal{A}$ is called an **algebra**[18] if

$$f+g,\ f\cdot g,\ cf \in \mathcal{A} \text{ whenever } f,g \in \mathcal{A},\ c \in \mathbb{C}.$$

(2) Algebra $\mathcal{A}$ is **uniformly closed** if

$$f_n \in \mathcal{A} \text{ and } f_n \xrightarrow{u} f \text{ then } f \in \mathcal{A}.$$

(3) For a given algebra $\mathcal{A}$,

$$\mathcal{B} = \{f : \exists\, f_n \in \mathcal{A} \text{ such that } f_n \xrightarrow{u} f\}$$

is called the **uniform closure** of $\mathcal{A}$. We write $\mathcal{B} = \overline{\mathcal{A}}^{\|\cdot\|} = \overline{\mathcal{A}}^{u}$.

Example. Examples of algebras. ($[a,b]$ can be changed to compact sets.)

(1) $\mathcal{A} = \{f : f \in C([a,b])\}$.

(2) $\mathcal{A} = \{f : f \text{ is bounded on } [a,b]\}$.

(3) $\mathcal{A} = \{f : f \text{ is a polynomial on } [a,b]\}$.

Also note that $\overline{\mathcal{A}}^{u} = C([a,b])$ by Theorem 7.12 + Weierstrass Theorem.

Question. We are interested in $C(E,\mathbb{R})$ and $C(E,\mathbb{C})$ when E is a compact metric space. *We want to find an algebra whose uniform closure is $C(E,\mathbb{R})$ or $C(E,\mathbb{C})$.*[19]

Theorem 7.29 Let $\mathcal{B}$ be the uniform closure of an algebra $\mathcal{A}$ of bounded functions. Then $\mathcal{B}$ is

[18] Real-valued functions의 경우 real algebra라고 부르고 $c \in \mathbb{R}$ 이다.

[19] 찾게 되면, 그 algebra의 원소들로 연속함수를 uniformly 근사할 수 있게 된다!

a uniformly closed algebra.

Proof. $\mathcal{B}$ is uniformly closed by definition. Suppose that $f, g \in \mathcal{B}$ and $c \in \mathbb{C}$. By definition,

$$\exists\, f_n, g_n \in \mathcal{A} \text{ such that } f_n \xrightarrow{u} f,\ g_n \xrightarrow{u} g \text{ on } E.$$

Then $f_n + g_n \xrightarrow{u} f + g$, $f_n g_n \xrightarrow{u} fg$, $cf_n \xrightarrow{u} cf$ on E.[20] Thus $f+g,\ fg,\ cf \in \mathcal{B}$, which makes $\mathcal{B}$ a uniformly closed algebra.

Definition 7.30 Let $\mathcal{A}$ be a family of functions on a set E.

(1) $\mathcal{A}$ is said to **separate points** on E if

$$\text{for every distinct } x_1, x_2 \in E,\ \exists\, f \in \mathcal{A} \text{ such that } f(x_1) \neq f(x_2).$$

(2) $\mathcal{A}$ **vanishes at no point** of E if

$$\forall x \in E,\ \exists\, f \in \mathcal{A} \text{ such that } f(x) \neq 0.$$[21]

Example.

(1) Set of polynomials on $\mathbb{R}$ separates points on E and vanishes at no point of E.

(2) Even polynomials on $[-1, 1]$ does not separate points on E. $(f(x) = f(-x))$

Theorem 7.31 Suppose that

(1) $\mathcal{A}$ is an algebra of functions on a set E,

(2) $\mathcal{A}$ separates points on E,

(3) $\mathcal{A}$ vanishes at no point of E.

Then for any distinct points $x_1, x_2 \in E$ and for all $c_1, c_2 \in \mathbb{C}$,

$$\text{there exists } f \in \mathcal{A} \text{ such that } f(x_1) = c_1 \text{ and } f(x_2) = c_2.$$

Proof. We want to find $f(x) = c_1 f_1(x) + c_2 f_2(x)$ where

$$f_1(x_1) = 1,\ f_1(x_2) = 0,\ f_2(x_1) = 0,\ f_2(x_2) = 1.$$

From the given assumptions, we can find $g, h, k \in \mathcal{A}$ such that

(1) $g(x_1) \neq g(x_2)$ (separates points),

[20] $f_n g_n \xrightarrow{u} fg$ works because the functions are bounded.
[21] 모든 함수가 0인 점은 없다!

(2) $h(x_1) \neq 0$, $k(x_2) \neq 0$ (vanishes at no point).

Let

$$u(x) = g(x)k(x) - g(x_1)k(x), \quad v(x) = g(x)h(x) - g(x_2)h(x).$$

Then $u(x_1) = v(x_2) = 0$, $u(x_2) \neq 0$, $v(x_1) \neq 0$. Therefore setting

$$f_1(x) = \frac{v(x)}{v(x_1)}, \quad f_2(x) = \frac{u(x)}{u(x_2)} \implies f(x) = \frac{c_1 v(x)}{v(x_1)} + \frac{c_2 u(x)}{u(x_2)} \in \mathcal{A}$$

will give the desired result.

Theorem 7.32 (Stone-Weierstrass) Let $\mathcal{A}$ be a real algebra of real continuous functions on a compact set K. (i.e. $\mathcal{A} \subseteq C(K, \mathbb{R})$) If

(1) $\mathcal{A}$ separates points on K,

(2) $\mathcal{A}$ vanishes at no point of K,

the uniform closure of $\mathcal{A}$ consists of all real continuous functions on K. (i.e. $\overline{\mathcal{A}}^u = C(K, \mathbb{R})$)

Proof. Let $\mathcal{B} = \overline{\mathcal{A}}^u$. We know that $\mathcal{B} \subseteq C(K, \mathbb{R})$. So we only need to show $\mathcal{B} \supseteq C(K, \mathbb{R})$.

(Step 1) $f \in \mathcal{B} \implies |f| \in \mathcal{B}$.

Let $a = \|f\| = \sup_{x \in K} |f(x)|$. Given $\epsilon > 0$, there exists a polynomial approximating $|x|$.

$$\exists\, c_1, \ldots, c_n \in \mathbb{R} \text{ such that } \sup_{y \in [-a,a]} \left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \epsilon.$$

Define $g = \sum_{i=1}^{n} c_i f^i \in \mathcal{B}$. Then (plugging $f(x)$ into y gives)

$$|g(x) - |f(x)|| = \left| \sum_{i=1}^{n} c_i (f(x))^i - |f(x)| \right| < \epsilon, \quad (x \in K).$$

Since $\mathcal{B}$ is uniformly closed, $|f| \in \mathcal{B}$.

(Step 2) $f_1, \ldots, f_n \in \mathcal{B} \implies \max\{f_1, \ldots, f_n\}, \min\{f_1, \ldots, f_n\} \in \mathcal{B}$.

For $f, g \in \mathcal{B}$, $f + g, f - g \in \mathcal{B}$. Also by Step 1, $|f + g|, |f - g| \in \mathcal{B}$. Thus

$$\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}, \quad \min\{f, g\} = \frac{f+g}{2} - \frac{|f-g|}{2} \in \mathcal{B}.$$

By induction, $\max\{f_1, \ldots, f_n\}, \min\{f_1, \ldots, f_n\} \in \mathcal{B}$.

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Now we fix $\epsilon > 0$ and $f \in C(K, \mathbb{R})$.

(Step 3) *For each $x \in K$, there exists a function $g_x \in \mathcal{B}$ such that*

$$g_x(x) = f(x) \text{ and } g_x(y) > f(y) - \epsilon \text{ for all } y \in K.$$

Since $\mathcal{A} \subseteq \mathcal{B}$, we can use Theorem 7.31. For every $y \in K$, $\exists\, h_y \in \mathcal{B}$ such that

$$h_y(x) = f(x) \text{ and } h_y(y) = f(y).$$

By (uniform) continuity of f and h_y, there exists $\delta_y > 0$ such that

$$d(y,t) < \delta_y \implies |h_y(t) - h_y(y)| + |f(t) - f(y)| < \epsilon.$$

Therefore,

$$|h_y(t) - f(t)| \le |h_y(t) - f(y)| + |f(y) - f(t)| < \epsilon$$

for all $d(y,t) < \delta_y$. Now for all $t \in J_y = \{t \in K : d(y,t) < \delta_y\}$,

$$h_y(t) > f(t) - \epsilon. \tag{$*$}$$

Since K is compact, there exists a finite subcover.

$$\exists\, y_1, \dots, y_n \in K \text{ such that } K \subseteq \bigcup_{j=1}^{n} J_{y_j}.$$

We can rewrite $(*)$ as

$$h_{y_j}(t) > f(t) - \epsilon, \quad (t \in J_{y_j}).$$

Now take maximum of these,

$$g_x(t) = \max\{h_{y_1}(t), \dots, h_{y_n}(t)\} > f(t) - \epsilon$$

for all $t \in K$. By Step 2, $g_x(t) \in \mathcal{B}$.

(Step 4) *For all $f \in C(K, \mathbb{R})$, given $\epsilon > 0$, there exists $g \in \mathcal{B}$ such that $\|f - g\| < \epsilon$. (i.e. $g_n \xrightarrow{u} f$ on K) Since $\mathcal{B}$ is uniformly closed, $f \in \mathcal{B}$.*[22]

For each $x \in K$, let $g_x \in \mathcal{B}$ be the function defined in Step 3. By continuity of g_x and f,

$$\exists\, \delta_x > 0 \text{ such that } t \in V_x \implies |g_x(t) - g_x(x)| + |f(t) - f(x)| < \epsilon$$

[22] 임의의 $f \in C(K, \mathbb{R})$ 로 수렴하는 $g_n \in \mathcal{B}$ 를 잡을 수 있다는 뜻이므로, uniform closure의 정의에 의해 $f \in \mathcal{B}$ 이다.

where $V_x = \{t \in K : d(x,t) < \delta_x\}$. Therefore,

$$|g_x(t) - f(t)| \le |g_x(t) - g_x(x)| + |g_x(x) - f(t)| < \epsilon, \quad (t \in V_x)$$

Since K is compact, there exists a finite subcover.

$$\exists\, x_1, \dots, x_m \in K \text{ such that } K \subseteq \bigcup_{j=1}^{m} V_{x_j}.$$

Now for $t \in V_{x_j}$, (as we did in Step 3)

$$g_{x_j}(t) < f(t) + \epsilon,$$

so $g(t) = \min\{g_{x_1}(t), \dots, g_{x_m}(t)\} < f(t) + \epsilon$. By Step 2, $g \in \mathcal{B}$, and by Step 3,

$$g(t) > f(t) - \epsilon.$$

Thus we have found a function $g \in \mathcal{B}$ such that $\|g - f\| < \epsilon$.

Dense 하면서 nice(?)한 함수공간을 찾을 수 있을까? 여러분이 자세히 봐야 알 수 있겠지만 real algebra인 것을 가정한 거죠? 이 chapter의 마지막 부분은, real이 아니고 complex라면 같은 내용이 성립하는가 입니다. 단, 한 조건이 더 필요합니다. 그 전에 정의 하나 하고 갑니다.

Definition. We call a complex algebra $\mathcal{A}$ **self-adjoint** if $f \in \mathcal{A} \implies \overline{f} \in \mathcal{A}$.

Theorem 7.33 Let $\mathcal{A}$ be a self-adjoint algebra of complex continuous functions on a compact set K. (i.e. $\mathcal{A} \subseteq C(K, \mathbb{C})$) If

(1) $\mathcal{A}$ separates points on K,

(2) $\mathcal{A}$ vanishes at no point of K,

the uniform closure of $\mathcal{A}$ consists of all complex continuous functions on K. (i.e. $\overline{\mathcal{A}}^u = C(K, \mathbb{C})$)

Proof. Let $\mathcal{A}_{\mathbb{R}} = \{f \in \mathcal{A} : f(K) \subseteq \mathbb{R}\}$. Then it is easy to see that $\mathcal{A}_{\mathbb{R}}$ is a real algebra, and subset of $C(K, \mathbb{R})$.

Claim. $\mathcal{A}_{\mathbb{R}}$ separates points on K.

Proof. Let $x_1 \neq x_2$. There exists $f \in \mathcal{A}$ such that $f(x_1) = 1$, $f(x_2) = 0$. Write $f = u + iv$ where $u, v \in C(K, \mathbb{R})$. Since $\overline{f} \in \mathcal{A}$, we have

$$u = \frac{f + \overline{f}}{2} \in \mathcal{A}_{\mathbb{R}} \subseteq \mathcal{A}.$$

Moreover, $u(x_1) = f(x_1) = 1$, $u(x_2) = f(x_2) = 0$.

Claim. $\mathcal{A}_{\mathbb{R}}$ vanishes at no point of K.

Proof. Fix $x \in K$. Choose non-zero $g \in \mathcal{A}$. Choose $\lambda \in \mathbb{C}$ such that $\lambda g(x) > 0$, we define $f = \lambda g$ which can be written $f = u + iv$ where $u, v \in C(K, \mathbb{R})$. Similarly, $u \in \mathcal{A}_{\mathbb{R}}$ and $u(x) = \lambda g(x) > 0$.

From the 2 claims above, we know that $\overline{\mathcal{A}_{\mathbb{R}}}^u = C(K, \mathbb{R}) \subseteq \overline{\mathcal{A}}^u$, by Theorem 7.32. Suppose $f \in C(K, \mathbb{C})$, and write $f = u + iv$. Fix $\epsilon > 0$. There exists $\tilde{u}, \tilde{v} \in \mathcal{A}_{\mathbb{R}}$ such that

$$\|u - \tilde{u}\| + \|v - \tilde{v}\| < \epsilon.$$

Define $\tilde{f} = \tilde{u} + i\tilde{v} \in \mathcal{A}$. Then $\left\|f - \tilde{f}\right\| < \epsilon$, proving that $f \in \overline{\mathcal{A}}^u$.[23]

[23]Denseness를 보인 것입니다.

Chapter 8

Some Special Functions

Special Functions 라는 이론이 따로 있어요. 8장에서는 여러분들이 많이 아는 부분이 있어서 골라가면서 하고, 그래도 봐야할 것들은 조교님께 얘기해서 하라고 할게요.

첫 번째로 다룰 부분은 power series 인데 전에 얘기했던 내용입니다.

Recall. (Root Test) Given $\sum a_n$, let

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

(1) If $\alpha < 1$, $\sum a_n$ converges absolutely.

(2) If $\alpha > 1$, $\sum a_n$ diverges.

Definition. A **power series** in $\mathbb{R}$ about the point a is a series in the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n, \quad (x \in \mathbb{R}). \tag{$\star$}$$

c_n's are called the **coefficients** of the series.

Definition. (Radius of Convergence) $R = (0, \infty]$ is called the **radius of convergence** if the series $(\star)$ converges absolutely for $|x-a| < R$, and diverges for $|x-a| > R$. Here,

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}.$$ [1]

Theorem 8.1 Suppose the series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

[1] Theorem 3.39.

converges for $|x-a| < R$.

(1) f converges uniformly on $|x-a| \leq R-\epsilon$ for all $\epsilon \in (0, R)$.

(2) f is continuous and differentiable on $|x-a| < R$. Moreover,

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

on $|x-a| < R$.

Proof.

(1) WLOG, let $a = 0$, and fix $\epsilon \in (0, R)$. Observe that

$$|c_n x^n| \leq |c_n| (R-\epsilon)^n$$

for $|x| \leq R-\epsilon$. By the root test, $\sum |c_n| (R-\epsilon)^n < \infty$. Now we know that $\sum c_n x^n$ converges uniformly on $|x| \leq R-\epsilon$ by Weierstrass M-test.

(2) Using $n^{1/n} \to 1$, we see that

$$\limsup_{n\to\infty} \sqrt[n]{n |c_n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}.$$

Thus the two series have the same radius of convergence. So $\sum_{n=1}^{\infty} n c_n x^{n-1}$ converges on $|x| < R$, and converges uniformly on $|x| \leq R-\epsilon$ by (1).

Now by Theorem 7.17,[2] f is differentiable, and therefore continuous. Also,

$$f'(x) = \lim_{n\to\infty} \frac{d}{dx}\left(\sum_{k=1}^{n} c_k x^k\right) = \sum_{n=1}^{\infty} n c_n x^{n-1},$$

for any $|x| < R$. (We can always choose ϵ such that $|x| \leq R-\epsilon$.)

Corollary. Let $f : \mathbb{R} \to \mathbb{R}$. Under the assumptions of Theorem 8.1, f has derivatives of all orders in $(-R, R)$, and

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) c_n (x-a)^{n-k}.$$

In particular,

$$f^{(k)}(a) = k! \cdot c_k.$$

[2] f_n이 미분 가능하고, f_n, f_n'이 모두 고르게 수렴하면 f가 미분 가능하며 $f' = \lim f_n'$.

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Definition. (Real Analytic) A real valued function f is called **real-analytic**[3] on $|x-a|<R$ if there exists $c_n \in \mathbb{R}$ such that

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n$$

for $|x-a|<R$.

Remark. $f \in C^{\infty}$ does not imply that f is real-analytic.

Lemma. Suppose $\sum c_n y^n$ converges for $y \neq 0$. Then $\sum c_n x^n$ converges absolutely on $|x| < |y|$.

Proof. Since $|c_n y^n| \to 0$, choose $N \in \mathbb{N}$ such that

$$|c_n y^n| < 1 \text{ for all } n \geq N.$$

Now for all $n \geq N$,

$$|c_n x^n| \leq |c_n|\,|y|^n \left|\frac{x}{y}\right|^n < \left|\frac{x}{y}\right|^n.$$

Thus $\sum |c_n x^n|$ converges for $|x|<|y|$. (Comparison)

Theorem 6.2 (Abel) Suppose $\sum c_n$ converges. Let $f(x)=\sum c_n x^n$ for $x \in (-1,1)$. Then

$$\lim_{x \to 1^-} f(x) = \sum_{n=0}^{\infty} c_n.$$

Proof. Let $s_{-1}=0$, $s_n=\sum_{k=0}^{n} c_k$ for $n \geq 0$ and $s=\lim_{n\to\infty} s_n = \sum_{n=0}^{\infty} c_n$. We know that $|s_n|+|s| < M < \infty$. We have $c_n = s_n - s_{n-1}$ for $n \geq 0$,

$$\begin{aligned}
\sum_{n=0}^{m} c_n x^n &= \sum_{n=0}^{m} s_n x^n - x\sum_{n=1}^{m} s_{n-1}x^{n-1} \\
&= s_m x^m + \sum_{n=0}^{m-1} s_n x^n - x \sum_{n=0}^{m-1} s_n x^n \\
&= s_m x^m + (1-x)\sum_{n=0}^{m-1} s_n x^n.
\end{aligned}$$

Since $|x|<1$, $|s_m x^m| \to 0$ as $m \to \infty$. We only consider the second term, so

$$f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n.$$

[3] 복소함수의 경우 analytic의 의미가 미분가능성이기 때문에 real이라고 구분 해주는 것이 좋다.

Let $\epsilon > 0$, we can choose $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon/2$ for $n \geq N$. Therefore,

$$\begin{aligned}
|f(x) - s| &= \left|(1-x)\sum_{n=0}^{\infty} s_n x^n - s(1-x)\sum_{n=0}^{\infty} x^n\right| \\
&= \left|(1-x)\sum_{n=0}^{\infty}(s_n - s)x^n\right| \\
&\leq (1-x)\sum_{n=0}^{N}|s_n - s|\,|x|^n + (1-x)\sum_{n=N+1}^{\infty}|s_n - s|\,|x|^n \\
&\leq (1-x)\sum_{n=0}^{N}(|s_n| + |s|)\,|x|^n + \frac{\epsilon}{2}(1-x)\sum_{n=N+1}^{\infty}|x|^n \\
&\leq M(1-x)\sum_{n=0}^{N}|x|^n + \frac{\epsilon}{2} \\
&\leq MN(1-x) + \frac{\epsilon}{2} \leq \epsilon,
\end{aligned}$$

if we choose small enough $\delta > 0$ so that for $1-\delta < x < 1$, $MN(1-x) \leq \epsilon/2$. Thus $|f(x) - s| < \epsilon$, proving the result.

Theorem 3.51 (Cauchy Product) Suppose $\sum a_n$, $\sum b_n$, $\sum c_n$ converges, where

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

Then

$$\sum c_n = \left(\sum a_n\right)\left(\sum b_n\right)$$

Proof. On $0 \leq x \leq 1$, let

$$f(x) = \sum a_n x^n, \quad g(x) = \sum b_n x^n.$$

For $0 \leq x < 1$, these series converge absolutely, so we can multiply them.[4] Therefore

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n, \quad (0 \leq x < 1).$$

Now by Abel's Theorem, setting $x \to 1^-$ gives the desired result.

[4]곱한 뒤 재배열해야 하는데, 절대수렴하기 때문에 재배열할 수 있다.

아래 정리는 언제 무한급수의 더하는 순서를 바꿀 수 있는지 말해줍니다.

Theorem 8.3 (Fubini for Infinite Series) Given a double sequence (a_{ij}), suppose that

$$\text{Either } \sum_i \sum_j |a_{ij}| < \infty \text{ or } \sum_j \sum_i |a_{ij}| < \infty.$$

Then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Proof. Let $x_\infty = 0, x_n = 1/n$ for $n \geq 1$. Suppose

$$E = \{x_\infty, x_1, \dots, x_n, \dots\} \subseteq \mathbb{R}$$

and $x_n \to x_\infty$ as $n \to \infty$. For each i, define a function f_i on E such that

$$f_i(x) = \sum_{j=1}^{n} a_{ij} \text{ for } x = x_n \quad \text{and} \quad f_i(x_\infty) = \sum_{j=1}^{\infty} a_{ij}.$$

We have $f_i(x_n) \to f_i(x_\infty)$ as $x_n \to x_\infty$. Therefore f_i is continuous at x_∞ on E. Let

$$g(x) = \sum_{i=1}^{\infty} f_i(x), \quad (x \in E).$$

For all $x \in E$,

$$\sum_{i=1}^{\infty} |f_i(x)| \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}| < \infty.$$

By Weierstrass M-test, $g(x)$ converges uniformly on E. So $g(x)$ is continuous at x_∞.

$$\begin{aligned}\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} &= g(x_\infty) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \sum_{i=1}^{\infty} f_i(x_n) \\ &= \lim_{n\to\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{n} a_{ij} = \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.\end{aligned}$$

by continuity of g.

여기서 극한의 순서를 굉장히 조심해야 합니다.

$$\sum^{M} \sum^{N} a_{mn} \overset{N\to\infty}{\longrightarrow} \sum^{M} \sum^{\infty} a_{mn} \overset{M\to\infty}{\longrightarrow} \sum^{\infty} \sum^{\infty} a_{mn}$$

은 가능하지만,

$$\sum^{M} \sum^{N} a_{mn} \overset{M\to\infty}{\longrightarrow} \sum^{\infty} \sum^{N} a_{mn} \overset{N\to\infty}{\longrightarrow} \sum^{\infty} \sum^{\infty} a_{mn} (?)$$

는 전혀 다른 문제입니다.

Theorem 8.4 Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

which converges on $|x| < R$. If $a \in (-R, R)$, f can be expanded in a power series about the point $x = a$ which converges on $|x - a| < R - |a|$, as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad (|x-a| < R - |a|).$$

Proof. Fix $a \in (-R, R)$. We have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n \big((x-a)+a\big)^n \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} c_n \binom{n}{k}(x-a)^k a^{n-k} \overset{(*)}{=} \sum_{k=0}^{\infty}\left[\sum_{n=k}^{\infty}\binom{n}{k} c_n a^{n-k}\right](x-a)^k. \end{aligned}$$

This is the desired expansion about the point $x = a$. We only need to prove $(*)$, where the summation was switched. Meanwhile,

$$\begin{aligned} \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left|c_n\binom{n}{k}(x-a)^k a^{n-k}\right| &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}|c_n|\binom{n}{k}|x-a|^k\,|a|^{n-k} \\ &= \sum_{n=0}^{\infty}|c_n|\big(|x-a|+|a|\big)^n < \infty \end{aligned}$$

by the root test.

$$\limsup_{n\to\infty}\sqrt[n]{|c_n|}\big(|x-a|+|a|\big) < \frac{1}{R}\cdot R = 1$$

because f converges on $|x| < R$ and $|x-a| + |a| < R$. Now we calculate the coefficients,

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty}\left[\sum_{n=k}^{\infty}\binom{n}{k}c_n a^{n-k}\right](x-a)^k \\ &= \sum_{k=0}^{\infty}\left[\sum_{n=k}^{\infty} c_n n(n-1)\ldots(n-k+1)a^{n-k}\right]\frac{(x-a)^k}{k!} \\ &= \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k, \end{aligned}$$

because differentiating f k-times and plugging in a gives the exact from in the brackets.

Theorem 8.5 Suppose

$$h_1(x) = \sum a_n x^n \text{ and } h_2(x) = \sum b_n x^n$$

converge on $(-R, R)$. Let

$$E = \{x \in (-R, R) : h_1(x) = h_2(x)\}.$$

If E has a limit point in $(-R, R)$, then $a_n = b_n$ for all n, so $h_1 = h_2$ on $(-R, R)$.

Proof. We treat $(-R, R)$ as a metric space. We know that E' is closed in $(-R, R)$ and $E' \neq \varnothing$ by assumption. Since h_1, h_2 are continuous, $E' \subseteq E$.[5] Now define $B = (-R, R) \setminus E'$. Since E' is closed, B is open in $(-R, R)$. Now we show that E' is open, to show that $E' = (-R, R)$.[6]

Claim. E' is open.

Proof. Let $x_0 \in E'$. Let $f(x) = h_1(x) - h_2(x)$. f is 0 on E, and $f(x_0) = 0$ due to the continuity of f. For x in $|x - x_0| < R - |x_0|$, we can use Theorem 8.4 to expand the series at x_0,

$$f(x) = \sum_{n=0}^{\infty} d_n (x - x_0)^n,$$

which is continuous on $|x - x_0| < R - |x_0|$. Suppose $d_n \neq 0$ for some $n \geq 1$, choose smallest k such that $d_k \neq 0$. Then

$$f(x) = \sum_{n=k}^{\infty} d_n (x - x_0)^n = (x - x_0)^k \sum_{n=k}^{\infty} d_n (x - x_0)^{n-k}.$$

Let $g(x) = \sum_{n=k}^{\infty} d_n (x - x_0)^{n-k}$, then $g(x_0) = d_k \neq 0$. Since g is continuous near x_0, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies g(x) \neq 0.$$

Thus $f(x) \neq 0$ on $|x - x_0| < \delta$. But this is a contradiction, because $x_0 \in E'$. There exists a sequence (x_n) in E such that $x_n \to x_0$ and $f(x_n) = f(x_0) = 0$. Therefore $d_n = 0$ and $f(x) = 0$ for x in $|x - x_0| < R - |x_0|$. Thus

$$B_{R-|x_0|}(x_0) \subseteq E',$$

which proves that E' is open.

[5] 연속성으로 인해 극한점에서도 $h_1 = h_2$ 가 되기 때문이다.
[6] Open 이면서 동시에 closed 면 $\varnothing$ 이거나 전체집합 이거나.

October 4th, 2022

Theorem 8.8 (Algebraic Completeness of $\mathbb{C}$) Let $a_0, \dots, a_n \in \mathbb{C}$, $n \geq 1$, $a_n \neq 0$. Define

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

Then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

Proof. WLOG, let $a_n = 1$. We consider $|p(z)|$, and we are interested in its minimum. Let

$$\mu = \inf_{z \in \mathbb{C}} |p(z)|.$$

We want to show that (1) $\exists\, z_0 \in \mathbb{C}$ such that $\mu = |p(z_0)|$, and (2) $\mu = 0$.

