

해석개론 및 연습 1 과제 #4

2017-18570 컴퓨터공학부 이성찬

1. It is trivial that x is continuous on \mathbb{R} , thus finite dimensional polynomials are also continuous on \mathbb{R} , since polynomials are composed of sums and products of x .

(1) $(x+1)^3$ is continuous on $(-1, 1)$, and never 0 in this interval. Thus $f(x) = 1/(x+1)^3$ is continuous on $(-1, 1)$.

Consider $x_n = 1/n - 1$. We immediately observe that $-1 < 1/n - 1 \leq 0$, and $\lim_{n \rightarrow \infty} x_n = -1$. Thus $\langle x_n \rangle$ is a Cauchy sequence in $(-1, 1)$. But $f(x_n) = n^3$, and the sequence diverges. Therefore $f(x)$ is not uniformly continuous on X .

(2) $x+3$ is continuous on $(0, \infty)$ and never 0 in this interval. Thus $f(x) = 1/(x+3)$ is continuous on $(0, \infty)$.

For given $\epsilon > 0$, Set $\delta = 9\epsilon$. For $x, y \in (0, \infty)$, we observe that

$$9 < xy + 3x + 3y + 9 = (x+3)(y+3)$$

then if $|x - y| < \delta$,

$$|f(x) - f(y)| = \left| \frac{1}{x+3} - \frac{1}{y+3} \right| = \frac{|x-y|}{(x+3)(y+3)} < \frac{9\epsilon}{9} = \epsilon$$

Thus $f(x)$ is uniformly continuous on X .

(3) $x^2 + 1$ is continuous on \mathbb{R} , and never 0. Thus $f(x) = 1/(x^2 + 1)$ is continuous on \mathbb{R} .

Given $\epsilon > 0$, set $\delta = \epsilon$. Since

$$x^2 y^2 + x^2 + y^2 + 1 - x - y = x^2 y^2 + \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \frac{1}{2} > 0$$

, the following directly follows.

$$\frac{x+y}{(x^2+1)(y^2+1)} < 1$$

Now, if $|x - y| < \delta = \epsilon$,

$$|f(x) - f(y)| = \left| \frac{1}{x^2+1} - \frac{1}{y^2+1} \right| = \frac{|x+y||x-y|}{(x^2+1)(y^2+1)} < |x-y| < \epsilon$$

Therefore $f(x)$ is uniformly continuous on X .

(4) $x^2 + 1$ is continuous on $(0, \infty)$, and \sqrt{x} is continuous on $(1, \infty)$. Thus their composition, $f(x) = \sqrt{x^2 + 1}$ is continuous on $(0, \infty)$.

It is trivial that

$$\sqrt{x^2 + 1} + \sqrt{y^2 + 1} - x - y > 0 \implies \frac{x+y}{\sqrt{x^2+1} + \sqrt{y^2+1}} < 1$$

so given $\forall \epsilon > 0$, if $|x - y| < \delta = \epsilon$, we have

$$|f(x) - f(y)| = \left| \sqrt{x^2 + 1} - \sqrt{y^2 + 1} \right| = \frac{(x+y)|x-y|}{\sqrt{x^2+1} + \sqrt{y^2+1}} < |x-y| < \epsilon$$

Therefore $f(x)$ is uniformly continuous on X .

2. (1) (\implies) Suppose $x_0 \in X' \setminus X$. Then there exists a sequence x_n in X that converges to x_0 . Because f is uniformly continuous and x_n is a Cauchy sequence, $f(x_n)$ is also a Cauchy sequence. Thus $\lim_{n \rightarrow \infty} f(x_n) = \alpha \in \mathbb{R}$ exists. Define the continuous extension g of f by setting $g(x_0) = \alpha$.

Now we must check if $g(x_0)$ is well-defined. For any two sequence $\langle x_n \rangle, \langle y_n \rangle$ that converge to x_0 , consider $\langle z_n \rangle = x_1, y_1, x_2, y_2, \dots$. It is trivial that $z_n \rightarrow x_0$. Since $\langle z_n \rangle$ is a Cauchy sequence, $\langle f(z_n) \rangle$ is also a Cauchy sequence by uniform continuity of f . Let its limit be γ . Then $\langle f(x_n) \rangle, \langle f(y_n) \rangle$ is a subsequence of $\langle f(z_n) \rangle$, thus they both must converge to γ .

