Comprehensive Calibration, Bias Modeling, and Bayesian Filtering for Piezoelectric Pressure Transducers in Closed-Loop Control

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1 Introduction

Piezoelectric pressure transducers (PTs) produce analog voltages proportional to applied pressure, but calibration is fragile due to mounting torque, mechanical shift, wiring strain, and environmental drift. Current workflows relying on ridge regression with human-in-the-loop gauge readings are insufficient for closed-loop control applications requiring:

- Robust uncertainty quantification across transducer variability
- Automatic drift detection and bias correction
- Extrapolation confidence beyond calibration ranges
- Real-time adaptation to mounting and environmental changes

We develop a mathematically rigorous framework unifying Bayesian regression, total least squares (TLS), recursive least squares (RLS), generalized likelihood ratio (GLR) testing, and Extended Kalman Filter (EKF) integration for adaptive online bias correction with full uncertainty propagation.

2 Physical and Statistical Measurement Model

2.1 Complete Measurement Model

The PT measurement model accounts for multiple sources of uncertainty:

$$p_{\text{true}} = f(v; \boldsymbol{\theta}) + b(t) + \epsilon_{\text{meas}} + \epsilon_{\text{temp}} + \epsilon_{\text{aging}} \tag{1}$$

where:

- v is the transducer voltage
- $f(\cdot; \boldsymbol{\theta})$ is the deterministic calibration map
- $b(t)$ is time-varying bias from mounting and environmental effects
- $\epsilon_{\text{meas}} \sim \mathcal{N}(0, \sigma^2_{\text{meas}})$ is measurement noise
- $\epsilon_{\text{temp}} \sim \mathcal{N}(0, \sigma^2_{\text{temp}})$ is temperature-induced noise
- $\epsilon_{\text{aging}} \sim \mathcal{N}(0, \sigma^2_{\text{aging}})$ is aging/drift noise

2.2 Robust High-Fidelity Calibration Maps

2.2.1 Physically-Informed Polynomial Models

For piezoelectric transducers, the calibration map should capture the inherent non-linear behavior:

$$f(v;\boldsymbol{\theta}) = \theta_0 + \theta_1 v + \theta_2 v^2 + \theta_3 v^3 + \theta_4\sqrt{v} + \theta_5 \log(1+v) \tag{2}$$

2.2.2 Environmental-Robust Calibration Map

A unified calibration map that intrinsically handles all environmental variations:

$$f(v,\mathbf{e};\boldsymbol{\theta}) = \sum_{k=0}^{n} \theta_k \phi_k(v,\mathbf{e}) \tag{3}$$

where $\mathbf{e} = [T, \text{humidity}, \text{vibration}, \text{aging\_factor}]^T$ represents environmental state, and ϕ_k are robust basis functions:

$$\phi_0(v,\mathbf{e}) = 1 \tag{4}$$

$$\phi_1(v,\mathbf{e}) = v \tag{5}$$

$$\phi_2(v,\mathbf{e}) = v^2 + \alpha_1 T v + \alpha_2 \text{humidity} v \tag{6}$$

$$\phi_3(v,\mathbf{e}) = v^3 + \beta_1 T v^2 + \beta_2 \text{vibration} v \tag{7}$$

$$\phi_4(v,\mathbf{e}) = \sqrt{v} + \gamma_1 \text{aging\_factor} \log(v) \tag{8}$$

$$\phi_5(v,\mathbf{e}) = \log(1+v) + \delta_1 T + \delta_2 \text{humidity} \tag{9}$$

