Modern Algebra I

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Part I

Groups and Subgroups

Introduction

Section 1. Introduction and Examples

수 체계의 확장.

$$\mathbb{N} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{R} \longrightarrow \mathbb{C}$$

학부 현대대수학의 최종 목표는 다음 정리를 증명하는 것.

Theorem. n차 방정식의 일반해는 존재하지 않는다. $(n \geq 5)$

March 6th, 2023

추상적인 개념을 공부하는 이유는 구체적인 example 때문이다. Example이 곧 motivation이 되기 때문이다.[1]

- Complex numbers $\mathbb{C}$. $a + bi \in \mathbb{C}$, $a, b \in \mathbb{R}$ and $+, \times$ defined on them.
- The unit circle $U = \{a + bi : a^2 + b^2 = 1, a, b \in \mathbb{R}\} = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$. U is not closed under addition, but closed under multiplication.

  Note that the above two representations are intrinsically the 'same' representations of the unit circle. We write

  $$(U, \cdot) \approx \big([0, 2\pi), +_{2\pi}\big)$$

  and say that these two are **isomorphic**.

[1] 이인석 교수님: 추상화는 구체적인 example이 없으면 의미 없다.

- **Roots of Unity**: $U_n = \{\xi \in \mathbb{C} : \xi^n = 1\}$. We can see that

  $$\xi_k = e^{\frac{2\pi k}{n} i}, \quad (k = 0, \dots, n-1).$$

  When we multiply two elements for example, we do the following.

  $$\xi_1 \cdot \xi_2 = e^{\frac{2\pi}{n} i} \cdot e^{\frac{4\pi}{n} i} = e^{\frac{6\pi}{n} i} = \xi_3$$

  If we look closely, we see that we have transformed elements of $(U, \cdot)$ to $\big([0, 2\pi), +_{2\pi}\big)$. This can be done because the two sets are **isomorphic**!

Section 2. Binary Operations

March 8th, 2023

Definition. (Binary Operation) A **binary operation** $*$ on a set S is defined as a function

$$*: S \times S \to S$$

We write $a * b$ instead of $*(a, b)$.

Remark. If you consider the number systems like $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, we already know operations defined on them. We are just naming them *binary operations.* The examples came first, and the definitions came afterwards. We can also see that the definitions are really useful, and it will serve as a tool for formalizing our theory.

Example. Examples of binary operations.

(1) Addition $+$ and multiplication $\cdot$ on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

(2) Subtraction $-$ is not a binary operation on $\mathbb{N}$. We extend $\mathbb{N}$ to $\mathbb{Z}$, so that $-$ is a binary operation on $\mathbb{Z}$.

(3) The set of functions $f : \mathbb{R} \to \mathbb{R}$, and $+, -, \times, \circ$ defined on it.

Definition. (Closure) For a set S and a binary operation $*$ on S, suppose that $H \subset S$. We restrict the domain of $*$ to $H \times H$. Then

$$* |_{H\times H} : H \times H \to S.$$

If the image of $* |_{H\times H} \subset H$, then we say that H **is closed under** $*$.

Remark. When we learn new definitions, it's very important to think about examples that we already know. Often books give trivial examples first, and then show some non-trivial examples, motivating us to study. Books are written in that way!

Definition. Let $(S, *)$ be given. We say that

(1) $*$ is **commutative** if $a * b = b * a$ for all $a, b \in S$.

(2) $*$ is **associative** if $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

Example. Consider these binary operations on $\mathbb{Z}$.

(1) $a * b = a$, $*$ is not commutative but associative.

(2) $a * b = a + 2$, $*$ is not commutative nor associative.

Remark. To study these binary operations, we first start with finite sets where we can write tables with the results of the binary operation.

$$S = \{a\} \implies \begin{array}{c|c} * & a \\ \hline a & a \end{array}, \qquad S = \{a, b\} \implies \begin{array}{c|c|c} * & a & b \\ \hline a & & a * b \\ \hline b & b * a & \end{array}$$

Consider the relation between binary operation on a finite set S and tables. Binary operation is a function, so we have the existence and uniqueness of $a * b$. In terms of tables, we see that each cell in the table should have a value and it should be uniquely determined. So we conclude that *we can describe a binary operation with a table.*

March 13th, 2023

Section 3. Isomorphic Binary Structures

Consider $S = \{a, b, c\}, S' = \{1, 2, 3\}$ and binary operations $*, *'$, defined as the following.

$*$	a	b	c
a	b	a	c
b	c	a	b
c	a	b	c

$*'$	1	2	3
1	2	1	3
2	3	1	2
3	1	2	3

We see that if we rename a, b, c to $1, 2, 3$ respectively, we see that the tables are actually *equivalent*. How do we formalize the notion of equivalence of binary structures $(S, *), (S', *')$?

Definition. (Isomorphism) Let $(S, *)$ and $(S', *')$ be binary structures. If there exists a bijection $\varphi : S \to S'$ such that

$$\varphi(a * b) = \varphi(a) *' \varphi(b), \quad \forall a, b \in S,$$

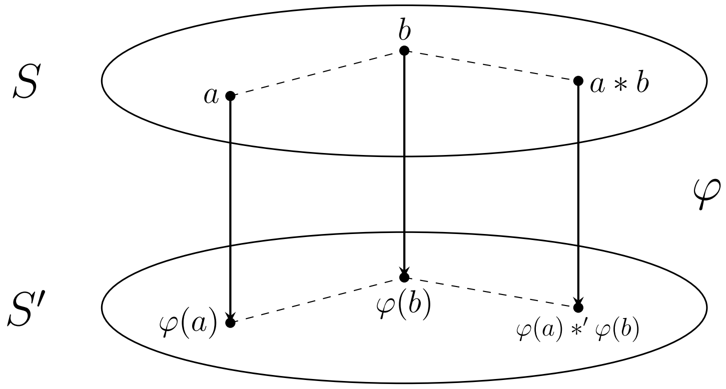
then φ is called an **isomorphism** between $(S, *)$ and $(S', *')$. We say that $(S, *)$ and $(S', *')$ are **isomorphic** to each other.

Definition. (Homomorphism) Let $(S, *)$ and $(S', *')$ be binary structures. A map $\varphi : S \to S'$ such that

$$\varphi(a * b) = \varphi(a) *' \varphi(b), \quad \forall a, b \in S$$

is called a **homomorphism** between $(S, *)$ and $(S', *')$.

Remark. Isomorphisms are *renaming functions* that preserve the structure of a set. Isomorphisms are homomorphisms that are bijective. Also, homomorphisms can be seen as a somewhat confusing(?) renaming functions, since they aren't bijective.



Example. Examples of isomorphisms.

(1) Let $\varphi : (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$ with $\varphi(x) = e^x$ for $x \in \mathbb{R}$.

(2) Let $\varphi : (\mathbb{Z}, +) \to (2\mathbb{Z}, +)$ with $\varphi(n) = 2n$ for $n \in \mathbb{Z}$.

(3) Non-example: $(\mathbb{Z}, *) \not\simeq (\mathbb{R}, *)$ (different cardinality)

Remark. Structural properties: properties preserved by isomorphisms

(1) Number of elements (cardinality, for infinite sets)

(2) Commutativity, associativity

(3) The equation $a * x = b$, $\forall a, b \in S$ has a solution in S

We can disprove the existence of an isomorphism by showing that any of the structural properties do not hold.

Example. For $(\mathbb{Z}, \cdot)$ and $(\mathbb{Z}^+, \cdot)$ we want to show that these two are not isomoprhic. Consider the equation $x^2 = x$. In $\mathbb{Z}$, the solutions are $x = 0, 1$, but in $\mathbb{Z}^+$, the solution is unique, $x = 1$.

Proof. Suppose that $\mathbb{Z}$ and $\mathbb{Z}^+$ are isomorphic, and let φ be the isomorphism. Let $\varphi(0) = a$, $\varphi(1) = b$. Then $a \neq b$, since φ is a bijection. However,

$$a = \varphi(0 \cdot 0) = \varphi(0) \cdot \varphi(0) = a \cdot a, \quad b = \varphi(1 \cdot 1) = \varphi(1) \cdot \varphi(1) = b \cdot b$$

but in $\mathbb{Z}^+$ there is only one solution to $x^2 = x$. This is a contradiction, so φ is not an isomorphism.

Definition. (Identity) Let $(S, *)$ be a binary structure. If $e \in S$ satisfies

$$e * a = a * e = a, \quad \forall a \in S,$$

then e is called the **identity** element of S.

Theorem. Identities are unique, if it exists.

Proof. Let $e, e' \in S$ be identities of S. Then,

$$e = e * e' = e' * e = e'$$

so $e = e'$.

Theorem. If $\varphi : S \to S'$ is an isomorphism, φ maps the identity to the identity.

Proof. Let $e \in S$ be the identity of S. For any $t \in S'$, there exists $s \in S$ such that $\varphi(s) = t$. Then

$$t *' \varphi(e) = \varphi(s * e) = \varphi(s) = t, \quad \varphi(e) *' t = \varphi(e * s) = \varphi(s) = t,$$

so $\varphi(e)$ is the identity of S'.

March 15th, 2023

Section 4. Groups

Definition. (Inverse) Let $(S, *)$ be a binary structure with identity $e \in S$. If $x \in S$ satisfies

$$a * x = x * a = e \text{ for some } a \in S,$$

then x is an **inverse** of a, and we write $x = a^{-1}$.

Definition. (Group) Let $(G, *)$ be a binary structure, with the following properties.

(1) $*$ is associative.

(2) G has an identity element.

(3) For all $x \in G$, there exists an inverse of x in G.

Then $G = (G, *)$ is called a **group**.