Now we must check if g is continuous on \overline{X} . For $x_0 \in \overline{X}$, there exists a sequence x_n in X that converges to x_0 . Since $g(x_n) = f(x_n)$, $f(x_n)$ converges to $f(x_0)$ by continuity of f . Thus $g(x)$ is continuous extension of f to \overline{X} .

(\impliedby) Since X is bounded, there exists a closed ball B such that $X \subset B$. Because \overline{X} is the smallest closed set containing X , $\overline{X} \subset B$, and \overline{X} is bounded. We know that \overline{X} is closed, thus \overline{X} is compact. By Heine's Theorem, the continuous extension g of f is uniformly continuous on \overline{X} . Now f is uniformly continuous since it is defined on a subset of the domain of g .

- (2) Since $\overline{X} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 3\}$, define the continuous extension g of f by

$$g(x, y) = \begin{cases} f(x, y) & (x^2 + y^2 < 3) \\ \sqrt{x^{2020} + y^{2020} + x^2 + 1} & (x^2 + y^2 = 3, x = \sqrt{3} \cos \theta, y = \sqrt{3} \sin \theta) \end{cases}$$

(Such $\theta \in [0, 2\pi)$ exists) Now we show that $g(x, y)$ is continuous on \overline{X} . For $(x_0, y_0) \in X' \setminus X$, $x_0^2 + y_0^2 = 3$, set $x_0 = \sqrt{3} \cos \theta, y_0 = \sqrt{3} \sin \theta$. Define a sequence in X by

$$(x_n, y_n) = \left(\left(\sqrt{3} - \frac{1}{n} \right) \cos \theta, \left(\sqrt{3} - \frac{1}{n} \right) \sin \theta \right)$$

($x_n^2 + y_n^2 < 3$ can be easily checked), and it converges to (x_0, y_0) . It can be easily seen that $g(x_n, y_n) \rightarrow g(x_0, y_0)$, because $1/n \rightarrow 0$. Thus $g(x)$ is a continuous extension of f to \overline{X} and therefore f is uniformly continuous on X .

- (3) (\implies) Since f is uniformly continuous on (a, b) , there exists a continuous extension g of f to $[a, b]$. Since $g(x)$ is continuous at $x = a$, $\forall \epsilon > 0, \exists \delta$ s.t. $x \in \overline{X}, |x - a| < \delta \implies |g(x) - g(a)| < \epsilon$. Observe that $x \in \overline{X}, |x - a| < \delta$ is equivalent to $x \in [a, a + \delta)$. Thus $(a, a + \delta) \subset [a, b]$, and we have

$$x \in (a, a + \delta) \implies x \in [a, a + \delta) \implies |g(x) - g(a)| = |f(x) - g(a)| < \epsilon$$

Now by definition, $\lim_{x \rightarrow a^+} f(x) = g(a)$.

Similarly, since $g(x)$ is continuous on $x = b$, $\forall \epsilon > 0, \exists \delta$ s.t. $x \in \overline{X}, |x - b| < \delta \implies |g(x) - g(b)| < \epsilon$. Observe that $x \in \overline{X}, |x - b| < \delta$ is equivalent to $x \in (b - \delta, b]$. Thus $(b - \delta, b) \subset [a, b]$, and we have

$$x \in (b - \delta, b) \implies x \in (b - \delta, b] \implies |g(x) - g(b)| = |f(x) - g(b)| < \epsilon$$

Now by definition, $\lim_{x \rightarrow b^-} f(x) = g(b)$.

3. Let the domain be $X = \mathbb{R} \setminus \{0\}$.

- (1) $\lim_{x \rightarrow 0^-} \frac{\max\{x, 0\}}{x} = 0$.
 $\forall \epsilon > 0$, set $\delta = \epsilon$, $(-\delta, 0) \subset X$, and since $\max\{x, 0\} = 0$ for all x in this interval,
 $\frac{\max\{x, 0\}}{x} = 0 < \epsilon$.
 $\lim_{x \rightarrow 0^+} \frac{\max\{x, 0\}}{x} = 1$.
 $\forall \epsilon > 0$, set $\delta = \epsilon$, $(0, \delta) \subset X$, and since $\max\{x, 0\} = x$ for all x in this interval,
 $\left| \frac{\max\{x, 0\}}{x} - 1 \right| = 0 < \epsilon$.

Thus the wanted limit is -1 .

- (2) Given $\epsilon > 0$, set $\delta = \sqrt{\epsilon}$. $(0, \delta) \subset X$, and if $0 < x < \delta$, $0 < x^2 < \delta^2 = \epsilon$, then $0 < x^3/|x| < \epsilon$.
Since $x > 0$, $|x^3/|x| - 0| < \epsilon$. $\therefore \lim_{x \rightarrow 0^+} \frac{x^3}{|x|} = 0$.

