HW Set 1 Solution.

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded (See Definition 7.19).

   Solution. Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of bounded functions, with bounds $|f_n(x)| \le M_n$ for every $x \in E$. Because $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy, for some $\varepsilon > 0$, say $\varepsilon = 1$, there exists $N \in \mathbb{N}$ s.t.

   $$n,\, m \ge N \Rightarrow |f_n(x) - f_m(x)| \le 1 \text{ for all } x \in E.$$

   From this, we get $|f_n(x)| \le |f_n(x) - f_N(x)| + |f_N(x)| \le M_N + 1$ and $M_n \le M_N + 1$ for every $n \ge N$. Now Let $M := \max\{M_1, \cdots, M_{N-1}, M_N + 1\}$, then clearly $|f_n(x)| \le M$ for any $x \in E$, $n \in \mathbb{N}$. $\square$

   Remark. Slightly modifying above argument, we can show that $(M_n)_{n=1}^{\infty}$ is a Cauchy sequence, so it is convergent sequence of non-negative reals and should be bounded.

2. If complex-valued (f_n) and (g_n) converge uniformly on a metric space E, prove that $(f_n + g_n)$ converges uniformly on E. If, in addition, (f_n) and (g_n) are sequences of bounded complex-valued functions, prove that $(f_n g_n)$ converges uniformly on E.

   Solution. Let f, g denote uniform limits of f_n and g_n respectively. Let $\varepsilon > 0$ be given, and choose $N_1, N_2 \in \mathbb{N}$ such that for all $x \in E$, $n \ge N_1$ implies $|f_n(x) - f(x)| \le \frac{\varepsilon}{2}$ and $n \ge N_2$ implies $|g_n(x) - g(x)| \le \frac{\varepsilon}{2}$. Then $n \ge \max\{N_1, N_2\}$ implies

   $$\begin{aligned} |(f_n + g_n)(x) - (f + g)(x)| &\le |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall x \in E, \end{aligned}$$

   showing that $f_n + g_n \to f + g$ uniformly.

   Now if we assume in addition that f_n, g_n are bounded, from Problem 1 we see that they are both uniformly bounded and thus there exises $M > 0$ for which $|f_n(x)|, |g_n(x)| \le M$, for all $n \in \mathbb{N}$ and $x \in E$. Then with $N \in \mathbb{N}$

chosen so that $n \geq N$ implies $|f_n(x) - f(x)|$, $|g_n(x) - g(x)| < \frac{\varepsilon}{2M}$ for all $x \in E$, we have

$$
\begin{aligned}
|(f_n g_n)(x) - (fg)(x)| &\leq |f_n(x)\{g_n(x) - g(x)\} + g(x)\{f_n(x) - f(x)\}| \\
&\leq |f_n(x)| \cdot |g_n(x) - g(x)| + |g(x)| \cdot |f_n(x) - f(x)| \\
&\leq M \cdot \frac{\varepsilon}{2M} + M \cdot \frac{\varepsilon}{2M} = \varepsilon
\end{aligned}
$$

for all $x \in E$ if $n \geq N$. Note that here we have used $|g(x)| = \lim_{n\to\infty} |g_n(x)| \leq M$. This proves that $f_n g_n \to fg$ uniformly $\square$

3. Let $f_n(x) = \begin{cases} 0 & \text{if } x < \frac{1}{n+1}, \\ \sin^2 \frac{\pi}{x} & \text{if } \frac{1}{n+1} \leq x \leq \frac{1}{n}, \\ 0 & \text{if } \frac{1}{n} < x. \end{cases}$

   Show that (f_n) converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

   Solution. It is clear that for any $x \leq 0$ or $x \geq 1$, $f(x) := \lim_{n\to\infty} f_n(x) = 0$. If $0 < x < 1$, then there exists $n_x \in N$ s.t. $\frac{1}{n_x} < x$, so $f_n(x) = 0$ if $n \geq n_x$ and this shows that $f(x) = 0$ for $0 < x < 1$ too.
   However, $f_n(\frac{1}{n+\frac{1}{2}}) = \sin^2(n + \frac{1}{2})\pi = 1$ holds for all $n \in N$ and this means $\text{Sup}_{x\in\mathbb{R}}|f_n(x) - f(x)| \geq 1$ for all $n \in N$. In other words, f_n doesn't converges to f uniformly.