2.2.3 Adaptive Spline Calibration

For maximum fidelity, use adaptive cubic splines with environmental-dependent knots:

$$f(v,\mathbf{e};\boldsymbol{\theta}) = \sum_{j=1}^{J} \theta_j B_j(v,\mathbf{e}) \tag{10}$$

where $B_j(v,\mathbf{e})$ are B-spline basis functions with knots that adapt to environmental conditions:

$$t_j(\mathbf{e}) = t_{j,\text{nom}} + \mathbf{w}_j^T \mathbf{e} \tag{11}$$

2.2.4 Physics-Informed Neural Network Calibration

For maximum flexibility and fidelity:

$$f(v,\mathbf{e};\boldsymbol{\theta}) = \mathcal{N}(v,\mathbf{e};\boldsymbol{\theta}) + \mathcal{P}(v,\mathbf{e}) \tag{12}$$

where $\mathcal{N}$ is a neural network and $\mathcal{P}$ enforces physical constraints:

$$\mathcal{P}(v,\mathbf{e}) = \lambda_1 \frac{\partial f}{\partial v} + \lambda_2 \frac{\partial^2 f}{\partial v^2} + \lambda_3 \int f\, dv \tag{13}$$

2.3 Gauge and Reference Uncertainty

During calibration, reference pressures contain measurement errors:

$$p_{\text{obs}} = p_{\text{true}} + \eta_{\text{gauge}} + \eta_{\text{drift}} \tag{14}$$

where $\eta_{\text{gauge}} \sim \mathcal{N}(0, \sigma^2_{\text{gauge}})$ and $\eta_{\text{drift}} \sim \mathcal{N}(0, \sigma^2_{\text{drift}})$.

3 Robust Regression Approaches

3.1 Errors-in-Variables and Total Least Squares

Since both voltage and pressure measurements contain noise, ordinary least squares is biased. Total least squares minimizes orthogonal distances:

$$\min_{\boldsymbol{\theta}} \sum_i \frac{(p_{\text{obs},i} - f(v_i; \boldsymbol{\theta}))^2}{\sigma^2_{\text{total},i}} \tag{15}$$

where the total uncertainty is:

$$\sigma^2_{\text{total},i} = \sigma^2_{\text{gauge}} + \sigma^2_{\text{drift}} + \sigma^2_{\text{meas}} + \sigma^2_{\text{temp}} + \sigma^2_{\text{aging}} + \sigma^2_{v} \tag{16}$$

3.2 Bayesian Regression with Hierarchical Priors

We employ a hierarchical Bayesian approach to model transducer-to-transducer variability:

3.2.1 Population-Level Priors

$$\boldsymbol{\theta} \sim \mathcal{N}(\boldsymbol{\mu}_{\text{pop}}, \Sigma_{\text{pop}}) \tag{17}$$

3.2.2 Individual Transducer Priors

For transducer j:

$$\boldsymbol{\theta}^{(j)} | \boldsymbol{\theta} \sim \mathcal{N}(\boldsymbol{\theta}, \Sigma_{\text{ind}}) \tag{18}$$

3.2.3 Posterior Distribution

The posterior for transducer j given calibration data $\mathcal{D}^{(j)}$ is:

$$p(\boldsymbol{\theta}^{(j)} | \mathcal{D}^{(j)}, \boldsymbol{\theta}) \propto p(\mathcal{D}^{(j)} | \boldsymbol{\theta}^{(j)}) p(\boldsymbol{\theta}^{(j)} | \boldsymbol{\theta}) p(\boldsymbol{\theta}) \tag{19}$$

3.2.4 Predictive Distributions

The predictive distribution for new measurements is:

$$p(p | v, \mathcal{D}^{(j)}) = \int p(p | v, \boldsymbol{\theta}^{(j)}) p(\boldsymbol{\theta}^{(j)} | \mathcal{D}^{(j)}) d\boldsymbol{\theta}^{(j)} \tag{20}$$

For Gaussian priors and likelihoods, this yields:

$$p(p | v, \mathcal{D}^{(j)}) = \mathcal{N}(\hat{p}, \sigma^2_{\text{pred}}) \tag{21}$$

where:

$$\hat{p} = f(v; \hat{\boldsymbol{\theta}}^{(j)}) \tag{22}$$

$$\sigma^2_{\text{pred}} = \sigma^2_{\text{meas}} + J_{\boldsymbol{\theta}} \Sigma^{(j)}_{\boldsymbol{\theta}} J^T_{\boldsymbol{\theta}} + \sigma^2_{\text{extrapolation}} \tag{23}$$