(1) For $|z| = R$,

$$|p(z)| \geq |z|^n - |a_{n-1}| |z|^{n-1} - \cdots - |a_0| = R^n (1 - |a_{n-1}| R^{-1} - \cdots - |a_0| R^{-n})$$

The above expression approaches ∞ as $R \to \infty$. Therefore $\mu = \inf_{|z| \leq R} |p(z)|$ for some $R_0 > 0$.[7] Since $p(z)$ is continuous on $|z| \leq R_0$ (compact), there exists $z_0 \in \mathbb{C}$ such that $\mu = |p(z_0)|$.

(2) Now suppose $\mu \neq 0$. ($p(z_0) \neq 0$) Define

$$Q(z) = \frac{p(z + z_0)}{p(z_0)}.$$

Then $Q(0) = 1$ and $|Q(z)| \geq 1$ for all $z \in \mathbb{C}$, since $p(z_0)$ was minimum.

There exists $k \leq n$ such that $b_k \neq 0$ and

$$Q(z) = 1 + b_k z^k + \cdots + b_n z^n,$$

because $\deg Q = n$, and $Q(0) = 1$.[8] We will take $z = re^{i\theta}$.[9] There exists $\theta \in \mathbb{R}$ such that

$$e^{-ik\theta} b_k = -|b_k|.$$

Take r small enough so that $0 < r^k |b_k| < 1$. Then,

$$\left|1 + b_k r^k e^{ik\theta}\right| = 1 - |b_k| r^k > 0.$$

[7] 어차피 이 범위 밖에서는 무한대로 발산할 것이다!
[8] 0에서는 1이고, n차 다항식이니 0이 아닌 항이 존재할 것이다.
[9] Idea: 적당히 돌리고 줄이는 작업을 하면 $Q(z)$를 1보다 작아지게 할 수 있다!

Now

$$|Q(z)| = \left|Q(re^{i\theta})\right| \le \left|1 + b_k r^k e^{ik\theta}\right| + \left|b_{k+1} r^{k+1} e^{i(k+1)\theta}\right| + \cdots + \left|b_n r^n e^{in\theta}\right|$$
$$= 1 - |b_k| r^k + |b_{k+1}| r^{k+1} + \cdots + |b_n| r^n$$
$$= 1 - r^k \left(|b_k| - |b_{k+1}| r - \cdots - |b_n| r^{n-k}\right).$$

Choose r smaller so that the expression in the parentheses is positive. Then $|Q(z)| < 1$, which contradicts $|Q(z)| \ge 1$.

이제 푸리에 급수를 공부할 차례입니다. 지금부터 다룰 함수들은 closed interval에서 정의된 함수인데, 항상 **주기함수**로 놓고 진행하도록 하겠습니다.

Definition. (Periodic Function) $f : \mathbb{R} \to \mathbb{C}$ is **periodic** if there exists $p > 0$ such that

$$f(x+p) = f(x), \quad (\forall x \in \mathbb{R}),$$

and p is called the **period** of f.[10]

Definition. (Trigonometric Polynomial)

$$f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$$

where $a_0, \ldots, a_N, b_0, \ldots, b_N \in \mathbb{C}$.[11] This is also written as

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx} = \sum_{-N}^{N} c_n e^{inx},$$

where

$$c_0 = \frac{1}{2} a_0, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}, \quad a_n = c_n + c_{-n}.$$

Remark. We know that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx}\, dx = \begin{cases} 1 & (n = 0) \\ 0 & (n \ne 0) \end{cases}.$$

Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx}\, dx = \sum_{n=-N}^{N} \frac{c_n}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)x}\, dx = \begin{cases} c_m & (|m| \le N) \\ 0 & (|m| > N) \end{cases}.$$

Therefore,

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx}\, dx, \quad (|m| \le N).$$

[10] 여기서는 가장 작은 p일 필요는 없다고 할게요.
[11] a_0 가 아니라 $a_0/2$를 쓰기도 합니다.

Remark. It can be checked that $c_{-n} = \overline{c_n}$ for $|n| \le N \iff f$ is real-valued.

Definition. (Trigonometric Series)

$$\sum_{n\in\mathbb{Z}} c_n e^{inx} = \sum_{-\infty}^{\infty} c_n e^{inx}, \quad (x \in \mathbb{R}, c_n \in \mathbb{C}).$$

f가 trigonometric polynomial일 때는 c_n이 어떤 형태로 나오는지 알았는데, f가 일반적인 주기 함수라면 c_n이 어떻게 표현될까?

Definition. (Fourier Series) Given $f \in \mathcal{R}$ on $[-\pi, \pi]$. Let

$$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx, \quad (m \in \mathbb{Z}).$$

Then we call

$$\sum_{-\infty}^{\infty} c_n e^{inx}$$

a **Fourier series** of f, and c_n is called the **Fourier coefficients** of f.[12] We write

$$f \sim \sum_{-\infty}^{\infty} c_n e^{inx}.$$[13]

Definition 8.10 (Orthogonal System) Let $(\phi_n)_{n=1}^{\infty}$ be a sequence of complex-valued functions on $[a, b]$. If

$$\int_a^b \phi_n(x)\overline{\phi_m(x)}\,dx = 0 \text{ for all } n \ne m,$$

we call $(\phi_n)_{n=1}^{\infty}$ an **orthogonal system of functions** on $[a, b]$. Moreover, if

$$\int_a^b |\phi_n(x)|^2\,dx = 1$$

holds additionally, we call $(\phi_n)_{n=1}^{\infty}$ an **orthonormal system of functions** on $[a, b]$.

Example.

(1) $\left(\frac{1}{\sqrt{2\pi}}e^{inx}\right)$ is an orthonormal system of functions on $[-\pi, \pi]$.

(2) The following functions form an orthonormal system on $[-\pi, \pi]$.

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots\right\}$$

[12] 사실은 relative to $\left(e^{inx}\right)_{n\in\mathbb{Z}}$.

[13] 푸리에 급수가 이렇게 주어진다는 것이다. 같다는 의미는 절대 아니다.

Definition. (Fourier Series (Generalized)) Given an orthonormal system of functions $(\phi_n)_{n=1}^{\infty}$ and $f : [a, b] \to \mathbb{C}$ where $f \in \mathcal{R}$ on $[a, b]$. Then

$$c_n = \int_a^b f(t)\overline{\phi_n(t)}\, dt, \quad n = 1, 2, \dots$$

is the **Fourier coefficient** of f relative to $(\phi_n)_{n=1}^{\infty}$. Also,

$$\sum_{n=1}^{\infty} c_n \phi_n$$

is called the **Fourier series** of f relative to $(\phi_n)_{n=1}^{\infty}$, and we write

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n.$$

Theorem 8.11 Let $(\phi_n)_{n=1}^{\infty}$ be an orthonormal system of functions on $[a, b]$, and $f \in \mathcal{R}$ on $[a, b]$. Let

$$c_m = \int_a^b f(x)\overline{\phi_m(x)}\, dx \quad \text{and} \quad s_n(x) = \sum_{m=1}^{n} c_m \phi_m(x).$$

Suppose $t_n(x) = \sum_{m=1}^{n} \gamma_m \phi_m(x)$ where $\gamma_m \in \mathbb{C}$. Then

(1) $\int_a^b |f - s_n|^2\, dx \le \int_a^b |f - t_n|^2\, dx$. ($\mathcal{R}^2[a, b]$-norm)

(2) Equality holds if and only if $c_m = \gamma_m$ for $m = 1, 2, \dots, n$.

Remark. s_n is the best approximation of f with respect to the norm of $\mathcal{R}^2[a, b]$.

Proof. Remember these identities!

$$\int_a^b f\overline{t_n}\, dx = \sum_{m=1}^{n} \overline{\gamma_m} \int_a^b f\overline{\phi_m}\, dx = \sum_{m=1}^{n} \overline{\gamma_m} c_m,$$

and

$$\int_a^b f\overline{s_n}\, dx = \sum_{m=1}^{n} \overline{c_m} \int_a^b f\overline{\phi_m}\, dx = \sum_{m=1}^{n} |c_m|^2.$$

Note that

$$\begin{aligned}\int_a^b |t_n|^2\, dx = \int_a^b t_n\overline{t_n}\, dx &= \int_a^b \left(\sum_{m=1}^{n} \gamma_m \phi_m\right)\left(\sum_{k=1}^{n} \overline{\gamma_k}\overline{\phi_k}\right) dx \\ &= \sum_{m=1}^{n}\sum_{k=1}^{n} \gamma_m \overline{\gamma_k} \int_a^b \phi_m \overline{\phi_k}\, dx \overset{(*)}{=} \sum_{m=1}^{n} |\gamma_m|^2.\end{aligned}$$

$(*)$: The integral is 1 if $m = k$ and 0 otherwise.

Similarly, we get $\int_a^b |s_n|^2 \, dx = \sum_{m=1}^{n} |c_m|^2$.

Therefore,

$$\begin{aligned}
\int_a^b |f - t_n|^2 = \int_a^b (f - t_n)(\overline{f} - \overline{t_n}) &= \int_a^b |f|^2 - \int_a^b f\overline{t_n} - \int_a^b \overline{f} t_n + \int_a^b |t_n|^2 \\
&= \int_a^b |f|^2 - \sum_{m=1}^{n} c_m \overline{\gamma_m} - \sum_{m=1}^{n} \overline{c_m}\gamma_m + \sum_{m=1}^{n} |\gamma_m|^2 \\
&= \int_a^b |f|^2 - \sum_{m=1}^{n} |c_m|^2 + \sum_{m=1}^{n} |\gamma_m - c_m|^2 . \qquad (**)
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
\int_a^b |f - s_n|^2 &= \int_a^b |f|^2 - \int_a^b f\overline{s_n} - \int_a^b \overline{f} s_n + \int_a^b |s_n|^2 \\
&= \int_a^b |f|^2 - 2\sum_{m=1}^{n} |c_m|^2 + \sum_{m=1}^{n} |c_m|^2 \\
&= \int_a^b |f|^2 - \sum_{m=1}^{n} |c_m|^2 . \qquad (\star)
\end{aligned}$$

Upon comparing this with $(**)$, we see that

$$\int_a^b |f - t_n|^2 = \int_a^b |f - s_n|^2 + \sum_{m=1}^{n} |\gamma_m - c_m|^2 \geq \int_a^b |f - s_n|^2 .$$

Thus (1) holds, and equality holds when $\gamma_m = c_m$.

Theorem 8.12 (Bessel's Inequality) With the hypotheses of Theorem 8.11,

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f|^2 \, dx < \infty.$$

In particular,

$$\lim_{n\to\infty} c_n = 0.$$

Proof. From $(\star)$,

$$\int_a^b |f|^2 - \sum_{m=1}^{n} |c_m|^2 \geq 0.$$

Let $n \to \infty$ to get the desired inequality.

여기서부터 trigonometric series는 Fourier series relative to

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}, \quad (n \in \mathbb{Z})$$

를 의미합니다.

We assume that $f : \mathbb{R} \to \mathbb{C}$ is a periodic function with period 2π, f is Riemman integrable on $[-\pi, \pi]$, and

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx.$$

Definition. Define

$$s_N = s_N(f; x) = \sum_{-N}^{N} c_n e^{inx} = \sum_{-N}^{N} (\sqrt{2\pi} c_n) \left(\frac{1}{\sqrt{2\pi}} e^{inx} \right).$$ [14]

We call s_N the N-th partial sum of the Fourier series of f.

We can calculate that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |s_N(x)|^2 \, dx = \sum_{-N}^{N} |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx.$$ [15]

So, when does this series converge? To be continued.

[14] 마지막 표현은 orthonormal임을 강조하기 위한 것이다.

[15] Check $\frac{1}{2\pi} \int_{-\pi}^{\pi} |f - s_N|^2 \, dx \geq 0$.

October 6th, 2022

When does the Fourier series converge? Can continuous functions be expressed as a trigonometric series?

Definition. (Dirichlet Kernel) Let $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.[16]

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \begin{cases} \dfrac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{x}{2}} & (x \neq 0) \\ 2N+1 & (x=0) \end{cases}.$$

The last expression comes from the fact that

$$(e^{ix}-1)D_N(x) = e^{i(N+1)x} - e^{-iNx},$$

when $D_N(x)$ is viewed as a geometric series.

Therefore,

$$\begin{aligned} s_N(f;x) = \sum_{n=-N}^{N} c_n e^{inx} &= \sum_{n=-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \cdot e^{inx} \\ &= \sum_{n=-N}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{in(x-t)}\,dt \\ &= \sum_{n=-N}^{N} \int_{-\pi}^{\pi} f(t)D_N(x-t)\,dt \\ &= \sum_{n=-N}^{N} \int_{-\pi}^{\pi} f(x-u)D_N(u)\,du, \quad (u=x-t) \end{aligned}$$

where the integration bounds does not change since f is periodic with period 2π.

Theorem 8.14 [17] For some $x \in \mathbb{R}$, if there exists $M > 0$ and $\delta > 0$ such that

$$|f(x+t) - f(x)| \leq M\,|t|$$

for all $|t| < \delta$, then

$$\lim_{N\to\infty} s_N(f;x) = f(x).$$

Proof. Let x be fixed. Note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1.$$

[16] 이 조건은 왜 필요하지?

[17] Note that we have the assumption: f is periodic and $f \in \mathcal{R}$.

We want $s_N(f;x) - f(x)$ to be small enough. So,

$$\begin{aligned} s_N(f;x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \big(f(x-t) - f(x)\big) D_N(t)\, dt \\ &\overset{(?)}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x-t)-f(x)}{\sin\frac{t}{2}} \sin\left(N+\frac{1}{2}\right) t\, dt. \end{aligned}$$

But $\sin\frac{t}{2}$ is undefined for $t=0$, so we define

$$g(t) = \begin{cases} \dfrac{f(x-t)-f(x)}{\sin\frac{t}{2}} & (0 < |t| \le \pi) \\ 0 & (t = 0) \end{cases},$$

and write

$$\begin{aligned} s_N(f;x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t) \sin\left(N+\frac{1}{2}\right) t\, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\cos\frac{t}{2}\sin Nt\, dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\sin\frac{t}{2}\cos Nt\, dt. \quad (*) \end{aligned}$$

Recall that $\{\phi_0, \phi_{2n}, \phi_{2n-1}\}$ where

$$\phi_0 = \frac{1}{\sqrt{2\pi}}, \quad \phi_{2n} = \frac{\sin nt}{\sqrt{\pi}}, \quad \phi_{2n-1} = \frac{\cos nt}{\sqrt{\pi}}$$

is an orthonormal system. Since $g(t)\cos\frac{t}{2}$ and $g(t)\sin\frac{t}{2}$ are Riemann integrable (bounded and continuous except for $t=0$), $(*) = \widehat{c_{2N}}(h_1) + \widehat{c_{2N-1}}(h_2)$ where $\widehat{c_{2N}}, \widehat{c_{2N-1}}$ are Fourier coefficients of

$$h_1(t) = \frac{\sqrt{\pi}}{2\pi} g(t)\cos\frac{t}{2}, \quad h_2(t) = \frac{\sqrt{\pi}}{2\pi} g(t)\sin\frac{t}{2},$$

relative to $\{\phi_n\}$. By Theorem 8.12,

$$s_N(f;x) - f(x) = \widehat{c_{2N}}(h_1) + \widehat{c_{2N-1}}(h_2) \to 0$$

as $N \to \infty$.

Theorem. Suppose f is continuous. Then for all $\epsilon > 0$, there exists a trigonometric polynomial P such that,

$$|P(x) - f(x)| < \epsilon, \quad (x \in \mathbb{R}).$$

Remark. The statement is equivalent to:

$\exists$ a sequence of trigonometric polynomials P_n such that $\sup_{x\in\mathbb{R}} |P_n(x) - f(x)| \to 0$ as $n \to \infty$.

Proof. We consider the domain of f as $[0, 2\pi)$, and define

$$T = \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 1\} \approx \{e^{ix} : x \in [0, 2\pi)\}.$$

Then $h(x) = e^{ix}$ is a bijection from $[0, 2\pi)$ to T, so h^{-1} exists. Now let

$$g = f \circ h^{-1}$$

so that $f(x) = g(e^{ix})$. Then $g : T \to \mathbb{C}$, and T is compact in $\mathbb{R}^2$. Let

$$\mathcal{A} = \left\{ \sum_{n=-N}^{N} c_n z^n : z \in T, c_n \in \mathbb{C} \right\}.$$

Then $\mathcal{A}$ is a self-adjoint algebra.[18] Using the identity function of T, we see that $\mathcal{A}$ separates points of T and $\mathcal{A}$ vanishes at no point of T. By Theorem 7.33, $\mathcal{A}$ is dense in $C(T, \mathbb{C})$. Therefore, for all $\epsilon > 0$, there exists

$$\widehat{p} \in \mathcal{A} \text{ such that } \sup_{z \in T} |\widehat{p}(z) - g(z)| < \epsilon.$$

Using $p(x) = \widehat{p}(e^{ix})$, we recover $f(x)$ and get

$$\sup_{x \in [0, 2\pi)} |p(x) - g(h(x))| = \sup_{x \in [0, 2\pi)} |p(x) - f(x)| < \epsilon.$$

Our last question is, what if f is not continuous?

Theorem 8.16 (Parseval) Suppose that $f, g \in \mathcal{R}^2[-\pi, \pi]$ and periodic with period 2π. If

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad g \sim \sum_{n=-\infty}^{\infty} \gamma_n e^{inx},$$

the following holds.

(1) $\displaystyle \lim_{N \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2 \, dx = 0.$

(2) (Parseval's Identity) $\displaystyle \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\overline{g(x)} \, dx = \sum_{n=-\infty}^{\infty} c_n \overline{\gamma_n}.$

(3) $\displaystyle \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2.$ (Directly follows from (2))

Proof. We will use the notation

$$\|h\|_2 = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(x)|^2 \, dx \right\}^{1/2}.$$

[18] Check! The 'magic bars' will do the work.

(1) Let $\epsilon > 0$ be fixed. From Problem 6.12,[19] there exists a continuous function h with period 2π such that

$$\|f - h\|_2 < \frac{\epsilon}{3}.$$

By Theorem 8.15, there exists a trigonometric polynomial p such that

$$\|h - p\|_2 < \frac{\epsilon}{3}.$$

We observe the degree of p, and let $N_0 = \deg p$. Then by Theorem 8.11,[20]

$$\|h - s_N(h)\|_2 \le \|h - p\|_2 < \frac{\epsilon}{3}$$

holds for $N \ge N_0$. Finally the following holds, $\|s_N(h-f)\|_2 \le \|h - f\|_2 < \epsilon$.[21]

Now we use Problem 6.11. (Triangle Inequality for Norms)

$$\begin{aligned}\|f - s_N(f)\|_2 &\le \|f - h\|_2 + \|h - s_N(h)\|_2 + \|s_N(h) - s_N(f)\|_2 \\ &= \|f - h\|_2 + \|h - s_N(h)\|_2 + \|s_N(h-f)\|_2 < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

(2) By Problem 6.10, (Schwarz Inequality for Integrals)

$$\left|\int_{-\pi}^{\pi} f\overline{g} - \int_{-\pi}^{\pi} s_N(f)\overline{g}\right| \le \int_{-\pi}^{\pi} |f - s_N(f)|\,|g| \le \left(\int_{-\pi}^{\pi} |f - s_N(f)|^2\right)^{1/2} \left(\int_{-\pi}^{\pi} |g|^2\right)^{1/2} \to 0$$

as $N \to \infty$, by (1). Now we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} s_N(f)\overline{g(x)}\,dx &= \sum_{n=-N}^{N} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}\overline{g(x)}\,dx \\ &= \sum_{n=-N}^{N} c_n \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} g(x)\,dx} = \sum_{n=-N}^{N} c_n\overline{\gamma_n},\end{aligned}$$

which converges to $\displaystyle\sum_{n=-\infty}^{\infty} c_n\overline{\gamma_n}$.

---

[19] $f \in \mathcal{R}$ 이면 f를 근사하는 연속함수 h를 잡을 수 있다. (과제 #2)

[20] Degree를 높이면 근사가 더 정확해진다.

[21] $s_N(f;x)$의 정의 참고.

$$\int_{-\pi}^{\pi} |s_N(f;x)|^2\,dx \le \int_{-\pi}^{\pi} |f(x)|^2\,dx$$

October 11th, 2022

Definition 8.17 (Gamma Function)

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad (0 < x < \infty).$$

Remark. Gamma function can be also written as

$$\Gamma(x) = \lim_{N\to\infty}\lim_{\epsilon\to 0^+}\int_\epsilon^N t^{x-1}e^{-t}\,dt.$$

Let

$$g_{\epsilon,N}(x) = \int_\epsilon^N t^{x-1}e^{-t}\,dt.$$

Then $g_{\epsilon,N}$ is continuous on $(0,\infty)$.[22] (Check!)

Fix $a, b \in (0,\infty)$ where $0 < a < 1 < b < \infty$.

$$\sup_{x\in[a,b]}\left|e^{-t}t^{x-1}\right| \le \begin{cases} t^{a-1} & (t\in(0,1]) \\ e^{-t/2}e^{-t/2}t^{b-1} \le e^{-t/2}\sup\limits_{s\in[1,\infty)} e^{-s/2}s^{b-1} & (t\in[1,\infty)) \end{cases}$$

Therefore $g_{\epsilon,N}$ converges uniformly to Γ on $[a,b]$ because

$$\sup_{x\in[a,b]}\left|\int_n^m e^{-t}t^{x-1}\,dt\right| \le M\int_n^m e^{-t/2}\,dt \to 0$$

as $n, m\to\infty$. ($M = \sup_{s\in[1,\infty)} e^{-s/2}s^{b-1}$) and

$$\sup_{x\in[a,b]}\left|\int_{1/n}^{1/m} e^{-t}t^{x-1}\,dt\right| \le \int_{1/n}^{1/m} t^{a-1}\,dt \to 0$$

as $n, m\to\infty$. Therefore $\Gamma(x)$ is well-defined and continuous on $(0,\infty)$. Note that $\Gamma(1) = 1$, and $\Gamma(x)$ is infinitely differentiable.

Theorem 8.18

(1) $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

(2) $\Gamma(n+1) = n!$ for $n\in\mathbb{N}$.

(3) $\log\Gamma$ is convex on $(0,\infty)$. (log-convex)

[22] 나중에 Lebesgue 이론을 공부하면 엄청 쉬워집니다!

Proof. (1)

$$\int_\epsilon^N e^{-t}t^x\,dt = \left[-e^{-t}t^x\right]_\epsilon^N + x\int_\epsilon^N e^{-t}t^{x-1}\,dt.$$

As $\epsilon \to 0$ and $N \to \infty$, $\Gamma(x+1) = x\Gamma(x)$. Now (2) directly follows by induction.

(3) We just need to show that for $p > 1$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$,

$$\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) \le \Gamma(x)^{1/p}\Gamma(y)^{1/q}.$$

$$\begin{aligned}
\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) &= \int_0^\infty t^{\frac{x}{p}+\frac{y}{q}-\frac{1}{p}-\frac{1}{q}} e^{-\frac{t}{q}-\frac{t}{q}}\,dt \\
&= \int_0^\infty (t^{x-1}e^{-t})^{1/p}(t^{y-1}e^{-t})^{1/q}\,dt \\
&\le \left(\int_0^\infty t^{x-1}e^{-t}\,dt\right)^{1/p}\left(\int_0^\infty t^{x-1}e^{-t}\,dt\right)^{1/q} = \Gamma(x)^{1/p}\Gamma(y)^{1/q}.
\end{aligned}$$

Remark. (Problem 4.23) If f is convex on (a, b), then for $a < s < t < u < b$,

$$\frac{f(t) - f(s)}{t - s} \le \frac{f(u) - f(s)}{u - s} \le \frac{f(u) - f(t)}{u - t}.$$

Theorem 8.19 Suppose $f : (0, \infty) \to (0, \infty)$. If f satisfies

(1) $f(x + 1) = xf(x)$ for $x > 0$.

(2) $f(1) = 1$.

(3) $\log f$ is convex on $(0, \infty)$.

Then $f(x) = \Gamma(x)$ for $x > 0$.

Proof. We only need to show that (1), (2), (3) uniquely determines $\varphi = \log f$ for $x \in (0, 1)$. We consider $x \in (0, 1)$ and $n \in \mathbb{N}$. By Problem 4.23,

$$\log n = \varphi(n + 1) - \varphi(n) \le \frac{\varphi(n + 1 + x) - \varphi(n + 1)}{x} \le \varphi(n + 2) - \varphi(n + 1) = \log(n + 1).$$

Therefore

$$x \log n \le \varphi(n + 1 + x) - \varphi(n + 1) \le x \log(n + 1),$$

and

$$0 \le \varphi(n + 1 + x) - \log n! - x \log n \le x \log\left(1 + \frac{1}{n}\right).$$

We know that $\varphi(n + 1 + x) = \varphi(x + n) + \log(x + n)$. By induction,

$$\varphi(n + 1 + x) = \varphi(x) + \log x(x + 1)\cdots(x + n).$$

Therefore,

$$0 \le \varphi(x) - \log\left(\frac{n! \cdot n^x}{x(x+1)\cdots(x+n)}\right) \le x\log\left(1+\frac{1}{n}\right)$$

and the right-hand side goes to 0 as $n \to \infty$.

$$\varphi(x) = \lim_{n\to\infty} \log\left(\frac{n! \cdot n^x}{x(x+1)\cdots(x+n)}\right), \quad (0 < x < 1).$$

Therefore f is determined uniquely as

$$f(x) = \lim_{n\to\infty} \frac{n! \cdot n^x}{x(x+1)\cdots(x+n)} = \Gamma(x).$$

Definition. (Beta Function) For $x > 0$, $y > 0$,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

Remark. The following are properties of the Beta function.

(1) $B(x,y) = B(y,x)$ for $x, y > 0$.

(2) $B(x,y) = \displaystyle\int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,du$ $(t = \frac{u}{1+u})$.

(3) $B(1,y) = \displaystyle\int_0^1 (1-t)^{y-1}\,dt = \frac{1}{y}$ for $y > 0$.

(4) $B(x+1,y) = \dfrac{x}{x+y}B(x,y)$. (integration by parts)

Theorem 8.20 For $x > 0$, $y > 0$,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. Fix $y > 0$ and define

$$f(x) = \frac{\Gamma(x+y)}{\Gamma(y)}B(x,y).$$

We check the 3 conditions in Theorem 8.19.

$$f(1) = \frac{\Gamma(1+y)}{\Gamma(y)}B(1,y) = y \cdot \frac{1}{y} = 1$$

by (3) in the above remark.

$$f(x+1) = \frac{\Gamma(x+y+1)}{\Gamma(y)}B(x+1,y) = \frac{(x+y)\Gamma(x+y)}{\Gamma(y)} \cdot \frac{x}{x+y}B(x,y) = xf(x)$$

by (4) in the above remark. Now to show convexity,

$$\log f(x) = \log\Gamma(x+y) + \log B(x,y) - \log\Gamma(y),$$

we only have to check convexity for $\log B(x,y)$.

