Lecture Notes for EE/CS/EST 135

Power System Analysis

A Mathematical Approach

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These are *draft* lecture notes. Corrections, comments, questions will be appreciated - please send them to slow@caltech.edu

Major changes:

Feb 5, 2022: Corrected error in external models of Δ-configured voltage source and impedance (Chapter 7.3.3)

Feb 12, 2022: Revised Chapter 9.4 on symmetrical components and sequence networks.

April 10, 2022: Revised Chapter 14 on semidefinite relaxations: BIM.

June 26, 2022:

- Added 3-phase transformer section 8.2. Split Chapters **??** and 9.
- Added Chapter 11.2.2 on Newton-Raphson algorithm and Chapter 11.2.3 on interior-point method.

October 5, 2022:

- Revised Chapter 9.2 on three-phase analysis, especially the solution strategy in Chapter 9.2.3.
- Revised Chapter 9.3 on balanced networks, especially the structural result (Theorem 9.7) in Chapter 9.3.3 and per-phase analysis in Chapter 9.3.4.
- Re-organize Part I to be on Single-phase networks and Part II on Unbalanced multiphase networks.

October 21, 2022: Revise/re-organize Chapter 4.1.

November 29, 2022: Revise Chapters 4, 5, 9, 10 to expand line models in BIM and BFM, single and three-phase, to allow general transformer models where series admittances y_{kj}^s and y_{kj}^s may not be equal and admittance matrices Y may not be symmetric (single-phase) or block symmetric (3-phase). Also added a 3-phase BFS for DistFlow model.

January 3, 2023:

- Revised BFS in Chapter 10.4.2 for 3-phase DistFlow model.
- Revised three-phase OPF formulation in Chapters 13.1 and 13.2.

January 26, 2023: Revised Chapter 14 on semidefinite relaxations in BIM.

February 7, 2023:

- Added Chapter 14.5 semidefinite relaxation of three-phase OPF in BIM.
- Revised Chapters 15 semidefinite relaxation of single and three-phase OPF in BFM.

February 27, 2023: Clarified SVD and corrected mistakes in Takagi factorization in Chapter 25.1.5, as well as the psdudo-inverse of admittance matrix Y in Chapter 4.2.5.

September 20, 2023:

- Revised Chapter 3.1 with the addition of T equivalent circuit and unitary voltage network models of single-phase transformer. Added Chapter 8.3 on three-phase transformer models with unitary voltage networks.
- Revised invertibility conditions and properties of admittance matrix Y, principal submatrices Y_{22}, and Schur complement Y/Y_{22} for single-phase networks (Chapters 4.2.5 and 4.2.7) and three-phase networks (Chapter 9.1.3).
- Split original chapter on three-phase Component Models into two chapters, Chapter 7 on devices (which absorbs voltage regulators originally in a separate chapter) and Chapter 8 on line and transformers.
- Added Chapter 11 on convex optimization at the beginning of Part III.

Acknowledgments

Preface

Some notes.

1. A key feature of this book is its extensive and systematic treatment of unbalanced three-phase modeling and power flow analysis. A three-phase network consists of three-phase devices connected by three-phase lines and transformers. Motivated by emerging applications in secondary distribution circuits, our perspective is that most controllable devices are the single-phase devices that make up three-phase devices in Y or Δ configurations. It is therefore important to model carefully the internal voltages, currents, and powers across these single-phase devices and how they determine the terminal voltages, currents, and powers that are externally observable and that interact over the network. This is developed in Part II of the book and used to formulate three-phase optimal power problems in Part III (Chapters 13.1 and 13.2). It will become clear that the difference between single-phase and three-phase systems mainly lies in the device models, not in network equations that relate the terminal variables.

Contents

Chapter 0

Introduction

0.1 Notations

Let $\mathbb{C}$ denote the set of complex numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{N}$ the set of integers. We use $\mathbf{i}$ to denote $\sqrt{-1}$. For $a \in \mathbb{C}$, $\mathrm{Re}\, a$ and $\mathrm{Im}\, a$ denote its real and imaginary parts respectively, and $\bar{a}$ or a^{H} denotes its complex conjugate. For any set $A \subseteq \mathbb{C}^n$, $\mathrm{conv} A$ denotes the convex hull of A. For $a \in \mathbb{R}$, $[a]^+ := \max\{a, 0\}$. For $a, b \in \mathbb{C}$, $a \leq b$ means $\mathrm{Re}\, a \leq \mathrm{Re}\, b$ and $\mathrm{Im}\, a \leq \mathrm{Im}\, b$. We sometimes abuse notation to use the same symbol a to denote either a complex number $\mathrm{Re}\, a + \mathbf{i}\,\mathrm{Im}\, a$ or a size 2 real vector $a =(\mathrm{Re}\, a, \mathrm{Im}\, a)$ depending on the context. The empty set is denoted $\emptyset$.

In general scalar or vector variables are in small letters, e.g. u, w, x, y, z. Most power system quantities however are in capital letters, e.g. $S_{jk}, P_{jk}, Q_{jk}, I_j, V_j$. Unless otherwise specified, a vector is a column vector and is written interchangeably as

$$V \;=\; \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} \quad \text{or} \quad V \;=\; (V_a, V_b, V_c)$$

A variable without a subscript usually denotes a vector with appropriate components, e.g. $s := (s_j, j = 0, \ldots, n)$, $S := (S_{jk}, (j,k) \in E)$. For a vector $a = (a_1, \ldots, a_k)$, a_{-i} denotes $(a_1, \ldots, a_{i-1}, a_{i+1}, a_k)$ without the a_i entry. For a subset $A \subsetneq \{1, \ldots, k\}$, $a_{-A} := (a_i, i \notin A)$. For vectors x, y, $x \leq y$ denotes componentwise inequality. We freely refer to x as singular if we mean the vector x or as plural if we mean its components $x_1, \ldots, x_n$. For example we may refer to λ^* as a locational marginal price or locational marginal prices.

Matrices are usually in capital letters. Let M, N be index sets with $m := |M|$, $n := |N|$. An $m \times n$ matrix with $a_{ij} \in \mathbb{C}$ as its (i, j)-th entry for $i \in M, j \in N$, can be written as $A = (a_{ij}, i \in M, j \in N)$. Given $k := \min\{m, n\}$ and scalars $a_1, \ldots, a_k$, $\mathrm{diag}(a_1, \ldots, a_k)$ is a $k \times k$ diagonal matrix with a_i on its diagonal. Given an $m \times n$ matrix A, $\mathrm{diag}(A) := \mathrm{diag}(A_{11}, \ldots, A_{kk})$. We use $\bar{A}$ to denote the componentwise complex conjugate of a matrix A. The transpose of a matrix A is denoted by A^{T} and its Hermitian (or conjugate) transpose by $A^{\mathsf{H}} := \bar{A}^{\mathsf{T}}$. Sometimes we also use A^* to denote A^{H}. If a is a scalar then $a^{\mathsf{H}} = a^*$ is its complex conjugate. We use interchangeably $\left(I^Y\right)^{\mathsf{H}}$ and $A^{Y\mathsf{H}}$. A matrix A is Hermitian if $A = A^{\mathsf{H}}$. A is positive semidefinite (or psd), denoted by $A \succeq 0$, if A is Hermitian *and* $x^{\mathsf{H}} A x \geq 0$ for all $x \in \mathbb{C}^n$; in particular if $A \succeq 0$ then by definition $A = A^{\mathsf{H}}$. A is negative semidefinite (nsd) if $-A$ is psd. For matrices A, B, $A \succeq B$ means $A - B$ is psd. Let $\mathbb{S}^n$ be the set of all $n \times n$ Hermitian matrices, $\mathbb{S}^n_+$ the set of $n \times n$ psd matrices, and $\mathbb{S}^n_-$ the set of $n \times n$ nsd matrices.

A graph $G = (N, E)$ consists of a set N of nodes and a set $E \subseteq N \times N$ of edges. If G is undirected then $(j,k) \in E$ if and only if $(k,j) \in E$. If G is directed then $(j,k) \in E$ only if $(k,j) \notin E$; in this case we will use (j,k) and $j \rightarrow k$ interchangeably to denote an edge pointing from j to k. Therefore, for an undirected graph, $\sum_{(j,k)\in E} x_{jk}$ includes both x_{jk} and x_{kj} for each edge $(j,k) \in E$, whereas, for a directed graph, $\sum_{(j,k)\in E} x_{jk}$ includes a single term x_{jk} for each directed edge $j \rightarrow k$. Sometimes, we write $\sum_{(j,k)\in E} \left(x_{jk} + x_{kj}\right)$ instead of $\sum_{(j,k)\in E} x_{jk}$ to emphasize the undirected nature of the graph. We sometimes use $\tilde{G} = (N, \tilde{E})$ to denote a directed graph. By "$j \sim k$" we mean an edge (j,k) if G is undirected and either $j \rightarrow k$ or $k \rightarrow j$ if G is directed. Sometimes we write $j \in G$ or $(j,k) \in G$ to mean $j \in N$ or $(j,k) \in E$ respectively. A *path* $p := (j_1, \ldots, j_K)$ is an ordered set of nodes $j_k \in N$ so that $(j_k, j_{k+1}) \in E$ for $k = 1, \ldots, K-1$. In that case we refer to a link or a node in the cycle by $(j_k, j_{k+1}) \in p$ or $j_k \in p$ respectively. A *cycle* is a path where $j_K = j_1$. A *simple cycle* is a cycle that visits every node at most once. Unless specified otherwise, we refer to j interchangeably as a node or a bus and $j \sim k$ interchangeably as a link, an edge, or a line.

Given a function $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$, $\frac{\partial f}{\partial x}$ is the $m \times n$ matrix whose (j,k) entry is

$$\left[\frac{\partial f}{\partial x}\right]_{jk} := \frac{\partial f_j}{\partial x_k}(x), \qquad j = 1, \ldots, m, \;\; k = 1, \ldots, n$$

and $\nabla f(x) := \left(\frac{\partial f}{\partial x}\right)^{\mathsf{T}}$ is its transpose. In particular if $m = 1$ then $\frac{\partial f}{\partial x}$ is a row vector and $\nabla f(x)$ is a column vector.

We use e to denote the constant $\lim_n (1 + 1/n)^n$ and $e_j \in \{0,1\}^n$ the unit vector of appropriate size n with a single 1 in the jth position. For the study of three-phase power systems, both balanced and unbalanced, $e^a := (1,0,0)$, $e^b := (0,1,0)$, $e^c := (0,0,1)$, and $e^\phi_j \in \{0,1\}^{3n}$ is the unit vector with a single 1 in the $j\phi$th position. The vector $\mathbf{1}$ usually denotes the vector of all 1s of size 3 and $\mathbb{I}$ usually denotes the identity matrix of size 3; they sometimes denote the vector of all 1s and the identity matrix respectively of other sizes depending on context. We often use $\alpha := e^{-\mathbf{i}2\pi/3}$. The standard balanced vector in positive sequence is $\alpha_+ := (1, \alpha, \alpha^2)$ and that in negative sequence is $\alpha_- := (1, \alpha^2, \alpha)$. The following conversion matrices are key to the understanding of three-phase power systems:

$$\Gamma := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad \Gamma^{\mathsf{T}} := \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Its properties are explained in Theorems 1.3 and 7.2. The similarity transformation to obtain symmetrical components due to Fortescue is defined by the eigenvectors $(\mathbf{1}, \alpha_+, \alpha_-)$ of Γ.

0.2 Units

The unit of a quantity is specified usually the first time the quantity is introduced. Commonly used units in this book are collected here for convenience. We often overload notations so that the same symbol may refer to different quantities depending on the context, e.g., I may denote a vector of current phasors $I = (I_i, i = 1, \ldots, n)$ or the identity matrix of appropriate size, V may denote a vector of voltage phasors $V = (V_i, i = 1, \ldots, n)$ or their unit volt.

1. voltage $v(t), V$: volt (V).
2. current $i(t), I$: ampere (A).
3. real power P : watt (W); reactive power Q : volt-ampere reactive (var); complex power $S := P + \mathbf{i}Q$, apparent power $|S|$: volt-ampere (VA).
4. resistance R, reactance $X = \mathbf{i}\omega L$ or $1/\mathbf{i}\omega C$, impedance $Z := R + \mathbf{i}X$: ohm (Ω).
5. conductance $G := R^{-1}$, susceptance $B := G^{-1}$, admittance $Y := Z^{-1} =: G + \mathbf{i}B$: mho (Ω^{-1}).
6. inductance L: henry (H); magnetic flux linkage $\lambda(t) = Li(t)$: weber-turn (Wb-turn).
7. capacitance C: farad (F); electric charge $q(t) = Cv(t)$: coulomb (C)

Part I

Single-phase networks

Chapter 1

Basic concepts

This chapter introduces basic concepts in modeling the steady-state behavior of an alternating current (AC) power system where voltages and currents are sinusoidal functions of time. For us, steady state means that the frequencies of voltages and currents in the entire network are at their nominal value (e.g., 60 Hz in the US, 50 Hz in China and Europe). In Chapter 1.1 we describe phasor representation of sinusoidal voltages and currents, and introduce circuit models of devices that make up a single-phase system. In Chapter 1.2 we explain balanced three-phase systems and how to simplify their analysis using per-phase models. In Chapter 1.3 we define the concept of complex power for single-phase and three-phase systems, and illustrate through an example that a three-phase system saves power and conductors compared with a single-phase system serving the same load.

1.1 Single-phase systems

An AC system consists of generators and loads connected by transmission or distribution lines and transformers. Their behavior can be described using quantities such as voltages, currents, and power which are sinusoidal functions of time. These quantities obey laws of physics. For our purposes they are the Kirchhoff's current law (KCL), Kirchhoff's voltage law (KVL), and Ohm's law. These laws allow us to analyze or simulate system behavior in the time domain. For steady-state behavior it is often easier to transform these quantities to the phasor domain, apply the corresponding physical laws in the phasor domain to analyze the steady state of a power network, and then translate the results back to the time domain, as illustrated in Figure 1.1.

In this section we define voltage and current phasors, present simple models of generators, loads, and lines using voltage sources, current sources, and impedances. We also summarize KCL, KVL and Ohm's law in the phasor domain. They can be used to analyze a network of these circuit elements.

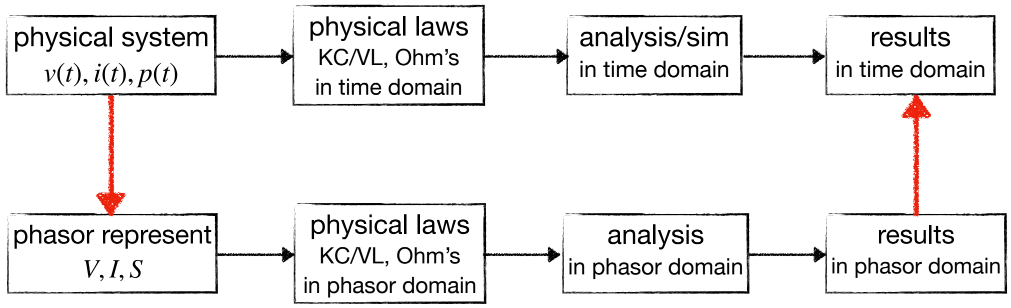


Figure 1.1: Phasor representation and analysis.

1.1.1 Voltage and current phasors

The quantities of interest, voltage $v(t)$, current $i(t)$, and power $p(t)$, are physical and can be empirically measured. The potential energy gained in moving a unit of charge from point k to point j is called the *voltage*, or *electric potential difference*, between j and k, denoted by v_{jk}. Its SI unit (International Systems of Units) is volt (V), or equivalently, joule/coulomb. Usually we arbitrarily fix a reference point 0 for all voltages in the system under study. In that case we refer to the voltage at point j with respect to the reference point simply as the *voltage at j* and denote v_{j0} simply by v_j. Then the voltage between two points j and k is $v_{jk} := v_j - v_k$ and represents the energy required to move a unit of charge from point k to point j. The flow rate of electric charge through a point is called the *current* through that point. Its SI unit is *ampere* (A), or equivalently, coulomb/second. The rate of energy transfer when a unit of charge is moved through an electric potential difference (voltage) between two points is called electric *power*. Its SI unit is watt (W), or equivalently, joule/second. It is equal to the product of voltage and current between these two points.

A sinusoidal voltage function is

$$v(t) \quad = \quad V_{\max}\cos(\omega t + \theta_V) \;=\; \mathrm{Re}\left\{V_{\max}e^{\mathbf{i}\theta_V}\cdot e^{\mathbf{i}\omega t}\right\}$$

where $V_{\max}$ is the amplitude (i.e., maximum magnitude) of the voltage $v(t)$, ω is the steady-state frequency in radian, and θ_V is the phase angle. In steady state, ω is assumed fixed systemwide, and hence a voltage function is fully specified by two parameters $(V_{\max}, \theta_V)$. This motivates the definition of voltage *phasor*

$$V \quad := \quad \frac{V_{\max}}{\sqrt{2}}\, e^{\mathbf{i}\theta_V} \qquad \text{volt (V)}$$

such that

$$v(t) \quad = \quad \mathrm{Re}\{\sqrt{2}V\cdot e^{\mathbf{i}\omega t}\} \tag{1.1}$$

A period of $v(t)$ is $T := 2\pi/\omega$. The magnitude of the voltage phasor

$$|V| \quad := \quad \frac{V_{\max}}{\sqrt{2}}$$

is equal to the root-mean-square (RMS) value of the voltage, defined as

$$\sqrt{\frac{1}{T}\int_0^{\mathrm{T}} v^2(t)dt} \quad = \quad \sqrt{\frac{1}{T}\int_0^{\mathrm{T}} V_{\max}^2\cos^2(\omega t+\theta_V)\,dt} \quad = \quad \frac{V_{\max}}{\sqrt{2}}$$

where we have used $\cos^2\phi = (1+\cos 2\phi)/2$.

Similarly let the sinusoidal current function be

$$i(t) \;=\; I_{\max}\cos(\omega t + \theta_I) \qquad \text{ampere (A)}$$

with the corresponding current phasor

$$I \;:=\; \frac{I_{\max}}{\sqrt{2}}\, e^{\mathbf{i}\theta_I}$$

such that

$$i(t) \;=\; \mathrm{Re}\{\sqrt{2}I\cdot e^{\mathbf{i}\omega t}\} \tag{1.2}$$

The RMS value of the current is $|I| := I_{\max}/\sqrt{2}$.

1.1.2 Single-phase devices

Basic building blocks of an AC power system are generators that generate power, loads that consume power, transmission and distribution lines and transformers that connect generators and loads. These devices can be modeled by circuit elements such as impedances, voltage sources, current sources, and (later) power sources, as we now explain.

Impedance Z. The voltage and current across a resistor R in ohm (Ω), an ideal inductor L in henry (H), or an ideal capacitor C in farad (F) satisfy a linear relation, both in the time domain and in the phasor domain. We now derive Ohm's law in the phasor domain from its representation in the time domain.

Consider the circuit in Figure 1.2. The voltage $v(t)$ across the resistor R and the current $i(t)$ through it

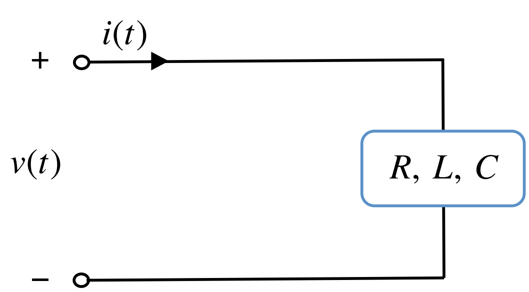


Figure 1.2: In phasor domain the voltage V and current I across a linear circuit element Z are related by $V = ZI$ where the impedances for resistor R, inductor L, capacitor C are $Z = R, \mathbf{i}\omega L, (\mathbf{i}\omega C)^{-1}$ respectively.

are related by Ohm's law:

$$v(t) \;=\; R\,i(t)$$

Using (1.1)(1.2), this is equivalent to:

$$\mathrm{Re}\left\{V\cdot\sqrt{2}e^{\mathbf{i}\omega t}\right\} \;=\; \mathrm{Re}\left\{RI\cdot\sqrt{2}e^{\mathbf{i}\omega t}\right\}$$

Hence Ohm's law in the phasor domain for a resistor is:

$$V = RI$$

The current across a resistor is called *in phase* with the voltage.

An ideal inductor L is characterized by

$$v(t) = L\frac{di(t)}{dt}$$

Substituting (1.1) and

$$\frac{di(t)}{dt} = -\omega I_{\max}\sin(\omega t+\theta_I) = \omega I_{\max}\cos(\omega t+\theta_I+\pi/2)$$

we have

$$\mathrm{Re}\left\{V\cdot\sqrt{2}e^{\mathbf{i}\omega t}\right\} = \mathrm{Re}\left\{\mathbf{i}\omega LI\cdot\sqrt{2}e^{\mathbf{i}\omega t}\right\}$$

or in the phasor domain:

$$V = (\mathbf{i}\omega L)I$$

The current across an inductor is said to *lag* the voltage by $\pi/2$ radian.

Similarly an ideal capacitor C is characterized by

$$i(t) = C\frac{dv(t)}{dt}$$

Substituting (1.2) and

$$\frac{dv(t)}{d(t)} = -\omega V_{\max}\sin(\omega t+\theta_V) = \omega V_{\max}\cos(\omega t+\theta_V+\pi/2)$$

we have

$$\mathrm{Re}\left\{I\cdot\sqrt{2}e^{\mathbf{i}\omega t}\right\} = \mathrm{Re}\left\{\mathbf{i}\omega CV\cdot\sqrt{2}e^{\mathbf{i}\omega t}\right\}$$

or in the phasor domain:

$$V = \frac{1}{\mathbf{i}\omega C}I$$

The current across a capacitor is said to *lead* the voltage by $\pi/2$ radian.

In summary we define the *impedances* of these elements, a resistor R, an ideal inductor L, and an ideal capacitor C in the phasor domain as respectively:

$$Z_R := R, \qquad Z_L := \mathbf{i}\omega L, \qquad Z_C := \frac{1}{\mathbf{i}\omega C}$$

Instead of impedance Z, sometimes it is convenient to use its inverse, called the *admittance* $Y := Z^{-1}$. The voltage V across an impedance Z (or admittance Y) and the current I through it are related in the phasor domain by

$$V = ZI \quad \text{and} \quad I = YV$$

An important advantage of phasor representation of an AC circuit is that circuit analysis involves only algebraic operations rather than differential equations in the time domain.

Example 1.1. A voltage $v(t)$ is applied to a resistor R and an inductor L in series and the current through these devices is $i(t)$. Derive the dynamic equation that relates $(v(t), i(t))$ in the time domain and the corresponding equation that relates their phasors (V, I).

Solution. Let $v_1(t) = Ri(t)$ denote the voltage drop across the resistor and $v_2(t)$ the voltage drop across the inductor that satisfies $v_2(t) = L\frac{d}{dt}i(t)$. Then the relation between $(v(t), i(t))$ is given by KVL: $v(t) = v_1(t) + v_2(t)$ or

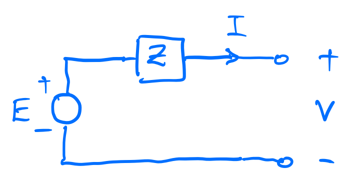
$$v(t) = Ri(t) + L\frac{d}{dt}i(t)$$

Noting that $v(t) = \text{Re}\left\{\sqrt{2}V\, e^{\mathbf{i}\omega t}\right\}$ and $i(t) = \text{Re}\left\{\sqrt{2}I\, e^{\mathbf{i}\omega t}\right\}$, we multiply both sides of the equation above by $e^{\mathbf{i}\omega t}$ to get

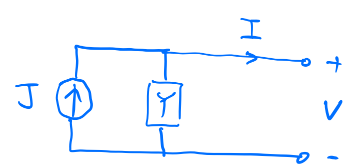
$$\begin{aligned} \sqrt{2}V\, e^{\mathbf{i}\omega t} &= R\,\sqrt{2}I\, e^{\mathbf{i}\omega t} + L\left(\mathbf{i}\omega\sqrt{2}I\, e^{\mathbf{i}\omega t}\right) \\ V &= (R + \mathbf{i}\omega L)\, I \end{aligned}$$

□

Voltage source (E, Z)**.** In the phasor domain, a voltage source is a circuit model with a constant *internal voltage* E in series with an impedance Z, as shown in Figure (a). Its external behavior is described by the



(a) Voltage source



(b) Current source

Figure 1.3: A voltage source (E, Z) and a current source (J, Y). An ideal voltage source has $Z = 0$ and an ideal current source has $Y = 0$.

relation between its *terminal voltage and terminal* (V, I):

$$V = E - ZI$$

Hence the open-circuit (terminal) voltage V equals the internal voltage E. We often adopt an ideal voltage source with $Z = 0$. In this case $V = E$.

Current source (J,Y). In the phasor domain, a current source is a circuit model with a constant *internal current* J in parallel with an admittance Y, as shown in Figure (b). Its external behavior is described by the relation between its terminal voltage and current (V,I):

$$I = J - YV$$

Hence the closed-circuit (terminal) current I equals the internal current J. We often adopt an ideal current source with $Y = 0$. In this case $I = J$.

Remark 1.1.

1. A nonideal voltage source (E,Z) and a current source (J,Y) are equivalent, i.e., have the same terminal voltage and current relationship if their parameters satisfy

$$J = \frac{E}{Z} \qquad \text{(closed-circuit equivalent)}$$
$$Y := Z^{-1} \qquad \text{(open-circuit equivalent)}$$

2. Ideal voltage or current sources are reasonable models as their series impedances or shunt admittances can be combined with the series impedance and shunt admittances of a transmission or distribution line to which they are connected, as we will see in Chapter 2. We will therefore often use ideal voltage and current sources in this book with series series impedances and shunt admittances.

□

Single-phase devices. Basic devices in a power system are generators, loads, transmission and distribution lines, and transformers. A generator can be modeled by a voltage source or current source. A load can be modeled by an impedance (or admittance), a voltage source, or a current source. A line can be modeled by a series impedance, possibly with a shunt admittance at each end of the line; the details are described in Chapter 2. A transformer can be modeled by a series impedance and a shunt admittance followed by voltage and current gains; the details are described in Chapter 3. We will introduce in Chapter 1.3 the concept of complex power. This leads to a device we will call a *power source* that generates or draws a constant power which we will study in Chapter 7.1.2. These are summarized in Table 1.1. This book

Device	Circuit model
Generator	Voltage source, current source, power source
Load	Impedance, voltage source, current source, power source
Line	Impedance (Chapter 2)
Transformer	Impedance, voltage/current gain (Chapter 3)

Table 1.1: Circuit elements commonly used for modeling generators, loads, lines, and transformers.

develops techniques for analyzing power system models constructed from these circuit elements.

A common load model is called a ZIP load where Z models a load by a constant impedance Z or its reciprocal $Y := 1/Z$, I models a load by a constant current source (J,Y), or equivalently, by a constant voltage source (E,Z), and P models a load by a constant complex power injection/withdrawal, e.g., a PQ bus in Chapter 4.3.4. All three types of loads can be represented by a relationship between the power S consumed by the load and the voltage V across the load:

$$S \;:=\; S_0\left(a_2|V|^2 \,+\, a_1|V| \,+\, a_0\right)$$

where

- S_0 is the nominal power consumption of the load;
- $a_2|V|^2$ represents a constant impedance load whose power is proportional to voltage magnitude $|V|$ quadratically.
- $a_1|V|$ represents a constant current load whose power is proportional to $|V|$.
- a_0 represents a constant power load.

For instance $a_0 = 1/3$, $a_1 := |V_0|^{-1}/3$, and $a_2 := |V_0|^{-2}/3$ where V_0 is the nominal voltage of the load. In this case the load power is a combination of ZIP and $S = S_0$ when $V = V_0$. The nominal power S_0 may depend also on frequency. During transient, this dependence can be made explicit by the time-domain model

$$s(t) \;:=\; s_0\left(a_2|v(t)|^2 \,+\, a_1|v(t)| \,+\, a_0\right)(1+a_3\Delta\omega(t))$$

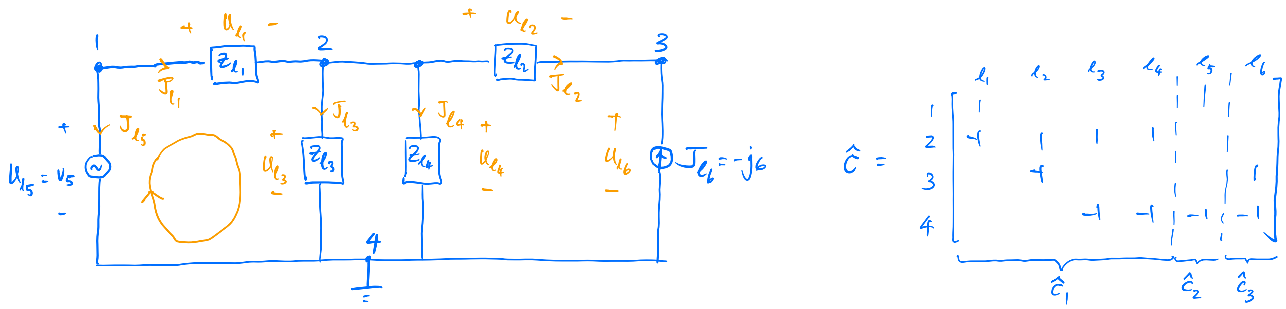
where $\Delta\omega(t)$ is the deviation from the nominal frequency during transient.

1.1.3 KVL, KCL, Ohm's Law, Tellegen's theorem

Consider a circuit consisting of an interconnection of resisters, inductors, capacitors, and voltage and current sources. An ideal voltage source between two points enforces a given voltage between these two points. An ideal current source between two points enforces a given current between them. We now describe KVL, KCL, Ohm's law for a general circuit and derive a result called Tellegen's theorem.

We represent a circuit by a connected *directed* graph $\hat{G} := (\hat{N}, \hat{E})$ with an arbitrary orientation where $\hat{N}$ is a set of nodes and $\hat{E} \subseteq \hat{N} \times \hat{N}$ is a set of links. We abuse notation and use $\hat{N}$ to denote both the set of nodes and the number of nodes in $\hat{N}$; the meaning should be clear from the context. We allow multiple links between two nodes j and k. A link l that points from node j to node k is represented by $l = (j,k)$ or $l = j \rightarrow k$. Multiple links $l_1, \ldots, l_k$ between nodes j and k may have different orientations, e.g., $l_1 = j \rightarrow k$ and $l_2 = k \rightarrow j$. There are two variables associated with each link $l = (j,k)$ between nodes j and k. The voltage across link l is denoted by U_l in the direction of l and the branch current over link l from j to k is denoted by J_l.

A link l represents either an impedance, a voltage source, or a current source. If link l represents an impedance then its value z_l is given and the voltage U_l and branch current J_l across link l satisfies $U_l = z_l J_l$ (Ohm's law). If link l represents a voltage source then $U_l = u_l$ is given, and if it represents a current source then $J_l = j_l$ is given. These notations are illustrated in Figure 1.4a.



(a) Circuit (b) Incidence matrix

Figure 1.4: A circuit represented as a directed graph where each link l is either an impedance z_l, a voltage source U_l, or a current source J_l. The voltage source $U_{l_5} = u_5$ and current source $J_{l_6} = -j_6$ are given. Its incidence matrix $\hat{C}$ is partitioned into $\hat{C}_1$ corresponding to the impedances, $\hat{C}_2$ corresponding to the voltage source, and $\hat{C}_3$ corresponding to the current source.

KCL, KVL. Kirchhoff's current law (KCL) states that the incident currents at any node j sum to zero, i.e.,

$$-\sum_{i:i\to j\in\hat{E}} J_{ij} + \sum_{k:j\to k\in\hat{E}} J_{jk} \;=\; 0 \tag{1.3a}$$

For the example in Figure 1.4 this means $-J_{l_1} + J_{l_2} + J_{l_3} + J_{l_4} \;=\; 0$ at node 2. Kirchhoff's voltage law (KVL) states that voltage drops around any cycle c sum to zero. Consider a cycle c in the graph with an arbitrary orientation, say, clockwise. A link l in the cycle that is in the same direction as c is denoted by $l \in c$ and a link l that is in the opposite direction to c is denoted by $-l \in c$. Then KVL states that for any cycle c

$$\sum_{l\in c} U_l - \sum_{-l\in c} U_l \;=\; 0 \tag{1.3b}$$

For the cycle indicated in Figure 1.4(a) we have $U_{l_1} + U_{l_3} - U_{l_5} \;=\; 0$.

We can represent (1.3) compactly in vector notation. Let $U := \left(U_l, l\in\hat{E}\right)$ and $J := \left(J_l, l\in\hat{E}\right)$ denote the vectors of voltages and currents respectively across these lines. Let $\hat{C} \in \{-1,0,1\}^{|\hat{N}|\times|\hat{E}|}$ be the node-by-link *incidence matrix* defined by:

$$\hat{C}_{jl} \;:=\; \begin{cases} 1 & \text{if } l = j\to k \text{ for some bus } k \\ -1 & \text{if } l = i\to j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}, \qquad j\in\hat{N},\ l\in\hat{E}$$

See Figure 1.4. Then Kirchhoff's current law (1.3a) states that

$$\text{KCL:} \qquad \hat{C}J \;=\; 0 \tag{1.4a}$$

Kirchhoff's voltage law is equivalent to the condition that there exist *nodal* voltages $V \in \mathbb{C}^{|\hat{N}|}$ (with respect to the common reference point node 0) such that

$$\text{KVL:} \qquad U \;=\; \hat{C}^{\mathsf{T}} V \tag{1.4b}$$

i.e., given line voltages U, there must exist nodal voltages such that $U_l = V_j - V_k$ where $l = j\to k$, from which (1.3b) follows. This seems intuitive and can be proved mathematically using concepts in algebraic graph theory (Exercise 1.1).

Circuit analysis. Consider a circuit represented by an incidence matrix $\hat{C}$. The system (1.4) consists of $|\hat{N}| + |\hat{E}|$ linear complex equations in $|\hat{N}| + 2|\hat{E}|$ complex variables (V, U, J). The $|\hat{N}| \times |\hat{E}|$ incidence matrix $\hat{C}$ is of rank $|\hat{N}| - 1$ since $\hat{G}$ is connected, with span($\mathbf{1}$) as its null space (see Chapter 25.2 for more details). Without loss of generality we use node $\hat{N}$ as the reference point for all voltages, i.e., $V_{\hat{N}} := 0$ by definition. Therefore (1.4) represents $|\hat{N}| + |\hat{E}| - 1$ linearly independent equations in $|\hat{N}| + 2|\hat{E}| - 1$ variables (V, U, J). To obtain another $|\hat{E}|$ linearly independent equations we note that across every link l is exactly one of the following devices:

1. *impedance* with a given z_l: Its behavior is described by Ohm's law
$$U_l = z_l J_l \tag{1.5a}$$
2. *voltage source* with a given u_l: Its behavior is described by
$$U_l = u_l \tag{1.5b}$$
3. *current source* with a given j_l: Its behavior is described by
$$J_l = i_l \tag{1.5c}$$

In other words (1.4)(1.5) specify $|\hat{N}| + 2|\hat{E}| - 1$ linearly independent equations in $|\hat{N}| + 2|\hat{E}| - 1$ variables $(V_{-\hat{N}}, U, J)$ where $V_{-\hat{N}} := (V_j, j \neq \hat{N})$ (recall that $V_{\hat{N}} := 0$). A circuit analysis problem is to solve for these variables. A sufficient condition is given in Theorem 1.1 for the existence and uniqueness of solution. A necessary condition for the existence of a solution is that the given voltage and current vectors (v, j) are consistent, e.g., if only current sources are incident on a node k, then these given currents must satisfy KCL at node k, or if a set of voltage sources form a cycle c then these given voltages must satisfy KVL on c.

The system (1.4)(1.5) of equations can be simplified, as follows. Partition the set E of links into three disjoint sets $E =: E_1 \cup E_2 \cup E_3$ where E_1 is the set of impedances, E_2 voltage sources, and E_3 current sources. Order the links such that the incidence matrix decomposes into submatrices $\hat{C}_1, \hat{C}_2, \hat{C}_3$ corresponding to impedances, voltage sources, and current sources respectively (see Figure 1.4b):

$$\hat{C} =: \begin{bmatrix} \hat{C}_1 & \hat{C}_2 & \hat{C}_3 \end{bmatrix}$$

Partitioned the branch voltages U and branch currents J accordingly:

$$U := \begin{bmatrix} U_1 \\ u \\ U_3 \end{bmatrix}, \qquad J := \begin{bmatrix} J_1 \\ J_2 \\ j \end{bmatrix}$$

where v and j are the given vectors of voltage and current sources respectively. Let Z be the diagonal matrix whose entries are the given impedances z_l. Then KCL and KVL are

$$\hat{C}_1 J_1 + \hat{C}_2 J_2 = -\hat{C}_3 j$$
$$U_1 = \hat{C}_1^{\mathsf{T}} V, \quad u = \hat{C}_2^{\mathsf{T}} V, \qquad U_3 = \hat{C}_3^{\mathsf{T}} V$$

for some nodal voltages V. Use Ohm's law $U_1 = ZJ_1$ to eliminate U_1 to obtain

$$\begin{bmatrix} 0 & \hat{C}_1 & \hat{C}_2 & 0 \\ \hat{C}_1^{\mathsf{T}} & -Z & 0 & 0 \\ \hat{C}_2^{\mathsf{T}} & 0 & 0 & 0 \\ \hat{C}_3^{\mathsf{T}} & 0 & 0 & -\mathbb{I}_{U_3} \end{bmatrix} \begin{bmatrix} V \\ J_1 \\ J_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} -\hat{C}_3 j \\ 0 \\ u \\ 0 \end{bmatrix} \tag{1.6}$$

where $\mathbb{I}_{U_3}$ is the identity matrix of compatible size with U_3. The desired quantities (V, U_3, J_1, J_2) are solutions of (1.6) if they exist. Given J_1, U_1 is given by $U_1 = ZJ_1$.

Recall that we take without loss of generality node $\hat{N}$ as the common reference point for nodal voltages and assign $V_{\hat{N}} := 0$. We can consider the $(|\hat{N}|-1) \times |\hat{E}|$ *reduced incidence matrix* C obtained from $\hat{C}$ by deleting the last row corresponding to the reference node $\hat{N}$. The advantage of using C is that it has a full row rank of $|\hat{N}|-1$. Let $\hat{V} := \left(V_j, j \neq \hat{N}\right)$ be the vector of all non-reference nodal voltages. Similarly partition C into $C =: [C_1 \;\; C_2 \;\; C_3]$. Then (1.6) is equivalent to the following equation:

$$\underbrace{\begin{bmatrix} 0 & C_1 & C_2 & 0 \\ C_1^{\mathsf{T}} & -Z & 0 & 0 \\ C_2^{\mathsf{T}} & 0 & 0 & 0 \\ C_3^{\mathsf{T}} & 0 & 0 & -\mathbb{I}_{U_3} \end{bmatrix}}_{M} \begin{bmatrix} \hat{V} \\ J_1 \\ J_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} -C_3 j \\ 0 \\ u \\ 0 \end{bmatrix} \tag{1.7}$$

The key feature of this model, compared with (1.6), is that it does not contain the reference node $\hat{N}$.

Example 1.2. Consider the circuit in Figure 1.4 represented by the directed graph $\hat{G} = (\hat{N}, \hat{E})$ with

$$\begin{aligned} \hat{N} &:= \{1,2,3,4\} \\ \hat{E} &:= \{l_1 := 1 \to 2,\ l_2 := 2 \to 3,\ l_3 := 2 \to 4,\ l_4 := 2 \to 4,\ l_5 := 1 \to 4,\ l_6 := 3 \to 4\} \end{aligned}$$

The incidence matrix $\hat{C}$ can be partitioned into submatrices

$$\hat{C}_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \qquad \hat{C}_2 := \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \qquad \hat{C}_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

The reduced incidence submatrices are then

$$C_1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad C_2 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C_3 := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The equation (1.7) becomes:

$$
\left[\begin{array}{ccc|cccc|c|c}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
\hline
1 & -1 & 0 & -z_{l_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -z_{l_2} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -z_{l_3} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -z_{l_4} & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}\right]
\left[\begin{array}{c}
V_1 \\ V_2 \\ V_3 \\ \hline J_{l_1} \\ J_{l_2} \\ J_{l_3} \\ J_{l_4} \\ \hline J_{l_5} \\ \hline U_{l_6}
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ 0 \\ j_6 \\ \hline 0 \\ 0 \\ 0 \\ 0 \\ \hline u_5 \\ \hline 0
\end{array}\right]
$$

□

We now discuss the existence and uniqueness of solution to (1.7).

Theorem 1.1. The matrix M in (1.7) is invertible if both of the following square matrices of sizes $\hat{N}-1$ and $|E_2|$ respectively are invertible:

$$C_1 Z^{-1} C_1^{\mathsf{T}}, \qquad C_2^{\mathsf{T}} \left(C_1 Z^{-1} C_1^{\mathsf{T}} \right)^{-1} C_2$$

where E_2 is the set of voltage sources. □

If z_l are real and positive then $C_1 Z^{-1} C_1^{\mathsf{T}}$ is invertible since $Z := \mathrm{diag}(z_l)$ is positive definite and C and hence its submatrix C_1 are both of full row rank. When Z is complex, $C_1 Z^{-1} C_1^{\mathsf{T}}$ may not be invertible even if z_l are all nonzero and C_1 is of full row rank (see discussions in Chapter 4.2.5). The matrix C_2^{T} is of full row rank if and only if no voltage sources form a cycle in the circuit.

The proof of Theorem 1.1 relies on the following fact. Let $M \in \mathbb{C}^{n \times n}$ and partition it into blocks:

$$M = \begin{bmatrix} A & B \\ D & C \end{bmatrix}$$

such that $C \in \mathbb{C}^{k \times k}$, $k < n$, is invertible and the other submatrices are of matching dimensions. The $(n-k) \times (n-k)$ matrix $M/C := A - BC^{-1}D$ is called the *Schur complement of block* C of matrix M. If A is invertible then the $k \times k$ matrix $M/A := C - DA^{-1}B$ is called the *Schur complement of block* A of matrix M. Then M is nonsingular if and only if C and M/C are nonsingular, or if and only if A and M/A are nonsingular; see Theorem 25.1 in Appendix 25.1.2.

Proof of Theorem 1.1. We can interchange the second and third rows and interchange the second and third column write (1.7) equivalently in terms of the matrix

$$
\tilde{M} = \left[\begin{array}{cc|cc}
0 & C_2 & C_1 & 0 \\
C_2^{\mathsf{T}} & 0 & 0 & 0 \\
\hline
C_1^{\mathsf{T}} & 0 & -Z & 0 \\
C_3^{\mathsf{T}} & 0 & 0 & -\mathbb{I}_{U_3}
\end{array}\right]
$$

The matrix M is nonsingular if and only if $\tilde{M}$ is. Since both Z and $\mathbb{I}_{U_3}$ are both nonsingular, $\tilde{M}$ is nonsingular if and only if the Schur complement of $\text{diag}\left(-Z, -\mathbb{I}_{U_3}\right)$:

$$S \;:=\; \begin{bmatrix} 0 & C_2 \\ C_2^{\mathsf{T}} & 0 \end{bmatrix} + \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z^{-1} & 0 \\ 0 & \mathbb{I}_{U_3} \end{bmatrix} \begin{bmatrix} C_1^{\mathsf{T}} & 0 \\ C_3^{\mathsf{T}} & 0 \end{bmatrix} \;=\; \begin{bmatrix} C_1 Z^{-1} C_1^{\mathsf{T}} & C_2 \\ C_2^{\mathsf{T}} & 0 \end{bmatrix}$$

The Schur complement S is a square matrix of size $(\hat{N}-1)+|\hat{E}_2|$ where E_2 is the set of voltage sources. By assumption the $(\hat{N}-1)\times(\hat{N}-1)$ matrix $C_1 Z^{-1} C_1^{\mathsf{T}}$ is nonsingular. Therefore M is nonsingular if and only if the Schur complement

$$S/0 \;:=\; -C_2^{\mathsf{T}} \left(C_1 Z^{-1} C_1^{\mathsf{T}} \right)^{-1} C_2$$

of the zero submatrix of S is nonsingular. □

Tellegen's theorem An important result in circuit theory is Tellegen's theorem that expresses a relation between voltage drops across links and currents on these links. It is a simple consequence of Kirchhoff's laws and algebraic graph theory (see Chapter 25.2 for more details). Since the rank of the $|\hat{N}| \times |\hat{E}|$ incidence matrix $\hat{C}$ is $|\hat{N}|-1$ assuming $\hat{G}$ is connected, the rank of the range space $\text{range}\left(\hat{C}^{\mathsf{T}}\right)$ is $|\hat{N}|-1$ and the rank of the null space $\text{null}(\hat{C})$ is $|\hat{E}|-|\hat{N}|+1$. Recall that the subspaces $\text{null}(\hat{C})$ and $\text{range}\left(\hat{C}^{\mathsf{T}}\right)$ are orthogonal complements of each other and they span $\mathbb{C}^{|\hat{E}|}$, i.e., $\mathbb{C}^{|\hat{E}|} = \text{null}(\hat{C}) \oplus \text{range}\left(\hat{C}^{\mathsf{T}}\right)$. The KCL and KVL (1.3a)(1.3b) say that the branch currents satisfy $J \in \text{null}(\hat{C})$ and the branch voltages satisfy $U \in \text{range}\left(\hat{C}^{\mathsf{T}}\right)$ respectively. Therefore

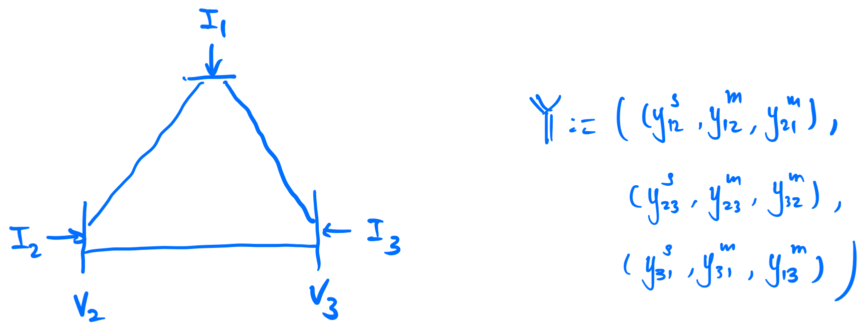
$$\text{Tellegen's theorem:} \qquad J^{\mathsf{H}} U \;=\; 0$$

It is remarkable that this relation holds for any branch current vector J and branch voltage vector U, even if they are from different networks as long as these networks have the same incidence matrix $\hat{C}$.

