Power System Analysis

Chapter 4 Bus injection models

Steven Low EE/CS/EST 135 Caltech (Oct 16, 2024)

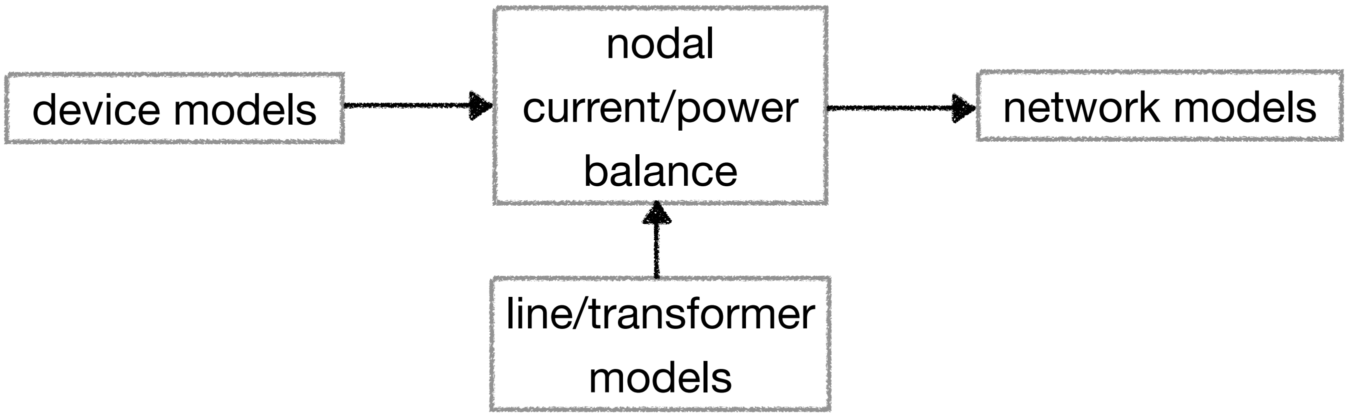
Outline

1. Component models
2. Network model: VI relation
3. Network model: Vs relation
4. Computation methods

Outline

1. Component models
   - Sources, impedance
   - Transmission or distribution line
   - Transformer
2. Network model: VI relation
3. Network model: Vs relation
4. Computation methods

Overview



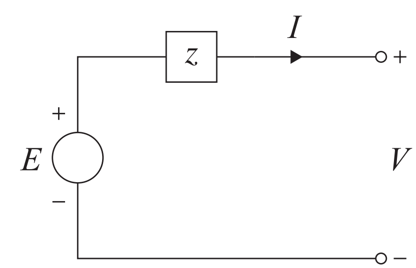
single-phase or 3-phase

Single-phase devices

1. Single-terminal device j
    - Voltage source $\left(E_j, z_j\right)$, current source $\left(J_j, y_j\right)$, power source $\left(\sigma_j, z_j\right)$, impedance z_j
    - Terminal variables $\left(V_j, I_j, s_j\right)$
    - External model: relation between $\left(V_j, I_j\right)$ or $\left(V_j, s_j\right)$
2. Two-terminal device (j, k)
    - Line $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$, transformer $\left(K_{jk}(n), \tilde{y}_{jk}^s, \tilde{y}_{jk}^m\right)$
    - Terminal variables $\left(V_j, I_{jk}, S_{jk}\right)$ and $\left(V_k, I_{kj}, S_{kj}\right)$
    - External model: relation between $\left(V_j, V_k, I_{jk}, I_{kj}\right)$ or $\left(V_j, V_k, S_{jk}, S_{kj}\right)$

Single-phase devices

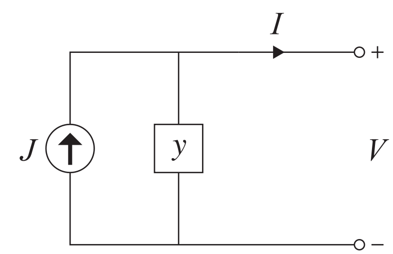
1. Voltage source $\left(E_j, z_j\right)$
   - Constant internal voltage E_j with series impedance z_j
   - Models for Thevenin equivalent circuit of a balanced synchronous machine, secondary side of transformer, grid-forming inverter
   - External model: $V_j = E_j - z_j I_j$
   - External model: $s_j \ = \ V_j I_j^{\mathsf{H}} \ = \ y_j^{\mathsf{H}} V_j \left(E_j - V_j\right)^{\mathsf{H}}$



Single-phase devices

2. Current source $\left(J_j, y_j\right)$

- Constant internal current J_j with shunt admittance y_j
- Models for Norton equivalent circuit of a synchronous generator, load (e.g. electric vehicle charger), grid-following inverter
- External model: $I_j = J_j - y_j V_j$
- External model: $s_j \;=\; V_j I_j^{\mathsf{H}} \;=\; V_j \left(J_j - y_j V_j\right)^{\mathsf{H}}$



Single-phase devices

3. Power source $\left(\sigma_j, z_j\right)$

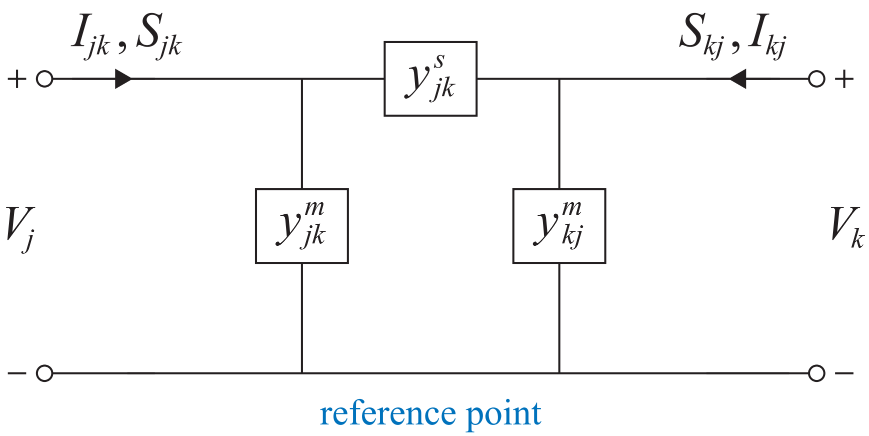
- Constant internal power σ_j in series with impedance z_j
- Models for load, generator, secondary side of transformer
- External model: $\sigma_j = \left(V_j - z_j I_j\right) I_j^{\mathsf{H}}$
- External model: $s_j \ = \ V_j I_j^{\mathsf{H}} \ = \ \sigma_j + z_j I_j I_j^{\mathsf{H}}$

Single-phase devices

4. Impedance z_j
   - Constant impedance z
   - Models for load
   - External model: $V_j = z_j I_j$
   - External model: $s_j \;=\; V_j I_j^{\mathsf{H}} \;=\; \dfrac{|V_j|^2}{z_j^{\mathsf{H}}}$

Single-phase line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$

VI relation: Π circuit and admittance matrix Y_{line}



$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} = \underbrace{\begin{bmatrix} y^s_{jk} + y^m_{jk} & -y^s_{jk} \\ -y^s_{jk} & y^s_{jk} + y^m_{kj} \end{bmatrix}}_{Y_{\text{line}}} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

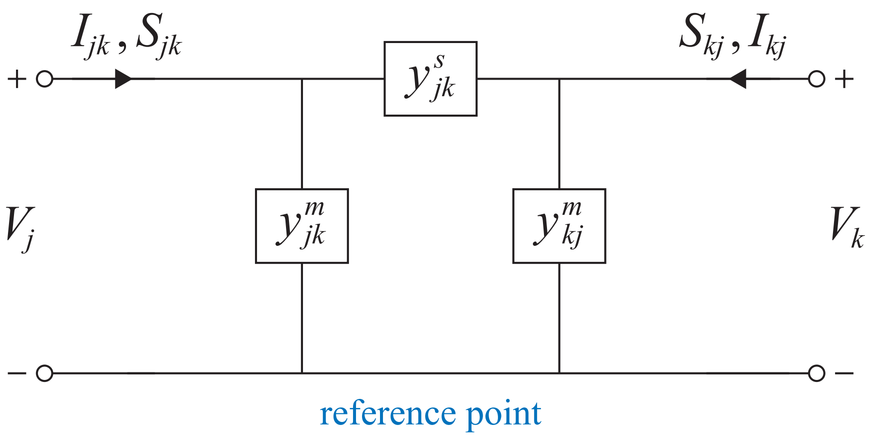
$$\begin{aligned} I_{jk} &= y^s_{jk}(V_j - V_k) + y^m_{jk} V_j, \\ I_{kj} &= y^s_{jk}(V_k - V_j) + y^m_{kj} V_k \end{aligned}$$

admittance matrix Y_{line} :

- complex symmetric
- $[Y]_{jk} = -$ series admittance

Single-phase line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$

VI relation: Π circuit and admittance matrix Y_{line}



$$I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk} V_j,$$
$$I_{kj} = y^s_{jk}(V_k - V_j) + y^m_{kj} V_k$$

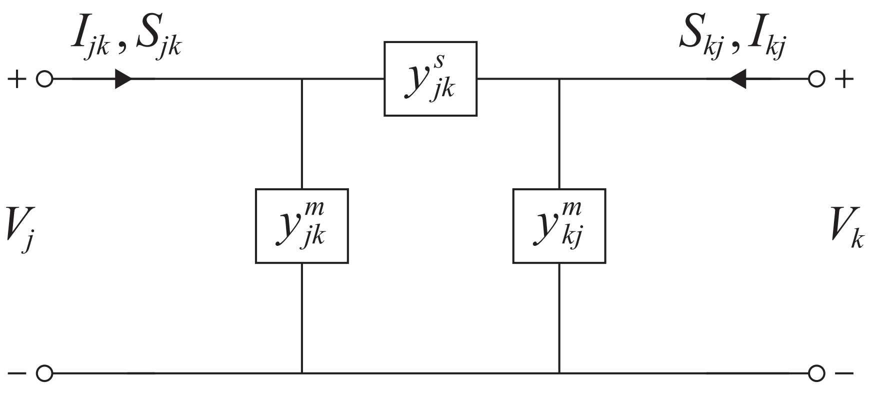
Their sum is total line current loss

$$I_{jk} + I_{kj} = y^m_{jk} V_j + y^m_{kj} V_k \neq 0$$

If $y^m_{jk} = y^m_{kj} = 0$, then $I_{jk} = -I_{kj}$

Single-phase line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$

Vs relation



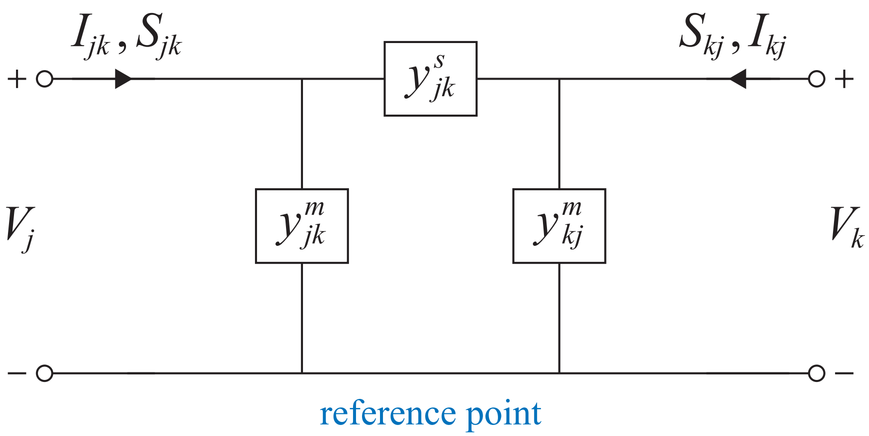
$$S_{jk} \;:=\; V_j I^H_{jk} \;=\; \left(y^s_{jk}\right)^H \left(|V_j|^2 - V_j V_k^H\right) + \left(y^m_{jk}\right)^H |V_j|^2$$

$$S_{kj} \;:=\; V_k I^H_{kj} \;=\; \left(y^s_{jk}\right)^H \left(|V_k|^2 - V_k V_j^H\right) + \left(y^m_{kj}\right)^H |V_k|^2$$

quadratic equations

Single-phase line $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$

Vs relation

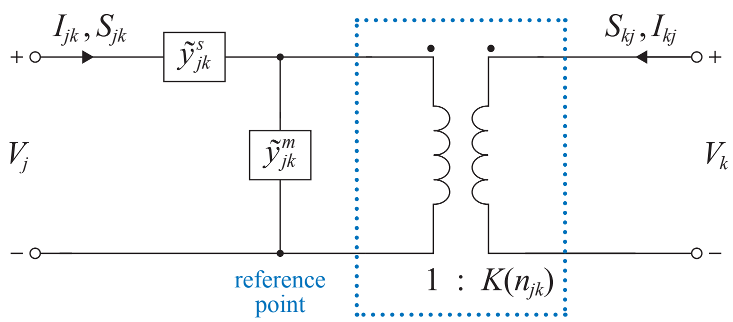


Line loss

$$S_{jk} + S_{kj} = \underbrace{\left(y_{jk}^s\right)^H \left|V_j - V_k\right|^2}_{\text{series loss}} + \underbrace{\left(y_{jk}^m\right)^H |V_j|^2 + \left(y_{kj}^m\right)^H |V_k|^2}_{\text{shunt loss}}$$