3.3 Recursive Least Squares with Forgetting

For online calibration updates:

$$K_k = P_{k-1}\phi_k(\lambda + \phi_k^T P_{k-1}\phi_k)^{-1} \tag{24}$$

$$\hat{\boldsymbol{\theta}}_k = \hat{\boldsymbol{\theta}}_{k-1} + K_k(p_{\text{obs},k} - \phi_k^T \hat{\boldsymbol{\theta}}_{k-1}) \tag{25}$$

$$P_k = \lambda^{-1}(P_{k-1} - K_k \phi_k^T P_{k-1}) \tag{26}$$

$$\Sigma_{\boldsymbol{\theta},k} = P_k + \Sigma_{\text{forgetting}} \tag{27}$$

where $\lambda \in (0, 1]$ is the forgetting factor and $\Sigma_{\text{forgetting}}$ accounts for parameter drift.

4 Bias and Drift Modeling

4.1 Unified Environmental Variance Model

Instead of separate bias terms, we model all environmental variations through a unified variance structure that adapts the calibration map itself:

4.1.1 Environmental State Vector

$$\mathbf{e} = [T, H, V, A, M]^T \tag{28}$$

where:

- T: Temperature
- H: Humidity
- V: Vibration level
- A: Aging factor (time-dependent)
- M: Mounting torque factor

4.1.2 Adaptive Variance Model

The measurement variance adapts to environmental conditions:

$$\sigma^2_{\text{total}}(v, \mathbf{e}) = \sigma^2_{\text{base}} + \sigma^2_{\text{env}}(v, \mathbf{e}) + \sigma^2_{\text{nonlinear}}(v, \mathbf{e}) \tag{29}$$

where:

$$\sigma^2_{\text{env}}(v, \mathbf{e}) = \mathbf{e}^T \mathbf{Q}_{\text{env}} \mathbf{e} + v^2 \mathbf{e}^T \mathbf{Q}_{\text{interaction}} \mathbf{e} \tag{30}$$

$$\sigma^2_{\text{nonlinear}}(v, \mathbf{e}) = \alpha_1 v^4 + \alpha_2 \|\mathbf{e}\|^2 v^2 + \alpha_3 \|\mathbf{e}\|^4 \tag{31}$$

4.1.3 Process Variance Evolution

The calibration parameters themselves evolve with environmental conditions:

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k + \mathbf{w}_{\theta,k}(\mathbf{e}_k) \tag{32}$$

where the process noise is environment-dependent:

$$\mathbf{w}_{\theta,k}(\mathbf{e}_k) \sim \mathcal{N}(0, \mathbf{Q}_\theta(\mathbf{e}_k)) \tag{33}$$

with:

$$\mathbf{Q}_\theta(\mathbf{e}) = \mathbf{Q}_{\theta,\text{base}} + \sum_{i=1}^{n_e} e_i \mathbf{Q}_{\theta,i} + \sum_{i=1}^{n_e} \sum_{j=i}^{n_e} e_i e_j \mathbf{Q}_{\theta,ij} \tag{34}$$

4.1.4 Residual Bias Model

A minimal residual bias term captures effects not captured by the environmental calibration map:

$$b_{\text{residual},k+1} = \rho b_{\text{residual},k} + w_b, \quad w_b \sim \mathcal{N}(0, Q_b(\mathbf{e}_k)) \tag{35}$$

where:

$$Q_b(\mathbf{e}) = Q_{b,\text{base}} \left(1 + \|\mathbf{e}\|^2 / \|\mathbf{e}_{\text{ref}}\|^2\right) \tag{36}$$