1.1.4 One-line diagram and equivalent circuit

In this subsection we formally define one-line diagram widely used in power system modeling and justify it using circuit analysis of Chapter 1.1.3.

One-line diagram. A power system is often described by what is called a one-line diagram. A simplest one line diagram can be specified by a pair $(G, \mathbb{Y})$ where $G := (\overline{N}, E)$ is a graph and $\mathbb{Y} := \left(y_{jk}^s, y_{jk}^m, y_{kj}^m,\ l=(j,k)\in E\right)$ is a set of line parameters for every line $l \in E$. Each node $j \in \overline{N}$ represents a bus in the power system. We will therefore refer to j as a bus or a node interchangeably. Associated with each node j are a nodal voltage $V_j \in \mathbb{C}$ with respect to an arbitrary but common reference point and a nodal current injection $I_j \in \mathbb{C}$. Each link $l \in E$ represents a transmission or distribution line or a transformer. We will therefore refer to l as a line, a link or a branch interchangeably. There can be multiple lines between two buses, though for notational simplicity we often assume there is a single line between each pair of buses in which case a line l between buses j and k can be identified by (j,k). The line parameter $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right) \in \mathbb{C}^3$ on each



(a) Graph $G = (\overline{N}, E)$ (b) Line parameters $\mathbb{Y}$

Figure 1.5: Three-bus network $(G, \mathbb{Y})$.

line $l = (j,k) \in E$ represents its series and shunt admittances (we assume here a single-phase system and $y^s_{jk} = y^s_{kj}$). This model is illustrated in Figure 1.5 for a three-bus network. We will explain below how to interpret line parameters in terms of a circuit model.

Let $V := (V_j, j \in \overline{N})$ and $I := (I_j, j \in \overline{N})$ denote the vectors of nodal voltages and current injections respectively. The main equation that describes the behavior of the one-line diagram defined by $(G, \mathbb{Y})$ is the nodal current balance equation:

$$I = YV \tag{1.8}$$

The matrix Y is called an admittance matrix whose entries consist of series and shunt admittances of the lines in E. All other power flow quantities can be derived in terms of V. To specify Y in terms of $(G, \mathbb{Y})$, define C_3 as the $|\overline{N}| \times |E|$ incidence matrix for G with an arbitrary orientation (the reason for the subscript 3 will become clear below). Let $Y^s := \text{diag}\left(y^s_{jk}, (j,k) \in E\right)$ denote the diagonal matrix of series admittances on the lines. Let $Y^m := \text{diag}\left(y^m_{jj}, j \in \overline{N}\right)$ denote the diagonal matrix of total shunt admittances $y^m_{jj} := \sum_{k:(j,k)\in E} y^m_{jk}$ incident on bus j. We specify Y in (1.8) as a theorem.

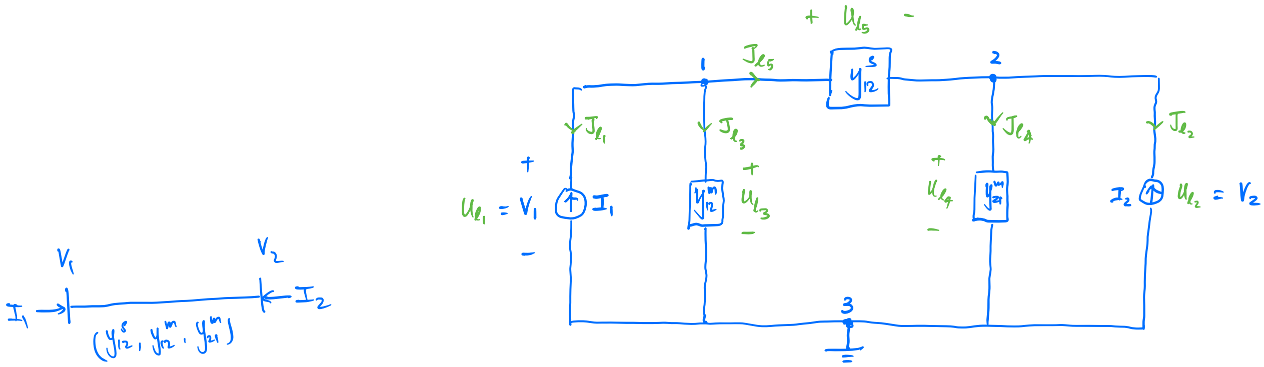
Theorem 1.2. The nodal currents I and voltages V satisfy $I = YV$ where

$$Y = C_3 Y^s C_3^{\mathsf{T}} + Y^m$$

□

The VI relation (1.8) including the invertibility of Y will be studied in detail in Chapter 4. In the rest of this subsection we interpret the line parameter $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$ as defining a Π circuit model for the line, and explain how the Π circuit for each line induces an equivalent circuit for the one-line diagram $(G, \mathbb{Y})$. Finally we apply KCL, KVL, and Ohm's law to the equivalent circuit to prove Theorem 1.2. This serves as a formal justification for one-line diagrams and (1.8).

Equivalent Π circuit of line $l \in E$. Consider the line $l = (1,2)$ in Figure 1.6(a) with nodal voltages and currents (V_1, I_1) and (V_2, I_2) at nodes 1 and 2 in $\overline{N}$ respectively. The line parameter $\left(y^s_{12}, y^m_{12}, y^m_{21}\right)$ defines



(a) One-line diagram

(b) Equivalent Π circuit

Figure 1.6: One-line diagram for line $l = (j,k)$ and its equivalent Π circuit. The nodal current injections (I_1, I_2) and the nodal voltages (V_1, V_2) in the one-line diagram become current sources and branch voltages respectively between nodes $(1,2)$ and the reference node 3 in the Π circuit.

the equivalent circuit in Figure 1.6(b) called the Π circuit of line $l = (1,2)$. (We will explain the origin of the equivalent circuit in Chapter 2.) The application of KVL, KCL, and Ohm's law on the Π circuit leads to a relation between (V_1, I_1) and (V_2, I_2), as we now explain.

The Π circuit can be analyzed using the method of Chapter 1.1.3. Let the *directed* graph $\hat{G} := (\hat{N}, \hat{E})$ represent the Π circuit where

$$\hat{N} \ := \ \{1,2,3\}, \qquad \hat{E} \ := \ \{l_1 := 1 \to 3,\ l_2 := 2 \to 3,\ l_3 := 1 \to 3,\ l_4 := 2 \to 3,\ l_5 := 1 \to 2\}$$

as shown in Figure 1.6b. Note that the graph G of the one-line diagram has 2 nodes while the graph $\hat{G}$ of its equivalent circuit has 3 nodes with node 3 being the voltage reference point. The key feature is that the nodal current injections (I_1, I_2) and the nodal voltages (V_1, V_2) in the one-line diagram become current sources and branch voltages respectively, between nodes 1, 2 and the reference node 3 in the Π circuit (see Figure 1.6b).

For each link $l \in \hat{E}$ let U_l and J_l denote the voltage and current across line l in the direction of l. Let $U := (U_l, l \in \hat{E})$ and $J := (J_l, l \in \hat{E})$. The devices on the links $l \in \hat{E}$ are:

l_1 : current source I_1 with $J_{l_1} = -I_1$, $\qquad$ l_2 : current source I_3 with $J_{l_2} = -I_2$

I_3 : admittance y^m_{12} with $J_{l_3} = y^m_{12} U_{l_3}$, $\qquad$ I_4 : admittance y^m_{21} with $J_{l_4} = y^m_{21} U_{l_4}$

l_5 : admittance y^s_{12} with $J_{l_5} = y^s_{12} U_{l_5}$

The node-by-link incidence matrix $\hat{C}$ of the Π circuit is

$$\hat{C} \ := \ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

The KCL, KVL and Ohm's law in terms of C, U, J for the Π circuit in Figure 1.6(b) are:

$$\text{KCL}: \quad \hat{C} J \ = \ 0 \tag{1.9a}$$

$$\text{KVL}: \quad \exists\, V := (V_1, V_2, V_3) \quad \text{s.t.} \quad U \ = \ \hat{C}^{\mathsf{T}} V \tag{1.9b}$$

$$\text{Ohm's law}: \quad J_{l_2} = y^m_{12} U_{l_2},\ \ J_{l_4} = y^m_{21} U_{l_4},\ \ U_{l_5} = y^s_{12} U_{l_5} \tag{1.9c}$$

We will set the nodal voltage V_3 implied by KVL to $V_3 := 0$ since node 3 in $\hat{N}$ is chosen to be the voltage reference point. Using

$$J_{l_1} \;=\; -I_1, \qquad\qquad J_{l_2} \;=\; -I_2, \qquad\qquad V_3 \;:=\; 0$$

to eliminate branch variables (U, J) from the set (1.9) of equations leads to a relation between the nodal currents $I := (I_1, I_2)$ and voltages $V := (V_1, V_2)$:

$$I_1 \;=\; y^s_{12}\,(V_1 - V_2) + y^m_{12} V_1, \qquad\qquad I_1 \;=\; y^s_{12}\,(V_2 - V_1) + y^m_{21} V_2$$

In vector form this is $I = YV$ in (1.8) with the admittance matrix

$$Y \;:=\; \begin{bmatrix} y^s_{12} + y^m_{12} & -y^s_{12} \\ -y^s_{12} & y^s_{12} + y^m_{21} \end{bmatrix}$$

Equivalent circuit of one-line diagram. Given a one-line diagram $(G = (\overline{N}, E), \mathbb{Y})$ the Π circuit model of each line $l \in E$ induces an equivalent circuit for the entire network described by a directed graph $\hat{G} = (\hat{N}, \hat{E})$ constructed from $G = (\overline{N}, E)$, as follows. The set $\hat{N}$ of nodes in the equivalent circuit is

$$\hat{N} \;:=\; \overline{N} \cup \{|\overline{N}| + 1\}$$

where the additional node $\hat{N} := |\overline{N}| + 1$ is the reference point for all voltages, i.e., $V_{\hat{N}} := 0$. For each node $j \in \overline{N}$ in the one-line diagram, there is a link $l = j \rightarrow \hat{N}$ in the equivalent circuit. Each such link corresponds to a current source with branch current $J_l = -I_j$. Denote this set of links by $\hat{E}_1 \subset \hat{E}$.

For each line $\lambda = (j,k) \in E$ parametrized by $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$ in the one-line diagram, there are 3 links $\left(l_{\lambda_1}, l_{\lambda_2}, l_{\lambda_3}\right)$ in $\hat{E}$ in the equivalent circuit, corresponding to

$$\begin{aligned} l_{\lambda_1} = j \rightarrow \hat{N} \;&:\; \text{shunt admittance } y^m_{jk} \text{ with } J_{\lambda_1} \;=\; y^m_{jk} U_{\lambda_1} \\ l_{\lambda_2} = k \rightarrow \hat{N} \;&:\; \text{shunt admittance } y^m_{kj} \text{ with } J_{\lambda_2} \;=\; y^m_{kj} U_{\lambda_2} \\ l_{\lambda_3} = j \rightarrow k \;&:\; \text{series admittance } y^s_{jk} \text{ with } J_{\lambda_3} \;=\; y^s_{jk} U_{\lambda_3} \end{aligned}$$

Let $\hat{E}_2$ denote the set of links corresponding to shunt admittances and $\hat{E}_3$ denote the set of links corresponding to series admittances. Like links in $\hat{E}_1$, links in $\hat{E}_2$ are of the form $l = j \rightarrow \hat{N}$ and connect nodes $j \in \overline{N}$ to the reference node $\hat{N}$. Links in $\hat{E}_3$ are of the form $l = j \rightarrow k$ and connect two non-reference nodes j, k. If bus $j \in \overline{N}$ is connected to k_j other buses $k \in \overline{N}$ in the one-line diagram, then there will be k_j links $l_k = j \rightarrow \hat{N}$ in the equivalent circuits, for $k = 1, \ldots, k_j$, all between nodes j and $\hat{N}$, representing shunt admittances y^m_{jk} on these lines. The set $\hat{E}$ is the disjoint union of these three types of links:

$$\hat{E} \;=\; \hat{E}_1 \cup \hat{E}_2 \cup \hat{E}_3$$

Their sizes are $|\hat{E}_1| = |\overline{N}|$, $|\hat{E}_2| = 2\,|E|$, and $|\hat{E}_3| = |E|$. See the two-bus network in Figure 1.6 and its equivalent Π circuit for an example.

Proof of Theorem 1.2. Let $\hat{C}$ denote the $|\hat{N}| \times |\hat{E}|$ incidence matrix of the equivalent circuit and C denote the $|\overline{N}| \times |\hat{E}|$ reduced incidence matrix obtained from $\hat{C}$ by removing the last row corresponding to the reference node $\hat{N}$. We can always order the links in $\hat{E}$ so that $\hat{C}$ and C can be partitioned according to links in $\hat{E}_1$, $\hat{E}_2$, $\hat{E}_3$:

$$\hat{C} \;=:\; \begin{bmatrix}\hat{C}_1 \; \hat{C}_2 \; \hat{C}_3\end{bmatrix}, \qquad\qquad C \;=:\; \begin{bmatrix}C_1 \; C_2 \; C_3\end{bmatrix}$$

For both submatrices C_1, C_2, every column is the $|\overline{N}|$-dimensional unit vector e_j when it corresponds to a link $l = j \to \hat{N}$ in the equivalent circuit that connects node j to node $\hat{N}$, i.e., it has a single 1 as its jth entry and 0 as all other entries. For C_2, there can be multiple columns that are e_j if j has multiple neighbors. The submatrix C_3 describes the connectivity between buses $j \in \overline{N}$ in the one-line diagram $(G, \mathbb{Y})$ and is the incidence matrix in Theorem 1.2. By construction, we can always order the nodes so that $C_1 = \mathbb{I}_{|\overline{N}|}$, the identity matrix of size $|\overline{N}|$. Moreover

$$\hat{C}_1 \;=\; \begin{bmatrix}\mathbb{I}_{|\overline{N}|} \\ \hline -1^{\mathsf{T}}_{|\overline{N}|}\end{bmatrix}, \qquad\qquad \hat{C}_2 \;=\; \begin{bmatrix}C_2 \\ -1^{\mathsf{T}}_{2|E|}\end{bmatrix}, \qquad\qquad \hat{C}_3 \;=\; \begin{bmatrix}C_3 \\ 0^{\mathsf{T}}_{|E|}\end{bmatrix}$$

where 1_n is the column vector of n 1s and 0_n is the column vector of n 0s.

Similarly partition the branch currents and voltages:

$$J \;=:\; \begin{bmatrix}J_1 & J_2 & J_3\end{bmatrix}, \qquad\qquad U \;=:\; \begin{bmatrix}U_1 & U_2 & U_3\end{bmatrix}$$

By construction $J_1 = -I$, the nodal current injections.

Applying KCL $CJ = 0$ to the equivalent circuit at all non-reference nodes, we have (since $C_1 = \mathbb{I}_{|\overline{N}|}$)

$$I \;=\; C_2 J_2 + C_3 J_3 \tag{1.10a}$$

Applying KVL to the equivalent circuit: there exist nodal voltages $V \in \mathbb{C}^{|\overline{N}|}$ and $V_{\hat{N}} := 0$ such that

$$U \;=\; \hat{C}^{\mathsf{T}} \begin{bmatrix}V \\ V_{\hat{N}}\end{bmatrix} \;=\; C^{\mathsf{T}} V$$

Since $C_1 = \mathbb{I}_{|\overline{N}|}$ we have

$$U_1 \;=\; V, \qquad\qquad U_2 \;=\; C_2^{\mathsf{T}} V, \qquad\qquad U_3 \;=\; C_3^{\mathsf{T}} V \tag{1.10b}$$

To apply Ohm's law to the shunt and series admittances, let $\tilde{Y}^m := \text{diag}\left(y^m_{jk}, y^m_{kj}\, l = (j,k) \in E\right)$ and $Y^s := \text{diag}\left(y^s_{jk},\, l = (j,k) \in E\right)$. Then

$$J_2 \;=\; \tilde{Y}^m U_2, \qquad\qquad J_3 \;=\; Y^s U_3 \tag{1.10c}$$

The current balance equation (1.8) can be derived by eliminating the branch variables (U, J) from (1.10). Specifically

$$I \;=\; \underbrace{\left(C_2 \tilde{Y}^m C_2^{\mathsf{T}} + C_3 Y^s C_3^{\mathsf{T}}\right)}_{Y} V$$

Recall that every column of C_2 is the $|\overline{N}|$-dimensional unit vector e_j when it corresponds to a link $l = j \to \hat{N}$ in the equivalent circuit. Suppose there are $k_j \geq 1$ links $l_1, \ldots, l_{k_j}$ that connect nodes j and $\hat{N}$. Then the jth row $\rho_j^\top$ of C_2 is a $2|E|$ dimensional $0/1$ row vector with k_j 1s, each corresponding to a link $l = j \to \hat{N}$ and 0 otherwise. Hence jth row of $C_2 \tilde{Y}^m C_2^\top$ is

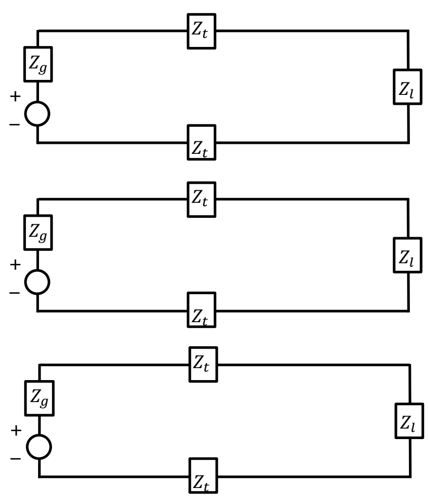
$$\rho_j^\top \tilde{Y}^m C_2^\top \;=\; y_{jj}^m e_j^\top$$

where $y_{jj}^m := \sum_{k:(j,k)\in E} y_{jk}^m$. This means that $C_2 \tilde{Y}^m C_2^\top = \text{diag}\left(y_{jj}^m,\, j \in \overline{N}\right)$. This completes the proof of Theorem 1.2. $\square$

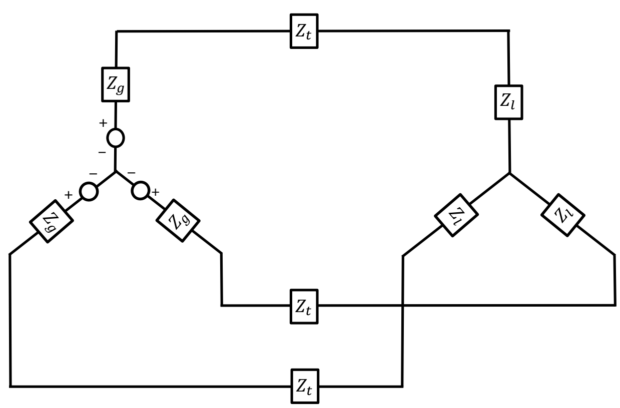
Since we can identify a one-line diagram $(G, \mathbb{Y})$ with its equivalent circuit, we often refer to the one-line diagram itself as a circuit model.

1.2 Three-phase systems

To motivate three-phase systems, consider the single-phase system in Figure 1.7(a) composed of three identical circuits each consisting of a generator modeled as a voltage source in series with an impedance Z_g, a forward conductor and a return conductor each modeled as an impedance Z_t, and a load modeled as an impedance Z_l. The same loads can also be supplied by a three-phase system shown in Figure 1.7(b). As we will illustrate in Chapter 1.3.3, such a three-phase system needs half as much the conductor and



(a) Single-phase system



(b) Balanced three-phase system

Figure 1.7: A single-phase system and a balanced three-phase system that transfer power from generators through transmission lines to loads.

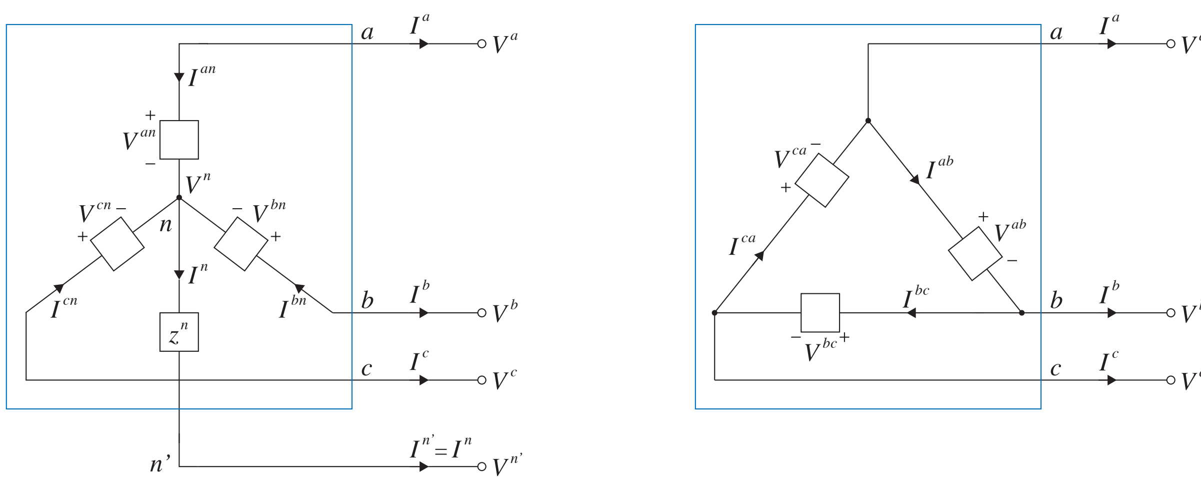
incurs half as much the thermal loss as the single-phase system. In this section we explain the operation of three-phase systems.

Three-phase sources and loads can be arranged in Y (Wye) or Δ (Delta) configurations. This is explained in Chapter 1.2.1. A three-phase system is balanced if all the sources are balanced, loads are identical, and transmission lines are identical and have symmetric geometry. A balanced three-phase system has several simplifying properties. In Chapter 1.2.2 we prove a theorem that summarizes the mathematical structure of balanced three-phase systems that underlies these properties. We apply this theorem to balanced system in Y configuration (Chapter 1.2.3) and Δ configuration (Chapter 1.2.4). This leads to per-phase analysis of a balanced system described in Chapter 1.2.5. Finally we present in Chapter 1.2.6 example configurations common in a power distribution system.

Even though power systems are generally multiphased, single-phase models are widely used as per-phase models of balanced three-phase systems, especially for transmission system applications. Unbalanced three-phase systems are studied in Part II of this book.

1.2.1 Y and Δ configurations

Three single-phase devices can be arranged in either an Y or a Δ configuration as shown in Figure 1.8. They



(a) Y configuration

(b) Δ configuration

Figure 1.8: Three-phase systems, not necessarily balanced, in Y and Δ configurations.

can be three voltage sources, three current sources, or three impedances and they may not be identical, e.g., the three impedances may have different values.

Y configuration. For the Y configuration, the internal voltage (vector) is $V^Y := (V^{an}, V^{bn}, V^{cn})$. These voltages are called *phase-to-neutral* or *phase* voltages. The internal current (vector) $I^Y := (I^{an}, I^{an}, I^{an})$ is defined to flow from each terminal to the neutral as shown in Figure 1.8(a). The external behavior of a three-phase device is described by what is measurable on the terminal of the device. The *terminal* (or *nodal* or *bus*) voltage $V := (V^a, V^b, V^c)$ are voltages with respect to an arbitrary but common point, and the *terminal* (or *line*) current $I := \left(I^a, I^b, I^c\right)$ is defined to be the current coming out of the device as shown in the figure. If the common reference point is taken to be the neutral of this device then $V = V^Y$, i.e., the terminal voltage is the same as the phase voltage for Y configuration. Otherwise $V = V^Y - V^n\mathbf{1}$ where $\mathbf{1}$ is three-dimensional vector of all 1s. As we will see in Chapters 1.2.3 and 1.2.4, for a balanced systems, the neutrals of all Y-configured devices are at the same voltage and therefore can serve as the common reference point. This is not necessarily the case for an unbalanced system, which we will study in Part II of this book.

Hence, for Y configuration, the terminal voltage and current (V,I) are determined by the internal voltage and current (V^Y, I^Y) according to (when the common reference point for V is the neutral so that $V^n := 0$):

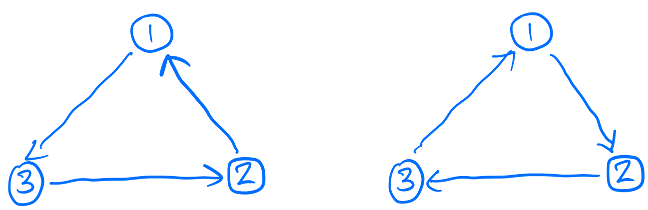
$$V \;=\; V^Y, \qquad I \;=\; -I^Y \tag{1.11}$$

When the common reference is not the neutral of this device, we have $V = (V^Y - V^n\mathbf{1})$.

Instead of the terminal voltage V it is also common to describe the behavior of the three-phase device in terms of its *line-to-line* or *line* voltage $V^{\text{line}} := (V^{ab}, V^{bc}, V^{ca})$. To relate V^{line} to V or to V^Y, define the matrices Γ and its transpose Γ^{T}:

$$\Gamma \;:=\; \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad \Gamma^{\mathsf{T}} \;:=\; \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{1.12}$$

We call Γ and Γ^{T} *conversion matrices*. They can be interpreted as the bus-by-line incidence matrices of the directed graphs shown in Figure 1.9 (properties of general incidence matrices are summarized in



(a) Γ (b) Γ^{T}

Figure 1.9: Directed graphs of which Γ and Γ^{T} are incidence matrices.

Appendix 25.2). Then

$$\begin{bmatrix} V^{ab} \\ V^{bc} \\ V^{ca} \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V^{a} \\ V^{b} \\ V^{c} \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V^{an} \\ V^{bn} \\ V^{cn} \end{bmatrix}$$

or in vector form:

$$V^{\text{line}} \;=\; \Gamma V \;=\; \Gamma V^Y \tag{1.13}$$

This holds for both Y and Δ configurations and whether or not the common reference point for V is the neutral of a Y configured device (since $\Gamma\mathbf{1} = 0$).

Δ configuration. For the Δ configuration in Figure 1.8(b), the internal voltage (vector) is the line-to-line voltage $V^\Delta := (V^{ab}, V^{bc}, V^{ca}) = V^{\text{line}}$, and the internal current $I^\Delta := (I^{ab}, I^{ac}, I^{aa})$ is the line-to-line current. As for the Y configuration, the terminal voltage $V := (V^a, V^b, V^c)$ are voltages with respect to an arbitrary but common reference point. The terminal current is $I := \left(I^a, I^b, I^c\right)$ as shown in Figure 1.8(b). The terminal voltage and current (V, I) is determined by the internal voltage and current $\left(V^\Delta, I^\Delta\right)$ according to

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V^a \\ V^b \\ V^c \end{bmatrix} = \begin{bmatrix} V^{ab} \\ V^{bc} \\ V^{ca} \end{bmatrix}, \qquad \begin{bmatrix} I^a \\ I^b \\ I^c \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\Gamma^\mathsf{T}} \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix}$$

or in vector form (for arbitrary common reference point for V):

$$\Gamma V = V^\Delta, \qquad I = -\Gamma^\mathsf{T} I^\Delta \tag{1.14}$$

Equivalent Y configuration. For any Δ configuration with given internal voltage $V^\Delta := (V^{ab}, V^{bc}, V^{ca})$ and current $I^\Delta := (I^{ab}, I^{ac}, I^{aa})$, an equivalent Y configuration is one that has the same external behavior. This means that, if $V^Y := (V^{an}, V^{bn}, V^{cn})$ and $I^Y := (I^{an}, I^{an}, I^{an})$ are the internal voltage and current of the Y-equivalent then they are related to $\left(V^\Delta, I^\Delta\right)$ according to (from (1.13) (1.14)):

$$\Gamma V^Y = V^\Delta, \qquad I^Y = \Gamma^\mathsf{T} I^\Delta \tag{1.15}$$

Summary. The external behavior (1.11) and (1.14) for Y and Δ configurations respectively as well as their equivalence (1.15) hold for any three-phase system whether or not it is balanced. The relationship (1.13) between line-to-line voltage V^{line} and terminal voltage V holds for Y and Δ configurations whether or not the system is balanced.

The behavior of a three-phase system is determined by the mathematical properties of the conversion matrices Γ and Γ^T. When a system is balanced the conversion becomes particularly simple because the transformation of balanced vectors under Γ and Γ^T preserves their balanced nature (Corollary 1.4). We now explain these mathematical properties and then apply them to analyze balanced systems in Chapters 1.2.3 and 1.2.4.

1.2.2 Balanced vectors and conversion matrices $\Gamma, \Gamma^\mathsf{T}$

Definition 1.1 (Balanced vector)**.** A vector $x := (x_1, x_2, x_3)$ with $x_j = |x_j| e^{\mathbf{i}\theta_j} \in \mathbb{C}$, $j = 1, 2, 3$, is called *balanced* if x_j have the same magnitude and they are separated by $120°$, i.e.,

$$|x_1| = |x_2| = |x_3|$$

and either

$$\theta_2 - \theta_1 = -\frac{2\pi}{3} \quad \text{and} \quad \theta_3 - \theta_1 = \frac{2\pi}{3} \qquad \text{(positive sequence)} \tag{1.16a}$$

or

$$\theta_2 - \theta_1 = \frac{2\pi}{3} \quad \text{and} \quad \theta_3 - \theta_1 = -\frac{2\pi}{3} \qquad \text{(negative sequence)} \tag{1.16b}$$

In this chapter we focus on single-phase equivalent circuits of balanced systems. In Part II of this book we study unbalanced systems and generalize the definition of balance to allow a nonzero bias (see Definition 7.1), i.e., we will call $\hat{x}$ a (generalized) balanced vector if it is of the form $\hat{x} = x + \gamma \mathbf{1}$ and x is balanced according to Definition 1.1, for some possibly nonzero $\gamma \in \mathbb{C}$. The bias γ may models a common reference voltage or the internal loop flow in a Δ configuration. We assume $\gamma = 0$ in this chapter which amounts to the assumption that loop flows are zero and that all neutrals are grounded directly and voltages are defined with respect to the ground.

A balanced vector x is said to be in a *positive sequence* if x satisfies (1.16a) and in a *negative sequence* set if x satisfies (1.16b). Let

$$\alpha \;:=\; e^{-\mathbf{i}2\pi/3}$$

Clearly $\alpha^2 = e^{\mathbf{i}2\pi/3}$, $\alpha^3 = 1$; see Figure 1.10. (Also see Exercise 1.4 for more properties of α.) Define

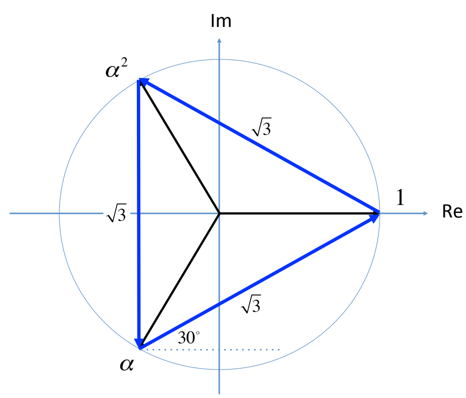


Figure 1.10: Phase shift $\alpha := e^{-\mathbf{i}2\pi/3}$ in Theorem 1.3.

the vectors

$$\alpha_+ \;:=\; \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}, \qquad \alpha_- \;:=\; \begin{bmatrix} 1 \\ \alpha^2 \\ \alpha \end{bmatrix} \tag{1.17a}$$

Then α_+ is a balanced vector in a positive sequence and α_- is a balanced vector in a negative sequence. Moreover the set of all balanced positive-sequence vectors is $\text{span}(\alpha_+)$ and the set of all balanced negative-sequence vectors is $\text{span}(\alpha_-)$, i.e., x is a balanced vector in a positive sequence and y a balanced vector in a negative sequence if and only if

$$x \;=\; x_1\alpha_+, \qquad y \;=\; y_1\alpha_-, \qquad x_1, y_1 \;\in\; C \tag{1.17b}$$

Note that $\overline{\alpha}_+ = \alpha_-$ where for any vector x, $\bar{x}$ is its complex conjugate componentwise. All main properties of balanced three-phase systems originate from the mathematical properties of the vectors α_+, α_- and their transformation under the matrices $\Gamma, \Gamma^{\mathsf{T}}$ defined in (1.12). These properties include: balanced voltages and currents sum to zero, phases are decoupled, and if all the voltage sources are in a balanced positive sequence then all phase and line voltages and currents are in balanced positive sequences. We now summarize these mathematical properties.

Define the matrix F whose columns are α_+, α_- as well as $\mathbf{1}$ normalized:

$$F \;:=\; \frac{1}{\sqrt{3}} \begin{bmatrix} \mathbf{1} & \alpha_+ & \alpha_- \end{bmatrix} \;=\; \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \tag{1.18}$$

The follow theorem describes the transformation of balanced vectors under $\Gamma, \Gamma^{\mathsf{T}}$ and underlies all the main properties of balanced three-phase systems. Its proof is left as Exercise 1.5. The theorem implies in particular that the transformations Γ and Γ^{T} preserve the balanced nature of a vector and hence ensures that the entire network stays balanced. The key enabling property is that the voltages and currents from balanced sources are in $\text{span}(\alpha_+)$ or $\text{span}(\alpha_-)$ and (α_+, α_-) are eigenvectors of $\Gamma, \Gamma^{\mathsf{T}}$ (according to (1.19a)(1.20a)).

Theorem 1.3 (Transformation of balanced vectors by $\Gamma, \Gamma^{\mathsf{T}}$)**.** Let $\alpha := e^{-\mathbf{i}2\pi/3}$. Recall the balanced vectors (α_+, α_-) defined in (1.17a), the matrices F in (1.18) and $\Gamma, \Gamma^{\mathsf{T}}$ in (1.12).

1. Suppose the entries x_j of $x := (x_1, x_2, x_3) \in \mathbb{C}^3$ have the same magnitude. Then x is balanced if and only if $x_1 + x_2 + x_3 = 0$.

2. The columns of F are orthonormal. Both F and $\overline{F}$ are complex symmetric, i.e., $F^{\mathsf{T}} = F$ and $\overline{F}^{\mathsf{T}} = \overline{F}$, where $\overline{F}$ is the complex conjugate of F componentwise. Hence

$$F^{-1} \;=\; F^{\mathsf{H}} \;=\; \overline{F} \;=\; \frac{1}{\sqrt{3}} \begin{bmatrix} \mathbf{1} & \alpha_- & \alpha_+ \end{bmatrix}$$

3. Γ is a normal matrix, $\Gamma\Gamma^{\mathsf{T}} = \Gamma^{\mathsf{T}}\Gamma$. (Note that $\Gamma\Gamma^{\mathsf{T}} = \Gamma^{\mathsf{T}}\Gamma$ are Laplacian matrices of the graphs in Figure 1.9.)

4. *Spectral decomposition of* Γ:

   (a) The eigenvalues and eigenvectors of Γ are

$$\Gamma\mathbf{1} \;=\; 0, \qquad \Gamma\alpha_+ \;=\; (1-\alpha)\alpha_+, \qquad \Gamma\alpha_- \;=\; (1-\alpha^2)\alpha_- \tag{1.19a}$$

   where $1-\alpha = \sqrt{3}e^{\mathbf{i}\pi/6}$ and $1-\alpha^2 = \sqrt{3}e^{-\mathbf{i}\pi/6}$.

   (b) Therefore the spectral decomposition of Γ is:

$$\Gamma \;=\; F \begin{bmatrix} 0 & & \\ & 1-\alpha & \\ & & 1-\alpha^2 \end{bmatrix} \overline{F} \tag{1.19b}$$

5. *Spectral decomposition of* Γ^{T}:

   (a) The eigenvalues and eigenvectors of Γ^{T} are

   $$\Gamma^{\mathsf{T}}\mathbf{1} \;=\; 0, \qquad \Gamma^{\mathsf{T}}\alpha_- \;=\; (1-\alpha)\alpha_-, \qquad \Gamma^{\mathsf{T}}\alpha_+ \;=\; (1-\alpha^2)\alpha_+ \tag{1.20a}$$

   where $1-\alpha = \sqrt{3}e^{\mathbf{i}\pi/6}$ and $1-\alpha^2 = \sqrt{3}e^{-\mathbf{i}\pi/6}$.

   (b) Therefore the spectral decomposition of Γ^{T} is:

   $$\Gamma^{\mathsf{T}} \;=\; \overline{F}\begin{bmatrix} 0 & & \\ & 1-\alpha & \\ & & 1-\alpha^2 \end{bmatrix} F \tag{1.20b}$$

The following corollary of the theorem is repeatedly used in the analysis of balanced systems. It says that the transformation of a balanced vector x under Γ and Γ^{T} reduces to a scaling by $(1-\alpha)$ and $(1-\alpha^2)$ respectively.

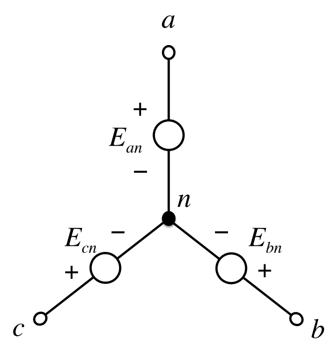
Corollary 1.4. For any balanced positive-sequence vector $x \in \mathbb{C}^3$ and $\gamma \in \mathbb{C}$, we have

1. $\Gamma(x+\gamma\mathbf{1}) \;=\; (1-\alpha)x$.
2. $\Gamma^{\mathsf{T}}(x+\gamma\mathbf{1}) \;=\; (1-\alpha^2)x$.
3. $\Gamma\Gamma^{\mathsf{T}}(x+\gamma\mathbf{1}) \;=\; \Gamma^{\mathsf{T}}\Gamma(x+\gamma\mathbf{1}) \;=\; 3x$.

Informally a three-phase system is called *balanced* if all voltages and currents are balanced vectors in, say, positive-sequence sets. The main consequence of the corollary is the following. A three-phase system consists of voltage sources, current sources, and impedances connected by lines. The voltage and current at any point in the system are induced by the internal voltages of voltage sources and the internal currents of current sources. When these sources are balanced positive-sequence sets, their internal voltages and currents are in $\text{span}(\alpha_+)$ and α_+ is an eigenvector of Γ and Γ^{T}. This means that the transformation of balanced voltages and currents under $\Gamma, \Gamma^{\mathsf{T}}$ reduces to a scaling of these variables by their eigenvalues $1-\alpha$ and $1-\alpha^2$ respectively. Since the voltage and current at every point in the system are linear combinations of transformed source voltages and source currents, transformed by Γ, Γ^T and line admittance matrices, they remain in $\text{span}(\alpha_+)$ when the sources are balanced and the lines are identical and phase-decoupled. This is the key property that enables balanced sources to induce balanced voltages and currents throughout the network, leading to per-phase analysis of three-phase systems. A formal statement and its proof have to wait till Chapter 9 (Theorem 9.7) when we develop a general model of unbalanced three-phase system. In this chapter we will use the corollary to analyze example circuits to build intuition.

1.2.3 Balanced systems in Y configuration

Figure 1.11 shows the Y configuration of voltage sources and impedance loads. The loads are said to be *balanced* if their impedances Z are identical. An ideal three-phase voltage source in Y configuration is



(a) Balanced sources (b) Balanced loads

Figure 1.11: Balanced three-phase (a) voltage source E^Y and (b) impedance Z^Y in Y configuration.

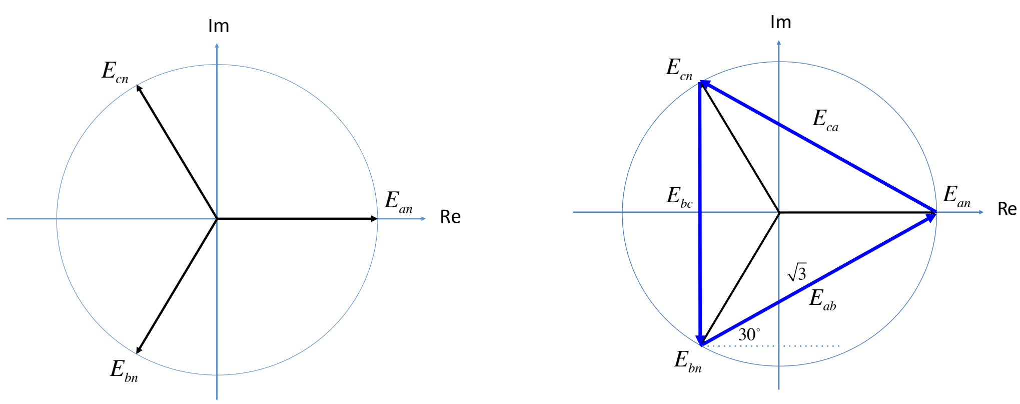
specified by its internal voltage (vector) $E^Y := \left(E^{an}, E^{bn}, E^{cn}\right)$ in the phasor domain between the terminals a, b, c and the neutral n respectively. It is called *balanced* if E^Y is a balanced vector according to Definition 1.1, i.e.,

$$\text{positive sequence:} \qquad E^{an} = 1\angle\theta, \quad E^{bn} = 1\angle\theta - 120^\circ, \quad E^{cn} = 1\angle\theta + 120^\circ$$

or

$$\text{negative sequence:} \qquad E^{an} = 1\angle\theta, \quad E^{bn} = 1\angle\theta + 120^\circ, \quad E^{cn} = 1\angle\theta - 120^\circ$$

where their magnitudes are normalized to 1. See Figure 1.12(a) where $\theta = 0$. For a balanced voltage source



(a) Phase voltages (b) Phase and line voltages

Figure 1.12: A balanced three-phase source in Y configuration. (a) Its phase voltage (vector) $E^Y := \left(E^{an}, E^{bn}, E^{cn}\right)$ is a balanced vector. (b) Its line voltage $E^{\text{line}} = \Gamma E^Y = (1-\alpha)E^Y$.

in a positive sequence, the instantaneous voltages in the time domain reach their maximum values in the order *abc*. We sometimes call *abc* in such an order a *positive sequence* and the voltages $\left\{E^{an}, E^{bn}, E^{cn}\right\}$ a (balanced) positive-sequence set. Whether a voltage source is in a positive or negative sequence depends only on how one labels the wires. Therefore, unless otherwise specified, we will always consider *abc* to be a positive sequence. If there are multiple three-phase sources connected to the same network their phase sequences must be the same.

Theorem 1.3 implies the following properties of a balanced positive-sequence voltage source:

1. Sum to zero: $E^{an} + E^{bn} + E^{cn} = 0$
2. All voltages and currents are in a balanced positive sequence, i.e., all are in $\text{span}(\alpha_+)$.
3. Phases are decoupled.

Sum to zero. The first property follows from Theorem 1.3.1, or more directly, $E^Y = \alpha_+ E^{an}$ and hence $\mathbf{1}^\mathsf{T} E^Y = \left(\mathbf{1}^\mathsf{T} \alpha_+\right) E^{an} = 0$.

Line voltage V^{line} is balanced. The second properties is due to the fact that α_+ is an eigenvector of $\Gamma, \Gamma^\mathsf{T}$. Specifically the line voltage $E^{\text{line}} := \left(E^{ab}, E^{bc}, E^{ca}\right)$ across the terminals is given by $E^{\text{line}} = \Gamma E^Y$ (from (1.13)), i.e.,

$$E^{ab} \;=\; E^{an} - E^{bn}, \quad E^{bc} \;=\; E^{bn} - E^{cn}, \quad E^{ca} \;=\; E^{cn} - E^{an}$$

This implies $\mathbf{1}^\mathsf{T} E^{\text{line}} = E^{ab} + E^{bc} + E^{ca} = 0$. Moreover Corollary 1.4 implies

$$E^{\text{line}} \;=\; \Gamma E^Y \;=\; (1-\alpha) E^Y$$

Hence E^{line} is in a balanced positive sequence if E^Y is:

$$E^{bc} \;=\; e^{-\mathbf{i}2\pi/3} E^{ab}, \quad E^{ca} \;=\; e^{\mathbf{i}2\pi/3} E^{ab}$$

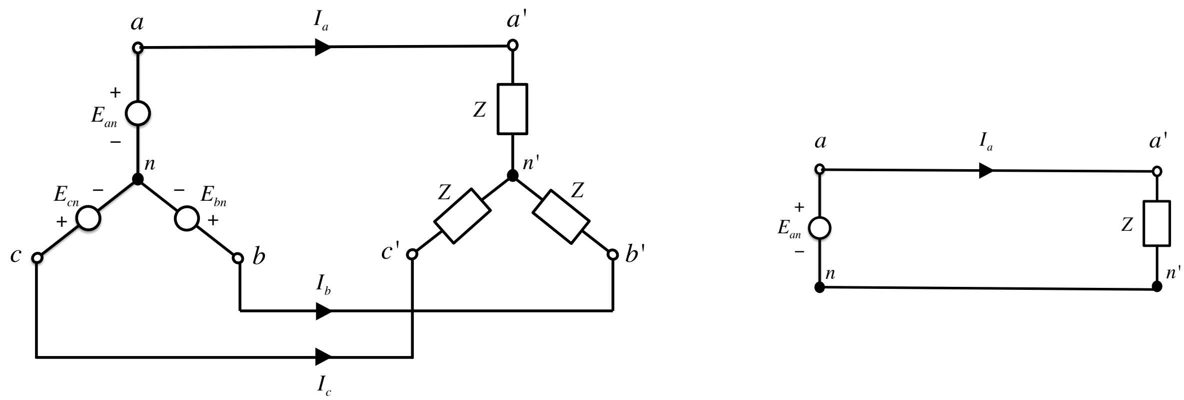
Since $1 - \alpha = \sqrt{3} e^{\mathbf{i}\pi/6}$ we have

$$E^{ab} \;=\; \sqrt{3} e^{\mathbf{i}\pi/6} E^{an}, \qquad E^{bc} \;=\; \sqrt{3} e^{\mathbf{i}\pi/6} E^{bn}, \qquad E^{ca} \;=\; \sqrt{3} e^{\mathbf{i}\pi/6} E^{cn}$$

This is illustrated in Figure 1.12(b).

Balanced systems are phase-decoupled. We start by analyzing the simple circuit in Figure 1.13(a) when a balanced three-phase load is connected to a balanced three-phase positive-sequence voltage source in Y configuration. We will show that

1. The neutral-to-neutral voltage is zero, $V_{nn'} = 0$.
2. The internal voltage and current across the impedances are in a balanced positive sequence.



(a) Balanced three-phase system (b) Equivalent per-phase system

Figure 1.13: Balanced three-phase source and load in Y configuration and its per-phase model.