4. (1) **True.** Since f, g are uniformly continuous on X , $\forall \epsilon > 0$, $\exists \delta$ s.t. if $\|x - y\| < \delta$ for all $x, y \in X \implies \|f(x) - f(y)\| < \epsilon/2$ and $\|g(x) - g(y)\| < \epsilon/2$. We immediately have

$$\|f(x) + g(x) - f(y) - g(y)\| \leq \|f(x) - f(y)\| + \|g(x) - g(y)\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

Thus $f + g$ is uniformly continuous on X .

- (2) **False.** (Counterexample) $f(x) = g(x) = x$ defined on \mathbb{R} . f, g are uniformly continuous, but x^2 is not uniformly continuous (proof in textbook).

- (3) **True.** We use the fact $\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}$. For a constant c , since cf is uniformly continuous if f is uniformly continuous, it is sufficient to show that $|f - g|$ is uniformly continuous, then the result directly follows by (1). Furthermore, because $f - g$ and $\|x\|$ are uniformly continuous, $(\forall \epsilon > 0$, set $\delta = \epsilon$. Then $\|x - y\| < \delta \implies \| \|x\| - \|y\| \| \leq \|x - y\| < \epsilon$) we show that their composition is uniformly continuous.

Claim. Suppose $f : X \rightarrow Y$, $g : Y \rightarrow Z$ are uniformly continuous. Then $g \circ f$ is uniformly continuous.

Proof. For given $\epsilon > 0$, $\exists \delta_1$ s.t. $\|x' - y'\| < \delta_1 \implies \|g(x') - g(y')\| < \epsilon$, for all $x', y' \in Y$.
For this δ_1 , $\exists \delta$ s.t. $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \delta_1$, for all $x, y \in X$. Then if $\|x - y\| < \delta \implies \|f(x) - f(y)\| < \delta_1 \implies \|g(f(x)) - g(f(y))\| < \epsilon$. Thus $g \circ f$ is uniformly continuous.

5. Define $g(x) = x^{2016} - f(x)$ on $[0, 1]$. Then since $x^{2016}, f(x)$ are continuous, its difference $g(x)$ is also continuous. Thus we have

$$g(0) = 0 - f(0) \leq 0 \leq 1 - f(1) = g(1)$$

and by IVT, there exists $x_0 \in [0, 1]$ s.t. $g(x_0) = 0$. For this x_0 , $f(x_0) = x_0^{2016}$.

6. (1) Suppose f is Hölder continuous. Given $\forall \epsilon > 0$, set $\delta = \left(\frac{\epsilon}{M}\right)^{1/\alpha}$. Since $y = x^\alpha$ ($\alpha > 0$) is increasing, for all $x, y \in X$, if $\|x - y\| < \delta$, $M \|x - y\|^\alpha < \epsilon$. By the Hölder continuity condition, $\|f(x) - f(y)\| < M \|x - y\|^\alpha < \epsilon$, and f is uniformly continuous on X .

(2) For fixed $x, y \in X$, suppose $x < y$. Given $N \in \mathbb{N}$, define x_0, \dots, x_N as follows.

$$x_0 = x, x_1 = x_0 + 1 \cdot \frac{y-x}{N}, \dots, x_i = x_0 + i \cdot \frac{y-x}{N}, \dots, x_N = y$$

Then we have

$$\|f(x) - f(y)\| \leq \sum_{i=0}^{N-1} \|f(x_i) - f(x_{i+1})\| \leq \sum_{i=0}^{N-1} M \|x_i - x_{i+1}\|^\alpha = M \frac{\|x - y\|^\alpha}{N^{\alpha-1}}$$

As $N \rightarrow \infty$, $0 \leq \|f(x) - f(y)\| \leq \lim_{N \rightarrow \infty} M \frac{\|x - y\|^\alpha}{N^{\alpha-1}} = 0$. Thus $\|f(x) - f(y)\| = 0$ for all $x, y \in X$. Thus $f(x) = f(0)$, merely a constant function.