Single-phase transformer $\left(K\left(n_{jk}\right), \tilde{y}^s_{jk}, \tilde{y}^m_{jk}\right)$

Complex $K\left(n_{jk}\right)$

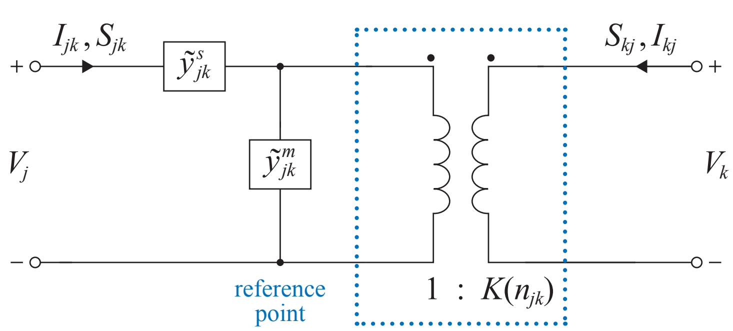


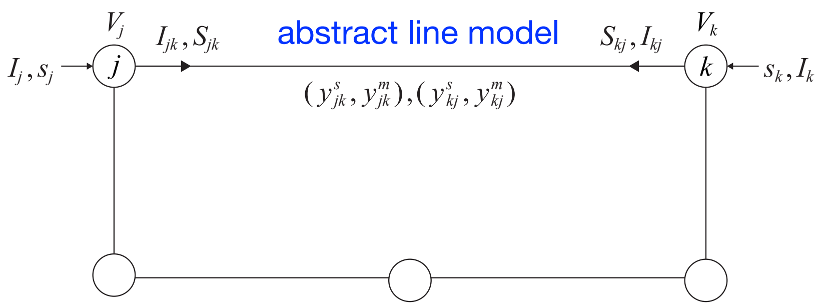
$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} = \underbrace{\begin{bmatrix} y^s_{jk} & -y^s_{jk}/K_{jk}(n) \\ -y^s_{jk}/\bar{K}_{jk}(n) & \left(y^s_{jk} + y^m_{jk}\right)/\,|\,K_{jk}(n)\,|^2 \end{bmatrix}}_{Y_{\text{transformer}}} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

- $Y_{\text{transformer}}$: *not* symmetric
- Has no equivalent Π circuit
- Use admittance or transmission matrix for analysis

Single-phase transformer $\left(K\left(n_{jk}\right), \tilde{y}^s_{jk}, \tilde{y}^m_{jk}\right)$

Complex $K\left(n_{jk}\right)$





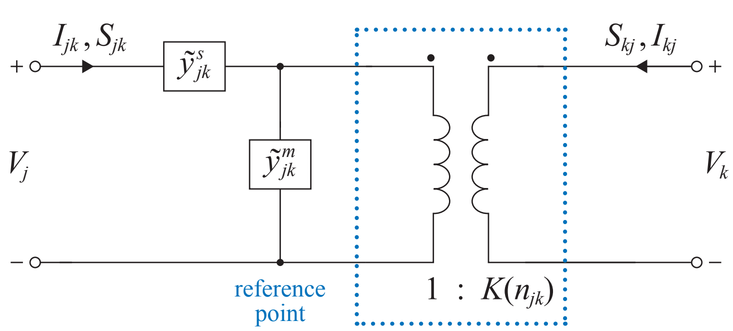
$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} = \underbrace{\begin{bmatrix} y^s_{jk} & -y^s_{jk}/K_{jk}(n) \\ -y^s_{jk}/\bar{K}_{jk}(n) & \left(y^s_{jk} + y^m_{jk}\right)/\left|K_{jk}(n)\right|^2 \end{bmatrix}}_{Y_{\text{transformer}}} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

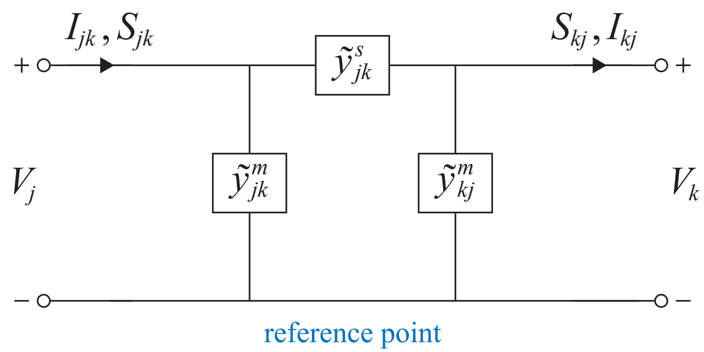
$$y^s_{jk} := \frac{\tilde{y}^s_{jk}}{K_{jk}(n)}, \qquad y^m_{jk} := \left(1 - \frac{1}{K_{jk}(n)}\right)\tilde{y}^s_{jk}$$

$$y^s_{kj} := \frac{\tilde{y}^s_{jk}}{\bar{K}_{jk}(n)}, \qquad y^m_{kj} := \frac{1 - K_{jk}(n)}{\left|K_{jk}(n)\right|^2}\tilde{y}^s_{jk} + \frac{1}{\left|K_{jk}(n)\right|^2}\tilde{y}^m_{jk}$$

Single-phase transformer $\left(K\left(n_{jk}\right), \tilde{y}^s_{jk}, \tilde{y}^m_{jk}\right)$

Real $K\left(n_{jk}\right) = n_{jk}$





$$I_{jk} = y^s_{jk}\left(V_j - a_{jk}V_k\right)$$

$$I_{jk} = y^m_{jk}\, a_{jk}V_k + n_{jk}(-I_{kj})$$

$$y^s_{jk} := a_{jk}\,\tilde{y}^s_{jk} = y^s_{kj}$$

$$y^m_{jk} := (1 - a_{jk})\tilde{y}^s_{jk}$$

$$\tilde{y}^m_{jk} \neq \tilde{y}^m_{kj}$$

$$y^m_{kj} := a_{jk}(a_{jk} - 1)\tilde{y}^s_{jk} + a^2_{jk}\,\tilde{y}^m_{jk}$$

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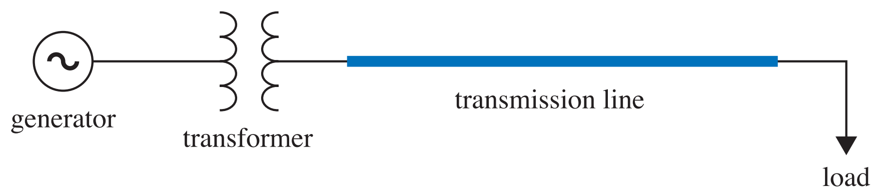
   - Example and network model
   - Admittance matrix Y and properties
   - Kron reduction Y/Y_{22} and properties
   - Radial network
3. Network model: Vs relation
4. Computation methods

Example



System

- Generator: current source (I_1, y_1)
- Transformer $(n, \tilde{y}^s, \tilde{y}^m)$
- Transmission line with series admittance y
- Load: current source (I_2, y_2)

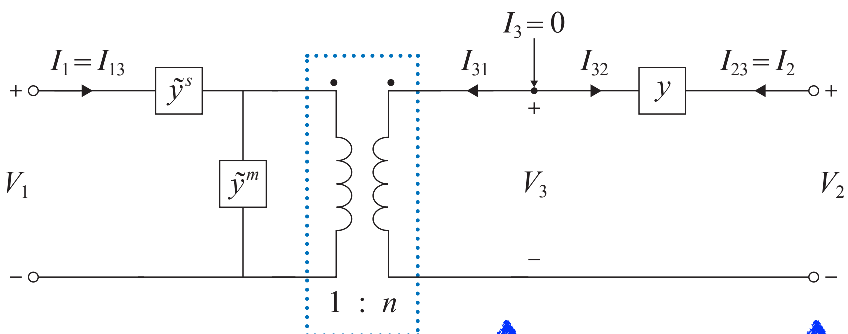
Derive

- Derive network model (admittance matrix Y)

Derive Y in 2 steps

Example

Step 1: transformer + line



$$\begin{bmatrix} I_{13} \\ I_{31} \end{bmatrix} = \begin{bmatrix} \tilde{y}^s & -a\,\tilde{y}^s \\ -a\,\tilde{y}^s & a^2\left(\tilde{y}^s + \tilde{y}^m\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix}$$

relate branch currents with nodal voltages

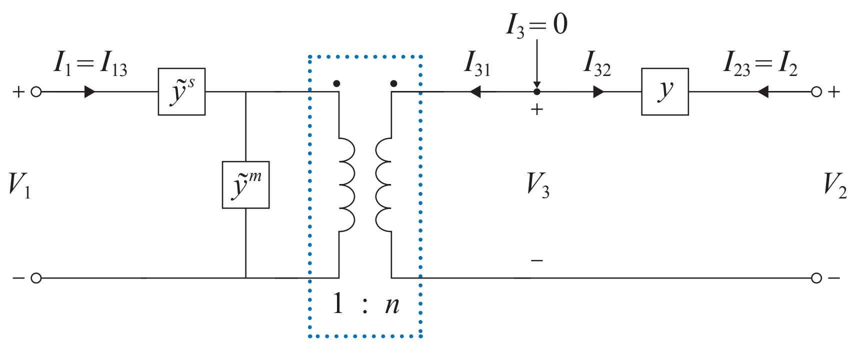
$$\begin{bmatrix} I_{32} \\ I_{23} \end{bmatrix} = \begin{bmatrix} y & -y \\ -y & y \end{bmatrix} \begin{bmatrix} V_3 \\ V_2 \end{bmatrix}$$

Nodal current balance (KCL):

$$\begin{aligned} I_1 &= I_{13} \\ I_3 &= I_{31} + I_{32} &= 0 \\ I_2 &= I_{23} \end{aligned}$$

Example

Step 1: transformer + line



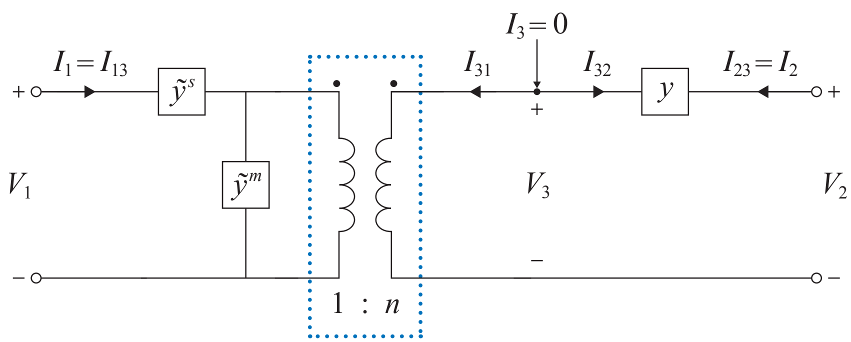
Eliminate branch currents:

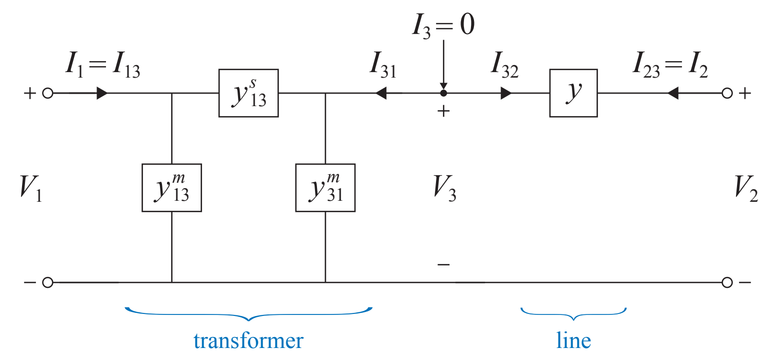
$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{y}^s & 0 & -a\tilde{y}^s \\ 0 & y & -y \\ -a\tilde{y}^s & -y & y + a^2\left(\tilde{y}^s + \tilde{y}^m\right) \end{bmatrix}}_{Y_1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