The most important implication is that the phases are decoupled, i.e., the variables in each phase depend on quantities only in that phase, and can be analyzed separately. We will illustrate through examples that these conclusions hold in more general balanced systems than the simple circuit in Figure 1.13(a). A full understanding of phase decoupling and per-phase analysis is postponed till Part II of this book where a balanced system is studied in the context of general unbalanced systems.

Referring to Figure 1.13(a) let

- $E^Y := \left(E^{an}, E^{bn}, E^{cn}\right)$ and $V'^Y := \left(V^{a'n'}, V^{b'n'}, V^{c'n'}\right)$ denote the internal voltages from terminals to neutrals, and $I'^Y := \left(I^{a'n'}, I^{b'n'}, I^{c'n'}\right)$ denote the internal current between the terminals a', b', c' and the neutral n' across the identical impedances $Z^Y := z\mathbf{1}$;
- $V := \left(V^a, V^b, V^c\right)$ denote the terminal voltage (vector), with respect to an arbitrary and common reference point, not necessarily the neutral n or n';
- V^n and $V^{n'}$ denote the neutral voltages with respect to the common reference point.

Given the balanced positive-sequence voltage E^Y and balanced impedance Z^Y, we wish to show that $V^n = V^{n'}$, that V'^Y, I'^Y are in a balanced positive sequence, and that phases are decoupled.

Solution. KVL, KCL, and Ohm's law imply

$$E^Y \;=\; V - V^n\mathbf{1}, \qquad V'^Y \;=\; V - V^{n'}\mathbf{1}, \qquad V'^Y \;=\; zI'^Y, \qquad \mathbf{1}^\mathsf{T} I'^Y \;=\; 0 \tag{1.22}$$

Therefore $E^Y - V'^Y = \left(V^{n'} - V^n\right)\mathbf{1}$ and hence (since $\mathbf{1}^\mathsf{T} E^Y = 0$)

$$\mathbf{1}^\mathsf{T}\left(E^Y - V'^Y\right) \;=\; \left(V^{n'} - V^n\right)\mathbf{1}^\mathsf{T}\mathbf{1} \quad\Longrightarrow\quad 3\left(V^{n'} - V^n\right) \;=\; -\mathbf{1}^\mathsf{T} V'^Y \;=\; -z\left(\mathbf{1}^\mathsf{T} I'^Y\right) \;=\; 0$$

showing that the voltage across the neutrals $V_{nn'} = 0$. Substituting it into (1.22) yields (denoting $y := z^{-1}$)

$$V'^{Y} \;=\; E^{Y} + \left(V^{n} - V^{n'}\right)\mathbf{1} \;=\; E^{Y}, \qquad\qquad I'^{Y} \;=\; yV'^{Y} \;=\; yE^{Y}$$

Hence both V'^{Y} and I'^{Y} are in a balanced positive sequence. Moreover the phases are decoupled in that $V_{\phi n'}$ and $I_{\phi n'}$, $\phi = a', b', c'$, depend only on $E_{\phi n}$ but not on voltages on other phases.

In view of Theorem 1.3.1, the terminal voltage V is not balanced unless $V^{n} = V^{n'} = 0$, i.e., the neutral is taken as the common reference point for voltages, because

$$\mathbf{1}^{\mathsf{T}} V \;=\; \mathbf{1}^{\mathsf{T}}\left(E^{Y} + V^{n}\mathbf{1}\right) \;=\; 3V^{n}$$

□

Remark 1.2.

1. Since $V_{nn'} = 0$, even if n and n' are connected, the current on that wire will be zero. We can therefore either assume n and n' are connected or disconnected in our analysis, whichever is more convenient.
2. Since the currents are balanced, $I^{a} + I^{b} + I^{c} = 0$ or $i^{a}(t) + i^{b}(t) + i^{c}(t) = 0$ at all times t, the currents flow from and return to the sources only via the wires connecting the sources to the loads, and no additional physical wires are necessary for return currents. This halves the amount of required wire compared with three separate single-phase circuits; see Chapter 1.3.3.

As a consequence, each phase of the balanced system is decoupled and equivalent to the circuit in Figure 1.13(b). We can therefore analyze the phase a equivalent circuit; see Chapter 1.2.5. The voltages and currents in phase b and phase c circuits will be the corresponding phase a quantities shifted by -120° and 120° respectively, assuming the three-phase source is of positive sequence.

These conclusions hold for more general circuits than that in Figure 1.13(a), as Example 1.3 shows.

Example 1.3 (Balanced three-phase system in Y configuration)**.** Figure 1.14 shows a balanced three-phase source of positive sequence supplies two sets of balanced three-phase loads in parallel through balanced transmission lines. The transmission lines have a common admittance t and all loads have a constant admittance l, as shown in the figure . Suppose the neutrals are connected by lines with a common admittance y. Denote the internal voltages and currents in stage $k = 1, 2$, by $V_k^Y := (V^{a_k n_k}, V^{b_k n_k}, V^{c_k n_k})$ and $I_k^Y := (I^{a_k n_k}, I^{a_k n_k}, I^{a_k n_k})$ respectively. Denote the terminal voltages and currents from stage $k-1$ to stage k, $k = 1, 2$, by $V_k := (V^{a_{k-1} a_k}, V^{b_{k-1} b_k}, V^{c_{k-1} c_k})$ and $I_k := (I^{a_{k-1} a_k}, I^{a_{k-1} b_k}, I^{a_{k-1} c_k})$ respectively.

Suppose $y \neq 0$, $t = y/\mu$, and $l = y/\mu^2$ for some real number $\mu \neq 0$. Prove that

1. $V_{n_0 n_1} = V_{n_1 n_2} = 0$.
2. For $k = 1, 2$, all voltages and currents V_k^Y, V_k, I_k^Y, I_k are balanced positive-sequence sets.
3. The phases are decoupled, i.e.,

$$\begin{aligned} E_0^Y &= V_1 + V_1^Y \\ V_1^Y &= V_2 + V_2^Y \end{aligned}$$

where $E_0^Y := (E^{a_0 n_0}, E^{b_0 n_0}, E^{c_0 n_0})$.

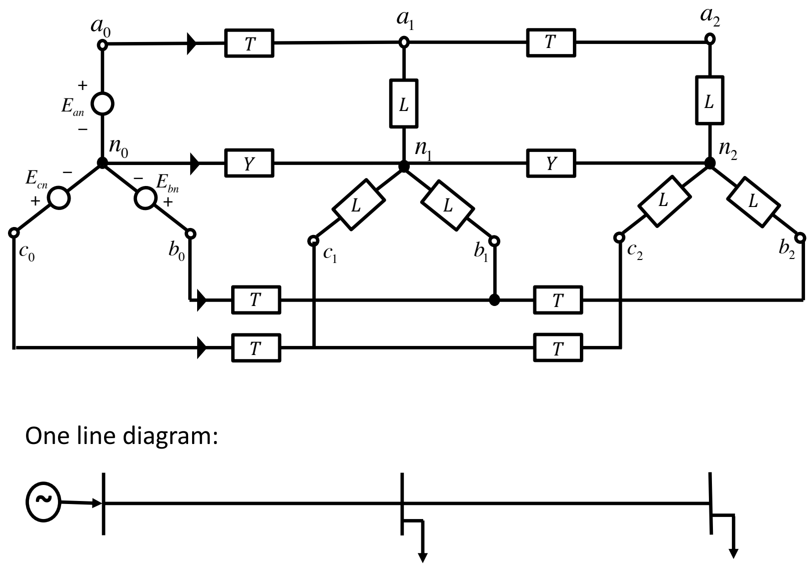


Figure 1.14: Balanced three-phase system in Y configuration (Example 1.3).

This implies that the three phases of the balanced system in Figure 1.14 are decoupled and can be studied by analyzing the per-phase circuit shown in Figure 1.15 where the line admittances connecting the neutrals are set to zero.

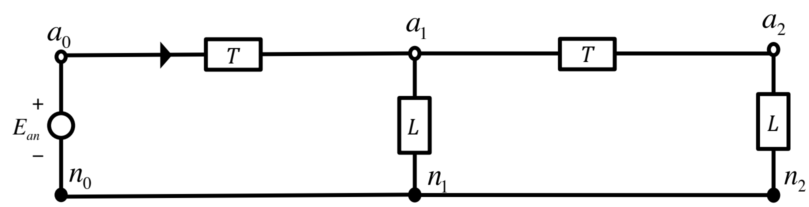


Figure 1.15: The per-phase equivalent circuit of the balanced system in Figure 1.14 in Y configuration.

Solution:

1. We will apply Ohm's law and Kirchhoff's current and voltage laws (KCL and KVL) to derive two linear equations in $(V_{n_0n_1}, V_{n_1n_2})$ and show that $V_{n_0n_1} = V_{n_1n_2} = 0$ is the only solution to these equations. By Ohm's law across each admittance, the currents are in terms of voltages:

$$I_k^Y \;=\; lV_k^Y, \qquad I_k \;=\; tV_k, \qquad k = 1,2 \tag{1.23}$$

   This allows us to eliminate currents I_k^Y, I_k and express KCL and KVL in the following in terms only of voltages V_k^Y, V_k.

   Making use of (1.23), apply KCL at node (a_1, b_1, c_1) to obtain

$$tV^{a_0a_1} \;=\; lV^{a_1n_1} + tV^{a_1a_2}, \;\; tV^{b_0b_1} \;=\; lV^{b_1n_1} + tV^{b_1b_2}, \;\; tV^{c_0c_1} \;=\; lV^{c_1n_1} + tV^{c_1c_2}$$

and similarly for KCL at nodes (a_2, b_2, c_2). This in vector form is

$$tV_1 \;=\; l\,V_1^Y \,+\, tV_2 \tag{1.24a}$$
$$tV_2 \;=\; l\,V_2^Y \tag{1.24b}$$

Apply KCL at nodes (n_0, n_1, n_2) to obtain

$$\begin{aligned} t\left(\mathbf{1}^\mathsf{T} V_1\right) \,+\, yV^{n_0n_1} &= 0 \\ l\left(\mathbf{1}^\mathsf{T} V_1^Y\right) \,+\, yV^{n_0n_1} &= yV^{n_1n_2} \\ l\left(\mathbf{1}^\mathsf{T} V_2^Y\right) \,+\, yV^{n_1n_2} &= 0 \end{aligned}$$

where $\mathbf{1} := (1,1,1)$ is the column vector of all 1's. Hence, since $y/t = \mu$ and $y/l = \mu^2$, we have

$$\mathbf{1}^\mathsf{T} V_1 \;=\; -\mu V^{n_0n_1}, \quad \mathbf{1}^\mathsf{T} V_1^Y \;=\; -\mu^2 V^{n_0n_1} + \mu^2 V^{n_1n_2}, \quad \mathbf{1}^\mathsf{T} V_2^Y \;=\; -\mu^2 V^{n_1n_2} \tag{1.25}$$

Finally, apply KVL around the loops from stage 0 to stage 1 to obtain

$$E^{a_0n_0} \;=\; V^{a_0a_1} + V^{a_1n_1} - V^{n_0n_1}, \;\; E^{b_0n_0} \;=\; V^{b_0b_1} + V^{b_1n_1} - V^{n_0n_1}, \;\; E^{c_0n_0} \;=\; V^{c_0c_1} + V^{c_1n_1} - V^{n_0n_1}$$

and similarly for loops from stage 1 to stage 2. This in vector form is

$$E_0^Y \;=\; V_1 \,+\, V_1^Y \,-\, V^{n_0n_1}\mathbf{1} \tag{1.26a}$$
$$V_1^Y \;=\; V_2 \,+\, V_2^Y \,-\, V^{n_1n_2}\mathbf{1} \tag{1.26b}$$

where $E_0^Y := (E^{a_0n_0}, E^{b_0n_0}, E^{c_0n_0})$. Substitute (1.24b) into the last equation to eliminate V_2:

$$V_1^Y \;=\; \left(\frac{1}{\mu} + 1\right) V_2^Y \,-\, V^{n_1n_2}\mathbf{1} \tag{1.26c}$$

To obtain a system of equations that involves only $(V^{n_0n_1}, V^{n_1n_2})$, multiply (1.26) by $\mathbf{1}^\mathsf{T}$ and apply (1.25) to obtain (using $\mathbf{1}^\mathsf{T} E_0 = 0$ since the sources are balanced):

$$\begin{bmatrix} \mu^2 + \mu + 3 & -\mu^2 \\ -\mu^2 & 2\mu^2 + \mu + 3 \end{bmatrix} \begin{bmatrix} V^{n_0n_1} \\ V^{n_1n_2} \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.27}$$

We now argue that the determinant of the matrix in (1.27) is nonzero, and hence $V^{n_0n_1} = V^{n_1n_2} = 0$. Let $B := \mu^2 + \mu + 3$. Then

$$\text{determinant} \;=\; B(B + \mu^2) - \mu^4$$

If determinant is zero then

$$B \;=\; -\frac{\mu^2}{2}\left(1 \pm \sqrt{5}\right)$$

By the definition of $B := \mu^2 + \mu + 3$ we therefore have

$$(3 \pm \sqrt{5})\mu^2 \,+\, 2\mu \,+\, 6 \;=\; 0$$

It is easy to check that no real number μ satisfies this equation, and hence $V^{n_0n_1} = V^{n_1n_2} = 0$.

2. We now prove that (V_k^Y, V_k) are balanced positive-sequence sets. Since $V^{n_1n_2} = 0$, (1.26c) implies

$$V_2^Y \;=\; \frac{\mu}{\mu+1} V_1^Y \tag{1.28}$$

Substitute this and (1.24b) into (1.24a) to obtain

$$V_1 \;=\; \frac{1}{\mu} V_1^Y + \frac{1}{\mu} V_2^Y \;=\; \frac{2\mu+1}{\mu(\mu+1)} V_1^Y$$

Substitute into (1.26a) to get

$$E_0^Y \;=\; \frac{2\mu+1}{\mu(\mu+1)} V_1^Y + V_1^Y$$

Hence

$$V_1^Y \;=\; \frac{\mu(\mu+1)}{\mu^2+3\mu+1} E_0 \quad \text{and} \quad V_1 \;=\; \frac{\mu(2\mu+1)}{\mu^2+3\mu+1} E_0$$

Hence V_1, V_1^Y are balanced positive-sequence sets since E_0 is. Furthermore V_2, V_2^Y are balanced positive-sequence sets from (1.28) and (1.24b). Then (1.23) implies that all currents (I_k^Y, I_k) are balanced positive-sequence sets.

3. To show that the phases are decoupled, substitute $V^{n_0n_1} = V^{n_1n_2} = 0$ in (1.26a)(1.26b).

This completes the proof. □

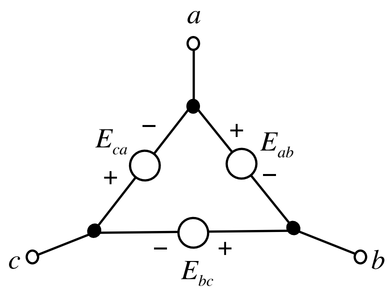
Remark 1.3 (Phase-decoupling of lines)**.** 1. A key enabling property that allows the balanced nature of voltages and currents to propagate from one node to the next is the assumption that three-phase lines are phase-decoupled (see Example 1.3 and Exercise 1.8). This assumption is valid only if the lines are symmetric and the sources and loads are balanced such that currents and charges both sum to zero in these lines across phases; see Chapter 2.1.4. Otherwise an unbalanced three-phase model of transmission lines should be used; see Part II of this book.

2. If the lines are symmetric but the sources or loads are unbalanced then variables of different phases are coupled. A similarity transformation can be used to transform the system to a so called sequence coordinate in which the lines become decoupled and single-phase analysis can then be applied in the sequence coordinate; see Chapter 9 in Part II of this book.

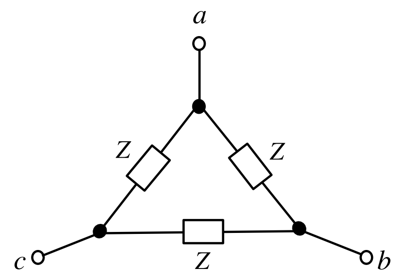
1.2.4 Balanced systems in Δ configuration

Figure 1.16 shows the Δ configuration of a balanced voltage source and a balanced impedance. An ideal voltage source in Δ configuration is specified by its line voltage $E^\Delta := \left(E^{ab}, E^{bc}, E^{ca}\right)$. It is *balanced* if E^Δ is a balanced vector according to Definition 1.1, i.e., assuming positive sequence:

$$E^{bc} \;=\; e^{-\mathbf{i}2\pi/3}\, E^{ab}, \quad E^{ca} \;=\; e^{\mathbf{i}2\pi/3}\, E^{ab}$$



(a) Balanced source



(b) Balanced load

Figure 1.16: Balanced three-phase (a) voltage source E^{Δ} and (b) impedance Z^{Δ} in Δ configuration.

A balanced three-phase system in Δ configuration enjoys the same properties as such a system in Y configuration in Chapter 1.2.3 does. In particular the line voltages sum to zero (see Figure 1.12(b)):

$$E^{ab} + E^{bc} + E^{ca} \quad = \quad 0$$

The three-phase voltages and currents in a balanced system in Δ configuration driven by balanced three-phase positive-sequence sources are balanced positive sequences. Moreover the phases are decoupled. We illustrate this in the next example.

Example 1.4 (Balanced three-phase system in Δ configuration)**.** Figure 1.17 shows a balanced three-phase source connected to a balanced three-phase load through balanced transmission lines in Δ configuration. The transmission lines have identical admittance $t \neq 0$ and the loads are of constant admittance $l \neq 0$. Sup-

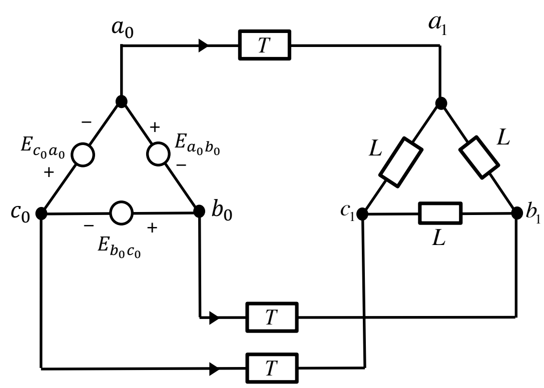


Figure 1.17: Example 1.4

pose the internal voltage $E^{\Delta} := (E^{a_0b_0}, E^{b_0c_0}, E^{c_0a_0})$ is in a positive sequence. Denote the terminal current by $I := (I^{a_0a_1}, I^{a_0b_1}, I^{a_0c_1})$, the terminal voltage by $V := (V^{a_0a_1}, V^{b_0b_1}, V^{c_0c_1})$, and the line-to-line voltage by $U := (V^{a_1b_1}, V^{b_1c_1}, V^{c_1a_1})$. We will show that I, V, U are in balanced positive sequences, provided the ratio

$$\mu \quad := \quad \frac{t}{l} \quad \neq \quad -3$$

Solution. Apply KCL at nodes a_1, b_1, c_1 to get

$$tV^{a_0a_1} \;=\; I^{a_0a_1} \;=\; lV^{a_1b_1} - lV^{c_1a_1}, \quad tV^{b_0b_1} \;=\; I^{a_0b_1} \;=\; lV^{b_1c_1} - lV^{a_1b_1}, \quad tV^{c_0c_1} \;=\; I^{a_0c_1} \;=\; lV^{c_1a_1} - lV^{b_1c_1}$$

or in vector form (cf. (1.14)):

$$I \;=\; l\Gamma^{\mathsf{T}}U \;=\; tV \tag{1.29}$$

where Γ^{T} is defined in (1.12). Apply KVL to get

$$E^{a_0b_0} \;=\; V^{a_0a_1} + V^{a_1b_1} - V^{b_0b_1}, \quad E^{b_0c_0} \;=\; V^{b_0b_1} + V^{b_1c_1} - V^{c_0c_1}, \quad E^{c_0a_0} \;=\; V^{c_0c_1} + V^{c_1a_1} - V^{a_0a_1}$$

This is in vector form:

$$E^{\Delta} \;=\; U \,+\, \Gamma V \tag{1.30}$$

where Γ is defined in (1.12). Eliminate V from (1.29) and (1.30) to get

$$E^{\Delta} \;=\; \frac{1}{\mu}\left(\mu\mathbb{I} \,+\, \Gamma\Gamma^{\mathsf{T}}\right)U \;=\; \frac{1}{\mu}\begin{bmatrix} \mu+2 & -1 & -1 \\ -1 & \mu+2 & -1 \\ -1 & -1 & \mu+2 \end{bmatrix} U \tag{1.31}$$

where $\mu := t/l$ and $\mathbb{I}$ is the identity matrix of an appropriate size. The matrix $\mu\mathbb{I} + \Gamma\Gamma^{\mathsf{T}}$ has a determinant of $\mu(\mu+3)^2$ and hence is nonsingular provided $\mu \neq 0, -3$. Since E^{Δ} is a balanced positive-sequence matrix we have

$$\left(\mu\mathbb{I} \,+\, \Gamma\Gamma^{\mathsf{T}}\right)U \;=\; \mu\, E^{ab}\, \alpha_+$$

It therefore suffices to show that α_+ is an eigenvector of $\mu\mathbb{I} + \Gamma\Gamma^{\mathsf{T}}$ with an associated eigenvalue λ, for then

$$U \;=\; \mu\, E^{ab}\left(\mu\mathbb{I} \,+\, \Gamma\Gamma^{\mathsf{T}}\right)^{-1}\alpha_+ \;=\; \frac{\mu\, E^{ab}}{\lambda}\,\alpha_+$$

showing that U is also a balanced positive-sequence voltage (note that if $Ax = \lambda x$ for a nonsingular matrix A then $A^{-1}x = \frac{1}{\lambda}x$). To show that α_+ is an eigenvector of $\mu\mathbb{I} + \Gamma\Gamma^{\mathsf{T}}$, we apply Theorem 1.3 to get

$$\left(\mu\mathbb{I} \,+\, \Gamma\Gamma^{\mathsf{T}}\right)\alpha_+ \;=\; \mu\,\alpha_+ \,+\, \Gamma(1-\alpha^2)\,\alpha_+ \;=\; \left(\mu \,+\, (1-\alpha)\left(1-\alpha^2\right)\right)\alpha_+ \;=\; \underbrace{(\mu \,+\, 3)}_{\lambda}\alpha_+$$

as desired. This shows that U is indeed a balanced positive-sequence voltage. Indeed

$$U \;=\; \frac{\mu}{\mu+3}\,E^{\Delta}$$

To show that phase voltages V are also a balanced positive sequence and decoupled, use (1.29) and Corollary 1.4 to get

$$V \;=\; \frac{1}{\mu}\Gamma^{\mathsf{T}}U \;=\; \frac{1}{\mu}\left(1-\alpha^2\right)U \;=\; \frac{1-\alpha^2}{\mu+3}\,E^{\Delta}$$

Hence V is in a balanced positive sequence. The expression $I = tV$ from (1.29) then implies that the phase current I is also in a balanced positive sequence and that the phases are decoupled. □

Δ and Y transformation. A balanced Δ-configured system also has a per-phase equivalent circuit. A theory for general systems is discussed in Chapter 9.3.4. For our purposes here we will first transform a balanced system in Δ configuration into an equivalent system in Y configuration, analyze the per-phase circuit of the Y-equivalent, and then translate the result back to the Δ configuration. We now explain how to transform between Δ and Y configuration. We will use this transformation to perform per-phase analysis in Chapter 1.2.5.

As explained in Chapter 1.2.1, given any balanced internal voltage $V^\Delta := (V^{ab}, V^{bc}, V^{ca})$ and current $I^\Delta := (I^{ab}, I^{ac}, I^{aa})$ in Δ configuration, an equivalent Y configuration is one that has the same external behavior, i.e., the internal voltage $V^Y := (V^{an}, V^{bn}, V^{cn})$ and current $I^Y := (I^{an}, I^{an}, I^{an})$ of the Y-equivalent satisfy (1.15) reproduced here

$$\Gamma V^Y \;=\; V^\Delta, \qquad\qquad I^Y \;=\; \Gamma^{\mathsf{T}} I^\Delta$$

Assume the neutral of the Y equivalent voltage source is the reference for all voltages and $V^n = 0$. Since V^Y and I^Δ are balanced vectors, Corollary 1.4 implies

$$(1-\alpha)V^Y \;=\; V^\Delta, \qquad\qquad I^Y \;=\; (1-\alpha^2)I^\Delta$$

Hence the Y-equivalent of $\left(Y^\Delta, I^\Delta\right)$ is

$$V^Y \;=\; \frac{1}{1-\alpha}V^\Delta \;=\; \frac{1}{\sqrt{3}\,e^{\mathbf{i}\pi/6}}V^\Delta, \qquad\qquad I^Y \;=\; \left(1-\alpha^2\right)I^\Delta \;=\; \frac{\sqrt{3}}{e^{\mathbf{i}\pi/6}}I^\Delta \tag{1.32a}$$

This implies in particular that a voltage source E^Δ in Δ configuration has an equivalent Y-configured voltage source with $E^Y := (1-\alpha)^{-1}E^\Delta$.

Consider a balanced three-phase impedance $z^\Delta \in \mathbb{C}$ in Δ configuration as shown in Figure 1.18(a). An Y-equivalent is a balanced impedance $z^Y \in \mathbb{C}$ as shown in Figure 1.18(b) so that their external behavior is the same, i.e., the terminal currents I are the same when the same line-to-line voltage V^{line} is applied to both impedances. Let $V^\Delta \in \mathbb{C}^3$ and $I^\Delta \in \mathbb{C}^3$ be the internal voltage and current across the impedance z^Δ in

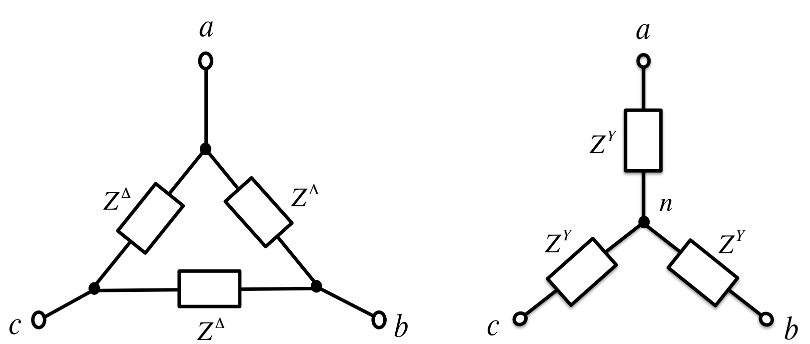


Figure 1.18: Δ-Y transformation of balanced loads: $Z^Y = Z^\Delta/3$.

Δ configuration. Let $Z^\Delta := \text{diag}\left(z^\Delta, z^\Delta, z^\Delta\right)$. Then $V^\Delta = Z^\Delta I^\Delta$ and

$$V^{\text{line}} \;=\; V^\Delta \;=\; Z^\Delta I^\Delta, \qquad\qquad I \;=\; -\Gamma^{\mathsf{T}} I^\Delta \;=\; -(1-\alpha^2)I^\Delta$$

where the last equality follows from Corollary 1.4. Hence, for Δ-configured impedance, the line-to-line voltage V^{line} is related to the terminal current I according to

$$V^{\text{line}} \;=\; -\frac{1}{1-\alpha^2} Z^{\Delta} I$$

For the Y-equivalent, let $V^Y \in \mathbb{C}^3$ and $I^Y \in \mathbb{C}^3$ be its internal voltage and current across the impedance z^Y in Y configuration. Let $Z^Y := \text{diag}\left(z^Y, z^Y, z^Y\right)$. Then $V^Y = Z^Y I^Y$ and Corollary 1.4 implies

$$V^{\text{line}} \;=\; \Gamma V^Y \;=\; (1-\alpha) Z^Y I^Y, \qquad\qquad I \;=\; -I^Y$$

Hence, for Y-configured impedance, the line-to-line voltage V^{line} is related to the terminal current I according to

$$V^{\text{line}} \;=\; -(1-\alpha) Z^Y I$$

The relationships between the line-to-line voltage V^{line} and the terminal current I for both the Δ-configured impedance and its Y-equivalent will be identical if and only if

$$z^Y \;=\; \frac{z^{\Delta}}{(1-\alpha)(1-\alpha^2)} \;=\; \frac{z^{\Delta}}{3} \tag{1.32b}$$

The corresponding admittances $y^Y := \left(z^Y\right)^{-1}$ and $y^{\Delta} := \left(z^{\Delta}\right)^{-1}$ are related by $y^Y = 3y^{\Delta}$.

1.2.5 Per-phase analysis

A balanced three-phase system consists of balanced three-phase sources and loads connected by balanced (identical) transmission lines. Given a balanced three-phase system with all sources and loads *in Y configuration*, assuming there is no mutual inductance between phases, then

- all the neutrals are at the same potential;
- all phases are decoupled;
- all corresponding network variables are in balanced sets of the same sequence as the sources.

These properties lead to equivalent per-phase circuits, as explained in Chapter 1.2.3. Even though we have only illustrated these properties for simple systems, they hold more generally. They allow us to study such a system by analyzing a single phase, say, phase a. The corresponding variables in phases b and c lags those in phase a by 120° and 240° respectively when abc is a positive sequence, and by 240° and 120° respectively when abc is a negative sequence.

When some or all of the sources and loads are in Δ configuration, the phases are still decoupled and can be analyzed separately. To obtain the equivalent per-phase circuit, however, we first transform each Δ configuration into an equivalent Y configuration using the transformation (1.32a) for sources and (1.32b) for loads. We then analyze the equivalent circuit that consists of only Y configurations. Finally we translate the results for equivalent Y configurations back to quantities in Δ configuration.

We emphasize that these transformations hold only in the balanced case with balanced sources, identical impedances, and symmetric transmission lines. Moreover the equivalence of these two configurations is with respect to their external behavior (V^{ab}, I^a, etc); for internal behavior, we have to analyze the original circuit; see Example 1.5.

In summary, the procedure for per-phase analysis is:

1. Convert all sources and loads in Δ configuration into their equivalent Y configurations using (1.32a) for sources and (1.32b) for loads.

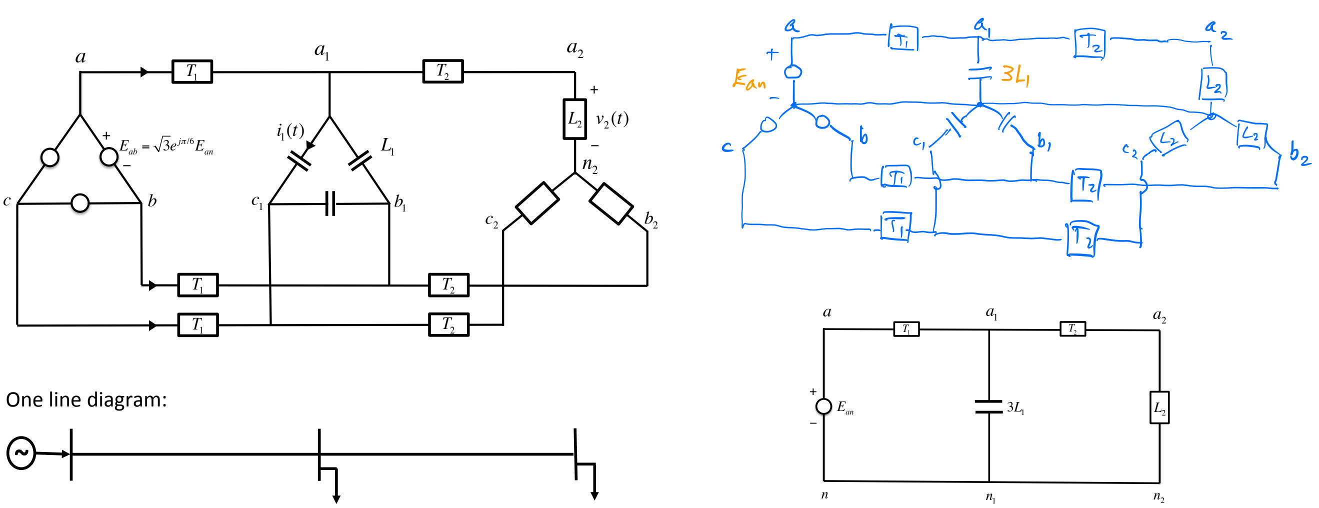
2. Solve for the desired phase a variables using phase a circuit with all neutrals connected.

3. For positive-sequence sources, the phase b and c variables are determined by subtracting 120° and 240° respectively from the corresponding phase a variables. For negative-sequence sources, add 120° and 240° instead.

4. If variables in the internal of a Δ configuration are desired, derive them from the original circuits.

This procedure is formally justified in Chapter 9.3.4. We illustrate it with an example.

Example 1.5 (Per-phase analysis)**.** Consider the balanced three-phase system shown in Figure 1.19. The three-phase sources are a balanced positive sequence in Δ configuration with line voltage $E^{ab} = \sqrt{3}e^{\mathbf{i}\pi/6}E^{an}$, etc. The Δ-configured loads are balanced with identical admittances L_1, and the Y-configured loads are balanced with identical admittances L_2. The transmission lines are modeled by admittances T_1 and T_2. Find the current $i_1(t)$ and voltage $v_2(t)$ in the diagram. Assume $3L_1L_2 + 3L_1T_2 + L_2(T_1 + T_2) + T_1T_2 \neq 0$.



(a) Balanced three-phase system (b) Equivalent per-phase system

Figure 1.19: Balanced three-phase system and its per-phase equivalent circuit. The balanced three-phase loads have admittances L_1 and L_2, and the transmission lines have admittances T_1 and T_2.

Solution. First we convert the Δ sources to their equivalent Y sources using (1.32a) and Δ loads to their equivalent Y loads using (1.32b). The result is shown in the upper panel of Figure 1.19(b). Then we construct the equivalent per-phase circuit with all neutrals n, n_1, n_2 connected, as shown in the lower panel of Figure 1.19(b).

We analyze the per-phase circuit to solve for voltages

$$V_1 \;:=\; V^{a_1n_1} \quad \text{and} \quad V_2 \;:=\; V^{a_2n_2}$$

Applying KCL to nodes a_1 and a_2 we get

$$\begin{aligned} T_1\left(E^{an} - V_1\right) &= 3L_1V_1 \;+\; T_2\left(V_1 - V_2\right) \\ T_2\left(V_1 - V_2\right) &= L_2V_2 \end{aligned}$$

Hence

$$\begin{bmatrix} 3L_1 + T_1 + T_2 & -T_2 \\ T_2 & -(L_2+T_2) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \;=\; \begin{bmatrix} T_1E^{an} \\ 0 \end{bmatrix}$$

By assumption, the determinant

$$\Delta \;:=\; -\left(3L_1L_2 + 3L_1T_2 + L_2(T_1+T_2) + T_1T_2\right)$$

is nonzero. Hence

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \;=\; \frac{1}{\Delta} \begin{bmatrix} -(L_2+T_2) & T_2 \\ -T_2 & 3L_1 + T_1 + T_2 \end{bmatrix} \begin{bmatrix} T_1E^{an} \\ 0 \end{bmatrix} \;=\; \frac{-T_1E^{an}}{\Delta} \begin{bmatrix} L_2 + T_2 \\ T_2 \end{bmatrix} \tag{1.33}$$

Since $V^{a_2n_2} = V_2$, we get:

$$v_2(t) \;=\; \sqrt{2}\,|V_2|\cos\left(\omega t + \angle V_2\right)$$

where ω is the steady-state system frequency and V_2 is given by (1.33). To calculate

$$i_1(t) \;=\; \sqrt{2}\,|I^{a_1c_1}|\cos\left(\omega t + \angle I^{a_1c_1}\right) \tag{1.34}$$

we use (1.32a) to first get

$$V^{a_1b_1} \;=\; \sqrt{3}\,e^{\mathbf{i}\pi/6}\,V_1$$

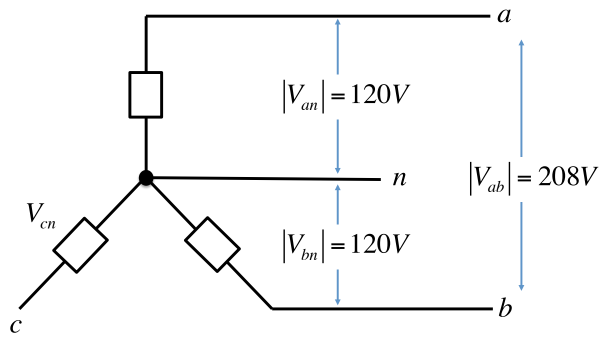
where V_1 is given by (1.33). Hence

$$I^{a_1b_1} \;=\; L_1V^{a_1b_1} \;=\; \sqrt{3}\,L_1\,e^{\mathbf{i}\pi/6}\,V_1$$

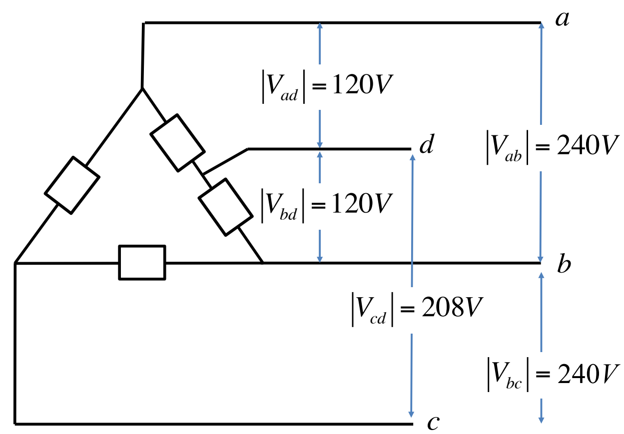
Since the sources are a positive sequence we have

$$I^{a_1c_1} \;=\; -I^{a_1a_1} \;=\; -I^{a_1b_1}\,e^{\mathbf{i}2\pi/3} \;=\; -\sqrt{3}\,e^{\mathbf{i}5\pi/6}\,3L_1V_1 \;=\; 3\sqrt{3}\,e^{-\mathbf{i}\pi/6}\,L_1V_1$$

where V_1 is given by (1.33). Substituting $I^{a_1c_1}$ into (1.34) yields $i_1(t)$. $\square$



(a) $208Y/120V$ 3-phase Y



(b) $240V$ split phase Δ

Figure 1.20: Common distribution transformer configurations.

1.2.6 Example configurations and line limits

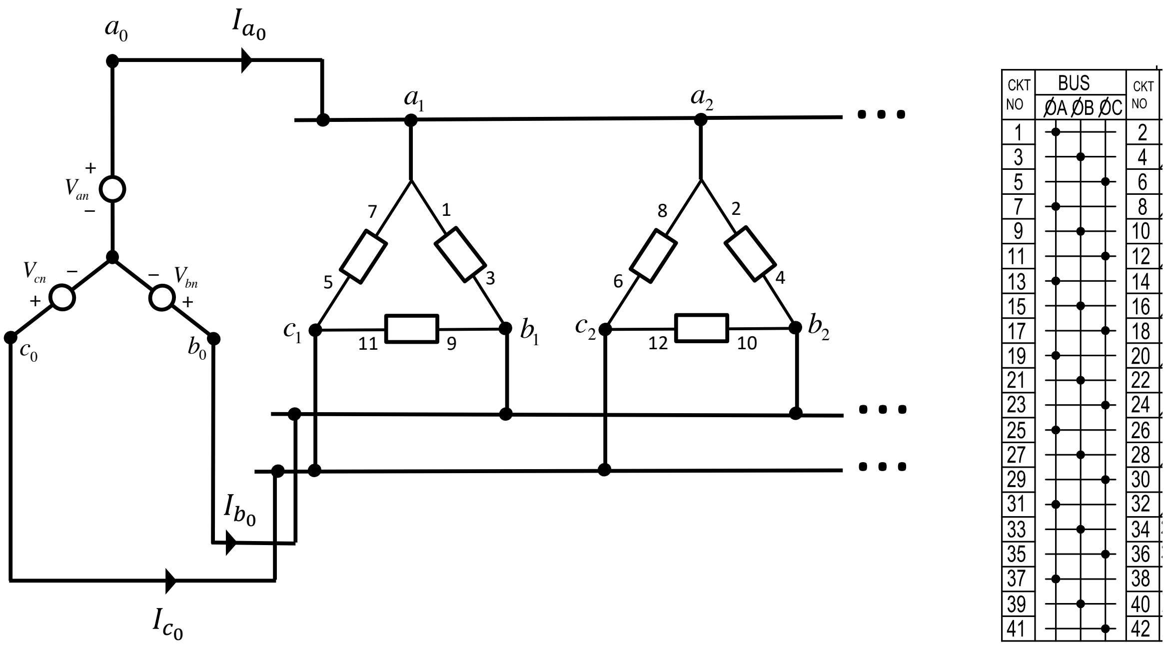
The secondary sides of three-phase distribution transformers are commonly configured in practice as shown in Figure 1.20. For our purposes we can treat them as balanced three-phase sources. Figure 1.20(a) shows a 5-wire three-phase transformer in Y configuration. Three phase wires (labeled a, b, c) and a neutral wire (labeled n) are shown. The fifth wire, not shown, is the earth ground wire, typically connected to neutral. A different voltage magnitude can be supplied to a load depending on how it is connected. The voltage magnitude between a phase wire and the neutral is $120V$ and that between a pair of phase wires is $120\sqrt{3}V = 208V$.

Figure 1.20(b) shows a 5-wire transformer in Δ configuration with one of the phases center-tapped to provide three voltage levels. Four phase wires (labeled a, b, c, d) are shown but an earth ground wire is not shown. The voltage magnitude between wires ad or bd is $120V$, whereas that between wire cd is $208V$ (derive this). The line-to-line voltage magnitude is $240V$.

Line limits. Figure 1.21(a) shows a Y-configured voltage source connected to a set of loads in Δ configuration. The voltage source is the secondary side of a three-phase $208Y/120V$ transformer shown in Figure 1.20(a). The voltage magnitude across each load is the line-to-line voltage $208V$. Figure 1.21(b) shows the electric panel arrangement to connect the loads to the voltage source. The dot in the first row indicates that the wires numbered 1 and 2 are connected to phase a, the dot in the second row indicates that the wires numbered 3 and 4 are connected to phase b, the dot in the third row indicates that the wires numbered 5 and 6 are connected to phase c, and so on. Therefore the load connected between wires 1 and 3 is connected between phase a and phase b lines (see the corresponding labels on the loads in Figure 1.21(a)). Similarly for the load connected between wires 2 and 4, and other loads connected between different phases.

We are interested in the currents $J_0 := (I^{a_0a_1}, I^{a_0b_1}, I^{a_0c_1})$ supplied by the three-phase source to the loads. Suppose the wires connecting the three-phase source to the loads are rated at $I^{\max}$. Then we require that the current magnitude in each phase be bounded by $I^{\max}$:

$$|I^{p_0p_1}| \quad \leq \quad I^{\max}, \qquad p = a, b, c \tag{1.35}$$



(a) Voltage source supplying loads in parallel

(b) Panel arrangement

Figure 1.21: (a) Three-phase voltage source connected to loads in parallel. (b) Three-phase panel used to connect loads in parallel to the voltage source.

Suppose the loads are not impedance loads, but constant current loads that draw specified currents. Let the current drawn by the load in Figure 1.21(a) between wires 1 and 3 be $I^{a_1b_1}$, that between wires 9 and 11 be $I^{a_1c_1}$, that between wires 5 and 7 be $I^{a_1a_1}$. In general, let the load currents in the kth three-phase load be

$$I_k \quad := \quad \begin{bmatrix} I^{a_kb_k} \\ I^{a_kc_k} \\ I^{a_ka_k} \end{bmatrix}$$

We now derive bounds on the load currents $(I_k, k = 1, \ldots, K)$ that enforce the line limits (1.35). Recall that the magnitudes $(|I^{a_kb_k}|, |I^{a_kc_k}|, |I^{a_ka_k}|)$ of the current phasors are the root-mean-square (RMS) values of the corresponding sinusoidal functions. In practice, load currents are usually specified in terms of RMS values, and hence we are interested in bounding the magnitudes $(|I^{a_kb_k}|, |I^{a_kc_k}|, |I^{a_ka_k}|)$ of the current phasors, not their phases.

Before proceeding, we mention as an example application the smart charging of electric vehicles where each load is a vehicle. We are to design an algorithm that determines the charging rate, i.e., current magnitude $|I^{p_kq_k}|$, for each vehicle to optimize certain objective subject to capacity constraints such as (1.35) and other constraints. Such an algorithm can be applied periodically, e.g., every minute, to update the charging rates. Note that in this kind of applications, the system is unbalanced since the loads $|I^{p_kq_k}|$ are generally not identical across phases, but here we ignore the effect of wires connecting these devices.

Applying KCL at nodes (a_1, b_1, c_1) we have

$$\underbrace{\begin{bmatrix} I^{a_0a_1} \\ I^{a_0b_1} \\ I^{a_0c_1} \end{bmatrix}}_{J_0} = \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\Gamma^{\mathsf{T}}} \underbrace{\begin{bmatrix} I^{a_1b_1} \\ I^{a_1c_1} \\ I^{a_1a_1} \end{bmatrix}}_{I_1} + \underbrace{\begin{bmatrix} I^{a_1a_2} \\ I^{a_1b_2} \\ I^{a_1c_2} \end{bmatrix}}_{J_1}$$

where $J_k := (I^{a_ka_{k+1}}, I^{a_kb_{k+1}}, I^{a_kc_{k+1}})$, $k = 0, \ldots, K-1$, are the line currents from stage k to stage $k+1$. In general we have

$$J_k = \Gamma^{\mathsf{T}} I_k + J_{k+1}, \qquad k = 0, \ldots, K-1$$

Hence the total supply currents are given by

$$J_0 = \Gamma^{\mathsf{T}} (I_0 + I_1 + \cdots + I_K) \tag{1.36}$$

when there are K three-phase constant current loads. Note that this expression does not require that the loads are balanced. In particular, if a load (say) $I^{a_kb_k}$ is absent, then we set $I^{a_kb_k} = 0$ in (1.36).

