해석개론 및 연습 2 과제 #2

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1. Suppose that f is not a constant function. Then there exists $\alpha, \beta \in [0, \infty)$ such that $f(\alpha) \neq f(\beta)$. Now we show by contradiction that $\{f_n\}$ cannot be equicontinuous on $[0, 1]$. Set $\epsilon = |f(\alpha) - f(\beta)|$. There should be a $\delta > 0$ such that for all $n \in \mathbb{N}$,

$$x, y \in [0, 1], |x - y| < \delta \implies |f_n(x) - f_n(y)| < \epsilon.$$

However, we can always choose $N \in \mathbb{N}$ large enough ($N > \alpha, \beta$) so that $|\alpha - \beta| / N < \delta$. For $x = \alpha/N, y = \beta/N$, $x, y \in [0, 1]$ and $|x - y| = |\alpha - \beta| / N < \delta$ but

$$|f_N(x) - f_N(y)| = \left| f_N\left(\frac{\alpha}{N}\right) - f_N\left(\frac{\alpha}{N}\right) \right| = |f(a) - f(b)| = \epsilon.$$

Thus such $\delta > 0$ cannot exist and $\{f_n\}$ cannot be equicontinuous, leading to a contradiction. f has to be a constant function.

2. Let $\epsilon > 0$ be given. Equicontinuity of $\{f_n\}$ lets us choose $\delta > 0$ such that

$$x, y \in K, d(x, y) < \delta \implies |f_n(x) - f_n(y)| < \frac{\epsilon}{3} \tag{$*$}$$

for all $n \in \mathbb{N}$. The set $\displaystyle\bigcup_{x \in K} B_\delta(x)$ is an open cover of K, and there exists a finite subcover because K is a compact set.

$$\exists\, x_1, x_2, \dots, x_r \in K \text{ such that } K \subseteq \bigcup_{i=1}^{r} B_\delta(x_i).$$

For all $x \in K$, there exists $s \leq r$ such that $x \in B_\delta(x_s)$. By $(*)$,

$$|f_n(x) - f_n(x_s)| < \frac{\epsilon}{3}, \quad (\forall n \in \mathbb{N})$$

Also since $\{f_n\}$ converges pointwise, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$,

$$|f_n(x_s) - f_m(x_s)| < \frac{\epsilon}{3}.$$

Therefore, for $n, m \geq N$,

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_s)| + |f_n(x_s) - f_m(x_s)| + |f_m(x_s) - f_m(x)| < \frac{\epsilon}{3} \cdot 3 = \epsilon$$

for all $x \in K$. f_n converges uniformly on K.

(Counterexample) Take $f_n(x) = x/n$ over $\mathbb{R}$.

3. We first prove the following claim.

Claim. *Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)\,dx = 0$. Then $f(x) = 0$ on $[a,b]$.*

Proof. Suppose there exists some $x_0 \in [a,b]$ such that $f(x_0) > 0$. Since f is continuous, we can find $\delta > 0$ such that

$$x \in [x_0 - \delta, x_0 + \delta] \implies |f(x) - f(x_0)| < \frac{f(x_0)}{2}.$$

We can conclude that $f(x) > \dfrac{f(x_0)}{2}$ on $[x_0 - \delta, x_0 + \delta]$. Then,

$$0 = \int_a^b f(x)\,dx \geq \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx \geq \int_{x_0-\delta}^{x_0+\delta} \frac{f(x_0)}{2}\,dx = \delta f(x_0) > 0,$$

which is a contradiction. Therefore $f(x) = 0$ on $[a,b]$.

Since f is continuous on $[0,1]$ and bounded (Extreme Value Theorem), there exists a sequence of polynomials $\{P_n\}$ such that $P_n \to f$ uniformly on $[0,1]$. Let $P_n(x) = \sum_{k=0}^{\deg P_n} p_{n,k}x^k$. Then

$$\int_0^1 f(x)P_n(x)\,dx = \int_0^1 f(x) \sum_{k=0}^{\deg P_n} p_{n,k}x^k\,dx = \sum_{k=0}^{\deg P_n} p_{n,k} \int_0^1 f(x)x^k\,dx = 0.$$

Note that P_n is uniformly bounded by its uniform convergence. Since P_n and f are both bounded, $f(x)P_n(x)$ converges uniformly on $[0,1]$. Therefore,

$$\int_0^1 f(x)^2\,dx = \int_0^1 \lim_{n\to\infty} f(x)P_n(x)\,dx = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)\,dx = 0.$$

(Uniform convergence lets us switch the order of integration and limit.) By the claim above, $f(x)^2 = 0$ on $[0,1]$, which implies $f(x) = 0$ on $[0,1]$.