$(\mathbb{N}, +)$ is not a group. The equation $n+x = m$, $(n, m \in \mathbb{N})$ does not have a solution if $n \leq m$. So we extend the number system to $\mathbb{Z}$ and consider $n + x = m$ for $n, m \in \mathbb{Z}$. This equation always has a solution, this is due to the fact that $(\mathbb{Z}, +)$ is a group. The operation $+$ is associative, $\mathbb{Z}$ has an identity 0, and also has an inverse for any $n \in \mathbb{Z}$.

So if $(G, *)$ is a group, equations of the from $a * x = b$ for given $a, b \in G$ can be solved by multiplying a^{-1} on the left.

$$\begin{aligned} a^{-1} * (a * x) &= a^{-1} * b \\ (a^{-1} * a) * b &= a^{-1} * b \\ e * b &= a^{-1} * b \\ x &= a^{-1} * b. \end{aligned}$$

Note that all three properties of the group were used!

Example.

(1) $(\mathbb{Z}, +), (\mathbb{Q}, +), (\mathbb{R}, +), (\mathbb{C}, +)$ are all groups.

(2) $(\mathbb{N}, +), (\mathbb{Z}, \cdot), (\mathbb{Q}, \cdot), (\mathbb{R}, \cdot), (\mathbb{C}, \cdot)$ are not groups, since they don't have an inverse for 0.

(3) $(\mathbb{Q}^{\times}, \cdot), (\mathbb{R}^{\times}, \cdot), (\mathbb{C}^{\times}, \cdot)$ are groups.

(4) The roots of unity with multiplication form a group.

Definition. (Commutative Group) A group G is **commutative/abelian** if

$$a * b = b * a \text{ for all } a, b \in G.$$

Proposition. (Basic properties of groups) Let G be a group.

(1) G has a unique identity.

(2) For $a \in G$, its inverse a^{-1} is unique.

(3) Left and right cancellation laws hold.

Remark. $(G, *)$ is a group $\iff *$ is associative, has left identity, has left inverse.

Proof. ($\Longleftarrow$) Let e be a left identity of G. For any $a \in G$, let a' be a left inverse of a. Then, $a' * a * e = e * e = e = a' * a = a' * e * a$. Let a'' be a left inverse of a', then multiplying a'' on the left gives

$$a'' * a' * a * e = a'' * a' * e * a \implies a * e = e * a = a,$$

so $a * e = a$, proving that e is also a right identity.[2]

Let a' be a left inverse of a, and let a'' be a left inverse of a'. Then $a''*a'*a*a' = e*a*a' = a*a'$, also $a'' * a' * a * a' = a'' * e * a' = a'' * a' = e$. Therefore $a * a' = e$, proving that a' is also a right inverse.[3]

Remark. For finite groups, the elements in a single row or column should be unique. For example, if some two elements $a * x$, $a * y$ in the same row (but different column) are the same, we can use the left cancellation law to show that $x = y$. This is a contradiction.

So, let $G = \{e, a\}$ be a group with identity e. Then its operation table is determined uniquely, as the following.

$*$	e	a
e	e	a
a	a	e

As for $G = \{e, a, b\}$, it is also unique.

$*$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

[2] Alternatively, $ae = eae = (a''a')a(a'a) = a''ea'a = ea = a$.

[3] Alternatively, $aa' = (a''a')aa' = a''(a'a)a' = a''ea' = e$.

March 20th, 2023

Definition. (Equivalence Relation) A relation $\mathcal{R}$ on S is a **equivalence relation** if it satisfies the following.

(1) (Reflexive) $x\mathcal{R}x$. $(x \in S)$

(2) (Symmetric) If $x\mathcal{R}y$, then $y\mathcal{R}x$. $(x, y \in S)$

(3) (Transitive) If $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$. $(x, y, z \in S)$

Example.

(1) Relation '=' on $\mathbb{Q} = \left\{\frac{y}{x} : x \in \mathbb{Z}^\times, y \in \mathbb{Z}\right\}$. Defined as $\frac{y}{x} = \frac{y'}{x'} \iff xy' = yx'$. The second equality is equality in $\mathbb{Z}$. We are defining '=' in $\mathbb{Q}$ using '=' in $\mathbb{Z}$.

(2) Relation '>' on $\mathbb{Z}$ is not symmetric, so it is not an equivalence relation.

Theorem. Equivalence relation $\sim$ on a set S yields a partition of S.

Example. Let $S = \mathbb{Z}$, $x \sim y \iff x \equiv y \pmod 5$. Then

$$\mathbb{Z} = \overline{0} \sqcup \overline{1} \sqcup \overline{2} \sqcup \overline{3} \sqcup \overline{4},$$

where $\overline{x} = \{y \in \mathbb{Z} : x \sim y\}$.

Section 5. Subgroups

Definition. (Subgroup) Let $(G, *)$ be a group, $H \subset G$. H is a **subgroup** of G if $(H, * |_{H\times H})$ is also a group. We write $H \leq G$.

Example.

(1) $(\mathbb{Z}, +) \leq (\mathbb{Q}, +) \leq (\mathbb{R}, +)$.

(2) (Trivial Subgroup) $\{e\} \leq G$.

(3) (Improper Subgroup) $G \leq G$.

(4) $\{e\} \leq \mathbb{Z}_2 \leq \mathbb{Z}_4$.

(5) $V_4 \not\simeq \mathbb{Z}_4$ (different subgroup lattices).

(6) $\mathbf{SL}_n(\mathbb{R}) \leq \mathbf{GL}_n(\mathbb{R})$.

The following is a method to check that H is a subgrouop of G.

Theorem. Let G be a group and $H \subseteq G$. Then $H \leq G$ if and only if

(1) H is closed under the binary operation $*$ of G.

(2) Identity $e \in H$.

(3) For all $x \in H$, there exists an inverse $x^{-1} \in H$.

How can we find non-trivial subgroups? We include an element and generate elements, since the binary operations are always closed.

Theorem. Let G be a group, and let $a \in G$. Then

$$\{a^n : n \in \mathbb{Z}\} \leq G.$$

Proof. Let $H = \{a^n : n \in \mathbb{Z}\}$. Then for $a^n, a^m \in H$ $(n, m \in \mathbb{Z})$, $a^n a^m = a^{n+m} \in H$ (closed), $e = a^0 \in H$ (has an identity), $(a^n)^{-1} = a^{-n} \in H$ (has an inverse). So $H \leq G$.

Remark. Let $H = \{a^n : n \in \mathbb{Z}\}$.

(1) H is the smallest subgroup of G containing a.

(2) Any subgroup of G containing a has H as a subgroup.

(3) H is commutative.

Definition. (Cyclic) Let G be a group and let $H = \{a^n : n \in \mathbb{Z}\} \leq G$ for $a \in G$.

(1) H is called the **cyclic subgroup** generated by a, and we write $H = \langle a \rangle$.

(2) If there exists $x \in G$ such that $G = \langle x \rangle$, then G is called a **cylic group**.

Example. $U_n = \{\xi \in \mathbb{C} : \xi^n = 1\} = \langle \xi \rangle$, is a cyclic group where $\xi = e^{\frac{2\pi i}{n}}$. If we visualize this on the complex plane, the element ξ generates the whole group, in a cycle, hence the name cyclic group.

March 22nd, 2023

Section 6. Cyclic Groups

Theorem. Every cyclic group is commutative.

Always consider $U_n \simeq (\mathbb{Z}_n, +_n)$ as an example, when dealing with cyclic groups.

Theorem. A subgroup of a cyclic group is also cyclic.

Proof. Let $G = \langle g \rangle$ be a cyclic group, and let $H \leq G$. Choose the smallest positive $r \in \mathbb{N}$ such that $g^r \in H$. We show that $H = \langle g^r \rangle$. It is clear that $\langle g^r \rangle \leq H$, since H is a subgroup.

Suppose that there exists $s \in \mathbb{Z}$ such that $g^s \in H$, but $g^s \notin \langle g^r \rangle$. Then there exists unique quotient and remainder $q \in \mathbb{Z}, t \in \{1, \cdots, r-1\}$ such that $s = rq + t$. Then $g^s, g^{rq} \in H$, so $g^s(g^{rq})^{-1} = g^t \in H$, contradicting the minimality of r. Thus $H \leq \langle g^r \rangle$ and $H = \langle g^r \rangle$.

Example. $(\mathbb{Z}, +) = \langle 1 \rangle$. So any subgroup of $(\mathbb{Z}, +)$ should be $\langle n \rangle = n\mathbb{Z}$ for some $n \in \mathbb{Z}$.

Definition. Let $S \subset G$. Then $\langle S \rangle$ is the smallest subgroup of G generated by S.

If $H = \langle a, b \rangle \leq \mathbb{Z}$, we can rewrite $H = \langle d \rangle$ for some $d \in \mathbb{Z}$. We know that $d = \gcd(a, b)$.

Definition. (Greatest Common Divisor) Let $r, s \in \mathbb{N}$. The **greatest common divisor** of r, s is the generator d which generates $\langle r, s \rangle \leq \mathbb{Z}$. We write $d = \gcd(a, b)$ and if $\gcd(r, s) = 1$, we say that r, s are **relatively prime**.

Remark. $\gcd(r, s) = 1 \iff mr + ns = 1$ for some $m, n \in \mathbb{Z}$.

We want to classify all cyclic groups!

Theorem. (Classification of Cyclic Groups) Suppose that $G = \langle g \rangle$ is cyclic.

(1) $|G| = \infty \iff G \simeq \mathbb{Z}$.

(2) $|G| = n \iff G \simeq \mathbb{Z}_n$.