Let $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. Then

$$\begin{aligned} B\left(\frac{x_1}{p}+\frac{x_2}{q},y\right) &= \int_0^1 t^{\frac{x_1}{p}+\frac{x_2}{q}-\frac{1}{p}-\frac{1}{q}}(1-t)^{\frac{y}{p}+\frac{y}{q}-\frac{1}{p}-\frac{1}{q}}\,dt \\ &= \int_0^1 \left(t^{x_1-1}(1-t)^{y-1}\right)^{1/p}\left(t^{x_2-1}(1-t)^{y-1}\right)^{1/q}\,dt \\ &\leq \left(\int_0^1 t^{x_1-1}(1-t)^{y-1}\,dt\right)^{1/p}\left(\int_0^1 t^{x_2-1}(1-t)^{y-1}\,dt\right)^{1/q} \end{aligned}$$

Therefore $\log B(x,y)$ is convex (w.r.t x). By Theorem 8.19, $f(x)=\Gamma(x)$.

Remark. Consequences of Theorem 8.20.

(1) Using change of variables $t=\sin^2\theta$,

$$B(x,y)=2\int_0^{\pi/2}\sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta.$$

Let $y=x=\frac{1}{2}$.

$$2\int_0^{\pi/2}d\theta=\Gamma\left(\frac{1}{2}\right)^2.$$

(2) Using change of variables $t=s^2$,

$$\Gamma(x)=2\int_0^\infty s^{2x-1}e^{-s^2}\,ds.$$

Setting $s=\frac{1}{2}$,

$$\Gamma\left(\frac{1}{2}\right)=2\int_0^\infty e^{-s^2}\,ds=\int_{-\infty}^\infty e^{-s^2}\,ds=\sqrt{\pi}.$$

(3) We can show that

$$\Gamma(x)=\frac{2^{x-1}}{\sqrt{\pi}}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right).$$

Setting $f(x)=\text{RHS}$, check that $f(1)=1$, $f(x+1)=xf(x+1)$ and

$$\log\Gamma(x)=\log\Gamma\left(\frac{x}{2}\right)+\log\Gamma\left(\frac{x+1}{2}\right)+\log 2(x-1)-\log\sqrt{\pi}$$

is convex.

Chapter 11

The Lebesgue Theory

October 18th, 2022

르벡 적분 공부한다고 해서, 실수의 모든 집합에 대해서 하면 좋겠지만 그게 불가능해요. $\mathbb{R}$의 power set 위에서 measure를 정의하고 싶은데, 이게 불가능해요! 그러니 이보다 좀 작은 set들에 대해서 할건데, 좋은 구조를 가졌으면 좋겠어요. 그러면 집합들 간에 연산이 잘 정의되어 있고 닫혀 있었으면 좋겠어요. 여기서 연산은 교집합, 합집합, 차집합을 얘기합니다.

Algebra of Sets

Let $\mathcal{R}_0$ be a family of sets. We assume that $\mathcal{R}_0 \neq \varnothing$.

Definition. (Ring) $\mathcal{R}_0$ is a **ring** if for $A, B \in \mathcal{R}_0$,

$$A \cup B \in \mathcal{R}_0 \text{ and } A \setminus B \in \mathcal{R}_0.$$

Remark. Suppose $\mathcal{R}_0$ is a ring.

- Since $\mathcal{R}_0 \neq \varnothing$, $\exists\, A \in \mathcal{R}_0$. So $A \setminus A = \varnothing \in \mathcal{R}_0$.
- If $A, B \in \mathcal{R}_0$, then $A \cap B = A \setminus (A \setminus B) \in \mathcal{R}_0$.

Definition. (Power Set) Let X be a set. We define the **power set** of X as

$$\mathcal{P}(X) = \{A : A \subseteq X\}.$$

Definition. (Algebra) $\mathscr{F}_0 \subseteq \mathcal{P}(X)$ is called an **algebra** on X if

(1) $X \in \mathscr{F}_0$.

(2) $X \setminus A \in \mathscr{F}_0$ if $A \in \mathscr{F}_0$.

(3) $A \cup B \in \mathscr{F}_0$ if $A, B \in \mathscr{F}_0$.

Remark. $\mathscr{F}_0$ is an algebra on $X \iff \mathscr{F}_0$ is not empty + (2) + (3).[1]

Proof. ($\Longleftarrow$) $\exists A \in \mathscr{F}_0$. Then $X \setminus A \in \mathscr{F}_0$, so $X = (X \setminus A) \cup A \in \mathscr{F}_0$.

Remark. We write $A^C = X \setminus A$. We know that

$$A \cap B = (A^C \cup B^C)^C, \quad A \setminus B = A \cap B^C.$$

So if $A, B \in \mathscr{F}_0$, then $A \cap B, A \setminus B \in \mathscr{F}_0$.

Proposition.

(1) If $\mathcal{R}$ is an algebra on X then $\mathcal{R}$ is a ring.

(2) If $\mathcal{R} \subseteq \mathcal{P}(X)$ is a ring and $X \in \mathcal{R}$ then $\mathcal{R}$ is an algebra on X.[2]

Definition. (σ-ring) A ring $\mathcal{R}$ is called **σ-ring** if $\bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$ whenever $A_n \in \mathcal{R}$.[3]

그럼 마찬가지로 intersection도 성립하겠죠.

Remark. Check that

$$\bigcap_{n=1}^{\infty} A_n = A_1 \setminus \bigcup_{n=1}^{\infty} (A_1 \setminus A_n).$$

Therefore if $\mathcal{R}$ is a σ-ring and $A_n \in \mathcal{R}$, then $\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$.

Definition. (σ-algebra) An algebra $\mathscr{F}$ on X is called a **σ-algebra** if $\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$ whenever $A_n \in \mathscr{F}$.

Remark. Since an algebra is also a ring, $\bigcap_{n=1}^{\infty} A_n \in \mathscr{F}$ by the above remark.

Algebra와 σ-algebra를 소개했어요. Example로 생각하고 있을 것이 $\mathcal{P}(\mathbb{R})$...

[1] 교재에서는 공집합이 아닐 것을 가정하고 있기 때문에 (2), (3)이면 충분한 것입니다.
[2] 요즘 책들은 대부분 algebra 위에서 하고, ring 위에서 하면 진짜 조금 일반화 하는 거예요.
[3] Countable 한 집합의 union을 해도 안에 들어간다.

Definition 11.2 (Set Function) Let $\mathcal{R}_0$ be a (non-empty) ring.

$$\phi : \mathcal{R}_0 \to \overline{\mathbb{R}}$$

is called a **set function** on $\mathcal{R}_0$.

We assume that $\{-\infty, \infty\} \nsubseteq \text{Range}(\phi)$, and $\text{Range}(\phi)$ is not $\{\infty\}$ or $\{-\infty\}$. Therefore,

$$\exists\, A \in \mathcal{R}_0 \text{ such that } \phi(A) \in \mathbb{R}.$$

(1) ϕ is **additive** if

$$\phi(A \cup B) = \phi(A) + \phi(B)$$

for disjoint $A, B \in \mathcal{R}_0$.

(2) ϕ is **countably additive** (σ-additive) if

$$\phi\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \phi(A_i).$$

for pairwise disjoint $A_i \in \mathcal{R}_0$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{R}_0$.[4]

(3) (Measure) Given a σ-ring $\mathcal{R}_0$, we call μ a **measure** on $\mathcal{R}_0$ if μ is a set function on $\mathcal{R}_0$ which is countably additive and $\text{Range}(\mu) = [0, \infty]$.

Remark.

(1) If ϕ is additive,

$$\phi\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \phi(A_i) \qquad (*)$$

for pairwise disjoint $A_i \in \mathcal{R}_0$. We call the property $(*)$ *finite additivity* or we say that ϕ is *finitely additive*.

(2) $\phi(\varnothing) = 0$. Write $\phi(A) = \phi(A \cup \varnothing)$, and cancel $\phi(A) \in \mathbb{R}$. [5]

Definition. Let μ be a measure on σ-algebra $\mathscr{F} \subseteq \mathcal{P}(X)$.

(1) μ is **finite** $\iff \mu(X) < \infty$.

(2) μ is **σ-finite** $\iff \exists\, F_1 \subseteq F_2 \subseteq \cdots,\ \mu(F_i) < \infty$ and $\bigcup_{i=1}^{\infty} F_i = X$.

[4] σ-ring 이면 불필요한 조건이지만, 일반적인 ring에 대해서는 필요한 조건이다.
[5] 지금부터는 extended real number가 나오기 때문에 뺄셈에 조심하세요!

Some Basic Properties of Set Functions

Let ϕ be a set function.

- If ϕ is countably additive on a ring $\mathcal{R}_0$, then ϕ is additive.
- If ϕ is additive on $\mathcal{R}_0$, then for $A, B \in \mathcal{R}_0$,

$$\phi(A \cup B) + \phi(A \cap B) = \phi(A) + \phi(B).$$[6]

- If ϕ is additive on $\mathcal{R}_0$, then for $A_1, A_2 \in \mathcal{R}_0$ such that $A_1 \subseteq A_2$,

$$\phi(A_2) = \phi(A_2 \setminus A_1) + \phi(A_1).$$

  (1) If $\phi \geq 0$, then $\phi(A_1) \leq \phi(A_2)$. (Monotonicity)

  (2) If $|\phi(A_1)| < \infty$, then $\phi(A_2 \setminus A_1) = \phi(A_2) - \phi(A_1)$.

- If ϕ is additive and $\phi \geq 0$, then for $A, B \in \mathcal{R}_0$,

$$\phi(A \cup B) \leq \phi(A) + \phi(B).$$

  By induction,

$$\phi\left(\bigcup_{n=1}^{m} A_n\right) \leq \sum_{n=1}^{m} \phi(A_n) \tag{$*$}$$

  for all $A_i \in \mathcal{R}_0$.[7] The property $(*)$ is called *finite subadditivity*.

Theorem 11.3 Let ϕ be countably additive on $\mathcal{R}_0$. Suppose $A_n \in \mathcal{R}_0$, $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}_0$, and $A_1 \subseteq A_2 \subseteq \cdots$. Then

$$\lim_{n\to\infty} \phi(A_n) = \phi(A) = \phi\left(\bigcup_{n=1}^{\infty} A_n\right).$$[8]

Proof. Let $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$ for $n \geq 2$. We see that B_n are pairwise disjoint. Then

$$\phi(A_n) = \phi\left(\bigcup_{k=1}^{n} B_k\right) = \sum_{k=1}^{n} \phi(B_k).$$

---

[6] 고등학교 때 봤던 확률의 덧셈정리와 유사하죠?

[7] Disjoint일 필요는 없어요!

[8] 반대로 decreasing인 경우에는 안됩니다. 반례는 $[n, \infty)$를 생각해보면 됩니다.

Since $A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$,

$$\phi(A) = \sum_{n=1}^{\infty} \phi(B_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \phi(B_k) = \lim_{n\to\infty} \phi(A_n).$$

Corollary. Let μ be a measure on a σ-algebra $\mathscr{F}$. Suppose $A_n \in \mathscr{F}$ and $A_1 \subseteq A_2 \subseteq \cdots$. Then

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right).$$

이제 우리의 목표는 $\mathbb{R}^p$에서 measure를 construct하는 것인데, 일단 잴 수 있는 것부터 시작할 것입니다. 예를 들면 box 같은 것들. (경계는 고려하지 않고)

October 20th, 2022

오늘 할 부분의 목표는 르벡 적분을 정의하는 것입니다. 우리가 잴 수 있는 집합들부터 시작합니다. $\mathbb{R}^p$에서 논의할 건데, 이제 여기서부터는 $\mathbb{R}$의 interval이 open/closed에 관계 없습니다.

Definition. (Intervals in $\mathbb{R}^p$) $a_i, b_i \in \mathbb{R}$, $a_i \leq b_i$. An interval in $\mathbb{R}^p$ is defined as

$$\prod_{i=1}^{p} I_i = I_1 \times \cdots \times I_p,$$

where I_i is an interval in $\mathbb{R}$.

Definition. (Elementary Sets) A set is called an **elementary set** if it is a finite union of intervals. Let Σ denote the family of all elementary sets of $\mathbb{R}^p$.

Remark. It is trivial that elementary sets are bounded.

Proposition. Σ is a ring. But it is not σ-ring.[9]

Elementary set에서는 재는 방법을 아주 잘 알고 있죠?

Definition. Let $I = \prod_{i=1}^{p} I_i$ be an interval on $\mathbb{R}^p$, where a_i, b_i are endpoints of I_i. We define

$$m(I) = \prod_{i=1}^{p}(b_i - a_i).$$

Additionally, if $A = \bigcup_{i=1}^{n} I_i \in \Sigma$ and I_i are pairwise disjoint intervals on $\mathbb{R}^p$, then

$$m(A) = \sum_{i=1}^{n} m(I_i).$$

이 정의가 well-defined 인지 확인해야 합니다.

Remark. m is additive on Σ. So $m : \Sigma \to [0, \infty)$ is a set function on a ring and additive.

근데 여기서 regularity를 추가로 만족했으면 좋겠어요.[10]

Definition. (Regularity) Suppose $\mu : \Sigma \to [0, \infty]$ is additive. We say that μ is **regular** if for every $A \in \Sigma$ and $\epsilon > 0$,

$$\exists\, F_{\text{closed}}, G_{\text{open}} \in \Sigma \text{ such that } F \subseteq A \subseteq G \text{ and } \mu(G) - \epsilon \leq \mu(A) \leq \mu(F) + \epsilon.$$

[9] 전체 공간인 $\mathbb{R}^p$를 포함하고 있지 않아요.
[10] 이게 왜 필요한가요?

Remark. Check that m is regular!

Assume a finite set function $\mu : \Sigma \to [0, \infty)$ is regular and additive. (also finite!)

Definition. (Outer Measure) We define the **outer measure** $\mu^* : \mathcal{P}(\mathbb{R}^p) \to [0, \infty]$ of $E \in \mathcal{P}(\mathbb{R}^p)$ corresponding to μ as

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \mu(A_n) : \text{open set } A_n \in \Sigma \text{ such that } E \subseteq \bigcup_{n=1}^{\infty} A_n \right\}.$$

밖에서 길이를 재서 근사하는 거예요. 모든 power set에 대해서 정의할 수 있으니, 이런 것들로 다 잴 수 있으면 좋겠어요. 근데 이게 measure가 되려면 countably additive 해야하는데, 이게 제일 만족시키기 어려운 조건입니다. 그래서 이게 안되거든요? 그럼 되는 애들만 모아서 할거예요.

Remark.

- $\mu^* \geq 0$.
- $\mu^*(E_1) \leq \mu^*(E_2)$ if $E_1 \subseteq E_2$. (Monotonicity)

Theorem 11.8

(1) $\mu^*(A) = \mu(A)$ if $A \in \Sigma$.[11]

(2) Countable subadditivity holds.

$$\mu^* \left(\bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \mu^*(E_n), \quad (\forall E_n \in \mathcal{P}(\mathbb{R}^p))$$

Proof.
(1) Let $A \in \Sigma$ and $\epsilon > 0$. By regularity of μ on Σ, $\exists\, G_{\text{open}} \in \Sigma$ such that $A \subseteq G$ and

$$\mu^*(A) \leq \mu(G) \leq \mu(A) + \epsilon. \tag{$*$}$$

By definition of μ^*, there exists open sets $A_n \in \Sigma$ such that $A \subseteq \bigcup_{n=1}^{\infty} A_n$ and

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu^*(A) + \epsilon.$$

By regularity of μ on Σ, there exists a closed set $F \in \Sigma$ such that $F \subseteq A$ and $\mu(A) \leq \mu(F) + \epsilon$. Since $F \subseteq \mathbb{R}^p$ is closed and bounded, it is compact. So we can take a finite open cover,

[11] A가 open이 아니면 자명하지 않은 명제입니다.

$$\exists\, N \in \mathbb{N} \text{ such that } F \subseteq \bigcup_{i=1}^{N} A_i.$$

Therefore,

$$\mu(A) \le \mu(F) + \epsilon \le \sum_{i=1}^{N} \mu(A_i) \le \sum_{i=1}^{n} \mu(A_i) + \epsilon \le \mu^*(A) + 2\epsilon \qquad (**)$$

Now set $\epsilon \to 0$ for $(*)$, $(**)$ to see that $\mu(A) = \mu^*(A)$.

(2) Assume both sides are finite, that $\mu^*(E_n) < \infty$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ be given. For each $n \in \mathbb{N}$, there exists open sets $A_{n,k} \in \Sigma$ such that

$$E_n \subseteq \bigcup_{k=1}^{\infty} A_{n,k} \quad \text{and} \quad \sum_{k=1}^{\infty} \mu(A_{n,k}) \le \mu^*(E_n) + 2^{-n}\epsilon.$$

So,

$$\mu^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu(A_{n,k}) \le \sum_{n=1}^{\infty} \mu^*(E_n) + \epsilon$$

since μ^* is taken as the infimum. Now take $\epsilon \to 0$ and the inequality holds.

Notation. (Symmetric Difference) $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Definition. $d(A, B) = \mu^*(A \triangle B)$, $A_n \to A$ is defined as $d(A_n, A) \to 0$.

Remark.

- $d(A, B) \le d(A, C) + d(C, B)$ for $A, B, C \in \mathbb{R}^p$.
- For $A_1, B_2, B_1, B_2 \in \mathbb{R}^p$,

$$\left.\begin{aligned} &d(A_1 \cup A_2, B_1 \cup B_2) \\ &d(A_1 \cap A_2, B_1 \cap B_2) \\ &d(A_1 \setminus A_2, B_1 \setminus B_2) \end{aligned}\right\} \le d(A_1, B_1) + d(A_2, B_2).$$

Definition. (Finitely μ-measurable) A is **finitely μ-measurable** if $\exists\, A_n \in \Sigma$ such that $A_n \to A$. We write

$$\mathfrak{M}_F(\mu) = \{A : A \text{ is finitely } \mu\text{-measurable}\}.$$ [12]

Definition. (μ-measurable) A is **μ-measurable** if $A = \bigcup_{n=1}^{\infty} A_n$ where $A_n \in \mathfrak{M}_F(\mu)$.

We write

$$\mathfrak{M}(\mu) = \{A : A \text{ is } \mu\text{-measurable}\}.$$

[12] μ라는 set function에 의해 $\mu^*(A_n \triangle A) \to 0$ 이 되는 sequence of elementary sets A_n이 존재한다.

Remark. $\mu^*(A) = d(A, \varnothing) \le d(A,B) + \mu^*(B)$.

Proposition. If $\mu^*(A)$ or $\mu^*(B)$ is finite,

$$|\mu^*(A) - \mu^*(B)| \le d(A,B).$$

Corollary. If $A \in \mathfrak{M}_F(\mu)$ then $\mu^*(A) < \infty$.

Proof. There exists $A_n \in \Sigma$ such that $A_n \to A$, and $\exists\, N \in \mathbb{N}$ such that

$$\mu^*(A) \le d(A_N, A) + \mu^*(A_N) \le 1 + \mu^*(A_N) < \infty.$$

Corollary. If $A_n \to A$ and $A_n, A \in \mathfrak{M}_F(\mu)$, then $\mu^*(A_n) \to \mu^*(A) < \infty$.

Proof. $\mu^*(A), \mu^*(A_n)$ are finite, so $|\mu^*(A_n) - \mu^*(A)| \le d(A_n, A) \to 0$ as $n \to \infty$.

Theorem 11.10 $\mathfrak{M}(\mu)$ is a σ-algebra and μ^* is a measure on $\mathfrak{M}(\mu)$.[13]

Proof. $\mathfrak{M}(\mu)$가 σ-algebra이고 μ^*가 $\mathfrak{M}(\mu)$에서 countably additive임을 보이면 된다.

(Step 0) $\mathfrak{M}_F(\mu)$ *is a ring.*

Let $A, B \in \mathfrak{M}_F(\mu)$. Then $\exists\, A_n, B_n \in \Sigma$ such that $A_n \to A$, $B_n \to B$. Then,

$$\left.\begin{aligned} &d(A_n \cup B_n, A \cup B) \\ &d(A_n \cap B_n, A \cap B) \\ &d(A_n \setminus B_n, A \setminus B) \end{aligned}\right\} \le d(A_n, A) + d(B_n, B) \to 0.$$

Therefore $A_n \cup B_n \to A \cup B$, $A_n \setminus B_n \to A \setminus B$ and thus $\mathfrak{M}_F(\mu)$ is a ring.

(Step 1) μ^* *is additive on* $\mathfrak{M}_F(\mu)$.

By the corollary, we know that[14]

$$\begin{aligned} \mu(A_n) &\to \mu^*(A), \quad \mu(A_n \cup B_n) \to \mu^*(A \cup B), \\ \mu(B_n) &\to \mu^*(B), \quad \mu(A_n \cap B_n) \to \mu^*(A \cap B). \end{aligned}$$

Since $\mu(A_n) + \mu(B_n) = \mu(A_n \cup B_n) + \mu(A_n \cap B_n)$, let $n \to \infty$. Then

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B).$$

Setting $A \cap B = \varnothing$ shows that μ^* is additive.

[13] 정의역을 좀 좁히면 measure가 된다!
[14] Σ 위에서는 $\mu = \mu^*$ 였다!

(Step 2) $\mathfrak{M}_F(\mu) = \{A \in \mathfrak{M}(\mu) : \mu^*(A) < \infty\}$.[15]

Claim. $A \in \mathfrak{M}(\mu)$ can be written as a disjoint union of elements in $\mathfrak{M}_F(\mu)$.

Proof. Let $A = \bigcup A'_n$ with $A'_n \in \mathfrak{M}_F(\mu)$. Set

$$A_1 = A'_1 \text{ and } A_n = A'_n \setminus (A'_1 \cup \cdots \cup A'_{n-1}) \text{ for } n \geq 2.$$

Then we see that A_n are disjoint and $A_n \in \mathfrak{M}_F(\mu)$.

Write $A = \bigcup_{n=1}^{\infty} A_n$ where $A_n \in \mathfrak{M}_F(\mu)$.

(1) By Theorem 11.8 (2), $\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

(2) $\bigcup_{n=1}^{k} A_n \subseteq A$, $\sum_{n=1}^{k} \mu^*(A_n) \leq \mu^*(A)$ by Step 1. Set $k \to \infty$ to get $\mu^*(A) \geq \sum_{n=1}^{\infty} \mu^*(A_n)$.

By (1), (2), $\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n)$.[16][17]

Let $B_n = \bigcup_{k=1}^{n} A_k$. If we suppose that $\mu^*(A) < \infty$, by convergence we know that

$$d(A, B_n) = \mu^*\left(\bigcup_{k=n+1}^{\infty} A_k\right) = \sum_{k=n+1}^{\infty} \mu^*(A_i) \to 0 \text{ as } n \to \infty.$$

Since $B_n \in \mathfrak{M}_F(\mu)$, we can take $C_n \in \Sigma$ such that $d(B_n, C_n)$ is arbitrarily small for each $n \in \mathbb{N}$. Then $d(A, C_n) \leq d(A, B_n) + d(B_n, C_n)$ can be made arbitrarily small for large enough n, so we can conclude that $C_n \to A$ and hence $A \in \mathfrak{M}_F(\mu)$.

(Step 3) μ^* *is countably additive on* $\mathfrak{M}(\mu)$.

Suppose that $A_n \in \mathfrak{M}(\mu)$ is a partition of $A \in \mathfrak{M}(\mu)$. If $\mu^*(A_m) = \infty$ for some $m \in \mathbb{N}$, then countable additivity holds since

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \geq \mu^*(A_m) = \infty = \sum_{n=1}^{\infty} \mu^*(A_n).$$

If $\mu^*(A_n) < \infty$ for all $n \in \mathbb{N}$, then $A_n \in \mathfrak{M}_F(\mu)$ by Step 2, so

$$\mu^*(A) = \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu^*(A_n).$$

[15] A가 μ-measurable인데 $\mu^*(A) < \infty$이면 A는 finitely μ-measurable이다.
[16] A가 countable union of sets in $\mathfrak{M}_F(\mu)$이므로 μ^*도 각 set의 μ^*의 합이 된다.
[17] 아직 증명이 끝나지 않았습니다. A_n은 $\mathfrak{M}(\mu)$의 원소가 아니라 $\mathfrak{M}_F(\mu)$의 원소입니다.

(Step 4) $\mathfrak{M}(\mu)$ *is a σ-ring.*

If $A_n \in \mathfrak{M}(\mu)$ then there exists $B_{n,k} \in \mathfrak{M}_F(\mu)$ such that $A_n = \bigcup_k B_{n,k}$. Then

$$\bigcup_n A_n = \bigcup_{n,k} B_{n,k} \in \mathfrak{M}(\mu).$$

For $A, B \in \mathfrak{M}(\mu)$, $A = \bigcup A_n$, $B = \bigcup B_n$ where $A_n, B_n \in \mathfrak{M}_F(\mu)$. We see that

$$A \setminus B = \bigcup_{n=1}^{\infty} (A_n \setminus B) = \bigcup_{n=1}^{\infty} (A_n \setminus (A_n \cap B)),$$

so it is enough to show that $A_n \cap B \in \mathfrak{M}_F(\mu)$. We have

$$A_n \cap B = \bigcup_{k=1}^{\infty} (A_n \cap B_k) \in \mathfrak{M}(\mu)$$

by definition, and since $\mu^*(A_n \cap B) \leq \mu^*(A_n) < \infty$, $A_n \cap B \in \mathfrak{M}_F(\mu)$. Therefore $A \setminus B$ is a countable union of elements of $\mathfrak{M}_F(\mu)$, so $A \setminus B \in \mathfrak{M}(\mu)$.

Thus $\mathfrak{M}(\mu)$ is a σ-ring and also σ-algebra.

We extend the definition of μ on Σ to $\mathfrak{M}(\mu)$ (σ-algebra) by setting $\mu = \mu^*$ on $\mathfrak{M}(\mu)$. When $\mu = m$ on Σ, such extension m on $\mathfrak{M}(m)$ is called the **Lebesgue measure** on $\mathbb{R}^p$, and $A \in \mathfrak{M}(m)$ is called a Lebesgue measurable set.

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Remark. (11.11)

Proposition. If A is open, then $A \in \mathfrak{M}(\mu)$. Also, $A^C \in \mathfrak{M}(\mu)$.

Proof. Let $I(x,r)$ be an open box centered at x, with radius r. Then

$$A = \bigcup_{\substack{x\in\mathbb{Q}^p,\, r\in\mathbb{Q} \\ I(x,r)\subseteq A}} I(x,r) \quad \text{(countable union of } \mathfrak{M}_F(\mu))$$

Since $\mathfrak{M}(\mu)$ is a σ-algebra, $A^C \in \mathfrak{M}(\mu)$. Every closed set is also a member of $\mathfrak{M}(\mu)$.

Proposition. If $A \in \mathfrak{M}(\mu)$, there exists open set G and closed set F such that

$$F \subseteq A \subseteq G \text{ and } \mu(G \setminus A) < \epsilon \text{ and } \mu(A \setminus F) < \epsilon.$$[18]

Proof. Let $A = \bigcup_{n=1}^\infty A_n$ ($A_n \in \mathfrak{M}_F(\mu)$), and fix $\epsilon > 0$. For each $n \in \mathbb{N}$, there exists open sets $B_{n,k} \in \Sigma$ such that $A_n \subseteq \bigcup_{k=1}^\infty B_{n,k}$ and

$$\mu\left(\bigcup_{k=1}^\infty B_{n,k}\right) \le \sum_{k=1}^\infty \mu(B_{n,k}) < \mu(A_n) + 2^{-n}\epsilon.$$[19]

Let $G = \bigcup_{n=1}^\infty G_n$ where $G_n = \bigcup_{k=1}^\infty B_{n,k}$. Since $\mu(A_n) < \infty$, ($\because A_n \in \mathfrak{M}_F(\mu)$)

$$\mu(G \setminus A) = \mu\left(\bigcup_{n=1}^\infty G_n \setminus \bigcup_{n=1}^\infty A_n\right) \le \mu\left(\bigcup_{n=1}^\infty G_n \setminus A_n\right) \le \sum_{n=1}^\infty \mu(G_n \setminus A_n) \le \sum_{n=1}^\infty 2^{-n}\epsilon = \epsilon.$$

Using a similar argument, there exists an open set F^C such that $A^C \subseteq F^C$ and $\mu\left(F^C \setminus A^C\right) < \epsilon$. ($A$ had no conditions on open/closed) F is a closed set, and since $F^C \setminus A^C = F^C \cap A = A \setminus F$, $\mu(A \setminus F) < \epsilon$ and $F \subseteq A$. Therefore the proposition is proven.