4. (a) $\{f_n\}$ is uniformly bounded and $\mathbb{Q}$ is countable. By Theorem 7.23, there exists a subsequence $\{f_{m_i}\}$ of $\{f_n\}$ such that $\{f_{m_i}(r)\}$ converges for $\forall r \in \mathbb{Q}$. Now let $f(x) = \sup_{r \leq x} f(r)$.

Suppose that f is continuous at $x_0 \in \mathbb{R}$. Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{\epsilon}{2}.$$

Take $r, s \in \mathbb{Q}$ from the interval $(x_0 - \delta, x_0 + \delta)$ such that $r \leq x_0 \leq s$. Since f_n is increasing for all $n \in \mathbb{N}$, f should also be increasing. Then

$$f(x_0) - \frac{\epsilon}{2} < f(r) \leq f(x_0) \leq f(s) < f(x_0) + \frac{\epsilon}{2}. \qquad (*)$$

Since $f_{m_i}(r) \to f(r)$ and $f_{m_i}(s) \to f(s)$, set i large enough so that

$$|f_{m_i}(r) - f(r)| < \frac{\epsilon}{2}, \quad |f_{m_i}(s) - f(s)| < \frac{\epsilon}{2}.$$

Combining these inequalities with $(*)$ gives

$$f(x_0) - \epsilon < f(r) - \frac{\epsilon}{2} < f_{m_i}(r), \quad f_{m_i}(s) < f(s) + \frac{\epsilon}{2} < f(x_0) + \epsilon.$$

Since $f_{m_i}(r) \le f_{m_i}(x_0) \le f_{m_i}(s)$, we have that for large i,

$$f(x_0) - \epsilon < f_{m_i}(x_0) < f(x_0) + \epsilon \implies |f_{m_i}(x_0) - f(x_0)| < \epsilon.$$

Thus $\{f_{m_i}\}$ converges at points of continuity.

For a monotone function on $\mathbb{R}$, the set of discontinuities D is at most countable. We can apply Theorem 7.23 once again on $\{f_{n_i}\}$ to get a subsequence $\{f_{m_k}\}$ that converges for all $x \in D$.

Now we modify the definition of $f(x)$ so that for $x \in D$,

$$f(x) = \lim_{k\to\infty} f_{n_k}(x).$$

Since $\{f_{n_k}\}$ is a subsequence of $\{f_{m_i}\}$, $\{f_{n_k}\}$ also converges for all $x \in \mathbb{R} \setminus D$, so we can write

$$f(x) = \lim_{k\to\infty} f_{n_k}(x), \quad x \in \mathbb{R}.$$

(b) Let f be a continuous function on a compact set K. Then f is uniformly continuous.[1] Let $\epsilon > 0$ be given.

Take any $t \in K$. Since f is uniformly continuous, we can find $\delta > 0$ such that

$$x \in (t-\delta, t+\delta) \implies f(t) - \frac{\epsilon}{2} < f(x) < f(t) + \frac{\epsilon}{2}. \tag{$\star$}$$

Let $g_k = f_{n_k}$. Since g_k is increasing,

$$g_k\left(t - \frac{\delta}{2}\right) \le g_k(t) \le g_k\left(t + \frac{\delta}{2}\right). \tag{$\spadesuit$}$$

Also because $g_k \to f$ pointwise, there exists large enough $N \in \mathbb{N}$ such that

$$k \ge N \implies \left|g_k\left(t - \frac{\delta}{2}\right) - f\left(t - \frac{\delta}{2}\right)\right| < \frac{\epsilon}{2} \text{ and } \left|g_k\left(t + \frac{\delta}{2}\right) - f\left(t + \frac{\delta}{2}\right)\right| < \frac{\epsilon}{2}.$$

For $k \ge N$,

$$f\left(t - \frac{\delta}{2}\right) - \frac{\epsilon}{2} < g_k\left(t - \frac{\delta}{2}\right) \text{ and } g_k\left(t + \frac{\delta}{2}\right) < f\left(t + \frac{\delta}{2}\right) + \frac{\epsilon}{2}.$$

From $(\spadesuit)$,

$$f\left(t - \frac{\delta}{2}\right) - \frac{\epsilon}{2} < g_k(t) < f\left(t + \frac{\delta}{2}\right) + \frac{\epsilon}{2}.$$

Since $t - \delta/2, t + \delta/2 \in (t-\delta, t+\delta)$, by $(\star)$,

$$f(t) - \epsilon < f\left(t - \frac{\delta}{2}\right) - \frac{\epsilon}{2} < g_k(t) < f\left(t + \frac{\delta}{2}\right) + \frac{\epsilon}{2} < f(t) + \epsilon.$$

Thus for $k \ge N$, $|g_k(t) - f(t)| < \epsilon$ $(\forall t \in K)$. $f_{n_k} \to f$ uniformly on K.

[1] Heine-Cantor Theorem.