Proof.
(1) Consider the map $\varphi : G \to \mathbb{Z}$, defined as $\varphi(g) = 1$. We first check that φ is well-defined. This is clear, since $|G| = \infty$, so $n \neq m \in \mathbb{Z} \iff g^n \neq g^m$. Otherwise, $|G|$ would be finite. This also implies that φ is bijective. Also φ is a homomorphism, since $\varphi(g^n g^m) = n + m = \varphi(g^n) + \varphi(g^m)$. φ is an isomorphism and $G \simeq \mathbb{Z}$.

(2) Consider the map $\varphi_n : G \to \mathbb{Z}_n$, defined as $\varphi_n(g) = 1$. We can check that φ_n is a well-defined isomorphism.

Theorem. Let $G = \langle a \rangle$ be a cyclic group of order n.

(1) Let $b = a^s \in G$. Then $|\langle b \rangle| = \dfrac{n}{\gcd(n, s)}$.

(2) $\langle a^s \rangle = \langle a^t \rangle \iff \gcd(n, s) = \gcd(n, t)$.

Proof. (1) We want to find the smallest positive integer m such that $b^m = e$. ($a^{ms} = e$, so $n \mid ms$) Take $d = \gcd(n, s)$, then $\gcd\left(\frac{n}{d}, \frac{s}{d}\right) = 1$. Since $\frac{n}{d} \mid \frac{s}{d} \cdot m$, then $\frac{n}{d} \mid m$. Hence the smallest positive integer m is $\frac{n}{d}$.

(2) Directly follows from (1). May have to prove that $\langle a^s \rangle = \langle a^t \rangle$.

Corollary. If $G = \langle a \rangle$ with order n, other generators of G are the elements of the form a^r where $\gcd(r, n) = 1$.

Part II

Permutations, Cosets and Direct Products

March 27th, 2023

Section 8. Groups of Permutations

Definition. (Permutation) A **permutation** on a set A is a bijective function $\varphi : A \to A$.

Remark. Let S_A be the set of permutations on A. Then $f \circ g \in S_A$ for all $f, g \in S$, $\circ$ has associativity, and $id \in S_A$, $f^{-1} \in S_A$ for all $f \in S$. Therefore $(S, \circ)$ is a group.

We study the case when A is a finite set, i.e, $A = \{1, 2, \ldots, n\}$.

Definition. (Symmetric Group S_n) Let $A = \{1, 2, \ldots, n\}$. Let S_n be the set of all permutations on A. Then $(S_n, \circ)$ is called the **symmetric group on n letters**.

Let B be a set with n elements. We denote S_B be the set of all permutations on B. With the composition operation $\circ$, we see that $S_B \simeq S_n$ as groups.

Example.

(1) $S_2 = \{e, \tau\}$ where $\tau = (1, 2)$.

(2) On S_3, there are 6 permutations.

$$S_3 = \{e, \mu_1, \mu_2, \mu_3, \rho_1, \rho_2\}$$

where μ_i swaps other two elements other than i, and $\rho_1 = (1, 2, 3), \rho_2 = (1, 3, 2)$.

(3) Also we can see that S_3 is the group of symmetries on an equilateral triangle. Each ρ_i represents a rotation, and each μ_i represents a reflection.

Remark. S_3 is not commutative. Check that $\rho_1 \circ \mu_1 = \mu_3$, but $\mu_1 \circ \rho_1 = \mu_2$. In fact, S_3 is the non-commutative group having the smallest order.

A natural question arises here: *Can we get S_n from the symmetries of a regular n-gon?*

Example. We try this for S_4, but this doesn't work. Symmetries of a square consists of 4 rotations and 4 reflections, which is a total of 8 elements, but $|S_4| = 4! = 24$.

Definition. (Dihedral Group D_n) The group of symmetries of a regular n-gon is called the **n-th dihedral group D_n**.

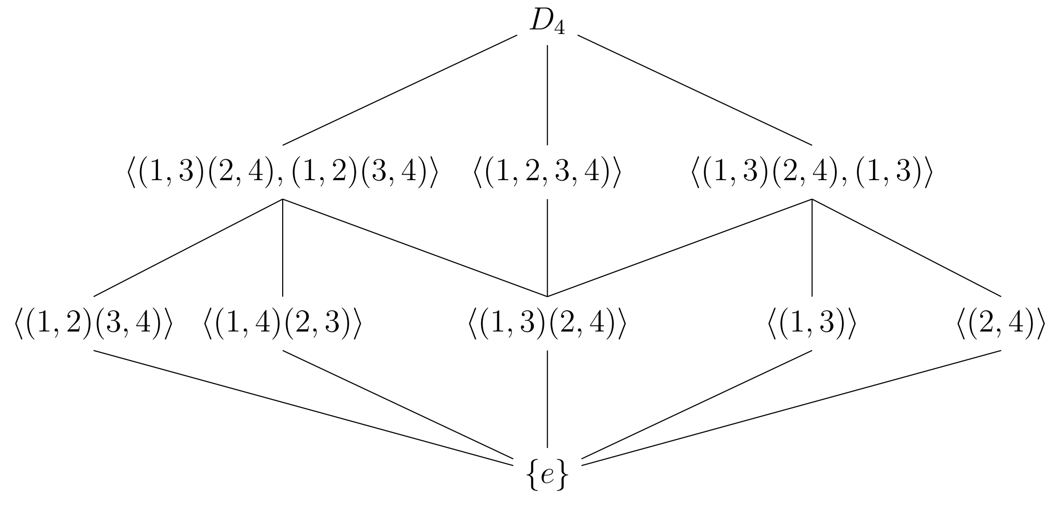
Remark.

(1) $D_3 \simeq S_3$, $D_4 < S_4$, D_4 is not commutative, so S_4 is not commutative.

(2) $|D_n| = 2n$.

(3) D_4 is generated by 2 elements. $D_4 = \langle \rho, \mu \rangle$, where ρ is a rotation by 90 degrees, and μ is some reflection.

(4) Subgroup lattice of D_4.


D_4
$\langle (1,3)(2,4), (1,2)(3,4) \rangle$
$\langle (1,2,3,4) \rangle$
$\langle (1,3)(2,4), (1,3) \rangle$
$\langle (1,2)(3,4) \rangle$
$\langle (1,4)(2,3) \rangle$
$\langle (1,3)(2,4) \rangle$
$\langle (1,3) \rangle$
$\langle (2,4) \rangle$
$\{e\}$

Lemma. For a group homomorphism $\varphi : G \to H$, $\operatorname{im}\varphi \leq H$. Additionally if φ is injective, $G \simeq \operatorname{im}\varphi \leq H$.

Proof. $\operatorname{im}\varphi$ is closed under the binary operation on H, since for any $a, b \in G$, $\varphi(a)\varphi(b) = \varphi(ab)$, and $ab \in G$, so $\varphi(a)\varphi(b) \in H$. $\varphi(e)$ is the identity, and $\varphi(a)^{-1} = \varphi(a^{-1})$. So $\operatorname{im}\varphi \leq H$. If φ is injective, restricting the range of φ to $\operatorname{im}\varphi$ gives an isomorphism, so $G \simeq \operatorname{im}\varphi$.

Why do we study symmetric groups? It is because of the following theorem. It states that all groups are isomorphic to some permutation group.

Theorem. (Cayley) Every group is isomorphic to some subgroup of S_n.

Proof. Consider $\varphi : G \to S_G$ such that $\varphi(g) = \lambda_g$, where $\lambda_g(x) = gx$ for $x \in G$. (left multiplication by g) We check that $\lambda_g \in S_G$, since groups have the cancellation law. Now check that φ is a monomorphism, then $G \simeq \operatorname{im} \varphi \leq S_G$.

Section 9. Orbits, Cycles and the Alternating Groups

Definition. Equivalence relation $\sim_\sigma$ on A with respect to $\sigma \in S_A$ is defined as

$$\text{For } a, b \in A,\ a \sim_\sigma b \iff \exists\, n \in \mathbb{Z} \text{ such that } a = \sigma^n(b).$$

Remark. Check that $\sim$ is indeed an equivalence relation.

Definition. (Orbit) Equivalence classes in A induced from the relation $\sim_\sigma$ are called the **orbits of** σ.

Example. Consider $\sigma = (1,3,6)(2,5,7,4)(8)$. Then $\{1,3,6\}, \{2,4,5,7\}, \{8\}$ are orbits.

If we represent orbits in circles, σ can be represented as 2 circles. $\tau = (1,2,3,4)$ would be represented as a circle, and $\tau' = (1,2,3)(4)$ would be represented as a circle.

Definition. (Cycles)

(1) A permutation $\sigma \in S_n$ is called a **cycle** if σ has at most 1 orbit with more than 1 element.

(2) The **length** of a cycle σ is the number of elements in its largest orbit.

τ is a cycle of length 4, τ' is a cycle of length 3, but σ is not a cycle.

Question. *For any $\sigma \in S_n$, can σ be represented as a composition of cycles?*

Theorem. Every permutation of a finite set is a product of disjoint cycles.

Proof. Take any $\sigma \in S_n$. Then the equivalence relation $\sim_\sigma$ induces a partition on $\{1, 2, \ldots, n\}$ as $\bigsqcup_{i=1}^k B_i$. Then $\sigma|_{B_i} \colon B_i \to B_i$ is a well-defined permutation. Now define

$$\mu_i(x) = \begin{cases} \sigma(x) & (x \in B_i), \\ x & (x \notin B_i). \end{cases}$$

Then μ_i is a cycle of B_i, and $\sigma = \mu_1 \circ \mu_2 \circ \cdots \circ \mu_k$.

Definition. (Transposition) A cycle of length 2 is called a **transposition**.