Definition. (Borel σ-algebra) $\mathfrak{B} = \mathfrak{B}(\mathbb{R}^p)$ is the σ-algebra containing all open sets and closed sets. The definition is equivalent to the smallest σ-algebra on $\mathbb{R}^p$ containing all open sets. Let O denote the collection of open sets of $\mathbb{R}^p$. Then

$$\mathfrak{B} = \bigcap_{O\subseteq G,\, G:\, \sigma\text{-algebra}} G.$$

Remark. E is a **Borel set** if $E \in \mathfrak{B}$. Also, $\mathfrak{B} \subseteq \mathfrak{M}(\mu)$ by definition.

Definition. (μ-measure zero set) $A \in \mathfrak{M}(\mu)$ is called a **μ-measure zero set** if $\mu(A) = 0$.

[18] μ is also regular on $\mathfrak{M}(\mu)$.
[19] 첫 번째 부등식은 countable subadditivity, 두 번째 부등식은 μ^*의 정의에서 나온다.

Proposition. If $A \in \mathfrak{M}(\mu)$, there exists Borel sets F, G such that $F \subseteq A \subseteq G$. Also, A can be written as a union of a Borel set and a set μ-measure zero set.

Proof. Take open sets $G_n \in \Sigma$, closed sets $F_n \in \Sigma$ such that

$$F_n \subseteq A \subseteq G_n \text{ and } \mu(G_n \setminus A) < \frac{1}{n} \text{ and } \mu(A \setminus F_n) < \frac{1}{n}.$$

Define $F = \bigcup_{n=1}^{\infty} F_n$, $G = \bigcap_{n=1}^{\infty} G_n$, then $F, G \in \mathfrak{B}$ and $F \subseteq A \subseteq G$. Also,

$$\left.\begin{aligned} \mu(G \setminus A) \leq \mu(G_n \setminus A) < \tfrac{1}{n} \\ \mu(A \setminus F) \leq \mu(A \setminus F_n) < \tfrac{1}{n} \end{aligned}\right\} \to 0 \text{ as } n \to \infty.$$

Now we can write $A = F \cup (A \setminus F)$, $G = A \cup (G \setminus A)$. So $A \in \mathfrak{M}(\mu)$ is a union of a Borel set, and a set of μ-measure zero. Unioning $A \in \mathfrak{M}(\mu)$ with some μ-measure zero set can make it a Borel set.

Proposition. For every μ, the set of μ-measure zero form a σ-ring.

Proof. Check countable subadditivity, and the rest is trivial. If $\mu(A_n) = 0$ for all $n \in \mathbb{N}$, then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n) = 0$.

Proposition. Countable sets have Lebesgue measure zero. Also, there are uncountable sets with measure zero.

Proof. Let A be a countable set. Then it is a countable union of points[20] which have measure zero. Thus $m(A) = 0$. As for the uncountable case, consider the Cantor set P. Define E_n as in 2.44 then $P = \bigcap_{n=1}^{\infty} E_n$, but $m(E_n) = \left(\frac{2}{3}\right)^n$ for all $n \in \mathbb{N}$. $P \subseteq E_n$ so $m(P) \leq m(E_n)$ and setting $n \to \infty$ shows that $m(P) = 0$.

Remark. $\mathfrak{M}(m) \subsetneq \mathcal{P}(\mathbb{R}^p)$. (증명은 어려워 우리의 범위를 넘는다)

Definition. (Measure Space) X is a **measure space** if there exists a σ-algebra/σ-ring $\mathfrak{M}$ on X and a measure μ on $\mathfrak{M}$. We write $(X, \mathfrak{M}, \mu)$.

Definition. (Measurable Space) X is called a **measurable space** if $\mathfrak{M}$ is a σ-algebra on X. ($X \in \mathfrak{M}$) We write $(X, \mathfrak{M})$.

Example.

(1) $(\mathbb{R}^p, \mathfrak{M}(m), m)$ is the Lebesgue measure space.

(2) $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where $\mu(E) = |E|$ for $E \in \mathcal{P}(\mathbb{N})$. (counting measure)

[20] Since points are closed, $A \in \mathfrak{B}(\mathbb{R}^p)$.

October 27th, 2022

We will discuss measurable functions on a general measurable space $(X, \mathscr{F})$ and integration on general measure space $(X, \mathscr{F}, \mu)$ where μ is a measure on $\mathscr{F}$.

Definition 11.13 (Measurable Function) Let $(X, \mathscr{F})$ be a measurable space where $\mathscr{F}$ is a σ-algebra on X. $f : X \to \overline{\mathbb{R}}$ is an **$\mathscr{F}$-measurable function** if the set

$$\{x \in X : f(x) > a\}$$

is measurable for all $a \in \mathbb{R}$.

We directly observe the following corollary.

Corollary. Every continuous function f on $\mathbb{R}^p$ is Lebesgue measurable.

Proof. The set $\{x : f(x) > a\}$ is open in $\mathbb{R}^p$, so it is an element of $\mathfrak{M}(m)$.

Theorem 11.15 Let f be a function defined on a measurable space X. The following are equivalent.

(1) $\{x : f(x) > a\}$ is measurable for all $a \in \mathbb{R}$.

(2) $\{x : f(x) \geq a\}$ is measurable for all $a \in \mathbb{R}$.

(3) $\{x : f(x) < a\}$ is measurable for all $a \in \mathbb{R}$.

(4) $\{x : f(x) \leq a\}$ is measurable for all $a \in \mathbb{R}$.

Proof. The following relations can be used to prove the result.
($1 \Longrightarrow 2$)

$$\{x : f(x) \geq a\} = f^{-1}([a, \infty)) = f^{-1}\left(\bigcup_{n=1}^{\infty}\left(a + \frac{1}{n}, \infty\right)\right) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left(a + \frac{1}{n}, \infty\right)\right).$$

($2 \Longrightarrow 3$) $\{x : f(x) < a\} = X \setminus \{x : f(x) \geq a\}$.

($3 \Longrightarrow 4$)

$$\{x : f(x) \leq a\} = f^{-1}((-\infty, a]) = f^{-1}\left(\bigcup_{n=1}^{\infty}\left(-\infty, a - \frac{1}{n}\right)\right) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left(-\infty, a - \frac{1}{n}\right)\right).$$

($4 \Longrightarrow 1$) $\{x : f(x) > a\} = X \setminus \{x : f(x) \leq a\}$.

Theorem 11.16 If f is measurable, then $|f|$ is also measurable.

Proof. $\{x : |f(x)| < a\} = \{x : f(x) < a\} \cap \{x : f(x) > -a\}$.

Remark. 역은 성립하지 않는다. Suppose that $S \subseteq (0,\infty)$ is not measurable. Define

$$g(x) = \begin{cases} x & (x \in S) \\ -x & (x \notin S) \end{cases}$$

then $|g(x)| = x$ for all $x \in \mathbb{R}$. Thus $|g(x)|$ is a measurable function, but $g(x)$ is not measurable since $\{x : g(x) > 0\} = \mathbb{R} \setminus (-\infty, 0] = S$ is not measurable.

Theorem 11.17 Let $\{f_n\}$ be a sequence of measurable functions. Then

$$\sup_{n\in\mathbb{N}} f_n(x), \quad \inf_{n\in\mathbb{N}} f_n(x), \quad \limsup_{n\to\infty} f_n(x), \quad \liminf_{n\to\infty} f_n(x) \quad (x \in X)$$

are measurable.

Proof. It suffices to prove the statement for $\sup f_n$, since

$$\inf f_n = -\sup\left(-f_n\right), \quad \limsup f_n = \inf_n \sup_{k\geq n} f_k, \quad \liminf f_n = -\limsup\left(-f_n\right).$$

Claim. $\{x : \sup_{n\in\mathbb{N}} f_n(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\} \in \mathscr{F}$.

Proof.

($\subseteq$) If $\sup f_n(x) > a$, $\exists N \in \mathbb{N}$ such that $f_N(x) > a$, so $x \in \bigcup_{n=1}^{\infty}\{x : f_n(x) > a\}$.

($\supseteq$) If $x \in \bigcup_{n=1}^{\infty}\{x : f_n(x) > a\}$, $\exists N \in \mathbb{N}$ such that $f_N(x) > a$. Then $\sup f_n(x) > a$.

Corollary. If f, g are measurable functions,

(1) $\max\{f, g\}$, $\min\{f, g\}$ are measurable.

(2) $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$ are measurable.

Proof. (1) $\{x : \max\{f, g\} > a\} = \{x : f(x) > a\} \cup \{x : g(x) > a\}$, and $\{x : \min\{f, g\} < a\} = \{x : f(x) < a\} \cup \{x : g(x) < a\}$. (2) is trivial from (1).

Corollary. The limit of a convergent sequence of measurable functions is measurable.

Proof. Consider $\lim f_n = \limsup f_n = \liminf f_n$.

Theorem 11.18 Let f, g be measurable real-valued functions on X. If $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $h(x) = F\big(f(x), g(x)\big)$ is measurable. In particular, $f+g$, fg are measurable.[21]

Proof. Since $G_a = \{(u,v) \in \mathbb{R}^2 : F(u,v) > a\}$ is open, we can write it as a union of open intervals.

$$G_a = \bigcup_{n=1}^{\infty} I_n \text{ where } I_n = (a_n, b_n) \times (c_n, d_n), \quad (a_n, b_n, c_n, d_n \in \mathbb{R})$$

Then

$$\begin{aligned}
\{x \in X : F\big(f(x), g(x)\big) > a\} &= \{x \in X : \big(f(x), g(x)\big) \in G_a\} \\
&= \bigcup_{n=1}^{\infty} \{x \in X : a_n < f(x) < b_n,\ c_n < g(x) < d_n\} \\
&= \bigcup_{n=1}^{\infty} \{x \in X : a_n < f(x) < b_n\} \cap \{x \in X : c_n < g(x) < d_n\} \in \mathscr{F}.
\end{aligned}$$

The class of measurable functions on X depends only on the σ-ring $\mathfrak{M}$. For example, Borel-measurable functions on $\mathbb{R}^p$ are functions of which $\{x : f(x) > a\}$ is always a Borel set.

Definition. (Characteristic Function) For $E \subseteq X$, the **characteristic function** χ_E is defined as[22]

$$\chi_E(x) = \begin{cases} 1 & (x \in E) \\ 0 & (x \notin E) \end{cases}$$

Definition. (Simple Function) $s : X \to \mathbb{R}$ is called a **simple function** if the range $s(X)$ is a finite set.[23]

Remark. If $s(X) = \{c_1, c_2, \dots, c_n\}$ where c_i are distinct and nonzero, define $E_i = s^{-1}(c_i)$. Then we can write

$$s(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x).$$

If $s^{-1}(0)$ is included as E_0, $X = \bigcup_{i=0}^{n} E_i$. All simple functions are a linear combination of χ_{E_i} where E_i are disjoint. If E_i is *measurable*, χ_{E_i} is a *measurable* function. So all *measurable* simple functions are a linear combination of *measurable* χ_{E_i} where E_i are disjoint.

[21] Note that we don't want the case $\infty - \infty$.
[22] Also known as the indicator function, also written as $\mathbf{1}_E, K_E$.
[23] Note that the codomain is not $\overline{\mathbb{R}}$.

All functions can be approximated by simple functions.

Theorem 11.20 Let $f : X \to \overline{\mathbb{R}}$. There exists a sequence s_n of simple functions such that $\lim_{n\to\infty} s_n(x) = f(x)$ for every $x \in X$, and $|s_n(x)| \le |f(x)|$.

(1) If f is measurable, s_n may be taken measurable.

(2) If $f \ge 0$, s_n may be chosen to be a monotonically increasing sequence, $\sup_{n\in\mathbb{N}} s_n = f$.

(3) If f is bounded, the convergence is uniform.

Proof. Suppose $f \ge 0$.

(1) For $n \in \mathbb{N}$, define

$$E_{n,i} = \begin{cases} \{x : i \cdot 2^{-n} \le f(x) < (i+1)\cdot 2^{-n}\} & (i = 0, 1, \ldots, n\cdot 2^n - 1) \\ \{x : f(x) \ge n\} & (i = n \cdot 2^n) \end{cases}$$

and let $s_n(x) = \sum_{n=0}^{n\cdot 2^n} \frac{i}{2^n}\chi_{E_{n,i}}(x)$ then s_n is simple. We also have $s_n(x) \le f(x)$, $|f(x) - s_n(x)| \le 2^{-n}$ for $x \in \{x : f(x) < n\}$.

(i) If $f(x) < \infty$, $f(x) < M$ for some $M > 0$. So setting $n > M$ large enough will give

$$|f(x) - s_n(x)| \le 2^{-n} \qquad (*)$$

for $x \in \{x : f(x) < n\}$.

(ii) If $f(x) \to \infty$, then $s_n(x) = n \to \infty$ on $\{x : f(x) \ge n\}$. $s_n \to f$ as $n \to \infty$.

Therefore $\lim_{n\to\infty} s_n(x) = f(x)$.

(2) If f is measurable, $E_{n,i}$ is measurable, so s_n is measurable.

(3) Since $s_n(x) \le s_{n+1}(x)$, $\sup_{n\in\mathbb{N}} s_n(x) = f(x)$.

(4) If f is bounded, $(*)$ gives uniform convergence.

For general f, write $f = f^+ - f^-$.[24] We can find simple functions g_n, h_n such that $g_n \nearrow f^+$, $h_n \nearrow f^-$. Then set $s_n = g_n - h_n$, then $|s_n(x)| \le |f(x)|$ and $|s_n(x)| \nearrow |f(x)|$.

Corollary. If f, g are measureable and $f+g$, fg are well-defined, then $f+g$, fg are measureable. (Approximated as a limit of measurable simple functions)

[24] Note that $\infty - \infty$ doesn't appear here.

November 1st, 2022

$(X, \mathscr{F}, \mu)$ 라고 계속 가정합니다. $\mathscr{F}$는 σ-algebra on X, μ는 measure on $\mathscr{F}$ 입니다.

If $E \in \mathscr{F}$, we can consider $(X, \mathscr{F}_E, \mu_E)$ where

$$\mathscr{F}_E = \{A \cap E : A \in \mathscr{F}\}, \quad \mu_E = \mu|_{\mathscr{F}_E},$$

and develop the integration theory with $\int = \int_E$.

Instead, we develop integration theory with $\int = \int_X$ and set $\displaystyle\int_E f\,d\mu = \int f\chi_E\,d\mu$.

(Step 1) For $A \in \mathscr{F}$, we define $\displaystyle\int \chi_A\,d\mu = \mu(A)$.

(Step 2) Let $f : X \to [0, \infty)$ be a measurable simple function.
Then there exists pairwise disjoint $(A_k)_{k=1}^n$ in $\mathscr{F}$ and $(a_k)_{k=1}^n$ in $[0, \infty)$[25] such that

$$f(x) = \sum_{k=1}^n a_k \chi_{A_k}.$$

Then we can define

$$\int f\,d\mu = \sum_{k=1}^n a_k \mu(A_k) \in [0, \infty]. \tag{$*$}$$

이런 정의를 보면 여러분이 제일 먼저 생각해야 하는 것이 well-definedness 입니다!

Remark. $(*)$ is well-defined for all measurable simple functions.

Proof. Let

$$f(x) = \sum_{k=1}^n a_k \chi_{A_k} = \sum_{i=1}^m b_i \chi_{B_i},$$

where $0 \le a_k, b_i < \infty$ and $A_k, B_i \in \mathscr{F}$ are partitions of X.[26] Let $C_{k,i} = A_k \cap B_i$. Then

$$\sum_{k=1}^n a_k \mu(A_k) = \sum_{k=1}^n a_k \mu\left(A_k \cap \bigcup_{i=1}^m B_i\right) = \sum_{k=1}^n \sum_{i=1}^m a_k \mu(C_{k,i}),$$

$$\sum_{i=1}^m b_i \mu(B_i) = \sum_{i=1}^m b_i \mu\left(B_i \cap \bigcup_{k=1}^n A_k\right) = \sum_{i=1}^m \sum_{k=1}^n b_i \mu(C_{k,i}).$$

If $C_{k,i} \neq \varnothing$, then $a_k = b_i$.[27] If $C_{k,i} = \varnothing$, then $\mu(C_{k,i}) = 0$. Therefore

$$b_i \mu(C_{k,i}) = a_k \mu(C_{k,i}), \quad (\forall k, i) \implies \int f\,d\mu = \sum_{k=1}^n a_k \mu(A_k) = \sum_{i=1}^m b_i \mu(B_i).$$

[25] 책에서는 $(0, \infty)$이긴 한데, 우리는 $0 \cdot \infty = 0$이라 생각하고 $\bigcup_{k=1}^n A_k = X$ 라고 생각할게요.
[26] Pairwise disjoint and their union is X.
[27] For $x \in C_{k,i}$, $f(x) = a_k = b_i$.

적분은 선형이고, monotonicity를 항상 유지합니다.

Remark. For $a, b \in [0, \infty)$ and measurable simple functions $f, g \geq 0$,

$$\int (af + bg)\, d\mu = a \int f\, d\mu + b \int g\, d\mu.$$

Proof. Let

$$f = \sum_{j=1}^{m} y_j \chi_{A_j}, \quad g = \sum_{k=1}^{n} z_k \chi_{B_k}$$

where A_j, B_k is a partition of X and $y_j, z_k \geq 0$. Let $C_{j,k} = A_j \cap B_k$. Then

$$\begin{aligned}
a \int f\, d\mu + b \int g\, d\mu &= \sum_j a y_j \mu(A_j) + \sum_k b z_k \mu(B_k) \\
&= \sum_j a y_j \sum_k \mu(A_j \cap B_k) + \sum_k b z_k \sum_j \mu(B_k \cap A_j) \\
&= \sum_j \sum_k a y_j \mu(C_{j,k}) + \sum_k \sum_j b z_k \mu(C_{j,k}) \\
&= \sum_{j,k} (a y_j + b z_k) \mu(C_{j,k}) = \int (af + bg)\, d\mu.
\end{aligned}$$

Remark. If $f \geq g \geq 0$ are measurable simple functions, $\displaystyle\int f\, d\mu \geq \int g\, d\mu$.

Proof. Check from definition, or check that $f - g \geq 0$ is simple and measurable.

$$\int f\, d\mu = \int [g + (f - g)]\, d\mu = \int g\, d\mu + \int (f - g)\, d\mu \geq \int g\, d\mu \geq 0.$$[28]

(Step 3) Let $f : X \to [0, \infty]$ be a measurable function. Define

$$\int f\, d\mu = \sup \left\{ \int h\, d\mu : 0 \leq h \leq f, h \text{ is measurable and simple} \right\}.$$

Note that if f is simple, this accords with the definition in Step 2.

For measurable $f \geq g \geq 0$,

$$\int g\, d\mu = \sup_{0 \leq h \leq g} \int h\, d\mu \leq \sup_{0 \leq h \leq f} \int h\, d\mu = \int f\, d\mu.$$

[28] 적분값이 무한대일 수 있으니 함부로 이항하면 안됩니다.

이걸 먼저 증명하면 유용해서 잠깐 이걸 먼저 할게요.

Theorem 11.28 (Monotone Convergence Theorem) Let $f_n : X \to [0, \infty]$ be measurable functions[29] and $f_n(x) \le f_{n+1}(x)$ for all $x \in X$. Let

$$\lim_{n\to\infty} f_n(x) = \sup_n f_n(x) = f(x).$$

Then,

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu = \sup_n \int f_n \, d\mu.$$

Proof.

($\ge$) Since $f_n(x) \le f(x)$, $\int f_n \, d\mu \le \int f \, d\mu$ for all $n \in \mathbb{N}$ by monotonicity. Therefore

$$\sup_n \int f_n \, d\mu \le \int f \, d\mu.$$

($\le$) Let $c \in (0,1)$ (we will let $c \nearrow 1$). Let $0 \le s \le f$ where s is simple and measurable. We have $c \cdot s(x) < f(x)$ for all $x \in X$. Let $E_n = \{x \in X : f_n(x) \ge cs(x)\} \in \mathscr{F}$.[30] Since f_n is increasing, $E_n \subseteq E_{n+1} \subseteq \cdots$, and since $f_n \to f$, $\bigcup_{n=1}^{\infty} E_n = X$. For every x, $\exists\, N$ such that $f(x) \ge f_n(x) > cs(x)$ for all $n \ge N$. Since $f_n \ge f_n \chi_{E_n} \ge cs\chi_{E_n}$,

$$\int f_n \, d\mu \ge \int f_n \chi_{E_n} \, d\mu \ge c \int s\chi_{E_n} \, d\mu, \tag{$\star$}$$

where s, χ_{E_n} are simple. So we can write $s = \sum_{k=0}^{m} y_k \chi_{A_k}$, then

$$s\chi_{E_n} = \sum_{k=0}^{m} y_k \chi_{A_k \cap E_n} \implies \int s\chi_{E_n} \, d\mu = \sum_{k=0}^{m} y_k \mu(A_k \cap E_n).$$

$A_k \cap E_n \nearrow A_k$ as $n \to \infty$. By continuity of measure, $\mu(A_k \cap E_n) \nearrow \mu(A_k)$ and

$$\lim_{n\to\infty} \int s\chi_{E_n} \, d\mu = \int s \, d\mu.$$

By ($\star$),

$$\lim_{n\to\infty} \int f_n \, d\mu \ge c \int s \, d\mu.$$

Let $c \nearrow 1$ and take supremum over $0 \le s \le f$, then

$$\lim_{n\to\infty} \int f_n \, d\mu \ge \sup_{0 \le s \le f} \int s \, d\mu = \int f \, d\mu.$$

[29] Non-negative인 것이 중요합니다!

[30] $f_n(x) - cs(x)$ is a measurable function.

Remark. If $f \geq 0$ is measurable, we have already constructed measurable simple functions s_n such that $s_n(x) \leq s_{n+1}(x)$. (Theorem 11.20) Let $f(x) = \lim_{n\to\infty} s_n(x)$. Observe that

$$\int_E s_n \, d\mu = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \mu\left(\left\{x \in E : \frac{i-1}{2^n} \leq f(x) \leq \frac{i}{2^n}\right\}\right) + n\mu(\{x \in E : f(x) \geq n\}).$$

Then by MCT,

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_E s_n \, d\mu.$$

Lebesgue integral이 Riemann이랑 어떻게 다른가? Riemann은 domain을 자르고 upper/lower sum을 고려해서 만든다면, Lebesgue의 경우 range를 잘라서, 자른 range의 preimage에 대한 measure를 이용해 preimage의 길이를 재는 것입니다. Riemann과 마찬가지로 domain의 길이와 높이를 곱해 모두 더하는 idea는 동일합니다.

Definition. If $E \in \mathcal{F}$, and $f \geq 0$ is measurable, then

$$\int_E f \, d\mu = \int f\chi_E \, d\mu.$$

Remark. If $0 \leq f_n \nearrow f$ on E, then $0 \leq f_n\chi_E \nearrow f\chi_E$. So MCT also holds on E.

$$0 \leq f_n \nearrow f \text{ on } E \implies \lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu.$$

Remark. Counterexample of MCT: $\chi_{[n,\infty)} \searrow 0$ as $n \to \infty$. For Lebesgue measure m,

$$\infty = \int \chi_{[n,\infty)} \, dm \neq \int 0 \, dm = 0.$$

Remark. Suppose $f, g \geq 0$ be measurable functions, and $\alpha, \beta \in [0, \infty)$. Then

$$\int_E (\alpha f + \beta g) \, d\mu = \alpha \int_E f \, d\mu + \beta \int_E g \, d\mu.$$

Proof. Take $0 \leq f_n \nearrow f$ and $0 \leq g_n \nearrow g$ where f_n, g_n are measurable simple functions. Then $\alpha f_n + \beta g_n \nearrow \alpha f + \beta g$ and is simple. So by MCT,

$$\int_E (\alpha f_n + \beta g_n) \, d\mu = \alpha \int_E f_n \, d\mu + \beta \int_E g_n \, d\mu \to \alpha \int_E f \, d\mu + \beta \int_E g \, d\mu.$$

Theorem 11.30 For measurable $f_n : X \to [0, \infty]$, $\displaystyle\sum_{n=1}^{\infty} f_n$ is measurable and by MCT,

$$\int_E \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu.$$

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$f_n \geq 0$ 일 때 성립하는 것부터 살펴봅시다.

Theorem 11.31 (Fatou) If $f_n \geq 0$ and measurable and E is measurable. Then

$$\int_E \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_E f_n \, d\mu.$$ [31]

Proof. Let $E = X$. Let $g_n = \inf_{k\geq n} f_k$. Then $\lim_{n\to\infty} g_n = \liminf_{n\to\infty} f_n$, and we know that g_n is increasing and non-negative. By definition, $g_n \leq f_k$ for all $k \geq n$. Therefore,

$$\int g_n \, d\mu \leq \inf_{k\geq n} \int f_k \, d\mu.$$

Letting $n \to \infty$ gives

$$\int \liminf_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int g_n \, d\mu \leq \lim_{n\to\infty} \inf_{k\geq n} \int f_k \, d\mu = \liminf_{n\to\infty} \int f_n \, d\mu,$$

where the first equality holds by the MCT.

Remark. The above theorem doesn't work for $\limsup$.

$$\int_E \limsup_{n\to\infty} f_n \, d\mu \ngeq \limsup_{n\to\infty} \int_E f_n \, d\mu.$$

Consider $\chi_{[n,\infty)}$. LHS $= 0$, but RHS $= \infty$.[32]

(Step 4) If f is measurable, then $f^+, f^- \geq 0$ are measurable. So for $E \in \mathscr{F}$, define

$$\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu,$$

except for the case of $\infty - \infty$.[33]

Definition. (Lebesgue Integrable) f is **Lebesgue integrable on E with respect to μ** if f is measurable and

$$\int_E |f| \, d\mu = \int_E f^+ \, d\mu + \int_E f^- \, d\mu < \infty.$$

다음 주에 함수들의 class를 볼텐데, 르벡 적분 가능한 함수를 다음과 같이 표기합니다.

Notation. f is Lebesgue integrable $\iff f \in \mathcal{L}^1(E,\mu)$. If $\mu = m$, $f \in \mathcal{L}^1(E)$.

Note that $f \in \mathcal{L}^1(E,\mu) \iff f^+, f^- \in \mathcal{L}^1(E,\mu) \iff |f| \in \mathcal{L}^1(E,\mu)$.

[31] ILLLI: integral of limit less than limit of integral.
[32] 뒤에서 하겠지만, $|f_n| \leq g$인 $g \in \mathcal{L}^1$ 가 있어야 합니다.
[33] 둘 중 하나는 유한해야 한다.

Remark.