- Y_1 : complex symmetric
- Hence: admittance matrix with Π circuit
- Unequal shunt elements (even if $\tilde{y}^m = 0$)

Example

Step 1: transformer + line





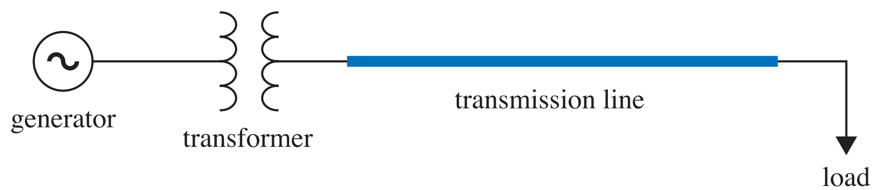
Eliminate branch currents:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \tilde{y}^s & 0 & -a\tilde{y}^s \\ 0 & y & -y \\ -a\tilde{y}^s & -y & y + a^2\left(\tilde{y}^s + \tilde{y}^m\right) \end{bmatrix}}_{Y_1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\begin{aligned} y_{13}^s &:= a\tilde{y}^s \\ y_{13}^m &:= (1-a)\tilde{y}^s \\ y_{31}^m &:= a(a-1)\tilde{y}^s + a^2\tilde{y}^m \end{aligned}$$

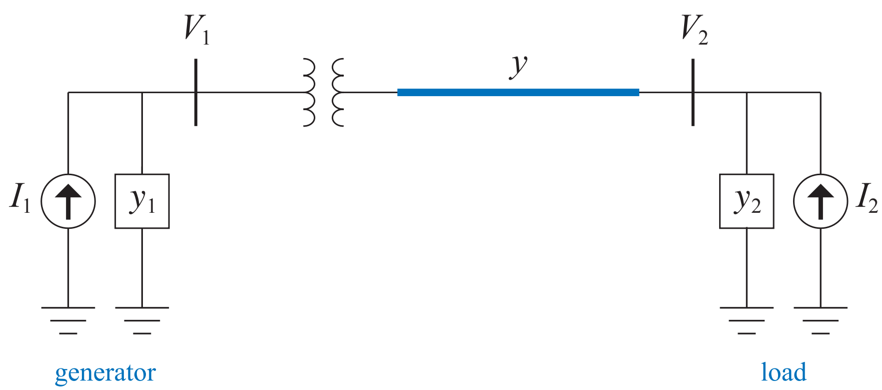
Example

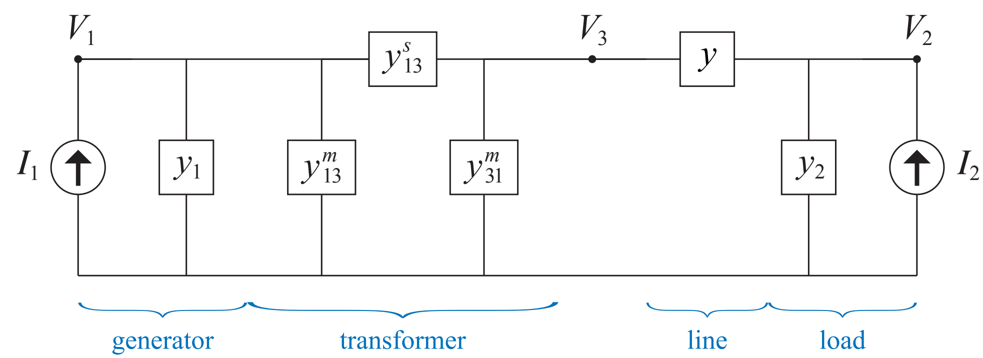
Step 2: overall system



generator/load admittances

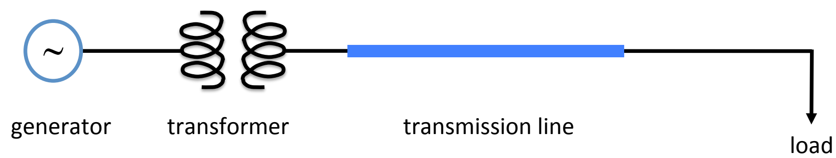
$$\begin{bmatrix} I_1 \\ I_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{y}^s + \boxed{y_1} & 0 & -a\tilde{y}^s \\ 0 & y + \boxed{y_2} & -y \\ -a\tilde{y}^s & -y & y + a^2\left(\tilde{y}^s + \tilde{y}^m\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$





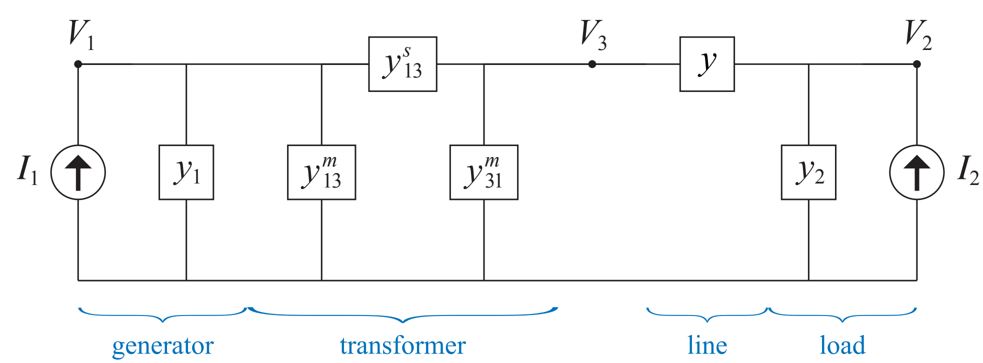
Example

Step 2: overall system



generator/load admittances

$$\begin{bmatrix} I_1\\ I_2\\ 0\end{bmatrix} = \begin{bmatrix} \tilde{y}^s + \boxed{y_1} & 0 & -a\tilde{y}^s \\ 0 & y+\boxed{y_2} & -y \\ -a\tilde{y}^s & -y & y + a^2\left(\tilde{y}^s+\tilde{y}^m\right)\end{bmatrix}\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix}$$



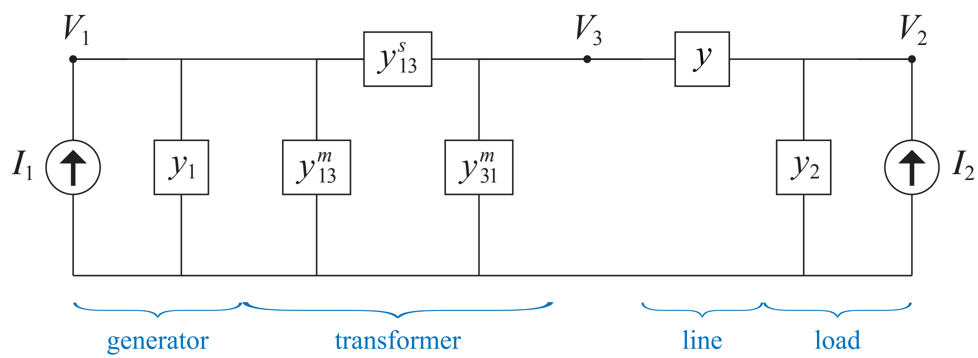
- Overall network model: **ideal** current sources connected by network
- Network: admittance matrix Y
- Y includes admittances of non-ideal current sources

Example

Step 2: overall system

generator/load admittances

$$
\begin{bmatrix} I_1 \\ I_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{y}^s + \boxed{y_1} & 0 & -a\tilde{y}^s \\ 0 & y + \boxed{y_2} & -y \\ -a\tilde{y}^s & -y & y + a^2\left(\tilde{y}^s + \tilde{y}^m\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}
$$



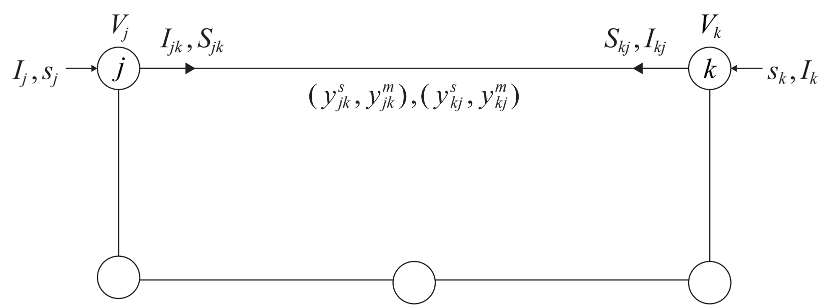
Kron reduction (see below)

- Internal bus has zero injection $I_3 = 0$
- Can eliminate (V_3, I_3)
- External behavior: relation between (I_1, I_2) and (V_1, V_2)

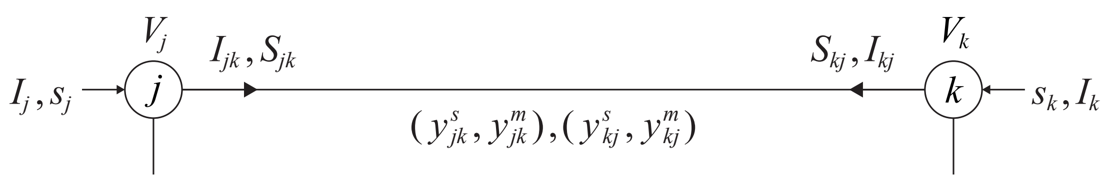
Line model

1. Network $G := (\overline{N}, E)$
   - $\overline{N} := \{0\} \cup N := \{0\} \cup \{1,\ldots,N\}$: buses/nodes/terminals
   - $E \subseteq \overline{N} \times \overline{N}$: lines/branches/links/edges

2. Each line (j, k) is parameterized by $\left(y_{jk}^s, y_{jk}^m\right)$ and $\left(y_{kj}^s, y_{kj}^m\right)$
   - (y_{jk}^s, y_{jk}^m) : series and shunt admittances from j to k
   - (y_{kj}^s, y_{kj}^m) : series and shunt admittances from k to j
   - Models transmission or distribution lines, single-phase transformers



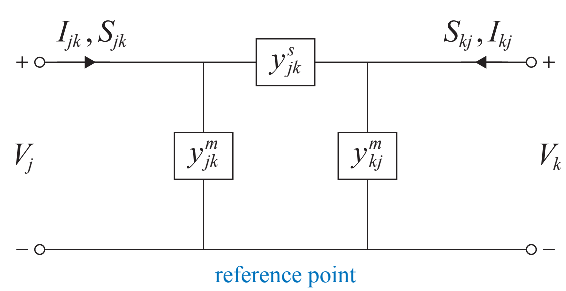
Line model



Sending-end currents

$$I_{jk} \;=\; y^s_{jk}(V_j - V_k) \;+\; y^m_{jk}\, V_j, \qquad I_{kj} \;=\; y^s_{kj}(V_k - V_j) \;+\; y^m_{kj}\, V_k,$$

If $y^s_{jk} = y^s_{kj}$: same relation but equivalent to Π circuit:



Network model

Nodal current balance

$$I_j = \sum_{k:j\sim k} I_{jk}$$

Network model

Nodal current balance

$$I_j = \sum_{k:j\sim k} I_{jk} \quad = \quad \left(\sum_{k:j\sim k} y^s_{jk} \; + \; y^m_{jj} \right) V_j \; - \sum_{k:j\sim k} y^s_{jk} V_k$$

total shunt admittance: $y^m_{jj} \; := \; \sum_{k:j\sim k} y^m_{jk}$

Network model

Admittance matrix Y

$$I_j = \sum_{k:j\sim k} I_{jk} \quad = \quad \left(\sum_{k:j\sim k} y^s_{jk} \; + \; y^m_{jj} \right) V_j \; - \; \sum_{k:j\sim k} y^s_{jk} V_k$$

In vector form:

$$I \; = \; Y V \text{ where } Y_{jk} = \begin{cases} -y^s_{jk}, & j \sim k \quad (j \neq k) \\ \sum_{l:j\sim l} y^s_{jl} \; + \; y^m_{jj}, & j = k \\ 0 & \text{otherwise} \end{cases}$$