Remark. $(1,2,3) = (1,3)(1,2)$, $(a_1, a_2, \ldots, a_n) = (a_1, a_n)(a_1, a_{n-1}) \cdots (a_1, a_2)$.

Corollary. Any permutation is a product of transpositions, since disjoint cycles can be decomposed into a product of transpositions by the above remark.

Note that this representation is not unique. Since $\tau^2 = id$ for any transposition τ, we can always multiply two same transpositions at the end.

Theorem. No permutation in S_n can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.

Proof.

(Step 1) Take $\sigma \in S_n$, transposition $\tau \in S_n$. Then for σ and $\tau\sigma$, their number of orbits differ by 1.

- Case 1. $\tau = (i, j)$ where i, j are not in the same orbit.
  Let $\sigma = (b, j, \dots)(a, i, \dots)(\dots)$. Then $\tau\sigma = (b, i, \dots, a, j, \dots)(\dots)$. So the number of orbits differ by 1.
- Case 2. $\tau = (i, j)$ where i, j are in the same orbit.
  Left as exercise.

(Step 2) Suppose we could write $\sigma = \tau_1\tau_2 \cdots \tau_t = \tau_1'\tau_2' \cdots \tau_s'$ where τ_i, τ_j' are transpositions. Then the number of orbits of σ, t and s have the same parity. This follows directly from Step 1.

So this definition is well-defined!

Definition. A permutation $\sigma \in S_n$ is called

(1) **even** if σ is a product of even number of transpositions.

(2) **odd** if σ is a product of odd number of transpositions.

Definition. (Alternating Group A_n) We define the **alternating group** A_n as

$$A_n = \{\sigma \in S_n : \sigma \text{ is even}\}.$$

Theorem.

$$|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}.$$

Proof. Consider a map $\lambda_\tau : A_n \to S_n \setminus A_n$ defined as $\lambda_\tau(\sigma) = \tau \circ \sigma$ where τ is any transposition. Then λ_τ is a bijection.

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Section 10. Cosets and the Theorem of Lagrange

Example. Motivation.

(1) $A_n \leq S_n$. We call $S_n \setminus A_n$ a coset of A_n.

(2) $3\mathbb{Z} \leq \mathbb{Z}$. We call $3\mathbb{Z}$, $3\mathbb{Z}+1$, $3\mathbb{Z}+2$ are cosets of $3\mathbb{Z}$.

We saw that A_n and $S_n \setminus A_n$ have the same number of elements. This was done by constructing a bijection between two sets. But we know that a bijection between $3\mathbb{Z}, 3\mathbb{Z}+1, 3\mathbb{Z}+2$ exist. We guess that the number of elements would be the same for each coset.

So if G is a finite group and $H \leq G$, we conjecture that we can partition G by cosets of H, and each cosets have the same number of elements. So $|H| \mid |G|$.

Definition. Let G be a group, $H \leq G$. Define a relation $\sim_L$ on G as

$$a \sim_L b \iff a^{-1}b \in H.$$

Remark. We can also define $\sim_R$ on G as $a \sim_R b \iff ba^{-1} \in H$.

Theorem. $\sim_L$ is an equivalence relation.

Proof.

- $\forall a \in G$, $a^{-1}a = e \in H$. $a \sim_L a$.
- $\forall a, b \in G$, if $a^{-1}b \in H \implies (a^{-1}b)^{-1} = b^{-1}a \in H$. $b \sim_L a$.
- $\forall a, b, c \in G$, if $a^{-1}b, b^{-1}c \in H \implies (a^{-1}b)(b^{-1}c) = a^{-1}c \in H$. $a \sim_L c$.

Definition. (Coset) Let G, H be groups and $H \leq G$. For $a \in G$, we define

(1) The **left coset** of H as $aH = \{ah : h \in H\}$.

(2) The **right coset** of H as $Ha = \{ha : h \in H\}$.

We see that $a \neq b$ does not imply $aH \neq bH$. If $a, b \in H$, then $aH = bH = H$. So when would we get the same coset?

Remark. $aH = bH \iff H = a^{-1}bH \iff a \sim_L b$.

Example.

(1) For a transposition $\tau \in S_n$, τA_n is a coset of A_n with odd permutations. So it is different from A_n. So for any odd permutation $\sigma \in S_n$, $\sigma A_n = \tau A_n$, and $\sigma \sim_L \tau$.

(2) $3\mathbb{Z} \leq \mathbb{Z}$. $3\mathbb{Z}$, $3\mathbb{Z}+1$, $3\mathbb{Z}+2$ are cosets. Since $\mathbb{Z}$ is commutative, the left and right cosets are equal to each other.

Remark. If G is commutative and $H \leq G$, $aH = Ha$ for $a \in G$.

Check for non-commutative groups that left and rights cosets need not be equal!

Theorem. (Lagrange) Let G be a finite group with $H \leq G$. Then $|H| \mid |G|$.

Proof. If we construct a bijection between any two left cosets, $|H|$ would be $|G|$ divided by the number of left cosets. This would imply $|H| \mid |G|$.

Recall that left cosets are defined as the equivalence classes with respect to the relation $\sim_L$. So G is a disjoint union of cosets, $G = \bigsqcup aH$. Therefore, $|G| = (\text{number of left cosets}) \cdot |H|$.

Lemma. $|aH| = |bH|$ for $a \in G$, $H \leq G$.

Proof. $\varphi : aH \to bH$ is a bijection. Check by yourself!

Corollary. The number of left cosets and the number of right cosets are equal.

Corollary. Every group of prime order is cyclic.

Proof. Let $|G| = p$, where p is prime. Take $a \in G$ which is not the identity. Then the cyclic subgroup $\langle a \rangle \leq G$ must have order 1 or p. But a must have order p because it is not the identity. So $|\langle a \rangle| = p$, which implies that $G = \langle a \rangle$.

This is a very important result related to the classification of finite (simple) groups. We have seen all groups of order 2, 3, 4. But for larger orders, we can't enumerate them all. With this result, we directly know that for groups with prime order p, the group is isomorphic to $\mathbb{Z}_p$.

This is also a direct result of Lagrange's theorem, since $\langle a \rangle \leq G$.

Theorem. The order of an element in a finite group divides the order of the group. In other words, $|\langle a \rangle| \mid |G|$, for $a \in G$.

Definition. (Index) Let G be a group, (not necessarily finite) and $H \leq G$. We define the **index**

of H in G as

$$(G:H) = \text{number of left cosets of } H = \text{number of right cosets of } H.$$

If $|G| < \infty$, $(G:H) = |G| \,/\, |H|$.

Theorem. For a group G, suppose that $K \leq H \leq G$. If $(G:H)$, $(H:K)$ are finite,

$$(G:K) = (G:H)\,(H:K)\,.$$

Proof. Write $G = \bigsqcup_{i=1}^{n} a_i H$, $H = \bigsqcup_{j=1}^{m} b_j K$, and show that $G = \bigsqcup_{i,j} a_i b_j K$. Check by yourself!

Section 11. Direct Products, Finitely Generated Abelian Groups

Definition. (Cartesian Product) For sets $S_1, \dots, S_n$, define

$$\prod_{i=1}^{n} S_i = S_1 \times S_2 \times \cdots \times S_n = \{(a_1, \dots, a_n) : a_i \in S_i\}.$$

What if S_i already have a group structure?

Definition. (Direct Product) Let $G_1, \dots, G_n$ be groups. Define a binary operation $\cdot$ as

$$(a_1, a_2, \dots, a_n) \cdot (b_1, b_2, \dots, b_n) = (a_1 b_1, a_2 b_2, \dots, a_n b_n).$$

Then the **direct product** $\prod_{i=1}^{n} G_i$ is a group with this binary operation.

Notation. We also write the direct sum as $\bigoplus_{i=1}^{n} G_i$ for additive groups.

Example. Compare the Klein 4-group V_4 with $\mathbb{Z}_2 \times \mathbb{Z}_2$. They have the exact same structure! $V_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. From this example, we found out that order 4 group is either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Now we know all groups of order up to 5. How about order 6? We know S_3 and $\mathbb{Z}_6$.

Example. Consider $\mathbb{Z}_2 \times \mathbb{Z}_3$. We can check that $(1, 1) \in \mathbb{Z}_2 \times \mathbb{Z}_3$ has order 6, so $\langle (1, 1) \rangle = \mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$.

Why was it that $\mathbb{Z}_4 \not\simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, but $\mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$?

Theorem. $\mathbb{Z}_m \times \mathbb{Z}_n \simeq \mathbb{Z}_{mn}$ if and only if $\gcd(m, n) = 1$.

Proof.
($\Longleftarrow$) We need to find a generator of $\mathbb{Z}_m \times \mathbb{Z}_n$ with order mn. Take $a = (1, 1) \in \mathbb{Z}_m \times \mathbb{Z}_n$. The order of this element should be divisible by m, n. So mn is the smallest positive integer, and $|\langle (1, 1) \rangle| = mn$.

($\Longrightarrow$) Suppose that $d = \gcd(a, b) > 1$. Then for all $(a, b) \in \mathbb{Z}_m \times \mathbb{Z}_n$, $\frac{mn}{d}(a, b) = (0, 0)$. So (a, b) cannot generate the entire group (which has mn elements). Therefore $\mathbb{Z}_m \times \mathbb{Z}_n$ is not cyclic.

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Corollary. $\prod_{i=1}^{n} \mathbb{Z}_{m_i} = \mathbb{Z}_{m_1} \times \cdots \mathbb{Z}_{m_n} \simeq \mathbb{Z}_{m_1 m_2 \cdots m_n} \iff \gcd(m_i, m_j) = 1$ for all $i \neq j$.