Let the total load current in each leg of the Δ configuration be denoted by

$$I^{ab} := \sum_{k=1}^{K} I^{a_kb_k}, \quad I^{bc} := \sum_{k=1}^{K} I^{a_kc_k}, \quad I^{ca} := \sum_{k=1}^{K} I^{a_ka_k} \tag{1.37}$$

Then (1.36) can be written in terms of the total load currents as:

$$\begin{bmatrix} I^{a_0a_1} \\ I^{a_0b_1} \\ I^{a_0c_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix}$$

The line limits (1.35) are therefore

$$\begin{aligned} |I^{a_0a_1}| &= |I^{ab} - I^{ca}| \leq I^{\max} \\ |I^{a_0b_1}| &= |I^{bc} - I^{ab}| \leq I^{\max} \\ |I^{a_0c_1}| &= |I^{ca} - I^{bc}| \leq I^{\max} \end{aligned}$$

Enforcing line limits requires one to know not just the magnitudes of the load currents, but also their phases in order to compute their sums. As explained in the caption of Figure 1.22, these inequalities are equivalent to:

$$|I^{ab}|^2 + |I^{ca}|^2 - 2|I^{ab}||I^{ca}| \cos\phi_{a_0a_1} \leq (I^{\max})^2 \tag{1.39a}$$
$$|I^{bc}|^2 + |I^{ab}|^2 - 2|I^{bc}||I^{ab}| \cos\phi_{b_0b_1} \leq (I^{\max})^2 \tag{1.39b}$$
$$|I^{ca}|^2 + |I^{bc}|^2 - 2|I^{ca}||I^{bc}| \cos\phi_{c_0c_1} \leq (I^{\max})^2 \tag{1.39c}$$

If we know the angles $\phi_{p_0p_1}$, $p = a, b, c$, between the total load currents (I^{ab}, I^{bc}, I^{ca}) in each leg of the Δ configuration, then (1.39) are convex quadratic constraints on the magnitudes of (I^{ab}, I^{bc}, I^{ca}). We next

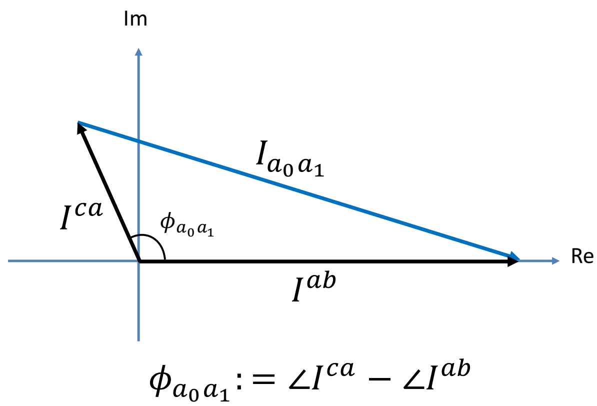


Figure 1.22: $I^{a_0 a_1} = I^{ab} - I^{ca}$. Hence by the cosine rule $|I^{a_0a_1}|^2 = \left|I^{ab}\right|^2 + |I^{ca}|^2 - 2\left|I^{ab}\right| |I^{ca}| \cos\phi$ where $\phi_{a_0a_1} := \angle I^{ca} - \angle I^{ab}$ is the angle between I^{ab} and I^{ca}.

consider several special cases and derive simple bounds on the magnitudes $(|I^{a_kb_k}|, |I^{a_kc_k}|, |I^{a_ka_k}|)$ of the individual load currents that will enforce (1.39).

Assumption 1: Current phasors $I^{a_kb_k}$ have the same, and known, phase angle θ_{ab} for all k; similarly for $I^{a_kc_k}$ and $I^{a_ka_k}$. From (1.37) we have

$$I^{ab} := e^{\mathbf{i}\theta_{ab}} \sum_{k=1}^{K} \left|I^{a_kb_k}\right|, \quad I^{bc} := e^{\mathbf{i}\theta_{bc}} \sum_{k=1}^{K} |I^{a_kc_k}|, \quad I^{ca} := e^{\mathbf{i}\theta_{ca}} \sum_{k=1}^{K} |I^{a_ka_k}|$$

and constraints (1.39a) become

$$\left(\sum_{k=1}^{K} \left|I^{a_kb_k}\right|\right)^2 + \left(\sum_{k=1}^{K} |I^{a_ka_k}|\right)^2 - 2\left(\sum_{k=1}^{K} \left|I^{a_kb_k}\right|\right)\left(\sum_{k=1}^{K} |I^{a_ka_k}|\right) \cos\phi_{a_0a_1} \leq (I^{\max})^2 \tag{1.40}$$

where $\cos\phi_{a_0a_1} := \theta_{ca} - \theta_{ab}$ is known. Similarly for constraints (1.39b) and (1.39c). These are quadratic constraints in the magnitudes $(|I^{a_kb_k}|, |I^{a_kc_k}|, |I^{a_ka_k}|)$ of the individual load currents that will enforce (1.39), given the angles $\phi_{p_0p_1}$, $p = a, b, c$, between the load currents in different legs of the Δ configuration.

Assumption 2: In addition to Assumption 1, the angles $\phi_{p_0p_1} = 120°$, for $p = a, b, c$. Then $\cos\phi_{p_0p_1} = -1/2$ and (1.40) becomes

$$\left(\sum_{k=1}^{K} \left|I^{a_kb_k}\right|\right)^2 + \left(\sum_{k=1}^{K} |I^{a_ka_k}|\right)^2 + \left(\sum_{k=1}^{K} \left|I^{a_kb_k}\right|\right)\left(\sum_{k=1}^{K} |I^{a_ka_k}|\right) \leq (I^{\max})^2 \tag{1.41}$$

Similarly for constraints (1.39b) and (1.39c).

Assumption 3 (balanced case): All load currents have the same magnitude and the phases of currents on different legs of the Δ differ by $120°$. That is, assuming positive sequence, for all $k = 1, \ldots, K$, we have

$$I^{a_kb_k} = I\, e^{\mathbf{i}\theta_{ab}}, \quad I^{a_kc_k} = I\, e^{\mathbf{i}\theta_{bc}}, \quad I^{a_ka_k} = I\, e^{\mathbf{i}\theta_{ca}}$$

where I is the common magnitude of the load currents, and

$$\theta_{ab} - \theta_{bc} \;=\; 120^\circ, \quad \theta_{bc} - \theta_{ca} \;=\; 120^\circ, \quad \theta_{ca} - \theta_{ab} \;=\; 120^\circ$$

Then the constraint (1.41) reduces to $3K^2I^2 \leq (I^{\max})^2$, or a bound on the common magnitude I of individual load currents

$$I \;\leq\; \frac{I^{\max}}{\sqrt{3}K} \tag{1.42}$$

Linear bounds. Many applications operate in unbalanced conditions, e.g., adaptive electric vehicle charging where the magnitudes $\left|I_{p_k q_k}\right|$ of the load currents are to be determined and generally different. In these cases there are two difficulties with the line limits (1.40) and (1.41). First the angles $(\theta_{ab}, \theta_{bc}, \theta_{ca})$ may not be known. Second even when these angles are known, the constraints are quadratic which can be computationally too expensive to implement in real time in inexpensive devices. In this case, we can impose linear constraints which are simpler but more conservative.

Take phase a as an example. Since $|I^{a_0 a_1}| \;=\; |I^{ab} - I^{ca}| \;\leq\; |I^{ab}| \,+\, |I^{ca}|$, a simple limit on the load currents that enforces $|I^{a_0 a_1}| \leq I^{\max}$ is to require

$$|I^{ab}| \,+\, |I^{ca}| \;\;\leq\;\; I^{\max}$$

i.e., the sum of the magnitudes of the total load currents in legs ab and ca should be less than the current rating $I^{\max}$. From (1.37) we have $\left|I^{ab}\right| \;=\; \left|\sum_k I^{a_k b_k}\right| \;\leq\; \sum_k |I^{a_k b_k}|$. Hence a simple linear bound on the load current magnitudes is:

$$\sum_{k=1}^{K} \left(|I^{a_k b_k}| \,+\, |I^{a_k a_k}| \right) \;\;\leq\;\; I^{\max} \tag{1.43}$$

The constraints on phases b and c are similar.

For a balanced system we can easily assess how conservative the bound (1.43) is compared with the exact limit (1.42) on the load currents. In the balanced case the bound (1.43) reduces to

$$I \;\leq\; \frac{I^{\max}}{2K}$$

Hence it is $\sqrt{3}/2 \sim 87\%$ of that in (1.42), i.e., it is conservative by $\sim 13\%$ for a balanced system.

1.3 Complex power

1.3.1 Single-phase power

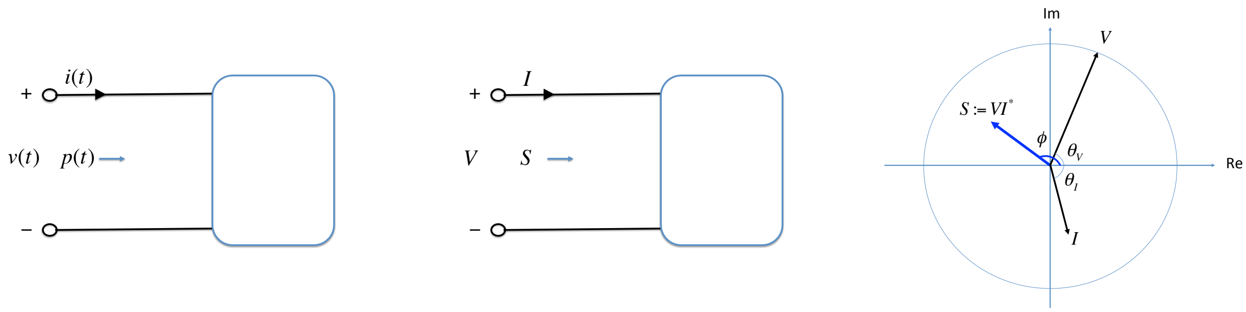
Instantaneous power. When a voltage $v(t)$ is applied across two ports and a current $i(t)$ flows between them, as shown in Figure 1.23(a), energy is delivered to the network that connects the ports. We define the *instantaneous power* supplied as:

$$p(t) \;\;:=\;\; v(t)i(t) \;=\; \frac{V_{\max}I_{\max}}{2}\left(\cos(\theta_V - \theta_I) + \cos(2\omega t + \theta_V + \theta_I)\right) \tag{1.44}$$

Since the last term inside the bracket of (1.44) is sinusoidal with twice the nominal frequency ω the average power delivered is

$$\frac{1}{T}\int_0^T p(t)dt \quad = \quad \frac{V_{\max}I_{\max}}{2}\cos(\theta_V - \theta_I)$$

where $T := 2\pi/\omega$.



(a) Instantaneous power (b) Complex power (c) $S = VI^*$

Figure 1.23: Definition of power

Complex power. Define the *complex power* in terms of the voltage and current phasors as:

$$S \quad := \quad VI^* \;=\; \frac{V_{\max}I_{\max}}{2}\, e^{\mathbf{i}(\theta_V - \theta_I)} \;=\; |V||I|e^{\mathbf{i}\phi} \tag{1.45}$$

where I^* denotes the complex conjugate of I. See Figures 1.23(b) and (c). Here $\phi := \theta_V - \theta_I$ is called the *power factor angle* and $\cos\phi$ is called the *power factor (PF)*. Power engineers often says *leading* or *lagging* power factor: here *lagging* means *current I lags voltage V* so that $\phi > 0$. A leading power factor has $\phi < 0$. A unity power factor means $\phi = 0$. Figure 1.24 shows four complex powers

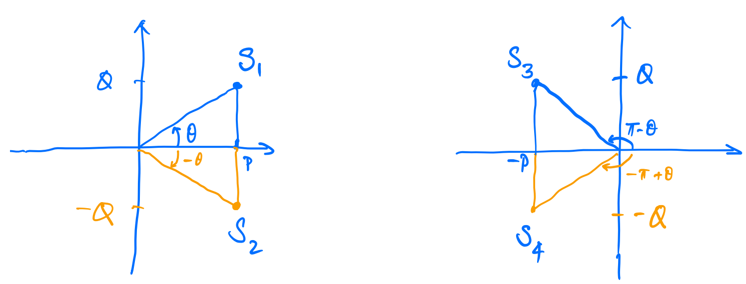


Figure 1.24: Power factor angles ϕ and power factor $\cos\phi$.

$$S_1 \;:=\; P + \mathbf{i}Q, \qquad S_2 \;:=\; P - \mathbf{i}Q, \qquad S_3 \;:=\; -P + \mathbf{i}Q, \qquad S_2 \;:=\; -P - \mathbf{i}Q$$

with power factor angles $\phi_1 := \theta$, $\phi_2 := -\theta$, $\phi_3 := \pi - \theta$, and $\phi_4 := -\pi + \theta$ respectively. Here $P, Q > 0$ and $\theta \in [0, \pi]$. Their power factors are

$$\cos\phi_1 \;=\; \frac{P}{\sqrt{P^2 + Q^2}} \;=\; \cos\phi_2, \qquad\qquad \cos\phi_3 \;=\; \frac{-P}{\sqrt{P^2 + Q^2}} \;=\; \cos\phi_4$$

Therefore power factor $\cos\phi_i$ does not differentiate between S_1 and S_2. Power engineers specify S_1 as power factor $\cos\theta$ lagging ($\phi_1 > 0$ and therefore $Q_1 := Q > 0$) and S_2 as power factor $\cos\theta$ leading ($\phi_2 < 0$ and $Q_2 := -Q < 0$). Similarly S_3 has a power factor $-\cos\theta$ lagging ($\phi_3 > 0$ and $Q_3 := Q > 0$) and S_4 has a power factor $-\cos\theta$ leading ($\phi_4 < 0$ and $Q_4 := -Q < 0$). For example "a load draws 100kW at a power factor of 0.707 leading" means that the real power Re(S) = 100 kW and $\cos\phi = \frac{1}{\sqrt{2}}$. Since the power factor is leading, $\phi = -45^\circ$ and $S = 100 - j100$ kVA .

Note that S is *not* a phasor because $\sqrt{2}|S|\cos(\omega t + \phi)$ is not the instantaneous power in the time domain. This complex quantity is important in power flow analysis in the phasor domain, as we will see. The real part of S

$$P \;\; := \;\; |V||I|\cos\phi$$

is called the *active* or *real power* and its unit is W (watt). The imaginary part of S

$$Q \;\; := \;\; |V||I|\sin\phi$$

is called the *reactive power* and its unit is var (volt-ampere reactive). We write both $S = P + jQ$ and $S = |V||I|e^{\mathbf{i}\phi}$. The magnitude $|S| = |V||I|$ is called the *apparent power* and its unit is VA (volt-ampere). Given an active power P and a power factor $\cos\phi$, the complex power S is given by (since $P = |S|\cos\phi$)

$$S \;\; = \;\; \frac{P}{\cos\phi}\, e^{\mathbf{i}\phi}$$

i.e. the complex power is completely determined by the active power P and the power factor angle ϕ. Power is balanced at every node in a network. Referring to Figure 1.25, if I_{jk} and S_{jk} are sending-end current and power respectively from node j to node k, then power balance at node j means $\sum_k S_{jk} = 0$.

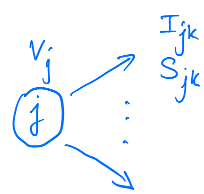


Figure 1.25: Power balance at a node.

This is a consequence of KCL $\sum_k I_{jk} = 0$ and the definition of branch power $S_{jk} := V_j I_{jk}^*$.

Relationship between instantaneous and complex power. The complex power S in the phasor domain is related to the instantaneous power in the time domain as follows. We can use (1.44) to express the instantaneous power $p(t)$ in terms of active power P and reactive power Q as (Problem 1.9):

$$p(t) \;\; = \;\; P \; + \; P\cos 2(\omega t + \theta_I) - Q\sin 2(\omega t + \theta_I) \tag{1.46}$$

It is then clear that the active power P is equal to the average power delivered (in the time domain):

$$P \;\; = \;\; \frac{1}{T}\int_0^T p(t)dt$$

as the last two terms in (1.46) average to zero over a cycle T. The reactive power Q determines the magnitude of the instantaneous power $p(t)$.

Power delivered to an impedance. The current and voltage across an impedance Z is related by Ohm's law, $V = ZI$ and hence

$$|Z| \;=\; \frac{|V|}{|I|} \quad \text{and} \quad \angle Z \;=\; \theta_V - \theta_I \;=:\; \phi$$

Therefore from (1.45)

$$S \;=\; Z|I|^2 \;=\; |Z||I|^2 e^{\mathbf{i}\phi}$$

and

$$P \;=\; |Z||I|^2 \cos\phi \quad \text{and} \quad Q \;=\; |Z||I|^2 \sin\phi$$

The active and reactive power for the three passive elements are given in Table 1.2.

	$\lvert Z\rvert$	$\phi = \angle Z$	P	Q
Resistor $Z = R$	R	0	$R\lvert I\rvert^2$	0
Inductor $Z = \mathbf{i}\omega L$	ωL	$\pi/2$	0	$\omega L\lvert I\rvert^2$
Capacitor $Z = (\mathbf{i}\omega C)^{-1}$	$(\omega C)^{-1}$	$-\pi/2$	0	$-(\omega C)^{-1}\lvert I\rvert^2$

Table 1.2: Power delivered to RLC elements.

Therefore the power delivered to a resistor is active $(Q = 0)$. The instantaneous power $p(t) := v(t)i(t)$ is

$$p(t) \;:=\; Ri^2(t) \;=\; RI_{\max}^2 \cos^2(\omega t + \theta_I) \;=\; P\left(1 + \cos 2(\omega t + \theta_I)\right)$$

which is (1.46). Table 1.2 also implies that the complex power delivered to an inductor or a capacitor is reactive. Substituting into (1.46), the instantaneous power $p(t)$ to a purely reactive load depends only on the reactive power Q:

$$p(t) \;=\; \begin{cases} -Q\sin 2(\omega t + \theta_I) & \text{for inductor } Z = j\omega L \\ Q\sin 2(\omega t + \theta_V) & \text{for capacitor } Z = (j\omega C)^{-1} \end{cases}$$

i.e., the net (average) power delivered to the load is zero and the instantaneous power is sinusoidal with twice the frequency and has an amplitude Q.

Example 1.6. Suppose $Z = j\omega L$ (inductance) or $Z = (j\omega C)^{-1}$ (capacitance). Prove directly in time domain that the average delivered power is 0 and the amplitude of the instantaneous power is Q.

Solution: Suppose power is delivered to an inductor $Z = j\omega L$. Let the current be $i(t) = I_{\max}\cos(\omega t + \theta_I)$. Then the voltage $v(t)$ across the inductor is given by

$$v(t) \;=\; L\frac{di}{dt}(t) \;=\; -\omega L\, I_{\max}\sin(\omega t + \theta_I)$$

and therefore

$$\begin{aligned} p(t) &= v(t)i(t) \;=\; -\omega L\, I_{\max}^2 \;\sin(\omega t + \theta_I)\cos(\omega t + \theta_I) \\ &= -\omega L\, \frac{I_{\max}^2}{2} \;\sin 2(\omega t + \theta_I) \;=\; -\omega L\, |I|^2 \;\sin 2(\omega t + \theta_I) \\ &= -Q\; \sin 2(\omega t + \theta_I) \end{aligned}$$

where the last equality follows from $Q = |Z||I|^2 \sin\angle Z = \omega L|I|^2$ since $\angle Z = \frac{\pi}{2}$. Moreover the average power delivered is

$$P \;=\; \frac{1}{T}\int_0^{\mathsf{T}} p(t)dt \;=\; 0$$

The case of capacitor load $Z = (j\omega C)^{-1}$ is similar and omitted (see Exercise 1.10). □

1.3.2 Three-phase power

Under balanced three-phase operation, the total instantaneous power delivered is constant and the total complex power is 3 times the per-phase complex power.

Indeed, for a balanced three-phase positive-sequence source, we have

$$V^{bn} = V^{an}\, e^{-\mathbf{i}2\pi/3}, \;\; I^{an} = I^{an}\, e^{-\mathbf{i}2\pi/3} \quad \text{and} \quad V^{cn} = V^{an}\, e^{\mathbf{i}2\pi/3}, \;\; I^{an} = I^{an}\, e^{\mathbf{i}2\pi/3}$$

Hence

$$S_{3\phi} \;=\; V^{an}I^{an\mathsf{H}} + V^{bn}I^{an\mathsf{H}} + V^{cn}I^{an\mathsf{H}} \;=\; 3\,V^{an}I^{an\mathsf{H}} \;=\; 3\,S$$

where $S := V^{an}I^{an\mathsf{H}}$ is the per-phase complex power.

For instantaneous power, we have from (1.44), for a balanced three-phase positive-sequence source,

$$\begin{aligned} p_{3\phi}(t) &:= v^a(t)i^a(t) + v^b(t)i^b(t) + v^c(t)i^c(t) \\ &= |V^a||I^a|\,(\cos\phi + \cos(2\omega t + \theta_V + \theta_I)) \\ &\qquad + |V^a||I^a|\,(\cos\phi + \cos(2\omega t + (\theta_V - 2\pi/3) + (\theta_I - 2\pi/3))) \\ &\qquad + |V^a||I^a|\,(\cos\phi + \cos(2\omega t + (\theta_V + 2\pi/3) + (\theta_I + 2\pi/3))) \\ &= 3|V^a||I^a|\cos\phi \;+\; |V^a||I^a|\,(\cos\theta(t) + \cos(\theta(t) - 4\pi/3) + \cos(\theta(t) + 4\pi/3)) \\ &= 3P \end{aligned}$$

where $\theta(t) := 2\omega t + \theta_V + \theta_I$ and P is the per-phase active power. Here the last equality follows from

$$\cos x + \cos(x - 4\pi/3) + \cos(x + 4\pi/3) \;=\; \mathrm{Re}\left(e^{\mathbf{i}x} + e^{\mathbf{i}(x-4\pi/3)} + e^{\mathbf{i}(x+4\pi/3)}\right)$$

and

$$\left(e^{\mathbf{i}x} + e^{\mathbf{i}(x-4\pi/3)} + e^{\mathbf{i}(x+4\pi/3)}\right) \;=\; \left(e^{\mathbf{i}x} + e^{\mathbf{i}(x+2\pi/3)} + e^{\mathbf{i}(x-2\pi/3)}\right) \;=\; 0$$

where the last equality follows from Theorem 1.3.

1.3.3 Advantages of three-phase power

There are two main advantages of balanced three-phase systems over a system with a single phase or that with other polyphases.

First it offers several benefits to motor operation. The total instantaneous power $p_{3\phi}(t) = 3P$ delivered is constant over time in a balanced three-phase system. On a generator or motor this produces a constant mechanical torque, reducing vibrations, noise, wear and tear, and other mechanical issues. A three-phase system can also self-start an induction motor.

In contrast, the instantaneous power

$$p_{1\phi}(t) \;=\; P \,+\, |V||I|\cos(2\omega t + \theta_V + \theta_I) \;=:\; P + |V||I|\cos\theta(t)$$

in a single-phase system, where $\theta(t) := 2\omega t + \theta_V + \theta_I$, is a sinusoidal signal with twice the system frequency. This is the case also with a two-phase system where the instantaneous power is

$$\begin{aligned}
p_{2\phi}(t) &= |V^a||I^a|\left(\cos\phi + \cos(2\omega t + \theta_V + \theta_I)\right) + |V^a||I^a|\left(\cos\phi + \cos(2\omega t + (\theta_V + \pi) + (\theta_I + \pi))\right) \\
&= |V^a||I^a|\left(2\cos\phi + \cos\theta(t) + \cos(\theta(t) + 2\pi)\right) \\
&= P \,+\, 2|V^a||I^a|\cos\theta(t)
\end{aligned}$$

For a four-phase system

$$\begin{aligned}
p_{4\phi}(t) &= |V^a||I^a|\left(4\cos\phi + \cos\theta(t) + \cos\left(\theta(t) + 2\left(\frac{\pi}{2}\right)\right) + \cos(\theta(t) + 2\pi) + \cos\left(\theta(t) + 2\left(\frac{3\pi}{2}\right)\right)\right) \\
&= 2|V^a||I^a|\left(2\cos\phi + 2\cos\theta(t) + 2\cos(\theta(t) + \pi)\right) \\
&= 4P
\end{aligned}$$

since $\cos\theta(t) + \cos(\theta(t) + \pi) = 0$ for all t. Even though a balanced four-phase system also has time-invariant instantaneous power, its design is more complex than a three-phase system.

Second a three-phase system typically saves materials and thermal loss ($|I^2|R$) compared with a single-phase system that serves the same load. For example, it is clear that the single-phase system that consists of three identical subsystems shown in Figure 1.7(a) needs twice as much transmission line and incurs twice as much thermal loss in transmission as the balanced three-phase system in Figure 1.7(b), since the balanced three-phase system has zero return current and hence does not need a neutral line.

The following example compares a balanced three-phase system with a single one-phase circuit with a higher ampacity, as opposed to three identical subcircuits in Figure 1.7(a), to supply the same load. The same conclusion holds that the three-phase system needs half as much conductor and incurs half as much transmission loss.

Example 1.7 (Single-phase vs three-phase systems)**.** Consider two systems that deliver a specified apparent power $|S|$ at a specified voltage magnitude $|V|$ to a constant power load, as shown in Figure 1.26. The distance between the generation and the load is d. The first system is single-phased and the second system is balanced three-phased. Compare the required amount of wire and thermal loss in the line in these systems.

The line has an impedance $z := r + jx$ per unit length where the resistance r per unit length is inversely proportional to the area of the line with proportionality constant ρ. The current density limit of the line is δ in ampere per unit area.

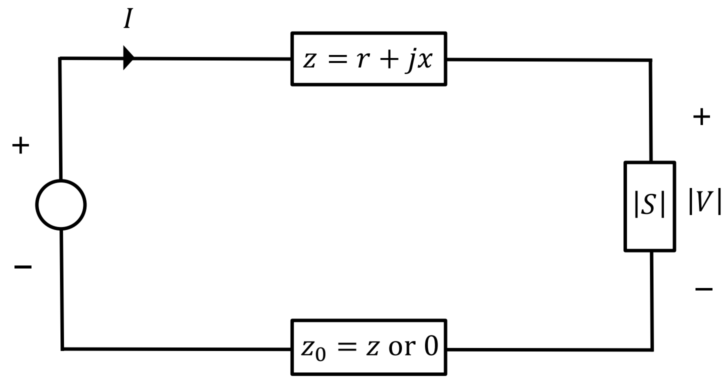


Figure 1.26: A system that delivers power $|S|$ to a load at voltage $|V|$. The distance between the generation and the load is d. The line has an impedance $z := r + jx$ per unit length.

Solution. A single-phase system requires two cables, one for return current, each carrying a current of magnitude $|I_{1\phi}| = |S|/|V|$. This is illustrated in Figure 1.26 with $z_0 = z$. A balanced three-phase system requires three cables, each carrying a per-phase apparent power of $|S|/3$ and a per-phase current of magnitude $|I_{3\phi}| = |S|/(3|V|)$. The per-phase equivalent circuit is illustrated in Figure 1.26 with $z_0 = 0$.

For the single-phase system the required cross-sectional area of the cable is

$$A_{1\phi} := \frac{|I_{1\phi}|}{\delta} = \frac{|S|}{\delta|V|}$$

Hence the amount of material (volume of the cable) required is

$$m_{1\phi} := 2A_{1\phi}d = 2\frac{d|S|}{\delta|V|}$$

Moreover the resistance per-unit length of the cable is

$$r_{1\phi} := \frac{\rho}{A_{1\phi}} = \frac{\rho\delta|V|}{|S|}$$

and hence the active power loss in the cable is

$$l_{1\phi} := 2r_{1\phi}|I_{1\phi}|^2 d = \frac{2\rho\delta|V|}{|S|}\cdot\frac{d|S|^2}{|V|^2} = 2\frac{\rho\delta d|S|}{|V|}$$

For the balanced three-phase system the required cross-sectional area of the cable in each phase is

$$A_{3\phi} := \frac{|I_{3\phi}|}{\delta} = \frac{|S|}{3\delta|V|}$$

Hence the amount of material required is

$$m_{3\phi} \;:=\; 3A_{3\phi}d \;=\; \frac{d|S|}{\delta|V|} \;=\; \frac{1}{2}m_{1\phi}$$

Moreover the resistance $r_{3\phi}$ per unit length of the cable is

$$r_{3\phi} \;:=\; \frac{\rho}{A_{3\phi}} \;=\; \frac{3\rho\delta|V|}{|S|}$$

and hence the active power loss in the cable is

$$l_{3\phi} \;:=\; 3r_{3\phi}|I_{3\phi}|^2 d \;=\; \frac{9\rho\delta|V|}{|S|}\cdot\frac{d|S|^2}{9|V|^2} \;=\; \frac{\rho\delta d|S|}{|V|} \;=\; \frac{1}{2}l_{1\phi}$$

i.e., the balanced three-phase system uses half as much material and incurs half as much loss as the single-phase system. □

Remark 1.4. 1. Example 1.7 also shows that thermal loss $r|I|^2$ is inversely proportional to $|V|$. Intuitively a higher load voltage $|V|$ requires a smaller load current $|I|$ to deliver the same amount of power $|S|$, resulting in a smaller thermal loss in the grid.

2. It is shown in Exercise 2.7 that, given a desired load power, the active line loss is inversely proportional to the square $|V|^2$ of the load voltage magnitude, rather than $|V|$ derived here. This is because, in Exercise 2.7, the line resistance is given and independent of load power and voltage $|V|$, whereas, here, the line resistance $r_{3\phi}$ is chosen to be proportional to $|V|$ (reducing the dependence of line loss $r_{3\phi}|I_{3\phi}|^2$ from $|V|^2$ to $|V|$).

3. Note that V is the voltage drop across the load, not the voltage drop across transmission line z which is $zdI = zdS^*/V^*$. In the case of balanced three-phase system (where $z_0 = 0$ in Figure 1.26), if the load power S and voltage V are specified then the required squared voltage magnitude at the source is

$$|zdI + V|^2 \;=\; \left|zd\frac{S^*}{V^*} + V\right|^2 \;=\; |V|^2 + d|z|^2\frac{|S|^2}{|V|^2} + 2d\mathrm{Re}\left(z^*S\right)$$

4. In practice most three-phase systems do include a grounded neutral line to carry unbalanced current during asymmetrical conditions, e.g., due to line faults, and reduce voltage transients during line switching or lightning events. Since the unbalanced current is much smaller than the phase currents, the neutral line is typically much smaller in size and ampacity and therefore much cheaper.

1.4 Bibliographical notes

There are many excellent textbooks on basic power system concepts, e.g., [1, 2, 3, 4]. Many materials in this chapter follow [1]. The example comparing the savings of single-phase and three-phase systems is from [4]. Circuit theory is a well established field. For general circuit analysis using KCL and KVL, see, e.g., [5, Chapter 12]. The connection with algebraic graph theory is recently surveyed in [6].

1.5 Problems

Chapter 1.1.

Exercise 1.1 (KVL)**.** Prove that Kirchhoff's voltage law (1.3b) is equivalent to (1.4b). (Hint: See Appendix 25.2 and use Theorem 25.30.1 and Theorem 25.30.2.)

Exercise 1.2 (Circuit analysis)**.** Consider a 3-node 3-link circuit specified by:

$$\text{incidence matrix } \hat{C} := \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad \text{impedances } z_{12} = z_{23} = 1, \quad \text{voltage source } v_{13}$$

Use (**??**) to determine the currents $J_1 := (J_{12}, J_{23}, J_{13})$, voltages $U_1 := (U_{12}, U_{23})$ and nodal voltages $V := (V_1, V_2)$, assuming without loss of generality that node 3 is the reference node with $V_3 := 0$.

Exercise 1.3 (Circuit analysis)**.** For the three-bus network in Figure 1.5, derive the current balance equation (1.8) by analyzing the equivalent circuit using KCL, KVL, and Ohm's law, as explained in Chapter 1.1.4.

Chapter 1.2.

Exercise 1.4 ($\alpha := e^{-\mathbf{i}2\pi/3}$)**.** Prove the following properties of $\alpha := e^{-\mathbf{i}\angle 120^\circ}$ (see Figure 1.27):

1. $\alpha^2 = \overline{\alpha}$, $\alpha^3 = 1$, $\alpha^4 = \alpha$, $\alpha^k = \alpha^{k \mod 3}$ where $\overline{a}$ denotes the complex conjugate of a.
2. $1 + \alpha + \alpha^2 = 0$.
3. $1 - \alpha = \sqrt{3}\angle 30^\circ$, $1 - \alpha^2 = \sqrt{3}\angle -30^\circ$.
4. $1 + \alpha = -\alpha^2 = 1\angle -60^\circ$, $1 + \alpha^2 = -\alpha = 1\angle 60^\circ$.
5. $\overline{\alpha}_+ = \alpha_-$, $\overline{\alpha}_- = \alpha_+$.

Exercise 1.5 (Proof of Theorem 1.3)**.** Let $\alpha := e^{-\mathbf{i}2\pi/3}$. Recall the matrices F defined in (1.18) and Γ in (1.12), reproduced here:

$$F := \frac{1}{\sqrt{3}} \begin{bmatrix} \mathbf{1} & \alpha_+ & \alpha_- \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}, \qquad \Gamma := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

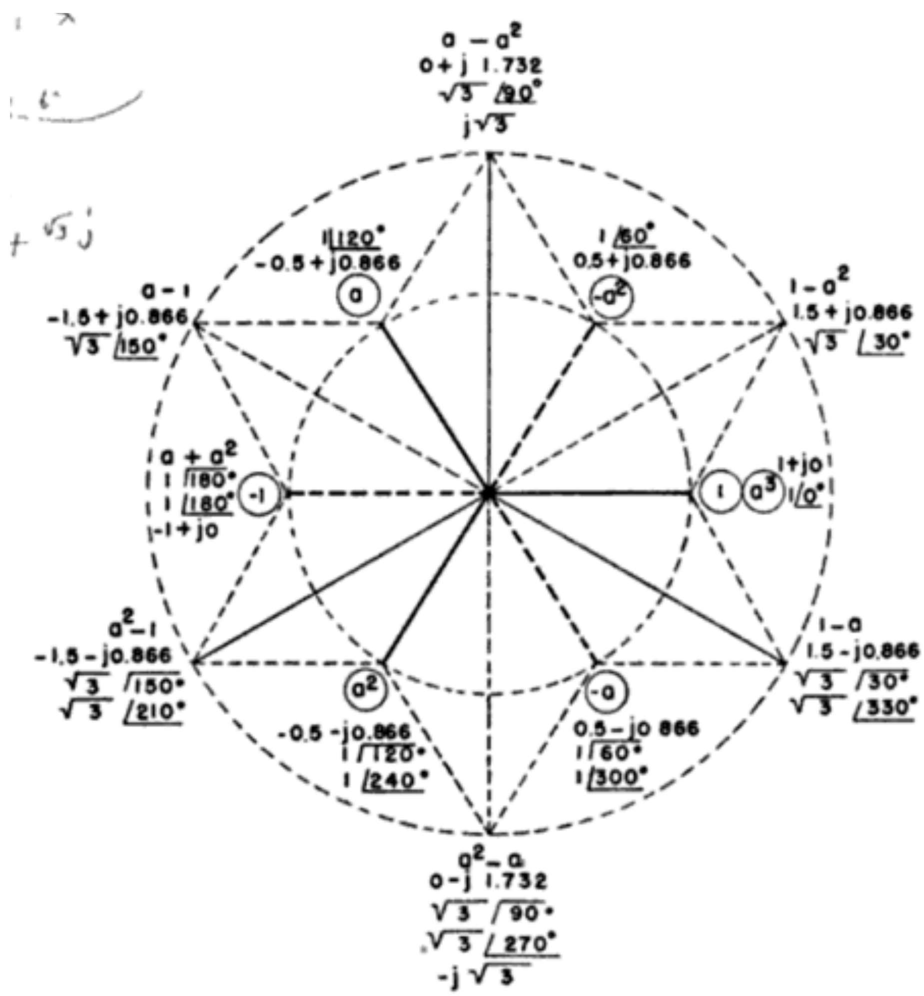


FIG. 3. Graphical representation of functions of the operator *a*.

Figure 1.27: Properties of α from [7, Fig. 3, p.9].

1. Suppose the entries x_j of $x := (x_1, x_2, x_3) \in \mathbb{C}^3$ have the same magnitude. Then x is balanced if and only if $x_1 + x_2 + x_3 = 0$.

2. The columns of F are orthonormal. Both F and $\overline{F}$ are complex symmetric, i.e., $F^{\mathsf{T}} = F$ and $\overline{F}^{\mathsf{T}} = \overline{F}$, where $\overline{F}$ is the complex conjugate of F componentwise. Hence

$$F^{-1} \; = \; F^{\mathsf{H}} \; = \; \overline{F} \; = \; \frac{1}{\sqrt{3}} \begin{bmatrix} \mathbf{1} & \alpha_- & \alpha_+ \end{bmatrix}$$

3. Γ is a normal matrix, $\Gamma\Gamma^{\mathsf{T}} = \Gamma^{\mathsf{T}}\Gamma$.

4. *Spectral decomposition of* Γ:

   (a) The eigenvalues and eigenvectors of Γ are

   $$\Gamma\mathbf{1} \; = \; 0, \qquad \Gamma\alpha_+ \; = \; (1-\alpha)\alpha_+, \qquad \Gamma\alpha_- \; = \; (1-\alpha^2)\alpha_- \tag{1.47}$$

   where $1 - \alpha = \sqrt{3}e^{\mathbf{i}\pi/6}$ and $1 - \alpha^2 = \sqrt{3}e^{-\mathbf{i}\pi/6}$.

   (b) Therefore the spectral decomposition of Γ is:

   $$\Gamma \; = \; F \begin{bmatrix} 0 & & \\ & 1-\alpha & \\ & & 1-\alpha^2 \end{bmatrix} \overline{F}$$

5. *Spectral decomposition of* Γ^{T}:

   (a) The eigenvalues and eigenvectors of Γ^{T} are

   $$\Gamma\mathbf{1} \; = \; 0, \qquad \Gamma\alpha_- \; = \; (1-\alpha)\alpha_-, \qquad \Gamma\alpha_+ \; = \; (1-\alpha^2)\alpha_+ \tag{1.48}$$

   where $1 - \alpha = \sqrt{3}e^{\mathbf{i}\pi/6}$ and $1 - \alpha^2 = \sqrt{3}e^{-\mathbf{i}\pi/6}$.

   (b) Therefore the spectral decomposition of Γ^{T} is:

   $$\Gamma^{\mathsf{T}} \; = \; \overline{F} \begin{bmatrix} 0 & & \\ & 1-\alpha & \\ & & 1-\alpha^2 \end{bmatrix} F \tag{1.49}$$

Exercise 1.6. Show that the voltage magnitude $|V^{cd}| = 208V$ in the split-phase Delta transformer in Figure 1.20(b), assuming the system is a balanced three-phase positive sequence.

Exercise 1.7. Consider the balanced three-phase system in Y configuration shown in Figure 1.28. Show that $V^{n_0 n_1} = 0$ provided $Z \neq -(Z_1 + L_1)/3$.[1]

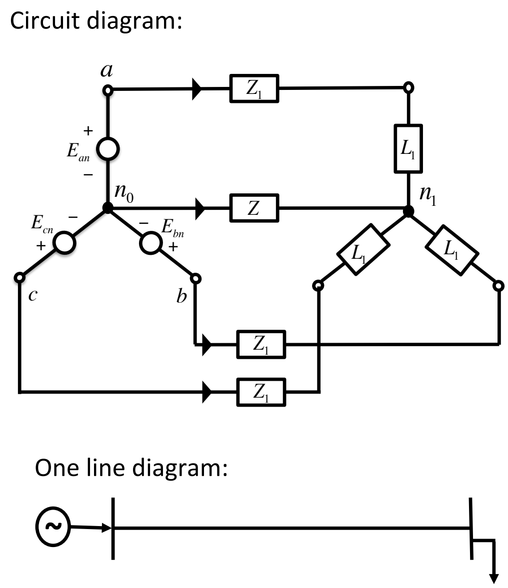


Figure 1.28: Balanced three-phase system in Y configuration where the impedances Z, Z_1, L_1 are given.

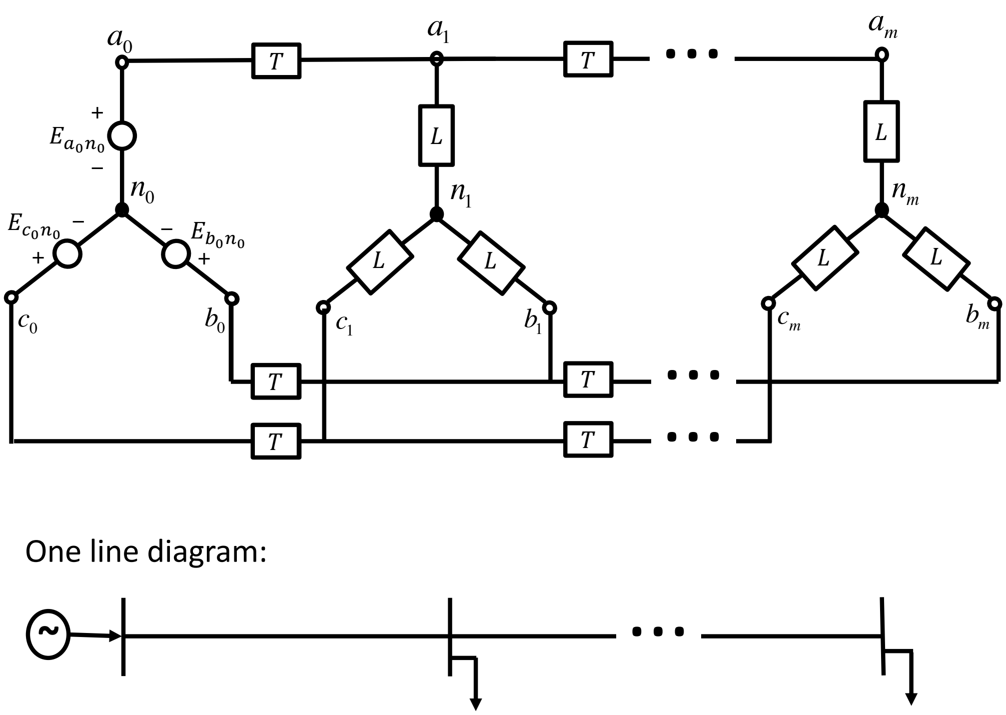


Figure 1.29: Balanced three-phase system in Y configuration where a three-phase voltage source in positive sequence supplies m three-phase loads in parallel.

Exercise 1.8 (Balanced Y loads)**.** Consider the balanced three-phase system in Y configuration shown in Figure 1.29 where a three-phase voltage source in positive sequence supplies m three-phase loads in parallel. All transmission lines have a common admittance $T = 1$ and all loads have a common admittance L. Consider the following $10m$ variables:

- a voltage and a current for each phase at each stage $k = 1, \ldots, m$:

$$\tilde{V}_k \;:=\; \begin{bmatrix} V^{a_k n_k} \\ V^{b_k n_k} \\ V^{c_k n_k} \end{bmatrix} \quad \text{and} \quad \tilde{I}_k \;:=\; \begin{bmatrix} I_{a_k n_k} \\ I_{b_k n_k} \\ I_{c_k n_k} \end{bmatrix}, \qquad k = 1, \ldots, m$$

  for a total of $6m$ variables.

- a current for each phase from stage $k-1$ to stage k:

$$\tilde{J}_{k-1,k} \;:=\; \begin{bmatrix} I_{a_{k-1} a_k} \\ I_{b_{k-1} b_k} \\ I_{c_{k-1} c_k} \end{bmatrix}, \qquad k = 1, \ldots, m$$

  for a total of $3m$ currents.

- a voltage between neutrals from stage $k-1$ to stage k: $V^{n_{k-1} n_k}$, $k = 1, \ldots, m$, for a total of m voltages.

1. Show that $V^{n_{k-1} n_k} = 0$ for $k = 1, \ldots, m$.
2. Show that

$$V^{a_k n_k} \;=\; \beta_k E^{a_0 n_0}, \qquad V^{b_k n_k} \;=\; \beta_k E^{b_0 n_0}, \qquad V^{c_k n_k} \;=\; \beta_k E^{c_0 n_0}, \qquad k = 1, \ldots, m$$

   where β_k is:

$$\beta_k \;:=\; \frac{r_1^k r_2^m (r_2 - 1) - r_2^k r_1^m (r_1 - 1)}{r_2^m (r_2 - 1) - r_1^m (r_1 - 1)}$$

   and r_1, r_2 are given by:

$$r_{1,2} \;=\; \frac{1}{2}\left((L+2) \pm \sqrt{L(L+4)} \right) \tag{1.50}$$

   (Hint: Derive a recursion on $\tilde{V}_k$ across stages k and solve the difference equation for each phase a, b, c separately.)

3. Show that $\tilde{V}_k, \tilde{I}_k, \tilde{J}_{k-1,k}$ are balanced positive-sequence sets for $k = 1, \ldots, m$.

---

[1] Suppose the impedances Z, Z_1, L_1 all have positive resistance, which is the case in practice. Then this condition is automatically satisfied. If $3Z = -(Z_1 + L_1)$ holds, however, then $V^{n_0 n_1}$ can take any value and Kirchhoff's laws will be satisfied because $I^{n_0 n_1} + I_a + I_b + I_c = 0$ will always be satisfied for any value of $V^{n_0 n_1}$.

Chapter 1.3.

Exercise 1.9. Consider power delivered to an impedance load Z. Show that

$$\begin{aligned} p(t) &= |V||I|\left(\cos\phi + \cos(2\omega t + \theta_V + \theta_I)\right) \\ &= P\ (1 + \cos 2(\omega t + \theta_I)) - Q\ \sin 2(\omega t + \theta_I) \end{aligned}$$

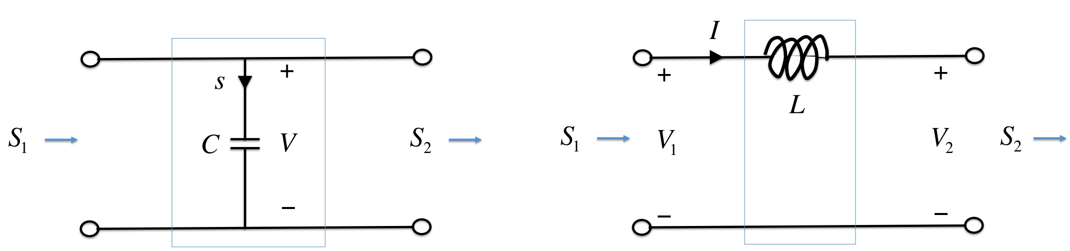
where $\phi := \theta_V - \theta_I$ is the power factor angle, $P := |V||I|\cos\phi$ is the real power and $Q := |V||I|\sin\phi$ is the reactive power.

Exercise 1.10. Suppose $Z = 1/\mathbf{i}\omega C$ (capacitance). Prove directly in time domain that the average delivered power is 0 and the magnitude of the instantaneous power is Q.

Exercise 1.11 (Power meter)**.** A power meter measures voltage and current magnitudes (rms values) $(|V|, |I|)$ and instantaneous power $p(t)$ over 1 or more period T. In addition to reporting $(|V|, |I|)$, it usually reports real and reactive power (P, Q), apparent power $|S|$, and power factor as well. Explain how to calculate these quantities.