   It is also clear that $F_n(x) := \sum_{k=1}^{n} f_n(x)$ converges to $F(x) := \sum_{k=1}^{\infty} f_n(x) = 0$ if $x \leq 0$ or $x \geq 1$. For $0 < x < 1$, choose (unique) $m_x \in \mathbb{N}$ s.t.
   $\frac{1}{m_x+1} \leq x < \frac{1}{m_x}$. Because $f_n(x)$ is zero at $(-\infty, \frac{1}{n+1}) \cup [\frac{1}{n}, \infty)$, we have $F(x) = \sum_{k=1}^{\infty} f_n(x) = f_{m_x}(x) = \sin^2 \frac{\pi}{x}$. Now we have

$$
F(x) = \begin{cases} 0 & (x \leq 0 \text{ or } x \geq 1) \\ \sin^2 \frac{\pi}{x} & (0 < x < 1) \end{cases}
$$

   and since each $f_n(x)$ is non-negative, $\sum_{k=1}^{\infty} f_n(x)$ absolutely converges. But this series doesn't converge uniformly, since $\text{Sup}_{x\in R}|F_{n+1}(x) - F_n(x)| =$

$\text{Sup}_{x\in\mathbb{R}}|f_{n+1}(x)| \geq 1$ for all $n \in \mathbb{N}$ and $(F_n)_{n=1}^{\infty}$ is not uniformly Cauchy.

(Alternative proof) After computing $F(x)$, Observe that $F(x)$ isn't continuous at $x = 0,\, 1$. If F_n converges uniformly to $F(x)$, since each F_n is continuous everywhere, $F(x)$ should be continuous everywhere but this is contradiction.

$\square$

4. Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Solution. Let $M > 0$ s.t. $I \subseteq [-M,\, M]$. Define $A_n := \sum\limits_{k=1}^{n} \frac{(-1)^n}{n^2}$, $B_n := \sum\limits_{k=1}^{n} \frac{(-1)^n}{n}$. We already know each A_n, B_n converges. Given $\varepsilon > 0$, take $N \in \mathbb{N}$ s.t. $m \geq n \geq N$ implies $|A_n - A_m| < \frac{\varepsilon}{2M^2}$, $|B_n - B_m| < \frac{\varepsilon}{2}$. Then we have

$$|\sum_{k=1}^{m}(-1)^k \frac{x^2+k}{k^2} - \sum_{k=1}^{n}(-1)^k \frac{x^2+k}{k^2}| = |\sum_{k=n+1}^{m}(-1)^k \frac{x^2+k}{k^2}|$$

$$= |x^2 \sum_{k=n+1}^{m} \frac{(-1)^k}{k^2} + \sum_{k=n+1}^{m} \frac{(-1)^k}{k}| \leq M^2|A_m - A_n| + |B_m - B_n|$$

$$\leq M^2 \cdot \frac{\varepsilon}{2M^2} + \frac{\varepsilon}{2} = \varepsilon.$$

However, $\sum\limits_{k=1}^{\infty}|(-1)^n \frac{x^2+k}{k^2}| \geq \sum\limits_{k=1}^{\infty} \frac{1}{k} = \infty$ and it does not converges absolutely at any points.

Alternative proof. since I is bounded interval, each x^2, $\sum\limits_{k=1}^{n}(-1)^n \frac{1}{n^2}$, $\sum\limits_{k=1}^{n}(-1)^n \frac{1}{n}$ are uniformly converging sequence of bounded functions. Now apply the result of Problem 2. $\square$

Supplement. Above series cannot be converge uniformly on any unbounded subset $A \subseteq \mathbb{R}$. If not, for some ε there exists $N \in \mathbb{N}$ s.t. $n \geq m \geq N$ implies $|\sum\limits_{k=n+1}^{m} (-1)^k \frac{x^2+k}{k^2}| < \varepsilon$. Specifically, choose $m = n+1 = N+1$ then $|\frac{x^2+(N+1)}{(N+1)^2}| < \varepsilon$ must holds for all $x \in A$. But we know $\lim\limits_{t\to\infty} \frac{t+(N+1)}{(N+1)^2} = \infty$ and A is unbounded by assumption, this is contradiction.