(1) (11.23) If f is measurable and bounded on E and $\mu(E) < \infty$,

$$\int_E |f| \, d\mu \le \int_E M \, d\mu = M\mu(E) < \infty \text{ and } f \in \mathcal{L}^1(E, \mu).$$

(2) If $f, g \in \mathcal{L}^1(E, \mu)$ and $f \le g$ on E, then

$$\chi_E(x) f^+(x) \le \chi_E(x) g^+(x) \text{ and } \chi_E(x) g^-(x) \le \chi_E(x) f^-(x),$$

which implies

$$\int_E f^+ \, d\mu \le \int_E g^+ \, d\mu < \infty \text{ and } \int_E g^- \, d\mu \le \int_E f^- \, d\mu < \infty.$$

Therefore monotonicity holds without the non-negative condition, i.e.

$$\int_E f \, d\mu \le \int_E g \, d\mu.$$

(3) If $f \in \mathcal{L}^1(E, \mu)$ and $c \in \mathbb{R}$, then $cf \in \mathcal{L}^1(E, \mu)$ since

$$\int_E |c| \, |f| \, d\mu = |c| \int_E |f| \, d\mu < \infty.$$

If $c < 0$, $(cf)^+ = -cf^-$, $(cf)^- = -cf^+$. Thus,

$$\int_E cf \, d\mu = \int_E (cf)^+ - \int_E (cf)^- \, d\mu = -c \int_E f^- \, d\mu - (-c) \int_E f^+ \, d\mu = c \int_E f \, d\mu.$$

(4) For measurable f, if $a \le f(x) \le b$ on E and $\mu(E) < \infty$, (integrable since bounded)

$$\int_E a\chi_E \, d\mu \le \int_E f\chi_E \, d\mu \le \int_E b\chi_E \, d\mu \implies a\mu(E) \le \int_E f \, d\mu \le b\mu(E).$$

(5) If $f \in \mathcal{L}^1(E, \mu)$ and for $A \in \mathscr{F}$ such that $A \subseteq E$ then $f \in \mathcal{L}^1(A, \mu)$ since

$$\int_A |f| \, d\mu \le \int_E |f| \, d\mu < \infty.$$

(6) Suppose that E is measurable and $\mu(E) = 0$. If f is measurable, $\min\{|f|, n\}\chi_E$ is measurable and $\min\{|f|, n\}\chi_E \nearrow |f| \chi_E$ as $n \to \infty$. By MCT,

$$\int_E |f| \, d\mu = \lim_{n\to\infty} \int_E \min\{|f|, n\} \, d\mu = 0$$

since $\int_E \min\{|f|, n\} \, d\mu \le \int_E n \, d\mu = n\mu(E) = 0$. Thus $f \in \mathcal{L}^1(E, \mu)$ and $\int_E f \, d\mu = 0$.[34]

[34] Even if $f \equiv \infty$. We defined $0 \cdot \infty = 0$.

적분 입장에서 보면, measure가 0인 곳에서 적분을 하면, 의미가 없다고 생각할 수 있겠죠? 그러면 앞으로 그런걸 무시해도 된다고 해버리죠.

Definition. (Almost Everywhere) Let $P = P(x)$ be a property.[35] We say that P holds **almost everywhere on E with respect to μ** if

$$\exists\, N \in \mathscr{F} \text{ such that } \mu(N) = 0 \text{ and } P \text{ holds for all } x \in E \setminus N.$$

Notation. We write p holds μ-a.e. on E, and if $E = X$, we omit 'on E'.

Theorem. (Markov Inequality) Let $u \in \mathcal{L}^1(E, \mu)$. For all $c > 0$,

$$\mu\left(\{|u| \geq c\} \cap E\right) \leq \frac{1}{c} \int_E |u|\ d\mu.$$

Proof. $\displaystyle\int_E |u|\ d\mu \geq \int_{E \cap \{|u| \geq c\}} |u|\ d\mu \geq \int_{E \cap \{|u| \geq c\}} c\, d\mu = c\mu\left(\{|u| \geq c\} \cap E\right).$

Theorem. Let $u \in \mathcal{L}^1(E, \mu)$. The following are equivalent.

(1) $\displaystyle\int_E |u|\ d\mu = 0.$

(2) $u = 0$ μ-a.e. on E.

(3) $\mu\left(\{x \in E : u(x) \neq 0\}\right) = 0.$

Proof.

(2 $\Longleftrightarrow$ 3) Clear since $E \cap \{u \neq 0\} \in \mathscr{F}$.

(2 $\Longrightarrow$ 1) $\displaystyle\int_E |u|\ d\mu = \int_{E \cap \{|u| > 0\}} |u|\ d\mu + \int_{E \cap \{|u| = 0\}} |u|\ d\mu = 0 + 0 = 0.$

(1 $\Longrightarrow$ 3) By Markov inequality,

$$\mu\left(\left\{|u| \geq \frac{1}{n}\right\} \cap E\right) \leq n \int_E |u|\ d\mu = 0.$$

Let $n \to \infty$, by continuity of measure, $\mu\left(\{|u| > 0\} \cap E\right) = 0$.

Remark. Let $A, B \in \mathscr{F}$. If $B \subseteq A$ and $\mu\left(A \setminus B\right) = 0$, then

$$\int_A f\, d\mu = \int_B f\, d\mu \text{ for all } f \in \mathcal{L}^1(A, \mu).$$

[35] Ex. $f(x)$ is continuous.

Theorem. If $u \in \mathcal{L}^1(E,\mu)$ then $u(x) \in \mathbb{R}$ μ-a.e. on E.[36]

Proof. $\mu(\{|u| \geq 1\} \cap E) \leq \int_E |u| \, d\mu < \infty$.[37] So,

$$\mu(\{|u| = \infty\} \cap E) = \mu\left(\bigcap_{n=1}^{\infty} \{x \in E : |u(x)| \geq n\}\right)$$
$$= \lim_{n\to\infty} \mu(\{|u| \geq n\} \cap E) \leq \limsup_{n\to\infty} \frac{1}{n} \int_E |u| \, d\mu = 0.$$

Corollary. If $u \in \mathcal{L}^1(E,\mu)$, then $\int_E u \, d\mu = \int_{E\cap\{|u|<\infty\}} u \, d\mu$.

Theorem. If $f_1, f_2 \in \mathcal{L}^1(E,\mu)$, then $f_1 + f_2 \in \mathcal{L}^1(E,\mu)$ and

$$\int_E (f_1 + f_2) \, d\mu = \int_E f_1 \, d\mu + \int_E f_2 \, d\mu.$$

Proof. Since $|f_1 + f_2| \leq |f_1| + |f_2|$, $f_1 + f_2 \in \mathcal{L}^1(E,\mu)$. Define $f = f_1 + f_2$ and

$$N = \left\{x : \max\left\{f_1^+, f_1^-, f_2^+, f_2^-, f^+, f^-\right\} = \infty\right\}.$$

Then by the above theorem, $\mu(N) = 0$. So on $E \setminus N$,

$$f^+ - f^- = f_1^+ - f_1^- + f_2^+ - f_2^- \implies f^+ + f_1^- + f_2^- = f^- + f_1^+ + f_2^+.$$

Then

$$\int_{E\setminus N} f^+ \, d\mu + \int_{E\setminus N} f_1^- \, d\mu + \int_{E\setminus N} f_2^- \, d\mu = \int_{E\setminus N} f^- \, d\mu + \int_{E\setminus N} f_1^+ \, d\mu + \int_{E\setminus N} f_2^+ \, d\mu.$$

Now using the fact that $\mu(N) = 0$,

$$\int_{E\setminus N} f \, d\mu = \int_{E\setminus N} f_1 \, d\mu + \int_{E\setminus N} f_2 \, d\mu \implies \int_E f \, d\mu = \int_E f_1 \, d\mu + \int_E f_2 \, d\mu.$$

[36] $u(x) = \infty$ 인 집합의 measure가 0이다.
[37] Continuity of measure를 사용하기 위해서는 첫 번째 집합의 measure가 유한해야 한다.

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Remark. $f \geq 0$ and measurable, $E \in \mathscr{F}$. Then we defined $\int_E f\,d\mu = \int f\chi_E\,d\mu$. If $E, F \in \mathscr{F}$ and $E \cap F = \varnothing$,

$$\int_{E\cup F} f\,d\mu = \int f(\chi_E + \chi_F)\,d\mu = \int_E f\,d\mu + \int_F f\,d\mu.$$

So if $f \in \mathcal{L}^1(E\cup F, \mu)$,

$$\int_{E\cup F} f\,d\mu = \int_E f\,d\mu + \int_F f\,d\mu.$$

Remark. If $A, B \in \mathscr{F}$, $B \subseteq A$ and $\mu(A \setminus B) = 0$ then

$$\int_A f\,d\mu = \int_B f\,d\mu \text{ if } f \in \mathcal{L}^1(A,\mu) \text{ or } f \text{ is measurable.}$$

Theorem 11.28 (Monotone Convergence Theorem) Suppose f_n are measurable and $0 \leq f_n(x) \leq f_{n+1}(x)$ μ-a.e. Then

$$\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu.$$[38]

Theorem 11.31 (Fatou) Suppose f_n are measurable and $f_n(x) \geq 0$ μ-a.e. Then

$$\int_E \liminf_{n\to\infty} f_n\,d\mu \leq \liminf_{n\to\infty} \int_E f_n\,d\mu.$$

Remark. Let f, g be measurable functions on $E \in \mathscr{F}$. If $|f| \leq |g|$ μ-a.e. on E. From

$$\int |f|\;d\mu \leq \int |g|\;d\mu,$$

we see that if $g \in \mathcal{L}^1(E,\mu)$ then $f \in \mathcal{L}^1(E,\mu)$.

Definition. Fix $E \in \mathscr{F}$, and consider a relation $\sim$ on the functions of $\mathcal{L}^1(E,\mu)$. We define $f \sim g$ if and only if $f = g$ μ-a.e. on E. Then $\sim$ is an equivalence relation, so we can write

$$[f] = \{g \in \mathcal{L}^1(E,\mu) : f \sim g\}.$$

Equivalence class의 대표에 대해서만 생각해도 충분하다!

Theorem 11.32 (Lebesgue's Dominated Convergence Theorem) Suppose that $E \in \mathscr{F}$ and f is measurable. Let (f_n) be a sequence of measurable functions such that $f(x) = \lim\limits_{n\to\infty} f_n(x)$ exists in $\overline{\mathbb{R}}$ μ-a.e. on E. (pointwise convergence) If there exists

$$g \in \mathcal{L}^1(E,\mu) \text{ such that } |f_n| \leq g\ (\forall n \geq 1)\ \mu\text{-a.e. on } E,$$

[38]증명은 $f_n \leq f_{n+1}$이 성립하지 않는 집합을 빼고 증명하면 됩니다.

then

$$\lim_{n\to\infty}\int_E |f_n - f|\, d\mu = 0.$$

Remark.

(1) Note that $f_n, f \in \mathcal{L}^1(E,\mu)$.

(2) Since

$$\left|\int f_n\, d\mu - \int f\, d\mu\right| \le \int |f_n - f|\, d\mu,$$

the conclusion implies that $\displaystyle\lim_{n\to\infty}\int f_n\, d\mu = \int f\, d\mu$.

Proof. Let

$$A = \left\{x \in E : \lim_{n\to\infty} f_n(x) \text{ exists and is real}, f_n(x), f(x), g(x) \in \mathbb{R}, |f_n(x)| \le g(x)\right\}.$$

Then $E \setminus A$ has measure zero. Now we only consider $x \in A$. Then

$$2g - |f_n - f| \ge 2g - \big(|f_n| + |f|\big) \ge 0.$$

Since $|f_n - f| \to 0$, $2g - |f_n - f| \to 2g$. By Fatou's lemma,

$$\begin{aligned}
2\int_E g\, d\mu = \int_A 2g\, d\mu &= \int_A \liminf_{n\to\infty}\big(2g - |f_n - f|\big)\, d\mu \\
&\le \liminf_{n\to\infty}\left(2\int_A g\, d\mu - \int_A |f_n - f|\, d\mu\right) \\
&= 2\int_A g\, d\mu - \limsup_{n\to\infty}\int_A |f_n - f|\, d\mu \le 2\int_A g\, d\mu.
\end{aligned}$$

So we conclude that

$$2\int_A g\, d\mu - \limsup_{n\to\infty}\int_A |f_n - f|\, d\mu = 2\int_A g\, d\mu,$$

and since $0 \le \displaystyle\int_A g\, d\mu < \infty$, $\displaystyle\limsup_{n\to\infty}\int_A |f_n - f|\, d\mu = 0$.

We suppose $(X, \mathscr{F}, \mu)$.

Theorem 11.24 Let f be a measurable function such that $f \ge 0$ μ-a.e. Define a set function on $\mathscr{F}$ as

$$\nu(A) = \int_A f\, d\mu, \quad (A \in \mathscr{F}).$$

Then ν is a measure on $\mathscr{F}$.

Proof. $\nu(\varnothing) = 0$. If $(A_n) \subseteq \mathscr{F}$ is disjoint,

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \int \left(\chi_{\bigcup_{n=1}^{\infty} A_n}\right) f\, d\mu = \int \sum_{n=1}^{\infty} \chi_{A_n} f\, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f\, d\mu = \sum_{n=1}^{\infty} \nu(A_n),$$

by MCT.

Remark. If $f \in \mathcal{L}^1$, ν is countably additive. Hint: Set $\chi_{\cup A_n} |f| \leq |f|$ and use LDCT.

Comparison with the Riemann Integral

For Lebesgue measure m, we write

$$\int_{[a,b]} f\, dm = \int_{[a,b]} f\, dx = \int_a^b f\, dx,$$

and denote the Riemann integral as $\mathcal{R}\displaystyle\int_a^b f\, dx$.

Theorem 11.33 Let $a, b \in \mathbb{R}$, $a < b$, f be bounded.

(1) If $f \in \mathcal{R}[a,b]$, then $f \in \mathcal{L}^1[a,b]$ and $\displaystyle\int_a^b f\, dx = \mathcal{R}\int_a^b f\, dx$.

(2) $f \in \mathcal{R}[a,b] \iff f$ is continuous a.e. on $[a,b]$.[39]

Proof. Choose partitions $P_k = \{a = x_0^k < x_1^k < \cdots < x_{n_k}^k = b\}$ on $[a,b]$ such that $P_k \subseteq P_{k+1}$ (refinement) and $\left|x_i^k - x_{i-1}^k\right| < \frac{1}{k}$. Then

$$\lim_{k\to\infty} L(P_k, f) = \mathcal{R}\underline{\int} f\, dx, \quad \lim_{k\to\infty} U(P_k, f) = \mathcal{R}\overline{\int} f\, dx.$$

Define a sequence of measurable simple functions U_k, L_k.

$$U_k = \sum_{i=1}^{n_k} \sup_{x_{i-1}^k \leq y \leq x_i^k} f(y)\chi_{(x_{i-1}^k, x_i^k]}, \quad L_k = \sum_{i=1}^{n_k} \inf_{x_{i-1}^k \leq y \leq x_i^k} f(y)\chi_{(x_{i-1}^k, x_i^k]}.$$

We have $L_k \leq f \leq U_k$,

$$\int_a^b L_k\, dx = L(P_k, f), \quad \int_a^b U_k\, dx = U(P_k, f),$$

with L_k increasing, U_k decreasing. (By refinement) Let

$$L(x) = \lim_{k\to\infty} L_k(x), \quad U(x) = \lim_{k\to\infty} U_k(x).$$

[39] $\mathcal{L}^1$의 equivalence를 고려하면 사실상 연속함수에 대해서만 리만적분할 수 있다는 뜻입니다.

The limits exist, and since f, L_k, U_k are bounded,

$$\int_a^b L\,dx = \lim_{k\to\infty}\int_a^b L_k\,dx = \mathcal{R}\underline{\int} f\,dx < \infty, \int_a^b U\,dx = \mathcal{R}\overline{\int} f\,dx < \infty$$

by LDCT. Thus $L, U \in \mathcal{L}^1[a,b]$, and

$$f \in \mathcal{R}[a,b] \iff \int_a^b (U-L)\,dx = 0 \iff U = L \text{ a.e. on } [a,b].$$

(1) If $f \in \mathcal{R}[a,b]$, we have $f = U = L$ a.e. on E. Thus f is measurable, and

$$\int_a^b f\,dx = \mathcal{R}\int_a^b f\,dx < \infty \implies f \in \mathcal{L}^1([a,b]).$$

(2) Suppose $x \notin \bigcup_{k=1}^\infty P_k$, then for every $\epsilon > 0$,

$$\exists\, n, j_0 \text{ such that } x \in (t_{j_0-1}^n, t_{j_0}^n) \text{ and } |L_n(x) - L(x)| + |U_n(x) - U(x)| < \epsilon.$$

Then for all $y \in (t_{j_0-1}^n, t_{j_0}^n)$,

$$|f(x) - f(y)| \le M_{j_0}^n - m_{j_0}^n = M_{j_0}^n - U(x) + U(x) - L(x) + L(x) - m_{j_0}^n \le U(x) - L(x) + \epsilon.$$

Therefore $\{x : U(x) = L(x)\} \setminus \bigcup_{k=1}^\infty P_k \subseteq \{x : f(x) \text{ is continuous}\} \subseteq \{x : U(x) = L(x)\}$.

Since $\bigcup_{k=1}^\infty P_k$ has measure zero, $U = L$ a.e. $\iff$ f is continuous a.e. Therefore,

$$f \in \mathcal{R}[a,b] \iff U = L \text{ a.e.} \iff f \text{ is continuous a.e.}$$

Remark.

(1) If $x \notin P_k$ for all k, f is continuous at $x \iff f(x) = U(x) = L(x)$.

(2) $L(x) \le f(x) \le U(x)$ and $L(x), U(x)$ are measurable.[40]

(3) Since $|f| \le M$, we can assume that $f \ge 0$, because we can consider $f + M$.

[40] Limit of measurable functions.

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Remark. If $A \subseteq B$, $B \in \mathfrak{M}$ and $m(B) = 0$ then B should be finitely measurable. So $\exists\, E_n \in \Sigma$ such that $m^*(B \triangle E_n) \to 0$. Thus, $m^*(A \triangle E_n) \le m^*(B \triangle E_n) \to 0$. Therefore $A \in \mathfrak{M}$.

Definition. (Complete Measure Space) $(X, \mathscr{F}, \mu)$ is called a **complete measure space** if every subset of measurable measure zero set is measurable. i.e,

$$\text{if } A \subseteq B,\ B \in \mathscr{F} \text{ and } \mu(B) = 0 \text{ then } A \in \mathscr{F}.$$

Example. $f(x) = x^\alpha e^{-\beta x}$ on $x > 0$, $\alpha > -1$, $\beta > 0$.[41]

$$f(x) = x^\alpha e^{-\beta x} \le \begin{cases} x^\alpha & (0 < x < 1) \\ c_N x^{\alpha - N} & (1 \le x < \infty) \end{cases}$$

since we can take $N \ge 3 + \alpha$ such that $e^{-\beta x} \le c_N x^{-N}$ if $x \ge 1$. Therefore

$$f(x) \le c\big(x^\alpha \chi_{(0,1)} + x^{-2}\chi_{[1,\infty)}\big)$$

for some constant c and $f \in \mathcal{L}^1(0, \infty)$.

Remark. 리만적분의 유용한 성질들을 가지고 와서 사용할 수 있다!

(1) If $f \ge 0$ and measurable, set $f_n = f\chi_{[0,n]}$. Then by MCT,

$$\int_0^\infty f\,dx = \lim_{n\to\infty} \int_0^\infty f_n\,dx = \lim_{n\to\infty} \int_0^n f\,dx.$$

(2) If $f \in \mathcal{R}(I)$ for any closed, finite interval $I \subseteq (0, \infty)$, $f \in \mathcal{L}^1(I)$. Setting $f_n = f\chi_{[0,n]}$ and using LDCT with dominator f gives

$$\int_0^\infty f\,dx = \lim_{n\to\infty} \int_0^\infty f_n\,dx = \lim_{n\to\infty} \int_0^n f\,dx = \lim_{n\to\infty} \mathcal{R}\int_0^n f\,dx.$$

Similarly, setting $f_n = f\chi_{(1/n,1)}$ and using LDCT with dominator f gives

$$\int_0^1 f\,dx = \lim_{n\to\infty} \int_0^1 f_n\,dx = \lim_{n\to\infty} \int_{1/n}^1 f\,dx = \lim_{n\to\infty} \mathcal{R}\int_{1/n}^1 f\,dx.$$

[41] 엄밀하게는 $((0,\infty), \mathfrak{M}, m)$ 입니다.

Recall. (Gamma Function) For $t > 0$,

$$\Gamma(t) = \int_{(0,\infty)} x^{t-1}e^{-x}\,dx = \int_0^\infty x^{t-1}e^{-x}\,dx.$$

(1) Γ is continuous at t for all $t \in (0,\infty)$.

(2) $\Gamma \in C^\infty(0,\infty)$.

Proof.

(1) We show that Γ is continuous at $t \in (a,b) \subseteq (0,\infty)$. Let $u(t,x) = x^{t-1}e^{-x}$ and $g(x) = x^{a-1}\chi_{(0,1)} + c_b x^{-2}\chi_{[1,\infty)}$. Then $u(t,x) \leq g(x)$ for all $x > 0$, $t \in [a,b]$.[42] For any given sequence $t_n \to t$, choose large enough N_0 so that $t_n, t \in (a,b)$ for $n \geq N_0$. Then $f_n(x) = u(t_n,x) \leq g(x)$. Since $g(x) \in \mathcal{L}^1(0,\infty)$ and $u(t_n,x) \to u(t,x)$, by LDCT,

$$\lim_{n\to\infty} \Gamma(t_n) = \lim_{n\to\infty} \int_0^\infty u(t_n,x)\,dx = \int_0^\infty u(t,x)\,dx = \Gamma(t).$$

(2) We just show that Γ is differentiable at $t \in (a,b) \subseteq (0,\infty)$.

$$\frac{\partial u}{\partial t}(t,x) = x^{t-1}e^{-x}\ln x$$

We also try to bound $\frac{\partial u}{\partial t}$. For $0 < x < 1$, take $cx^{a/2-1}$ where $c = \sup_{0<x<1} x^{a/2}\,|\ln x|$, for $1 \leq x < \infty$, take $x^b e^{-x}$. Therefore $\sup_{x\in[a,b]} |\partial_t u| \leq c(x^{a/2-1}\chi_{(0,1)} + x^b e^{-x}\chi_{[1,\infty)}) \in \mathcal{L}^1(0,\infty)$.

Take h small enough that $t, t+h \in (a,b)$. Then

$$\frac{u(t+h,x) - u(t,x)}{h} \leq \sup_{x\in[a,b]} \left|\frac{\partial u}{\partial t}\right| \in \mathcal{L}^1(0,\infty)$$

by the mean value theorem. For any $h_n \to 0$, choose n_0 large enough so that $t, t+h_n \in (a,b)$ for $n \geq n_0$. Then

$$\frac{u(t+h_n,x) - u(t,x)}{h_n} \leq \sup_{x\in[a,b]} \left|\frac{\partial u}{\partial t}\right| \in \mathcal{L}^1(0,\infty).$$

Since $\frac{u(t+h_n,x)-u(t,x)}{h_n} \to \frac{\partial u}{\partial t}$, by LDCT,

$$\lim_{n\to\infty} \int_0^\infty \frac{u(t+h_n,x) - u(t,x)}{h_n}\,dx = \int_0^\infty \frac{\partial u}{\partial t}(t,x)\,dx.$$

Γ is differentiable at t, and $\Gamma'(t) = \displaystyle\int_0^\infty x^{t-1}e^{-x}\ln x\,dx.$

Remark. Write $\Gamma(t) = \lim_{n\to\infty} \mathcal{R}\displaystyle\int_{1/n}^n x^{t-1}e^{-x}\,dx$. We can prove other properties of $\Gamma(t)$.

[42] c_b is chosen to satisfy this. Refer to the example above.

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Theorem 8.22 (Stirling's Formula)

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{(x/e)^x\sqrt{2\pi x}} = 1.$$

Proof.

$$\begin{aligned}
\Gamma(x+1) &= \int_0^\infty t^x e^{-t}\,dt \\
&= e^{-x}\int_{-x}^\infty (x+v)^x e^{-v}\,dv && (t = x+v) \\
&= e^{-x}x\int_{-1}^\infty x^x(1+u)^x e^{-xu}\,du = x^{x+1}e^x\int_{-1}^\infty (1+u)^x e^{-xu}\,du && (v = xu) \\
&= x^x e^{-x}\sqrt{2x}\int_{-\sqrt{x/2}}^\infty \left[1+s\sqrt{\frac{2}{x}}\right]^x \exp\left(-xs\sqrt{\frac{2}{x}}\right) ds && \left(u = s\sqrt{\frac{2}{x}}\right) \\
&= x^x e^{-x}\sqrt{2x}\int_{-\sqrt{x/2}}^\infty \exp\left(-x\left[s\sqrt{\frac{2}{x}} - \ln\left(1+s\sqrt{\frac{2}{x}}\right)\right]\right) ds
\end{aligned}$$

Now we try to approximate that the last integral $(*)$ is $\sqrt{\pi}$.

$$(*) = \int_{-\sqrt{x/2}}^\infty \exp\left(-x\left[s\sqrt{\frac{2}{x}} - \ln\left(1+s\sqrt{\frac{2}{x}}\right)\right]\right) ds$$

Let $h(u) = \dfrac{2}{u^2}\left(u - \ln(1+u)\right)$ and define $\psi_x(s)$ as follows so that

$$\psi_x(s) = \begin{cases} \exp\left(-s^2 h\left(s\sqrt{\frac{2}{x}}\right)\right) & \left(s > -\sqrt{\frac{x}{2}}\right) \\ 0 & \left(s \le -\sqrt{\frac{x}{2}}\right) \end{cases}$$

$$\implies \Gamma(x+1) = x^x e^{-x}\sqrt{2x}\int_{-\infty}^\infty \psi_x(s)\,ds.$$

We will show that $(*) \to \displaystyle\int_{-\infty}^\infty e^{-s^2}\,ds = \sqrt{\pi}$ as $x\to\infty$. We can check that

- $h(u) \to \infty$ as $u \to -1$.
- By l'Hôpital's rule, $h(0) \to 1$.
- $h(u) \to 0$ as $u \to \infty$.

$$h'(u) = \frac{2}{(1+u)u^3}\left(u^2 - 2(1+u)(u - \ln(1+u))\right)$$

Let $a(u) = -u^2 - 2u + 2(1+u)\ln(1+u)$. Then $a(0) = 0$ and

$$a'(u) = -2\left(u - \ln(1+u)\right) \leq 0.^{43}$$

Therefore $a(u) \leq 0$ on $u \geq 0$, $a(u) \geq 0$ on $-1 < u < 0$. So $h'(u) \leq 0$ and h is decreasing on $(-1, \infty)$.

If $s > 0$, $s\sqrt{\frac{2}{x}} \searrow 0$ as $x \to \infty$. So $h\left(s\sqrt{\frac{2}{x}}\right) \nearrow 1$, and $\exp\left(-s^2 h\left(s\sqrt{\frac{2}{x}}\right)\right) \searrow e^{-s^2}$.
If $s < 0$, $\exp\left(-s^2 h\left(s\sqrt{\frac{2}{x}}\right)\right) \nearrow e^{-s^2}$ as $x \to \infty$.