Exercise 1.12. Consider Figure 1.30.

1. *Shunt capacitor is VAR source:* Prove that in Figure 1.30(a), $S_2 = S_1 + \mathbf{i}\omega C|V|^2$.
2. *Short transmission line is inductive:* Prove that in Figure 1.30(b), if $|V_2| = |V_1|$ then $S_2 = S_1^{\mathsf{H}}$.



(a) Shunt capacitor is VAR source (b) Short transmission line is inductive

Figure 1.30: Conservation of power

Chapter 2

Transmission line models

An electric network consists of transmission lines that transfer power from generators to loads. In this chapter we develop models for terminal behavior of a three-phase transmission line that map the voltage and current at one end of the line to those at the other end, in two steps. In Chapter 2.1 we derive inductance and capacitance parameters of a transmission line as functions of line geometry. In Chapter 2.2 we use these parameters to develop circuit models for short, medium, and long-distance transmission lines. These line models are building blocks for network models developed in later chapters.

2.1 Line characteristics

The alternating currents in the conductors of a three-phase transmission line create electromagnetic interactions among them that couple the voltages on, and currents and charges in these conductors. In a balanced operation however the interactions are as if the phases are decoupled. This allows per-phase analysis where, in each phase, the line can be characterized as a combination of a series impedance and a shunt admittance parameterized by:

$$\begin{array}{rrcll} \text{series impedance per meter} & z & := & r + \mathbf{i}\omega l & \Omega/\text{m} \\ \text{shunt admittance per meter to neutral} & y & := & g + \mathbf{i}\omega c & \Omega^{-1}/\text{m} \end{array}$$

In this section we present models for these per-meter line parameters (r, l) and (g, c). In the next section we will use these parameters to derive lumped-circuit models of the line. A three-phase line consists of multiple wires and therefore we need to derive the series inductance l and shunt capacitance c due to currents and charges in multiple wires. The key property that will be important in our derivation is that the set of wires carry currents in both directions so that the currents and charges in all the wires sum to zero at all times, as expressed in (2.2) and (2.5) below.

2.1.1 Series resistance r and shunt conductance g

The direct current (dc) resistance of a conductor is

$$r_{\text{dc}} \;:=\; \frac{\rho_T}{A} \qquad \Omega/\text{m}$$

where ρ_T is called the conductor resistivity at temperature T and A is the cross-sectional area of the conductor. Hence the per-meter resistance is inversely proportional to the size of the line. The alternating current (ac) resistance (or effective resistance) of a conductor is defined to be

$$r_{\text{ac}} \;:=\; \frac{P_{\text{loss}}}{|I|^2} \qquad \Omega/\text{m}$$

where P_{loss} is the real power loss in W and $|I|$ is the root-mean-square of the current in A in the conductor. The current distributes uniformly throughout the conductor's cross-sectional area for dc. For ac, the current density is lower at the conductor center and higher near the conductor surface. This is called the skin effect and is more pronounced at higher ac frequencies. As frequency increases, the real power loss, and hence the ac resistance, also increase. At 60 Hz the ac resistance is at most a few percent higher than dc resistance. These effects are modeled by the series resistance r in Ω/m in transmission line models.

Shunt conductance g in Ω^{-1}/m accounts for real power loss between conductors or between conductors and ground, typically due to either leakage currents at insulators or to corona. Insulator loss depends on the environment such as moisture level. Corona occurs when a strong electric field at a conductor surface ionizes the air, causing it to conduct. It depends on meteorological conditions such as rain. Losses due to insulator leakage and corona are typically negligible compared to resistance loss $r|I|^2$. It is therefore common to assume zero shunt conductance g in transmission line models.

2.1.2 Series inductance l

Roughly, the per-meter series inductance l in henrys/m of a wire is the proportionality constant between the current i in a meter of the wire and the total magnetic flux linkages λ, i.e., $\lambda(t) = li(t)$, where $i(t)$ is in ampere and λ is in webers. We now study how the per-meter series inductance l of a wire depends on the geometry of the transmission lines.

Single conductor. Consider a straight infinitely long wire of radius r with uniform current density in the wire with a total current i (dropping t from the notation for simplicity). The total flux linkages λ_R per meter of the wire within a radius R of the wire is related to the current i and the geometry by:

$$\lambda_R \;=\; \frac{\mu_0}{2\pi}\left(\frac{\mu_r}{4} + \ln\frac{R}{r}\right) i$$

where $\mu_0 := 4\pi \times 10^{-7}$ weber/ampere-meter is the permeability of free space, and μ_r is the relative permeability of the wire. If the conductor is nonmagnetic (e.g. copper or aluminum), then $\mu_r \approx 1$. The first term is due to flux linkages inside the wire and the second term is due to flux linkages outside the wire up to radius R. The details are explained in [1, pp.54–59].

Multiple conductors. We will calculate approximately the per-meter total flux linkages λ_1 of conductor 1 that carries a current i_1. The total flux linkages λ_1 is determined not only by current i_1, but also by currents i_k from other conductors $k = 2, \ldots, n$, that carry currents i_k and are at distances d_{1k} from the center of conductor 1. See Figure 2.1.

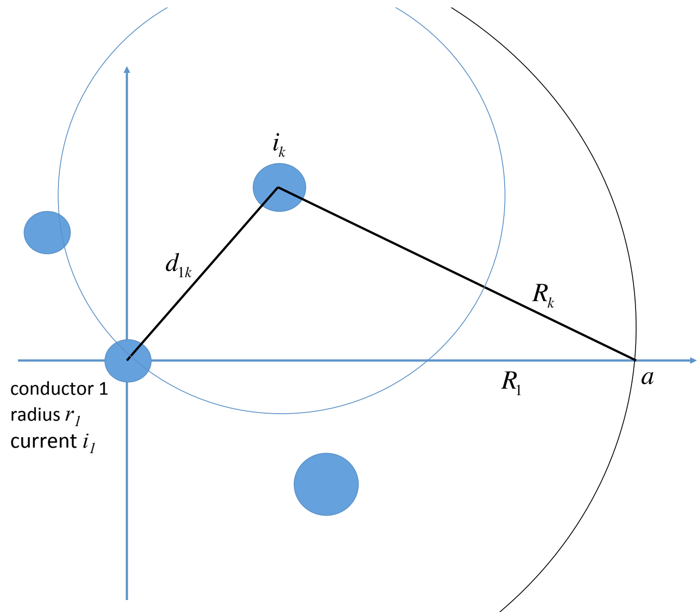


Figure 2.1: Per-meter total flux linkages in a volume within a radius R_1 from the center of conductor 1 due to all conductors. Conductors k carry currents i_k and their centers are distances d_{1k} from the center of conductor 1 and R_k from point a.

Denote by R_1 the distance of point a from the origin (center of conductor 1) and by R_k the distance of the center of conductor k from point a. Then the total flux linkages of conductor 1 is

$$\lambda_1 = \lim_{R_1 \to \infty} \frac{\mu_0}{2\pi} \left(i_1 \left(\frac{\mu_r}{4} + \ln \frac{R_1}{r_1} \right) + \sum_{k=2}^{n} i_k \ln \frac{R_k}{d_{1k}} \right) \tag{2.1}$$

where ln denotes the natural log. We make the key assumption

$$\sum_{k=1}^{n} i_k(t) = 0 \qquad \text{at all times } t \tag{2.2}$$

This is a reasonable assumption as in practice the lines carrying power from generation to load and the lines carrying the return currents follow the same physical path by design. The implication is that the magnetic inductances due to all the lines cancel each other at infinity. Formally, we add $-\ln R_1 \sum_{k=1}^{n} i_k$ into the bracket on the right-hand side of (2.1) to get

$$\lambda_1 = \lim_{R_1 \to \infty} \frac{\mu_0}{2\pi} \left(i_1 \left(\frac{\mu_r}{4} + \ln \frac{1}{r_1} \right) + \sum_{k=2}^{n} i_k \ln \frac{1}{d_{1k}} \right) + \frac{\mu_0}{2\pi} \sum_{k=1}^{n} i_k \ln \frac{R_k}{R_1}$$

As $R_1 \to \infty$, $\ln(R_k/R_1) \to 0$. Hence

$$\lambda_1 \;=\; \frac{\mu_0}{2\pi}\left(i_1 \,\ln\frac{1}{r_1'} \;+\; \sum_{k=2}^{n} i_k \,\ln\frac{1}{d_{1k}}\right)$$

where $r_1' := r_1 e^{-\mu_r/4}$ is the radius of an equivalent hollow conductor with the same flux linkages as the solid conductor of radius r. For a nonmagnetic wire, $\mu_r \approx 1$ and $r_1' \approx 0.78 r_1$.

In general the total flux linkages λ_k of conductor k depends not only on current i_k but currents $i_{k'}$ in other conductors as well, and is given by

$$\lambda_k \;=\; \left(\frac{\mu_0}{2\pi}\ln\frac{1}{r_k'}\right) i_k \;+\; \sum_{k' \neq k}\left(\frac{\mu_0}{2\pi}\ln\frac{1}{d_{kk'}}\right) i_{k'} \tag{2.3}$$

where $r_k' := r_k e^{-\mu_r/4}$. In vector form this is

$$\lambda \;=\; Li$$

where $\lambda := (\lambda_k, k = 1, \ldots, n)$, $i := (i_k, i = 1, \ldots, n)$, and the (k,k')-th entry of the $n \times n$ matrix L is

$$l_{kk'} \;=\; \begin{cases} \frac{\mu_0}{2\pi}\ln\frac{1}{r_k'} & \text{if} \quad k = k' \\ \frac{\mu_0}{2\pi}\ln\frac{1}{d_{kk'}} & \text{if} \quad k \neq k' \end{cases}$$

The voltage drop $v_k(t)$ between two points on conductor k that are separated by an infinitesimal distance is related to the rate of change of the total flux linkages $\lambda_k(t)$ (Faraday's law), i.e.,

$$v_k(t) \;=\; \frac{d}{dt}\lambda_k(t) \;=\; \sum_{k'} l_{kk'} \frac{d}{dt} i_{k'}(t)$$

This relation, in the phasor domain, is used in Chapter 2.2.1 to derive a circuit model of a transmission line. In a circuit model, the term

$$l_{kk} \;:=\; \frac{\mu_0}{2\pi}\,\ln\frac{1}{r_k'} \qquad \text{henrys/m}$$

is called the *self-inductance* per meter of conductor k and the term

$$l_{kk'} \;:=\; \frac{\mu_0}{2\pi}\,\ln\frac{1}{d_{kk'}} \qquad \text{henrys/m}$$

is called the *mutual inductances* per meter between conductors k and k'. The larger the conductor r_k the smaller the self-inductance l_k.

2.1.3 Shunt capacitance c

Roughly, the per-meter shunt capacitance c, in farads/m, of a wire is the proportionality constant between the charge q, in coulombs/m, in a meter of the wire and the voltage v on the surface of the wire, i.e.,

$q(t) = cv(t)$. We now study how the per-meter shunt capacitance c of a wire depends on the geometry of the transmission lines.

Consider the situation in Figure 2.1 with multiple conductors. A similar analysis to that in Chapter 2.1.2 shows that the voltage, with respect to a reference at infinity, at a point on the surface of conductor k is

$$v_k \;=\; \left(\frac{1}{2\pi\varepsilon}\ln\frac{1}{r_k}\right) q_k \;+\; \sum_{k'\neq k}\left(\frac{1}{2\pi\varepsilon}\ln\frac{1}{d_{kk'}}\right) q_{k'} \tag{2.4}$$

where ε is the permittivity of the medium ($\varepsilon = 8.854\times 10^{-12}$ farads/meter in free space and $\varepsilon \approx 1$ farad/meter in dry air). As before, r_k is the radius of conductor k and $d_{kk'}$ is the distance between the centers of conductors k and k'. Here q_k is the total charge per unit length of wire k in coulombs/m. In vector form this is

$$v \;=\; Fq$$

where $v := (v_k, k = 1,\ldots,n)$, $q := (q_k, k = 1,\ldots,n)$, and the (k,k')-th entry of the $n\times n$ matrix F is

$$f_{kk'} \;=\; \begin{cases} \frac{1}{2\pi\varepsilon}\ln\frac{1}{r_k} & \text{if} \quad k = k' \\ \frac{1}{2\pi\varepsilon}\ln\frac{1}{d_{kk'}} & \text{if} \quad k \neq k' \end{cases}$$

Taking time derivatives relates the currents in the conductors to the rate of change in a voltage on the surface of the conductor relative to the reference, $\dot{v} = Fi(t)$. Let $C := F^{-1}$. The diagonal entries c_{kk} of C are called self-capacitances per meter of conductor k and the off-diagonal entries $c_{kk'}$ of C are called mutual capacitances per meter between conductors k and k', in farads/m. The larger the conductor r_k the larger the self-capacitance c_{kk}.

The key assumption (among others) in deriving (2.4) is

$$\sum_{k=1}^{n} q_k(t) \;=\; 0 \qquad \text{at all times } t \tag{2.5}$$

Compare this assumption with the assumption (2.2), and the expressions (2.3) and (2.4).

Example 2.1. The voltage v_k in (2.4) is the potential, or voltage with respect to the reference at infinity, at a point on the surface of conductor k. The voltage difference v_{jk} between two points on the surfaces of two parallel conductors j and k that are on a plane perpendicular to conductor j is:

$$v_{jk} \;:=\; v_j - v_k \;=\; \frac{1}{2\pi\varepsilon}\left(q_j \ln\frac{d_{kj}}{r_j} - q_k \ln\frac{d_{jk}}{r_k} + \sum_{k'\neq j,k} q_{k'} \ln\frac{d_{kk'}}{d_{jk'}}\right)$$

2.1.4 Balanced three-phase line

Consider the simplest model of a symmetric three-phase transmission line in balanced operation, as shown in Figure 2.2, with the assumptions:

1. the conductors are equally spaced at D and have equal radii r;[1]
2. $i_a(t) + i_b(t) + i_c(t) = 0$ at all times t;
3. $q_a(t) + q_b(t) + q_c(t) = 0$ at all times t.

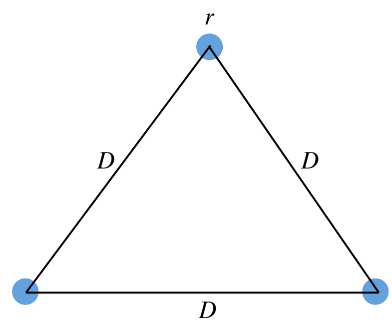


Figure 2.2: Per-meter inductance and capacitance of a symmetric three-phase transmission line in balanced operation.

It can be shown (see Exercise 2.1) that in this symmetric arrangement the effect of mutual inductances and capacitances among the transmission lines is particularly simple, resulting in the following equal per-phase inductance for each line:

$$l \quad = \quad \frac{\mu_0}{2\pi} \ln \frac{D}{r'} \qquad \text{H/m}$$

where $r' := re^{-\mu_r/4}$, and equal per-phase capacitance for each line:

$$c \quad = \quad \frac{2\pi\varepsilon}{\ln(D/r)} \qquad \text{F/m}$$

Note that l and c include not only the self-inductance and self-capacitance of the line, but also mutual inductances and capacitances. Two implications are as follows.

1. Although there is magnetic coupling between phases, the conditions $i_a(t) + i_b(t) + i_c(t) = 0$, $q_a(t) + q_b(t) + q_c(t) = 0$ and the symmetry (equal radii r and distances D) reduce the effect of the magnetic coupling to the term $\ln D$. This allows us to model the magnetic effect *as if* it consists of only self-inductance and electric effect *as if* it consists of only self-capacitance. Moreover, the inductances and capacitances are equal for each phase, permitting per-phase analysis.
2. To reduce the impedance per meter due to inductance or capacitance, we can reduce the spacing D or increase the wire radius r. Both have limitations. Other techniques are used in practice to approximate condition 1 above on the symmetry of line geometry, e.g., conductor bundling and transposition of the transmission lines.

[1]We use r to denote both the per-meter series resistance and the radius of the conductor; the meaning should be clear from the context.

Consider any point p that is equidistance from the centers of the conductors a,b,c, e.g., the point at the center of the triangle in Figure 2.2. The potential, or the voltage relative to the reference point at infinity, at this point p can be shown to be

$$v_p \;=\; \frac{1}{2\pi\varepsilon}\left(q_a \ln\frac{1}{d_{pa}} + q_b \ln\frac{1}{d_{pb}} + q_c \ln\frac{1}{d_{pc}}\right) \tag{2.6}$$

where $d_{pa} = d_{pb} = d_{pc}$ are the distances between p and the centers of the conductors. Since $q_a + q_b + q_c = 0$ we have $v_p = 0$, and hence p has the same potential as the reference point at infinity and can therefore be taken as the reference point. We will construct an imaginary geometric line parallel to the conductors pass through the equidistance point from these conductors. Every point on this line is the reference potential. By default we will pick this as the neutral potential that defines the phase-to-neutral voltages. The current supplied to the transmission line capacitance is called the *charging current* and the corresponding capacitance is also called the *line charging*. Figure 2.3 shows the corresponding circuit model of a transmission line. When the phase a line-to-neutral voltage is V_{an} the phase a charging current

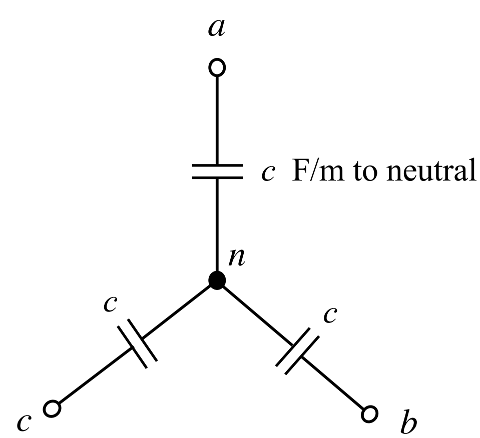


Figure 2.3: Circuit model of the cross section of a balanced three-phase transmission line.

is

$$I_{a,\text{charging}} \;=\; \mathbf{i}\omega c V_{an} \qquad \text{A/m}$$

from phase a conductor to neutral.

2.2 Line models

Consider a three-phase transmission line in balanced operation in sinusoidal steady state, modeled as in Figure 2.3. A key conclusion of Chapter 2.1.4 is that for balanced three-phase lines, we can analyze each phase separately. Consider now a transmission line on one of the phases. Let

$$\begin{aligned}
\text{series impedance per meter } z &:= r + \mathbf{i}\omega l \qquad \Omega/\text{m} \\
\text{shunt admittance per meter to neutral } y &:= g + \mathbf{i}\omega c \qquad \Omega^{-1}/\text{m}
\end{aligned}$$

where the per-meter resistance $r > 0$ and conductance $g > 0$ depend on the material and size of the line, and the per-meter inductance $l > 0$ and parameter $c > 0$ of the line can be calculated as in Chapters 2.1.2–2.1.4. In this section we derive two equivalent models of a balanced three-phase transmission line. The first model represents the terminal behavior, i.e., the mapping of the voltage and current between one end of the line and those at the other end, by a transmission matrix in (2.9) below. The second model represents the terminal behavior of the line by a linear circuit with series impedance and shunt admittances given in (2.14) below.

2.2.1 Transmission matrix

Distributed-element model. We start by deriving the V-I relations between two ends of a transmission line. Figure 2.4 shows a per-phase model of a balanced three-phase line of length ℓ. The voltages are phase (line-to-neutral) voltages as illustrated in Figure 2.3. We will call the left end the sending end and the right end the receiving end. When we apply a voltage V_1, with respect to neutral, at the sending end driving a current I_1 towards the receiving end, the voltage drops and the current leaks from the sending end to the receiving end so that the voltage $V(x)$ and current $I(x)$ at each point x of the line vary. We will derive a relation between the sending end (V_1, I_1) and the receiving end (V_2, I_2) by solving for $(V(x), I(x))$ in terms of (V_2, I_2) for all $0 \leq x \leq \ell$.

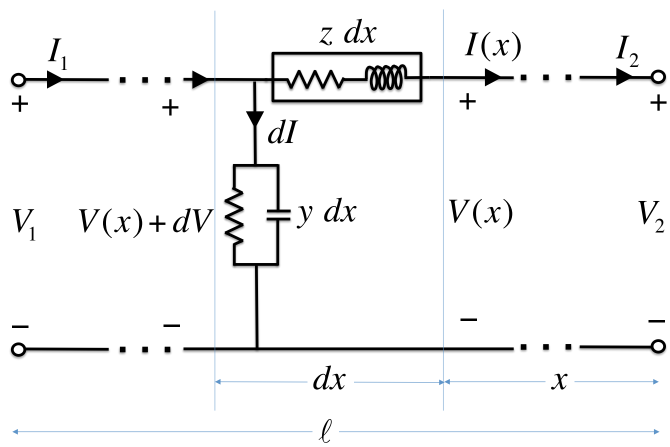


Figure 2.4: Per-phase model of a balanced three-phase line of length ℓ with impedance parameters z, y.

To this end consider the infinitesimal segment of length dx at a distance x from the receiving end. This segment is modeled by the circuit with series impedance zdx and shunt admittance ydx to neutral as shown in Figure 2.4. Let the voltage and current at point x be $V := V(x)$ and $I := I(x)$ respectively. Let the corresponding quantities at point $x + dx$ be $V(x) + dV$ and $I(x) + dI$. Applying Kirchhoff's laws to the segment, we have

$$\begin{aligned} dV &= zI(x)\, dx \\ dI &= (V(x) + dV)y\, dx \;\approx\; yV(x)\, dx \end{aligned}$$

where the approximation results from ignoring the second-order term $dV\,dx$. Hence we have

$$\begin{bmatrix} \frac{dV}{dx} \\ \frac{dI}{dx} \end{bmatrix} = \begin{bmatrix} 0 & z \\ y & 0 \end{bmatrix} \begin{bmatrix} V \\ I \end{bmatrix} \tag{2.7}$$

Transmission matrix. The ordinary differential equation (2.7) can be easily solved using standard method (see below for details), and the general solution is:

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = U \begin{bmatrix} e^{\gamma x} & 0 \\ 0 & e^{-\gamma x} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \tag{2.8a}$$

for some constants k_1, k_2, where

$$U := \begin{bmatrix} Z_c & -Z_c \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad U^{-1} := \frac{1}{2Z_c} \begin{bmatrix} 1 & Z_c \\ -1 & Z_c \end{bmatrix} \tag{2.8b}$$

Here

$$Z_c := \sqrt{\frac{z}{y}} \quad \Omega m^{-1} \qquad \text{and} \qquad \gamma := \sqrt{zy} \quad m^{-1} \tag{2.8c}$$

are called the *characteristic impedance* and *propagation constant* of the line respectively. At $x = 0$, $V(0) = V_2$ and $I(0) = I_2$. From (2.8) we have

$$\begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = U \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

and hence

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = U \begin{bmatrix} e^{\gamma x} & 0 \\ 0 & e^{-\gamma x} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = U \begin{bmatrix} e^{\gamma x} & 0 \\ 0 & e^{-\gamma x} \end{bmatrix} U^{-1} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

The sending-end voltage and current are therefore related to the receiving-end (V_2, I_2) as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = U \begin{bmatrix} e^{\gamma \ell} & 0 \\ 0 & e^{-\gamma \ell} \end{bmatrix} U^{-1} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Expanding, we have

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh(\gamma\ell) & Z_c \sinh(\gamma\ell) \\ Z_c^{-1} \sinh(\gamma\ell) & \cosh(\gamma\ell) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{2.9}$$

where $\cosh x := (e^x + e^{-x})/2$ and $\sinh x := (e^x - e^{-x})/2$. This defines a linear mapping that maps the voltage and current (V_2, I_2) at the receiving end to the voltage and current (V_1, I_1) at the sending end. The matrix in (2.9) is called a *transmission matrix*.

The ratio V_1/I_1 at the sending end is called the *driving-point impedance*. It is the equivalent impedance across the two sending-end terminals.

Example 2.2 (Driving-point impedance)**.** Consider the terminal model (2.9) of a transmission line. Suppose the receiving end is connected to an impedance load Z_l. Show that the driving-point impedance V_1/I_1 is equal to the characteristic impedance Z_c of the line under one of the following conditions:

- if the load is matched to the line, i.e., $Z_l = Z_c$; or
- if the line length ℓ grows to infinity, since the line parameters satisfy $r, x, g, c > 0$.

The second condition implies that as the line grows in length its impedance comes to dominate the load impedance Z_l.

Solution. Since $V_2 = Z_l I_2$, we have from (2.9) that when $Z_l = Z_c$

$$\frac{V_1}{I_1} \;=\; Z_c \frac{\cosh(\gamma\ell) + \sinh(\gamma\ell)}{\sinh(\gamma\ell) + \cosh(\gamma\ell)} \;=\; Z_c$$

For the second case, we have from (2.9)

$$\frac{V_1}{I_1} \;=\; Z_c \frac{Z_l \cosh(\gamma\ell) + Z_c \sinh(\gamma\ell)}{Z_l \sinh(\gamma\ell) + Z_c \cosh(\gamma\ell)} \;=\; Z_c \frac{Z_l + Z_c \tanh(\gamma\ell)}{Z_l \tanh(\gamma\ell) + Z_c}$$

Now $\gamma = \sqrt{zy} =: \sqrt{\hat{\gamma}}$ where $\hat{\gamma} := (rg - \omega^2 lc) + \mathbf{i}\omega(rc + gl)$. Note that $\operatorname{Im}\hat{\gamma} > 0$ and hence $\angle\hat{\gamma} \in (0, \pi)$ and $\gamma \in (0, \pi/2)$. If we write $\gamma =: \alpha + \mathbf{i}\beta$ then $\alpha > 0$. Hence

$$\begin{aligned}
\cosh(\gamma\ell) &= \frac{1}{2}\left(e^{\gamma\ell} + e^{-\gamma\ell}\right) = \frac{1}{2}\left(e^{(\alpha+\mathbf{i}\beta)\ell} + e^{-(\alpha+\mathbf{i}\beta)\ell}\right) \\
\sinh(\gamma\ell) &= \frac{1}{2}\left(e^{\gamma\ell} - e^{-\gamma\ell}\right) = \frac{1}{2}\left(e^{(\alpha+\mathbf{i}\beta)\ell} - e^{-(\alpha+\mathbf{i}\beta)\ell}\right)
\end{aligned}$$

and

$$\tanh(\gamma\ell) \;=\; \frac{e^{(\alpha+\mathbf{i}\beta)\ell} - e^{-(\alpha+\mathbf{i}\beta)\ell}}{e^{(\alpha+\mathbf{i}\beta)\ell} + e^{-(\alpha+\mathbf{i}\beta)\ell}} \;=\; \frac{1 - e^{-2(\alpha+\mathbf{i}\beta)\ell}}{1 + e^{-2(\alpha+\mathbf{i}\beta)\ell}} \;\to\; 1 \quad \text{as} \quad \ell \to \infty$$

Hence $V_1/I_1 \to Z_c$ as $\ell \to \infty$. $\square$

Example 2.3 (Matched load)**.** Suppose the line is terminated in its characteristic impedance Z_c, i.e., $V_2 = Z_c I_2$. Then (2.9) yields

$$\begin{aligned}
V_1 &= (\cosh(\gamma\ell) + \sinh(\gamma\ell))\, V_2 = V_2\, e^{\gamma\ell} \\
I_1 &= (\cosh(\gamma\ell) + \sinh(\gamma\ell))\, I_2 = I_2\, e^{\gamma\ell}
\end{aligned}$$

Therefore the driving-point impedance V_1/I_1 is also the characteristic impedance Z_c of the line. Moreover the ratio of the receiving to sending end voltages and currents are

$$\frac{V_2}{V_1} = \frac{I_2}{I_1} = e^{-\gamma\ell}$$

The ratio of the receiving power to the sending power is:

$$\frac{-S_{21}}{S_{12}} \;=\; \frac{V_2 I_2^*}{V_1 I_1^*} = e^{-\gamma\ell}\left(e^{-\gamma\ell}\right)^*$$

Writing $\gamma = \sqrt{zy} \; = \; \sqrt{(rg - \omega^2 lc) + \mathbf{i}\omega(rc + gl)} \; =: \; \alpha + \mathbf{i}\beta$, we have

$$\frac{-S_{21}}{S_{12}} \;=\; e^{-2\alpha\ell}$$

Since $e^{-2\alpha\ell}$ is real, the powers have the same phase angle $\angle(-S_{21}) = \angle S_{12} =: \theta$. This implies that the transmission efficiency has the same ratio in terms of real power $-P_{21}$ received and real power P_{12} sent:

$$\frac{-P_{21}}{P_{12}} \;=\; \frac{-S_{21}\cos\theta}{S_{12}\cos\theta} \;=\; e^{-2\alpha\ell}$$

Hence for an impedance load that is matched to the line impedance Z_c, the transmission efficiency η decreases exponential in the line length ℓ. For high-voltage transmission lines, $\alpha \approx 0$ so the loss is small and $\eta \approx 1$.

Indeed, for a lossless line, $r = g = 0$. Then $z = \mathbf{i}\omega l$ and $y = \mathbf{i}\omega c$. Hence

$$Z_c \;=\; \sqrt{\frac{z}{y}} = \sqrt{\frac{l\ell}{c\ell}} = \sqrt{\frac{L}{C}}$$

is real, where L is the total inductance of the line and C the total capacitance of the line, and

$$\gamma \;=\; \sqrt{zy} \;=\; \mathbf{i}\omega\sqrt{lc}$$

is purely imaginary ($\alpha = 0$). The transmission efficiency is $\eta = -P_{21}/P_{12} = 1$. We will study lossless lines in more detail in Chapter 2.2.4. $\square$

Solution of (2.7)**.** First we note that even though (V, I) and the parameters (y, z) are complex variables, the variable x (distance from terminal 2) is a real variable. Hence the ordinary differential equation (ode) (2.7) can be solved in the same way as an ode in the real domain. To see this consider a general ode:

$$\dot{z} \;:=\; \frac{dz}{dt} \;=\; Mz \tag{2.10}$$

where $z := x + jy \in \mathbb{C}^n$ with x, y in $\mathbb{R}^n$ and $M := A + jB \in \mathbb{C}^{n\times n}$ with A, B in $\mathbb{R}^{n\times n}$, with the interpretation $\dot{x} + j\dot{y} = (A + jB)(x + jy)$. Rewrite this in the real domain:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} A & -B \\ B & A \end{bmatrix}}_{\tilde{M}} \begin{bmatrix} x \\ y \end{bmatrix} \tag{2.11}$$

Two matrices

$$M \;=\; A + jB \qquad \text{and} \qquad \tilde{M} \;=\; \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

are *equivalent*, written $M \leftrightarrow \tilde{M}$, in the sense that for any $z = x + \mathbf{i}y$ with $x, y \in \mathbb{R}^n$,

$$\begin{bmatrix} \mathrm{Re}(Mz) \\ \mathrm{Im}(Mz) \end{bmatrix} \;=\; \tilde{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

Since

$$M^2 \;=\; \left(A^2 - B^2\right) \;+\; j\,(AB + BA) \qquad \text{and} \qquad \tilde{M}^2 \;=\; \begin{bmatrix} A^2 - B^2 & -(AB+BA) \\ AB+BA & A^2 - B^2 \end{bmatrix}$$

we have $\tilde{M}^2 \leftrightarrow M^2$, and by induction $\tilde{M}^k \leftrightarrow M^k$ for all k. Hence $e^{\tilde{M}} \leftrightarrow e^{M}$. This implies that a trajectory $z(t) \in \mathbb{C}^n$ is a solution of (2.10) if and only if $(x(t), y(t)) \in \mathbb{R}^{2n}$ with $z(t) =: x(t) + \mathbf{i}y(t)$ is a solution of (2.11). Hence solving (2.11) using $\tilde{M}$ in the real domain is equivalent to solving (2.10) using M directly in the complex domain.

We now solve the ode (2.7). Let

$$A \;:=\; \begin{bmatrix} 0 & z \\ y & 0 \end{bmatrix}$$

Then the eigvenvalues of A are $\pm\gamma$ where $\gamma := \sqrt{yz}$ is the propagation constant defined in (2.8c). Recall the characteristic impedance of the line $Z_c := \sqrt{\frac{z}{y}}$ also defined in (2.8c). The corresponding eigvenvectors are (any vectors proportional to) the columns of the matrix U defined in (2.8b). Let U^{-1} be its inverse. Since $AU = U\text{diag}(\gamma, -\gamma)$, if we define

$$\begin{bmatrix} \tilde{V}(x) \\ \tilde{I}(x) \end{bmatrix} \;:=\; U^{-1} \begin{bmatrix} V(x) \\ I(x) \end{bmatrix} \tag{2.12}$$

then

$$\frac{d}{dx}\begin{bmatrix} \tilde{V} \\ \tilde{I} \end{bmatrix} \;=\; U^{-1}\frac{d}{dx}\begin{bmatrix} V \\ I \end{bmatrix} \;=\; U^{-1}A\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} \;=\; U^{-1}AU\left(U^{-1}\begin{bmatrix} V(x) \\ I(x) \end{bmatrix}\right) \;=\; \text{diag}(\gamma, -\gamma)\begin{bmatrix} \tilde{V}(x) \\ \tilde{I}(x) \end{bmatrix}$$

i.e., $\tilde{V}$ and $\tilde{I}$ are decoupled. Hence

$$\tilde{V}(x) \;=\; k_1 e^{\gamma x} \quad \text{and} \quad \tilde{I}(x) \;=\; k_2 e^{-\gamma x}$$

for some constants k_1, k_2. Then (2.12) implies that the general solution of (2.7) is (2.8). □

2.2.2 Lumped-element Π-circuit model

If we are only interested in the terminal voltages and currents of a line, then we can represent the line by a lumped-circuit model as shown in Figure 2.5 that consists of a series impedance Z' and a shunt admittance $Y'/2$ at each end of the line. This is called the Π *model* or Π-*circuit model* of a transmission line. We now derive the parameters (Z', Y') in the Π model in terms of line characteristics (Z_c, γ).

Applying Kirchhoff's laws we have

$$\begin{aligned} I_1 &= \frac{Y'}{2}V_1 + \frac{Y'}{2}V_2 + I_2 \\ V_1 - V_2 &= Z'\left(\frac{Y'}{2}V_2 + I_2\right) \end{aligned}$$

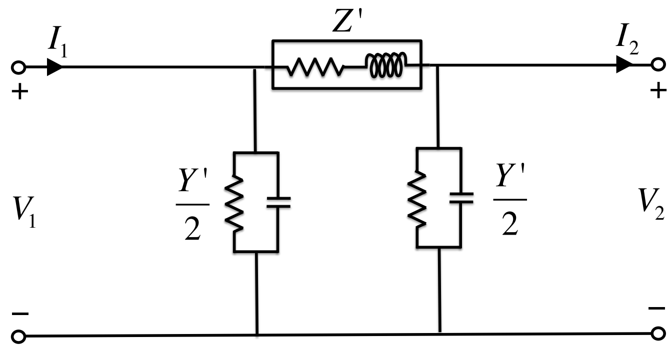


Figure 2.5: Lumped-circuit Π model of a transmission line.

Hence

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1+Z'Y'/2 & Z' \\ Y'(1+Z'Y'/4) & 1+Z'Y'/2 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{2.13}$$

Comparing (2.13) and (2.9) we find that the Π model in Figure 2.5 is given by:

$$Z' = Z_c \sinh(\gamma\ell) = \sqrt{\frac{z}{y}} \sinh(\gamma\ell) = Z\,\frac{\sinh(\gamma\ell)}{\gamma\ell} \tag{2.14a}$$

$$\frac{Y'}{2} = \frac{1}{Z_c}\,\frac{\cosh(\gamma\ell)-1}{\sinh(\gamma\ell)} = \frac{1}{Z_c}\,\frac{\sinh(\gamma\ell/2)}{\cosh(\gamma\ell/2)} = \frac{Y}{2}\,\frac{\tanh(\gamma\ell/2)}{\gamma\ell/2} \tag{2.14b}$$

where $Z := z\ell$ is the total series impedance of the line and $Y := y\ell$ is the total shunt admittance to neutral of the line.

When $|\gamma\ell| \ll 1$ then $\sinh(\gamma\ell)/(\gamma\ell) \approx 1$ and $\tanh(\gamma\ell/2)/(\gamma\ell/2) \approx 1$, in which case the Π model in Figure 2.5 can be approximated by the total series impedance Z and total shunt admittance Y to neutral of the line.

In summary each phase of a balanced three-phase transmission line can be modeled as follows:

- *Long line ($\ell > 150$ miles approximately):* Use either (2.9) or the Π circuit model with Z' and Y' given by (2.14).
- *Medium line ($50 < \ell < 150$ miles approximately):* Use the Π circuit model with $Z := z\ell$ and $Y := y\ell$ instead of Z' and Y'. Here $Z = R + \mathbf{i}\omega L$ is the total series impedance of the line and $Y = \mathbf{i}\omega C$ is the total shunt admittance to neutral of the line. In particular, for medium lines, the shunt resistance is negligible.
- *short line ($\ell < 50$ miles approximately):* Use the Π circuit model with Z only and neglect Y.

2.2.3 Real and reactive line losses

The power injected at terminal 1 towards terminal 2 and that at terminals 2 towards 1 are (from Kirchhoff's laws):

$$
\begin{aligned}
S_{12} &:= V_1 I_1^H = \left(\frac{1}{Z'}\right)^H \left(|V_1|^2 - V_1 V_2^H\right) + \left(\frac{Y'}{2}\right)^H |V_1|^2 \\
S_{21} &:= V_2(-I_2)^H = \left(\frac{1}{Z'}\right)^H \left(|V_2|^2 - V_2 V_1^H\right) + \left(\frac{Y'}{2}\right)^H |V_2|^2
\end{aligned}
$$

They are not negatives of each other because of power loss along the line. Indeed the total complex power loss is their sum:

$$
S_{12} + S_{21} = \left(\frac{1}{Z'}\right)^H |V_1 - V_2|^2 + \left(\frac{Y'}{2}\right)^H \left(|V_1|^2 + |V_2|^2\right) = Z'|I_{12}^s|^2 + \left(\frac{Y'}{2}\right)^H \left(|V_1|^2 + |V_2|^2\right)
$$

where I_{12}^s denotes the current through the series impedance Z'. The first term on the right-hand side is loss due to series impedance and the last term are losses due to shunt admittances of the line. Suppose $Z' = R^s + \mathbf{i}X^s$ and the shunt admittance is purely capacitive, i.e., $Y' = \mathbf{i}B^m$ with $R^s, X^s, B^m > 0$. Then, over the transmission line,

$$
\begin{aligned}
\text{real power loss} \quad \mathrm{Re}\,(S_{12} + S_{21}) &= R^s |I_{12}^s|^2 \\
\text{reactive power loss} \quad \mathrm{Im}\,(S_{12} + S_{21}) &= X^s |I_{12}^s|^2 - \frac{B^m}{2}\left(|V_1|^2 + |V_2|^2\right)
\end{aligned}
$$

Remark 2.1 (High voltage reduces line loss)**.** Consider a load supplied by a source through a transmission line modeled by a series impedance $R + \mathbf{i}X$ and zero shunt admittances. Suppose the load draws an active power P_{load} with power factor $\cos\phi$ at a specified voltage magnitude $|V_{\text{load}}|$. It can be shown that, given a desired active load power P_{load}, the active line loss P_{line} is inversely proportional to the square of the load voltage magnitude $|V_2|$ and its power factor $\cos\phi$ (Exercise 2.7):

$$
P_{\text{line}} = R|I_{\text{load}}|^2 = R\,\frac{P_{\text{load}}^2}{|V_2|^2 \cos^2\phi}
$$

Therefore a higher voltage (magnitude) reduces line loss.

Note that the higher voltage refers to the voltage $|V_2|$ across the load (and eventually the source voltage $|V_1|$), not the voltage across the transmission line which is $|V_1 - V_2|$; see Figure 2.5. It is derived in Example 1.7 that, given a desired load power, the active line loss is inversely proportional to the load voltage magnitude $|V_2|$, rather than $|V_2|^2$. This is because, in Exercise 2.7, the line resistance R is given and independent of load power and voltage $|V_2|$, whereas, in Example 1.7, the line resistance R is chosen to be proportional to $|V_2|$ (reducing the dependence of line loss $R|I_{\text{load}}|^2$ from $|V_2|^2$ to $|V_2|$). □

2.2.4 Lossless line

In this subsection we look at some properties of a lossless line, i.e., when $r = g = 0$. A lossless line is an important model because a high-voltage transmission line typically has very small power loss compared with the power flow on the line, and can be modeled as a lossless line. As noted above we have

$$\begin{aligned} Z_c &= \sqrt{\frac{z}{y}} = \sqrt{\frac{\mathbf{i}\omega l}{\mathbf{i}\omega c}} = \sqrt{\frac{l}{c}} \quad \Omega \\ \gamma &= \sqrt{zy} = \sqrt{(\mathbf{i}\omega l)(\mathbf{i}\omega c)} = \mathbf{i}\omega\sqrt{lc} =: \mathbf{i}\beta \quad m^{-1} \end{aligned}$$

with $\beta := \omega\sqrt{lc}$. Therefore the characteristic impedance Z_c is purely resistive while the propagation constant γ is purely reactive. The characteristic impedance Z_c is called a *surge impedance* for a lossless line. This implies

$$\cosh(\gamma x) = \cos(\beta x) \quad \text{and} \quad \sinh(\gamma x) = \mathbf{i}\sin(\beta x)$$

Π-circuit model. Substituting Z_c and γ into (2.9) the transmission matrix reduces to

$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = \begin{bmatrix} \cosh(\gamma x) & Z_c\sinh(\gamma x) \\ Z_c^{-1}\sinh(\gamma x) & \cosh(\gamma x) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} \cos(\beta x) & \mathbf{i}Z_c\sin(\beta x) \\ \mathbf{i}Z_c^{-1}\sin(\beta x) & \cos(\beta x) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{2.15}$$

for $x \in [0, \ell]$. The circuit elements Z' and Y' in the Π circuit model of a transmission line reduces to (from (2.14)):

$$Z' = Z_c\sinh(\gamma\ell) = \mathbf{i}Z_c\sin(\beta\ell) =: \mathbf{i}X \quad \Omega \tag{2.16a}$$

$$\frac{Y'}{2} = \frac{Y}{2}\frac{\tanh(\gamma\ell/2)}{\gamma\ell/2} = \frac{Y}{2}\frac{\tan(\beta\ell/2)}{\beta\ell/2} =: \mathbf{i}\frac{\omega C'}{2} \quad \Omega^{-1} \tag{2.16b}$$

where $Y := \mathbf{i}\omega c\ell$ and $C' := c\ell\left(\tan(\beta\ell/2)/(\beta\ell/2)\right)$. If ℓ is small then $C' \approx c\ell$. When $\beta\ell < \pi$ radian, both $Z' > 0$ and $Y' > 0$, i.e., the series impedance is purely inductive and the shunt admittances are purely capacitive. In practice, for overhead lines, $1/\sqrt{lc} \approx 3 \times 10^8$ ms^{-1}. At 60 Hz (using $\beta := \omega\sqrt{lc}$)

$$\frac{\pi}{\beta} = \frac{\pi}{2\pi(60)\sqrt{lc}} \approx 2,500 \text{ km}$$

Hence a lossless overhead transmission line less than 2,500 km can be modeled by the simple circuit in Figure 2.6 where X and C' are given in (2.16). It is a model for either a single-phase line or the phase-to-

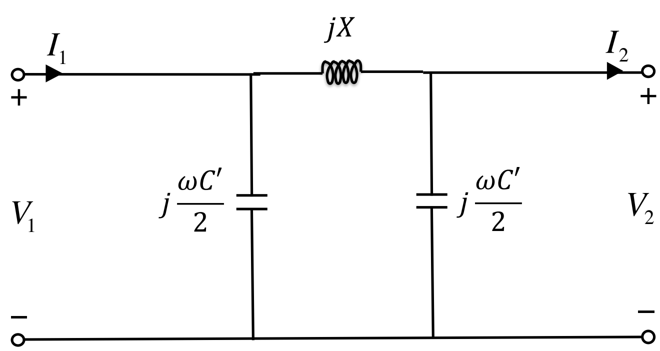


Figure 2.6: Π circuit model for a lossless line with length $\ell < \pi/\beta$.

neutral of a balanced three-phase line.

Voltage profile. Usually power must be delivered to a load at a specified nominal voltage magnitude $|V_2|$ at the load. To see how the voltage magnitude changes along a line from the source $x = \ell$ to the load $x = 0$, we determine the voltage $V(x)$ for $x \in [0, \ell]$ using (2.15):

$$V(x) \quad = \quad V_2 \cos(\beta x) \, + \, \mathbf{i} Z_c I_2 \sin(\beta x) \tag{2.17}$$

Suppose the line terminates at an impedance load $Z_{\text{load}} := R_{\text{load}} + \mathbf{i}X_{\text{load}}$. Then the voltage $V(x)$ at each point x depends on the load impedance because $V_2 = Z_{\text{load}} I_2$. There are four cases of load impedance:

1. *No load* $I_2 = 0$: $V(x) = V_2 \cos(\beta x)$ is real. Hence the voltage magnitude $V(x)$ increases from the source at $x = \ell$ to the end of the line at $x = 0$ as long as $\beta\ell < \pi/2$ radian.
2. *Surge impedance load* $Z_{\text{load}} = Z_c$: The voltage magnitude $|V(x)|$ is constant. Moreover the power delivered $S(x)$ at every point $x \in [0, \ell]$ is real and constant $|V_2|^2/Z_c$, so only real power is delivered. See Exercise 2.4.
3. *Full load*: Since $I_2 = V_2/Z_{\text{load}}$ we have
$$\begin{aligned} V(x) &= \left(\cos(\beta x) \, + \, \mathbf{i}\frac{Z_c}{Z_{\text{load}}} \sin(\beta x)\right) V_2 \\ &= \left(\cos(\beta x) + \frac{Z_c X_{\text{load}}}{|Z_{\text{load}}|^2} \sin(\beta x) \, + \, \mathbf{i}\frac{Z_c R_{\text{load}}}{|Z_{\text{load}}|^2} \sin(\beta x)\right) V_2 \end{aligned} \tag{2.18}$$
In Exercise 2.5 we derive for special cases sufficient conditions under which the voltage magnitude $|V(x)|$ decreases from the source at $x = \ell$ to the load Z_{load} at $x = 0$.
4. *Short circuit* $V_2 = 0$: $V(x) = \mathbf{i} Z_c I_2 \, \sin(\beta x)$. Hence the voltage magnitude $|V(x)|$ decreases from the source at $x = \ell$ to the load at $x = 0$ as long as $\beta\ell < \pi/2$ radian.