Example. Let $n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ where p_i are distinct primes, $r_i \in \mathbb{N}$. Then

$$\mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$$

since $p_i^{r_i}$ are pairwise coprime.

Definition. (Least Common Multiple) Let $r_1, r_2, \ldots, r_m \in \mathbb{N}$. The **least common multiple** l of $r_1, r_2, \ldots, r_m$ is defined as the positive l such that

$$\langle l \rangle = \langle r_1 \rangle \cap \langle r_2 \rangle \cap \cdots \cap \langle r_m \rangle$$

in $\mathbb{Z}$. We write $l = \operatorname{lcm}(r_1, r_2, \ldots, r_n)$.

Theorem. Let $(a_1, \ldots, a_n) \in \prod_{i=1}^{n} G_i$, where each a_i has finite order r_i. Then

$$|(a_1, \ldots, a_n)| = \operatorname{lcm}(r_1, \ldots, r_n).$$

Proof. Homework! □

Example. $(8, 4, 10) \in \mathbb{Z}_{12} \times \mathbb{Z}_{60} \times \mathbb{Z}_{24}$. Then $r_1 = 3$, $r_2 = 15$, $r_3 = 12$. We want to find the smallest positive N such that $N(8, 4, 10) = (0, 0, 0)$. Then $3, 15, 12 \mid N$, so $N = \operatorname{lcm}(3, 5, 12) = 60$.

Structure of Finitely Generated Abelian Groups

A group is cyclic if it can be generated by a single element. Consider the Klein 4-group $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. This group has no elements of order 4, so it is not cyclic. But we see that $V_4 = \langle (1,0), (0,1) \rangle$, so V_4 is generated by 2 elements. We extend this definition.

Definition. (Finitely Generated) A group G is **finitely generated** if there exists a finite subset $S \subset G$ such that $G = \langle S \rangle$.

Remark. If G is finite, $G = \langle G \rangle$ so it is finitely generated. But the converse is not true, since $\mathbb{Z}$ is infinite but $\mathbb{Z} = \langle 1 \rangle$.

We started learning from the simplest groups. They were generated by a single element and we classified them. We also learned about permutation groups and Cayley's theorem. Next, we classify a large family of groups. They are finitely generated abelian groups.

Theorem. (Fundamental Theorem of Finitely Generated Abelian Groups) Let G be a finitely generated abelian group. Then

$$G \simeq \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}} \times \overbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}^{\text{finite}}$$

where p_i are primes (not necessarily distinct), $r_i \in \mathbb{N}$, and this representation is unique up to order of products.

We only know cyclic groups, since they were the only groups that we classified completely. We studied permutation groups, but we saw that they are complex! So we don't know them very well, which leaves us to try a lot of things with cyclic groups. So we construct new groups from cyclic groups using direct products. Then we get this result that finitely generated abelian groups are actually a product of cyclic groups!

Remark. $\mathbb{Z}_4 \not\simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. If these two groups were isomorphic, then it contradicts the uniqueness of product representation. Similarly, $\mathbb{Z}_9 \not\simeq \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_9 \times \mathbb{Z}_3 \not\simeq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

This is a typical exercise after learning this theorem.

Example. Classify all abelian groups of order 360.

Proof. This group is finitely generated. We know that $360 = 2^3 \times 3^2 \times 5$.

$$\begin{array}{cc}
\mathbb{Z}_{2^3} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5 & \mathbb{Z}_{2^3} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
\mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5 & \mathbb{Z}_2 \times \mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \\
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2} \times \mathbb{Z}_5 & \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5
\end{array}$$

So these are all possible cases. □

Definition. (Decomposable) A group G is **decomposable** if $G \simeq H \times K$ for $H, K \leq G$.

This implies two things: that we can understand G by studying H, K, and that it is important to study indecomposable groups.

Theorem. A finite indecomposable abelian group is a cyclic group of order p^r where p is prime and $r \in \mathbb{N}$.

Theorem. Let G be a finite abelian group. If $m \mid |G|$, then there exists a subgroup of G with order m.

Proof. Let $G \simeq \mathbb{Z}_{p_1^{r_1}} \times \mathbb{Z}_{p_2^{r_2}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$. We use the fact that $\mathbb{Z}_{p_i^{s_i}} \leq \mathbb{Z}_{p_i^{r_i}}$ if $s_i \leq r_i$. Since $m \mid |G|$, we can write $m = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}$ for some $s_i \leq r_i$. Take $H_i \leq \mathbb{Z}_{p_i^{r_i}}$ such that $|H_i| = p_i^{s_i}$. Then $H_1 \times H_2 \times \cdots H_k \leq G$, and it has order m. □

Recall that from Lagrange's theorem, for finite group G, if $H \leq G$, $|H| \mid |G|$. We could ask if the converse is true. *Is there a subgroup of order m that divides $|G|$?* This is not true in general, but if G is abelian this is true.

Theorem. Let m be a square-free integer.[1] Then an abelian group of order m is cyclic.

Proof. Let $m = p_1 \cdots p_k$ where p_i are distinct primes. Using the fundamental theorem, an abelian group of order m can be written as $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_k}$, which is isomorphic to $\mathbb{Z}_m$. □

[1] $\nexists p$ prime such that $p^2 \mid m$.

Part III

Homomorphisms and Factor Groups

Section 13. Homomorphisms

We already learned about homomorphisms.

Recall. Let G, G' be groups. A map $\varphi : G \to G'$ such that

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2), \quad \forall g_1, g_2 \in G$$

is called a **group homomorphism**. If φ is a bijection, φ is a **group isomorphism**.

Definition. Let $\varphi : G \to G'$ be a group homomorphism.

(1) (Kernel) The **kernel** of φ is

$$\ker \varphi = \{x \in G : \varphi(x) = e'\} = \varphi^{-1}(\{e'\}),$$

where e' is the identity of G'.

(2) (Image) The **image** of φ is

$$\operatorname{im} \varphi = \varphi(G) = \{\varphi(x) : x \in G\}.$$

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Theorem. Let $\varphi : G \to G'$ be a group homomorphism, $H = \ker \varphi$. For $a \in G$,

$$\varphi^{-1}(\varphi(a)) = aH = Ha.$$

Proof. ($\subset$) Take $g \in \varphi^{-1}(\varphi(a))$. Then $\varphi(g) = \varphi(a)$, and $e' = \varphi(e) = \varphi(ag^{-1})$, so $ag^{-1} \in \ker \varphi$. $ag^{-1} = h$ for some $h \in H$, and $g = h^{-1}a \in Ha$. Similarly $g \in aH$.

($\supset$) Let $g = ha \in Ha$ for some $h \in H$. Then $\varphi(g) = \varphi(ha) = \varphi(a)$, so $g \in \varphi^{-1}(\varphi(a))$. For $g \in aH$, it can be shown similarly. □

Corollary. φ is a monomorphism if and only if $\ker \varphi = \{e\}$.

Proof. ($\implies$) Trivial. ($\impliedby$) For $a \in G$, $\varphi^{-1}(\varphi(a)) = a\{e\} = \{a\}$. φ is injective. □

Here is an alternative trivial proof.

Proof. ($\impliedby$) If $\varphi(x) = \varphi(y)$, then $\varphi(xy^{-1}) = e$, $xy^{-1} \in \ker \varphi$. So $xy^{-1} = e$ and $x = y$. □

Definition. (Normal Subgroup) Let $H \leq G$. If $aH = Ha$ for any $a \in G$, then H is called a **normal subgroup** of G. We write $H \trianglelefteq G$.

Corollary. $\ker \varphi \trianglelefteq G$ for any group homomorphism φ, since the left and right cosets coincide.

For injectivity, we can show that $\ker \varphi = \{e\}$ instead. We are on our way to define factor groups.

Section 14. Factor Groups

If $H \leq G$, $\{aH : a \in G\}$ were left cosets. We want to give a group structure on the cosets. Not just any structure, but a structure that naturally arises from the structure of G. $(aH)(bH) = abH$ is a natural candidate, but we have a problem. *Is this operation well-defined?* If $aH = a'H$ and $bH = b'H$, is it true that $abH = a'b'H$? Sadly, this is not true in general. But it is true when $aH = Ha$.[1]

Definition. (Factor Group) If $H \leq G$, G/H is defined as

$$G/H = \{aH : a \in G\}.$$

If G/H has a group structure with the binary operation $(aH)(bH) = abH$, we call G/H a **factor group**.

[1] Not math: $(aH)(bH) = a(Hb)H = a(bH)H = abH$, so we want $bH = Hb$!

Example. $\mathbb{Z}/3\mathbb{Z} = \{3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}\}$.

Theorem. Let $\varphi : G \to G'$ be a group homomorphism.

(1) $G/\ker\varphi$ is a factor group.

(2) (1st Isomorphism Theorem) $G/\ker\varphi \simeq \operatorname{im}\varphi$, with isomorphism $\mu(a\ker\varphi) = \varphi(a)$.

Proof.
(1) Well-definedness! If $a\ker\varphi = a'\ker\varphi$ and $b\ker\varphi = b'\ker\varphi$, then $\varphi(a) = \varphi(a')$ and $\varphi(b) = \varphi(b')$. Since φ is a homomorphism, $\varphi(ab) = \varphi(a'b')$. So $ab\ker\varphi = a'b'\ker\varphi$. Associativity directly follows, eH is the identity, $a^{-1}H = (aH)^{-1}$ can be checked.

(2) Well-definedness! If $a\ker\varphi = a'\ker\varphi$, then $\varphi(a) = \varphi(a')$, so $\mu(a\ker\varphi) = \mu(a'\ker\varphi)$. The fact that μ is an isomorphism can be checked easily. □

If we prove the well-definedness part, the rest is pretty automatic.