5 Change Detection and Calibration Validation

5.1 Generalized Likelihood Ratio Test

The GLR test evaluates whether new data is consistent with the current calibration:

$$\Lambda = \frac{\sup_{\boldsymbol{\theta}} L(\boldsymbol{\theta}; \mathcal{D}_{\text{new}})}{L(\hat{\boldsymbol{\theta}}; \mathcal{D}_{\text{new}})} \tag{37}$$

5.1.1 Windowed GLR Test

For a sliding window of size N:

$$\Lambda_k = \frac{\max_{j \in [k-N+1,k]} \sup_{\boldsymbol{\theta}} L(\boldsymbol{\theta}; \mathcal{D}_{j:k})}{L(\hat{\boldsymbol{\theta}}; \mathcal{D}_{j:k})} \tag{38}$$

5.1.2 Threshold Selection

The threshold γ is selected based on desired false alarm rate:

$$P(\Lambda > \gamma | H_0) = \alpha \tag{39}$$

5.2 Cumulative Sum (CUSUM) Test

For detecting gradual drift:

$$S_k = \max(0, S_{k-1} + \log \frac{p(z_k | \hat{\boldsymbol{\theta}}_{\text{new}})}{p(z_k | \hat{\boldsymbol{\theta}}_{\text{old}})}) \tag{40}$$

5.3 Extrapolation Confidence

For measurements outside calibration range, we compute extrapolation uncertainty:

$$\sigma^2_{\text{extrapolation}} = \sigma^2_{\text{model}} + \sigma^2_{\text{range}} \tag{41}$$

where:

$$\sigma^2_{\text{model}} = \sum_k \frac{\partial^2 f}{\partial \theta_k^2} \Sigma_{\theta_k} \tag{42}$$

$$\sigma^2_{\text{range}} = \alpha_{\text{range}} \left(\frac{v - v_{\text{cal,min}}}{v_{\text{cal,max}} - v_{\text{cal,min}}} \right)^2 \tag{43}$$

6 EKF Integration for Online Adaptation

6.1 State Augmentation

The EKF state vector includes physical states, calibration parameters, and environmental state:

$$\mathbf{x}_k = \begin{bmatrix} \mathbf{x}_{\text{phys},k} \\ \boldsymbol{\theta}_k \\ \mathbf{e}_k \\ b_{\text{residual},k} \end{bmatrix} \tag{44}$$

6.2 Process Model

$$\mathbf{x}_{k+1} = \begin{bmatrix} \mathbf{F}_{\text{phys}} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \\ 0 & 0 & \mathbf{F}_{\text{env}} & 0 \\ 0 & 0 & 0 & \rho \end{bmatrix} \mathbf{x}_k + \mathbf{w}_k \tag{45}$$

where the process noise is environment-dependent:

$$\mathbf{w}_k \sim \mathcal{N}(0, \mathbf{Q}(\mathbf{e}_k)) \tag{46}$$

with:

$$\mathbf{Q}(\mathbf{e}) = \begin{bmatrix} \mathbf{Q}_{\text{phys}} & 0 & 0 & 0 \\ 0 & \mathbf{Q}_\theta(\mathbf{e}) & 0 & 0 \\ 0 & 0 & \mathbf{Q}_{\text{env}} & 0 \\ 0 & 0 & 0 & Q_b(\mathbf{e}) \end{bmatrix} \tag{47}$$

6.3 Measurement Model

$$h(\mathbf{x}, v) = f(v, \mathbf{e}; \boldsymbol{\theta}) + b_{\text{residual}} \tag{48}$$

6.4 Jacobian Computation

$$\mathbf{H} = \frac{\partial h}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial h}{\partial \mathbf{x}_{\text{phys}}} & \frac{\partial f}{\partial \boldsymbol{\theta}} & \frac{\partial f}{\partial \mathbf{e}} & 1 \end{bmatrix} \tag{49}$$

where:

$$\frac{\partial f}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \phi_0(v, \mathbf{e}) & \phi_1(v, \mathbf{e}) & \cdots & \phi_n(v, \mathbf{e}) \end{bmatrix} \tag{50}$$

$$\frac{\partial f}{\partial \mathbf{e}} = \sum_{k=0}^{n} \theta_k \frac{\partial \phi_k}{\partial \mathbf{e}} \tag{51}$$