Network model

Admittance matrix Y

Y can be written down by inspection of network graph

- Off-diagonal entry: $-$ series admittance
- Diagonal entry: $\sum$series admittances $+$ total shunt admittance

In vector form:

$$I = YV \text{ where } Y_{jk} = \begin{cases} -y_{jk}^s, & j \sim k \quad (j \neq k) \\ \sum_{l:j\sim l} y_{jl}^s + y_{jj}^m, & j = k \\ 0 & \text{otherwise} \end{cases}$$

Network model

Admittance matrix Y

A matrix Y has a Π circuit representation

- if it is complex symmetric $\left(y^s_{jk} = y^s_{kj}\right)$

In vector form:

$$I \ = \ Y\,V \text{ where } Y_{jk} = \begin{cases} -y^s_{jk}, & j \sim k \quad (j \neq k) \\ \displaystyle\sum_{l:j\sim l} y^s_{jl} \ + \ y^m_{jj}, & j = k \\ 0 & \text{otherwise} \end{cases}$$

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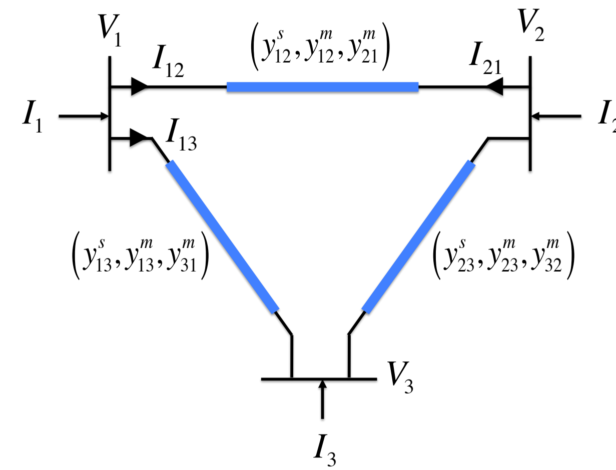
Network model

Example



$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} y_{12}^s + y_{13}^s + y_{11}^m & -y_{12}^s & -y_{13}^s \\ -y_{12}^s & y_{12}^s + y_{23}^s + y_{22}^m & -y_{23}^s \\ -y_{13}^s & -y_{23}^s & y_{13}^s + y_{23}^s + y_{33}^m \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

total shunt admittance: $y_{jj}^m \;:=\; \sum_{k:j\sim k} y_{jk}^m$

Admittance matrix Y

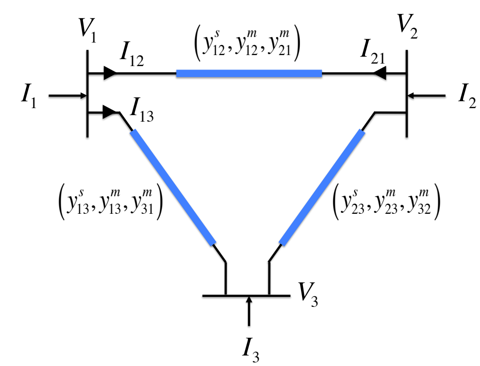
In terms of incidence matrix C

bus-by-line incidence matrix

$$C_{jl} = \begin{cases} 1 & \text{if } l = j \rightarrow k \text{ for some bus } k \\ -1 & \text{if } l = i \rightarrow j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

example:

$$C = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$



Admittance matrix Y

In terms of incidence matrix C

bus-by-line incidence matrix

$$C_{jl} = \begin{cases} 1 & \text{if } l = j \rightarrow k \text{ for some bus } k \\ -1 & \text{if } l = i \rightarrow j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

$$Y = C D_y^s C^{\mathsf{T}} + D_y^m$$

where $D_y^s := \text{diag}\left(y_l^s, l \in E\right)$, $D_y^m := \text{diag}\left(y_{jj}^m, j \in \overline{N}\right)$

Y is a complex Laplacian matrix when $Y^m = 0$

Properties of Y

1. The inverse $Z := Y^{-1}$, if exists, is called a bus impedance matrix or an impedance matrix
   - Useful for fault analysis
   - Solving $I = YV$ for V
   - Advantages of Y: Y can be constructed by inspection of one-line diagram and inherits sparsity structure of G. Z can/does not.
2. Next: study existence of Z
   - Derive (Schur complement) expressions for Z, when Y is nonsingular
   - 4 sufficient conditions for Y to be nonsingular based on the expressions for Z

Inverse of Y

If exists

Let $Y := G + iB,\ Z := R + iX$

Y nonsingular $\iff \exists (R, X)$ s.t. $YZ = ZY = \mathbb{I}$

$$\iff YZ \ = \ (GR - BX) \ + \ i(GX + BR) \ = \ \mathbb{I}$$

$$\iff \underbrace{\begin{bmatrix} G & -B \\ B & G \end{bmatrix}}_{M} \begin{bmatrix} R \\ X \end{bmatrix} \ = \ \begin{bmatrix} I \\ 0 \end{bmatrix}$$

Suppose G is nonsingular. Then Y nonsingular $\iff$ Schur complement $M/G := G + BG^{-1}B$ nonsingular

Then $M^{-1} \ = \ \begin{bmatrix} (M/G)^{-1} & (M/G)^{-1}BG^{-1} \\ -G^{-1}B(M/G)^{-1} & G^{-1} - G^{-1}B(M/G)^{-1}BG^{-1} \end{bmatrix}$ and hence $\begin{bmatrix} R \\ X \end{bmatrix} \ = \ \begin{bmatrix} (M/G)^{-1} \\ -G^{-1}B(M/G)^{-1} \end{bmatrix}$

Invertibility of Y

Theorem 1

Suppose Y is complex symmetric ($y^s_{jk} = y^s_{kj}$).

If Re$(Y) \succ 0$, then Y^{-1} exists, is symmetric, and Re $\left(Y^{-1}\right) \succ 0$

Proof

Let $Y = G + iB$ with $G \succ 0$. Then $M/G := G + BG^{-1}B \succ 0$ because $G, G^{-1} \succ 0$ and $B = B^\mathsf{T}$.

Therefore both G and M/G are nonsingular, which implies that Y is nonsingular (from previous slide).

Moreover $\begin{bmatrix} R \\ X \end{bmatrix} = \begin{bmatrix} (M/G)^{-1} \\ -G^{-1}B(M/G)^{-1} \end{bmatrix}$ implies Re $\left(Y^{-1}\right) = (M/G)^{-1} \succ 0$ since $M/G \succ 0$.

Finally, to prove $Z := Y^{-1}$ is symmetric: substitute $Z^\mathsf{T} Y^\mathsf{T} = Z^\mathsf{T} Y$ and $Y^\mathsf{T} Z^\mathsf{T} = YZ^\mathsf{T}$ into (transpose of) $ZY = YZ = \mathbb{I}$ to get:

$$Z^\mathsf{T} Y = Z^\mathsf{T} Y^\mathsf{T} = Y^\mathsf{T} Z^\mathsf{T} = YZ^\mathsf{T} = \mathbb{I} \quad \text{i.e.,} \quad Z^\mathsf{T} = Y^{-1} = Z$$

Invertibility of Y

Let $y^s_{jk} =: g^s_{jk} + ib^s_{jk}, \quad y^m_{jk} =: g^m_{jk} + ib^m_{jk}, \quad y^m_{kj} =: g^m_{kj} + ib^m_{kj}$

Conditions

1. $g^s_{jk}, g^m_{jk}, g^m_{kj} \geq 0$ for all lines $(j,k) \in E$, i.e., nonnegative conductances
2. $\sum_{k:k\sim j} g^m_{jk} \neq 0$ for all buses $j \in \overline{N}$, i.e., there is a shunt conductance incident on every bus
3. $g^s_{jk} \neq 0$ for all lines $(j,k) \in E$, and $\exists (j',k') \in E$ s.t. $g^m_{j'k'} \neq 0$, i.e., all series conductances are nonzero and there is at least one nonzero shunt conductance

<u>Theorem 2</u>

Suppose G is connected and Y is complex symmetric $(y^s_{jk} = y^s_{kj})$. If conditions 1 and either 2 or 3 are satisfied, then

1. Re$(Y) \succ 0$
2. Y^{-1} exists, is symmetric, and Re $\left(Y^{-1}\right) \succ 0$

Invertibility of Y

<u>Theorem 2</u>

Suppose G is connected and Y is complex symmetric ($y^s_{jk} = y^s_{kj}$). If conditions 1 and either 2 or 3 are satisfied, then

1. Re$(Y) \succ 0$
2. Y^{-1} exists, is symmetric, and Re $\left(Y^{-1}\right) \succ 0$

Proof

For any nonzero $\rho \in \mathbb{R}^{N+1}$, these conditions imply

$$\begin{aligned}
\rho^{\mathsf{T}} G \rho = \sum_j \sum_k \rho_j \rho_k G_{jk} &= \sum_j \left(\sum_{k:j\sim k} -\rho_j \rho_k g^s_{jk} + \rho_j^2 \sum_{i:j\sim i} (g^s_{ji} + g^m_{ji}) \right) \\
&= \sum_{(j,k)\in E} \left(\rho_j^2 - 2\rho_j\rho_k + \rho_k^2\right) g^s_{jk} + \sum_{j\in\overline{N}} \rho_j^2 \sum_{i:j\sim i} g^m_{ji} \\
&= \sum_{(j,k)\in E} \left(\rho_j - \rho_k\right)^2 g^s_{jk} + \sum_{j\in\overline{N}} \rho_j^2 \sum_{i:j\sim i} g^m_{ji} > 0
\end{aligned}$$

Inverse of Y

If exists

Let $Y := G + iB, \; Z := R + iX$

Y nonsingular $\iff \underbrace{\begin{bmatrix} G & -B \\ B & G \end{bmatrix}}_{M} \begin{bmatrix} R \\ X \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ which is the same as: $\underbrace{\begin{bmatrix} B & G \\ G & -B \end{bmatrix}}_{M'} \begin{bmatrix} R \\ X \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$

Suppose B is nonsingular. Then Y nonsingular $\iff$ Schur complement $M/B := -(B + GB^{-1}G)$ nonsingular

Then $M'^{-1} = \begin{bmatrix} B^{-1} + B^{-1}G(M'/B)^{-1}GB^{-1} & -B^{-1}G(M'/B)^{-1} \\ -(M'/B)^{-1}GB^{-1} & (M'/B)^{-1} \end{bmatrix}$ and hence $\begin{bmatrix} R \\ X \end{bmatrix} = \begin{bmatrix} -B^{-1}G(M'/B)^{-1} \\ (M'/B)^{-1} \end{bmatrix}$

This leads to 2 analogous sufficient conditions in terms of Im(Y) and $\left(b^s_{jk}, b^m_{jk}, b^m_{kj}\right)$ with similar proofs.

Invertibility of Y

Theorem 3

Suppose Y is complex symmetric ($y^s_{jk} = y^s_{kj}$).