This is illustrated in Figure 2.7. The general trend of decreasing voltage magnitude towards the load (case 3 above) can be problematic because loads are generally designed to work with specific voltages. As mentioned above low load voltage also increases line loss in the network. Voltages are regulated tightly around their nominal values through various voltage compensation devices in generating units and inside the network.

Example 2.4 (Steady-state stability limit)**.** To derive the power delivered to a generic load we have from (2.16) that

$$I_2 \quad = \quad \frac{V_1 - V_2}{\mathbf{i}X} \, - \, \mathbf{i}\frac{\omega C'}{2} V_2$$

Hence the complex power delivered is

$$-S_{21} \quad = \quad V_2(I_2^*) \quad = \quad -\left(\frac{|V_2|^2 - V_2 V_1^*}{-\mathbf{i}X} \, - \, \mathbf{i}\frac{\omega C'}{2}|V_2|^2\right)$$

and the real power delivered is

$$-P_2 \quad = \quad \frac{|V_1||V_2|}{X} \sin\delta$$

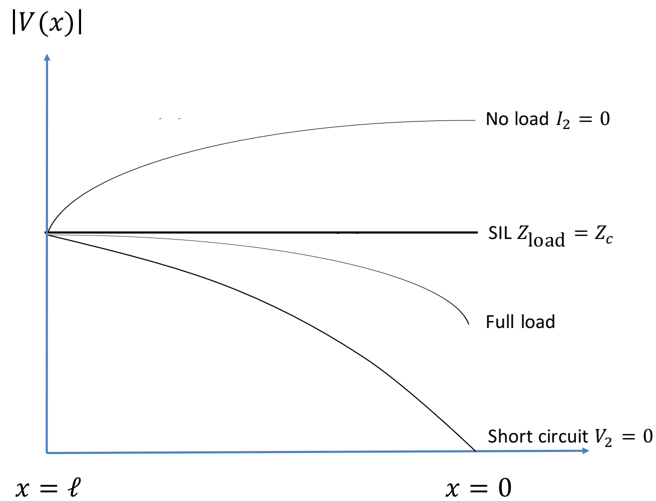


Figure 2.7: Voltage magnitude $|V(x)|$ on a lossless line.

where $\delta := \angle V_1 - \angle V_2$ is the angle difference between V_1 and V_2. Hence the maximum power is delivered on a lossless line if $\delta = \pi/2$ and the maximum power would have been $|V_1||V_2|/X$. This $\delta = \pi/2$ is called the steady-state stability limit. If the load exceeds this limit, there is no solution for δ for this equation. In practice a transmission network operates with $\delta \ll \pi/2$ because a line is typically limited by three other factors. First the voltage drop from the source to the load must be small, e.g., $|V_2|/|V_1| \geq 95\%$. Second δ is usually limited to 30° or 35° by transient stability. Third δ can be limited by the thermal rating of the conductor insulation materials. □

2.2.5 Short line

Consider a three-phase transmission line connecting two buses in balanced operation so we can analyze each phase separately. Assume the line is short and can be modeled by a Π equivalent circuit with only a series impedance $Z = R + \mathbf{i}X$ and no shunt admittances. We explain some properties of complex power transfer over this line.

Let V_i and I_i be the voltages and currents at buses $i = 1, 2$. Let S_{ij}, $i, j = 1, 2$, be the sending-end complex power from bus i to bus j, $i \neq j$, and I_{ij} be the complex current from bus i to bus j. Then

$$S_{ij} \;=\; V_i I_{ij}^* \;=\; V_i \frac{V_i^* - V_j^*}{Z^*} \;=\; \frac{1}{Z^*}\left(|V_i|^2 - V_i V_j^*\right) \tag{2.19}$$

If the voltage magnitudes $|V_i|$, $i = 1, 2$, are fixed, the branch powers depend only on the *power angle* $\theta_{ij} := \theta_i - \theta_j$:

$$S_{ij} \;=\; \frac{1}{Z^*}\left(|V_i|^2 - |V_i||V_j|e^{j\theta_{ij}}\right)$$

Taking the sum of the branch powers in (2.19), the complex loss over the line is

$$S_{12} + S_{21} \;=\; \frac{|V_1 - V_2|^2}{Z^*} \;=\; Z\,|I_{12}|^2$$

where I_{12} is the current from buses 1 to 2. In particular the real power loss is $P_{12} + P_{21} = R|I_{12}|^2$.

Nose curve and voltage collapse. Suppose bus 1 has a generator with a fixed $V_1 := |V_1|\angle 0^\circ$ supplying a load at bus 2 through a line with impedance Z. Let the power supplied to the load be $-S_{21} = |S_{21}|(\cos\phi + \mathbf{i}\sin\phi) =: P(1 + \mathbf{i}\tan\phi)$ where $P > 0$ is the active load power and ϕ is the power factor angle. The power flow equation (2.19) hence becomes

$$P(1+\mathbf{i}\tan\phi) \quad = \quad -\frac{1}{Z^*}\left(|V_2|^2 - |V_2|\,|V_1|\,e^{\mathbf{i}\theta_{21}}\right) \tag{2.20}$$

where $\theta_{21} := \angle V_2 - \angle V_1 = \angle V_2$. Voltage support is typically available on the generator side, so we assume $|V_1|$ is fixed even when the load power varies.[2] Voltage support may not be available on the load side and we are interested in the behavior of the load voltage $|V_2|$ as the active load power P increases while keeping the power factor angle ϕ constant.

Fix V_1 and ϕ. For each P, (2.20) defines two real equations in two variables $|V_2|$ and θ_{21}. For this simple system we can analytically solve for $|V_2|$ for each P. Depending on the value of P, there may be zero, one, or two solutions for $|V_2|$. As P varies, the solutions $|V_2|$ trace out a curve called a *nose curve*. As P increases from zero with fixed power factor angle ϕ, there are exactly two solutions for $|V_2|$, one with a high voltage and the other with a low voltage. The difference between the high-voltage solution and the low-voltage solution of $|V_2|$ decreases until they coincide. This is the point where the active load power $P = P_{\max}$ is maximum and represents the limit of power transfer from the voltage source V_1 through the transmission line Z to the load. If P increases further, real solutions for $|V_2|$ cease to exist. This phenomenon is called *voltage collapse*. This is studied in Exercise 2.9. See Chapter **??** for discussions on voltage collapse beyond the infinite bus model.

Short and lossless line $R = 0$. Suppose the series resistance is negligible (which is a reasonable approximation for high voltage transmission lines), $Z = \mathbf{i}X$. Then (2.19) reduces to

$$S_{ij} \quad = \quad \mathbf{i}\frac{1}{X}\left(|V_i|^2 - V_i V_j^*\right)$$

Hence

$$\begin{aligned}
P_{12} &= \frac{|V_1||V_2|}{X}\sin\theta_{12} \;=\; -P_{21} \qquad (2.21)\\
Q_{12} &= \frac{1}{X}\left(|V_1|^2 - |V_1||V_2|\cos\theta_{12}\right)\\
Q_{21} &= \frac{1}{X}\left(|V_2|^2 - |V_1||V_2|\cos\theta_{12}\right)
\end{aligned}$$

where $\theta_{12} := \angle V_1 - \angle V_2$. This has the following implications.

1. *Transmission efficiency.* The transmission efficiency $\eta := -P_{21}/P_{12} = 1$ since there is zero real power loss. The maximum power transfer $|V_1||V_2|/X$ is proportional to voltage magnitude product. This is another reason why transmission networks tend to operate at very high voltage levels. Indeed doubling the voltage increases the maximum power transfer capability by fourfold.

[2] An ideal voltage source whose complex bus voltage is fixed regardless of its power generation is called an *infinite bus*.

2. *DC power flow model.* When voltage magnitudes are fixed, the real power depends only on the power angle θ_{12}. When the power angle is small $|\theta_{12}| \approx 0$, $\sin\theta_{12} \approx \theta_{12}$ and the real powers P_{ij} are roughly *linear* in the phase angles (θ_1, θ_2). These assumptions are called the *DC power flow* approximation ($R = 0$, fixed $|V_i|$, small $|\theta_{ij}|$, ignore Q_{ij}); see Chapter **??** for more details.

3. *Decoupling.* When $|\theta_{12}| \approx 0$, there is a decoupling between real and reactive powers:

$$\begin{aligned}\frac{\partial P_{12}}{\partial \theta_{12}} &= -\frac{\partial P_{21}}{\partial \theta_{12}} = \frac{|V_1||V_2|}{X}\cos\theta_{12} \approx \frac{|V_1||V_2|}{X}\\ \frac{\partial P_{12}}{\partial |V_i|} &= -\frac{\partial P_{21}}{\partial |V_i|} = \frac{|V_j|}{X}\sin\theta_{12} \approx 0\end{aligned}$$

Hence the real powers P_{ij} depend strongly on θ_{12} but not on the voltage magnitudes $|V_k|$.

On the other hand

$$\frac{\partial Q_{ij}}{\partial \theta_{12}} = \frac{|V_1||V_2|}{X}\sin\theta_{12} \approx 0$$

i.e., the reactive powers Q_{ij} depend weakly on the power angle θ_{12}. Moreover

$$\frac{\partial Q_{12}}{\partial |V_2|} = -\frac{|V_1|}{X}\cos\theta_{12} < 0, \qquad \frac{\partial Q_{21}}{\partial |V_2|} = \frac{1}{X}\left(2|V_2| - |V_1|\cos\theta_{12}\right)$$

Typically $|V_1| \approx |V_2|$ and hence the second expression above is positive. Hence to maintain a high load voltage $|V_2|$, we should increase Q_{21} and/or decrease Q_{12}, i.e., the load should supply reactive power and the generation should absorb reactive power. This motivates the use of reactive power to regulate voltage magnitudes. The decoupling property holds in a network setting as well and leads to a fast algorithm to solve power flow problems; see Chapter 4.5.3.

4. *Out-of-step generators.* When generators are not synchronized, i.e., they operate with slightly different frequencies, the long-run average active power transmitted across a lossless line is zero. To see this, consider voltages at buses 1 and 2 given by

$$\begin{aligned}v_1(t) &= \sqrt{2}|V_1|\cos(\omega' t + \theta_1)\\ v_2(t) &= \sqrt{2}|V_2|\cos(\omega t + \theta_2)\end{aligned}$$

where the frequency ω' at bus 1 is slightly out of step, with $\omega' \approx \omega$. Write

$$v_1(t) = \sqrt{2}|V_1|\cos(\omega t + \theta_1'(t))$$

with a slowly-varying phase $\theta_1'(t) := \theta_1 + (\omega' - \omega)t$. If the phase $\theta_1'(t)$ varies slowly enough, we can still use the steady-state expressions above as reasonable approximations of powers. Then the short-term active power is given by (from (2.21)):

$$P_{12} = \frac{|V_1||V_2|}{X}\sin\left((\omega' - \omega)t + \theta_{12}\right)$$

Hence the long-term average of active power transfer is zero. This is not only ineffective, but highly undesirable because the line current can be very large. In practice protective devices would remove the out-of-step generator.

2.3 Bibliographical notes

There are many excellent textbooks on basic power system concepts and many materials in this chapter follow [1]; see also [2, Chapter 4]. We develop line characteristics in Chapter 2.1 based on basic results in physics that we do not elaborate. For example, the derivation of shunt capacitance c of a transmission line in Chapter 2.1.3 is explained in [1, Chapters 3.7–3.8] or [2, Chapters 4.8–4.12]). The expression (2.6) for the potential v_p at the center of a balanced three-phase transmission line is from [1, Example 3.8, p. 79]. Some of the materials on lossless lines follow [2].

2.4 Problems

Chapter 2.1.

Exercise 2.1. Consider the simplest model of a symmetric three-phase transmission line in balanced operation, as shown in Figure 2.8, with the assumptions

- the conductors are equally spaced at D and have equal radii r;
- $i_a(t) + i_b(t) + i_c(t) = 0$ at all times t;
- $q_a(t) + q_b(t) + q_c(t) = 0$ at all times t.

where $i_k(t)$ are currents and q_k are the total charge per unit length of wire k in coulombs/meter. Show that

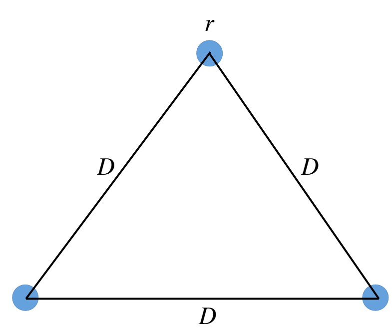


Figure 2.8: Per-meter inductance and capacitance of a symmetric three-phase transmission line in balanced operation.

the per-phase inductance per meter of the three-phase transmission line is

$$l = \frac{\mu_0}{2\pi} \ln \frac{D}{r'} \qquad \text{(in H/m)}$$

where $r' := re^{-\mu_r/4}$, and the per-phase capacitance per meter is

$$c = \frac{2\pi\varepsilon}{\ln(D/r)} \qquad \text{(in F/m)}$$

Chapter 2.2.

Exercise 2.2. Consider the per-phase transmission line model described by (2.9). We are to determine the line characteristic impedance Z_c and propagation constant $\gamma\ell$ from two measurements:

1. **Open-circuit test.** The load side is open-circuited so that $I_2 = 0$ and the driving-point impedance is measured as

$$Z_{oc} \quad := \quad \frac{V_1}{I_1}$$

2. **Short-circuit test.** The load side is short-circuited so that $V_2 = 0$ and the driving-point impedance is measured as

$$Z_{cc} \quad := \quad \frac{V_1}{I_1}$$

Derive Z_c and $\gamma\ell$ in terms of Z_{oc} and Z_{sc} (sign ambiguity is fine).

Exercise 2.3 (Lumped-circuit Π model)**.** Consider a general transmission matrix T that maps the receiving-end voltage and current (V_2, I_2) to those (V_1, I_1) at the sending-end:

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} \quad = \quad \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{T} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

1. Show that the transmission matrix T in (2.9) has the property $ad - bc = 1$.

2. Suppose $b \neq 0$ in T. Show that the condition $ad - bc = 1$ is necessary and sufficient for interpreting the transmission matrix T as a Π equivalent circuit consisting of a series impedance $Z \neq 0$ and shunt admittances (line charging) Y_1 and Y_2 at the sending and receiving ends respectively (note that Y_1 may not necessarily equal Y_2).

Exercise 2.4 (Surge impedance load (SIL) on lossless line.)**.** Consider a lossless line with $r = g = 0$ that terminates in an impedance load that is equal to the characteristic (surge) impedance $Z_{\text{load}} = Z_c = \sqrt{l/c}\ \Omega$ of the line. The power delivered by a lossless line to the resistive load Z_c is called the *surge impedance loading* (SIL).

1. Show that the voltage magnitude $|V(x)|$ is constant over $x \in [0, \ell]$.

2. Calculate SIL.

Exercise 2.5 (Voltage drop along lossless line)**.** We have derived in Chapter 2.2.4 the voltage $V(x)$ at each point $x \in [0, \ell]$ along a lossless line terminating at an impedance load $Z_{\text{load}} = R_{\text{load}} + \mathbf{i}X_{\text{load}}$ to be (from (2.18)):

$$V(x) \;=\; \left(\cos(\beta x) + \frac{Z_c X_{\text{load}}}{|Z_{\text{load}}|^2}\sin(\beta x) \;+\; \mathbf{i}\,\frac{Z_c R_{\text{load}}}{|Z_{\text{load}}|^2}\sin(\beta x)\right) V_2$$

Assume $\beta\ell < \pi/4$. Prove the following:

1. If the load is purely resistive $Z_{\text{load}} = R_{\text{load}}$ then $|V(x)|$ is an increasing function for all $x \in [0, \ell]$ (i.e., the voltage magnitude $|V(x)|$ drops from the source at $x = \ell$ to the load Z_{load} at $x = 0$) if and only if $R_{\text{load}} \le Z_c$.
2. If the load is purely inductive $Z_{\text{load}} = \mathbf{i}X_{\text{load}}$ with $X_{\text{load}} > 0$ then $|V(x)|$ is an increasing function for all $x \in [0, \ell]$ if and only if
$$X_{\text{load}} \;\le\; \frac{\sin(2\beta\ell)}{1 - \cos(2\beta\ell)}\, Z_c$$
3. If $Z_{\text{load}} = R_{\text{load}}(1 + \mathbf{i})$ then $|V(x)|$ is an increasing function for all $x \in [0, \ell]$ if and only if
$$R_{\text{load}} \;\le\; \left(\sqrt{1 + \frac{1}{\sin^2(2\beta\ell)}} \;-\; \cot(2\beta\ell)\right)^{-1} Z_c$$

Exercise 2.6 (Voltage, reactive power compensation)**.** Consider a generator with voltage and power injection (V_j, s_j) supplying a load with voltage and power *injection* (V_k, s_k) through a transmission line parametrized by series and shunt admittances $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$. Power balance at the load bus k is (with $y^s_{kj} = y^s_{jk}$)

$$s_k \;=\; \left(y^s_{kj}\right)^{\mathsf{H}} \left(|V_k|^2 - V_k V_j^{\mathsf{H}}\right) + \left(y^m_{kj}\right)^{\mathsf{H}} |V_k|^2 \tag{2.22}$$

Let $y^s_{kj} =: g^s_{kj} + \mathbf{i}b^s_{kj}$ and $y^m_{kj} =: g^m_{kj} + \mathbf{i}b^m_{kj}$ and suppose $g^s_{kj} \ge 0$, $b^s_{kj} < 0$ (inductive) and $g^m_{kj} \ge 0$, $b^m_{kj} \ge 0$ (capacitive). Let $s_k =: p_k + \mathbf{i}q_k$, and $V_i =: |V_i| e^{\mathbf{i}\theta_i}$, $i = j, k$. Use (2.22) to express the receiving real power $-p_k$ and receiving reactive power $-q_k$ in terms of the voltage magnitudes $|V_j|, |V_k|$, and the angle difference $\theta_{kj} := \theta_k - \theta_j$.

Suppose $y^m_{kj} = 0$ (zero shunt), $g^s_{jk} = 0$ (loss line), and $0 < |\theta_{kj}| \le \pi/2$ (power slow solution stability).

1. Show that real power is delivered to the load (i.e., $-p_k > 0$) if and only if $-\pi/2 \le \theta_{kj} < 0$.
2. The next few questions study the relation between load voltage magnitude $|V_k|$ and reactive power injection q_k. Show that:
   (a) For DC load (i.e., $q_k = 0$), we must have $|V_k| < |V_j|$, i.e., the load voltage magnitude must be smaller than the generator voltage magnitude.

(b) On the other hand, $|V_k| = |V_j|$ implies that $q_k > 0$, i.e., the load must inject reactive power to maintain a high load voltage magnitude.

(c) If $-q_k > 0$ (i.e., the load receives reactive power), then $|V_k| < |V_j| \cos \theta_{kj}$ (i.e., load voltage magnitude will be further suppressed).

3. The power factor angle is $\phi_k := \tan^{-1}(q_k/p_k)$ and the power factor PF is $\cos \phi_k$. Show that

$$1 + \tan \phi_k \tan \theta_{kj} = \frac{|V_k|}{|V_j| \cos \theta_{kj}}$$

When $|V_k| = |V_j| \cos \theta_{kj}$, what is the PF and is the load receiving or injecting real power?

4. Suppose further that $V_j := 1\angle 0°$ and $b^s_{jk} = -1$. Suppose that the load voltage magnitude $|V_k|$ must lie between $[1-\varepsilon, 1+\varepsilon]$.

(a) At unity power ($q_k = 0$), find the maximum received power $-p_k$ and the corresponding load voltage phasor $V_k = |V_k| e^{\mathbf{i}\theta_k}$. Conclude that the maximum received real power satisfies $-p_k \leq \frac{1}{2}$.

(b) Show that the maximum received real power is $-p_k = (1+\varepsilon)$ when the load must inject the reactive power $q_k = (1-\varepsilon)^2$.

Exercise 2.7 (Voltage, line loss and voltage drop)**.** Consider two buses 1 and 2 connected by a transmission line modeled by a per-phase Π circuit model with series impedance Z and shunt admittance (line charging) $Y/2$ at each end of the line, as shown in Figure 2.9. Let S_{12} be the sending-end complex power from buses

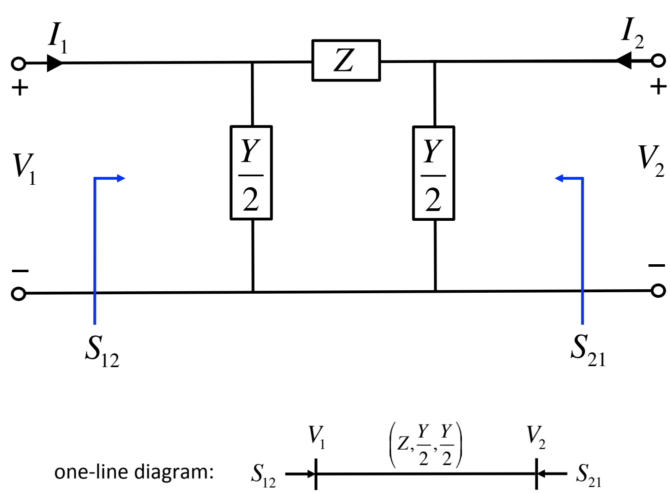


Figure 2.9: Two buses connected by a transmission line.

1 to 2 and S_{21} be the sending-end complex power from buses 2 to 1 (or, equivalently, $-S_{21}$ is the receiving-end complex power at bus 2). Note that the direction of load current I_2 is opposite to the convention we used in Chapter 2.2.2.

1. Calculate the complex line loss as a function of voltages (V_1, V_2). Can you express the complex line loss in terms of the load voltage and current (V_2, I_2) instead?
2. Suppose bus 2 is connected to a load that draws a fixed active power P_{load} with a fixed power factor $\cos\phi$ at a fixed voltage magnitude $|V_2|$. Suppose $Z = R + \mathbf{i}X$ and the shunt admittance $Y/2 = \mathbf{i}B/2$ is purely reactive (i.e., zero conductance). Calculate the active power loss P_{line} over the line in terms of the active load power P_{load}, the power factor angle ϕ, and the load voltage $|V_2|$.

For the following subproblems, assume $Y = \mathbf{i}B = 0$ (short transmission line).

3. Given the fixed active load power P_{load}, show that the active line loss P_{line} derived in part 2 of the problem is inversely proportional to the squared load voltage $|V_2|^2$ and to the squared power factor $\cos^2\phi$.
4. Suppose now the load at bus 2 is an electric vehicle that draws an active power of $P_{\text{load}} = 20$ kW with unity power factor at a voltage magnitude of $|V_2| = 200$V. Calculate the ratio of the active power loss to the active load power if $R = 0.04\Omega$ (wires with gauge number 6 at 100ft).
5. What is the magnitude of the voltage drop $|V_1 - V_2|$ across the transmission line (the series impedance Z), relative to the load voltage $|V_2|$, in terms of $Z, P_{\text{load}}, |V_2|, \cos\phi$?

Exercise 2.8. Consider the short-line model $S_{12} = (Z^*)^{-1}\left(|V_1|^2 - V_1V_2^*\right)$ of a transmission line with $Z := y^{-1}e^{\mathbf{i}\phi}$ that connects bus 1 and bus 2. Let V_1, V_2 be the complex voltages at buses 1 and 2 respectively and assume $|V_1| = |V_2| = 1$. Let $\theta_{12} := \angle V_1 - \angle V_2$.

1. For what value of θ_{12} is S_{12} real and nonzero?
2. What is the maximum real power $-P_{21}$ that can be received at bus 2 and what is θ_{12} that delivers it?

Exercise 2.9 (Nose curve and voltage collapse)**.** Consider a voltage source with a fixed magnitude $|V_1|$ supplying a load through a line modeled by a series impedance $z := |z|e^{\mathbf{i}\theta_z}$ with $|\theta_z| < \pi/2$. Let the power supplied to the load be $S_2 = |S_2|(\cos\phi + \mathbf{i}\sin\phi) =: P(1 + \mathbf{i}\tan\phi)$ where $P > 0$ is the active load power and ϕ is the power factor angle. The power flow equation is:

$$P(1 + \mathbf{i}\tan\phi) \quad = \quad -\frac{1}{z^*}\left(|V_2|^2 - |V_2||V_1|e^{\mathbf{i}\theta_{21}}\right) \tag{2.23}$$

where $\theta_{21} := \angle V_2 - \angle V_1$.

1. For each P, solve (2.23) for $|V_2|$ with $|V_1|$ and ϕ fixed.
2. Show that $|V_2|$ behaves as follows as P increases from $P = 0$ with the power factor angle ϕ kept constant: $|V_2|$ is a nonunique rool of a polynomial equation in P. As P increases, the resulting nonunique roots $|V_2|$ trace out a curve called the *nose curve*. As P keeps increasing, eventually, the polynomial equation has no real root, which is the phenomenon of *voltage collapse*.
3. Find the maximum power transfer $P = P_{\max}$ at which solutions for $|V_2|$ exist.

Chapter 3

Transformer models

A large electric network is composed of multiple areas that have different nominal voltage magnitudes. These areas are connected by transformers that convert between different voltage levels. The ease of converting between voltage levels is an important advantage of AC over DC transmission systems. It allows, for example, the transmission network to operate at $765kV$ to reduce power loss and household appliances to operate at $120V$ for safety. In this chapter we develop transformer models and explain how to analyze a balanced three-phase system that contains transformers.

We start in Chapter 3.1 with models of a single-phase transformer and use them in Chapter 3.2 to develop models of three-phase transformers in balanced operation. We describe in Chapter 3.3 how to refer impedances from one side of a transformer to the other side. We apply this method in Chapter 3.4 to simplify per-phase analysis of circuits that contain transformers. We explain in Chapter 3.5 per-unit normalization that further simplifies the analysis of balanced three-phase systems.

3.1 Single-phase transformer

We first model an ideal single-phase transformer by a transmission matrix and then describe circuit models of a nonideal single-phase transformer.

3.1.1 Ideal transformer

An *ideal transformer* has no loss (zero resistance), no leakage flux, and the magnetic core has infinite permeability. Let N_1 be the number of turns in the primary winding, N_2 that in the secondary winding, and

$$n := \frac{N_2}{N_1}, \qquad a := \frac{1}{n} = \frac{N_1}{N_2}$$

An ideal transformer is represented schematically in Figure 3.1. We will call n the *voltage gain* and its reciprocal a the *turns ratio*. The voltage gain n relates the voltages and currents in the primary and

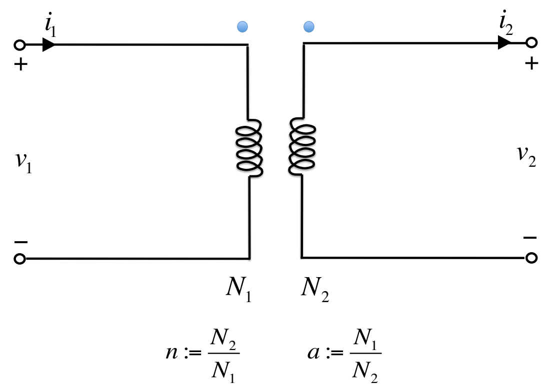


Figure 3.1: Single-phase ideal transformer.

secondary circuits, both at all times in the time domain:

$$\frac{v_2(t)}{v_1(t)} = n, \qquad \frac{i_2(t)}{i_1(t)} = a$$

and in the phasor domain:

$$\frac{V_2}{V_1} = n, \qquad \frac{I_2}{I_1} = a$$

This relation can also be written as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & n \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{3.1}$$

The matrix on the right-hand side is called a *transmission matrix* of an ideal transformer. It maps (V_2, I_2) to (V_1, I_1). The dot notation indicates that the currents I_1, I_2 are defined to be positive when one flows into and the other out of the dotted terminals, as indicated in Figure 3.1. This notation is convenient when we use single-phase transformers to construct three-phase transformers.

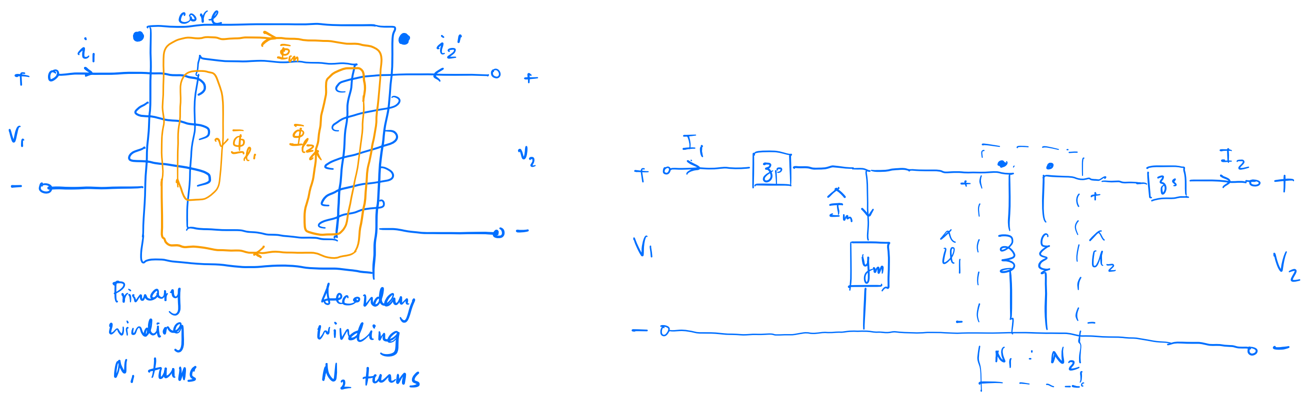
The ratio of the complex receiving-end to sending-end power is

$$\frac{-S_{21}}{S_{12}} := \frac{V_2 I_2^*}{V_1 I_1^*} = n \cdot a = 1$$

i.e., an ideal transformer has no power loss.

3.1.2 Nonideal transformer

A real transformer has power losses due to resistance in the windings $(r|I|^2)$, eddy currents and hysteresis losses. It also has nonzero leakage fluxes and finite permeability of the magnetic core. Figure 3.2(a) shows elements of a (nonideal) transformer. The primary winding has N_1 turns around the magnetic core and the secondary winding has N_2 turns. The mutual flux Φ_m due to the currents i_1 and i_2' links all the turns of the



(a) Nonideal transformer (b) Circuit model

Figure 3.2: Single-phase nonideal transformer. The dashed box represents an ideal transformer with $a := N_1/N_2$. (Megan: $\hat{U}_1 \to \hat{V}_1$, $\hat{U}_2 \to \hat{V}_2$.)

primary and secondary coils. The two dots indicate that the mutual flux components due to i_1 and i_2' add when these currents both enter (or exit) the dotted terminals according to the right-hand rule. The leakage fluxes Φ_{l1} and Φ_{l2} links the individual coils. The flux linkages $\lambda_{l1} =: L_{l1} i_1$ and $\lambda_{l2} =: L_{l2} i_2'$ due to Φ_{l1} and Φ_{l2} are proportional to the currents i_1 and i_2' respectively. The proportionality constants L_{l1}, L_{l2} are called inductances. Then the total flux linkages of the primary and secondary circuits are the sums of the leakage flux linkages and the mutual flux linkage:

$$\lambda_1 \;=\; \lambda_{l1} + N_1 \Phi_m, \qquad\qquad \lambda_2 \;=\; \lambda_{21} + N_2 \Phi_m$$

The voltages are

$$v_1 \;=\; r_1 i_1 + \frac{d\lambda_1}{dt} \;=\; r_1 i_1 + L_{l1}\frac{di_1}{dt} + N_1 \frac{d\Phi_m}{dt} \tag{3.2a}$$

$$v_2 \;=\; r_2 i_2' + \frac{d\lambda_2}{dt} \;=\; r_2 i_2' + L_{l2}\frac{di_2'}{dt} + N_2 \frac{d\Phi_m}{dt} \tag{3.2b}$$

where $r_1 i_1$ and $r_2 i_2'$ represent power losses in the core. The model for an ideal transformer neglects losses ($r_1 = r_2 = 0$) and leakage fluxes ($\lambda_{l1} = \lambda_{l2} = 0$) in (3.2) and hence $v_1 = N_1 \frac{d\Phi_m}{dt}$ and $v_2 = N_2 \frac{d\Phi_m}{dt}$, yielding $v_1/v_2 = N_1/N_2$.

The total magnetomotive force F due to the currents i_1 and i_2' is proportional to the mutual flux Φ_m:

$$F \;=\; N_1 i_1 + N_2 i_2' \;=\; R\Phi_m \tag{3.3}$$

where R is called the reluctance of the core. The model for an ideal transformer assumes infinite permeability of the magnetic core and hence $R = 0$, yielding $i_1/(-i_2') = N_2/N_1$. In practice the magnetic core has finite permeability, i.e., $R > 0$ and the the magnetomotive force F is nonzero. When the secondary circuit is open, $i_2' = 0$. The resulting primary current, denoted $\hat{i}_m$, is called the primary magnetizing current and satisfies $N_1 \hat{i}_m = R\Phi_m$ from (3.3).[1] Define

$$\hat{v}_1 \;:=\; N_1 \frac{d\Phi_m}{dt} \;=\; L_m \frac{d\hat{i}_m}{dt}, \qquad\qquad \hat{v}_2 \;:=\; N_2 \frac{d\Phi_m}{dt} \;=\; \frac{N_2}{N_1}\hat{v}_1$$

[1] Instead of $i_m := (R/N_1)\Phi_m$, we can define $i_m' := (R/N_2)\Phi_m$ as the secondary magnetizing current when the primary circuit is open $i_1 = 0$. In this case the shunt admittance y_m in Figure 3.4(a) will be in the secondary circuit.

where $L_m := N_1^2/R$. Substituting into (3.2) yields, denoting $i_2 := -i_2'$, we have

$$\begin{array}{llll} \text{Nonideal elements:} & v_1 \;=\; r_1 i_1 + L_{l1}\dfrac{di_1}{dt} + \hat{v}_1, & \hat{v}_1 \;=\; L_m \dfrac{d\hat{i}_m}{dt}, & v_2 \;=\; -r_2 i_2 - L_{l2}\dfrac{di_2}{dt} + \hat{v}_2 \\ \text{Ideal transformer:} & \hat{v}_2 \;=\; \dfrac{N_2}{N_1}\hat{v}_1, & i_2 \;=\; \dfrac{N_1}{N_2}\left(i_1 - \hat{i}_m\right) & \end{array}$$

where the last equality follows from substituting $R\Phi_m = N_1 \hat{i}_m$ into (3.3). This set of equations in the phasor domain is

$$\begin{array}{llll} \text{Nonideal elements:} & V_1 \;=\; z_p I_1 + \hat{V}_1, & \hat{I}_m \;=\; y_m \hat{V}_1, & \hat{V}_2 \;=\; z_s I_2 + V_2 \end{array} \tag{3.4a}$$

$$\begin{array}{lll} \text{Ideal transformer:} & \hat{V}_2 \;=\; \dfrac{N_2}{N_1}\hat{V}_1, & I_2 \;=\; \dfrac{N_1}{N_2}\left(I_1 - \hat{I}_m\right) \end{array} \tag{3.4b}$$

where the series impedances $z_p := r_1 + \omega L_{l1}$ and $z_s := r_2 + \omega L_{l2}$ model the core losses and leakage fluxes in the primary and secondary circuits respectively, and the shunt admittance $y_m := 1/(\omega L_m) = R/(\omega N_1^2)$ models the finite permeability of the core. The model (3.4) can be interpreted as the circuit in Figure 3.2(b). Variables with hats denote internal variables.

In the following we present three circuit models derived from that in Figure 3.2(b). Their relation is shown in 3.3. The circuit model in Figure 3.2(b) is equivalent to a T equivalent circuit (Chapter 3.1.3). The

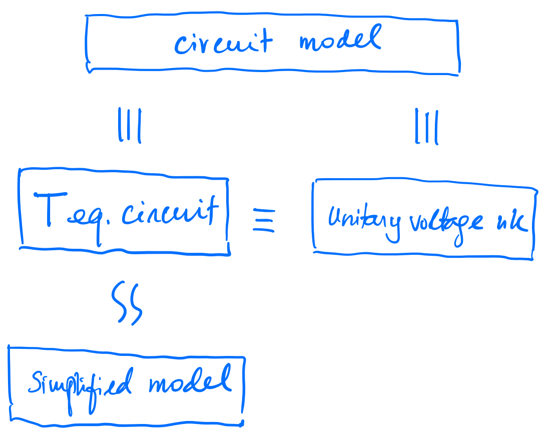


Figure 3.3: Relation between different circuit models of transformers.

T equivalent circuit can be approximated by a simplified model whose parameters can be determined by short-circuit and open-circuit tests (Chapter 3.1.4). The circuit model in Figure 3.2(b) is also equivalent to a circuit consisting of two ideal transformers connected by a unitary voltage network (Chapter 3.1.5). The unitary voltage network can be generalized to model nonstandard transformers with multiple windings, e.g., split-phase transformer. These models reduce to the same model when the shunt admittance y_m in Figure 3.2(b) is assumed zero (i.e., open-circuited). We emphasize that, by equivalence, we mean two circuits have the same *end-to-end behavior*, as described by a transmission matrix or an admittance matrix, but their internal variables may take different values (we elaborate on this in Chapter 3.1.3).

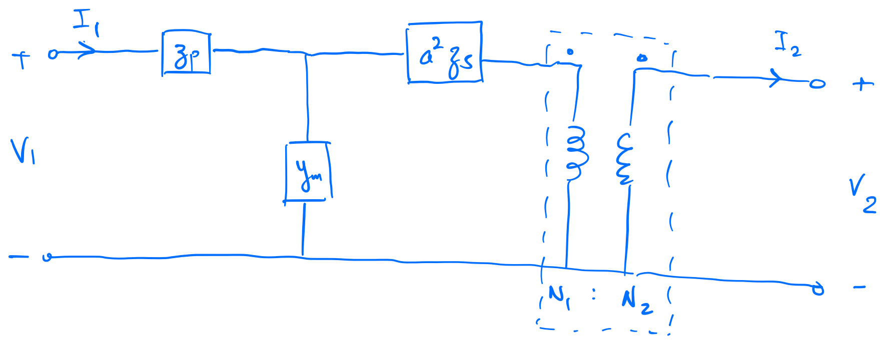


Figure 3.4: T equivalent circuit.

3.1.3 T equivalent circuit

We can refer the leakage impedance z_s in Figure 3.2(b) on the secondary side to the primary side using (3.14) in Chapter 3.3. The resulting model, shown in Figure 3.4, is called the T equivalent circuit of the transformer. It is equivalent in the sense that the transmission matrices that map (V_2, I_2) to (V_1, I_1) are the same in both models; see Chapter 3.3.1. Indeed the transmission matrix T of the T equivalent circuit is given by (Exercise 3.1)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \underbrace{\begin{bmatrix} a(1+z_p y_m) & a z_s (1+z_p y_m) + n z_p \\ a y_m & n + a z_s y_m \end{bmatrix}}_{T} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{3.5}$$

where $n := N_2/N_1$ and $a := N_1/N_2$.

Even though the circuit model in Figure 3.2(b) and the T equivalent circuit in Figure 3.4 have the same transmission matrix, their internal variables may not be equal because of the reference of z_s to the primary side. Indeed (3.4) describes the internal variables of the model in Figure 3.2(b), but not necessarily those in the T equivalent circuit in Figure 3.4. For instance, when the secondary circuit is shorted, i.e., setting $V_2 = 0$, the internal variables $\hat{V}_1$ and $\hat{V}_2$ are nonzero in general in Figure 3.2(b), as determined by (3.4), but these voltages are zero in Figure 3.4. This has implications on parameter determination as we now explain.

Parameter determination. Two simple tests are often used to determine transformer model parameters:

1. *Short-circuit test* $(V_2 = 0)$. With the secondary circuit short-circuited, the primary voltage V_{sc} and primary current I_{sc} are measured. The primary short-circuit voltage V_{sc} is called the *impedance voltage*.

2. *Open-circuit test* $(I_2 = 0)$. With the secondary circuit open, the primary voltage V_{oc} and primary current I_{oc} are measured.

To determine the parameters (z_p, z_s, y_m) of the transmission matrix T in (3.5), note that during the short-circuit test, the voltage on the primary side of the ideal transformer is zero. Hence

$$V_{sc} \;=\; \left(z_p + \left(y_m + \frac{1}{a^2 z_s}\right)^{-1}\right) I_{sc} \tag{3.6a}$$

During the open-circuit test, the secondary current $I_2 = 0$ and hence there is zero current on the primary side of the ideal transformer. Hence

$$V_{oc} \;=\; \left(z_p + \frac{1}{y_m}\right) I_{oc} \tag{3.6b}$$

Since there are three unknowns (z_p, z_s, y_m), they cannot be uniquely determined from the two equations in (3.6). Additional measurements will be needed to determine (z_p, z_s, y_m), e.g. measurements of separate dc resistances in the primary and secondary circuits. Sometimes y_m is assumed to be zero (open-circuited) so that (3.6a) becomes $V_{sc} = (z_p + a^2 z_s) I_{sc}$, yielding the total leakage impedance $z_p + z_s$. Alternatively assuming $z_p = \eta z_s$ with known η results in two nonlinear equations in two unknowns (z_s, y_m).

It may seem that we can measure the current I_2 in the T equivalent circuit in Figure 3.4 during a short-circuit test and use it to determine (z_p, z_s, y_m), but this is not the case because it will involve internal variables. Even though we have informally justified (3.6) using internal variables in the T equivalent circuit, e.g., the voltage and current on the primary side of the ideal transformer, we should be careful with this line of reasoning. A more rigorous derivation of (3.6) uses the circuit model in Figure 3.2(b), by setting $V_2 = 0$ in (3.4) (Exercise 3.2). In this case, even if the short-circuit current I_2 is also measured, there are 6 unknowns $(\hat{V}_1, \hat{V}_2, \hat{I}_m; z_p, z_s, y_m)$ but only 5 equations in (3.4) and hence these unknowns cannot be uniquely determined from just the short-circuit and open-circuit tests either. This implies that we cannot apply the measured value of short-circuit current I_2 to determine (z_p, z_s, y_m).

3.1.4 Simplified model

In practice the shunt admittance y_m is much smaller than the leakage admittances (see Example 3.1). Specifically when $|y_m| \ll 1/|a^2 z_s|$ or $|\varepsilon| := |a^2 z_s y_m| \ll 1$, we interchange y_m and $a^2 z_s$ to obtain the simplified model in Figure 3.5(a) with $z_l = z_p + a^2 z_s$. An even simpler model assumes $y_m = 0$, as shown in Figure 3.5(b).

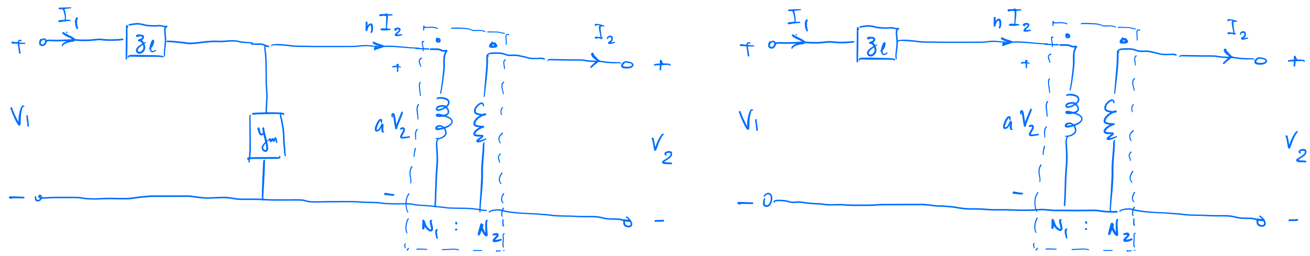
Transmission matrix. Apply KCL, KVL and Ohm's law to the model in Figure 3.5(a) to get:

$$V_1 \;=\; z_l I_1 + aV_2, \qquad\qquad I_1 \;=\; y_m(aV_2) + nI_2$$

Hence the transmission matrix $\hat{T}$ is given by

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} a\,(1 + z_l y_m) & n z_l \\ a y_m & n \end{bmatrix}}_{\hat{T}} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{3.7}$$

We mostly use the simplified model in Figure 3.5, or equivalently, in (3.7). When $z_l = y_m = 0$ the model (3.7) reduces to (3.1) for an ideal transformer.



(a) Simplified model (b) $y_m = 0$

Figure 3.5: (a) Simplified model of nonideal transformer including power losses, leakage flux and finite permeability of magnetic core. (b) Simplified model assuming infinite permeabilitiy.

Approximation to T equivalent circuit. We now justify the model in Figure 3.5(a) with $z_l = z_p + a^2 z_s$ as a reasonable approximation of the T equivalent circuit in Figure 3.4(b) when y_m is small. The difference in their transmission matrices is (from (3.5) and (3.7))

$$\hat{T} - T \;=\; \begin{bmatrix} a\,(1+z_l y_m) & n z_l \\ a y_m & n \end{bmatrix} - \begin{bmatrix} a\,(1+z_p y_m) & a z_s(1+z_p y_m)+n z_p \\ a y_m & n + a z_s y_m \end{bmatrix} \;=\; \varepsilon \begin{bmatrix} a & -n z_p \\ 0 & -n \end{bmatrix}$$

where $\varepsilon := a^2 z_s y_m$. The conductance in the shunt admittance is negligible in practice and hence the shunt admittance y_m due to the primary magnetizing current takes the form $y_m = (\mathbf{i} x_m)^{-1} = -\mathbf{i} b_m$ with $b_m > 0$. The leakage impedance z_p takes the form $z_p = r_p + \mathbf{i} x_p$ with $r_p > 0$ and $x_p > 0$; similarly for z_s. Suppose $z_p = \eta z_s$ for some real number $\eta > 0$ and $|\varepsilon| \ll 1$. Then the relative error can be shown to satisfy (Exercise 3.3)

$$\frac{\|\hat{T} - T\|}{\|T\|} \;<\; |\varepsilon| \;\ll\; 1$$

where the matrix norm $\|A\|$ is the sum norm $\|A\| := \sum_{i,j} |A_{ij}|$, or the l_1 vector norm when the $n \times n$ matrix A is treated as a vector in $\mathbb{C}^{n^2}$ (see Appendix 25.1.7.3 for matrix norms). Note that for $a < 1$, the model parameters (z_l, y_m) should be on the high voltage side. When the shunt admittance is neglected $y_m = 0$, these two models are the same, i.e., $\hat{T} = T$.

