HW Set 2. (Due day: September 24, 23:59pm)

1. Suppose f is a real valued continuous function on $\mathbb{R}$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \cdots$, and (f_n) is equicontinuous on $[0, 1]$. Show that f is a constant on $[0, \infty)$.

2. Suppose (f_n) is an equicontinuous sequence of functions on a compact set K, and (f_n) converges pointwise on K. Prove that (f_n) converges uniformly on K. Take a counterexample(without proof) when K is not compact.
   Hint: Review the proof of theorem 7.25.

3. If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)x^n \, dx = 0 \quad \text{for all } n = 0, 1, 2, \cdots ,$$

   prove that $f(x) = 0$ on $[0, 1]$.

   Hint: The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem (Theorem 7.26) to show that $\int_0^1 f^2(x) \, dx = 0$.

4. Assume that (f_n) is a sequence of monotonically increasing functions on $\mathbb{R}$ with $0 \leq f_n(x) \leq 1$ for all x and all n.

   (a) Prove that there is a function f and a sequence (n_k) such that

$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$

   for every $x \in \mathbb{R}$. (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem*.)

   (b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets.

   Hint: (i) Some subsequence (f_{n_l}) converges at all rational points r, say, to $f(r)$. (ii) Define $f(x)$, for any $x \in \mathbb{R}$, to be $\sup f(r)$, the sup being taken over all $r \leq x$. (iii) Show that $f_{n_l}(x) \to f(x)$ at every x at which f is continuous. (This is where monotonicity is strongly used.) (iv) A subsequence of (f_{n_l}) converges at every point of discontinuity of f since there are at most countably many such points. This proves (a). To prove (b), modify your proof of (iii) appropriately.

5. Recall that $\mathcal{R}(\alpha)$ denotes the family of Riemann-Stieltjes integrable functions with respect to α over $[a, b]$.

   Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 \, d\alpha \right\}^{1/2}.$$

   Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

   as a consequence of the Schwarz inequality (as in the proof of Theorem 1.37).

6. With the notations of 5, suppose $f \in \mathcal{R}(\alpha)$ and $\epsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \epsilon$.
   Hint: Let $P = \{x_0, \cdots, x_n\}$ be a suitable partition of $[a, b]$, define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

   if $x_{i-1} \leq t \leq x_i$.