If Im$(Y) \prec 0$, then Y^{-1} exists, is symmetric, and Im $\left(Y^{-1}\right) \succ 0$

Invertibility of Y

Let $y_{jk}^s =: g_{jk}^s + ib_{jk}^s, \quad y_{jk}^m =: g_{jk}^m + ib_{jk}^m, \quad y_{kj}^m =: g_{kj}^m + ib_{kj}^m$

Conditions

1. $b_{jk}^s, b_{jk}^m, b_{kj}^m \leq 0$ for all lines $(j,k) \in E$, i.e., nonpositive susceptances
2. $\sum_{k:k\sim j} b_{jk}^m \neq 0$ for all buses $j \in \overline{N}$, i.e., there is a shunt susceptances incident on every bus
3. $b_{jk}^s \neq 0$ for all lines $(j,k) \in E$, and $\exists (j', k') \in E$ s.t. $b_{j'k'}^m \neq 0$, i.e., all series susceptances are nonzero and there is at least one nonzero shunt susceptance

<u>**Theorem 4**</u>

Suppose G is connected and Y is complex symmetric $(y_{jk}^s = y_{kj}^s)$. If conditions 1 and either 2 or 3 are satisfied, then

1. $\text{Im}(Y) \prec 0$
2. Y^{-1} exists, is symmetric, and $\text{Im}\left(Y^{-1}\right) \succ 0$

Invertibility of Y

Sufficiency only

These conditions on are sufficient only

- Conditions $\left(g_{jk}^s, g_{jk}^m, g_{kj}^m\right)$ in Theorem 2 are usually satisfied by transmission/distribution lines
- … but not by transformers

Example:

Example 1 with node 3 at the primary side of the ideal transformer has an admittance matrix

$$Y = \begin{bmatrix} \tilde{y}^s & 0 & -\tilde{y}^s \\ 0 & y & -ny \\ -\tilde{y}^s & -ny & \tilde{y}^s + \tilde{y}^m + n^2 y \end{bmatrix}$$

Suppose $g^s, \tilde{g}^s > 0,\ b^s, \tilde{b}^s \leq 0,\ \tilde{b}^m \geq 0$. Then $g_{23}^m := (1-n)g^s$ and $g_{32}^m := n(n-1)g^s$ have opposite signs $(n \neq 1)$

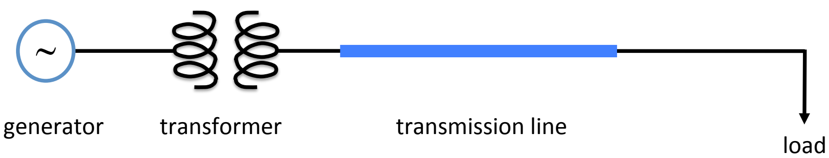
Hence Y does not satisfy conditions in Theorem 2. But Y is nonsingular if and only if $\tilde{b}_m > 0$

Outline

1. Component models
2. Network model: VI relation
   - Example and network model
   - Admittance matrix Y and properties
   - Kron reduction Y/Y_{22} and properties
   - Radial network
3. Network model: Vs relation
4. Computation methods

Example

Step 2: overall system

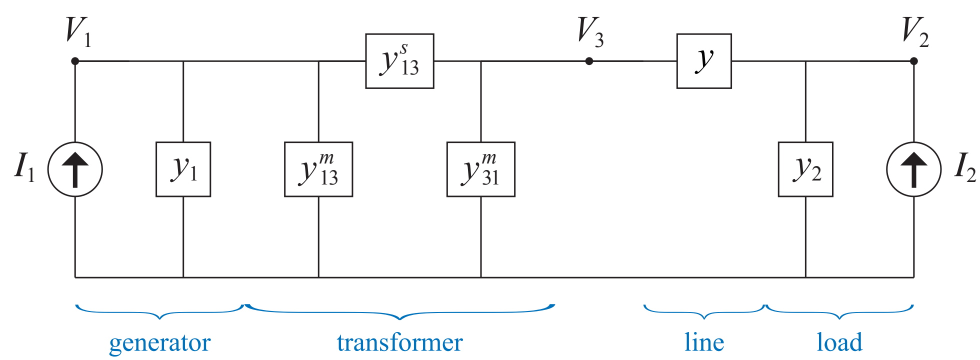


generator/load admittances

$$\begin{bmatrix} I_1 \\ I_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{y}^s + \boxed{y_1} & 0 & -a\tilde{y}^s \\ 0 & y + \boxed{y_2} & -y \\ -a\tilde{y}^s & -y & y + a^2\left(\tilde{y}^s + \tilde{y}^m\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Kron reduction (see below)

- Internal bus has zero injection $I_3 = 0$
- Can eliminate (V_3, I_3)
- External behavior: relation between (I_1, I_2) and (V_1, V_2)



Kron reduction

- $N_{\text{red}} \subseteq \overline{N}$: buses of interest, e.g., terminal buses
- Want to relate current injections and voltages at buses in N_{red}

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}_{Y} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{array}{l} \longleftarrow N_{\text{red}} \\ \longleftarrow \overline{N} \backslash N_{\text{red}} \end{array}$$

- Eliminate $V_2 = -Y_{22}^{-1} Y_{21} V_1 + Y_{22}^{-1} I_2$
- giving $\underbrace{\left(Y_{11} - Y_{12} Y_{22}^{-1} Y_{21}\right)}_{\text{Schur complement}} V_1 = I_1 - Y_{12} Y_{22}^{-1} I_2$

Kron reduction

If internal injections $I_2 = 0$:

$$Y/Y_{22} \;:=\; \left(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}\right) V_1 \;=\; I_1$$

Schur complement

- Describes effective connectivity and line admittances of reduced network

Example:

(a) Original network (b) Kron reduced network

Existence of Kron reduction

Admittance matrix $Y = CY^sC$ where $Y^s := \text{diag}\left(y^s_{jk}\right)$

When Y is real, it is called a real Laplacian matrix

- $(N+1) \times (N+1)$ real symmetric matrix
- Row sum = column sum = 0
- $\text{rank}(Y) = N$, $\text{null}(Y) = \text{span}(\mathbf{1})$ when all y^s_{jk} are (real &) of the same sign (otherwise $\text{rank}(Y)$ can be $< N$)
- Any principal submatrix is invertible, i.e., Y/Y_{22} always exists

When Y is a complex symmetric, but not Hermitian, these properties may not hold
In particular, Y_{22} may not be invertible and Y/Y_{22} may not exist

Existence of Kron reduction

Next: Properties of Y_{22} and Y/Y_{22}

- Conditions on $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$ for Y_{22} to be nonsingular, hence existence of Y/Y_{22}
- Conditions on $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$ for Y/Y_{22} to be nonsingular

Invertibility of Y_{22}

When $y_{jk}^s = y_{kj}^s$

Recall proof of Theorem 2:

$$\rho^{\mathsf{T}} G \rho = \sum_{(j,k)\in E} \left(\rho_j - \rho_k\right)^2 g_{jk}^s + \sum_{j\in\overline{N}} \rho_j^2 \sum_{i:j\sim i} g_{ji}^m \quad > \quad 0$$

term associated with lines

term associated with nodes

Invertibility of Y_{22}

When $y_{jk}^s = y_{kj}^s$

Recall proof of Theorem 2:

$$\rho^{\mathsf{T}} G \rho = \sum_{(j,k)\in E} \left(\rho_j - \rho_k\right)^2 g_{jk}^s + \sum_{j\in\overline{N}} \rho_j^2 \sum_{i:j\sim i} g_{ji}^m > 0$$

Similar structure for strict principal submatrix Y_{22}:

$$\mathrm{Re}\left(\alpha^{\mathsf{H}} Y_{22} \alpha\right) = \sum_i \left(\sum_{j,k\in C_i:(j,k)\in E} g_{jk}^s \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in C_i} G_j |\alpha_j|^2 \right)$$

$$\mathrm{Im}\left(\alpha^{\mathsf{H}} Y_{22} \alpha\right) = \sum_i \left(\sum_{j,k\in C_i:(j,k)\in E} b_{jk}^s \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in C_i} B_j |\alpha_j|^2 \right)$$

Invertibility of Y_{22}

Proof

For strict principal submatrix:

$$Y_{22}[j,j] \;=\; \sum_{k\notin A:(j,k)\in E} y^s_{jk} + \sum_{k\in A:(j,k)\in E} y^s_{jk} \;+\; y^m_{jj}$$

Hence

$$\begin{aligned}
\alpha^{\mathsf{H}} Y_{22}\alpha &= \sum_{j\in A}\left(\left(\sum_{k\notin A:(j,k)\in E} y^s_{jk} + \sum_{k\in A:(j,k)\in E} y^s_{jk} \;+\; y^m_{jj}\right)|\alpha_j|^2 \;-\; \sum_{k\in A:(j,k)\in E} y^s_{jk}\,\alpha_j^{\mathsf{H}}\alpha_k\right) \\
&= \sum_{j,k\in A:(j,k)\in E}\left(y^s_{jk}|\alpha_j|^2 - y^s_{jk}\alpha_j^{\mathsf{H}}\alpha_k - y^s_{kj}\alpha_k^{\mathsf{H}}\alpha_j + y^s_{kj}|\alpha_k|^2\right) \;+\; \sum_{j\in A}\left(\sum_{k\notin A:(j,k)\in E} y^s_{jk} \;+\; y^m_{jj}\right)|\alpha_j|^2 \\
&= \sum_{j,k\in A:(j,k)\in E} y^s_{jk}\left|\alpha_j - \alpha_k\right|^2 \;+\; \sum_{j\in A}\left(\sum_{k\notin A:(j,k)\in E} y^s_{jk} \;+\; y^m_{jj}\right)|\alpha_j|^2
\end{aligned}$$

Invertibility of Y_{22}

Proof

For strict principal submatrix:

$$Y_{22}[j,j] \;=\; \sum_{k \notin A:(j,k)\in E} y^s_{jk} + \sum_{k \in A:(j,k)\in E} y^s_{jk} \;+\; y^m_{jj}$$

Hence

$$\text{Re}\left(\alpha^{\mathsf{H}} Y_{22} \alpha\right) = \sum_i \left(\sum_{j,k\in C_i:(j,k)\in E} g^s_{jk} \left| \alpha_j - \alpha_k \right|^2 \;+\; \sum_{j\in C_i} G_j |\, \alpha_j |^2 \right)$$

$$\text{Im}\left(\alpha^{\mathsf{H}} Y_{22} \alpha\right) \;=\; \sum_i \left(\sum_{j,k\in C_i:(j,k)\in E} b^s_{jk} \left| \alpha_j - \alpha_k \right|^2 \;+\; \sum_{j\in C_i} B_j |\, \alpha_j |^2 \right)$$

Similar conditions to Theorem 2:

$$\rho^{\mathsf{T}} G \rho \,=\, \sum_{(j,k)\in E} \left(\rho_j - \rho_k\right)^2 g^s_{jk} \;+\; \sum_{j\in \overline{N}} \rho_j^2 \sum_{i:j\sim i} g^m_{ji} \;\;>\;\; 0$$

Invertibility of Y_{22}

When $y_{jk}^s = y_{kj}^s$

Let $y_{jk}^s =: g_{jk}^s + ib_{jk}^s, \quad y_{jk}^m =: g_{jk}^m + ib_{jk}^m, \quad y_{kj}^m =: g_{kj}^m + ib_{kj}^m$

Conditions

1. For all lines $(j, k) \in E$, $g_{jk}^s \geq 0$; for all buses $j \in \overline{N}$, $G_j \geq 0$
2. For all buses $j \in \overline{N}$, $G_j \neq 0$
3. For all lines $(j, k) \in E$, $g_{jk}^s \neq 0$; for each connected component C_i, $\exists j_i \in C_i$ s.t. $G_{j_i} \neq 0$

<u>**Theorem 5**</u>

Suppose G is connected and Y is complex symmetric $(y_{jk}^s = y_{kj}^s)$. If conditions 1 and either 2 or 3 are satisfied, then

1. $\text{Re}(Y_{22}) \succ 0$
2. Y_{22}^{-1} exists, is symmetric, and Re $\left(Y_{22}^{-1}\right) \succ 0$

Invertibility of Y_{22}

When $y^s_{jk} = y^s_{kj}$

Let $y^s_{jk} =: g^s_{jk} + ib^s_{jk}, \quad y^m_{jk} =: g^m_{jk} + ib^m_{jk}, \quad y^m_{kj} =: g^m_{kj} + ib^m_{kj}$

Conditions

1. For all lines $(j,k) \in E$, $b^s_{jk} \le 0$; for all buses $j \in \overline{N}$, $B_j \le 0$
2. For all buses $j \in \overline{N}$, $B_j \neq 0$
3. For all lines $(j,k) \in E$, $b^s_{jk} \neq 0$; for each connected component C_i, $\exists j_i \in C_i$ s.t. $B_{j_i} \neq 0$

<u>Theorem 6</u>

Suppose G is connected and Y is complex symmetric $(y^s_{jk} = y^s_{kj})$. If conditions 1 and either 2 or 3 are satisfied, then

1. $\text{Im}(Y_{22}) \prec 0$
2. Y_{22}^{-1} exists, is symmetric, and $\text{Im}\left(Y_{22}^{-1}\right) \succ 0$

Invertibility of Y_{22}

When $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$

Corollary 7

Suppose G is connected, Y is complex symmetric $(y^s_{jk} = y^s_{kj})$ and $y^m_{jk} = y^m_{kj} = 0$.