5. Knowing that $\|f\|_2 \geq 0$ for any $f \in \mathcal{R}(\alpha)$, we shall prove that for $f, g \in \mathcal{R}(\alpha)$,

$$\left(\|f\|_2 + \|g\|_2\right)^2 \geq \|f+g\|_2^2 . \tag{$\star$}$$

Let $I = [a, b]$.

$$\begin{aligned}
\left(\|f\|_2 + \|g\|_2\right)^2 - \|f+g\|_2^2 &= \|f\|_2^2 + \|g\|_2^2 + 2\,\|f\|_2\,\|g\|_2 + \|f+g\|_2^2 \\
&= \int_I |f|^2 \; d\alpha + \int_I |g|^2 \; d\alpha + 2\left\{\int_I |f|^2 \; d\alpha \cdot \int_I |g|^2 \; d\alpha\right\}^{1/2} \\
&\qquad - \int_I |f|^2 \; d\alpha - \int_I |g|^2 \; d\alpha - 2\int_I |fg| \; d\alpha \\
&= 2\left[\left\{\int_I |f|^2 \; d\alpha \cdot \int_I |g|^2 \; d\alpha\right\}^{1/2} - \int_I |fg| \; d\alpha\right]. \qquad (*)
\end{aligned}$$

Similarly, the integrals are all positive, so we instead prove the following.

Claim. $\displaystyle\int_I |f|^2 \; d\alpha \cdot \int_I |g|^2 \; d\alpha - \left(\int_I |fg| \; d\alpha\right)^2 \geq 0.$

Proof. Consider the function $(t\,|f| - |g|)^2$, where t is some real constant. Integrating this function with respect to α should be non-negative. Therefore,

$$\int_I \left(t\,|f| - |g|\right)^2 d\alpha = t^2 \int_I |f|^2 \; d\alpha - 2t\int_I |fg| \; d\alpha + \int_I |g|^2 \; d\alpha \geq 0.$$

Treating the above expression as a quadratic of t, the discriminant should be non-positive.

$$D/4 = \left(\int_I |fg| \; d\alpha\right)^2 - \int_I |f|^2 \; d\alpha \cdot \int_I |g|^2 \; d\alpha \leq 0,$$

giving the desired result. Equality holds when f, g are linearly dependent.

Hence $(*) \geq 0$, proving $(\star)$. For $f, g, h \in \mathcal{R}(\alpha)$, replacing f by $f - g$, g by $g - h$ gives

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2 ,$$

proving the original inequality.

6. For any given partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a, b]$, define

$$g(t) = \frac{x_i - t}{x_i - x_{i-1}} f(x_{i-1}) + \frac{t - x_{i-1}}{x_i - x_{i-1}} f(x_i)$$

for $t \in [x_{i-1}, x_i]$. $g(t)$ is piecewise continuous (linear) and for points in the partition except for the both ends, the left/right limit of $g(t)$ at $t = x_i$ are the same. Thus $g(t)$ is continuous on $[a, b]$.

Since $f \in \mathcal{R}(\alpha)$, f is bounded on $[a, b]$. Let $|f| < M$ for some positive real M.

For any interval $I_i = [x_{i-1}, x_i] \subseteq [a, b]$, $\inf_{x \in I_i} f(x) \leq f(x) \leq \sup_{x \in I_i} f(x)$ by definition. Also, since $g(x)$ is linear,

$$\inf_{x \in I_i} f(x) \leq \min\{f(x_{i-1}), f(x_i)\} \leq g(x) \leq \max\{f(x_{i-1}), f(x_i)\} \leq \sup_{x \in I_i} f(x).$$

Therefore,

$$0 \leq |f(x) - g(x)| \leq \sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x) \implies |f(x) - g(x)|^2 \leq \big(\sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x)\big)^2.$$

Additionally,

$$\sup_{x \in I_i} f(x) - \inf_{x \in I_i} f(x) < 2M.$$

Now let $\epsilon > 0$ be given. Since f is integrable, we can choose a partition P of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon^2}{2M}.$$

Then we can define a continuous function $g(x)$ as above, and

$$\begin{aligned}
\|f - g\|_2^2 &\leq U(P, |f - g|^2, \alpha) \\
&= \sum_{i=1}^{n} \sup_{x \in [x_{i-1}, x_i]} |f(x) - g(x)|^2 \big(\alpha(x_i) - \alpha(x_{i-1})\big) \\
&\leq \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right)^2 \big(\alpha(x_i) - \alpha(x_{i-1})\big) \\
&\leq 2M \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) \big(\alpha(x_i) - \alpha(x_{i-1})\big) \\
&= 2M\big(U(P, f, \alpha) - L(P, f, \alpha)\big) < 2M \cdot \frac{\epsilon^2}{2M} = \epsilon^2.
\end{aligned}$$

Therefore $\|f - g\|_2 < \epsilon$, as desired.