Recall. $N \trianglelefteq G \iff gN = Ng$ for all $g \in G \iff gNg^{-1} = N$ for all $g \in G$.

Theorem. For $H \leq G$, G/H is a factor group if and only if $H \trianglelefteq G$.

Proof. ($\Longleftarrow$) Trivial.
($\Longrightarrow$) Let $x \in aH$. Choose $x \in aH$, $a^{-1} \in a^{-1}H$. then $H = (aH)(a^{-1}H) = (xH)(a^{-1}H) = (xa^{-1})H$. So $xa^{-1} \in H$, showing that $x \in Ha$. Similarly, $Ha \subset aH$. $H \trianglelefteq G$. □

Definition. The G/H in the above theorem is called a **factor group** or a **quotient group**.

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This theorem implies that a normal subgroup is a kernel of some homomorphism.[2]

Theorem. (Fundamental Homomorphism Theorem) Let $\varphi : G \to G'$ be a group homomorphism. Then $\operatorname{im}\varphi$ is a group, and $\mu : G/\ker\varphi \to \operatorname{im}\varphi$ defined as

$$\mu(a\ker\varphi) = \varphi(a), \quad a \in G,$$

is an isomorphism.

$$\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[dr, "\gamma"'] & & \operatorname{im}\varphi \\
& G/H \arrow[ur, "\mu"'] \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}$$

Theorem. If $H \leq G$, The following are equivalent.

(1) $H \trianglelefteq G$.

(2) $gHg^{-1} = H$ for all $g \in G$.

(3) $ghg^{-1} \in H$ for all $g \in G$, $h \in H$.

(4) $gH = Hg$ for all $g \in H$.

Definition. (Automorphism)

(1) An isomorphism $\varphi : G \to G$ is called an **automorphism**.

(2) $\imath_g : G \to G$ defined as $\imath_g(x) = gxg^{-1}$ is called the **inner automorphism** by g.

(3) gxg^{-1} is called a **conjugation** of x by g.

(4) For $H \leq G$, $i_g(H) = gHg^{-1}$ is called the **conjugation subgroup** of H.

(5) If $\imath_g(H) = H$ then H is called **invariant**.

Remark. Normal subgroups of G are invariant under all inner automorphisms.

[2] Natural projection $\pi : G \to G/N$, $\ker\pi = N$.

Section 15. Factor Group Computations & Simple Groups

We want to see if some quotient group is a group we already know!

Example.

(1) $\mathbb{Z}_n \simeq \mathbb{Z}/n\mathbb{Z}$.

(2) $A_n \trianglelefteq S_n$, since $\tau A_n = A_n\tau$ for any transposition τ. (a permutation is either odd or even) Then $(S_n : A_n) = 2$, and since a group of order 2 is unique, we have $S_n/A_n \simeq \mathbb{Z}_2$.

Theorem. If $H \leq G$ with $(G : H) = 2$, $H \trianglelefteq G$ and $G/H \simeq \mathbb{Z}_2$.

Proof. Write $G = H \sqcup aH$ where $a \in G \setminus H$. Similarly, $G = H \sqcup Ha$. So $aH = Ha = G \setminus H$. Therefore $H \trianglelefteq G$, and G/H is a quotient group with order 2. Since $\mathbb{Z}_2$ is the only group of order 2, $G/H \simeq \mathbb{Z}_2$. □

Recall. (Lagrange) Let G be a finite group with $H \leq G$. Then $|H| \mid |G|$.

Proposition. The converse of Lagrange's Theorem does not hold.

Proof. Consider A_4. Suppose that there is a subgroup H of order 6. Then $A_4/H \simeq \mathbb{Z}_2$. Then for any $\sigma \in A_4$, $\sigma^2 \in H$. Since $(i,j,k)^2 = (i,k,j)$ and $(i,k,j)^2 = (i,j,k)$, every 3-cycle should be in H. There are 8 3-cycles in A_4, so $|H| \neq 6$. □

Theorem. For groups H, K, let $G = H \times K$. Let $\overline{H} = H \times \{e_K\} \leq G$. Then $\overline{H} \trianglelefteq G$ and $K \simeq G/\overline{H}$.

Proof.

(Method 1) For any $(h,k) \in G$, $(h,k)^{-1}(h', e_k)(h,k) \in \overline{H}$ for all $h, h' \in K$ and $k \in K$. Then the map $\varphi : K \to G/\overline{H}$ defined as $\varphi(k) = (\overline{e_H}, k)$ is an isomorphism. □

(Method 2) Consider $\varphi : G \to K$ defined as $\varphi(h,k) = k$, and show that φ is an epimorphism with $\ker\varphi = \overline{H}$. The result directly follows from the first isomorphism theorem. □

Example. $(\mathbb{Z}_4 \times \mathbb{Z}_6) / \langle(0,2)\rangle$. We see that $\langle(0,2)\rangle = \{(0,0),(0,2),(0,4)\}$, so the order of the quotient group should be $24/3 = 8$. By the fundamental theorem of FGAG, this group should be isomorphic to one of $\mathbb{Z}_8$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

- $\mathbb{Z}_4 \times \mathbb{Z}_6$ does not have an element of order greater than 8, so G/H doesn't either. $G/H \not\simeq \mathbb{Z}_8$.
- For $(1,0) \in \mathbb{Z}_4 \times \mathbb{Z}_6$, $2(1,0) = (2,0) \notin \langle(0,2)\rangle$. So $(1,0)$ has order greader than 2 in G/H.

So $G/H \not\simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Therefore $G/H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$.

There are lots of exercises like this. We see another example with infinite order. We cannot enumerate all cases like above in this example.

Example. $(\mathbb{Z} \times \mathbb{Z}) / \langle(1,1)\rangle$. Consider $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined as $\varphi(a,b) = a - b$. Next, show that φ is an epimorphism and that $\ker\varphi = \langle(1,1)\rangle$. Then $(\mathbb{Z} \times \mathbb{Z}) / \langle(1,1)\rangle \simeq \mathbb{Z}$ by the first isomorphism theorem.

Why do we study quotient groups? First, we get new kinds of groups. Second, for finitely generated abelian groups, we like to write them in their decomposed form since we know cyclic groups *very* well. So we can handle them. But for non-commutative groups, we cannot use this approach. So we want 'simple' groups, that will be a building block for larger groups.

Definition. (Simple Group) A group is **simple** if it does not have a nontrivial proper normal subgroup. i.e. only normal subgroups are $\{e\}$ and itself.

We state this theorem and use it without proof. The insolvability of the quintic comes from this statement.

Theorem. A_n is simple for $n \geq 5$.

If simple groups are the building blocks, there should be a way to derive simple groups from any group.

Definition. (Maximal Normal Subgroup) M is a **maximal normal subgroup** of G if for any *proper* $N \lhd G$, $M \subsetneq N$.

Theorem. M is a maximal normal subgroup of G if and only if G/M is simple.

April 19th, 2023

Proposition. Let $\varphi : G \to G'$ be a group homomorphism.

(1) $N \trianglelefteq G \implies \varphi(N) \trianglelefteq \varphi(G)$.

(2) $N' \trianglelefteq \varphi(G) \implies \varphi^{-1}(N') \trianglelefteq G$.

Proof. For any $\varphi(g) \in \varphi(G)$, $\varphi(g)\varphi(n)\varphi(g)^{-1} = \varphi(gng^{-1}) \in \varphi(N)$. □

Remark. $\varphi(N) \trianglelefteq G'$ is not true! For $K \trianglelefteq H \trianglelefteq G'$, $K \trianglelefteq H$ does not imply $K \trianglelefteq G'$. (not transitive) Think about the definition of normal subgroups. $K \trianglelefteq H \iff aK = Ka$ for all $a \in H$, but as for $K \trianglelefteq G'$, $aK = Ka$ for all $a \in G'$. So if we extend to a larger group, it may not be a normal subgroup.

Theorem. M is a maximal normal subgroup of G if and only if G/M is simple.

Proof. The projection $\varphi : G \to G/M$, $\varphi(g) = gM$ is a group homomorphism.

($\implies$) Suppose that M is a maximul normal subgroup and G/M is not simple. Then there exists proper $\overline{P} \lhd G/M = \varphi(G)$ which is not trivial. Then $\varphi^{-1}(\overline{P}) \lhd G$ is a normal subgroup which strictly contains M.

($\impliedby$) Suppose that G/M is simple and M is not maximal. Then there exists $P \lhd G$ such that $M \subsetneq P$. Then $\varphi(P) \lhd G/M$ is a nontrivial proper normal subgroup. □

Center & Commutator Subgroup

Definition. (Center) The center of a group G is defined as

$$Z(G) = \{z \in G : gz = zg,\ \forall g \in G\}.$$

Remark. $Z(G) \trianglelefteq G$.

Definition. (Commutator Subgroup)

(1) For $a, b \in G$, $aba^{-1}b^{-1}$ is called the **commutator** of G.

(2) $C = \langle aba^{-1}b^{-1} : a, b \in G\rangle \leq G$ is called the **commutator subgroup**.

The following theorem is the reason we use commutator subgroups. Think about the meaning. Center of a group is used to get a commutative subgroup of G. Commutator subgroup is used to quotient out the non-commutative elements, to get a commutative group.

Theorem. Let C be the commutator subgroup of G.

(1) $C \trianglelefteq G$.

(2) If $N \trianglelefteq G$, G/N is commutative if and only if $C \subset N$.

Proof. (1) Let $g \in G$, $aba^{-1}b^{-1} \in C$. We want to show that $g^{-1}aba^{-1}b^{-1}g \in C$.

$$g^{-1}aba^{-1}b^{-1}g = g^{-1}aba^{-1}(gb^{-1}bg^{-1})b^{-1}g = (g^{-1}a)b(g^{-1}a)^{-1}b^{-1}(bg^{-1}b^{-1}g) \in C.$$

(2) Left as exercise. □

Section 16. Group Action on a Set

This is a very important section! Groups appear on many branches of mathematics.