1. If $g^s_{jk} > 0$ for all $(j, k) \in E$, then Y_{22}^{-1} exists, is symmetric. Moreover Re$(Y_{22}) \succ 0$ and Re$(Y_{22}^{-1}) \succ 0$
2. If $b^s_{jk} < 0$ for all $(j, k) \in E$, then Y_{22}^{-1} exists, is symmetric. Moreover Im$(Y_{22}) \prec 0$ and Im$(Y_{22}^{-1}) \succ 0$

Theorem 8

Suppose G is connected, Y is complex symmetric $(y^s_{jk} = y^s_{kj})$ and $y^m_{jk} = y^m_{kj} = 0$. If $g^s_{jk} \geq 0$ and $b^s_{jk} \leq 0$ $\forall (j, k) \in E$ then

1. Re$(Y_{22}) \succeq 0,$ Im$(Y_{22}) \preceq 0,$ Re(Y_{22}) − Im$(Y_{22}) \succ 0$
2. Y_{22}^{-1} exists and is symmetric

Invertibility of Y/Y_{22}
When $y_{jk}^s = y_{kj}^s$

Theorem 9

Suppose Y_{22} is nonsingular.

1. If Re$(Y) \succ 0$, then $\left(Y/Y_{22}\right)^{-1}$ exists and is symmetric. Moreover Re$(Y/Y_{22}) \succ 0$ and Re $\left(\left(Y/Y_{22}\right)^{-1}\right) \succ 0$
2. If Im$(Y) \prec 0$, then $\left(Y/Y_{22}\right)^{-1}$ exists and is symmetric. Moreover Im$(Y/Y_{22}) \prec 0$ and Im $\left(\left(Y/Y_{22}\right)^{-1}\right) \succ 0$

Outline

1. Component models
2. Network model: VI relation
   - Example and network model
   - Admittance matrix Y and properties
   - Kron reduction Y/Y_{22} and properties
   - Radial network
3. Network model: Vs relation
4. Computation methods

Radial networks

When $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$

$(N+1) \times N$ incidence matrix C, $D^s_y :=$ diag $\left(y^s_l, l \in E\right)$:

$$Y = C D^s_y C^\top \qquad \text{admittance matrix}$$

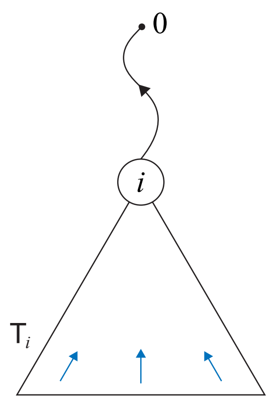
$N \times N$ reduced incidence matrix $\hat{C}$, $D^s_y :=$ diag $\left(y^s_l, l \in E\right)$:

$$\hat{Y} = \hat{C} D^s_y \hat{C}^\top \qquad \text{reduced admittance matrix}$$

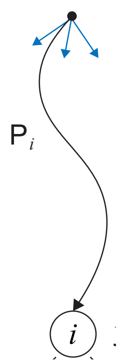
Main property: $\hat{C}$ and hence $\hat{Y}$ are always nonsingular. Moreover $\hat{Z} := \hat{Y}^{-1}$ has a simple and useful structure

Radial networks

When $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$



T_i : subtree rooted at bus i



T_i : unique path from 0 to i

<u>Theorem 10</u>

Suppose G is connected, Y is complex symmetric $(y^s_{jk} = y^s_{kj})$ and $y^m_{jk} = y^m_{kj} = 0$.

1. Reduced incidence matrix $\hat{C}$ is nonsingular

$$\left[\hat{C}^{-1}\right]_{lj} = \begin{cases} -1 & l \in \mathrm{P}_j \\ 1 & -l \in \mathrm{P}_j \\ 0 & \text{otherwise} \end{cases}$$

2. Reduced admittance matrix $\hat{Y}$ is nonsingular, and

$$\hat{Z} := \hat{Y}^{-1} = \hat{C}^{-\mathsf{T}} D^s_z \, \hat{C}^{-1}$$

$$\hat{Z}_{jk} = \sum_{l \in \mathrm{P}_j \cap \mathrm{P}_k} z^s_l$$

sum of $z^s_{jk} := 1/y^s_{jk}$ on common segment of paths from ref bus 0 to j and k

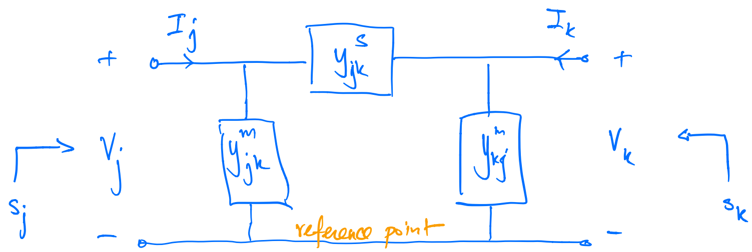
This property has been applied for topology identification, voltage control, …

Outline

1. Component models
2. Network model: VI relation
3. Network model: Vs relation
   - Complex form
   - Polar form
   - Cartesian form
   - Types of buses
   - Application: topology identification
4. Computation methods

General network

Branch currents



(b) Π equivalent circuit if $y^s_{jk} = y^s_{kj}$

Sending-end currents

$$I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk} V_j$$

$$I_{kj} = y^s_{kj}(V_k - V_j) + y^m_{kj} V_k$$

Power flow models
Complex form

Using $S_{jk} := V_j I_{jk}^H$:

$$S_{jk} = \left(y_{jk}^s\right)^H \left(|V_j|^2 - V_j V_k^H\right) + \left(y_{jk}^m\right)^H |V_j|^2$$

$$S_{kj} = \left(y_{kj}^s\right)^H \left(|V_k|^2 - V_k V_j^H\right) + \left(y_{kj}^m\right)^H |V_k|^2$$

Power flow models

Complex form

Bus injection model $s_j = \sum_{k:j\sim k} S_{jk}$:

$$s_j = \sum_{k:j\sim k} \left(y_{jk}^s\right)^H \left(|V_j|^2 - V_j V_k^H\right) + \left(y_{jj}^m\right)^H |V_j|^2$$

In terms of admittance matrix Y

$$s_j = \sum_{k=1}^{N+1} Y_{jk}^H V_j V_k^H$$

$N+1$ complex equations in $2(N+1)$ complex variables $\left(s_j, V_j, j \in \overline{N}\right)$

Power flow models

Polar form

Write $s_j \; =: \; p_j + iq_j$ and $V_j =: |V_j| e^{i\theta_j}$ with $y_{jk}^s =: g_{jk}^s + ib_{jk}^s$, $y_{jk}^m =: g_{jk}^m + ib_{jk}^m$:

$$p_j = \sum_{k:k\sim j} \left(g_{jk}^s + g_{jk}^m\right) |V_j|^2 \; - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \cos\theta_{jk} + b_{jk}^s \sin\theta_{jk}\right)$$

$$q_j = -\sum_{k:k\sim j} \left(b_{jk}^s + b_{jk}^m\right) |V_j|^2 \; - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \sin\theta_{jk} - b_{jk}^s \cos\theta_{jk}\right)$$

$2(N+1)$ real equations in $4(N+1)$ real variables $\left(p_j, q_j, \; |V_j|, \theta_j, \; j \in \overline{N}\right)$

Power flow models

Cartesian form

Write $s_j \;=:\; p_j + iq_j$ and $V_j =: c_j + id_j$ with $c_j = |V_j| \cos\theta_j$ and $d_j = |V_j| \sin\theta_j$:

$$p_j = \sum_{k:k\sim j} \left(g_{jk}^s + g_{jk}^m\right)\left(c_j^2 + d_j^2\right) - \sum_{k:k\sim j} \left(g_{jk}^s(c_jc_k + d_jd_k) + b_{jk}^s(d_jc_k - c_jd_k)\right)$$

$$q_j = -\sum_{k:k\sim j} \left(b_{jk}^s + b_{jk}^m\right)\left(c_j^2 + d_j^2\right) - \sum_{k:k\sim j} \left(g_{jk}^s(d_jc_k - c_jd_k) - b_{jk}^s(c_jc_k + d_jd_k)\right)$$

$2(N+1)$ real equations in $4(N+1)$ real variables $\left(p_j, q_j,\ c_j, d_j,\ j \in \overline{N}\right)$

Power flow models

Types of buses

Power flow equations specify $2(N+1)$ real equations in $4(N+1)$ real variables

- Power flow (load flow) problem: given $2(N+1)$ values, determine remaining vars

Types of buses

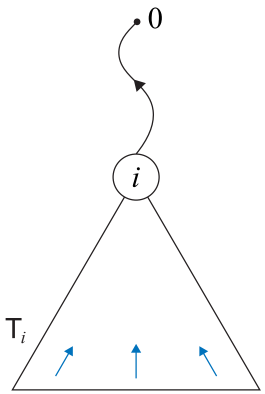
- PV buses : $\left(p_j, |V_j|\right)$ specified, determine $\left(q_j, \theta_j\right)$, e.g. generator
- PQ buses : $\left(p_j, q_j\right)$ specified, determine V_j, e.g. load
- Slack bus 0 : $V_0 := 1\angle 0°$ pu specified, determine $\left(p_0, q_0\right)$

Outline

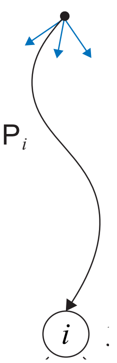
1. Component models
2. Network model: VI relation
3. Network model: Vs relation
    - Complex form
    - Polar form
    - Cartesian form
    - Types of buses
    - Application: topology identification
4. Computation methods

Radial networks

When $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$



T$_i$: subtree rooted at bus i



T$_i$: unique path from 0 to i

Theorem 10

Suppose G is connected, Y is complex symmetric $(y^s_{jk} = y^s_{kj})$ and $y^m_{jk} = y^m_{kj} = 0$.

1. Reduced incidence matrix $\hat{C}$ is nonsingular

$$\left[\hat{C}^{-1}\right]_{lj} = \begin{cases} -1 & l \in \mathrm{P}_j \\ 1 & -l \in \mathrm{P}_j \\ 0 & \text{otherwise} \end{cases}$$

2. Reduced admittance matrix $\hat{Y}$ is nonsingular, and

$$\hat{Z} := \hat{Y}^{-1} = \hat{C}^{-\mathsf{T}} D^s_z \, \hat{C}^{-1}$$

$$\hat{Z}_{jk} = \sum_{l \in \mathrm{P}_j \cap \mathrm{P}_k} z^s_l$$

Topology identification

1. Distribution grid typically consists of a meshed network with sectionalizing and tie switches on some lines
2. At any time switch are configured s.t. operational network is a spanning tree (substation at its root)
3. System operator knows the meshed network, but may not always know accurately switch status and hence operational network

Goal: Identify operational network from measurements of voltage magnitudes

Linearized power flow model

Linearization of polar form

Assumptions: For all $(j, k) \in E$

1. $y^s_{jk} = y^s_{kj} = g^s_{jk} + ib^s_{jk}$; $y^m_{jk} = y^m_{kj} = 0$
2. $g^s_{jk} > 0$ and $b^s_{jk} < 0$

Consider flat voltage profile: $V^{\text{flat}}_j = \mu e^{i\theta} \implies \left(p^{\text{flat}}, q^{\text{flat}}\right) = (0,0)$

- All voltages have same magnitude (e.g. $\mu = 1$ pu) and angle

Let

- $(|\hat{V}|, \hat{\theta})$: perturbation variable around V^{flat} at non-reference buses
- $(\hat{p}, \hat{q})$: perturbation variable around $\left(p^{\text{flat}}, q^{\text{flat}}\right) = (0,0)$ at non-reference buses

Linearized power flow model

Linearization of polar form

Polar form power flow model

$$p_j = \sum_{k:k\sim j} \left(g_{jk}^s + g_{jk}^m \right) |V_j|^2 - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \cos\theta_{jk} + b_{jk}^s \sin\theta_{jk} \right)$$

Linearized power flow model

Linearization of polar form

Polar form power flow model

$$p_j = \sum_{k:k\sim j} \left(g_{jk}^s + g_{jk}^m\right) |V_j|^2 - \sum_{k:k\sim j} |V_j||V_k| \left(g_{jk}^s \cos\theta_{jk} + b_{jk}^s \sin\theta_{jk}\right)$$

Linearize around $\left(V^{\text{flat}}, p^{\text{flat}}, q^{\text{flat}}\right)$ yields a linear model from $|\hat{V}|$ to $(\hat{p}, \hat{q})$ at non-reference buses:

$$\boxed{|\hat{V}| = \hat{R}\hat{p} + \hat{X}\hat{q} + \hat{v}_0}$$

where

$$\boxed{\hat{R} := \hat{C}^{-T} D_1 \hat{C}^{-1} \succ 0, \quad \hat{X} := -\hat{C}^{-T} D_2 \hat{C}^{-1} \succ 0}$$