6.5 Adaptive Measurement Covariance

$$\mathbf{R}_k = \sigma^2_{\text{total}}(v_k, \mathbf{e}_k) + \mathbf{J}_{\boldsymbol{\theta}}\Sigma_{\boldsymbol{\theta},k}\mathbf{J}_{\boldsymbol{\theta}}^T + \mathbf{J}_{\mathbf{e}}\Sigma_{\mathbf{e},k}\mathbf{J}_{\mathbf{e}}^T \tag{52}$$

7 Robustness Across Transducers

7.1 Population-Based Calibration

7.1.1 Multi-Transducer Calibration

Given calibration data from M transducers:

$$\mathcal{D}_{\text{pop}} = \{\mathcal{D}^{(1)}, \mathcal{D}^{(2)}, \ldots, \mathcal{D}^{(M)}\} \tag{53}$$

The population-level posterior is:

$$p(\boldsymbol{\theta}, \Sigma_{\text{ind}}|\mathcal{D}_{\text{pop}}) \propto \prod_{j=1}^{M} \int p(\mathcal{D}^{(j)}|\boldsymbol{\theta}^{(j)})p(\boldsymbol{\theta}^{(j)}|\boldsymbol{\theta}, \Sigma_{\text{ind}})d\boldsymbol{\theta}^{(j)} \tag{54}$$

7.1.2 Transfer Learning

For a new transducer with limited data, we use the population prior:

$$p(\boldsymbol{\theta}^{(j)}|\mathcal{D}^{(j)}, \mathcal{D}_{\text{pop}}) \propto p(\mathcal{D}^{(j)}|\boldsymbol{\theta}^{(j)})p(\boldsymbol{\theta}^{(j)}|\hat{\boldsymbol{\theta}}_{\text{pop}}, \hat{\Sigma}_{\text{ind}}) \tag{55}$$

7.2 Uncertainty Propagation

The total measurement uncertainty includes:

$$\sigma^2_{\text{total}} = \sigma^2_{\text{measurement}} + \sigma^2_{\text{calibration}} + \sigma^2_{\text{bias}} + \sigma^2_{\text{extrapolation}} \tag{56}$$

$$= \sigma^2_{\text{meas}} + \mathbf{J}_{\boldsymbol{\theta}}\Sigma_{\boldsymbol{\theta}}\mathbf{J}_{\boldsymbol{\theta}}^T + \Sigma_b + \sigma^2_{\text{extrapolation}} \tag{57}$$

8 High-Fidelity Calibration Map Design

8.1 Design Principles for Maximum Fidelity

8.1.1 Physical Constraint Integration

To achieve maximum fidelity, the calibration map must respect physical constraints:

$$\frac{\partial f}{\partial v} > 0 \quad \text{(monotonicity)} \tag{58}$$

$$\frac{\partial^2 f}{\partial v^2} \geq 0 \quad \text{(convexity for piezoelectric response)} \tag{59}$$

$$\lim_{v \to 0} f(v, \mathbf{e}; \boldsymbol{\theta}) = p_{\text{offset}} \quad \text{(zero-voltage offset)} \tag{60}$$

8.1.2 Multi-Resolution Calibration

Use hierarchical calibration with multiple resolution levels:

$$f(v, \mathbf{e}; \boldsymbol{\theta}) = f_{\text{coarse}}(v, \mathbf{e}; \boldsymbol{\theta}_{\text{coarse}}) + f_{\text{fine}}(v, \mathbf{e}; \boldsymbol{\theta}_{\text{fine}}) \tag{61}$$

where:

$$f_{\text{coarse}}(v, \mathbf{e}; \boldsymbol{\theta}_{\text{coarse}}) = \sum_{k=0}^{3} \theta_k \phi_k^{\text{coarse}}(v, \mathbf{e}) \tag{62}$$

$$f_{\text{fine}}(v, \mathbf{e}; \boldsymbol{\theta}_{\text{fine}}) = \sum_{k=4}^{n} \theta_k \phi_k^{\text{fine}}(v, \mathbf{e}) \tag{63}$$