Definition. (G-set) Let G be a group and X be a set. A **group action** of G on X is a map $G \times X \to X$ such that

(1) If e is the identity of G, $ex = x$ for all $x \in X$.

(2) $(g_1g_2)x = g_1(g_2x)$ for all $g_1, g_2 \in G$ and $x \in X$.

The set X is called a G-**set**.

Example.

(1) $X = G$, consider a map $G \times X \to X$, where $(g_1, g_2) \mapsto g_1g_2$. X is a G-set.

(2) Let $G = S_n$, $X = \{1, 2, \ldots, n\}$. Consider a map $G \times X \to X$ defined as $(\sigma, x) \mapsto \sigma(x)$. X is a S_n-set.

Theorem. Let X be a G-set. For each $g \in G$, the function $\sigma_g : X \to X$ defined by $\sigma_g(x) = gx$ is a permutation of X. Also, the map $\varphi : G \to S_X$ defined by $\varphi(g) = \sigma_g$ is a homomorphism with the property $\varphi(g)(x) = gx$.

Definition. Let X be a G-set.

(1) G acts **faithfully** on X if $\{a \in G : ax = x,\ \forall x \in X\} = \{e\}$.

(2) G acts **transitively** on X if for any $x_1, x_2 \in X$, $\exists\, g \in G$ such that $gx_1 = x_2$.

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We can consider G/N-action on X. If G is not transitive, we can make it transitive by quotienting it out by N. (Why?)

Example. Think about the definitions...

(1) Consider the numbered square and D_4. Then D_4 acts on X transitively.

(2) Refer to Example 16.8. There doesn't exist an element of D_4 such that $m_1 \mapsto d_1$. This action is not transitive.

Isotropy Subgroups

Definition. Given a G-set X, fix $g \in G$, $x \in X$.

(1) $X_g = \{x \in X : gx = x\} \subset X$. (Fixed points)

(2) (Isotropy Subgroup) $G_x = \{g \in G : gx = x\} \subset G$.

$X_g \subset X$ always, but is $G_x \leq G$?

Theorem. Let X be a G-set. For $x \in X$, $G_x \leq G$.

Proof. For $g_1, g_2 \in G_x$, $(g_1 g_2)x = g_1(g_2 x) = g_1 x = x$, so $g_1 g_2 \in G_x$. Also $ex = x$ so $e \in G_x$. If $g \in G_x$, $g^{-1}x = g^{-1}(gx) = (g^{-1}g)x = x$, so $g^{-1} \in G_x$. □

Can we partition or classify elements of X with respect to the actions of G?

Theorem. Given a G-set X, for $x_1, x_2 \in X$, define

$$x_1 \sim x_2 \iff \exists\, g \in G \text{ such that } gx_1 = x_2.$$

Then $\sim$ is an equivalence relation.

Proof. Left as exercise. □

So X has a partition induced by the action of G. Each equivalence class is an orbit.

Definition. (Orbit) Given a G-set X, the **orbit** of x under G as

$$Gx = \{gx : g \in G\}.$$

Remark. Do not get confused! $Gx \subset X$, $G_x \leq G$.

Theorem. (Orbit-Stabilizer Theorem) Given a G-set X, let $x \in X$. Then $|Gx| = (G : G_x)$.

Proof. Let H be the left cosets of G_x, define $\varphi : Gx \to H$ by $\varphi(gx) = gG_x$. We show that φ is a bijection. Well-definedness! Does $g_1x = g_2x$ imply $g_1G_x = g_2G_x$? The reverse direction shows that φ is injective. Then surjectivity is trivial if we have well-definedness. Left as exercise. □

By Lagrange's Theorem, $|Gx| = |G| \,/\, |G_x|$, so we have $|G| = |Gx| \cdot |G_x|$.

Section 17. Applications of G-sets to Counting

Theorem. (Burnside's Formula) Let $|G| < \infty$, X be a finite G-set. Let r be the number of orbits. Then

$$r\,|G| = \sum_{g \in G} |X_g|\,.$$

Proof. By double counting, note that

$$\sum_{g\in G} |X_g| = \sum_{x \in X} |G_x| = |\{(g,x) \in G \times X : gx = x\}|\,.$$

By the orbit-stabilizer theorem,

$$\sum_{g\in G} |X_g| = \sum_{x\in X} \frac{|G|}{|Gx|} = |G| \sum_{\text{orbit } \mathcal{O}} \sum_{x \in \mathcal{O}} \frac{1}{|Gx|} = r\,|G|\,.$$

Here, $\sum_{x\in\mathcal{O}} \frac{1}{|Gx|} = \sum_{x\in\mathcal{O}} \frac{1}{|\mathcal{O}|} = 1$, so the last equality holds. □

Example. We have a cube with 6 faces, where we want to mark the faces with 6 distinct marks. Then how many are distinguishable?

Proof. If a cube can be rotated to give the same markings, they are not distinguishable. Let X be the set of all distinct markings. Then $|X| = 6! = 720$. (We treat all 6 faces as distinct faces) Let G be the group of rotations of the cube, then $|G| = 6 \times 4 = 24$. Then the number of orbits of X under G is equal to the number of distinguishable cubes. By Burnside's formula, $r = \frac{1}{|G|}\sum_{g\in G} |X_g|$. We can check that $X_e = X$, and if $g \neq e$, $|X_g| = 0$. So $r = \frac{720}{24} = 30$. □

Part VII

Advanced Group Theory

April 26th, 2023

Section 34. Isomorphism Theorems

Theorem. (First Isomorphism Theorem) Let $\varphi : G \to G'$ be a group homomorphism. Then $\mu : G/\ker\varphi \to \operatorname{im}\varphi$ defined as

$$\mu(x\ker\varphi) = \varphi(x), \quad (x \in G)$$

is an isomorphism, and $G/\ker\varphi \simeq \operatorname{im}\varphi$.

$$\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowright"] & \operatorname{im}\varphi \\
 & G/\ker\varphi \arrow[ur, dashed, "\mu"'] &
\end{tikzcd}$$

Lemma. Let $N \trianglelefteq G$ and $\gamma : G \to G/N$ be a canonical homomorphism. Then

$$\varphi : \{M \trianglelefteq G : N \leq M\} \to \{K \trianglelefteq G/N\}$$

defined as $\varphi(M) = \gamma(M)$ is a bijection.

Proof. Since γ is a epimorphism, if $M \trianglelefteq G$, then $\varphi(M) = \gamma(M) \trianglelefteq \gamma(G) = G/N$. We next show that φ is a bijection. $\varphi(L) = \varphi(M)$, then $\gamma(L) = \gamma(M)$ so $L = M$. This works because $N \leq M$ and $\gamma^{-1}(\gamma(M)) = M$. Also, for $K \trianglelefteq G/N$, $\varphi(\gamma^{-1}(K)) = K$. □

Recall. We proved a part of this when we proved that if M is a maximal normal subgroup, then G/M is simple.

Notation. For $H, N \leq G$, define $H \vee N = \langle H, N \rangle$.

Lemma.

(1) If $N \trianglelefteq G$ and $H \leq G$, $HN = NH \leq G$. i.e. $H \vee N = HN = NH$.

(2) If $N \trianglelefteq G$ and $H \trianglelefteq G$, then $H \vee N = HN \trianglelefteq G$.

Proof.

(1) Let $h_1 n_1, h_2 n_2 \in HN$ for $h_i \in H, n_i \in N$. Since $N \trianglelefteq G$, $\exists n_3 \in N$ such that $h_1 n_1 h_2 n_2 = h_1 h_2 n_3 n_2$, which is an element of HN. Check that $e \in HN$, $(hn)^{-1} \in HN$.

(2) For $g \in G$, we show that $ghng^{-1} \in HN$ for $h \in H$, $n \in N$. Since both subgroups are normal, there exists $h' \in H$, $n' \in N$ such that $hng^{-1} = g^{-1}h'n'$. Then $ghng^{-1} = (gg^{-1})h'n' \in HN$. □

Theorem. (Second Isomorphism Theorem) If $H \leq G$ and $N \trianglelefteq G$,

$$\frac{HN}{N} \simeq \frac{H}{H \cap N}.$$

Proof. We first have to check that $N \trianglelefteq HN$ and $H \cap N \trianglelefteq H$. (Check!)

Consider $\gamma : G \to G/N$. It is clear that γ is surjective.

- $\gamma \mid_H : H \to \gamma(H)$
- $\gamma \mid_{HN} : HN \to \gamma(HN)$. Actually, $\gamma(HN) = \gamma(H)$.

By the first isomorphism theorem,

$$\frac{H}{\ker \gamma \mid_H} \simeq \gamma(H) \simeq \frac{HN}{\ker \gamma \mid_{HN}}.$$

Now we check that $\ker \gamma \mid_H = H \cap N$ and $\ker \gamma \mid_{HN} = N$. This is trivial. □

Example. Let $G = \mathbb{Z}_{24}$, $H = \langle 4 \rangle$, $N = \langle 6 \rangle$. Then

$$HN = \{n4 + m6 : n, m \in \mathbb{Z}\} = \langle 2 \rangle, \quad H \cap N = \langle 4 \rangle \cap \langle 6 \rangle = \langle 12 \rangle.$$

We see that $HN/N \simeq H/H \cap N \simeq \mathbb{Z}_3$.