Therefore $\psi_x(s) \to e^{-s^2}$. For $x \geq 1$,

$$\psi_x(s) \leq \begin{cases} \psi_1(s) & (s > 0) \\ e^{-s^2} & (s \leq 0) \end{cases}.$$

Also, $\psi_1(s)\chi_{(0,\infty)} + e^{-s^2}\chi_{(-\infty,0]} \in \mathcal{L}^1(\mathbb{R})$. By LDCT,

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} \psi_x(s)\, ds = \int_{-\infty}^{\infty} \lim_{x\to\infty} \psi_x(s)\, ds = \int_{-\infty}^{\infty} e^{-s^2}\, ds = \sqrt{\pi}.$$

Integration on Complex Valued Function

Let $(X, \mathscr{F}, \mu)$ be a measure space, and $E \in \mathscr{F}$.

Definition.

(1) A complex valued function $f = u + iv$, (u, v real function) is measurable if u and v are measurable.

(2) A complex function $f \in \mathcal{L}^1(E, \mu) \iff \displaystyle\int_E |f|\, d\mu < \infty \iff u, v \in \mathcal{L}^1(E, \mu)$.

(3) If $f = u + iv \in \mathcal{L}^1(E, \mu)$, define $\displaystyle\int_E f\, d\mu = \int_E u\, d\mu + i\int_E v\, d\mu$.

Remark.

(1) Linearity also holds for complex valued functions. $f_1, f_2 \in \mathcal{L}^1(\mu)$, $\alpha \in \mathbb{C}$

$$\int_E (f_1 + \alpha f_2)\, d\mu = \int_E f_1\, d\mu + \alpha \int_E f_2\, d\mu.$$

[43] Note that $u > -1$.

(2) Choose $c \in \mathbb{C}$ and $|c| = 1$ (rotation) such that $c \int_E f \, d\mu \geq 0$.

$$\begin{aligned}\left| \int_E f \, d\mu \right| &= c \int_E f \, d\mu = \int_E cf \, d\mu = \int_E u \, d\mu \\ &\leq \int_E (u^2 + v^2)^{1/2} \, d\mu = \int_E |cf| \, d\mu = \int_E |f| \, d\mu.\end{aligned}$$

Where $cf = u + vi$. (u, v real) (integral of v is 0)

Functions of Class $\mathcal{L}^p$

Assume that $(X, \mathscr{F}, \mu)$ is given and $X = E$.

Definition. ($\mathcal{L}^p$) A complex function $f \in \mathcal{L}^p(\mu)$ if f is measurable and $\displaystyle\int_E |f|^p \, d\mu < \infty$.

Definition. ($\mathcal{L}^p$-norm) **$\mathcal{L}^p$-norm** of f is defined as

$$\|f\|_p = \left[\int_E |f|^p \, d\mu \right]^{1/p}.$$

Recall. (Young Inequality) Problem 6.10 (p139) $u, v \geq 0$, $p > 1$, $1/p + 1/q = 1$ then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Therefore for $\mathscr{F}$-measurable f, g on X,

$$|fg| \leq \frac{|f|^p}{p} + \frac{|g|^q}{q} \implies \|fg\|_1 \leq \frac{\|f\|_p^p}{p} + \frac{\|g\|_q^q}{q}.$$

So if $\|f\|_p = \|g\|_q = 1$, then $\|fg\|_1 \leq 1$.

Theorem 11.35 (Hölder Inequality) Let $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. If f, g are measurable,

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q .$$ [44]

So if $f \in \mathcal{L}^p(\mu)$, $g \in \mathcal{L}^q(\mu)$, then $fg \in \mathcal{L}^1(\mu)$.

Proof. If $\|f\|_p = 0$ or $\|g\|_q = 0$ then $f = 0$ a.e. or $g = 0$ a.e. So $fg = 0$ a.e. and $\|fg\|_1 = 0$.

Now suppose that $\|f\|_p > 0$ and $\|g\|_q > 0$. Then the result directly follows from

$$\left\| \frac{f}{\|f\|_p} \cdot \frac{g}{\|g\|_q} \right\|_1 \leq 1.$$

[44] q를 p의 conjugate라고 합니다.

November 17th, 2022

Complex measurable function $f \in \mathcal{L}^p(X, \mu) = \mathcal{L}^p(\mu)$.

Theorem 11.36 (Minkowski Inequality) For $1 \leq p < \infty$, if f, g are measurable, then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

Proof. If $f, g \notin \mathcal{L}^p$, the right hand side is ∞ and we are done. For $p = 1$, the equality is equivalent to the triangle inequality. Also if $\|f+g\|_p = 0$, the inequality holds trivially. We suppose that $p > 1$, $f, g \in \mathcal{L}^p$ and $\|f+g\|_p > 0$.

Let $q = \frac{p}{p-1}$. Since

$$|f+g|^p = |f+g| \cdot |f+g|^{p-1} \leq \big(|f| + |g|\big)\, |f+g|^{p-1} ,$$

we have

$$\begin{aligned} \int |f+g|^p &\leq \int |f| \cdot |f+g|^{p-1} + \int |g| \cdot |f+g|^{p-1} \\ &\leq \left(\int |f|^p\right)^{1/p} \left(\int |f+g|^{(p-1)q}\right)^{1/q} + \left(\int |q|^p\right)^{1/p} \left(\int |f+g|^{(p-1)q}\right)^{1/q} \\ &= \left(\|f\|_p + \|g\|_p\right) \left(\int |f+g|^p\right)^{1/q} . \end{aligned}$$

Since $\|f+g\|_p^p > 0$, we have

$$\|f+g\|_p = \left(\int |f+g|^p\right)^{1/p} = \left(\int |f+g|^p\right)^{1-\frac{1}{q}} \leq \|f\|_p + \|g\|_p .$$

Definition. $f \sim g \iff f = g$ μ-a.e. and define

$$[f] = \{g : f \sim g\}.$$

We treat $[f]$ as an element in $\mathcal{L}^p(X, \mu)$, and write $f = [f]$.

Remark.

(1) We write $\|f\|_p = 0 \iff f = [0] = 0$ in the sense that $f = 0$ μ-a.e.

(2) Now $\|\cdot\|_p$ is a **norm** in $\mathcal{L}^p(X, \mu)$ so $d(f, g) = \|f-g\|_p$ is a **metric** in $\mathcal{L}^p(X, \mu)$.

잠시 11.41, 11.42로 갑니다. 함수 공간이 나왔으니 completeness가 궁금하죠.

Definition 11.41 (Convergence in $\mathcal{L}^p$) Let $f, f_n \in \mathcal{L}^p(\mu)$.

(1) $f_n \to f$ in $\mathcal{L}^p(\mu) \iff \|f_n - f\|_p \to 0$ as $n \to \infty$.

(2) $(f_n)_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{L}^p(\mu)$ if and only if

$$\forall \epsilon > 0,\ \exists\, N > 0 \text{ such that } n, m \geq N \implies \|f_n - f_m\|_p < \epsilon.$$

Lemma. Let (g_n) be a sequence of measurable functions. Then,

$$\left\| \sum_{n=1}^\infty |g_n| \right\|_p \leq \sum_{n=1}^\infty \|g_n\|_p .$$

Thus, if $\displaystyle\sum_{n=1}^\infty \|g_n\|_p < \infty$, then $\displaystyle\sum_{n=1}^\infty |g_n| < \infty$ μ-a.e. So $\displaystyle\sum_{n=1}^\infty g_n < \infty$ μ-a.e.

Proof. By MCT and Minkowski inequality,

$$\left\| \sum_{n=1}^\infty |g_n| \right\|_p = \lim_{m\to\infty} \left\| \sum_{n=1}^m |g_n| \right\|_p \leq \lim_{n\to\infty} \sum_{n=1}^m \|g_n\|_p = \sum_{n=1}^\infty \|g_n\|_p < \infty.$$

Thus $\sum_{n=1}^\infty |g_n| < \infty$ μ-a.e. and $\sum_{n=1}^\infty g_n < \infty$ μ-a.e. by absolute convergence.

Theorem 11.42 (Fischer) Suppose (f_n) is a Cauchy sequence in $\mathcal{L}^p(\mu)$. Then there exists $f \in \mathcal{L}^p(\mu)$ such that $f_n \to f$ in $\mathcal{L}^p(\mu)$.

Proof. We construct (n_k) by the following procedure.

$\exists\, n_1 \in \mathbb{N}$ such that $\|f_m - f_{n_1}\|_p < \frac{1}{2}$ for all $m \geq n_1$.

$\exists\, n_2 \in \mathbb{N}$ such that $\|f_m - f_{n_2}\|_p < \frac{1}{2^2}$ for all $m \geq n_2$.

Then, $\exists\, 1 \leq n_1 < n_2 < \cdots < n_k$ such that $\|f_m - f_{n_k}\|_p < \frac{1}{2^k}$ for $m \geq n_k$.

Since $\left\|f_{n_{k+1}-f_{n_k}}\right\|_p < \frac{1}{2^k}$, $\sum_{k=1}^\infty \left\|f_{n_{k+1}} - f_{n_k}\right\|_p < \infty$. By the above lemma, $\sum \left|f_{n_{k+1}} - f_{n_k}\right|$ and $\sum(f_{n_{k+1}} - f_{n_k})$ are finite. Let $f_{n_0} \equiv 0$. Then as $m \to \infty$,

$$f_{n_{m+1}} = \sum_{k=0}^m \left(f_{n_{k+1}} - f_{n_k}\right)$$

converges μ-a.e. Take $N \in \mathscr{F}$ with $\mu(N) = 0$ such that f_{n_k} converges on $X \setminus N$. Let

$$f(x) = \begin{cases} \displaystyle\lim_{k\to\infty} f_{n_k}(x) & (x \in X \setminus N) \\ 0 & (x \in N) \end{cases}$$

then f is measurable. Using the convergence,

$$\begin{aligned}\|f-f_{n_m}\|_p &= \left\|\sum_{k=m}^{\infty}\left(f_{n_{k+1}}(x)-f_{n_k}(x)\right)\right\|_p \le \left\|\sum_{k=m}^{\infty}\left|f_{n_{k+1}}(x)-f_{n_k}(x)\right|\right\|_p \\ &\le \sum_{k=m}^{\infty}\left\|f_{n_{k+1}}-f_{n_k}\right\|_p \le 2^{-m}\end{aligned}$$

by the choice of f_{n_k}. So $f_{n_k} \to f$ in $\mathcal{L}^p(\mu)$.[45] Also, $f = (f - f_{n_k}) + f_{n_k} \in \mathcal{L}^p(\mu)$.

Let $\epsilon > 0$ be given. Since (f_n) is a Cauchy sequence in $\mathcal{L}^p$, $\exists\, N \in \mathbb{N}$ such that for all $n, m \ge N$, $\|f_n - f_m\| < \frac{\epsilon}{2}$. Note that $n_k \ge k$, so $n_k \ge N$ if $k \ge N$. Choose $N_1 \ge N$ such that for $k \ge N$, $\|f - f_{n_k}\|_p < \frac{\epsilon}{2}$. Then for all $k \ge N_1$,

$$\|f - f_k\|_p \le \|f - f_{n_k}\|_p + \|f_{n_k} - f_k\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Remark. $\mathcal{L}^p$ is a complete normed vector space, a.k.a. **Banach space**.

Theorem 11.38 $C[a,b]$ is a dense subset of $\mathcal{L}^p[a,b]$. That is,

$$\text{for every } f \in \mathcal{L}^p[a,b] \text{ and } \epsilon > 0,\ \exists\, g \in C[a,b] \text{ such that } \|f-g\|_p < \epsilon.$$

Proof. Let A be a closed subset in $[a,b]$, and consider a distance function

$$d(x,A) = \inf_{y\in A} |x-y|, \quad x \in [a,b].$$

Since $d(x,A) \le |x-z| \le |x-y| + |y-z|$ for all $z \in A$, taking inf over $z \in A$ gives $d(x,A) \le |x-y| + d(y,A)$. So

$$|d(x,A) - d(y,A)| \le |x-y|,$$

and $d(x,A)$ is continuous. If $d(x,A) = 0$, $\exists\, x_n \in A$ such that $|x_n - x| \to d(x,A) = 0$. Since A is closed, $x \in A$. We know that $x \in A \iff d(x,A) = 0$.

Let $g_n(x) = \frac{1}{1+nd(x,A)}$. g_n is continuous, $g_n(x) = 1$ if and only if $x \in A$. Also for all $x \in [a,b]\setminus A$, $g_n(x) \to 0$ as $n \to \infty$.

$$\|g_n - \chi_A\|_p^p = \int_A |g_n - \chi_A|^p\, dx + \int_{[a,b]\setminus A} |g_n - \chi_A|^p\, dx = 0 + \int_{[a,b]\setminus A} |g_n|^p\, dx \to 0$$

by LDCT. ($|g_n|^p \le 1$) We have shown that characteristic functions of closed sets can be approximated by continuous functions in $\mathcal{L}^p[a,b]$.

[45] Pointwise 이면서 $\mathcal{L}^p$에서도 수렴한다.

For every $A \in \mathfrak{M}(m)$, $\exists\, F_{\text{closed}} \subseteq A$ such that $m(A \setminus F) < \epsilon$. Since $\chi_A - \chi_F = \chi_{A \setminus F}$,

$$\int |\chi_A - \chi_F|^p \, dx = \int \left|\chi_{A\setminus F}\right|^p \, dx = \int_{A\setminus F} dx = m(A \setminus F) < \epsilon.$$

Therefore, for every $A \in \mathfrak{M}$, $\exists\, g_n \in C[a,b]$ such that $\|g_n - \chi_A\|_p \to 0$ as $n \to \infty$. So characteristic functions of any measurable set can be approximated by continuous functions in $\mathcal{L}^p[a,b]$.

Next, for any measurable simple function $f = \sum_{k=1}^m a_k \chi_{A_k}$, we can find $g_n^k \in C[a,b]$ so that

$$\left\| f - \sum_{k=1}^m a_k g_n^k \right\|_p = \left\| \sum_{k=1}^m a_k \left(\chi_{A_k} - g_n^k\right) \right\|_p \to 0.$$

Next for $f \in \mathcal{L}^p$ and $f \geq 0$, there exists simple functions $f_n \geq 0$ such that $f_n \nearrow f$ in $\mathcal{L}^p$. Finally, any $f \in \mathcal{L}^p$ can be written as $f = f^+ - f^-$, which completes the proof.[46]

[46]이러한 확장을 몇 번 해보면 굉장히 routine해요.
χ_F for closed $F \to \chi_A$ for measurable $A \to$ measurable simple $f \to 0 \leq f \in \mathcal{L}^p \to f \in \mathcal{L}^p$.

November 22nd, 2022

지금부터는 $\mathcal{L}^2(\mu)$로 한정해서 논의를 전개합니다.

Check that

$$\langle f, g\rangle = \int_X f\overline{g}\, d\mu$$

is a inner product in $\mathcal{L}^2(X,\mu)$. Then $\mathcal{L}^2(X,\mu)$ is a complete inner product space, a.k.a. **Hilbert space** with scalar $\mathbb{C}$. Now that we have a new tool, we revisit the Fourier series.

Definition. (Orthonormal set) A sequence $(\phi_n)_{n=1}^{\infty} \subseteq \mathcal{L}^2(\mu)$ is an **orthonormal set of functions on X** if

$$\langle \phi_n, \phi_m\rangle = \begin{cases} 1 & (n = m) \\ 0 & (n \neq m) \end{cases}.$$

Definition. For $f \in \mathcal{L}^2(\mu)$, we define the **Fourier series of f** as

$$\sum_{n=1}^{\infty} c_n\phi_n \text{ where } c_n = \langle f, \phi_n\rangle \text{ and write } f \sim \sum_{n=1}^{\infty} c_n\phi_n.$$

Theorem 8.11 & 8.12 (in $\mathcal{L}^2$) Suppose that (ϕ_n) is an orthonormal set in $\mathcal{L}^2(\mu)$ and $f \in \mathcal{L}^2(\mu)$. Let

$$c_m = \langle f, \phi_m\rangle = \int_X f\overline{\phi_m}\, d\mu \quad \text{and} \quad s_n = \sum_{m=1}^{n} c_m\phi_m.$$

Suppose that $t_n = \sum_{m=1}^{n} \gamma_m\phi_m$ for some $\gamma_m \in \mathbb{C}$. Then

(1) $\|f - s_n\|_2 \le \|f - t_n\|_2$, and equality holds when $\gamma_m = c_m$ for all $m = 1, 2, \dots, n$.

(2) (Bessel Inequality) $\sum_{n=1}^{\infty} |c_n|^2 \le \|f\|_2^2 < \infty$, so $\lim_{n\to\infty} c_n = 0$.

Proof. Calculate!

$$\|f - t_n\|_2^2 = \|f\|_2^2 - \sum_{m=1}^{n} |c_m|^2 + \sum_{m=1}^{n} |\gamma_m - c_m|^2 = \|f - s_n\|_2^2 + \sum_{m=1}^{n} |\gamma_m - c_m|^2.$$

연속함수를 넘어 $\mathcal{L}^2(\mu)$ 에서 성립한다!

Theorem 11.40 (Parseval in $\mathcal{L}^2$) Suppose that $f \in \mathcal{L}^2([-\pi,\pi], m)$ and

$$s_n = \sum_{k=-n}^{n} c_k e^{ikx} \quad \text{where} \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\, dx,$$

(Lebesgue integral) the following holds.

(1) $f = \lim_{n\to\infty} s_n$ in $\mathcal{L}^2[-\pi,\pi]$. ($\lim_{n\to\infty} \|s_n - f\|_2 = 0$)

(2) $\displaystyle\sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2\, dx$.

Proof. (1) Let $\epsilon > 0$ be given. By Theorem 11.38, there exists a continuous function $\widetilde{g}$ such that $\|f-\widetilde{g}\|_2 < \frac{\epsilon}{4}$. Suppose that $\widetilde{g}(\pi) = a < b = \widetilde{g}(-\pi)$. Then take δ_0 small enough so that $|\widetilde{g}(\pi-\delta_0) - a| + |\widetilde{g}(-\pi+\delta_0) - b| < b - a$.[47] There exists continuous and periodic g with $g(\pi) = g(-\pi)$ and $\|f-g\|_2 \le \|f-\widetilde{g}\|_2 + \|\widetilde{g} - g\|_2 < \frac{\epsilon}{2}$.

By Theorem 8.15, we can approximate g with a trigonometric polynomial T with degree N, and

$$\|g-T\|_2^2 = \int_{-\pi}^{\pi} |g-T|^2\, dx \le 2\pi \sup_{x\in[-\pi,\pi]} |g(x) - T(x)|^2 < \frac{\epsilon^2}{4}.$$

Now by Theorem 8.11, if $n \ge N$, $\|s_n - f\|_2 \le \|T-f\|_2 \le \|T-g\|_2 + \|g-f\|_2 < \epsilon$.

(2) Note that as $n \to \infty$,

$$\left|\|f\|_2^2 - \langle s_n, f\rangle\right| = |\langle f - s_n, f\rangle| \le \|f-s_n\|_2 \|f\|_2 \to 0$$

by (1). Therefore,

$$\langle s_n, f\rangle = \int_{-\pi}^{\pi} s_n \overline{f}\, dx = \sum_{-n}^{n} c_k \int_{-\pi}^{\pi} e^{ikx}\overline{f}\, dx = 2\pi \sum_{-n}^{n} |c_n|^2 \to 2\pi \sum_{-\infty}^{\infty} |c_n|^2.$$

Corollary. If $f \in \mathcal{L}^2[-\pi,\pi]$ and $\displaystyle\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx = 0$ for all $n \in \mathbb{Z}$, then $\|f\|_2 = 0$.

Theorem 11.43 (Riesz-Fischer) Suppose (ϕ_n) is a orthonormal set in $\mathcal{L}^2(X,\mu)$ and $\sum_{n=1}^{\infty} |c_n|^2 < \infty$. Define $s_n = \sum_{k=1}^{n} c_k\phi_k$. Then there exists $f \in \mathcal{L}^2(\mu)$ such that $s_n \to f$ in $\mathcal{L}^2(\mu)$. Moreover, $c_n = \langle f, \phi_n\rangle = \int_X f\overline{\phi_n}\, d\mu$.

Proof. We show that s_n is a Cauchy sequence. WLOG, let $n > m$. Then

$$\|s_n - s_m\|_2^2 = \left\|\sum_{k=m+1}^{n} c_k\phi_k\right\|_2^2 = \sum_{k=m+1}^{n} |c_k|^2 \to 0$$

as $n, m \to \infty$. So s_n converges in $\mathcal{L}^2(\mu)$, let $f \in \mathcal{L}^2(\mu)$ be its limit.

For $k < n$, $c_k = \langle s_n, \phi_k\rangle$. So we see that $\langle f, \phi_k\rangle = c_k$ since as $n \to \infty$,

$$|\langle f, \phi_k\rangle - c_k| = |\langle f - s_n, \phi_k\rangle| \le \|f-s_n\|_2 \|\phi_k\|_2 = \|f-s_n\|_2 \to 0.$$

[47]???

November 29th, 2022

Does $\sum c_n \phi_n$ converge in $\mathcal{L}^2$?

- $\left\| \sum_{n=1}^{m} a_n \phi_n \right\|_2^2 = \sum_{n=1}^{m} |a_n|^2.$
- $\left\| f - \sum_{n=1}^{m} \langle f, \phi_n \rangle \phi_n \right\|_2^2 = \|f\|_2^2 - \sum_{n=1}^{m} |\langle f, \phi_n \rangle|^2.$
- (Bessel Inequality) $\sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \leq \|f\|_2^2.$

Definition 11.44 (Completeness) An orthonormal set (ϕ_n) is said to be **complete** if

$$\langle f, \phi_n \rangle = \int f \overline{\phi_n} \, d\mu = 0 \text{ for all } n \text{ and } f \in \mathcal{L}^2, \text{ then } f = 0 \text{ in } \mathcal{L}^2.$$ [48]

Example. In $\mathcal{L}^2[-\pi, \pi]$, we know that

$$\sum_{-\infty}^{\infty} \left| \langle f, e^{inx} \rangle \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 \, dx.$$

Therefore $\|f\|_2 = 0 \iff \langle f, e^{inx} \rangle = 0$ for all n.

Theorem 11.45 (Parseval) Suppose (ϕ_n) is a complete orthonormal set in $\mathcal{L}^2(\mu)$, and $f \in \mathcal{L}^2(\mu)$. Then

$$\int_X |f|^2 \, d\mu = \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2 \quad \text{and} \quad \sum_{n=1}^{m} \langle f, \phi_n \rangle \phi_n \to f \text{ in } \mathcal{L}^2(\mu).$$

Proof. Let $c_n = \langle f, \phi_n \rangle$. Then by Bessel inequality, $\sum_{n=1}^{\infty} |c_n|^2 \leq \|f\|_2^2 < \infty$. By Riesz-Fischer theorem, $\exists\, g \in \mathcal{L}^2$ such that $s_m = \sum_{n=1}^{m} c_n \phi_n \to g$ in $\mathcal{L}^2$, where $c_n = \langle g, \phi_n \rangle$.

Since $\|s_m\|_2^2 \to \|g\|_2^2$,

$$\int |g|^2 \, d\mu = \lim_{m \to \infty} \int |s_m|^2 \, d\mu = \lim_{m \to \infty} \sum_{n=1}^{m} |c_n|^2 = \sum_{n=1}^{\infty} |c_n|^2.$$

We have that for all $n \in \mathbb{N}$,

$$\int g \overline{\phi_n} \, d\mu = c_n = \int f \overline{\phi_n} \, d\mu \implies \int (g - f) \overline{\phi_n} \, d\mu = 0.$$

Therefore by completeness, $f \sim g$ in $\mathcal{L}^2$.

[48] Also known as *countable orthonormal basis*.

Corollary. Suppose that (ϕ_n) is a complete orthonormal set in $\mathcal{L}^2(\mu)$. Then

$$\mathcal{L}^2(\mu) = \left\{ \sum_{n=1}^{\infty} c_n \phi_n : \sum_{n=1}^{\infty} |c_n|^2 < \infty \right\}.$$

Proof. $(\supseteq)$ by 11.43, $(\subseteq)$ by 11.45.

Corollary. Suppose that (ϕ_n) is an orthonormal set in $\mathcal{L}^2(\mu)$. The following are equivalent.

(1) (ϕ_n) is complete.

(2) $\displaystyle\int |f|^2 \, d\mu = \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2$ for all $f \in \mathcal{L}^2$.

(3) For every $f \in \mathcal{L}^2$, $\displaystyle\sum_{n=1}^{m} \langle f, \phi_n \rangle \phi_n \to f$ in $\mathcal{L}^2(\mu)$.

Proof. (1) $\Longrightarrow$ (2), (3) by Theorem 11.45, (3) $\Longrightarrow$ (2) by Theorem 8.11, (2) $\Longrightarrow$ (1) is clear.

Therefore, $\mathcal{L}^2(\mu)$ may be regarded as an infinite-dimensional Euclidean vector space, in which a vector f has coordinates c_n with respect to the basis vector ϕ_n.

Supplementary Material

Definition. Let G_n be sets. We define

$$\limsup_{n\to\infty} G_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} G_n, \qquad \liminf_{n\to\infty} G_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} G_n.$$

Remark.

(1) $x \in \limsup_{n\to\infty} G_n \iff x \in \bigcup_{n=N}^{\infty} G_n$ for all $N \iff \forall N, \exists\, n \geq N$ such that $x \in G_n$. ($x \in G_n$ for infinitely many n)

(2) $x \in \liminf_{n\to\infty} G_n \iff \exists\, N$ such that $x \in G_n$ for all $n \geq N$.

(3) $\limsup_{n\to\infty} G_n = \{x : \limsup_{n\to\infty} \chi_{G_n}(x) = 1\}$.

(4) $\left(\limsup_{n\to\infty} G_n\right)^C = \liminf_{n\to\infty} G_n$.

In measure space $(X, \mathscr{F}, \mu)$, suppose that $G_n \in \mathscr{F}$.

If $\mu\left(\bigcup_{n=1}^{\infty} G_n\right) < \infty$, then $\mu\left(\limsup_{n\to\infty} G_n\right) = \lim_{N\to\infty} \mu\left(\bigcup_{n\geq N}^{\infty} G_n\right)$. If $\sum_{n=1}^{\infty} \mu(G_n) < \infty$, then $\mu\left(\bigcup_{n\geq N}^{\infty} G_n\right) \leq \sum_{n\geq N}^{\infty} \mu(G_n) \to 0$. Therefore $\mu\left(\limsup_{n\to\infty} G_n\right) = 0$.

Proposition. Let $f \in \mathcal{L}^1([a,b], m)$ and $\epsilon > 0$. Then, there exists $\delta > 0$ such that for every $A \in \mathfrak{M}$ with $m(A) < \delta$, $\int_A |f|\, dx < \epsilon$.

To prove this, we consider the reverse version of Fatou's lemma.

Lemma. Let f_n be measurable functions. If there exists $|f_n| \leq g \in \mathcal{L}^1[a,b]$, then

$$\int_X \limsup_{n\to\infty} f_n\, dx \geq \limsup_{n\to\infty} \int_X f_n\, dx.$$

Proof. of Lemma. Consider $g - f_n$.

Proof. Suppose not. Then for some $\epsilon_0 > 0$, there exists $F_n \in \mathfrak{M}$ such that $m(F_n) < 2^{-n}$ but $\int_{F_n} |f|\, dx \geq \epsilon_0$. We see that $\sum_{n=1}^{\infty} m(F_n) < \infty$, so $m\left(\limsup_{n\to\infty} F_n\right) = 0$. Then,

$$0 = \int_{\limsup_n F_n} |f|\, dx = \int_X |f| \limsup_{n\to\infty} K_{F_n} \geq \limsup_{n\to\infty} \int_{F_n} |f|\, dx \geq \epsilon_0,$$

which is a contradiction.