8.1.3 Adaptive Basis Function Selection

Automatically select optimal basis functions using information criteria:

$$\text{AIC} = 2k - 2\ln(L) + \frac{2k(k+1)}{N-k-1} \tag{64}$$

where k is the number of parameters and L is the likelihood.

8.2 Environmental Robustness Enhancement

8.2.1 Cross-Environmental Calibration

Calibrate across multiple environmental conditions simultaneously:

$$\min_{\boldsymbol{\theta}} \sum_{j=1}^{M} \sum_{i=1}^{N_j} \frac{(p_{\text{obs},ij} - f(v_{ij}, \mathbf{e}_j; \boldsymbol{\theta}))^2}{\sigma^2_{\text{total},ij}} \tag{65}$$

where M is the number of environmental conditions and N_j is the number of measurements in condition j.

8.2.2 Transfer Learning Between Environments

Use domain adaptation techniques to transfer calibration knowledge:

$$f(v, \mathbf{e}_{\text{new}}; \boldsymbol{\theta}) = f(v, \mathbf{e}_{\text{ref}}; \boldsymbol{\theta}) + \Delta f(\mathbf{e}_{\text{new}} - \mathbf{e}_{\text{ref}}; \boldsymbol{\theta}_\Delta) \tag{66}$$

8.3 Variance Model Sophistication

8.3.1 Heteroscedastic Variance Modeling

Model variance as a function of both voltage and environmental conditions:

$$\log \sigma^2(v, \mathbf{e}) = \beta_0 + \beta_1 v + \beta_2 v^2 + \mathbf{e}^T \boldsymbol{\beta}_{\text{env}} + \mathbf{e}^T \mathbf{B}_{\text{interaction}} \mathbf{e} \tag{67}$$

8.3.2 Non-parametric Variance Estimation

Use Gaussian Process regression for variance modeling:

$$\sigma^2(v, \mathbf{e}) \sim \mathcal{GP}(\mu_{\sigma^2}(v, \mathbf{e}), k_{\sigma^2}((v, \mathbf{e}), (v', \mathbf{e}'))) \tag{68}$$

with kernel:

$$k_{\sigma^2}((v, \mathbf{e}), (v', \mathbf{e}')) = k_v(v, v') \times k_{\mathbf{e}}(\mathbf{e}, \mathbf{e}') \tag{69}$$