Parameter determination. The parameters (z_l, y_m) of the simplified model in Figure 3.5(a), or equivalently, in (3.7), can be uniquely determined from two simple tests:

1. *Short-circuit test* ($V_2 = 0$). With the secondary circuit short-circuited, the primary voltage V_{sc} and current I_{sc} are measured. Then, from Figure 3.5,

$$z_l \;=\; \frac{V_{sc}}{I_{sc}}$$

   The primary short-circuit voltage V_{sc} is called the impedance voltage.

2. *Open-circuit test* ($I_2 = 0$). With the secondary circuit open, the primary voltage V_{oc} and current I_{oc} are measured. Then $V_{oc} = (z_l + 1/y_m) I_{oc}$ and hence

$$\frac{1}{y_m} \;=\; \frac{V_{oc}}{I_{oc}} - \frac{V_{sc}}{I_{sc}}$$

Example 3.1 (Parameter determination)**.** Consider a single-phase distribution (stepdown) transformer with the following ratings: 2.9 MVA, 7.2 kV / 240 V. Construct the equivalent circuit model in Figure 3.5 from the following test results:

1. *Short-circuit test* ($V_2 = 0$). With the secondary circuit short-circuited, a voltage $|V_{sc}| = 500\,\text{V}$ is applied to the primary circuit that causes the rated primary current $|I_1^s|$ to flow.
2. *Open-circuit test* ($I_2 = 0$). With the secondary circuit open, the rated voltage $|V_{oc}| = 7.2\,\text{kV}$ is applied to the primary circuit. This caused a current of $|I_{oc}| = 7\text{A}$ to flow in the primary circuit.

Assume $z_l = \mathbf{i}x_l$ and $y_m = (\mathbf{i}x_m)^{-1}$. Determine x_l and x_m.

Solution. In the short-circuit test the secondary voltage $V_2 = 0$. Hence the voltage on the primary side of the *ideal* transformer is zero and the shunt reactance x_m is effectively short-circuited, leaving only the leakage reactance x_l in the primary circuit. Since the rated primary current is $|I_{sc}| = 2.9\,\text{MVA}/7.2\,\text{kV} = 403\text{A}$, we have $|V_{sc}| = |I_{sc}z_l| = |I_{sc}|x_l$. Hence $x_l = 500\,\text{V}/403\,\text{A} = 1.24\ \Omega$.

In the open-circuit test the secondary current $I_2 = 0$ and hence there is zero current on the primary side of the *ideal* transformer (see Figure 3.5). Hence $|V_{oc}| = |I_{oc}(z_l + 1/y_m)| = |I_{oc}|(x_l + x_m)$, and $x_m = |V_{oc}|/|I_{oc}| - x_l = 7.2\,\text{kV}/7\,\text{A} - 1.24 = 1.03\,\text{k}\Omega$.

As expected, $|y_m| \ll 1/|z_l|$. □

In transformer ratings, the ratio of secondary open-circuit voltage to the primary open-circuit voltage is usually taken to be the voltage gain n, even though more precisely it should be

$$\frac{V_2}{V_1} = n \cdot \frac{1/y_m}{z_l + 1/y_m}$$

In practice the resistances due to core losses are much smaller than the reactances due to leakage fluxes and finite permeability of the core so that $z_l \approx \mathbf{i}x_l$ and $y_m \approx -\mathbf{i}b_m$. Moreover $b_m \ll 1/x_l$. For Example 3.1

$$\frac{V_2}{V_1} = n\,\frac{x_m}{x_l + x_m} = \frac{1.03\,\text{k}\Omega}{1.03\,\text{k}\Omega + 1.24\,\Omega} \simeq n$$

Parameter determination from transformer ratings when $y_m := 0$. If $y_m := 0$ then the model parameter is just the leakage impedance z_l in the primary circuit, which can be determined from the short-circuit test, $z_l = V_{sc}/I_{sc}$. Moreover its magnitude can be determined from typical transformer ratings, as follows.

A typical specification of a three-phase transformer includes:

- Three-phase power rating $|S_{3\phi}|$.
- Rated primary line-to-line voltage $|V_{\text{pri}}|$ and rated primary line current $|I_{\text{pri}}|$.
- Rated secondary line-to-line voltage $|V_{\text{sec}}|$ and rated secondary line current $|I_{\text{sec}}|$.
- Impedance voltage β on the primary side, per phase, as a percentage of the rated primary voltage. The shunt admittance is assumed zero.

As mentioned above, the impedance voltage is the voltage drop across the leakage impedance z_l on the primary side of each *single-phase* transformer in a short-circuit test. The β specification means that the voltage needed on the primary side to produce the rated primary current across each single-phase transformer is β, as a percentage of the rated primary voltage. We emphasize that the short-circuit voltage and current needed to derive z_l should be those across each single-phase transformer, which depends on the configuration of the primary circuit. If the primary circuit is in Δ configuration then the short-circuit voltage and current on the primary side of the single-phase transformer are (assuming balanced positive sequence):

$$\Delta \text{ configuration:} \qquad |V_{sc}| = |V_{ab}| = \beta |V_{\text{pri}}|, \qquad |I_{sc}| = |I_{ab}| = \left| \frac{I_{\text{pri}}}{\sqrt{3}} e^{\mathbf{i}\pi/6} \right|$$

If the primary circuit is in Y configuration then the short-circuit voltage and current on the primary side of the single-phase transformer are:

$$Y \text{ configuration:} \qquad |V_{sc}| = |V_{an}| = \beta \left| \frac{V_{\text{pri}}}{\sqrt{3}\, e^{\mathbf{i}\pi/6}} \right|, \qquad |I_{sc}| = |I_{an}| = |I_{\text{pri}}|$$

Since $z_l = V_{sc}/I_{sc}$ we therefore have,

$$\Delta \text{ configuration:} \quad |z_l| = \frac{\sqrt{3}\beta |V_{\text{pri}}|}{|I_{\text{pri}}|}; \qquad Y \text{ configuration:} \quad |z_l| = \frac{\beta |V_{\text{pri}}|}{\sqrt{3} |I_{\text{pri}}|} \tag{3.8a}$$

We reiterate that V_{pri} denotes the line-to-line voltage even for Y configuration; otherwise $|z_l| = \beta |V_{\text{pri}}|/|I_{\text{pri}}|$ for Y configuration if the rated voltage V_{pri} is line-to-neutral.

Sometimes the primary line current $|I_{\text{pri}}|$ is not specified directly. In that case z_l can be determined from the power and voltage ratings $(|S_{3\phi}|, |V_{\text{pri}}|)$, as follows. If the primary circuit is in Δ configuration then the short-circuit voltage and current on the primary side of the single-phase transformer are (assuming balanced positive sequence):

$$\begin{aligned} \Delta \text{ configuration:} \qquad & |S_{3\phi}| = 3|S_\phi| = 3|V_{ab}|\,|I_{ab}| \\ & |V_{sc}| = |V_{ab}| = \beta |V_{\text{pri}}|, \qquad |I_{sc}| = |I_{ab}| = \frac{|S_{3\phi}|}{3|V_{\text{pri}}|} \end{aligned}$$

Note that $\frac{|S_{3\phi}|}{3|V_{\text{pri}}|}$ is the rated primary current produced in the short-circuit test. If the primary circuit is in Y configuration then the short-circuit voltage and current on the primary side of the single-phase transformer are:

$$\begin{aligned} Y \text{ configuration:} \qquad & |S_{3\phi}| = 3|S_\phi| = 3|V_{an}|\,|I_{an}| \\ & |V_{sc}| = |V_{an}| = \beta \left| \frac{V_{\text{pri}}}{\sqrt{3}\, e^{\mathbf{i}\pi/6}} \right|, \qquad |I_{sc}| = |I_{an}| = \frac{|S_{3\phi}|}{3\left| \frac{V_{\text{pri}}}{\sqrt{3}\, e^{\mathbf{i}\pi/6}} \right|} = \frac{|S_{3\phi}|}{\sqrt{3}|V_{\text{pri}}|} \end{aligned}$$

Since $z_l = V_{sc}/I_{sc}$ we therefore have,

$$\Delta \text{ configuration:} \quad |z_l| = \frac{3\beta |V_{\text{pri}}|^2}{|S_{3\phi}|}; \qquad Y \text{ configuration:} \quad |z_l| = \frac{\beta |V_{\text{pri}}|^2}{|S_{3\phi}|} \tag{3.8b}$$

As mentioned above, V_{pri} denotes the line-to-line voltage even for Y configuration; otherwise $|z_l| = 3\beta |V_{\text{pri}}|^2/|S_{3\phi}|$ for Y configuration if the rated voltage V_{pri} is line-to-neutral.

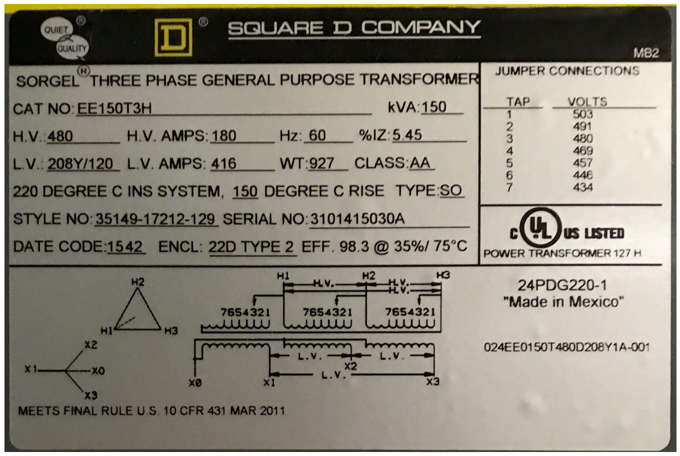


Figure 3.6: The transformer ratings.

Example 3.2 (Transformer ratings)**.** Figure 3.6 shows a typical specification of a three-phase transformer in ΔY configuration:

- Three-phase power rating $|S_{3\phi}| = 150\,\text{kVA}$.
- Rated primary line-to-line (high) voltage $|V_{\text{pri}}| = 480\,\text{V}$ in Δ configuration with rated primary line current $|I_{\text{pri}}| = 180\,\text{A}$.
- Rated secondary line-to-line (low) voltage $|V_{\text{sec}}| = 208Y/120\,\text{V}$ in Y configuration with rated secondary line current $|I_{\text{sec}}| = 416\,\text{A}$. This notation means that the secondary side is Y-configured with a line-to-line voltage of 208 V and line-to-neutral voltage of 120 V.
- Impedance voltage $\beta = 5.45\%$ on the primary side (the shunt admittance is assumed zero).

Verify that the rated line currents on the primary and secondary sides are consistent with the power rating and voltage ratings. Determine the magnitude $|z_l|$ of the leakage impedance of the transformer.

Solution. The primary side is in Δ configuration and hence we have

$$|S_{3\phi}| = 3\,|S_{ab}| \;=\; 3\,|V_{ab}\,\bar{I}_{ab}| \;=\; 3\,|V_{\text{pri}}|\,|I_{ab}|$$

Since (assuming balanced positive sequence)

$$I_a \;=\; I_{ab} - I_{ca} \;=\; I_{ab}\left(1 - e^{\mathbf{i}2\pi/3}\right) \;=\; I_{ab}\cdot\sqrt{3}\,e^{-\mathbf{i}\pi/6}$$

we have $|I_{\text{pri}}| = \sqrt{3}\,|I_{ab}|$. Hence

$$|S_{3\phi}| \;=\; \sqrt{3}\,|V_{\text{pri}}|\,|I_{\text{pri}}|$$

The rated line-to-line voltage $|V_{\text{pri}}| = |V_{ab}| = 480\text{V}$. The rated line current $|I_{\text{pri}}| = |I_a| = 180\text{A}$. Hence

$$\sqrt{3}\,|V_{\text{pri}}|\,|I_{\text{pri}}| \;=\; \sqrt{3}\cdot 480\cdot 180 \;=\; 149.65\text{ kVA}$$

which is approximately the power rating $|S_{3\phi}| = 150$ kVA.

The secondary side is in Y configuration and hence we have

$$|S_{3\phi}| \;=\; 3\,|S_{an}| \;=\; 3\,|V_{an}\,\bar{I}_{an}| \;=\; 3\,\left|\frac{V_{\text{sec}}}{\sqrt{3}e^{\mathbf{i}\pi/6}}\right|\,|I_{\text{sec}}| \;=\; \sqrt{3}\,|V_{\text{sec}}|\,|I_{\text{sec}}|$$

where the third equality follows since $V_{\text{sec}} = V_{ab} = V_{an}\left(\sqrt{3}e^{\mathbf{i}\pi/6}\right)$ is the line-to-end voltage. The rated secondary line-to-line voltage is $|V_{\text{sec}}| = 208\,\text{V}$ and the line current $|I_{\text{sec}}| = 416\,\text{A}$, and hence

$$\sqrt{3}\,|V_{\text{sec}}|\,|I_{\text{sec}}| \;=\; \sqrt{3}\cdot 208\cdot 416 \;=\; 149.87\text{kVA}$$

which is approximately the power rating 150 kVA.

From (3.8a) the magnitude $|z_l|$ of the leakage impedance of each single-phase transformer is (β is the impedance voltage on the *primary* side)

$$|z_l| \;=\; \frac{\sqrt{3}\beta\,|V_{\text{pri}}|}{|I_{\text{pri}}|} \;=\; \frac{\sqrt{3}\cdot 5.45\%\cdot 480\,\text{V}}{180\,\text{A}} \;=\; 0.2517\,\Omega$$

$\square$

Distribution system transformers. In the US, single-phase or three-phase stepdown transformers are typical in the distribution system. The most common three-phase system voltage on the primary side is 12.47 kV, serving more than 50% of loads. This is the line-to-line voltage (magnitude) and hence the line-to-neutral voltage is $|V_{an}| = 12.47/\sqrt{3} = 7.2\,\text{kV}$. A typical primary side current rating is $|I_{an}| = 400$A. Hence the total (three-phase) rated apparent power is $|S_{3\phi}| = 3|V_{an}||I_{an}| = (3)(7.2)(400) = 8.6\,\text{MVA}$. Other common distribution system voltages and their total power at 400A are shown in Table 3.1. The

line-to-line voltage (kV) $\lvert V_{ab}\rvert$	phase voltage (kV) $\lvert V_{an}\rvert$	total power (MVA) $\lvert S_{3\phi}\rvert$
4.8	2.8	3.3
12.47	7.2	8.6
22.9	13.2	15.9
34.5	19.9	23.9

Table 3.1: Typical distribution system voltages (line-to-line) and their total (three-phase) power rating at 400A current.

advantages of a higher-voltage system include:

- It can carry more power for a given ampacity.
- It has a smaller voltage drop for a given level of power flow, requiring fewer voltage regulators and capacitor banks for voltage support (see Exercise 2.7.5).
- It has a smaller line loss for a given level of power flow (see Exercise 2.7).

- It can cover a larger service area since it has a smaller voltage drop and a smaller line loss. Roughly, for the same load density, the area covered increases linearly with voltage.

- It requires fewer substations since it covers a larger service area, which can be a big cost saving.

The disadvantages of a higher-voltage system include:

- It may be less reliable, since a longer circuit can lead to more customer interruptions.

- Crew safety is a bigger concern with a higher voltage.

- Higher voltage equipment costs more, from transformers to cables to voltage regulators.

The 12.47 kV system seems to strike a good balance.

On the primary side, one end of the winding typically connects to one of the primary phases and the other end connects to the transformer case which is connected to the neutral wire of the three-phase system and also earth ground. On the secondary side, the 240V is center-tapped and the center neutral wire is grounded, making the two ends “hot” with respect to the center tap. These three wires run down the service drop to the meter and electric panel of a house. This is shown in Figure 3.7. Connecting a

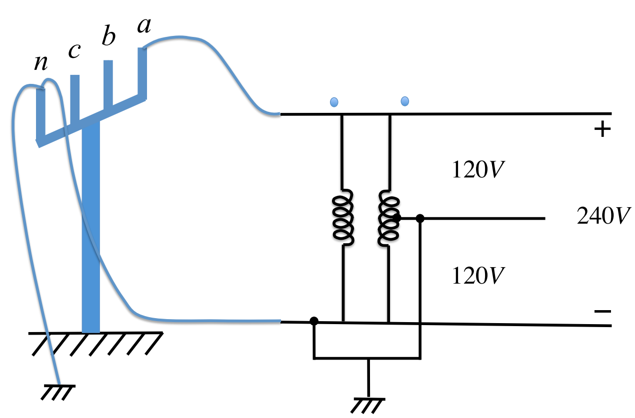
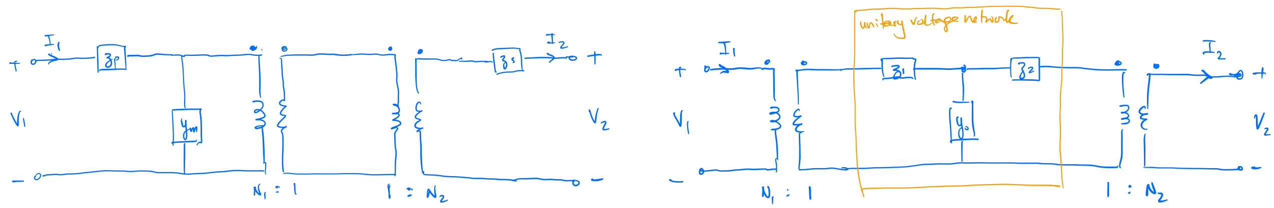


Figure 3.7: A common single-phase distribution transformer in the US.

load between either hot wire and the neutral gives 120V while connecting it between both hot wires gives 240V. Note that the transformer is single-phase. This is the split-phase 120/240 V system typical in the US.

3.1.5 Model with unitary voltage network

Single-phase two-winding transformer. As far as the end-to-end behavior is concerned, the transformer model in Figure 3.2(b) is equivalent to the model in Figure 3.8(a) where the ideal transformer with turns



(a) Equivalent model (b) Unitary voltage network

Figure 3.8: Models of nonideal transformer with unitary voltage network.

ratio N_1/N_2 is replaced by two ideal transformers in series with turns ratios N_1 and $1/N_2$. Referring the leakage impedances (z_p, z_s) and shunt admittance y_m to the other sides of the ideal transformers using (3.14) in Chapter 3.3, this model is equivalent to the one in Figure 3.8(b) where

$$y_0 := N_1^2 y_m, \qquad z_1 := \frac{z_p}{N_1^2}, \qquad z_2 := \frac{z_s}{N_2^2} \tag{3.9}$$

The network between the two ideal transformers is sometimes referred to as a unitary voltage network because the nominal voltage of the network is 1 pu if the scaled nominal voltages $V_1^{\text{nom}}/N_1 = V_2^{\text{nom}}/N_2$ on both sides of the (nonideal) transformer is used as the voltage base for per unit normalization. Note that no nodes in the transformer models may be grounded. The main advantage of modeling a nonideal transformer this way is that the unitary voltage network can be generalized from the simple network in Figure 3.8(b) to a more general network that can be used to model nonstandard transformers with multiple windings; see below.

We now derive the admittance matrix that maps (V_1, V_2) to $(I_1, -I_2)$. First focus on the unitary voltage network, shown in Figure 3.9, where $y_1 := 1/z_1 = N_1^2 y_p$, $y_2 := 1/z_2 = N_2^2 y_s$ with $y_p := 1/z_p$, $y_s := 1/z_s$. Variables with hats denote internal variables.[2] The variables $(\hat{V}_0, \hat{V}_1, \hat{V}_2)$ are defined as voltage drops as

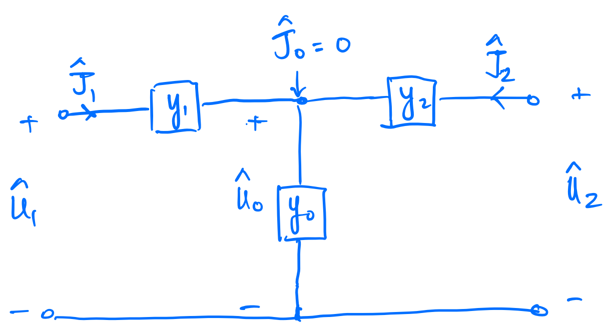


Figure 3.9: Unitary voltage network of the model in Figure 3.8(b). (Megan: Change $\hat{J}_0, \hat{J}_1, \hat{J}_2 \rightarrow \hat{I}_0, \hat{I}_1, \hat{I}_2$ and $\hat{U}_0, \hat{U}_1, \hat{U}_2 \rightarrow \hat{V}_0, \hat{V}_1, \hat{V}_2$. Same figure as Figure 8.17 except for labels.)

shown in the figure and $(\hat{I}_0, \hat{I}_1, \hat{I}_2)$ are the current injections at these nodes with $\hat{I}_0 := 0$. Then

$$\hat{I}_1 = y_1(\hat{V}_1 - \hat{V}_0), \qquad \hat{I}_2 = y_2(\hat{V}_2 - \hat{V}_0), \qquad \hat{I}_0 + \hat{I}_1 + \hat{I}_2 = y_0 \hat{V}_0 \tag{3.10}$$

[2] The explicit separation of internal variables (e.g., $\hat{V}_i, \hat{I}_i$) and terminal variables (e.g., V_i, I_i) may not be significant for single-phase devices but turns out to be crucial in modeling three-phase devices; see Chapters 7 and 8.

or in terms of admittance matrix (we will study admittance matrices in detail in Chapter 4)

$$\begin{bmatrix} \hat{I}_0 \\ \hat{I}_1 \\ \hat{I}_2 \end{bmatrix} = \begin{bmatrix} y_0+y_1+y_2 & -y_1 & -y_2 \\ -y_1 & y_1 & 0 \\ -y_2 & 0 & y_2 \end{bmatrix} \begin{bmatrix} \hat{V}_0 \\ \hat{V}_1 \\ \hat{V}_2 \end{bmatrix}$$

Since $\hat{I}_0 = 0$ we can eliminate $\hat{V}_0$ and derive the Kron-reduced admittance matrix Y_{uvn} that maps $(\hat{V}_1, \hat{V}_2)$ to $(\hat{I}_1, \hat{I}_2)$. Let $\hat{I} := (\hat{I}_1, \hat{I}_2)$ and $\hat{V} := (\hat{V}_1, \hat{V}_2)$. Then $\hat{I} = Y_{\text{uvn}}\hat{V}$ where Y_{uvn} is the Schur complement of $y_0+y_1+y_2$ (see Appendix 25.1.2.1 for details of Schur complement):

$$Y_{\text{uvn}} := \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} - \frac{1}{\sum_{i=0}^{2} y_i} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} y_1 & y_2 \end{bmatrix} = \frac{1}{\sum_i y_i} \begin{bmatrix} y_1(y_0+y_2) & -y_1y_2 \\ -y_1y_2 & y_2(y_0+y_1) \end{bmatrix} \tag{3.11a}$$

Next connect the two ideal transformers to each side of the unitary voltage network; see Figure 3.8(b). Let $I := (I_1, -I_2)$ and $V := (V_1, V_2)$. The conversion between internal variables $(\hat{V}, \hat{I})$ and terminal variables (V, I) is $\hat{V} = MV$ and $\hat{I} = M^{-1}I$ where

$$M := \begin{bmatrix} 1/N_1 & 0 \\ 0 & 1/N_2 \end{bmatrix} \tag{3.11b}$$

Substituting into $\hat{J} = Y_{\text{uvn}}\hat{U}$ we obtain the relation between the terminal variables V to I:

$$I = (MY_{\text{uvn}}M)\, V \tag{3.11c}$$

where $MY_{\text{uvn}}M$ is called the admittance matrix of the transformer. It can be shown that (3.11) is equivalent to the T equivalent circuit (3.5) (Exercise 3.4). As a consequence the model parameters (y_0, y_1, y_2) cannot be uniquely determined by just the short-circuit and open-circuit tests.

We often do not know the numbers N_1, N_2 of turns of the primary and secondary windings respectively, but can determine the turns ratio $a := N_1/N_2$ from the specified rated voltages. The admittance matrix $MY_{\text{uvn}}M$ can also be written in terms of the turns ratio a (Exercise 3.5):

$$Y_{YY} := MY_{\text{uvn}}M = \frac{y_p y_s}{a^2 y_m + a^2 y_p + y_s} \begin{bmatrix} 1 + a^2 y_m/y_s & -a \\ -a & a^2(1 + y_m/y_p) \end{bmatrix} \tag{3.11d}$$

If $y_0 = y_m = 0$ then both (3.5) and (3.11) are equivalent to the simplified model in Figure 3.5(b). In this case the model parameter is just the leakage impedance z_l in the primary circuit, which can be determined from standard power ratings as described above. Recall that $z_l = z_p + a^2 z_s$ and hence the leakage admittance in the simplified model is

$$y_l = \frac{1}{z_l} = \frac{1}{1/y_p + a^2 1/y_s} = \frac{y_p y_s}{a^2 y_p + y_s}$$

Indeed, when $y_m = 0$, the admittance matrix Y_{YY} is the same for both the simplified model and the unitary voltage network model, from (3.11d):

$$Y_{YY} = MY_{\text{uvn}}M = y_l \begin{bmatrix} 1 & -a \\ -a & a^2 \end{bmatrix}$$

Multi-winding transformers. The single-phase circuit model in Figure 3.8(b) can be generalized in two ways, or a combination. First, multiple copies of the single-phase model can be connected in Δ or Y configuration on each side to create models for three-phase transformers. This is derived in detail in Chapter 8.3 for unbalanced three-phase systems. Second, the unitary voltage network can be generalized to model nonstandard transformers with more than two windings. As an illustration we now use this approach to model a split-phase transformer.

Figure 3.10 shows a single-phase split-phase transformer. The internal voltages $(\hat{V}_0, \hat{V}_1, \hat{V}_2, \hat{V}_3)$ and

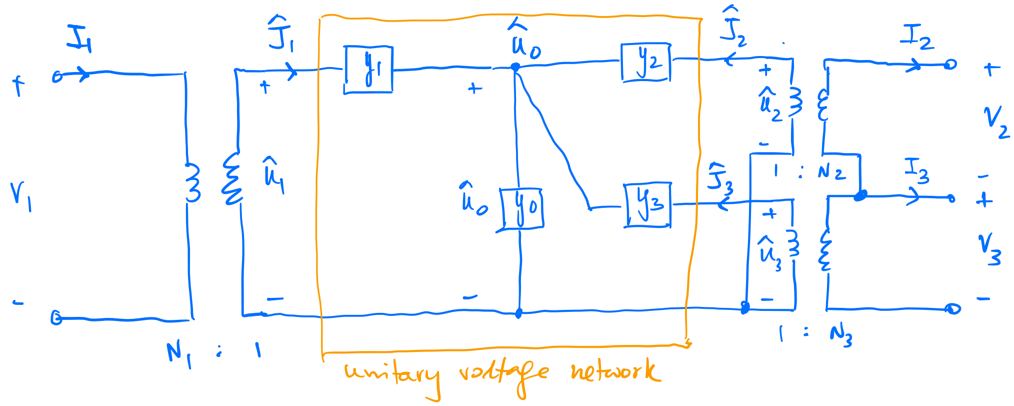


Figure 3.10: Single-phase split-phase transformer. (Megan: Change $\hat{J}_0, \hat{J}_1, \hat{J}_2, \hat{J}_3 \to \hat{I}_0, \hat{I}_1, \hat{I}_2, \hat{I}_3$ and $\hat{U}_0, \hat{U}_1, \hat{U}_2, \hat{U}_3 \to \hat{V}_0, \hat{V}_1, \hat{V}_2, \hat{V}_3$. Remove the $\hat{U}_0$ at the top.)

currents $(\hat{I}_0, \hat{I}_1, \hat{I}_2, \hat{I}_3)$ on the unitary voltage network are defined in the figure. The admittance matrix that maps these voltages to currents is given by:

$$\begin{bmatrix} \hat{I}_0 \\ \hat{I}_1 \\ \hat{I}_2 \\ \hat{I}_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^3 & -y_1 & -y_2 & -y_3 \\ -y_1 & y_1 & 0 & 0 \\ -y_2 & 0 & y_2 & 0 \\ -y_3 & 0 & 0 & y_3 \end{bmatrix} \begin{bmatrix} \hat{V}_0 \\ \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \end{bmatrix}$$

Let $\hat{V} := (\hat{V}_1, \hat{V}_2, \hat{V}_3)$ and $\hat{I} := (\hat{I}_1, \hat{I}_2, \hat{I}_3)$. Since $\hat{I}_0 = 0$ we can eliminate $\hat{V}_0$ to relate $\hat{I} = Y_{\text{uvn}} \hat{V}$ where Y_{uvn} is the Kron-reduced admittance matrix:

$$\begin{aligned} Y_{\text{uvn}} &:= \begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} - \frac{1}{\sum_{i=0}^3 y_i} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \\ &= \frac{1}{\sum_i y_i} \begin{bmatrix} y_1(y_0 + y_2 + y_3) & -y_1 y_2 & -y_1 y_3 \\ -y_2 y_1 & y_2(y_0 + y_1 + y_3) & -y_2 y_3 \\ -y_3 y_1 & -y_3 y_2 & y_3(y_0 + y_1 + y_2) \end{bmatrix} \end{aligned} \tag{3.12a}$$

This extends in a straightforward manner Y_{uvn} in (3.11) from two to three windings.

Next we connect ideal transformers to the unitary voltage network as shown in Figure 3.10. The terminal voltages $V := (V_1, V_2, V_3)$ and currents $I := (I_1, -I_2, -I_3)$, as well as the internal current $\hat{I}_3$ into the third winding, are defined in the figure. Let $M := \text{diag}(1/N_1, 1/N_2, 1/N_3)$. Then $\hat{V} = MV$ and, using

$I_2 + I_3 + \hat{I}_3 = 0$,

$$\hat{I} \;=\; M^{-1}\begin{bmatrix} I_1 \\ -I_2 \\ \hat{I}_3 \end{bmatrix} \;=\; M^{-1}\begin{bmatrix} I_1 \\ -I_2 \\ -I_2 - I_3 \end{bmatrix} \;=:\; M^{-1}AI$$

where

$$A \;:=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{3.12b}$$

Substituting into $\hat{I} = Y_{\text{uvn}}\hat{V}$ we obtain the relation between the terminal variables V to I:

$$I \;=\; A^{-1}\left(MY_{\text{uvn}}M\right)V \tag{3.12c}$$

3.2 Three-phase transformers

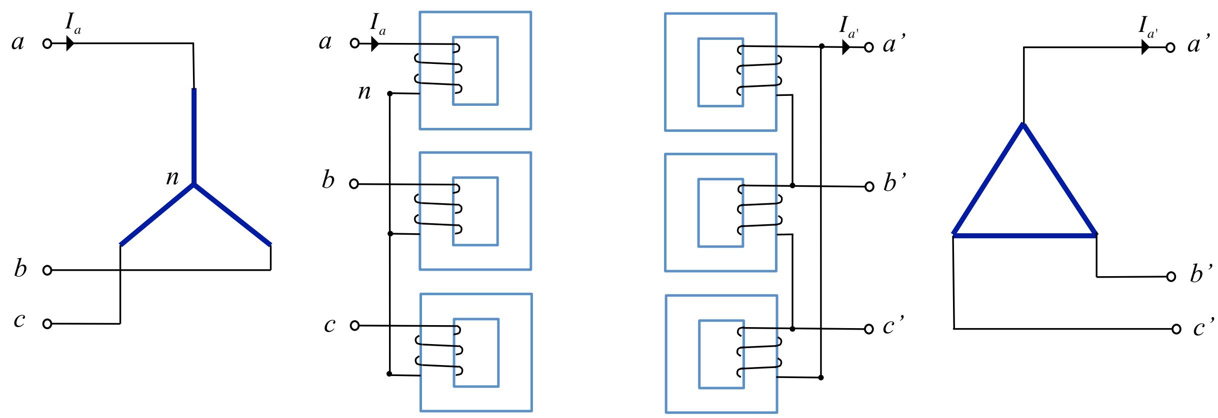
In this section we develop models for a balanced three-phase transformer and derive its single-phase equivalent.

3.2.1 Ideal transformers

The primary and secondary circuits of a three-phase transformer can be arranged in four different configurations: YY, $\Delta\Delta$, ΔY, $Y\Delta$. Figure 3.11(a) shows a primary three-phase winding in Y configuration and its schematic diagram. The winding on the first magnetic core goes from terminal a to neutral n and then connects with the neutral terminals on the second and third magnetic cores. It matches the connectivity in the schematic diagram where the windings are indicated by the thick lines. Figure 3.11(b) shows a secondary three-phase winding in Δ configuration and its schematic diagram. In both diagrams, the windings go from terminal a on the first magnetic core to terminal b on the second magnetic core to terminal c on the third magnetic core.

We now derive the properties of ideal three-phase transformers in $YY, \Delta\Delta, \Delta Y, Y\Delta$ configurations. We will first derive the terminal behavior. This means the line-to-line voltage gain and the line current gains in these configurations. We then derive the line-to-neutral voltage and current gains of their YY equivalent models, which yields their per-phase circuits. We will see that, as expected, the terminal behavior of a three-phase transformer has the same gains as those in its per-phase circuit. The derivation proceeds in three steps:

1. *Internal model*: Derive the voltage and current gains for each single-phase transformer.
2. *External model*: Derive the line-to-line voltage gains and line current gains for the three-phase transformer.
3. *YY equivalent*: Derive the YY equivalent circuit from which a per-phase circuit can be obtained.



(a) Primary winding in Y configuration (b) Secondary winding in Δ configuration

Figure 3.11: Primary and secondary windings in Y and Δ configurations respectively. The thick lines in the schematic diagrams represent transformer windings.

The voltage and current gains in step 1 are defined by the pairing of primary and secondary windings in each single-phase ideal transformer and depends on the configuration. Steps 2 and 3 apply the following relation between line voltages/currents and phase voltages/currents to the single-phase voltage and current gains:

$$Y \text{ configuration (line voltages):} \quad V_{ab} = \sqrt{3}\, e^{\mathbf{i}\pi/6} V_{an}, \qquad V_{a'b'} = \sqrt{3}\, e^{\mathbf{i}\pi/6} V_{a'n'} \tag{3.13a}$$

$$\Delta \text{ configuration (line currents):} \quad I_a = \sqrt{3}\, e^{-\mathbf{i}\pi/6} I_{ab}, \qquad I_{a'} = -\sqrt{3}\, e^{-\mathbf{i}\pi/6} I_{a'b'} \tag{3.13b}$$

where the signs of the current gains are different on the primary side (entering terminal a and the secondary sides (leaving terminal a') of Δ configuration. The relations (3.13) follows (1.13) (1.14), reproduced here

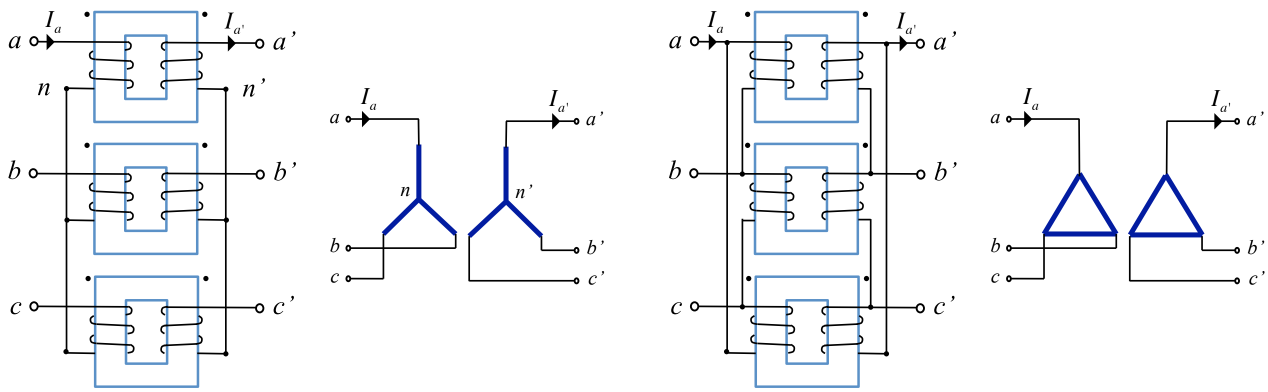
$$Y \text{ configuration:} \qquad V^{\text{line}} = \Gamma V^Y$$
$$\Delta \text{ configuration:} \qquad I = \pm\, \Gamma^{\mathsf{T}} I^{\Delta}$$

Assuming positive sequence, the balanced voltages V^Y and currents I^{Δ} are in span(α_+) and hence Corollary 1.4 implies

$$Y \text{ configuration:} \qquad V^{\text{line}} = (1-\alpha)\, V^Y = \sqrt{3} e^{\mathbf{i}\pi/6} V^Y$$
$$\Delta \text{ configuration:} \qquad I = \pm\, (1-\alpha^2)\, I^{\Delta} = \pm\, \sqrt{3} e^{-\mathbf{i}\pi/6} V^{\Delta}$$

In per-phase analysis later, we will convert each Δ configuration into an equivalent Y configuration. For YY configurations, line voltage/current gains are equal to line-to-neutral voltage/currents gains and therefore, with Y equivalents, we often use the ratios $V_{a'n'}/V_{an}$ and $-I_{a'n'}/I_{an}$ to represent both the internal and the external models.

***YY* configuration.** The winding of an ideal three-phase transformer in YY configuration and its schematic digram are shown in Figure 3.12(a). The parallel lines in the schematic diagram indicate corresponding



(a) YY configuration (b) $\Delta\Delta$ configuration

Figure 3.12: Ideal three-phase transformers in YY and $\Delta\Delta$ configurations. The parallel lines in the schematic diagram indicate corresponding primary and secondary windings.

primary and secondary windings in the single-phase transformers. From the figure, the YY configuration is characterized by the following voltage and current gains:

$$\frac{V_{a'n'}}{V_{an}} = n, \qquad \frac{I_{a'}}{I_a} = \frac{-I_{a'n'}}{I_{an}} = \frac{1}{n}$$

Note the opposite directions of the currents $I_{a'}$ and $I_{a'n'}$. The line voltages, being proportional to phase voltages from (3.13a), have the same ratio, i.e., its external model is the same as its internal model. The voltage and current gains are the same for phases b and c as well under balanced operation.

$\Delta\Delta$ configuration. The winding of an ideal three-phase transformer in $\Delta\Delta$ configuration and its schematic digram are shown in Figure 3.12(b). It is characterized by the following voltage and current gains from the single-phase transformer:

$$\frac{V_{a'b'}}{V_{ab}} = n, \qquad \frac{-I_{a'b'}}{I_{ab}} = \frac{1}{n}$$

Applying (3.13b) to both I_a on the primary side and $I_{a'} = -\sqrt{3}\,e^{-\mathbf{i}\pi/6} I_{a'b'}$ on the secondary side, the external model is

$$\frac{V_{a'b'}}{V_{ab}} = n, \qquad \frac{I_{a'}}{I_a} = \frac{1}{n}$$

Similarly for voltage and current gains on other lines.

Equivalent YY configuration. To calculate the ratio of line-to-neutral voltages of an equivalent YY configuration, we use (3.13a) to obtain

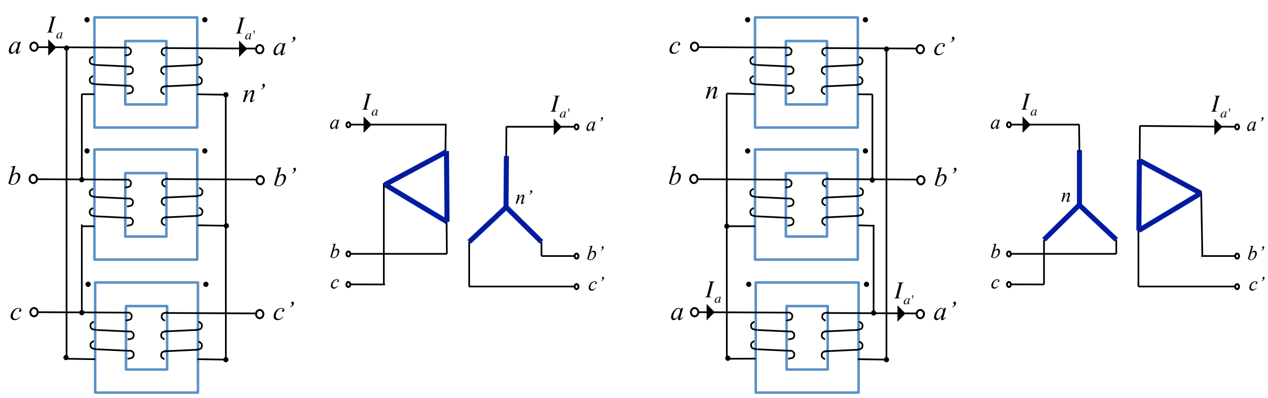
$$\frac{V^Y_{a'n'}}{V^Y_{an}} = \frac{\left(\sqrt{3}\,e^{\mathbf{i}\pi/6}\right)^{-1} V^Y_{a'b'}}{\left(\sqrt{3}\,e^{\mathbf{i}\pi/6}\right)^{-1} V^Y_{ab}} = \frac{V_{a'b'}}{V_{ab}} = n$$

since $V^Y_{a'b'} = V_{a'b'}$ and $V^Y_{ab} = V_{ab}$ by the definition of Y equivalence. To calculate the ratio of the phase currents in the equivalent YY configuration, we use the property that the terminal currents in the $\Delta\Delta$ configuration and its equivalent YY configuration are the same. Therefore $I^Y_{an} = I^Y_a = I_a$ and $-I^Y_{a'n'} = I^Y_{a'} = I_{a'}$. Hence

$$\frac{-I^Y_{a'n'}}{I^Y_{an}} \;=\; \frac{I_{a'}}{I_a} \;=\; \frac{1}{n}$$

Therefore the line voltages and currents in the $\Delta\Delta$ configuration and the phase voltages and currents in its equivalent YY configuration have the same ratios. The voltage and current gains for phases b and c are the same as for phase a under balanced operation.

ΔY configuration. The winding of an ideal three-phase transformer in ΔY configuration and its schematic digram are shown in Figure 3.13(a). It is characterized by the following voltage and current gains in the



(a) ΔY configuration (b) $Y\Delta$ configuration

Figure 3.13: Ideal three-phase transformers in ΔY and $Y\Delta$ configurations. The parallel lines in the schematic diagram indicate corresponding primary and secondary windings. Note the wiring of the $Y\Delta$ configuration.

single-phase transformer:

$$\frac{V_{a'n'}}{V_{ab}} = n, \qquad \frac{-I_{a'n'}}{I_{ab}} = \frac{1}{n}$$

To obtain the external model we have (assuming positive sequence)

$$\frac{V_{a'b'}}{V_{ab}} \;=\; \frac{\sqrt{3}e^{\mathbf{i}\pi/6}V_{a'n'}}{V_{ab}} \;=\; \sqrt{3}e^{\mathbf{i}\pi/6}n \;=:\; K_{\Delta Y}(n)$$

where the complex voltage gain

$$K_{\Delta Y}(n) \;:=\; \sqrt{3}n\, e^{\mathbf{i}\pi/6}$$

boosts the voltage gain by $\sqrt{3}$ and shifts the phase by 30°. The line current gain is (using (3.13b))

$$\frac{I_{a'}}{I_a} = \frac{-I_{a'n'}}{\sqrt{3}e^{-\mathbf{i}\pi/6}I_{ab}} = \frac{1}{\sqrt{3}e^{-\mathbf{i}\pi/6}n} = \frac{1}{K^*_{\Delta Y}(n)}$$

Similarly on other lines.

Equivalent YY configuration. We have on the primary side $V^Y_{an} = \left(\sqrt{3}\,e^{\mathbf{i}\pi/6}\right)^{-1} V^Y_{ab} = \left(\sqrt{3}\,e^{\mathbf{i}\pi/6}\right)^{-1} V_{ab}$ since $V^Y_{ab} = V_{ab}$ by definition of Y equivalence. Hence

$$\frac{V_{a'n'}}{V^Y_{an}} = \frac{V_{a'n'}}{\left(\sqrt{3}\,e^{\mathbf{i}\pi/6}\right)^{-1}\,V_{ab}} = \sqrt{3}\,e^{\mathbf{i}\pi/6}\,\frac{V_{a'n'}}{V_{ab}} = K_{\Delta Y}(n)$$

To calculate the phase currents in the equivalent Y configured primary circuit, use (3.13b) to get $I^Y_{an} = I_a = \sqrt{3}e^{-\mathbf{i}\pi/6}\,I_{ab}$. Hence

$$\frac{-I_{a'n'}}{I^Y_{an}} = \frac{-I_{a'n'}}{\sqrt{3}e^{-\mathbf{i}\pi/6}I_{ab}} = \frac{1}{\sqrt{3}n\,e^{-\mathbf{i}\pi/6}} = \frac{1}{K^*_{\Delta Y}(n)}$$

This also implies $I_a = K^*_{\Delta Y}(n)\,I_{a'}$. As expected the voltage and current gains in the ΔY configuration are the same as those in their YY equivalent. Hence the phase voltages and currents on the secondary and primary sides of the YY equivalent configuration are related as:

$$\begin{aligned}
\frac{V_{a'n'}}{V^Y_{an}} &= \frac{V_{b'n'}}{V^Y_{bn}} = \frac{V_{c'n'}}{V^Y_{cn}} = K_{\Delta Y}(n)\\
\frac{I_{a'}}{I_a} &= \frac{I_{b'}}{I_b} = \frac{I_{c'}}{I_c} = \frac{1}{K^*_{\Delta Y}(n)}
\end{aligned}$$

The ΔY connection has several advantages (e.g., a gain of $\sqrt{3}$ in addition to the gain n due to turns ratio) and is the most commonly adopted in practice.

***Y*Δ configuration.** The winding of an ideal three-phase transformer in $Y\Delta$ configuration and its schematic digram are shown in Figure 3.13(b). Note that the windings in phase *an* on the primary side are not paired with the windings in phase $a'b'$ on the secondary side, but with $a'c'$ instead. Otherwise a $Y\Delta$-configured transformer with primary on the left will be the same as a ΔY-configured transformer with primary on the right (see Example 3.6 and Exercise 3.6).

The $Y\Delta$ configuration in Figure 3.13(b) is characterized by the following voltage and current gains in a single-phase transformer:[3]

$$\frac{V_{a'c'}}{V_{an}} = n, \qquad \frac{I_{c'a'}}{I_{an}} = \frac{1}{n}$$

[3]Despite the connectivity, positive sequence still means

$$\begin{aligned}
V_{bn} &= e^{-\mathbf{i}2\pi/3}V_{an}, & V_{cn} &= e^{\mathbf{i}2\pi/3}V_{an}\\
V_{b'c'} &= e^{-\mathbf{i}2\pi/3}V_{a'b'}, & V_{c'a'} &= e^{\mathbf{i}2\pi/3}V_{a'b'}
\end{aligned}$$

It amounts to assumptions about the relative phases of voltages applied to the terminals a,b,c.