Definition. (Uniformly Integrable) Suppose that (f_α) is a collection of measurable functions on $[a,b]$.[49] (f_α) is called **uniformly integrable** on $[a,b]$ if

$$\forall \epsilon > 0,\ \exists\, \delta > 0 \text{ such that if } A \in \mathfrak{M} \text{ and } m(A) < \delta, \text{ then } \sup_\alpha \int_A |f_\alpha| < \epsilon.$$

Theorem. (Vitali Convergence Theorem) Suppose (f_n) is uniformly integrable on $[a,b]$ and $f_n \to f$ pointwise a.e. Then

$$\lim_{n\to\infty} \int_a^b f_n\, dx = \int_a^b f\, dx.$$

Proof. Set $\epsilon = 1$. Consider a partition $(I_i)_{i=1}^M$ of $[a,b]$. Then

$$\int_a^b |f|\, dx = \sum_{i=1}^M \int_{I_i} |f|\, dx \le \sum_{i=1}^M \liminf_{n\to\infty} \int_{I_i} |f_n|\, dx \le \sum_{i=1}^M \sup_n \int_{I_i} |f_n|\, dx \le M < \infty.$$

Therefore $f \in \mathcal{L}^1$.

Given $\epsilon > 0$, $\exists\, \delta > 0$ such that if $m(A) < \delta$ then $\int_A |f_n| < \frac{\epsilon}{3}$, $\int_A |f| < \frac{\epsilon}{3}$ for all n. Using HW problem, there exists $B \in \mathfrak{M}$ such that if $m(B) < \delta$ and $N \ge 1$, $\sup_{x\in[a,b]\setminus B} |f_n(x) - f(x)| < \frac{\epsilon}{3(b-a)}$ for all n.

Therefore,

$$\int_a^b |f_n - f| \le \int_{[a,b]\setminus B} |f_n - f| + \int_B |f_n| + \int_B |f| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Theorem. (Lebesgue's Theorem on Monotone Functions) Suppose that $f : (a,b) \to \mathbb{R}$ is monotone. Then f is differentiable a.e. on (a,b).

Corollary. If f is non-decreasing on $[a,b]$, then $\int_a^b f'\, dx \le f(b) - f(a)$.

Proof. Consider $g_n(x) = n\left(f\left(x + \frac{1}{n}\right) - f(x)\right)$ and $f(x) = f(b)$ for $x \ge b$. Then $g_n(x) \ge 0$ and $g_n \to f'$ a.e. by Lebesgue's Theorem. Note that

$$\begin{aligned}\int_a^b g_n\, dx &= n\left(\int_a^b f\left(x + \frac{1}{n}\right) dx - \int_a^b f(x)\, dx\right) = n\left(\int_b^{b+\frac{1}{n}} f\, dx - \int_a^{a+\frac{1}{n}} f\, dx\right)\\ &= f(b) - n\int_a^{a+\frac{1}{n}} f\, dx \le f(b) - f(a).\end{aligned}$$

By Fatou's lemma,

$$\int_a^b f'\, dx \le \liminf_{n\to\infty} \int_a^b g_n\, dx \le f(b) - f(a).$$

Remark. If f is non-increasing on (a,b), then $\int_a^b f'\, dx \ge f(b) - f(a)$.

[49] May be uncountable.

Definition. (Total Variation) **Total variation** of $f : [a,b] \to \mathbb{R}$ over $[a,b]$ is defined as

$$T_a^b(f) = \sup_{P \in \mathcal{P}[a,b]} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|\,.$$

Definition. (Bounded Variation) f is of **bounded variation** on $[a,b]$ if $T_a^b(f) < \infty$.

Remark.

(1) $T_a^b(f) \geq |f(b) - f(a)| \geq 0$.

(2) $T_a^b(f) = T_a^c(f) + T_c^b(f)$ if $a < c < b$.

(3) $T_a^b(f+g) \leq T_a^b(f) + T_a^b(g)$.

(4) $T_a^b(cf) = |c|\, T_a^b(f)$.

Proposition. Let $f(x) = \int_a^x \phi(t)\, dt$ on $a < x \leq b$, where ϕ is measurable. Then $T_a^x(f) \leq \int_a^x |\phi(t)|\, dt$. Thus if $\phi \in \mathcal{L}^1[a,b]$, then f is of bounded variation.

Proof. Observe that

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |\phi|\ dt = \int_a^x |\phi|\ dt.$$

Now take sup over all partitions of $[a,x]$.

Theorem. If f is of bounded variation if and only if $f = g - h$ where g, h are non-decreasing real-valued functions. In fact,

$$g(x) = \frac{1}{2}\left[T_a^x(f) + f(x)\right], \qquad h(x) = \frac{1}{2}\left[T_a^x(f) - f(x)\right].$$

Proof. ($\Longleftarrow$) $T_a^b(f) = T_a^b(g-h) \leq T_a^b(g) + T_a^b(h) = (g+h)(b) - (g+h)(a) < \infty$.
($\Longrightarrow$) For $y > x$, we show that g, h are non-decreasing.

$$T_a^y(f) \pm f(y) - (T_a^x(f) \pm f(x)) = T_x^y(f) \pm (f(y) - f(x)) \geq T_x^y(f) - |f(y) - f(x)| \geq 0.$$

Proposition. Suppose f is of bounded variation on $[a,b]$. Then by Lebesgue's Theorem, f is differentiable a.e. and $\int_a^b |f'|\ dx \leq T_a^b(f)$.

Proof. It suffices to show that $|f'(x)| \leq \frac{d}{dx} T_a^x(f)$ a.e. Let $h > 0$ and $x < x + h < b$.

$$\frac{1}{h}\left(T_a^{x+h}(f) - T_a^x(f)\right) = \frac{1}{h} T_x^{x+h}(f) \geq \frac{1}{h}\left|f(x+h) - f(x)\right|.$$

Let $h \to 0$ then the inequality holds a.e. Now we have

$$\int_a^b \frac{d}{dx} T_a^x(f)\,dx \le T_a^b(f) - T_a^a(f) = T_a^b(f).$$

Lemma. If $f \in \mathcal{L}^1[a,b]$ and $\int_{(c,d)} f\,dx \ge 0$ for all $a \le c < d \le b$ then $f \ge 0$ a.e. [50]

Corollary. Suppose $f \in \mathcal{L}^1[a,b]$. Let $F(x) = \int_a^x f\,dx$, then $|F'| \le |f|$ a.e.

Proof. We know that F is continuous and of bounded variation and differentiable a.e. For $a \le c < d \le b$,

$$\int_c^d |f| \ge T_c^d(F) \ge \int_c^d |F'|\,.$$

By Lemma, $|F'| \le |f|$ a.e.

Theorem. (1st Fundamental Theorem of Calculus for Lebesgue Integral) Suppose $f \in \mathcal{L}^1[a,b]$, let $F(x) = \int_a^x f\,dx$. Then $F'(x) = f(x)$ a.e.

Proof. For $n \in \mathbb{Z}$,

$$\int_a^x (f-n)\,dx = F(x) - n(x-a), \quad (x \in [a,b]).$$

By the corollary above, $|f-n| \ge |F'(x) - n|$ a.e. We see that

$$\pm f(x) = \lim_{n\to-\infty} |f(x)-n| \pm n \le \lim_{n\to-\infty} |F'(x)-n| \pm n = \pm F'(x)$$

holds a.e. Therefore $F'(x) \le f(x) \le F'(x)$ a.e. and $F'(x) = f(x)$ a.e.

Suppose that $f : [a,b] \to \mathbb{R}$.

Definition. (Absolute Continuity) $f : [a,b] \to \mathbb{R}$ is **absolutely continuous** on $[a,b]$ if $\forall \epsilon > 0$, $\exists\, \delta > 0$ such that

$$\text{whenever } (x_j, x_j') \text{ are disjoint and } \sum_{j=1}^n (x_j' - x_j) < \delta, \text{ we have } \sum_{j=1}^n \left|f(x_j') - f(x_j)\right| < \epsilon.$$

Remark.

(1) If f is absolutely continuous, then f is continuous.

(2) If f is absolutely continuous, then f is of bounded variation on $[a,b]$.

(3) If f is Lipschitz continuous, then f is absolutely continuous.

Corollary. If f is absolutely continuous, then f is differentiable a.e. [51]

[50] Use the fact that the Lebesgue measure is regular.
[51] Since it is of bounded variation.

Theorem. If f is absolutely continuous, then $f = g - h$ where g, h are non-decreasing and g, h are absolutely continuous.

Proof. f is of bounded variation, so we can write $f = g - h$ where g, h are non-decreasing. Considering the representation of g, h where

$$g(x) = \frac{1}{2}\left[T_a^x(f) + f(x)\right], \qquad h(x) = \frac{1}{2}\left[T_a^x(f) - f(x)\right],$$

it suffices to show that $T_a^x(f)$ is absolutely continuous.

Let $\{(c_k, d_k)\}_{k=1}^n$ be disjoint and $\sum_{k=1}^n (d_k - c_k) < \delta$. Now let $P_k = \{c_k = x_1^k < x_2^k < \cdots < x_n^k = d_k\}$ be a partition of (c_k, d_k). Then by absolute continuity,

$$\sum_{k=1}^n \sum_{j=1}^{n_k} \left|f(x_j^k) - f(x_{j-1}^k)\right| < \frac{\epsilon}{2}.$$

Take sup on partition P_k of each (c_k, d_k). Then

$$\sum_{k=1}^n T_{c_k}^{d_k}(f) = \sum_{k=1}^n \left|T_a^{d_k}(f) - T_a^{c_k}(f)\right| \le \frac{\epsilon}{2} < \epsilon.$$

Therefore $T_a^x(f)$ is absolutely continuous.

Theorem. Define $\Delta_h f(x) = \frac{1}{h}\left(f(x+h) - f(x)\right)$ for $h > 0$. Suppose that f is absolutely continuous on $[a, b]$. Then $(\Delta_h f)_{0<h<1}$ is uniformly integrable on $[a, b]$.

Proof. By the above theorem, we can assume that f is non-decreasing, so $\Delta_h f \ge 0$. By regularity of Lebesgue measure, we just need to show that $\sup_{0<h\le 1} \int_E \Delta_h f \, dx < \epsilon$ if $m(E) < \delta$ and $E = \bigcup_{k=1}^n (c_k, d_k)$ where (c_k, d_k) are disjoint. Assume that $f(y) = f(b)$ for $b < y \le b + 1$. If $\sum_{k=1}^n (d_k - c_k) < \delta$ (which implies $m(E) < \delta$) then $\sum_{k=1}^n \left[(d_k + t) - (c_k + t)\right] < \delta$ for any $0 < t < 1$. So $\sum_{k=1}^n \left[f(d_k + t) - f(c_k + t)\right] < \frac{\epsilon}{2}$ for all $0 < t < 1$. (f is non-decreasing) Then

$$\int_E \Delta_h f \, dt = \frac{1}{h} \sum_{k=1}^n \int_0^h \left[f(d_k + t) - f(c_k + t)\right] dt < \frac{\epsilon}{2}, \qquad (0 \le h \le 1).$$

Therefore $\sup_{0<h\le 1} \int_E \Delta_h f \, dt \le \frac{\epsilon}{2} < \epsilon$ and $(\Delta_h f)_{0<h<1}$ is uniformly integrable.

Theorem. (2nd Fundamental Theorem of Calculus for Lebesgue Integral) Suppose that f is absolutely continuous on $[a, b]$. Then $f' \in \mathcal{L}^1$ and

$$\int_a^b f' \, dx = f(b) - f(a).$$

Proof. Since f is differentiable a.e. , $\lim_{n\to\infty} \Delta_{\frac{1}{n}} f(x) = f'(x)$ a.e. and by the above theorem, $\left(\Delta_{\frac{1}{n}} f\right)_n$ is uniformly integrable. Extend f as $f(x) = f(b)$ for $x \ge b$. Then by Vitali convergence theorem,

$$\lim_{n\to\infty}\int_a^b \Delta_{\frac{1}{n}} f\, dx = \int_a^b f'(x)\, dx.$$

Note that the expression inside the limit on the left is equal to

$$\int_a^b \Delta_{\frac{1}{n}} f\, dx = \frac{1}{n}\int_a^b \left[f\left(x+\frac{1}{n}\right) - f(x)\right] dx = \frac{1}{n}\int_b^{b+\frac{1}{n}} f\, dx - \frac{1}{n}\int_a^{a+\frac{1}{n}} f\, dx.$$

Since f is continuous, the last two terms converge to $f(b), f(a)$ as $n \to \infty$ respectively.

함수공간의 Story

우리가 해석개론1에서 실수를 공부하기 위해서 어떻게 했었는지 떠올려 보면, 절대 실수 하나 하나를 개별적으로 보지 않았습니다. 실수의 모임을 두고, 실수열을 공부하고, 위상적인 구조를 주는 등의 작업을 하고 나서야 실수를 제대로 이해할 수 있었습니다.

함수도 마찬가지입니다. 우리는 함수를 이해하기 위해서 함수 하나를 개별적으로 보는 것이 아니라, 함수가 속해있는 공간을 공부하는 것입니다. 실수를 공부할 때와 마찬가지로, 함수열을 공부하고, 위상적인 구조를 공부하며 함수공간을 이해하게 됩니다. 그런데 $\mathbb{R}/\mathbb{R}^n$과 함수공간의 가장 큰 차이점은 좌표공간은 유한차원이지만, **함수공간은 무한차원**이라는 점입니다.

함수공간을 벡터공간으로 만들기는 했으나, 선형대수학의 대부분 정리들은 벡터공간의 차원이 유한이라는 가정이 필요하기 때문에, 공짜로 얻어지는 정리는 없습니다. 이로 인해 **norm을 도입**하게 됩니다.

Norm을 도입하게 되면 공간에 거리 개념이 생기므로, metric space를 논할 수 있게 되고, 자연스럽게 수렴성을 논할 수 있게 됩니다. 이와 동시에 Cauchy 수열의 개념도 생겨납니다. 그리고 open ball을 정의할 수 있고, open/closed set이 정의되고, compact set까지 정의하게 됩니다. 함수공간에서는 norm이 없으면 아무것도 할 수 없습니다. **함수공간에는 norm이 항상 존재합니다.**

그리고 마지막으로 이 norm과 수렴 개념을 바탕으로 Cauchy 수열이 수렴하는지 살펴봅니다. 함수공간을 너무 작게 잡으면, Cauchy 수열이 수렴하지 않을 수 있습니다. 그러면 더 함수를 넣어야 합니다. 함수를 넣다보면 또 새로운 Cauchy 수열이 생깁니다. 이 과정을 반복하여 다 넣게 되면 드디어 Cauchy 수열이 수렴하게 되고, 비로소 **completeness**(완비)를 만족하게 됩니다.[52]

해석개론2의 첫 장에서는 **연속함수열의 수렴**에 대해 공부했습니다. 연속함수의 공간 $C(X)$에서 점별수렴과 고른수렴이 있었는데, 극한함수 또한 연속이어야 하기 때문에 우리는 이 함수공간에서 **고른수렴**을 올바른 수렴의 정의로 선택했습니다. 그래야 $C(X)$의 Cauchy 수열이 수렴하여 $C(X)$가 completeness를 만족하게 되기 때문입니다.

또 고른수렴에 대해서 공부하면서 얻은 부산물로, 고른수렴이 언제 미분가능성과 적분가능성을 보존하는지 공부했습니다. 극한함수의 미분가능성에 대해서는 굉장히 까다로운 조건이 필요했지만, 적분가능성의 경우 잘 보존되는 것을 확인했습니다. 이를 기점으로 해석학은 적분에 주안점을 두고 가게 됩니다. 우리가 함수를 미분하면 함수가 나빠지는 반면, 적분을 통해 얻은 함수는 상대적으로 다루기 쉽습니다. 또한 현실 세계에서 미분가능한 함수를 만나기 쉽지 않기도 합니다. **결국 해석학은 적분을 발전시키는 방향으로 나아가게 됩니다.**

수렴을 공부한 이후, 본격적으로 함수공간 $C(X)$를 공부했습니다. 이제 우리의 관심사는 $C(X)$

[52] Complete normed space를 Banach space라고 부릅니다.

의 compact set 입니다. 따라서 **수렴하는 부분수열**에 대해 공부하게 됩니다.[53] 이는 마치 $\mathbb{R}^n$에서 Bolzano-Weierstrass 정리를 공부했던 것과 동일합니다. 그래서 $C(X)$의 수렴하는 부분수열을 찾기 위해 점별유계, 고른유계, 동등연속의 개념을 공부했으며 Arzela-Ascoli 정리를 공부했습니다. 그리고 실제로 주어진 연속함수로 수렴하는 연속함수열(특별히 다항함수)이 존재한다는 사실을 Weierstrass 정리를 통해 공부했습니다. 이는 곧 $\mathbb{Q}$가 $\mathbb{R}$에서 조밀(dense)했던 것처럼 다항함수가 $C(X)$에서 조밀함을 보여줍니다.

수렴하는 부분수열이 중요한 또 다른 이유는 **precompact** 개념 때문이기도 합니다. 집합 X의 수열이 수렴하는 부분수열을 가질 때, X를 precompact set이라고 합니다. Precompact 개념을 이용하면 (f_n)이 f로 수렴하는지 확인하려 할 때, 이를 2단계로 나눠 증명할 수 있게 됩니다. 먼저 $\{f_n : n \in \mathbb{N}\}$ 이 precompact임을 보이고, (f_n)의 부분수열이 g로 수렴하면 $f = g$ 임을 보이면 됩니다.

이후로는 수열공간 $\ell^p(\mathbb{N})$, 특이적분가능함수공간 $\mathcal{R}^p(I)$에 대해 공부하게 되는데, 여기서도 같은 story가 반복됩니다. 함수공간을 벡터공간으로 만들어 norm을 정의하고, Cauchy 수열이 수렴하는지 확인하여 completeness를 확인합니다. 그리고 compact set에 대해 조사하게 됩니다. 수열공간 $\ell^p(\mathbb{N})$의 경우 유계이고 닫힌 집합만으로 compact set이 되기에는 부족함을 확인하게 되고,[54] 특이적분가능함수공간 $\mathcal{R}^p(I)$는 complete 하지 않다는 것도 확인하게 됩니다. 이는 적분가능함수공간을 너무 작게 잡았다는 의미로, 후에 르벡 적분 등장의 배경이 되며 르벡적분가능함수공간 $L^p(E)$는 completeness를 만족하게 됩니다.

[53]당연히 $C(X)$의 수렴은 고른수렴입니다.

[54]실제로, 무한차원의 normed space는 Heine-Borel property (compact $\iff$ bounded & closed)를 가질 수 없음이 알려져 있습니다.

September 1st, 2022 (Practice)

해석개론 1 복습

1. Real Number System

Let $A \subseteq \mathbb{R}$.

- $b \in \mathbb{R}$ is an upper bound of A: $\forall a \in A \implies a \leq b$.
- $b \in \mathbb{R}$ is a lower bound of A: $\forall a \in A \implies a \geq b$.
- Least uppoer bound is denoted as $\sup A$.
- Greatest lower bound is denoted as $\inf A$.
- Least upper bound property: If $A \neq \varnothing$, $\exists \sup A$.
- Extended Real Numbers: $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$
- Now, if $\varnothing = A \subseteq \overline{\mathbb{R}}$, $\sup A = -\infty$.

2. Metric Spaces

Metric space: (X, d_X) where $d_X : X \times X \to \mathbb{R}$. For all $x, y, z \in X$ the following must hold.

(1) $d_X(x, y) = 0 \iff x = y$.

(2) $d_X(x, y) = d_X(y, x)$ (Symmetric)

(3) $d_X(x, y) + d_X(y, z) \geq d_X(x, z)$

Notation. (Neighborhood) Ball of radius r, centered at p is denoted as

$$B_r(p) = \{x \in X \mid d_X(x, p) < r\}$$

- $U \subseteq X$ is open $\iff \forall p \in U, \exists\, r > 0$ such that $B_r(p) \subseteq U$.
- $C \subseteq X$ is closed $\iff C$ contains every limit point of C. Or alternatively, C^C is open.
- Union of open sets is open, <u>finite</u> intersection of open sets is open.
- $p \in B \subseteq X$ is a limit point of $B \iff \forall r \geq 0, (B_r(p) \setminus \{p\}) \cap B \neq \varnothing$.[55]
- A' is the set of limit points of A.

[55] 임의의 근방에서 자기자신을 제외하고 B의 점이 존재한다.

- $\overline{A} = A \cup A'$, which is the smallest closed set containing A.
- $A \subseteq X$ is dense in $X \iff \overline{A} = X$.
- $A \subseteq X$ is bounded $\iff \exists\, r > 0$ such that $A \subseteq B_r(p)$ for some $p \in X$.
- Sets A and B are separated $\iff \overline{A} \cap B = \varnothing = A \cap \overline{B}$.
- Set C is disconnected $\iff \exists$ non-empty separated sets A, B such that $C \subseteq A \cup B$.

Suppose $\{U_\alpha\}$ is a collection of open sets in X.

- $\{U_\alpha\}$ is an open cover of $A \iff A \subseteq \bigcup_\alpha U_\alpha$.
- $K \subseteq X$ is compact $\iff$ for every open cover of K, there exists a finite subcover of K.

$$\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \text{ such that } K \subseteq \bigcup_{k=1}^{n} U_{\alpha_k}$$

- (Heine-Borel) In $\mathbb{R}^n$, compact $\iff$ bounded and closed.
- If K is compact and $A \subseteq K$ is closed, then A is also compact.
- If $\{K_\alpha\}$ is a collection of compact sets and $\bigcap_\alpha K_\alpha = \varnothing$, then

$$\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \text{ such that } \bigcap_{k=1}^{n} K_{\alpha_k} = \varnothing.$$ [56]

3. Sequences

A sequence $a : \mathbb{N} \to A$, is a function. We write $a(i) = a_i$, and we usually consider sequences in metric spaces.

- $\{a_n\}$ converges to $\alpha \iff \forall \epsilon > 0, \exists\, N \in \mathbb{N}$ such that $n \geq N \implies d_X(a_n, \alpha) < \epsilon$.
- (Cauchy Sequence) $\{a_n\}$ is Cauchy
$\iff \forall \epsilon > 0, \exists\, N \in \mathbb{N}$ such that $n, m \geq N \implies d_X(a_n, a_m) < \epsilon$.
- (X, d) is complete $\iff$ every Cauchy sequence converges.[57]
- $\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}$.
- $\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$.

[56] 정의로 쉽게 보일 수 있다?
[57] 수렴하면 코시 수열이지만, 모든 코시 수열이 수렴하지는 않는다. Consider any sequence of rational numbers converging to an irrational real number.

- $\lim a_n = \alpha \iff \limsup a_n = \liminf a_n = \alpha$ $(\alpha \in \mathbb{R})$.
- For power series $\sum a_n x^n$, the radius of convergence $R \in \overline{\mathbb{R}}$ is calculated as

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

- Absolute convergence implies convergence.

4. Limit of Functions

Given metric spaces X, Y, define a function $f : E \subseteq X \to Y$.

- If $p \in E'$[58] then we can define $\lim_{x\to p} f(x) = \alpha$ as

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } 0 < d_X(x,p) < \delta \implies d_Y(f(x), \alpha) < \epsilon.$$

  Or equivalently, for any sequence $\{a_n\}$ in X with $a_n \neq p$,

$$\text{if } \lim_{n\to\infty} a_n = p \text{ then } \lim_{n\to\infty} f(a_n) = \alpha.$$

- f is continuous at $p \in E$[59] $\iff$

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } x \in E, d_X(x,p) < \delta \implies d_Y(f(x), f(p)) < \epsilon.$$

  Or equivalently, for any sequence $\{a_n\}$ in X,[60]

$$\text{if } \lim_{n\to\infty} a_n = p \text{ then } \lim_{n\to\infty} f(a_n) = f(p).$$

- f is continuous $\iff$ for any open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.
- Suppose that f is continuous.
  - If $K \subseteq E$ is compact, $f(K)$ is also compact.
  - If $C \subseteq E$ is connected, $f(C)$ is also connected.
- (Extreme Value Theorem) Suppose $K \subseteq E$ is compact and $f : K \to \mathbb{R}$ is continuous. Because $f(K)$ is a compact set in $\mathbb{R}$, it is a closed interval. Hence f has a maximum/minimum.
- (Uniform Continuity) f is uniformly continuous on $E \iff$

$$\forall \epsilon > 0, \exists\, \delta > 0 \text{ such that } \forall x, \forall y \in E,\ d_X(x,y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

[58] 함수의 극한은 극한점에서 논한다! 다가갈 점들이 있어야 하지 않겠는가?
[59] Limit point가 아니어도 정의할 수 있으며, 고립점에서는 연속이다.
[60] 여기서는 $a_n \neq p$ 조건이 빠진다.

- If $f : E \subseteq X \to Y$ is continuous and E is compact, f is uniformly continuous.

5. Differentiation

Function $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b] \iff$

$$\text{the limit } f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \text{ exists.}$$

- If f is differentiable at $x = p$, then f is continuous at $x = p$.
- If f is differentiable at $x = p$ and $g : f([a, b]) \to \mathbb{R}$ is differentiable at $x = f(p)$ $\implies g \circ f$ is differentiable at $x = p$ and

$$(g \circ f)'(p) = g'\big(f(p)\big) f'(p).$$

- (Fermat) If f is differentiable and has a local extremum at $x = a$, then $f'(a) = 0$.
- (Mean Value Theorem) If f is continuous on $[a, b]$ and differentiable on (a, b), there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

6. Integration

Given a <u>bounded</u> function $f : [a, b] \to \mathbb{R}$, a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ and a monotonically increasing function $\alpha : [a, b] \to \mathbb{R}$, define

$$U(P, f, \alpha) = \sum_{i=0}^{n-1} \sup_{x \in [x_i, x_{i+1}]} f(x)\big(\alpha(x_{i+1}) - \alpha(x_i)\big)$$

$$L(P, f, \alpha) = \sum_{i=0}^{n-1} \inf_{x \in [x_i, x_{i+1}]} f(x)\big(\alpha(x_{i+1}) - \alpha(x_i)\big)$$

We define upper integral and lower integral as follows:

$$\overline{\int_a^b} f \, d\alpha = \inf_{P \in \mathcal{P}[a,b]} U(P, f, \alpha) \qquad \underline{\int_a^b} f \, d\alpha = \sup_{P \in \mathcal{P}[a,b]} L(P, f, \alpha).$$

f is Stieltjes integrable with respect to $\alpha \iff$

$$\forall \epsilon > 0, \exists\, P \in \mathcal{P}[a, b] \text{ such that } U(P, f, \alpha) - L(P, f, \alpha) < \epsilon.$$

Or equivalently, $\overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha$. We write $f \in \mathcal{R}(\alpha)$.

Supplementary Material

F is a field for this section.

Definition. (Vector Space) A set V with addition $+ : V \times V \to V$ and scalar multiplication $\cdot : F \times V \to V$ is a vector space over F if the following properties hold.

(1) (Associativity of $+$) $u + (v + w) = (u + v) + w$ for all $v, w, u \in V$.

(2) (Commutativity of $+$) $v + w = w + v$ for all $v, w \in V$.

(3) (Identity of $+$) $\exists\, 0_V \in V$ such that $v + 0 = 0 + v = v$ for all $v \in V$.

(4) (Inverse of $+$) For each $v \in V$, $\exists\, x \in V$ such that $v + x = x + v = 0_V$.