7. (1) Let $y \in f(\bar{A})$. Then there exists $x_0 \in \bar{A}$ s.t. $f(x_0) = y$. Given $\epsilon > 0$, since $x_0 \in \bar{A}$, there exists $\delta > 0$ s.t. $N_{\mathbb{R}^m}(x_0, \delta) \cap A \neq \emptyset$. Take an element x from $N_{\mathbb{R}^m}(x_0, \delta) \cap A$. Since $x \in A$, $f(x) \in f(A)$, and since $x \in N_{\mathbb{R}^m}(x_0, \delta)$, $f(x) \in N_{\mathbb{R}^n}(f(x_0), \epsilon)$ by continuity of f . Therefore we have $f(x) \in N_{\mathbb{R}^n}(f(x_0), \epsilon) \cap f(A) \neq \emptyset$, and $f(x_0) = y \in \overline{f(A)}$. $f(\bar{A}) \subset \overline{f(A)}$.

(2) False. Consider $f(x) = \frac{1}{x^2+1}$. f is continuous. Set $A = (0, \infty)$. Then $\bar{A} = [0, \infty)$, $f(\bar{A}) = (0, 1)$, while $\overline{f(A)} = [0, 1]$.

8. Define $f_A(x) = \text{dist}(\{x\}, A)$, $f_B(x) = \text{dist}(\{x\}, B)$. Then

$$f(x) = \frac{f_B(x)}{f_A(x) + f_B(x)}$$

is a function that satisfies the requirements. The following should be checked.

(i) Is $f : \mathbb{R}^d \rightarrow [0, 1]$?

The value of the dist function is always greater than equal to 0, and $f(x) \leq 1$ is trivial.

Also note that the denominator is never 0 since A, B are disjoint.

(ii) $x \in A \implies f(x) = 1$?

If $x \in A$, $f_A(x) = 0$, $f_B(x) > 0$. ($A \cap B = \emptyset$) Thus $f(x) = \frac{f_B(x)}{0+f_B(x)} = 1$.

(iii) $x \in B \implies f(x) = 0$?

If $x \in B$, $f_B(x) = 0$, $f_A(x) > 0$. ($A \cap B = \emptyset$) Thus $f(x) = \frac{0}{f_A(x)+0} = 0$.

(iv) Are f_A, f_B continuous ?

We will show that f_A is uniformly continuous. For $x, y \in \mathbb{R}^d$, and any $a \in A$, the following holds by triangle inequality.

$$\text{dist}(\{x\}, \{a\}) \leq \text{dist}(\{x\}, \{y\}) + \text{dist}(\{y\}, \{a\})$$

Taking infimum over all $a \in A$ gives

$$\text{dist}(\{x\}, A) \leq \text{dist}(\{x\}, \{y\}) + \text{dist}(\{y\}, A)$$

Switching roles for x, y will give us another inequality, and combining it with the above inequality will give us

$$|f_A(x) - f_A(y)| = |\text{dist}(\{x\}, A) - \text{dist}(\{y\}, A)| \leq \text{dist}(\{x\}, \{y\}) = \|x - y\|$$

Therefore, for all $\epsilon > 0$, setting $\delta = \epsilon$ will make f_A satisfy the definition of uniform continuity. Thus f_A is (uniformly) continuous. The proof is symmetric for f_B .

(v) Is f continuous ?

Since the denominator is never 0 and f_A, f_B are continuous, f is continuous.

9. (\implies) Define a function $F : X \rightarrow X \times \mathbb{R}$ as $F(x) = (x, f(x))$. Then $F(X) = E$. To show compactness of E , we will show that F is continuous.

For $x \in X$, consider a sequence $\langle x_n \rangle$ in X converging to x . (X is closed) By the continuity of f , $\langle f(x_n) \rangle$ converges to $f(x)$. Therefore $\langle F(x_n) \rangle$ converges to $(x, f(x)) = F(x)$. Thus F is continuous, and because X is compact, its image $E = F(X)$ is also compact.

(\impliedby) Since E is compact, it is closed, and consider a sequence $(x_n, f(x_n))$ in E that converges to $(x, f(x))$. Then x_n is a sequence in X , and if $x_n \rightarrow x$, $x \in X$. (X is closed) Then we know that $f(x_n)$ must converge to $f(x)$ (*), which implies continuity of f on X .

(*) If $(x_n, f(x_n)) \rightarrow (x, f(x))$ and $x_n \rightarrow x$, then $f(x_n) \rightarrow f(x)$.

