Assumption 1. $W : [0,1]^2 \to [0,1]$ is symmetric and L–Lipschitz in the product metric:

$$|W(x,y) - W(x',y')| \le L\big(|x-x'| + |y-y'|\big), \qquad \forall (x,y),(x',y') \in [0,1]^2.$$

In particular, for fixed y, the map $x \mapsto W(x,y)$ is L–Lipschitz. Define the *degree function*

$$\delta(x) := \int_0^1 W(x,y)\,dy.$$

Then δ is L–Lipschitz: $|\delta(x) - \delta(x')| \le L|x-x'|$.

Let $x_1, x_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}[0,1]$. Conditional on $X = (x_i)_{i=1}^n$, draw independent edges $(A_{ij})_{i<j}$ with

$$A_{ij} \sim \mathrm{Bernoulli}\big(W(x_i,x_j)\big), \qquad A_{ii} = 0, \qquad A_{ji} = A_{ij}.$$

Let $A = (A_{ij})$, $D = \mathrm{diag}(d_1, \ldots, d_n)$ with $d_i = \sum_{j \ne i} A_{ij}$, $L = D - A$, and set the rescaled Laplacian $\mathcal{L} := \frac{1}{n} L$. Denote the eigenvalues of $\mathcal{L}$ by $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}$ nondecreasing.

Lemma 1. *Under the previous assumption, for every polynomial* $P(t) = \sum_{m=0}^M c_m t^m$,

$$\frac{1}{n}\sum_{i=1}^n P\big(\lambda_i^{(n)}\big) \xrightarrow[n\to\infty]{\text{a.s.}} \int_0^1 P\big(\delta(x)\big)\,dx.$$

Proof. By linearity, it suffices to consider $P(t) = t^m$ with $m \ge 1$ (the case $m = 0$ is trivial).
For any fixed $m \ge 1$,

$$\frac{1}{n}\sum_{i=1}^n \big(\lambda_i^{(n)}\big)^m = \frac{1}{n}\operatorname{Tr}(\mathcal{L}^m) = \frac{1}{n^{m+1}}\operatorname{Tr}\big((D-A)^m\big).$$

Expanding $(D-A)^m$, the all-D word contributes

$$\frac{1}{n^{m+1}}\operatorname{Tr}(D^m) = \frac{1}{n}\sum_{i=1}^n \Big(\frac{d_i}{n}\Big)^m.$$

Claim 2. If $w(D,A)$ is a word of length m containing at least one A, then

$$\big|\operatorname{Tr} w(D,A)\big| \le n^m.$$

Proof. By cyclicity of trace, write

$$w(D,A) = D^{\alpha_1} A D^{\alpha_2} A \cdots A D^{\alpha_r},$$

where $r \ge 1$ is the number of A's and $\sum_{s=1}^r \alpha_s = m - r$. Then

$$\operatorname{Tr} w(D,A) = \sum_{i_1,\ldots,i_r=1}^n \Big(d_{i_1}^{\alpha_1} A_{i_1 i_2} d_{i_2}^{\alpha_2} \cdots A_{i_r i_1} d_{i_1}^{\alpha_r}\Big).$$

Each summand has absolute value at most n^{m-r} since $d_i \le n$ and $A_{ij} \in \{0,1\}$, and there are n^r tuples. Hence $|\operatorname{Tr} w(D,A)| \le n^m$. □

After dividing by n^{m+1}, all mixed words vanish as $O(1/n)$. Therefore

$$\frac{1}{n}\sum_{i=1}^{n}\big(\lambda_i^{(n)}\big)^m = \frac{1}{n}\sum_{i=1}^{n}\Big(\frac{d_i}{n}\Big)^m + o(1). \tag{0.1}$$

Define $m_i := \sum_{j\neq i} W(x_i, x_j)$. We claim that

$$\max_{1\le i\le n}\left|\frac{d_i}{n} - \delta(x_i)\right| \xrightarrow{\text{a.s.}} 0. \tag{0.2}$$

Conditional on $X = (x_j)$, d_i is a sum of independent Bernoullis with mean m_i. Hoeffding and a union bound yield

$$\mathbb{P}\Big(\max_i \left|\tfrac{d_i}{n} - \tfrac{m_i}{n}\right| > \varepsilon \;\Big|\; X\Big) \le 2n\, e^{-2\varepsilon^2 n}.$$