(5) (Identity of $\cdot$) $1v = v$ for $v \in V$, where $1 \in F$ is the multiplicative identity in F.

(6) (Distributive Property of $\cdot$ w.r.t. Vector $+$) For $a \in F$ and $v, w \in V$, $a(v + w) = av + aw$.

(7) (Distributive Property of $\cdot$ w.r.t. Field $+$) For $a, b \in F$ and $v \in V$, $(a + b)v = av + bv$.

(8) (Compatibility of $\cdot$ w.r.t. $+$) $a(bv) = (ab)v$ for $a, b \in F$, $v \in V$.

We write $V = (V, +, \cdot)$.

Definition. (Normed Vector Space) A vector space V with a norm $\|\cdot\| : V \to \mathbb{R}$ if a normed vector space if the following properties hold.

(1) $\|v\| \geq 0$ for all $v \in V$.

(2) $\|v\| = 0 \iff v = 0$.

(3) For all $\alpha \in F$ and $v \in V$, $\|\alpha v\| = |\alpha| \, \|v\|$.

(4) (Triangle Inequality) For all $v, w \in V$, $\|v + w\| \leq \|v\| + \|w\|$.

For inner product spaces, $F = \mathbb{C}$ or $F = \mathbb{R}$.

Definition. (Inner Product Space) A vector space V with an inner product $\langle \cdot, \cdot \rangle : V \times V \to F$ is an inner product space if the following properties hold.

(1) (Linearity in the first argument) For $x, y, z \in V$ and $a, b \in F$, $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$.

(2) (Conjugate Symmetry) For $x, y \in V$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

(3) (Positive Definiteness) If $0 \neq x \in V$, $\langle x, x \rangle > 0$.

Remark. An inner product can induce a norm by $\|v\| = \sqrt{\langle v, v\rangle}$. With norm as the distance metric, the following holds.

$$\text{Inner Product Space } \Longrightarrow \text{Normed Vector Space } \Longrightarrow \text{Metric Space}$$

If the inner product space is complete with respect to the distance metric, it is said to be a Hibert space.

September 8th, 2022 (Practice)

미분가능성이 잘 보존되지 않는다.

Example. $f_n(x) = \dfrac{\sin nx}{n}$. Converges to $f(x) = 0$ uniformly, but not differentiable.

$$f_n'(x) = \cos nx \neq f'(x) = 0$$

반례를 생각하는 방법: target limit function을 먼저 생각하고 걔로 수렴하는 함수열을 잡는다.

Example. Consider a triangular pulse

$$f_n(x) = \begin{cases} n^2 x & \left(0 \leq x \leq \frac{1}{n}\right) \\ -n^2 x + 2n & \left(\frac{1}{n} \leq x \leq \frac{2}{n}\right) \end{cases}.$$

Converges pointwise, but not the convergence is not uniform.

Example. $f_n : X \subseteq \mathbb{R} \to \mathbb{R}$. Suppose $f_n \xrightarrow{u} f$. Then if f_n is increasing, f is also increasing.

Proof. (Contradiction) Suppose f is not increasing...!

Example. $f_n : X \to \mathbb{R}$. Suppose $f_n \xrightarrow{u} f$. If f_n has a local maxima at $x = 0$, f need not have a local maxima at $x = 0$. Consider

$$f_n(x) = \begin{cases} 0 & \left(x < \dfrac{1}{n}\right) \\ x - \dfrac{1}{n} & \left(x \geq \dfrac{1}{n}\right) \end{cases} \to f(x) = \begin{cases} 0 & (x < 0) \\ x & (x \geq 0) \end{cases}.$$

Then each f_n has a local maximum at $x = 0$, but $f(x)$ has a local minimum at $x = 0$.

Problem 7.3 Product of uniformly convergent sequence of functions need not converge uniformly.

Proof. Let $f_n(x) = \dfrac{1}{n}$, $g(x) = g_n(x) = x$. Then

$$f_n g_n - fg = \frac{x}{n},$$

which does not converge uniformly.

Theorem 7.11 (Cases with ∞) Theorem 7.11 also holds when

$$\lim_{x \to a} f_n(x) = \pm\infty, \qquad \lim_{x \to \pm\infty} f_n(x) = A_n.$$

Proof. Consider a bijective, increasing, continuous function $g : (-1, 1) \to (-\infty, \infty)$, $g'(x) \geq$

1.

($a = \infty$ case) Then $x \to \infty$ with respect to f is equivalent to $x \to 1^-$ with respect to $f \circ g$. Observe that

$$\sup_{x\in(-1,1)} |f_n(g(x)) - f(g(x))| = \sup_{x\in\mathbb{R}} |f_n(x) - f(x)|,$$

thus $h_n = f_n \circ g$ will converge uniformly to $h = f \circ g$.

($\lim f_n(x) = \infty$ case) Similarly, consider $g^{-1} \circ f_n$ and $g^{-1} \circ f_n$. Then $\lim_{x\to a} g^{-1} \circ f_n = 1$. Now we show uniform convergence.

$$\lim_{n\to\infty} \sup_{x\in E} \left|g^{-1}(f_n(x)) - g^{-1}(f(x))\right| \leq \lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| \cdot \sup_{x\in E} \left|(g^{-1})'(x)\right| \to 0$$

Example. $f_n : X \to Y$, $A_1, \ldots, A_k \subseteq X$, $f_n \overset{u}{\to} f$ on $A_i \implies f_n \overset{u}{\to} f$ on $\bigcup A_i$.

Problem 7.4 Examine the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}.$$

Proof. We do not consider $x = -1/n^2$ for some $n \in \mathbb{N}$.

(Absolute Convergence) For $x \neq 0$, take large enough $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} \left|\frac{1}{1+n^2x}\right| = \sum_{n=N}^{\infty} \left|\frac{1}{n^2x\left(1+\dfrac{1}{n^2x}\right)}\right| \leq \sum_{n=N}^{\infty} \frac{1}{0.9n^2x} < \infty.$$

(Uniform Convergence) For $[k, \infty)$ $(k > 0)$,

$$\sum_{n=m}^{\infty} \left|\frac{1}{1+n^2x}\right| \leq \sum_{n=m}^{\infty} \left|\frac{1}{n^2x}\right| \leq \frac{1}{k} \sum_{n=m}^{\infty} \frac{1}{n^2},$$

thus f converges uniformly on $[k, \infty)$. Now for $(\infty, k]$ $(k < 0)$,

$$\frac{1}{xm^2} > -\frac{1}{2} \iff m > \sqrt{\frac{-2}{x}}$$

and now we can choose m so that

$$\sum_{n=m}^{\infty} \left|\frac{1}{1+n^2x}\right| \leq \sum_{n=m}^{\infty} \left|\frac{1}{n^2x \cdot (1/2)}\right| \leq \frac{2}{|k|} \sum_{n=m}^{\infty} \frac{1}{n^2}.$$

Thus f also converges uniformly on $(-\infty, k]$. Now how about $(-\infty, 0) \cup (0, \infty)$? Suppose the series converges uniformly. Then for $\epsilon = 1$, $\forall N \in \mathbb{N}$ such that

$$\left| \sum_{n=N}^{\infty} \frac{1}{1+n^2 x} \right| < 1.$$

As $x \to 0^+$, $\dfrac{1}{1+N^2x} + \dfrac{1}{1+(N+1)^2x} \to 2$. Thus does not converge uniformly.

$\therefore$ Converges uniformly on $(-\infty, -k] \cup [k, \infty)$, $(k > 0)$.

(Continuity) Follows directly from uniform convergence. $(-\infty, -k] \cup [k, \infty)$.

(Boundedness) No.

Problem 7.12 Since $|f| \leq g$,

$$\int_a^b f\,dx = \int_a^b \frac{|f|+f}{2}\,dx - \int_a^b \frac{|f|-f}{2}\,dx.$$

Since $\int_0^\infty g\,dx < \infty$, (bounded) we can set $a \to 0$, $b \to \infty$.

For all $\epsilon > 0$, choose $[a, b]$ such that

$$\left| \int_0^\infty f\,dx - \int_a^b f\,dx \right| < \epsilon \text{ and } \left| \int_0^\infty g\,dx - \int_a^b g\,dx \right| < \epsilon.$$

by uniform continuity, $\exists\, N \in \mathbb{N}$ such that $n \geq N$ then $\left| \int_a^b f_n\,dx - \int_a^b f\,dx \right| < \epsilon.$

Therefore,

$$\left| \int_0^\infty f_n\,dx - \int_0^\infty f\,dx \right| < 3\epsilon$$

and the theorem is proven.

September 15th, 2022 (Practice)

Uniform continuity는 하나의 함수에 대해서 하는 이야기이고, equicontinuity는 여러 함수에 대해서 하는 이야기 입니다. 둘 다 continuity의 확장입니다.

Definition. (고른연속) $f : X \to Y$ 가 **고른연속**이다. $\iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon.$$

직관적으로는 "함수의 기울기가 finite하다"라고 이해할 수 있습니다. 물론 미분가능하지 않으면 기울기를 생각한다는게 웃기긴 하지만... 미분은 불가능 하더라도

$$\sup \left\{ \left| \frac{f(x) - f(y)}{x - y} \right| : x, y \in X, x \neq y \right\}$$

를 생각해 볼 수는 있겠죠.

Definition. (동등연속) Family of functions $\mathscr{F} = \{f_\alpha\}_{\alpha \in I}$ 가 **동등연속**이다. $\iff$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_X(x, y) < \delta \implies d_Y(f_\alpha(x), f_\alpha(y)) < \epsilon \text{ for all } \alpha \in I.$$

동등연속이 아니다? 그렇다면 $\left| \frac{f_\alpha(x) - f_\alpha(y)}{x - y} \right|$ 를 원하는 만큼 크게 할 수 있다. 단, 기울기가 발산한다고 해서 동등연속인지 아닌지는 확인해봐야 한다.

Example. (f_n) where $f_n(x) = nx$ is not equicontinuous.

Problem 7.10 $\{x\}$ denotes the fractional part of x.

$$f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2}$$

$f(x)$ converges uniformly by the M-test. 따라서 함수항급수의 부분합이 리만적분 가능한지 확인하면 된다. 닫힌 구간에서는 불연속점이 유한개이므로, 부분합은 당연히 리만적분 가능하고 이에 따라 f도 리만적분 가능하다.

불연속점을 찾기 위해서는 각 항이 어디서 불연속인지 찾으면 되는데,

$$A_k = \left\{ \frac{b}{a} : a, b \in \mathbb{Z}, 1 \leq a \leq k \right\}$$

로 정의하면, f_k는 $\mathbb{R} \setminus A_k$ 에서 연속일 것이다. 따라서 $\mathbb{R} \setminus \mathbb{Q}$ 에서도 연속이고, f가 $\mathbb{R} \setminus \mathbb{Q}$ 에서 연속이다.

기약분수 $x = \frac{q}{p}$ 를 고정하자. $(p \geq 1)$ 그러면 x가 기약분수이므로

$$x \in A_p, A_{2p}, A_{3p}, \ldots$$

일 것이다. 이제 연속인지 살펴보면,

$$\lim_{h\to 0^+}(f_k(x+h)-f_k(x-h))=\sum_{n=1}^{k}\lim_{h\to 0^+}\left(\frac{\{n(x+h)\}-\{n(x-h)\}}{n^2}\right) \tag{$*$}$$

이다. 만약 $n=pl$ $(l\in\mathbb{Z})$ 이라고 하면, $(*)$의 분자는

$$\sum_{\substack{p|n\\1\le n\le k}}\frac{-1}{n^2}$$

이 된다. 만약 f가 연속이었다면, $k\to\infty$ 일 때 $(*)\to 0$ 이었어야 한다. 하지만 그렇지 않으므로, 불연속이다. $\mathbb{Q}$가 countably dense 임은 이미 알고 있다.

Problem 7.14 (Space-Filling Curve) Define $0\le f(t)\le 1$, $f(t)=f(t+2)$

$$f(t)=\begin{cases}0 & (t\in[0,\frac{1}{3}])\\ 1 & (t\in[\frac{2}{3},1])\end{cases}.$$

Also define, $\Phi(t)=(x(t),y(t))$ where

$$x(t)=\sum_{n=1}^{\infty}2^{-n}f(3^{2n-1}t),\quad y(t)=\sum_{n=1}^{\infty}2^{-n}f(3^{2n}t).$$

우선 M-test에 의해 $x(t),y(t)$가 고르게 수렴함을 안다. f가 연속이므로, Φ도 연속이다.

칸토어 집합 K은 compact, perfect, 길이가 양수인 구간을 포함하지 않음. 하지만 $\mu(K)>0$. 각 원소를 3진수로 썼을 때 모든 자리수가 0 또는 2.

Problem 7.18 F_n이 pointwise bounded 이고 equicontinuous임을 보이면 끝!

Problem 7.23 귀납법.

September 29th, 2022 (Practice)

삼각함수와 지수함수를 엄밀하게 construct 하는 방법.

Definition. Define the **exponential function** as

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad (z \in \mathbb{C}).$$

Check that

$$E(z)E(w) = E(z+w), \quad (z, w, \in \mathbb{C}).$$

Be careful when you switch infinite summations. We directly get

$$E(z)E(-z) = E(0) = 1, \quad (z \in \mathbb{C}).$$

This shows that $E(z) \neq 0$, $E(x) > 0$ even if $x < 0$. Also,

$$E(x) \to \infty \text{ as } x \to \infty \quad \text{ and } \quad E(x) \to 0 \text{ as } x \to -\infty.$$

Also check that $E'(x) = E(x) > 0$.

Definition. (Constant e) Define

$$e = E(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots$$

Then for $n \in \mathbb{N}$,

$$E(n) = E(\overbrace{1 + 1 + \cdots + 1}^{n \text{ times}}) = (E(1))^n = e^n.$$

Similar process can be done for $n \in \mathbb{Z}$. For $1/m \in \mathbb{Q}$,

$$E\left(\frac{1}{m}\right)^m = E\overbrace{\left(\frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}\right)}^{m \text{ times}} = E(1) = e,$$

thus $E\left(\frac{1}{m}\right) = \sqrt[m]{e}$. For $n/m \in \mathbb{Q}$,

$$E\left(\frac{n}{m}\right) = E\overbrace{\left(\frac{1}{m} + \frac{1}{m} + \cdots + \frac{1}{m}\right)}^{n \text{ times}} = E\left(\frac{1}{m}\right)^n = \left(\sqrt[m]{e}\right)^n = e^{n/m}.$$

For $r \in \mathbb{R}$,

$$e^r = \sup\{E(q) : q \in \mathbb{Q}, q < r\} = \inf\{E(q) : q \in \mathbb{Q}, q > r\} = \lim_{q \to r} E(q).$$

Using the monotonicity and continuity of $E(z)$, gives

$$E(x) = e^x, \quad (x \in \mathbb{R}).$$

Theorem 8.6 For every $n \in \mathbb{N}$,

$$\lim_{x \to \infty} \frac{x^n}{e^x} = 0.$$

Proof.

$$e^x > \frac{x^{n+1}}{(n+1)!} \implies \frac{x^n}{e^x} < \frac{(n+1)!}{x}.$$

Now take the limit $x \to \infty$.

The exponential function is strictly increasing and bijective, so it has an inverse function L.

Definition. Define the **logarithmic** function L as

$$L(E(x)) = x \text{ for } x \in \mathbb{R} \quad \text{or} \quad E(L(y)) = y \text{ for } y > 0.$$

We write

$$L(y) = \log y, \quad (y > 0).$$

Using the chain rule gives

$$L'(E(x))E'(x) = 1 \implies L'(y) = \frac{1}{y},$$

and

$$L(x) = \int_1^x \frac{1}{t}\, dt.$$

Check that

$$L(xy) = L(x) + L(y), \quad (x, y > 0)$$

$$E\left(\frac{1}{m}L(x)\right) = x^{1/m}, \quad E\left(\frac{n}{m}L(x)\right) = x^{n/m} \qquad (x > 0, n, m \in \mathbb{N})$$

Therefore for $\alpha \in \mathbb{Q}$,

$$x^\alpha = E(\alpha L(x)) = e^{\alpha \log x}, \quad (x > 0),$$

and differentiating gives

$$(x^\alpha)' = E(\alpha L(x)) \cdot \frac{\alpha}{x} = \alpha x^{\alpha - 1}.$$

Theorem. For every $\alpha > 0$,

$$\lim_{x \to \infty} \frac{\log x}{x^\alpha} = 0.$$

Proof. Take $0 < \epsilon < \alpha$ and $x > 1$. Then

$$\frac{\log x}{x^\alpha} = \frac{1}{x^\alpha}\int_1^x \frac{1}{t}\,dt < \frac{1}{x^\alpha}\int_1^x \frac{t^\epsilon}{t}\,dt = \frac{1}{x^\alpha}\frac{x^\epsilon - 1}{\epsilon} < \frac{1}{\epsilon x^{\alpha-\epsilon}}.$$

Now take the limit $x \to \infty$. We also have the series representation

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots .$$

Trigonometric Functions

Definition. Define

$$C(x) = \frac{E(ix) + E(-ix)}{2}, \quad S(x) = \frac{E(ix) - E(-ix)}{2i}.$$

From the series representation of $E(x)$, we see that

$$C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Also,

$$E(ix) = C(x) + iS(x), \quad (x \in \mathbb{R}).$$

and

$$|E(ix)|^2 = E(ix)\overline{E(ix)} = E(ix)E(-ix) = E(0) = 1$$

So we know that

$$C^2(x) + S^2(x) = 1$$

and from the power series representation,

$$S'(x) = C(x), \quad C'(x) = -S(x).$$

There exists positive numbers x such that $C(x) = 0$. (Proof in text) Let x_0 be the smallest positive number such that $C(x_0) = 0$.[61]

Definition. Define the number π by $\pi = 2x_0$.

We know that $C(x) > 0$ for $x \in [0, \pi/2]$, so $S(\pi/2) = 1$ (S is increasing on $(0, \pi/2)$). Thus

$$e^{2\pi i} = e^{i\frac{\pi}{2}\cdot 4} = i^4 = 1.$$

We want to show that E hs period $2\pi i$. If there exists $T \in (0, 2\pi)$ such that $e^{xi} = e^{(x+T)i}$,

[61] $C^{-1}(0) \cap \mathbb{R}^+$ 는 닫힌 집합. 아래로 유계이고 닫혀 있으므로, inf 가 존재.

$e^{Ti} = 1$. Then $e^{\frac{T}{4}i}$ is one of $\pm 1, \pm i$. Since $0 < T < 2\pi$, $0 < T/4 < \pi/2$. But $0 < C(\pi/4) < 1$, while

$$\Re(e^{\frac{\pi}{4}i}) = 1 \text{ or } 0,$$

which leads to a contradiction.

We can prove the trig identities by

$$C(x+y) + iS(x+y) = e^{i(x+y)} = e^{ix}e^{iy} = (C(x) + iS(x))(C(y) + iS(y))$$

and expanding the last expression.

Note that e^{ix} defined on $[0, 2\pi)$ is an injective function, and $|e^{ix}| = 1$. Consider the curve e^{ix} for $0 \leq x \leq \theta$. Then the length of this curve is

$$\int_0^\theta \left| \frac{d}{dx} e^{ix} \right| \, dx = \int_0^\theta \, dx = \theta.$$

This gives the definition of radian angles. On the unit circle with a point (x, y),

$$\cos\theta = x = \Re(e^{ix}) = C(x), \quad \sin\theta = y = \Im(e^{ix}) = S(x).$$

October 6th, 2022 (Practice)

Problem 8.1

$$f(x) = \begin{cases} e^{-1/x^2} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

First show that $\lim_{x\to 0} \dfrac{f(x)}{x^m} = 0$, from $\lim_{x\to\infty} \dfrac{x^k}{e^x} = 0$ for $k \geq 0$.

Next, prove by induction that

$$f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-1/x^2},$$

where $P_n(x)$ is a polynomial in x.

Now show that $f^{(n)}(0) = 0$ by induction.

Remark. 함수의 분류

$$\text{함수} \supseteq \text{잴 수 있는 함수} \supseteq \text{르벡 적분 가능 함수} \supseteq \text{리만 적분 가능 함수} \supseteq \text{연속 함수}$$

$$\text{연속 함수} \supseteq \text{미분 가능 함수} \supseteq C^k \supseteq \cdots \supseteq C^\infty \supseteq \text{해석 함수}$$

Problem 8.6 f is continuous, $f(x+y) = f(x)f(y)$.

(1) Show that $f(x)f(-x) = 1$ and that $f(x) > 0$ for all $x \in \mathbb{R}$.

(2) Let $g(x) = \log f(x)$, show that $g(x) = cx$ for some $c \in \mathbb{R}$. (Use continuity!)

Problem. Consider $z_1, \dots, z_n \in \mathbb{C}$. We want to maximize

$$|z_{a_1} + z_{a_2} + \cdots + z_{a_k}|$$

compared to $\sum_{i=1}^n |z_i|$.

Proof. Let $z_i = r_i e^{i\theta_i}$ and consider θ-direction, by using scalar projections. Define

$$(\theta\text{-direction sum}) = g(\theta) = \sum_{i=1}^n r_i \max\{\cos(\theta - \theta_i), 0\}$$

The desired value must be at least the 'mean' of g on $[0, 2\pi)$.

$$\frac{1}{2\pi}\int_0^{2\pi} g(\theta)\, d\theta = \frac{1}{2\pi}\sum_{i=1}^n r_i \int_0^{2\pi} \max\{\cos(\theta-\theta_i), 0\}\, d\theta = \frac{1}{\pi}\sum_{i=1}^n |z_i|$$

Problem 8.19 f is continuous with period 2π, α/π is irrational.

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt, \quad (x \in \mathbb{R}).$$

Remark. 만약 $\alpha/\pi \in \mathbb{Q}$ 이면 어느 순간부터 같은 점만 찍힐 것이다. 이 값이 무리수이기 때문에 $[-\pi,\pi]$ 를 거의 랜덤으로 채우는 것이다.

(1) First show for $f(x) = e^{inx}$.

(2) We can approximate f by a trigonometric polynomial P. Show that

$$\begin{aligned}\left|\frac{1}{N}\sum f - \frac{1}{2\pi}\int f\right| \le \left|\frac{1}{N}\sum f - \frac{1}{N}\sum P\right| + \left|\frac{1}{N}\sum P - \frac{1}{2\pi}\int P\right| \\ + \left|\frac{1}{2\pi}\int P - \frac{1}{2\pi}\int f\right| < \epsilon.\end{aligned}$$

Problem 8.17 f bounded and monotonic on $[-\pi,\pi)$. (Integrable)

(b) 푸리에 급수가 점별로 수렴한다!

October 20th, 2022 (Practice)

Problem 8.22 Newton's Binomial Theorem, 감마함수를 사용하면 복잡한 formula를 쉽게 표현할 수 있다.

Problem. $(X, \mathscr{F})$, $\mu : \mathscr{F} \to \overline{\mathbb{R}}$. Show that

$$\mu\text{: finite and subadditivity} \iff \text{countable additivity.}$$

Proof. $(\impliedby)$ Trivial by definition.

$(\implies)$ Let $\{E_i\}$ be a collection of disjoint sets in $\mathscr{F}$. It suffices to show that

$$\sum_{i=1}^{\infty} \mu(E_i) \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right).$$

Let $E = \bigcup_{i=1}^{\infty} E_i$. For any $k \in \mathbb{N}$, we observe that

$$\sum_{i=1}^{k} \mu(E_i) = \mu\left(\bigcup_{i=1}^{k} E_i\right) \leq \mu(E).$$

Let $k \to \infty$ to get the result.

Problem. Let S be a set containing x. Show that the following set functions $\mu : \mathcal{P}(S) \to \overline{\mathbb{R}}$ are countably additive.

$$(1) \quad \mu(A) = \begin{cases} 0 & (x \notin A) \\ 1 & (x \in A) \end{cases} \qquad (2) \quad \mu(A) = \begin{cases} |A| & (|A| < \infty) \\ \infty & \text{otherwise} \end{cases}$$

Proof. Suppose $\{A_i\}$ is a collection of disjoint sets.

(1) If no set contains x, the problem is trivial. If there exists a set A_i that contains x,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 = \sum_{i=1}^{\infty} \mu(A_i).$$

(2) If there exists a set with $|A_i| = \infty$, the equality holds. Now, for the case where all sets are finite but $\sum \mu(A_i) = \infty$, for all $K > 0$, there exists $i_k \in \mathbb{N}$ such that $\sum_{i=1}^{i_k} \mu(A_i) > K$. Therefore,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \mu\left(\bigcup_{i=1}^{i_k} A_i\right) = \sum_{i=1}^{i_k} \mu(A_i) > K$$

for all $K > 0$. Therefore $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \infty$.

October 27th, 2022 (Practice)

The gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

is the unique function that satisfies the following properties.

(1) $\Gamma(x+1) = x\Gamma(x)$.

(2) $\Gamma(1) = 1$.

(3) $\log\Gamma$ is convex.

Definition. (Borel Set) A is a **Borel set** if there exists a countable collection of open[62] sets $(A_n)_{n=1}^{\infty}$ such that $A = A_1 \circ A_2 \circ A_3 \circ \cdots$ where each $\circ$ is any of $\cup, \cap, \setminus$.

The collection $\mathfrak{B}$ of all Borel sets in $\mathbb{R}^p$ is a σ-ring. Borel σ-algebra has the following properties.

(1) $\mathbb{R}^p \in \mathfrak{B}$.

(2) $A \cup B \in \mathfrak{B}$ for any $A, B \in \mathfrak{B}$.

(3) $A \setminus B \in \mathfrak{B}$ for any $A, B \in \mathfrak{B}$.

Definition. (Borel σ-algebra) **Borel σ-algebra** is the smallest σ-algebra containing all open sets in $\mathbb{R}^p$.

Check that the two definitions are equivalent!

Problem. Show that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f' is measurable.

Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded, continuous and $f(x) \neq 0$. Show that $g(x) = \displaystyle\int_0^x f(t)\,dt$ is measurable.

[62] closed도 가능하다.

November 3rd, 2022

Question. 다음과 같이 감소하는 경우에도 MCT가 성립하는가?

(1) $f_n \geq f_{n+1} \geq \cdots$, and $|f_n| < M$. Provided that $\mu(X) < \infty$.

(2) $f_n \geq f_{n+1} \geq \cdots$, and $f_1 \in \mathcal{L}^1(\mu)$.

Proof. (1) Since f_1 is bounded and $\mu(X) < \infty$, $f_1 \in \mathcal{L}^1(\mu)$.

(2) Consider $g_n = f_1 - f_n$.

Question. Prove that Fatou's Lemma $\implies$ MCT.

Theorem 11.32 (Lebesgue's Dominated Convergence Theorem, LDCT)
$f_n \to f$ a.e. and $\exists\, g \in \mathcal{L}^1(\mu)$ such that $|f_n| \leq g$. Then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Remark. $f_n \to f$ a.e., $\exists\, g \in \mathcal{L}^1(\mu)$ such that $|f_n| \leq g_n$ and $\lim_{n\to\infty} \int |g_n - g| \, d\mu \to 0$. Then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Problem 11.4

Problem 11.6

Problem. Prove the following.

(1) $\displaystyle \lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin \frac{x}{n} \, dx = 0.$

(2) $\displaystyle \lim_{n\to\infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} \, dx = 0.$

(3) $\displaystyle \lim_{n\to\infty} \int_0^\infty n \sin \frac{x}{n} \frac{dx}{x(1 + x^2)} = \frac{\pi}{2}.$