$\hat{C}$ is reduced incidence matrix and

$$D_g := \text{diag}(g_l^s) \succ 0, \qquad D_b := \text{diag}(b_l^s) \prec 0$$

$$D_1 := \left(D_g + D_b D_g^{-1} D_b\right)^{-1} \succ 0, \qquad D_2 := \left(D_b + D_g D_b^{-1} D_g\right)^{-1} \prec 0$$

Covariance of voltages and powers

Define covariance and cross-covariance matrices

$$\Sigma_v := E[\,|\hat{V}| - E(|\hat{V}|)][(|V| - E(|V|)]^\mathsf{T}$$

$$\Sigma_p := E[\hat{p} - E\hat{p}][\hat{p} - E\hat{p}]^\mathsf{T}, \qquad \Sigma_q := E[\hat{q} - E\hat{q}][\hat{q} - E\hat{q}]^\mathsf{T}$$

$$\Sigma_{pq} := E[\hat{p} - E\hat{p}][\hat{q} - E\hat{q}]^\mathsf{T}, \qquad \Sigma_{qp} := E[\hat{q} - E\hat{q}][\hat{p} - E\hat{p}]^\mathsf{T}$$

Then

$$\Sigma_v \;=\; \hat{R}\Sigma_p\hat{R}^\mathsf{T} + \hat{X}\Sigma_q\hat{X}^\mathsf{T} + \hat{R}\Sigma_{pq}\hat{X}^\mathsf{T} + \hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}$$

Covariance of voltages and powers

Assumptions: power injections at same bus are positively correlated, those at different buses are uncorrelated

3. For all $j \in N$: $\Sigma_p[j,j] > 0, \;\; \Sigma_q[j,j] > 0, \;\; \Sigma_{pq}[j,j] = \Sigma_{qp}[j,j] > 0\,; \;\; y^m_{jk} = y^m_{kj} = 0$
4. For all $j \neq k$: $\Sigma_p[j,k] = \Sigma_q[j,k] = \Sigma_{pq}[j,k] = \Sigma_{qp}[j,k] = 0$

Covariance of voltages and powers

Assumptions: power injections at same bus are positively correlated, those at different buses are uncorrelated

3. For all $j \in N$: $\Sigma_p[j,j] > 0, \;\; \Sigma_q[j,j] > 0, \;\; \Sigma_{pq}[j,j] = \Sigma_{qp}[j,j] > 0\,; \;\; y^m_{jk} = y^m_{kj} = 0$

4. For all $j \neq k$: $\Sigma_p[j,k] = \Sigma_q[j,k] = \Sigma_{pq}[j,k] = \Sigma_{qp}[j,k] = 0$

Theorem

Under assumptions 1-4:

1. If a non-reference bus $j \in N$ is a descendant of bus i, then var$(\,|V_j|\,) >$ var$(\,|V_i|\,)$
2. If bus i is a parent of bus j then the variance of $|V_i| - |V_j|$ is given by:

$$E\left((|V_i| - |V_i|) - E(|V_j| - |V_j|)\right)^2 \;=\; \sum_{k \in \mathsf{T}_j} \left(r_{ij}^2 \text{var}(p_k) \;+\; x_{ij}^2 \text{var}(q_k) \;+\; 2 r_{ij} x_{ij} \text{cov}(p_k, q_k)\right)$$

Covariance of voltages and powers

<u>**Proof:**</u> **part 1**

Theorem 10 implies

$$\hat{R}_{jk} = \sum_{l \in \mathsf{P}_j \cap \mathsf{P}_k} r_l \ > \ 0, \qquad \hat{X}_{jk} = \sum_{l \in \mathsf{P}_j \cap \mathsf{P}_k} x_l \ > \ 0$$

Hence

$$\hat{R}_{jk} = \hat{R}_{ik} + r_{ij}, \qquad \hat{R}_{ik} = \sum_{l \in P_i} r_l, \qquad \text{if } k \in \mathsf{T}_j$$

$$\hat{R}_{ik} = \hat{R}_{jk}, \qquad \text{if } k \notin \mathsf{T}_j$$

Use these to evaluate the diagonal entries of $\text{var}(|V_j|) - \text{var}(|V_i|) = \Sigma_v[j,j] - \Sigma_v[i,i]$, for each of the four terms in

$$\Sigma_v \ = \ \hat{R}\Sigma_p\hat{R}^\mathsf{T} + \hat{X}\Sigma_q\hat{X}^\mathsf{T} + \hat{R}\Sigma_{pq}\hat{X}^\mathsf{T} + \hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}$$

Covariance of voltages and powers

Due to covariances Σ_p, Σ_q :

$$\left(\hat{R}\Sigma_p\hat{R}^{\mathsf{T}}\right)[j,j] \; - \; \left(\hat{R}\Sigma_p\hat{R}^{\mathsf{T}}\right)[i,i] \;\; = \;\; \sum_{k\in \mathsf{T}_j} \Sigma_p[k,k]\left(2\sum_{l\in \mathsf{P}_i} r_l + r_{ij}\right) r_{ij} \quad > \quad 0$$

similarly: $\left(\hat{X}\Sigma_q\hat{X}^{\mathsf{T}}\right)[j,j] \; > \; \left(\hat{X}\Sigma_q\hat{X}^{\mathsf{T}}\right)[i,i]$

Due to cross-covariances Σ_{pq}, Σ_{qp} :

$$\left(\hat{R}\Sigma_{pq}\hat{X}^{\mathsf{T}}\right)[j,j] \; - \; \left(\hat{R}\Sigma_{pq}\hat{X}^{\mathsf{T}}\right)[i,i] \;\; = \;\; \sum_{k} \Sigma_{pq}[k,k]\left(\hat{R}_{jk}\hat{X}_{jk} - \hat{R}_{ik}\hat{X}_{ik}\right) \quad > \quad 0$$

similarly: $\left(\hat{X}\Sigma_{qp}\hat{R}^{\mathsf{T}}\right)[j,j] \; > \; \left(\hat{X}\Sigma_{qp}\hat{R}^{\mathsf{T}}\right)[i,i]$

yielding: $\Sigma_v[j,j] > \Sigma_v[i,i]$

Covariance of voltages and powers

<u>Proof</u>: part 2

If bus i is a parent of bus j, then variance of $|V_i| - |V_j|$ is:

$$E\left((|V_i| - E|V_i|) - (|V_j|) - |V_j|)\right)^2 \;=\; \Sigma_v[i,i] + \Sigma_v[j,j] - 2\Sigma_v[i,j]$$

Again use

$$\hat{R}_{jk} \;=\; \hat{R}_{ik} + r_{ij}, \qquad \hat{R}_{ik} \;=\; \sum_{l \in P_i} r_l, \qquad \text{if } k \in \mathsf{T}_j$$

$$\hat{R}_{ik} \;=\; \hat{R}_{jk}, \qquad \text{if } k \notin \mathsf{T}_j$$

to show that the first term of

$$\Sigma_v \;=\; \hat{R}\Sigma_p\hat{R}^\mathsf{T} \;+\; \hat{X}\Sigma_q\hat{X}^\mathsf{T} \;+\; \hat{R}\Sigma_{pq}\hat{X}^\mathsf{T} \;+\; \hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}$$

yields a simple expression:

$$\sigma_1 \;:=\; \left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[i,i] \;+\; \left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[j,j] \;-\; 2\left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[i,j] \;=\; r_{ij}^2 \sum_{k \in \mathsf{T}_j} \Sigma_p[k,k]$$

Covariance of voltages and powers

Similarly, the other terms of

$$\Sigma_v \;=\; \hat{R}\Sigma_p\hat{R}^\mathsf{T} + \hat{X}\Sigma_q\hat{X}^\mathsf{T} + \hat{R}\Sigma_{pq}\hat{X}^\mathsf{T} + \hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}$$

yield

$$\sigma_1 := \left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[i,i] + \left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[j,j] - 2\left(\hat{R}\Sigma_p\hat{R}^\mathsf{T}\right)[i,j] \;=\; r_{ij}^2 \sum_{k\in\mathsf{T}_j} \Sigma_p[k,k]$$

$$\sigma_2 := \left(\hat{X}\Sigma_q\hat{X}^\mathsf{T}\right)[i,i] + \left(\hat{X}\Sigma_q\hat{X}^\mathsf{T}\right)[j,j] - 2\left(\hat{X}\Sigma_q\hat{X}^\mathsf{T}\right)[i,j] \;=\; x_{ij}^2 \sum_{k\in\mathsf{T}_j} \Sigma_q[k,k]$$

$$\sigma_3 := \left(\hat{R}\Sigma_{pq}\hat{X}^\mathsf{T}\right)[i,i] + \left(\hat{R}\Sigma_{pq}\hat{X}^\mathsf{T}\right)[j,j] - 2\left(\hat{R}\Sigma_{pq}\hat{X}^\mathsf{T}\right)[i,j] \;=\; r_{ij}x_{ij} \sum_{k\in\mathsf{T}_j} \Sigma_{pq}[k,k]$$

$$\sigma_4 := \left(\hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}\right)[i,i] + \left(\hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}\right)[j,j] - 2\left(\hat{X}\Sigma_{qp}\hat{R}^\mathsf{T}\right)[i,j] \;=\; r_{ij}x_{ij} \sum_{k\in\mathsf{T}_j} \Sigma_{qp}[k,k]$$

Covariance of voltages and powers

Summing:

$$\Sigma_v[i,i] - \Sigma_v[i,j] \;=\; \sum_{k=1}^{4} \sigma_k \;=\; \sum_{k \in \mathrm{T}_j} \left(r_{ij}^2 \Sigma_p[k,k] \;+\; x_{ij}^2 \Sigma_q[k,k] \;+\; 2 r_{ij} x_{ij} \Sigma_{pq}[k,k] \right)$$

Covariance of voltages and powers

Theorem

Under assumptions 1-4:

1. If a non-reference bus $j \in N$ is a descendant of bus i, then var$(|V_j|) >$ var$(|V_i|)$
2. If bus i is a parent of bus j then the variance of $|V_i| - |V_j|$ is given by:

$$E\left((|V_i| - |V_i|) - E(|V_j| - |V_j|)\right)^2 = \sum_{k \in \mathsf{T}_j} \left(r_{ij}^2 \text{var}(p_k) + x_{ij}^2 \text{var}(q_k) + 2 r_{ij} x_{ij} \text{cov}(p_k, q_k)\right)$$

Implications

Property 1 identifies a leaf node j as one with max var$(|V_j|)$

Property 2 identifies j's parent i as one that most closely satisfies the formula

Algorithm

1. Identify a leaf node j among unidentified nodes.
2. Identify j's parent.
3. Remove j from set of unidentified nodes and goto 1

Outline

1. Component models
2. Network model: VI relation
3. Network model: Vs relation
4. Computation methods
   - Gauss-Seidel algorithm
   - Newton-Raphson algorithm
   - Fast decoupled algorithm

Computation methods

Gauss-Seidel algorithm

Case 1: given V_0 and $\left(s_1, \ldots, s_N\right)$, determine s_0 and $(V_1, \ldots, V_N)$

Power flow equations

$$s_0 = \sum_k Y_{0k}^H V_0 V_k^H$$

$$s_j = \sum_k Y_{jk}^H V_j V_k^H, \qquad j \in N$$

- First compute $(V_1, \ldots, V_N)$
- Then compute s_0

Computation methods

Gauss-Seidel algorithm

Case 1: given V_0 *and* $\left(s_1, \ldots, s_N\right)$*, determine* s_0 *and* $(V_1, \ldots, V_N)$

Rearrange 2nd equation:

$$\frac{s_j^H}{V_j^H} = Y_{jj}V_j + \sum_{\substack{k=0 \\ k \neq j}}^{N} Y_{jk}V_k, \qquad j \in N$$

$$V_j = \frac{1}{Y_{jj}} \left(\frac{s_j^H}{V_j^H} - \sum_{\substack{k=0 \\ k \neq j}}^{N} Y_{jk}V_k \right) =: f_j\left(V_1, \ldots, V_N\right), \qquad j \in N$$

Computation methods

Gauss-Seidel algorithm

Case 1: given V_0 *and* $\left(s_1, \ldots, s_N\right)$*, determine* s_0 *and* $(V_1, \ldots, V_N)$

2nd power flow equation:

$$V = f(V)$$

where $V := \left(V_j, j \in N\right), \ f := \left(f_j, j \in N\right)$

Gauss algorithm is the fixed point iteration

$$V(t+1) = f(V(t))$$

Computation methods

Gauss-Seidel algorithm

Case 1: given V_0 and $\left(s_1, \ldots, s_N\right)$, determine s_0 and $(V_1, \ldots, V_N)$