5. For $n = 1, 2, 3, \cdots$, and any $x \in \mathbb{R}$, define

$$f_n(x) = \frac{x}{1+nx^2}.$$

Show that (f_n) converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

Solution. From AM-GM inequality, $1 + nx^2 \geq 2\sqrt{1 \cdot nx^2} = 2\sqrt{n}|x|$ and $|f_n(x)| \leq \frac{1}{2\sqrt{n}}$. This means that $f_n \to 0$ uniformly.
We can also easily compute $(f_n(x))' = \frac{1-nx^2}{(1+nx^2)^2}$.
For $x = 0$, $\lim\limits_{n\to\infty} f_n'(x) = \lim\limits_{n\to\infty} 1 = 1$.
For $x \neq 0$, $\lim\limits_{n\to\infty} f_n'(x) = \lim\limits_{n\to\infty} \frac{1}{n} \frac{\frac{1}{n}-x^2}{(\frac{1}{n}+x^2)^2} = 0 \times (-\frac{1}{x^2}) = 0$.
Surely $f' = (0)' = 0$.

$\square$

6. Define $I(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$ Suppose that (x_n) is a sequence of distinct points of (a, b) and that $\sum |c_n|$ converges. Prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \ \ (a \leq x \leq b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Solution. Note that $|I(x-a)| \leq 1$, so that $|c_n I(x-x_n)| \leq |c_n|$ and $f(x)$ converges uniformly from Weierstrass M-test.
For continuity, define $A := \mathbb{R} \setminus \{x_n \,|\, n \in \mathbb{N}\}$. Since $I(x)$ is continuous on $\mathbb{R} \setminus \{0\}$, $I(x-x_n)$ is continuous on $\mathbb{R} \setminus \{x_n\}$ and surely on A. Then $\sum_{k=1}^{n} c_k I(x-x_k)$ is continuous on A and its uniform limit $\sum_{n=1}^{\infty} c_k I(x-x_k)$ is also continuous on A. □

Bonus. Let's show that $\sum_{n=1}^{\infty} c_k I(x-x_k)$ is discontinuous at $x = x_m$ if $c_m \neq 0$. Let $f_n(x) := \sum_{k=1}^{n} c_k I(x-x_k)$, $f(x) := \sum_{n=1}^{\infty} c_k I(x-x_k)$. Since f_n converges uniformly to f, $f_n(x_m+x)-f_n(x_m-x)$ also uniformly converges to $f(x_m+x)-f(x_m-x)$. For $n \geq m$, consider

$$\begin{aligned} &\lim_{x\to 0+} \{f_n(x_m+x) - f_n(x_m-x)\} \\ = &\sum_{k=1}^{n} \lim_{x\to 0+} \{I(x_m - x_k + x) - I(x_m - x_k - x)\} = c_m. \end{aligned}$$

From limit interchanging theorem, we have

$$\lim_{x\to 0+} \{f(x_m+x) - f(x_m-x)\} = \lim_{n\to\infty} \begin{cases} c_m & (n \geq m) \\ 0 & (n < m) \end{cases} = c_m \neq 0.$$

7. Let (f_n) be a sequence of continuous functions which converges uniformly to a function f on a metric space E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x) \tag{1}$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$.
Is the converse of this true?(i.e. if a sequence of continuous functions (f_n) on a metric space E satisfies (1) for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$, then does (f_n) necessarily converge uniformly to f?)

Solution. f is also continuous since it is a uniform limit of continuous

functions, so $f(x_n)$ converges to $f(x)$. For given $\varepsilon > 0$, Let $N_1,\, N_2 \in \mathbb{N}$ s.t. $n \geq N_1$ implies
$|f(x_n) - f(x)| < \frac{\varepsilon}{2}$, and $n \geq N_2$ implies $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$. Then we get

$$\begin{aligned}|f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \ \ \forall n \geq \max\{N_1,\, N_2\}.\end{aligned}$$

For counterexample, Let $f_n(x) = \begin{cases} |1 - |x-n|| & (n-1 \leq x \leq n+1) \\ 0 & \text{otherwise} \end{cases}$.
We can easily check that $f_n(x) = 0$ if $n \geq x+1$, so its pointwise limit is 0 everywhere. Because $|f_n(n) - 0| = 1$, it does not converge uniformly. Let $(x_n)_{n=1}^{\infty}$ be a sequence which converges to $x \in \mathbb{R}$. There exists $N \in \mathbb{N}$ s.t. $n \geq N$ implies $|x - x_n| < 1$. Then if $n \geq \max\{|x|+2,\, N\}$, $x_n + 1 \leq x + 2 \leq |x| + 2 \leq n$ so $f_n(x_n) = 0$ and this means that $f_n(x_n)$ converges to $0 = f(x)$. $\square$