Theorem. (Third Isomorphism Theorem) If $H, K \trianglelefteq G$ and $K \leq H$,[1] then

$$G/H \simeq \frac{G/K}{H/K}.$$

[1] This implies that $K \trianglelefteq H$.

Proof. We first have to check that $K \trianglelefteq H$ and $H/K \trianglelefteq G/K$. (Check the first)

Let $\overline{g} \in G/K$, $\overline{h} \in H/K$. We show that $\overline{g} \cdot \overline{h} \cdot \overline{g^{-1}} \in H/K$, this is trivial. Define

$$\varphi : G \xrightarrow{\psi_1} G/K \xrightarrow{\psi_2} \frac{G/K}{H/K}.$$

We check that φ is a well-defined group homomorphism, and that φ is surjective, and $\ker \varphi = H$. Then the result directly follows from the first isomorphism theorem. □

Example. Let $G = \mathbb{Z}$, $H = 2\mathbb{Z}$, $K = 6\mathbb{Z}$. $G/K \simeq \mathbb{Z}_2$, $G/K \simeq \mathbb{Z}_6$, $H/K \simeq \mathbb{Z}_3$. Then $\mathbb{Z}_2 \simeq \mathbb{Z}_6/\mathbb{Z}_3$ (abuse of notation) by the third isomorphism theorem.

Section 35. Series of Groups

Given a group G, we want to decompose G with simpler groups, $G \rightsquigarrow G_1 * G_2 * \cdots * G_k$. We consider a series of normal subgroups,

$$\cdots \trianglelefteq N_3 \trianglelefteq N_2 \trianglelefteq N_1 \trianglelefteq G.$$

Then we can consider a sort of a situation like

$$G = G/N_1 * N_1/N_2 * N_2/N_3 * \cdots$$

and if each normal subgroup is maximal, then each term is simple! This leads us to the Jordan-Hölder theorem. We will use the second/third isomorphism theorems.

May 1st, 2023

Definition. Given a finite sequence of subgroups

$$\{e\} = H_0 < H_1 < \cdots < H_k = G,$$

(1) (Subnormal Series) If $H_i \lhd H_{i+1}$ for $i = 0, \ldots, k-1$, it is called a **subnormal series**.

(2) (Normal Series) If $H_i \lhd G$ for $i = 0, \ldots, k-1$, it is called a **normal series**.

Note that the *finite* condition is important!

Remark. Since $H_i \lhd G \implies H_i \lhd H_{i+1}$, a normal series is a subnormal series.

We will focus on subnormal series.

Example. Let $G = \mathbb{Z}$. Since G is abelian, all subgroups are normal, so

$$\{0\} = H_0 < 12\mathbb{Z} < 6\mathbb{Z} < 3\mathbb{Z} < \mathbb{Z} = G$$

is a subnormal series. Also,

$$\{0\} = H_0 < 18\mathbb{Z} < 9\mathbb{Z} < 3\mathbb{Z} < \mathbb{Z} = G.$$

We see that subnormal series are not unique, and note that we can find an infinite series of subgroups of $\mathbb{Z}$. But actually we want uniqueness, since we want to find the building blocks of a group. How can we obtain a *unique* subnormal series, up to isomorphism?

Definition. Subnormal (normal) series $\{H_i\}_{i\in I}$, $\{K_j\}_{j\in J}$ of G are **isomorphic** if there exists a bijection between

$$\{H_{i+1}/H_i\} \longleftrightarrow \{K_{j+1}/K_j\}$$

such that the corresponding quotient groups are isomorphic.

Example. Consider the following subnormal series,

$$\{H_i\} : \{0\} < \{0, 2, 4, 6\} < \mathbb{Z}_8, \quad \{K_j\} : \{0\} < \{0, 2\} < \mathbb{Z}_8.$$

We know that $H_2/H_1 \simeq \mathbb{Z}_2$, $H_1/H_0 \simeq \mathbb{Z}_4$, and $K_2/K_1 \simeq \mathbb{Z}_4$, $K_1/K_0 \simeq \mathbb{Z}_2$. So we will map H_2/H_1 to K_1/K_0, and H_1/H_0 to K_2/K_1. So these two series are isomorphic. The building blocks of $\mathbb{Z}_8$ obtained from the two series are equivalent!

Definition. (Refinement) For two subnormal (normal) series $\{H_i\}$ and $\{K_j\}$ of G, $\{K_j\}$ is a **refinement** of $\{H_i\}$ if $\{H_i\} \subset \{K_j\}$.

Example. Given a subnormal series $\{H_i\}$: $\{0\} < \langle 8\rangle < \langle 2\rangle < \mathbb{Z}_{24}$, we can insert $\langle 4\rangle$ to get a subnormal series

$$\{K_j\} : \{0\} < \langle 8\rangle < \langle \mathbf{4}\rangle < \langle 2\rangle < \mathbb{Z}_{24}.$$

Now, each K_{j+1}/K_j has no nontrivial proper subgroups.

We are one step closer to getting uniqueness.

Theorem. (Schrier) Any two subnormal (normal) series of G have isomorphic refinements.

Remark. This theorem does not state that for any two pairs of subnormal series, their isomorphic refinements are isomorphic. i.e. if $\{H_i\}, \{K_i\}, \{L_i\}, \{M_i\}$ are subnormal series of G, the isomorphic refinements obtained from $\{H_i\}, \{K_i\}$ and $\{L_i\}, \{M_i\}$ need not be isomorphic. So we don't have the exact uniqueness. As an example, consider the following series

$$\{0\} < 36\mathbb{Z} < 12\mathbb{Z} < 3\mathbb{Z} < \mathbb{Z}, \quad \{0\} < 72\mathbb{Z} < 8\mathbb{Z} < 4\mathbb{Z} < \mathbb{Z}.$$

We can obtain many isomoprhic refinements by appending some $n\mathbb{Z}$ to the end of the series. So the isomorphic refinements can change, depending on the last element of the series.

Example. Consider these subnormal series,

$$\{0\} < \langle 8\rangle < \langle 2\rangle < \mathbb{Z}_{24}, \quad \{0\} < \langle 12\rangle < \langle 6\rangle < \mathbb{Z}_{24}.$$

We can obtain a refinement by

$$\{0\} < \langle 8\rangle < \langle \mathbf{4}\rangle < \langle 2\rangle < \mathbb{Z}_{24}, \quad \{0\} < \langle 12\rangle < \langle 6\rangle < \langle \mathbf{3}\rangle < \mathbb{Z}_{24}.$$

These two series are isomorphic.

Definition. Let $\{H_i\}$ be a subnormal (normal) series of G. If H_{i+1}/H_i is simple for any i, $\{H_i\}$ is called a **composition (principal) series**.

Theorem. (Jordan-Hölder) Any two composition series of a group G are isomorphic.

Remark. We assumed something that wasn't an assumption in the Schrier theorem. Does G really have a composition series? Consider the series $\{0\} < n\mathbb{Z} < 6\mathbb{Z} < 3\mathbb{Z} < \mathbb{Z}$. This cannot be a composition series, since any multiple m of n, the quotient group $n\mathbb{Z}/\{0\} \simeq n\mathbb{Z}$ has a subgroup $m\mathbb{Z}$. So this series is infinite, thus $\mathbb{Z}$ has no composition series. In Jordan-Hölder theorem, we are assuming that G has a composition series, which gets rid of groups that don't have a composition series.

Proof. Composition series cannot have any further refinements. By Schrier theorem. □

Proof. (of Schrier, Idea sketch) Consider two subnormal series

$$\{0\} = H_0 \leq H_1 \leq \cdots \leq H_i \leq H_{i+1} \leq \cdots H_k = G,$$

$$\{0\} = K_0 \leq K_1 \leq \cdots \leq K_j \leq K_{j+1} \leq \cdots \leq K_l = G.$$

Then in between $H_i \leq H_{i+1}$, write

$$H_i(K_0 \cap H_{i+1}) \leq H_i(K_1 \cap H_{i+1}) \leq \cdots \leq H_i(K_l \cap H_{i+1}).$$

In between $K_j \leq K_{j+1}$, write

$$K_j(H_0 \cap K_{j+1}) \leq K_j(H_1 \cap K_{j+1}) \leq \cdots \leq K_j(H_k \cap K_{j+1}).$$

Our claim is that if we obtain a refienment by applying the above to all pairs, we can get an isomorphic subnormal series.

Claim. $H_i(K_{j+1} \cap H_{i+1})/H_i(K_j \cap H_{i+1}) \simeq K_j(H_{i+1} \cap K_{j+1})/K_j(H_i \cap K_{j+1})$.

Proof. By the following lemma. □

Lemma. (Zassenhaus) Suppose that $H^* \trianglelefteq H$, $K^* \trianglelefteq K$, $H, K \leq G$. Then the following hold.

(1) $H^*(H \cap K^*) \trianglelefteq H^*(H \cap K)$.

(2) $K^*(H^* \cap K) \trianglelefteq K^*(H \cap K)$.

(3) $H^*(H \cap K)/H^*(H \cap K^*) \simeq K^*(H \cap K)/K^*(H^* \cap K) \simeq H \cap K/(H^* \cap K)(H \cap K^*)$.

Proof. Consider $\varphi : H^*(H \cap K) \to H \cap K/(H^* \cap K)(H \cap K^*)$, that maps $\varphi(hx) = \overline{x}$. ($h \in H^*$, $x \in H \cap K$) We need to check the following.

- $H^*(H \cap K)$ is a group.
- $H \cap K/(H^* \cap K)(H \cap K^*)$ is a quotient group. ($H^* \cap K, H \cap K^* \trianglelefteq H \cap K$)
- φ is a well-defined epimorphism.
- $\ker \varphi = H^*(H \cap K^*)$.

Then the result follows from the first isomorphism theorem. □