The RHS is summable and independent of X, hence by Borel–Cantelli,

$$\max_i \left|\tfrac{d_i}{n} - \tfrac{m_i}{n}\right| \to 0 \quad \text{a.s.}$$

Set

$$Z_n := \sup_{x\in[0,1]}\left|\frac{1}{n}\sum_{j=1}^{n} W(x, x_j) - \delta(x)\right|.$$

Fix $\varepsilon > 0$, let $\eta := \varepsilon/(4L)$, and let $\mathcal{G}$ be a grid on $[0,1]$ with mesh η and size $|\mathcal{G}| \le 3/\eta$. For each x choose $u \in \mathcal{G}$ with $|x-u| \le \eta$.

Claim 3 (Lipschitzness of δ)**.** δ is L–Lipschitz: $|\delta(u) - \delta(x)| \le L|u-x|$.

Proof.

$$|\delta(u) - \delta(x)| = \left|\int_0^1 \big(W(u,y) - W(x,y)\big)\,dy\right| \le \int_0^1 L|u-x|\,dy = L|u-x|.$$

$\square$

By triangle inequality and the L–Lipschitz property of W in its first argument and of δ,

$$\begin{aligned}
\left|\frac{1}{n}\sum_{j=1}^{n} W(x,x_j) - \delta(x)\right| &\le \left|\frac{1}{n}\sum_{j=1}^{n}\big(W(x,x_j) - W(u,x_j)\big)\right| + \left|\frac{1}{n}\sum_{j=1}^{n} W(u,x_j) - \delta(u)\right| + |\delta(u) - \delta(x)| \\
&\le L|x-u| + \left|\frac{1}{n}\sum_{j=1}^{n} W(u,x_j) - \delta(u)\right| + L|x-u| \\
&\le \left|\frac{1}{n}\sum_{j=1}^{n} W(u,x_j) - \delta(u)\right| + 2L\eta.
\end{aligned}$$

Thus

$$Z_n \le \max_{u\in\mathcal{G}}\left|\frac{1}{n}\sum_{j=1}^{n} W(u,x_j) - \delta(u)\right| + \frac{\varepsilon}{2}.$$

For fixed $u \in \mathcal{G}$, the variables $W(u, x_j) \in [0, 1]$ are i.i.d. with mean $\delta(u)$, so Hoeffding yields

$$\mathbb{P}\left(\left| \tfrac{1}{n} \sum_{j=1}^{n} W(u, x_j) - \delta(u) \right| > \tfrac{\varepsilon}{2} \right) \leq 2e^{-\frac{\varepsilon^2}{2} n}.$$

A union bound over $\mathcal{G}$ gives

$$\mathbb{P}(Z_n > \varepsilon) \leq \frac{6}{\eta} e^{-\frac{\varepsilon^2}{2} n} = \frac{24L}{\varepsilon} e^{-\frac{\varepsilon^2}{2} n},$$

which is summable; hence $Z_n \to 0$ a.s. Moreover,

$$\left| \tfrac{m_i}{n} - \delta(x_i) \right| \leq Z_n + \tfrac{1}{n} \to 0 \quad \text{a.s.}$$

Combining previous results yields (0.2). Finally, since $t \mapsto t^m$ is m–Lipschitz on $[0, 1]$,

$$\left| \frac{1}{n} \sum_{i=1}^{n} \left(\tfrac{d_i}{n} \right)^m - \frac{1}{n} \sum_{i=1}^{n} \delta(x_i)^m \right| \leq m \max_i \left| \tfrac{d_i}{n} - \delta(x_i) \right| \xrightarrow{\text{a.s.}} 0. \tag{0.3}$$

By the strong law of large numbers,

$$\frac{1}{n} \sum_{i=1}^{n} \delta(x_i)^m \xrightarrow{\text{a.s.}} \int_0^1 \delta(x)^m \, dx. \tag{0.4}$$

We conclude from (0.1), (0.3), and (0.4), and triangle inequality

$$\frac{1}{n} \sum_{i=1}^{n} \left(\lambda_i^{(n)} \right)^m \xrightarrow{\text{a.s.}} \int_0^1 \delta(x)^m \, dx.$$

By linearity, the claim follows for all polynomials P. □