Gauss algorithm:

$$
\begin{aligned}
V_1(t+1) &= f_1\left(V_1(t), \ldots, V_N(t)\right) \\
V_2(t+1) &= f_2\left(V_1(t), \ldots, V_N(t)\right) \\
&\vdots \\
V_N(t+1) &= f_N\left(V_1(t), \ldots, V_{N-1}(t), V_N(t)\right)
\end{aligned}
$$

Computation methods

Gauss-Seidel algorithm

Case 1: given V_0 and $\left(s_1, \ldots, s_N\right)$, determine s_0 and $(V_1, \ldots, V_N)$

Gauss-Seidel algorithm:

$$
\begin{aligned}
V_1(t+1) &= f_1\left(V_1(t), \ldots, V_N(t)\right) \\
V_2(t+1) &= f_2\left(V_1(t+1), \ldots, V_N(t)\right) \\
&\vdots \\
V_N(t+1) &= f_N\left(V_1(t+1), \ldots, V_{N-1}(t+1), V_N(t)\right)
\end{aligned}
$$

Computation methods

Gauss-Seidel algorithm

Case 2: given $\left(V_0, \ldots, V_m\right)$ *and* $\left(s_{m+1}, \ldots, s_N\right)$*, determine* $\left(s_j, j \leq m\right)$ *and* $(V_j, j > m)$

Power flow equations

$$s_j = \sum_k Y_{jk}^H V_j V_k^H, \qquad j \leq m$$

$$s_j = \sum_k Y_{jk}^H V_j V_k^H, \qquad j > m$$

- First compute $\left(V_{m+1}, \ldots, V_N\right)$ from 2nd set of equations using the same algorithm
- Then compute $\left(s_j, j \leq m\right)$ from 1st set of equations

Computation methods

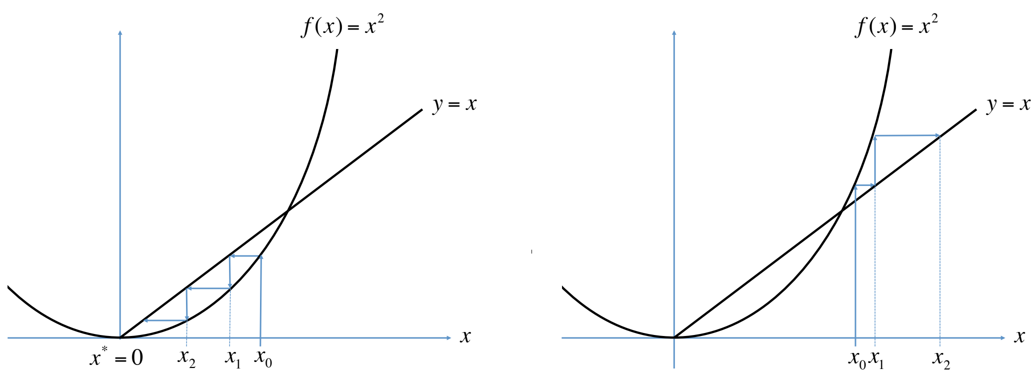
Gauss-Seidel algorithm

If algorithm converges, the limit is a fixed point and a power flow solution

Algorithm converges linearly to unique fixed point if f is a contraction mapping

- Contraction is sufficient, but not necessary, for convergence

In general, algorithm may or may not convergence depending on initial point



(a) Convergence (b) Divergence

Computational methods

Newton-Raphson algorithm

To solve $$f(x) \; = \; 0$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, e.g. $\nabla F(x) = 0$ for unconstrained optimization

Idea:

- Linear approximation

$$\hat{f}(x(t+1)) \; = \; f(x(t)) + J(x(t)) \, \Delta x(t)$$

- Choose $\Delta x(t)$ such that $\hat{f}(x(t+1)) = 0$, i.e., solve

$$J(x(t))\Delta x(t) \; = -f(x(t))$$

- Next iterate $x(t+1) \; := \; x(t) + \Delta x(t)$

$$J(x) := \frac{\partial f}{\partial x}(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{bmatrix}$$

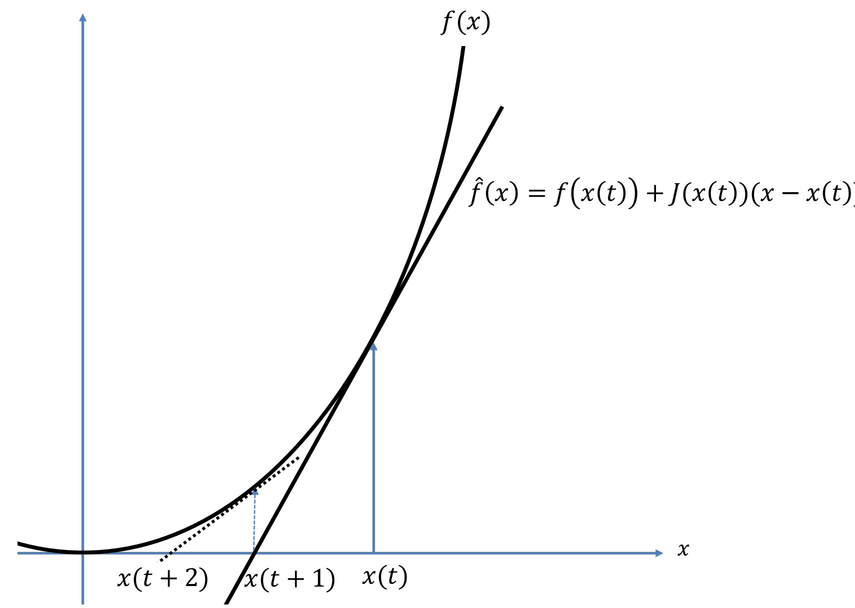
Computational methods

Newton-Raphson algorithm

To solve $$f(x) = 0$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, e.g. $\nabla F(x) = 0$ for unconstrained optimization

$$x(t+1) := x(t) - \left(J(x(t))\right)^{-1} f(x(t))$$



Computational methods

Newton-Raphson algorithm

Kantorovic Theorem

Consider $f : D \to \mathbb{R}^n$ where $D \subseteq \mathbb{R}^n$ is an open convex set. Suppose

- f is differentiable and ∇f is Lipschitz on D, i.e., $\|\nabla f(y) - \nabla f(x)\| \leq L\|y - x\|$
- $x_0 \in D$ and $\nabla f(x_0)$ is invertible

Let $\beta \geq \left\| \left(\nabla f(x_0) \right)^{-1} \right\|$, $\eta \geq \left\| \left(\nabla f(x_0) \right)^{-1} f(x_0) \right\|$ and

$$h := \beta\eta L, \quad r := \frac{1 - \sqrt{1 - 2h}}{h}\,\eta$$

Computational methods

Newton-Raphson algorithm

Kantorovic Theorem

Consider $f : D \to \mathbb{R}^n$ where $D \subseteq \mathbb{R}^n$ is an open convex set. Suppose

- f is differentiable and ∇f is Lipschitz on D, i.e., $\|\nabla f(y) - \nabla f(x)\| \ \leq \ L\|y - x\|$
- $x_0 \in D$ and $\nabla f(x_0)$ is invertible

If the closed ball $B_r(x_0) \subseteq D$ and $h \leq 1/2$, then Newton iteration

$$x(t+1) \ := \ x(t) - \left(\nabla f(x(t)) \right)^{-1} f(x(t))$$

converges to a solution $x^* \in B_r\left(x_0\right)$ of $f(x) = 0$

Newton-Raphson converges if it starts close to a solution, often quadratically

Computational methods

Newton-Raphson algorithm

Apply to power flow equations in polar form:

$$p_j(\theta, |V|) = p_j, \qquad j \in N$$

$$q_j(\theta, |V|) = q_j, \qquad j \in N_{pq}$$

where

$$p_j(\theta, |V|) := \left(\sum_{k=0}^{N} g_{jk}\right)|V_j|^2 - \sum_{k \neq j} |V_j||V_k|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right)$$

$$q_j(\theta, |V|) := -\left(\sum_{k=0}^{N} b_{jk}\right)|V_j|^2 - \sum_{k \neq j} |V_j||V_k|\left(g_{jk}\sin\theta_{jk} - b_{jk}\cos\theta_{jk}\right)$$

Computational methods

Newton-Raphson algorithm

Define $f : \mathbb{R}^{N+N_{qp}} \rightarrow \mathbb{R}^{N+N_{qp}}$

$$f(\theta, |V|) \ := \ \begin{bmatrix} \Delta p(\theta, |V|) \\ \Delta q(\theta, |V|) \end{bmatrix} \ := \ \begin{bmatrix} p(\theta, |V|) - p \\ q(\theta, |V|) - q \end{bmatrix}$$

with

$$J(\theta, |V|) := \begin{bmatrix} \frac{\partial p}{\partial \theta} & \frac{\partial p}{\partial |V|} \\ \frac{\partial q}{\partial \theta} & \frac{\partial q}{\partial |V|} \end{bmatrix}$$

Computational methods

Newton-Raphson algorithm

1. Initialization: choose $\left(\theta(0), |V(0)|\right)$

2. Iterate until stopping criteria

   (a) Determine $\left(\Delta\theta(t), \Delta|V|(t)\right)$ from

$$J\left(\theta(t), |V|(t)\right)\begin{bmatrix}\Delta\theta(t)\\ \Delta|V|(t)\end{bmatrix} = -\begin{bmatrix}\Delta p(\theta(t), |V|(t))\\ \Delta q(\theta(t), |V|(t))\end{bmatrix}$$

   (b) Set

$$\begin{bmatrix}\theta(t+1)\\ |V|(t+1)\end{bmatrix} := \begin{bmatrix}\theta(t)\\ |V|(t)\end{bmatrix} + \begin{bmatrix}\Delta\theta(t)\\ \Delta|V|(t)\end{bmatrix}$$

Computational methods

Fast Decoupled algorithm

Key observation: the Jacobian is roughly block-diagonal

$$J(\theta, |V|) := \begin{bmatrix} \frac{\partial p}{\partial \theta} & \frac{\partial p}{\partial |V|} \\ \frac{\partial q}{\partial \theta} & \frac{\partial q}{\partial |V|} \end{bmatrix} \approx \begin{bmatrix} \frac{\partial p}{\partial \theta} & 0 \\ 0 & \frac{\partial q}{\partial |V|} \end{bmatrix}$$

i.e., decoupling between p and $|V|$, and between q and θ

Computational methods

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This simplifies the computation of $\Delta x(t)$

$$\frac{\partial p}{\partial \theta}(\theta(t), |V|(t)) \ \Delta\theta(t) = -\Delta p(\theta(t), |V|(t))$$

$$\frac{\partial q}{\partial |V|}(\theta(t), |V|(t)) \ \Delta |V|(t) = -\Delta q(\theta(t), |V|(t))$$

Computational methods

Fast Decoupled algorithm

Decoupling assumption: $g_{jk} = 0, \sin\theta_{jk} = 0$

$$\frac{\partial p_j}{\partial |V_k|} = \begin{cases} -|V_j|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right), & j \neq k \\ \dfrac{p_j(\theta, |V|)}{|V_j|} + \left(\sum_i g_{ji}\right)|V_j|, & j = k \end{cases}$$

$$g_{jk} = 0,\ \sin\theta_{jk} = 0,\ p_j(\theta, |V|) = 0 \quad \Rightarrow \quad \frac{\partial p}{\partial |V|} = 0$$

Computational methods

Fast Decoupled algorithm

Decoupling assumption: $g_{jk} = 0, \sin\theta_{jk} = 0$

$$\frac{\partial q_j}{\partial \theta_k} = \begin{cases} |V_j||V_k|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right), & j \neq k \\ p_j(\theta, |V|) \; - \; \left(\sum_i g_{ji}\right)|V_j|^2, & j = k \end{cases}$$

$$g_{jk} = 0,\ \sin\theta_{jk} = 0,\ p_j(\theta, |V|) = 0 \quad \Rightarrow \quad \frac{\partial q}{\partial \theta} = 0$$

Summary

1. Component models
    - Single-phase devices, line, transformer
2. Network models
    - VI relation (admittance matrix Y), Vs relation (power flow equations)
    - Raidal network: inverse of reduced admittance matrix has simple structure
3. Computation methods
    - Gauss-Seidel algorithm, Newton-Raphson algorithm, Fast decoupled algorithm