9 Practical Implementation Workflow

9.1 High-Fidelity Calibration Phase

Algorithm 1 Environmental-Robust Bayesian Calibration with Adaptive TLS

Input: Calibration data $\{(v_i, p_{\text{obs},i}, \mathbf{e}_i)\}_{i=1}^N$
Input: Environmental uncertainties $\{\sigma_{\text{env},i}\}$
Input: Population priors $\boldsymbol{\mu}_{\text{pop}}, \Sigma_{\text{pop}}$
Initialize $\boldsymbol{\theta}^{(0)} = \boldsymbol{\mu}_{\text{pop}}$, $\Sigma_{\boldsymbol{\theta}}^{(0)} = \Sigma_{\text{pop}}$
Initialize environmental variance parameters $\mathbf{Q}_{\text{env}}^{(0)}, \mathbf{Q}_{\text{interaction}}^{(0)}$
for $k = 1$ to max_iterations **do**
    Step 1: Update environmental variance model
    $\sigma_{\text{total},i}^2 = \sigma_{\text{base}}^2 + \mathbf{e}_i^T \mathbf{Q}_{\text{env}}^{(k-1)} \mathbf{e}_i + v_i^2 \mathbf{e}_i^T \mathbf{Q}_{\text{interaction}}^{(k-1)} \mathbf{e}_i$
    $\sigma_{\text{total},i}^2 \mathrel{+}= \alpha_1 v_i^4 + \alpha_2 \|\mathbf{e}_i\|^2 v_i^2 + \alpha_3 \|\mathbf{e}_i\|^4$
    Step 2: Solve robust TLS with environmental calibration map
    $\min_{\boldsymbol{\theta}} \sum_i \frac{(p_{\text{obs},i} - f(v_i, \mathbf{e}_i; \boldsymbol{\theta}))^2}{\sigma_{\text{total},i}^2}$
    Step 3: Update calibration parameter posterior
    $\mathbf{H}_i = \frac{\partial f}{\partial \boldsymbol{\theta}}\Big|_{v_i, \mathbf{e}_i, \boldsymbol{\theta}^{(k-1)}}$
    $\Sigma_{\boldsymbol{\theta}}^{(k)} = \left(\Sigma_{\boldsymbol{\theta}}^{(k-1)^{-1}} + \sum_i \frac{\mathbf{H}_i^T \mathbf{H}_i}{\sigma_{\text{total},i}^2} \right)^{-1}$
    $\boldsymbol{\theta}^{(k)} = \Sigma_{\boldsymbol{\theta}}^{(k)} \left(\Sigma_{\boldsymbol{\theta}}^{(k-1)^{-1}} \boldsymbol{\theta}^{(k-1)} + \sum_i \frac{\mathbf{H}_i^T (p_{\text{obs},i} - f(v_i, \mathbf{e}_i; \boldsymbol{\theta}^{(k-1)}))}{\sigma_{\text{total},i}^2} \right)$
    Step 4: Update environmental variance parameters
    Estimate $\mathbf{Q}_{\text{env}}^{(k)}, \mathbf{Q}_{\text{interaction}}^{(k)}$ from residuals
    $\alpha_1^{(k)}, \alpha_2^{(k)}, \alpha_3^{(k)} \leftarrow$ nonlinear variance fitting
    if $\|\boldsymbol{\theta}^{(k)} - \boldsymbol{\theta}^{(k-1)}\| < \epsilon$ and $\|\mathbf{Q}_{\text{env}}^{(k)} - \mathbf{Q}_{\text{env}}^{(k-1)}\| < \epsilon$ **then**
        break
    end if
end for
Step 5: Validate calibration robustness
Test extrapolation confidence across environmental ranges
Compute cross-validation metrics
Assess population-level consistency
Output: $\hat{\boldsymbol{\theta}}$, $\Sigma_{\boldsymbol{\theta}}$, $\hat{\mathbf{Q}}_{\text{env}}$, $\hat{\mathbf{Q}}_{\text{interaction}}$, calibration quality metrics

9.2 Deployment Phase

9.3 Quality Metrics and Validation

9.3.1 Calibration Quality Metrics

- **Normalized Root Mean Square Error (NRMSE):**

$$\text{NRMSE} = \frac{\sqrt{\frac{1}{N} \sum_{i=1}^N (p_{\text{obs},i} - \hat{p}_i)^2}}{\max(p_{\text{obs}}) - \min(p_{\text{obs}})} \tag{70}$$