Bonus. If E is compact metric space, then its converse holds. Assume otherwise. For constant sequence $x_n = x$ equation (1) must holds. this means that pointwise limit of $(f_n)_{n=1}^{\infty}$ is f. Let $\varepsilon > 0$ s.t. for any $N \in \mathbb{N}$ there exists a positive integer $n, m > N$ and $x \in E$ which satiefies $|f_m(x) - f_n(x)| \geq \varepsilon$. Then we can construct a strictly increasing sequence of positive integers $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ and $(x_n)_{n=1}^{\infty}$ s.t.
$|f_{a_n}(x_n) - f_{b_n}(x_n)| \geq \varepsilon$. Since E is compact, there exists a strictly increasing sequence of natural numbers $(c_n)_{n=1}^{\infty}$ s.t. x_{c_n} converges to some $x \in E$. Now we have $|f_{a_{c_n}}(x_{c_n}) - f_{b_{c_n}}(x_{c_n})| \geq \varepsilon$. Consider a sequence

$$y_n = \begin{cases} x_{c_1} & (n < a_{c_1}) \\ x_{c_m} & (a_{c_m} \leq n < a_{c_m}) \end{cases}, \ z_n = \begin{cases} x_{c_1} & (n < b_{c_1}) \\ x_{c_m} & (b_{c_m} \leq n < b_{c_m}) \end{cases}.$$

Then we can check that $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n = x$, $f_{a_{c_n}}(x_{c_n}) = f_{a_{c_n}}(y_{a_{c_n}})$, $f_{b_{c_n}}(x_{c_n}) = f_{b_{c_n}}(z_{b_{c_n}})$. Since $f_n(y_n)$, $f_n(z_n)$ converges to $f(x)$, their subsequences $f_{a_{c_n}}(x_{c_n}) = f_{a_{c_n}}(y_{a_{c_n}})$, $f_{b_{c_n}}(x_{c_n}) = f_{b_{c_n}}(z_{b_{c_n}})$ also converges to $f(x)$ but is contradiction to $|f_{a_{c_n}}(x_{c_n}) - f_{b_{c_n}}(x_{c_n})| \geq \varepsilon$.

8. Suppose (f_n) and (g_n) are real-valued functions defined on a metric space E, and

- $\sum f_n$ has uniformly bounded partial sums;
- $g_n \to 0$ uniformly on E;
- $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E. (*Hint*: Compare with Theorem 3.42.)

Solution. Define $F_n(x) := \sum_{k=1}^{n} f_k(x)$ then $(F_n)_{n=1}^{\infty}$ is uniformly bounded, say by $M > 0$. For given $\varepsilon > 0$, there exists $N \in N$ s.t. $n \geq N$ implies $|g_n(x)| \leq \frac{\varepsilon}{5M}$. Then

$$
\begin{aligned}
m \geq n \geq N \Longrightarrow |\sum_{k=n+1}^{m} f_k(x)g_k(x)| &= |\sum_{k=n+1}^{m} \{F_k(x) - F_{k-1}(x)\}(x)g_k(x)| \\
&|\sum_{k=n+1}^{m-1} F_k(x)\{g_k(x) - g_{k+1}(x)\} + F_m(x)g_m(x) - F_n(x)g_{n+1}(x)| \\
\leq |\sum_{k=n+1}^{m-1} |F_k(x)| \cdot |g_k(x) - g_{k+1}(x)| &+ |F_m(x)| \cdot |g_m(x)| + |F_n(x)| \cdot |g_{n+1}(x)| \\
&\leq M \sum_{k=n+1}^{m-1} |g_k(x) - g_{k+1}(x)| + M|g_m(x)| + M|g_{n+1}(x)| \\
&= M\{g_{n+1}(x) - g_m(x) + |g_m(x)| + |g_{n+1}(x)|\} \\
&\leq M \cdot \frac{\varepsilon}{5M} < \varepsilon
\end{aligned}
$$

and $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ converges uniformly on E. $\square$