10. We first prove the following inequality.

Claim. For convex function $f : (a, b) \rightarrow \mathbb{R}$ and $a < x < y < z$,

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(y)}{z - y}$$

Proof. Since f is convex, for $t \in (0, 1)$,

$$f(tx + (1 - t)z) \leq tf(x) + (1 - t)f(z)$$

, and set $y = tx + (1 - t)z$. Multiply $z - x$ on both sides, then

$$(z - x)f(y) \leq t(z - x)f(x) + (1 - t)(z - x)f(z) = (z - y)f(x) + (y - x)f(z)$$

Rearranging the terms gives

$$(z - y)(f(y) - f(x)) \leq (y - x)(f(z) - f(y))$$

, which directly gives the inequality.

Suppose $x \in (a, b)$. Since (a, b) is open, select real numbers s.t. $x_0 < x_1 < x, y < x_2 < x_3$, and define $C = \max\left\{\frac{|f(x_1) - f(x_0)|}{x_1 - x_0}, \frac{|f(x_3) - f(x_2)|}{x_3 - x_2}\right\}$. Given $\epsilon > 0$, choose $\delta = \min\{\epsilon/C, x_2 - x_1\}$. If $x > y$, by **Claim**,

$$\frac{f(x) - f(y)}{x - y} \leq \frac{f(x_2) - f(x)}{x_2 - x} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

and

$$\frac{f(x) - f(y)}{x - y} \geq \frac{f(y) - f(x_1)}{y - x_1} \geq \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

, therefore

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq C = \max\left\{ \frac{|f(x_1) - f(x_0)|}{x_1 - x_0}, \frac{|f(x_3) - f(x_2)|}{x_3 - x_2} \right\}$$

and the inequality above can be shown similarly for $x < y$.

Therefore if $|x - y| < \delta$, $|f(x) - f(y)| \leq C|x - y| \leq C \frac{\epsilon}{C} = \epsilon$. $f(x)$ is continuous.

11. Since f is continuous, f has a maximum at $\alpha_x \in [a, x]$, by EVT. Then $f^*(x) = f(\alpha_x)$.

If $x_1 < x_2$, $f^*(x_2) = \sup\{f(y) : y \in [a, x_2]\} = \max\{f(\alpha_{x_1}), \sup\{f(y) : y \in [x_1, x_2]\}\} \geq f(\alpha_{x_1}) = f^*(x_1)$. Therefore f^* is increasing.

For continuity, let $x_0 \in X = [a, b]$, and let $f^*(x_0) = M (= \sup\{f(y) : y \in [a, x_0]\})$.

Case 1. $f(x_0) < M$

Since f is continuous, for $\epsilon = M - f(x_0)$, there exists $\delta_1 > 0$ s.t. $|x - x_0| < \delta_1, x \in X \implies |f(x) - f(x_0)| < \epsilon$. Then $f(x) - f(x_0) < \epsilon \implies f(x) < M$, in this interval. Thus $|x - x_0| < \delta_1, x \in X \implies f^*(x) = M$.

Case 2. $f(x_0) = M$

For all $\epsilon > 0$, $\exists \delta_2 > 0$ s.t. $|x - x_0| < \delta_2, x \in X \implies |f(x) - f(x_0)| < \epsilon \implies M - \epsilon < f(x) < M + \epsilon$. Therefore we have $M - \epsilon < f^*(x) < M + \epsilon$, by the continuity of f . Now we have $|f^*(x) - f^*(x_0)| < \epsilon$.

For both cases, setting $\delta = \min\{\delta_1, \delta_2\}$ will give us $|f^*(x) - f^*(x_0)| < \epsilon$. Thus f^* is continuous.

12. Consider

$$f(x) = \frac{1}{x} \sin \frac{1}{x} \quad x \in (0, 1]$$

Since $1/x$ and $\sin x$ are continuous on $(0, 1]$, \mathbb{R} respectively, f is also continuous on $(0, 1]$. Suppose f attains maximum value M at x_0 . It is trivial that $M > 0$, because $M \geq f(2/\pi) = \pi/2 > 0$. From this result we also know that $\sin \frac{1}{x_0} > 0$.

Then for $x' = (1/x_0 + 2\pi)^{-1} (< 1)$, $f(x') = \left(\frac{1}{x_0} + 2\pi\right) \sin\left(\frac{1}{x_0} + 2\pi\right) = M + 2\pi \sin \frac{1}{x_0} > M$, contradicting the choice of M . Therefore f has no maximum. Similarly, if f attains minimum value m at x_1 , we know that $\sin \frac{1}{x_1} < 0$, and setting $x' = (1/x_1 + 2\pi)^{-1}$ will let us arrive at a contradiction, which contradicts the choice of m . Thus f is continuous on $(0, 1]$ but has no minimum or maximum.