Algorithm 2 Online Environmental-Adaptive EKF with Change Detection

Input: Calibration parameters $\hat{\boldsymbol{\theta}}$, $\Sigma_{\boldsymbol{\theta}}$
Input: Environmental variance model $\hat{\mathbf{Q}}_{\text{env}}$, $\hat{\mathbf{Q}}_{\text{interaction}}$
Input: Initial environmental state $\hat{\mathbf{e}}_0$, $\Sigma_{\mathbf{e},0}$
Initialize EKF state: $\mathbf{x}_0 = [\mathbf{x}_{\text{phys},0}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{e}}_0, 0]^T$
Initialize EKF covariance: $\mathbf{P}_0 = \text{blkdiag}(\mathbf{P}_{\text{phys},0}, \Sigma_{\boldsymbol{\theta}}, \Sigma_{\mathbf{e},0}, \Sigma_{b,0})$
for each measurement $(v_k, p_{\text{obs},k}, \mathbf{e}_{\text{sensor},k})$ **do**
    Environmental State Update:
    $\hat{\mathbf{e}}_{k|k-1} = \mathbf{F}_{\text{env}}\hat{\mathbf{e}}_{k-1|k-1}$
    $\Sigma_{\mathbf{e},k|k-1} = \mathbf{F}_{\text{env}}\Sigma_{\mathbf{e},k-1|k-1}\mathbf{F}_{\text{env}}^T + \mathbf{Q}_{\text{env}}$
    Prediction:
    $\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}\hat{\mathbf{x}}_{k-1|k-1}$
    $\mathbf{Q}_k = \mathbf{Q}(\hat{\mathbf{e}}_{k|k-1})$ ▷ Environment-dependent process noise
    $\mathbf{P}_{k|k-1} = \mathbf{F}\mathbf{P}_{k-1|k-1}\mathbf{F}^T + \mathbf{Q}_k$
    Adaptive Variance Computation:
    $\sigma^2_{\text{total},k} = \sigma^2_{\text{base}} + \hat{\mathbf{e}}^T_{k|k-1}\hat{\mathbf{Q}}_{\text{env}}\hat{\mathbf{e}}_{k|k-1}$
    $\sigma^2_{\text{total},k} \mathrel{+}= v_k^2\hat{\mathbf{e}}^T_{k|k-1}\hat{\mathbf{Q}}_{\text{interaction}}\hat{\mathbf{e}}_{k|k-1}$
    $\sigma^2_{\text{total},k} \mathrel{+}= \alpha_1 v_k^4 + \alpha_2\|\hat{\mathbf{e}}_{k|k-1}\|^2 v_k^2 + \alpha_3\|\hat{\mathbf{e}}_{k|k-1}\|^4$
    GLR Test:
    Compute Λ_k using sliding window with environmental-robust likelihood
    if $\Lambda_k > \gamma$ **then**
        Trigger recalibration or increase uncertainty
        $\Sigma_{\boldsymbol{\theta},k|k-1} \leftarrow \Sigma_{\boldsymbol{\theta},k|k-1} + \Delta\Sigma_{\text{recal}}$
    end if
    Update:
    $\mathbf{H}_k = \frac{\partial h}{\partial \mathbf{x}}|_{\hat{\mathbf{x}}_{k|k-1}}$
    $\mathbf{R}_k = \sigma^2_{\text{total},k} + \mathbf{J}_{\boldsymbol{\theta}}\Sigma_{\boldsymbol{\theta},k|k-1}\mathbf{J}_{\boldsymbol{\theta}}^T + \mathbf{J}_{\mathbf{e}}\Sigma_{\mathbf{e},k|k-1}\mathbf{J}_{\mathbf{e}}^T$
    $\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_{k|k-1}\mathbf{H}_k^T + \mathbf{R}_k)^{-1}$
    $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(p_{\text{obs},k} - h(\hat{\mathbf{x}}_{k|k-1}, v_k))$
    $\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k|k-1}$
    Output: $\hat{p}_k = f(v_k, \hat{\mathbf{e}}_{k|k}; \hat{\boldsymbol{\theta}}_{k|k}) + \hat{b}_{k|k}$, $\sigma^2_{p,k}$
end for

- **Uncertainty Calibration:**

$$\text{Coverage} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}[|p_{\text{obs},i} - \hat{p}_i| \leq k\sigma_{p,i}] \tag{71}$$

- **Extrapolation Confidence:**

$$\text{Confidence} = \exp\left(-\frac{\sigma^2_{\text{extrapolation}}}{2\sigma^2_{\text{cal}}}\right) \tag{72}$$