To obtain the terminal behavior we have (assuming positive sequence)

$$\frac{V_{a'c'}}{V_{ac}} = \frac{V_{a'c'}}{V_{an} - V_{cn}} = \frac{V_{a'c'}}{\sqrt{3}e^{-\mathbf{i}\pi/6}V_{an}} = \frac{n}{\sqrt{3}}e^{\mathbf{i}\pi/6} =: K_{Y\Delta}(n)$$

The line current gain is

$$\frac{I_{a'}}{I_a} = \frac{I_{c'a'} - I_{a'b'}}{I_{an}} = \frac{\sqrt{3}\,e^{\mathbf{i}\pi/6}\,I_{c'a'}}{I_{an}} = \frac{\sqrt{3}\,e^{\mathbf{i}\pi/6}}{n} = \frac{1}{K^*_{Y\Delta}(n)}$$

Similarly on other lines.

Equivalent YY configuration. To obtain the Y equivalent $V^Y_{a'n'}$ of the Δ configuration we have

$$V_{a'c'} = V^Y_{a'c'} = V^Y_{a'n'} - V^Y_{c'n'} = V^Y_{a'n'}(1 - e^{\mathbf{i}2\pi/3}) = \sqrt{3}e^{-\mathbf{i}\pi/6}\,V^Y_{a'n'}$$

Hence

$$\frac{V^Y_{a'n'}}{V_{an}} = \frac{\left(\sqrt{3}\,e^{-\mathbf{i}\pi/6}\right)^{-1}\,V_{a'c'}}{V_{an}} = \left(\sqrt{3}\,e^{-\mathbf{i}\pi/6}\right)^{-1} n = K_{Y\Delta}(n)$$

Similar to (3.13b), we relate the line and phase currents on the secondary side, $-I^Y_{a'n'} = I_{a'} = \sqrt{3}e^{\mathbf{i}\pi/6}\,I_{c'a'}$. Hence

$$\frac{-I^Y_{a'n'}}{I_{an}} = \frac{\sqrt{3}e^{\mathbf{i}\pi/6}\,I_{c'a'}}{I_{an}} = \frac{\sqrt{3}e^{\mathbf{i}\pi/6}}{n} = \frac{1}{K^*_{Y\Delta}(n)}$$

This also implies $I_a = K^*_{Y\Delta}(n)\,I_{a'}$. As expected the line voltage and current gains in the $Y\Delta$ configuration are the same as those in their YY equivalent. Hence the phase voltages and currents on the secondary and primary sides of the YY equivalent configuration are related as:

$$\frac{V^Y_{a'n'}}{V_{an}} = \frac{V^Y_{b'n'}}{V_{bn}} = \frac{V^Y_{c'n'}}{V_{cn}} = K_{Y\Delta}(n)$$
$$\frac{I_{a'}}{I_a} = \frac{I_{b'}}{I_b} = \frac{I_{c'}}{I_c} = \frac{1}{K^*_{Y\Delta}(n)}$$

Property	Gain
Voltage gain	$K(n)$
Current gain	$\frac{1}{K^*(n)}$
Power gain	1
Sec Z_l referred to pri	$\frac{Z_l}{\lvert K(n)\rvert^2}$

Configuration	Gain
YY	$K_{YY}(n) := n$
$\Delta\Delta$	$K_{\Delta\Delta}(n) := n$
ΔY	$K_{\Delta Y}(n) := \sqrt{3}n\,e^{\mathbf{i}\pi/6}$
$Y\Delta$	$K_{Y\Delta}(n) := \frac{n}{\sqrt{3}}\,e^{\mathbf{i}\pi/6}$

Table 3.2: Ideal complex transformer properties.

Summary. These properties of an ideal three-phase transformer in balanced operation are summarized in Table 3.2. For each configuration, $K(n)$ denotes the complex voltage gain of an ideal three-phase transformer:

$$\begin{aligned} \text{voltage gain} \quad \frac{V_{\text{sec}}}{V_{\text{pri}}} &= K(n) \\ \text{current gain} \quad \frac{I_{\text{sec}}}{I_{\text{pri}}} &= \frac{1}{K^*(n)} \end{aligned}$$

As we have shown, these gains apply to both phase voltages/currents and line voltages/currents in both the original transformer and its YY equivalent. Hence the complex power gain is 1 for ideal transformers:

$$\frac{-S'}{S} \;:=\; \frac{V^Y_{a'n'}(-I^Y_{a'n'})^*}{V^Y_{an}\,(I^Y_{an})^*} \;=\; K(n)\,\frac{1}{K(n)} \;=\; 1$$

It often simplifies per-phase analysis of a balanced system to refer series impedances and shunt admittances on one side to the other side of a transformer. This is explained in Chapter 3.3. In particular, a secondary series impedance Z_l is referred to the primary as $Z_l/|K(n)|^2$ according to (3.14) below. When terminated in a symmetric three-phase impedance load Z_{load} on the secondary side so that $V^Y_{a'n'} = Z_{\text{load}} I^Y_{a'n'}$, the per-phase driving-point impedance on the primary side is

$$\frac{V^Y_{an}}{I^Y_{an}} \;=\; \frac{V^Y_{a'n'}/K(n)}{I^Y_{a'n'}K^*(n)} \;=\; \frac{Z_{\text{load}}}{|K(n)|^2}$$

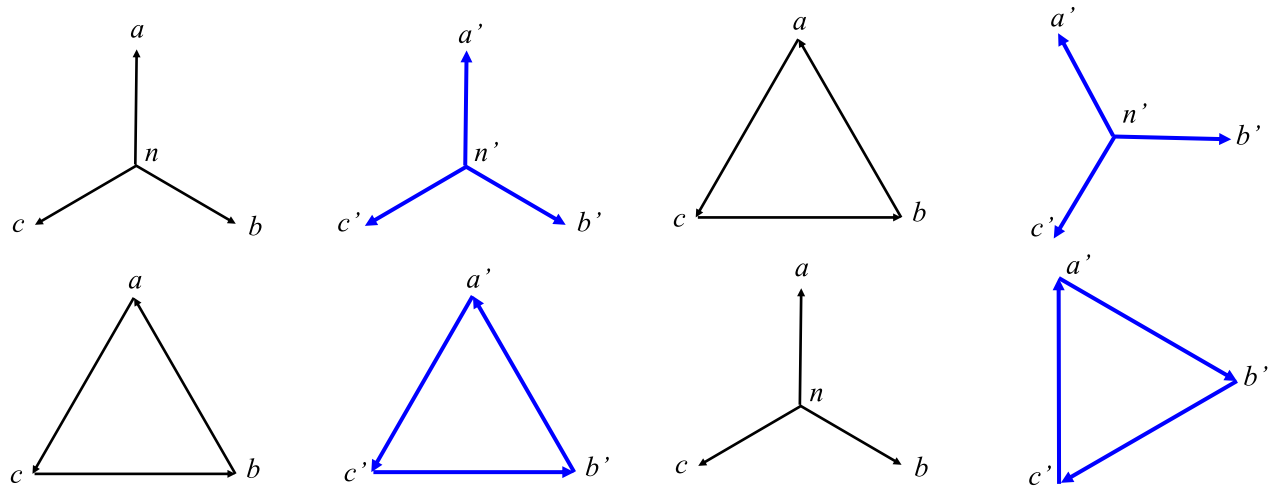
The different configurations of three-phase transformer banks can also be represented compactly as in Figure 3.14 (see its caption for details).

3.2.2 Nonideal transformers

In this section we first present circuit models of (nonideal) three-phase transformers and then their per-phase equivalent circuits after all Δ-configured transformers have been converted into their Y equivalents.

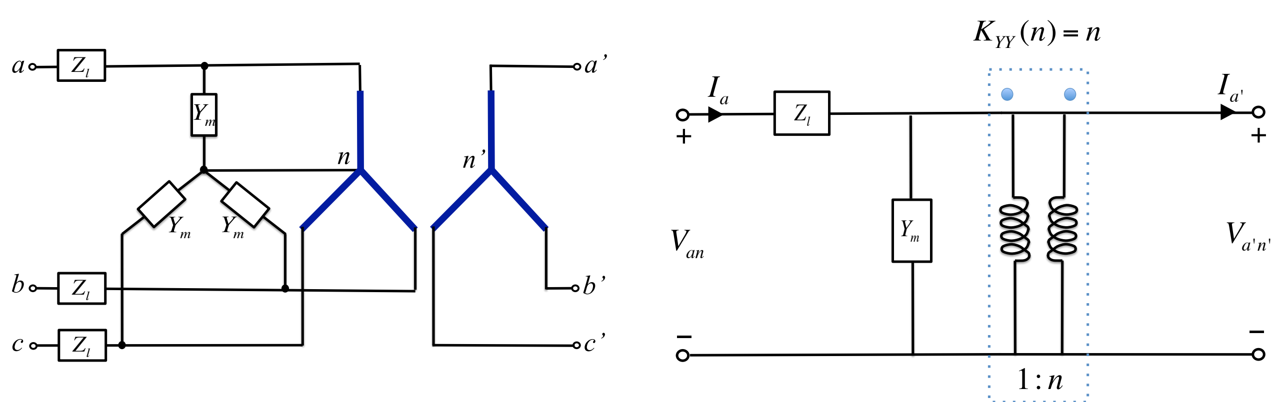
Per-phase equivalent circuits. Figure 3.15(a) shows a model of balanced three-phase (nonideal) transformers in YY configuration and Figure 3.15(b) shows its per-phase equivalent circuit. The per-phase circuit is identical to that in Figure 3.5(b). Figure 3.16(a) shows a model of balanced three-phase transformers in $\Delta\Delta$ configuration. Its YY equivalent and per-phase circuit are identical to those in Figure 3.15 except that the equivalent leakage impedance $Z_l/3$ is one-third of the value in the original $\Delta\Delta$ circuit and the shunt admittance $3Y_m$ is three times the value in the original $\Delta\Delta$ circuit. This can be verified by checking the secondary open-circuit equivalent and the secondary short-circuit equivalent of the original $\Delta\Delta$ circuit. Figure 3.17 shows a model of balanced three-phase transformers in ΔY configuration and its per-phase equivalent circuit. Finally Figure 3.18 shows the model for $Y\Delta$ configuration and its per-phase circuit.

Hence balanced three-phase transformers in YY, $\Delta\Delta$, ΔY and $Y\Delta$ configurations all have the same per-phase equivalent circuit, with appropriate values for their leakage impedance and shunt admittance and the corresponding (complex) transformer gains $K(n)$.



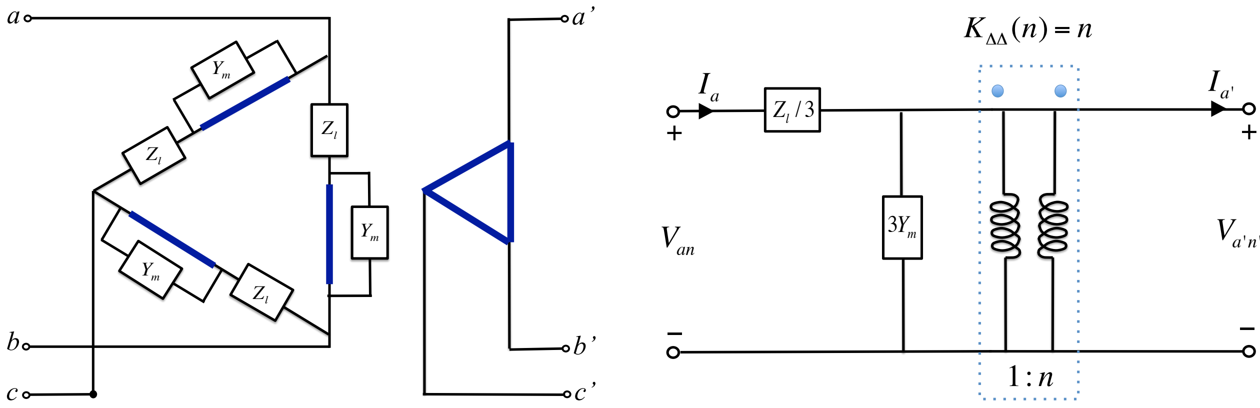
(a) YY and $\Delta\Delta$ configurations (b) ΔY and $Y\Delta$ configurations

Figure 3.14: Compact representation of ideal three-phase transformers in (a) YY, $\Delta\Delta$ configurations and (b) ΔY, $Y\Delta$ configurations. For instance, in the YY configuration, the vertical arrow represents the vector V_{an} in the complex plane. The arrow from b to a (not shown) represents the vector V_{ab}. The parallel lines in the diagram indicate corresponding primary and secondary windings.



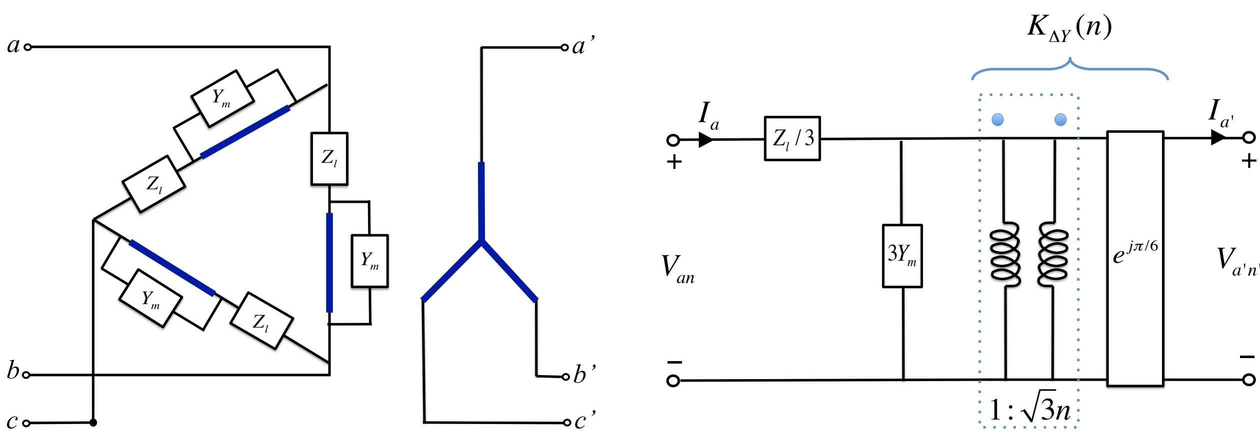
(a) YY configuration (b) Per-phase circuit

Figure 3.15: Model of three-phase transformers in YY configuration and its per-phase equivalent circuit.



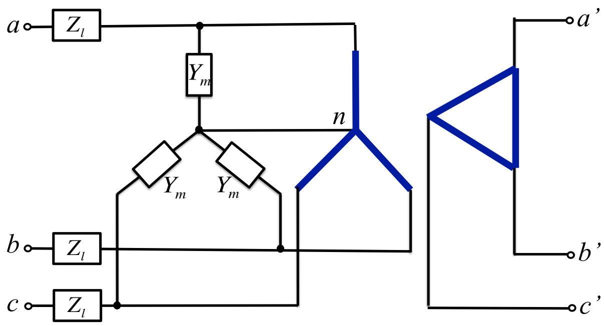
(a) $\Delta\Delta$ configuration (b) Per-phase circuit

Figure 3.16: Model of three-phase transformers in $\Delta\Delta$ configuration and its per-phase equivalent circuit.

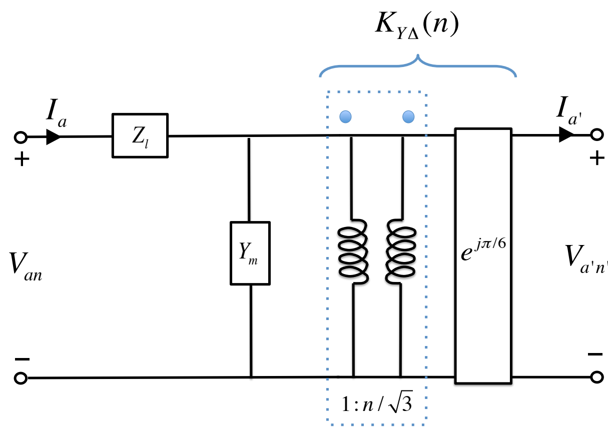


(a) ΔY configuration (b) Per-phase circuit

Figure 3.17: Model of three-phase transformers in ΔY configuration and its per-phase equivalent circuit.



(a) $Y\Delta$ configuration



(b) Per-phase circuit

Figure 3.18: Model of three-phase transformers in $Y\Delta$ configuration and its per-phase equivalent circuit.

3.3 Equivalent impedance in transformer circuit

In this subsection we explain how to derive an "equivalent" impedance when looking into the terminal, either on the primary side or on the secondary side of a transformer. Consider the singe-phase equivalent circuit of a balanced three-phase transformer. A series impedance Z_s in the secondary circuit of the transformer can be equivalently replaced by a series impedance Z_p in the primary circuit, and vice versa, provided they are related by:

$$Z_p \;=\; \frac{Z_s}{|K(n)|^2} \qquad \text{or equivalently} \qquad Z_s \;=\; |K(n)|^2\, Z_p \tag{3.14a}$$

The first operation in (3.14a) is called *referring Z_s in the secondary to the primary*. The second operation is called *referring Z_p in the primary to the secondary*. A shunt admittance Y_s in the secondary circuit of the transformer can be equivalently replaced by a shunt admittance Y_p in the primary circuit, and vice versa, provided they are related by:

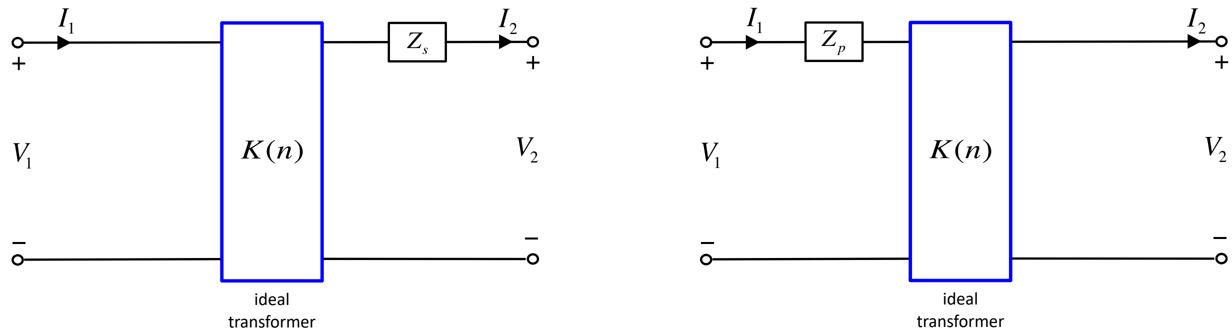
$$Y_p \;=\; |K(n)|^2\, Y_s \qquad \text{or equivalently} \qquad Y_s \;=\; \frac{Y_p}{|K(n)|^2} \tag{3.14b}$$

These operations will be used as a shortcut in the analysis of circuits that contain transformers the same way we use the Thévenin equivalent of impedances in series or in parallel; see Chapter 3.4.

Here "equivalence" means that the external behavior remains unchanged when a series impedance or a shunt admittance on one side is referred to the other. Specifically we consider two kinds of external behavior. In the first case, explained in Chapter 3.3.1, the external behavior is the transmission matrix that maps (V_2, I_2) to (V_1, I_1). In the second case, explained in Chapter 3.3.2, the external behavior is the driving-point impedance on one side of the transformer when the other side is connected to an impedance. We next derive (3.14) as a simple consequence of Kirchhoff's and Ohm's laws.

3.3.1 Transmission matrix

Consider the per-phase transformer circuits in Figure 3.19 of a balanced three-phase system, one with a series impedance in the secondary circuit and the other in the primary circuit. Let T_s and T_p denote the



(a) Series impedance Z_s in the secondary circuit. (b) Series impedance Z_p in the primary circuit.

Figure 3.19: Referring series impedance in the secondary to the primary.

transmission matrices that maps (V_2, I_2) to (V_1, I_1) in Figure 3.19(a) and Figure 3.19(b) respectively. We claim that the relation (3.14a) between series impedances Z_p and Z_s ensures that $T_s = T_p$. It is in this sense that we say these two circuits are equivalent.

To show that $T_s = T_p$ let V denote the voltage at the secondary terminal of the ideal transformer in Figure 3.19(a). Then

$$V = V_2 + Z_s I, \qquad I = I_2$$

or

$$\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} 1 & Z_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Then

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} K^{-1}(n) & 0 \\ 0 & K^*(n) \end{bmatrix} \begin{bmatrix} 1 & Z_s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \underbrace{\begin{bmatrix} K^{-1}(n) & K^{-1}(n) Z_s \\ 0 & K^*(n) \end{bmatrix}}_{T_s} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

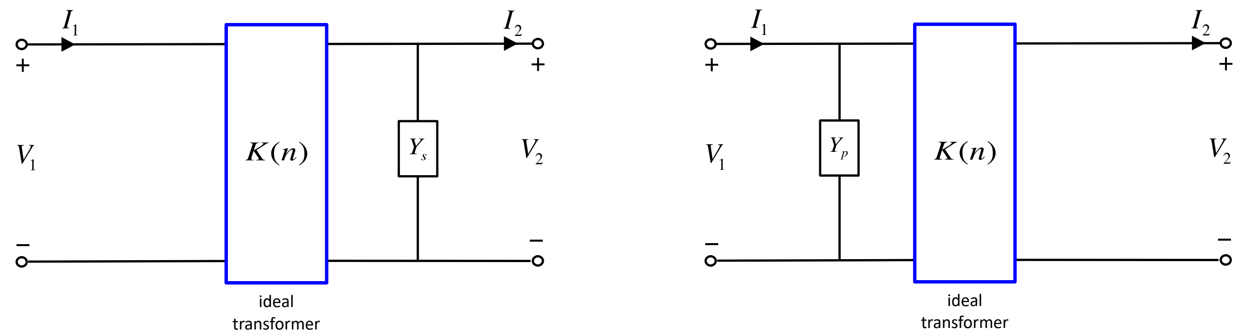
Similarly, for the circuit in Figure 3.19(b), we have

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 & Z_p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} K^{-1}(n) & 0 \\ 0 & K^*(n) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} = \underbrace{\begin{bmatrix} K^{-1}(n) & K^*(n) Z_p \\ 0 & K^*(n) \end{bmatrix}}_{T_p} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Hence $T_s = T_p$ if and only if (3.14a) holds.

The relation (3.14b) between shunt admittances Y_p and Y_s ensures that the transmission matrix for the circuit in Figure 3.20(a) is the same as that in Figure 3.20(b). This is left as Exercise 3.8.

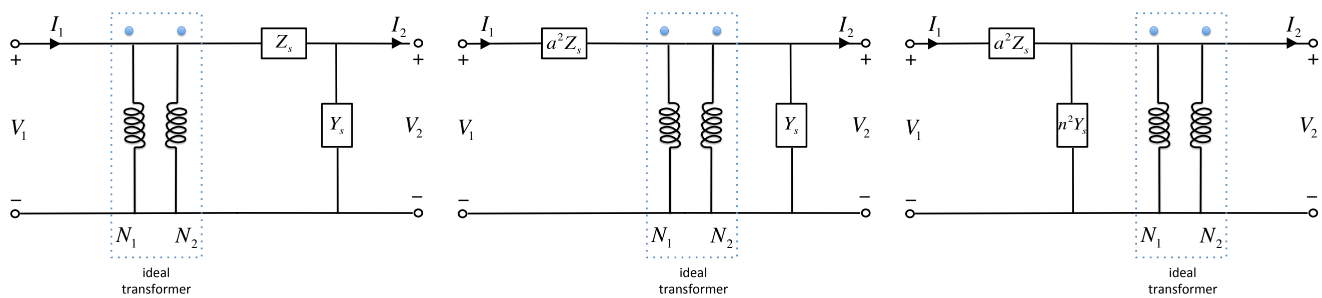
The operations in (3.14) can be repeatedly applied to a circuit involving multiple impedances and admittances, as illustrated in the next example.



(a) Shunt admittance Y_s in the secondary circuit. (b) Shunt admittance Y_p in the primary circuit.

Figure 3.20: Referring shunt admittance in the secondary to the primary.

Example 3.3. A combination of a series impedance Z_s and a shunt admittance Y_s in the secondary circuit, as shown in Figure 3.21(a), can be referred to the primary one element at a time, starting from the element that is *closest to* the ideal transformer. The transformer gain is $K(n) = n = 1/a$. Referring the series



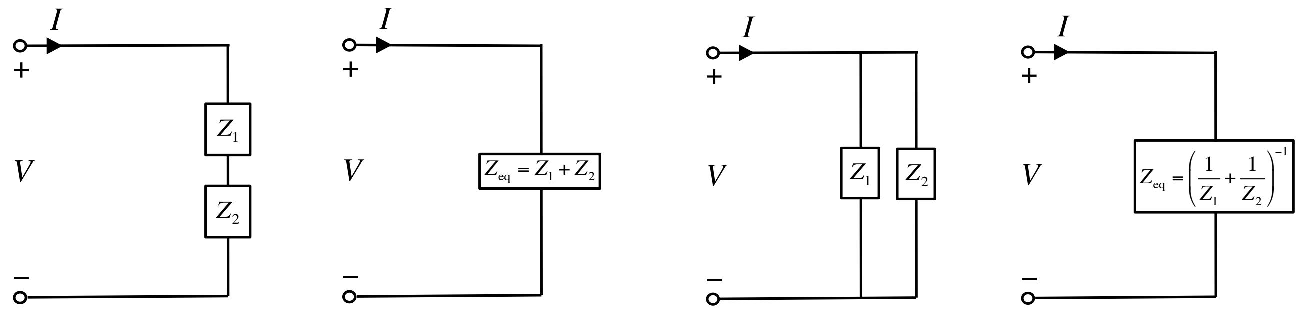
(a) (Z_s, Y_s) in the secondary circuit. (b) Refer Z_s to the primary. (c) Refer Y_s to the primary.

Figure 3.21: Referring (Z_s, Y_s) in the secondary to the primary.

impedance Z_s to the primary yields the equivalent circuit in Figure 3.21(b) with an equivalent primary impedance $a^2 Z_s$. Referring then the shunt admittance Y_s to the primary yields the equivalent circuit in Figure 3.21(c) with an equivalent shunt admittance $n^2 Y_s$. □

3.3.2 Driving-point impedance

In the second case the external behavior is the driving point impedances on one side of the transformer when the other side is connected to an impedance. In general suppose we apply a voltage V across two terminals that are connected to a network of impedances and transformers. Suppose a current I flows between these two terminals through the network. The ratio V/I is called the driving-point impedance at these terminals. For networks consisting of a cascade of impedances in series and in parallel, the driving-point impedance is also called the Thévenin equivalent impedance. The Thévenin equivalent impedance of such a network can be derived by repeatedly applying simple reduction rules for the two basic configurations shown in Figure 3.22. For two impedances Z_1, Z_2 in series depicted in Figure 3.22(a), the Thévenin equiv-



(a) Impedances in series

(b) Impedances in parallel

Figure 3.22: (a) Thévenin equivalent Z_{eq} of two impedances Z_1, Z_2 in series. (b) Thévenin equivalent Z_{eq} of two impedances Z_1, Z_2 in parallel.

alent impedance Z_{eq} is defined such that the two networks in Figure 3.22(a) have the same driving-point impedance:

$$\frac{V}{I} \;=\; Z_1 + Z_2 \;=:\; Z_{\text{eq}} \tag{3.15a}$$

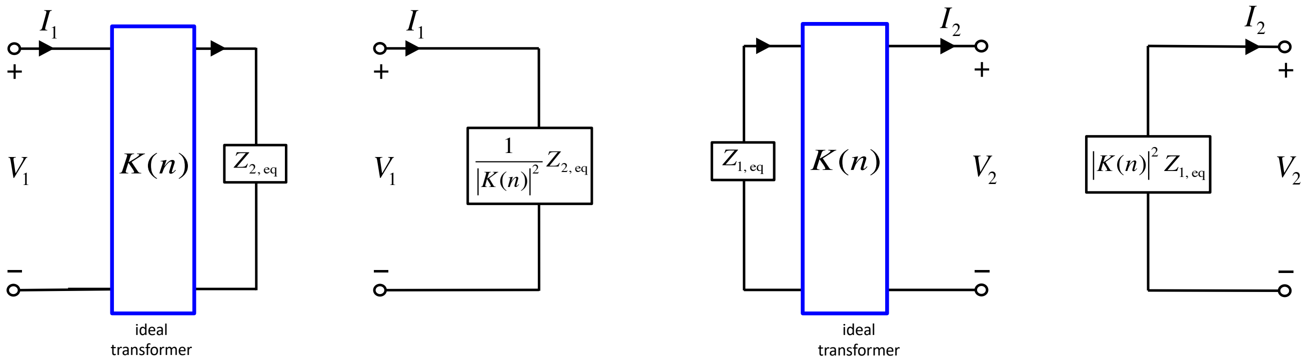
Similarly the Thévenin equivalent impedance of two impedances in parallel depicted in Figure 3.22(b) is defined to be:

$$\frac{V}{I} \;=\; \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)^{-1} \;=:\; Z_{\text{eq}} \tag{3.15b}$$

These are simple consequences of Kirchhoff's and Ohm's laws. Repeated application of (3.15) reduces a cascade of impedances in parallel and series into a single equivalent impedance that preserves the driving-point impedance.

When such a network contains not just impedances, but also transformers, the relation (3.14) allows us to reduce it to a single Thévenin equivalent impedance with the same driving-point impedance. As we explain below, the key element of this procedure is the driving-point impedance seen from two terminals of one side of a single-phase transformer when the other side is connected to an impedance Z_{eq} that may be the Thévenin equivalent of a network of impedances. This yields an equivalent network where the transformer and Z_{eq} is replaced by a scaled impedance and the number of transformer is reduced by 1. Repeated application of (3.14) and (3.15) can then be used to remove all transformers from the equivalent network, allowing the derivation of the Thévenin equivalent impedance of the original network. When applicable, this technique greatly simplifies per-phase analysis of a balanced system as we will see in Chapter 3.4.

We now explain the key building block of this procedure. When the secondary side of an ideal transformer is connected to an impedance $Z_{2,\text{eq}}$ as shown in Figure 3.23(a), the transformer and the impedance $Z_{2,\text{eq}}$ can be replaced by the Thévenin equivalent impedance $Z_{2,\text{eq}}/|K(n)|^2$ in the sense that the driving-point impedance V_1/I_1 on the primary side is the same in both circuits in Figure 3.23(a). This is the same operation that refers $Z_{2,\text{eq}}$ in the secondary to the primary expressed in (3.14a). It is a consequence of the Kirchhoff's and Ohm's laws and is derived in Exercise 3.10. Similarly when the primary side is connected to an impedance $Z_{1,\text{eq}}$ as shown in Figure 3.23(b), the transformer and the impedance $Z_{1,\text{eq}}$ can be replaced



(a) V_1/I_1 on the primary side

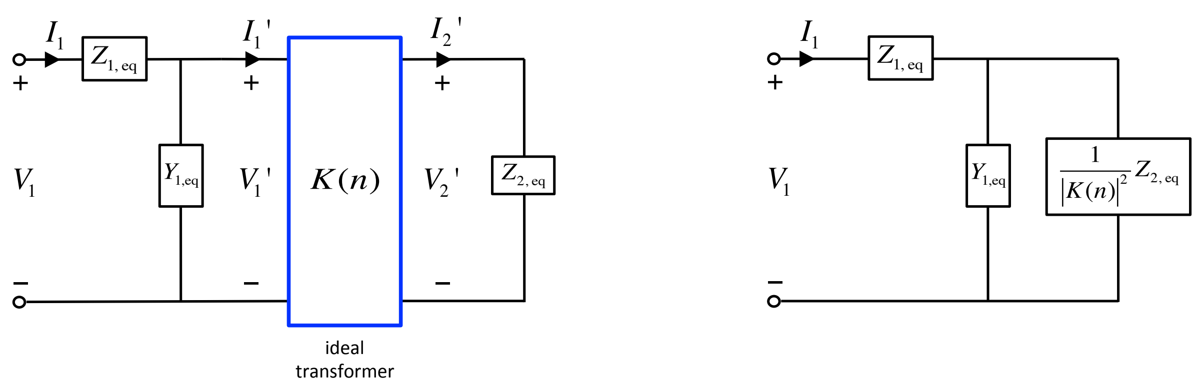
(b) V_2/I_2 on the secondary side

Figure 3.23: Driving-point impedances

by the Thévenin equivalent impedance $|K(n)|^2\ Z_{1,\text{eq}}$ in the sense that the driving-point impedance V_2/I_2 on the secondary side is the same in both circuits in Figure 3.23(b). This is the same operation that refers $Z_{1,\text{eq}}$ in the primary to the secondary expressed in (3.14a) (see Exercise 3.10).

We caution that the shortcut (3.14) and (3.15) are not always applicable. For example they may not be applied to a circuit that contains parallel paths; see Example 3.7 in Chapter 3.4.2. In that case we analyze the circuit using Kirchhoff's and Ohm's laws. The shortcut is usually applicable to a radial system that does not contain parallel paths. We now illustrate its application in the derivation of the driving-point impedances on the primary and the secondary side.

Example 3.4 (V_1/I_1 on the primary side.)**.** Consider the network in Figure 3.24(a) where the secondary side is connected to a network whose Thévenin equivalent is $Z_{2,\text{eq}}$. What is the driving-point impedance V_1/I_1? We first derive the driving-point impedance directly using Kirchhoff's and Ohm's laws. We then



(a) Transformer circuit

(b) Equivalent circuit seen on the primary side

Figure 3.24: Driving-point impedance V_1/I_1 on the primary side.

use the result to verify the shortcut expressed in (3.14) and (3.15).

Circuit analysis. We have for the primary circuit

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1+Z_{1,\text{eq}}Y_{1,\text{eq}} & Z_{1,\text{eq}} \\ Y_{1,\text{eq}} & 1 \end{bmatrix} \begin{bmatrix} V_1' \\ I_1' \end{bmatrix}$$

Hence

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1+Z_{1,\text{eq}}Y_{1,\text{eq}} & Z_{1,\text{eq}} \\ Y_{1,\text{eq}} & 1 \end{bmatrix} \begin{bmatrix} K^{-1}(n) & 0 \\ 0 & K^*(n) \end{bmatrix} \begin{bmatrix} V_2' \\ I_2' \end{bmatrix}$$

Substituting

$$V_2' = Z_{2,\text{eq}} I_2'$$

we have

$$\begin{aligned} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix} &= \begin{bmatrix} 1+Z_{1,\text{eq}}Y_{1,\text{eq}} & Z_{1,\text{eq}} \\ Y_{1,\text{eq}} & 1 \end{bmatrix} \begin{bmatrix} |K(n)|^{-2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Z_{2,\text{eq}} \\ 1 \end{bmatrix} K^*(n) I_2' \\ &= \begin{bmatrix} 1+Z_{1,\text{eq}}Y_{1,\text{eq}} & Z_{1,\text{eq}} \\ Y_{1,\text{eq}} & 1 \end{bmatrix} \begin{bmatrix} Z_{2,\text{eq}}/|K(n)|^2 \\ 1 \end{bmatrix} K^*(n) I_2' \end{aligned}$$

Hence the driving-point impedance is

$$\frac{V_1}{I_1} = \frac{(1+Z_{1,\text{eq}}Y_{1,\text{eq}})\left(Z_{2,\text{eq}}/|K(n)|^2\right) + Z_{1,\text{eq}}}{Y_{1,\text{eq}}\left(Z_{2,\text{eq}}/|K(n)|^2\right) + 1}$$

or equivalently

$$\frac{V_1}{I_1} = Z_{1,\text{eq}} + \left(Y_{1,\text{eq}} + \frac{1}{Z_{2,\text{eq}}/|K(n)|^2}\right)^{-1} \tag{3.16}$$

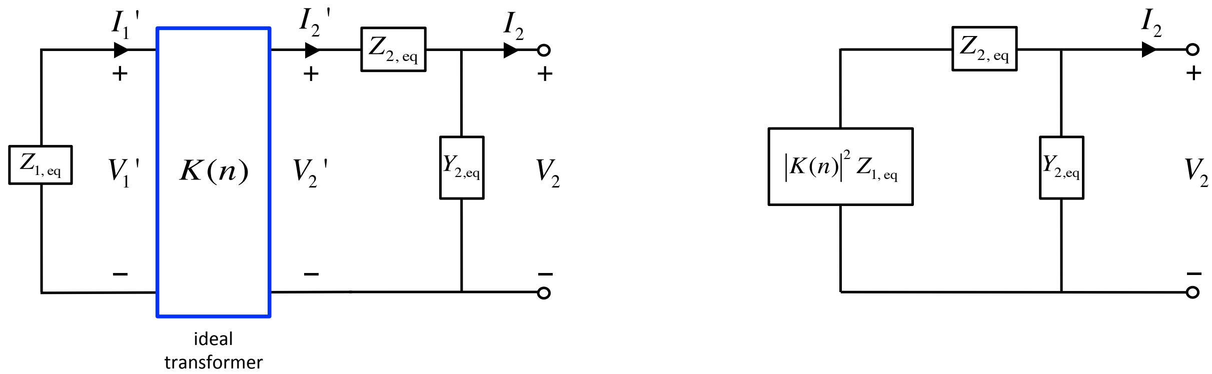
It is the Thévenin equivalent on the primary side of a network consisting of impedances, admittances, as well as an ideal transformer. The Thévenin equivalent (3.16) has a simple interpretation, as we now explain.

Shortcut.. Use (3.14a) to refer $Z_{2,\text{eq}}$ in the secondary to the primary, we can replace the ideal transformer and $Z_{2,\text{eq}}$ by the equivalent impedance $Z_{2,\text{eq}}/|K(n)|^2$ and arrive at the equivalent circuit in Figure 3.24(b) seen from the primary side. The application of (3.15) then yields the driving-point impedance (3.16). □

Example 3.5 (V_2/I_2 on the secondary side.)**.** Consider the circuit in Figure 3.25(a) where the primary side is connected to the impedance $Z_{1,\text{eq}}$. Use (3.14a) to refer $Z_{1,\text{eq}}$ in the primary to the secondary, we can replace the ideal transformer and $Z_{1,\text{eq}}$ by the equivalent impedance $|K(n)|^2 Z_{2,\text{eq}}$ and arrive at the equivalent circuit in Figure 3.25(b) seen from the secondary side. The application of (3.15) then yields the driving-point impedance:

$$\frac{V_2}{I_2} = \left(Y_{2,\text{eq}} + \frac{1}{Z_{2,\text{eq}} + |K(n)|^2 \cdot Z_{1,\text{eq}}}\right)^{-1} \tag{3.17}$$

□



(a) Transformer circuit (b) Equivalent circuit seen on the secondary side

Figure 3.25: Driving-point impedance V_2/I_2 on the secondary side.

3.4 Per-phase analysis

In this section we apply the techniques developed in the previous sections in the analysis of a balanced three-phase power system consisting of generators, transformers, transmission lines, and loads, in a mix of Y and Δ configurations. We first explain how to obtain a per-phase equivalent circuit of the system and then illustrate, through an example, the per-phase analysis using the shortcut (3.14) and (3.15). Finally we discuss a circuit that contains parallel paths to which the shortcut is not applicable. We explain why the end to end complex transformer gains on these paths should be equal.

3.4.1 Analysis procedure

We have explained in Chapter 1.2.5 how to convert all sources, series impedances, shunt admittances in Δ configurations into their equivalent Y configurations and obtain a per-phase equivalent circuit. Chapter 3.2.1 shows that an ideal balanced three-phase transformer has a per-phase equivalent model specified by a complex voltage gain $K(n)$ that relates the line-to-neutral voltages and the line currents on two sides of the transformer. Chapter 3.2.2 shows how to incorporate the transformer series impedance and shunt admittance into the per-phase model for both Y and Δ configurations. Chapter 3.3.1 explains how to refer series impedances and shunt admittances on one side to the other and Chapter 3.3.2 explains how to use this shortcut to simplify circuit analysis the same way we use Thévenin equivalent of impedances in series or in parallel. Putting everything together the procedure for per-phase analysis of a balanced three-phase system is as follows:

1. Convert all sources and loads in Δ configuration into their Y equivalents using (1.32a) for sources and (1.32b) for loads.

2. Convert all ideal transformers in Δ configuration into their Y equivalents with voltage gains $K(n)$ given in Table 3.2.

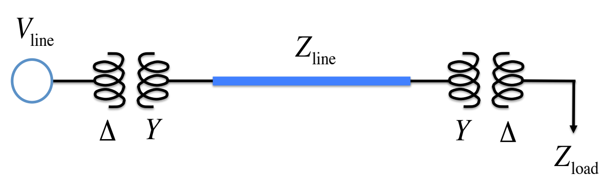
3. Obtain the phase a equivalent circuit by connecting all neutrals.

4. Solve for the desired phase a variables. Use Thévenin equivalent of series impedances and shunt admittances in a network containing transformers to simplify the analysis when applicable, e.g., for a radial system.

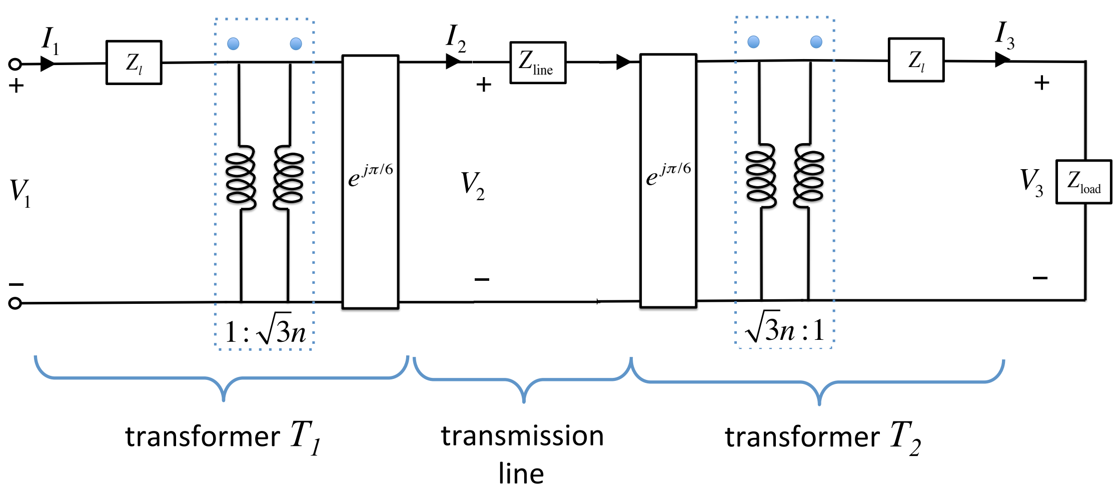
5. Obtain variables for phases b and c by subtracting (or adding) 120° and 240° from the phase a variables for positive-sequence (negative-sequence) sources. If variables in the internal of the Δ configurations are desired, derive them from the original circuits.

We illustrate this procedure in the next example.

Example 3.6. Consider the balanced system described by the one-line diagram in Figure 3.26(a) where a three-phase generator is connected to a stepup three-phase transformer bank (primary on the left) in ΔY configuration, which is connected through a three-phase transmission line to a stepdown transformer bank (primary on the right) in ΔY configuration, and then to a load. The terminal line voltage of the generator



(a) One-line diagram



(b) Per-phase circuit

Figure 3.26: Example 3.6.

is V_{line}. The transmission line is modeled by a series impedance Z_{line} and the load is assumed to be an impedance Z_{load}. The transformer banks are made up of identical single-phase transformers each specified by a series impedance of $3Z_l$ and a turns ratio of $a := 1/n$.

Find the generator current, the transmission line current, the load current, the load voltage, and the complex power delivered to the load in terms of the given parameters.

Solution. The per-phase equivalent circuit is shown in Figure 3.26(b). Note that the stepdown ΔY transformer near the load has its primary side on the right and secondary side on the left so that, going from left to right, the voltage (current) angle is shifted *down (down)* by 30° and their magnitudes scaled *down (up)* by $\sqrt{3}n$.[4] The primary sides of both the stepup and stepdown transformers have been converted from Δ to its Y equivalent, with an equivalent series impedance Z_l that is $1/3$ of the original impedance $3Z_l$. The phase voltage of the generator is

$$V_1 \quad := \quad \frac{V_{\text{line}}}{\sqrt{3}\, e^{j\pi/6}}$$

Our solution strategy is as follows. We will use (3.14) and (3.15) to refer all the (load, transformer, and transmission line) impedances to the primary side of the stepup transformer. This calculates the driving-point impedance seen at the generator. Given generator phase voltage V_1, we can derive the generator current I_1. We then propagate this towards the load to calculate the other quantities.

Let $K(n) := \sqrt{3}n\, e^{j\pi/6}$. Going from right to left, we cross the stepdown transformer T_2 from the primary to the secondary. Referring the impedance $Z_{1,\text{eq}} := Z_{\text{load}} + Z_l$ on the primary to the secondary (see Figure 3.23(b)), the equivalent impedance at the right-end of the transmission line is

$$|K(n)|^2 \left(Z_{\text{load}} + Z_l\right)$$

Hence the equivalent impedance at the secondary side of the stepup transformer T_1 is

$$Z_{2,\text{eq}} \quad := \quad Z_{\text{line}} \;+\; |K(n)|^2 \left(Z_{\text{load}} + Z_l\right)$$

Referring this impedance to the primary side of T_1 (see Figure 3.23(a)), the driving point impedance at the generator is:

$$\begin{aligned} \frac{V_1}{I_1} \quad &= \quad Z_l \;+\; \frac{1}{|K(n)|^2} \cdot \left(Z_{\text{line}} \;+\; |K(n)|^2 \left(Z_{\text{load}} + Z_l\right)\right) \\ &= \quad 2Z_l \;+\; \frac{Z_{\text{line}}}{|K(n)|^2} \;+\; Z_{\text{load}} \end{aligned}$$

Hence the primary side of T_1 sees the series impedance Z_l of the two transformers, a scaled down version of the line impedance Z_{line}, and the load Z_{load}, all in series. Note that, seen from the generator, the load Z_{load} goes through a stepdown transformer and a stepup transformer and therefore the scaling effects of these two transformers are canceled out.

Given the bus voltage V_1 of the generator, the generator current is then

$$I_1 \quad = \quad \frac{V_1}{2Z_l \;+\; \frac{Z_{\text{line}}}{|K(n)|^2} \;+\; Z_{\text{load}}}$$

[4] See Exercise 3.6.

The transmission line current is

$$I_2 \;=\; \frac{I_1}{K^*(n)} \;=\; \frac{V_1}{K^*(n)\left(2Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}}\right)}$$

The load current is

$$I_3 \;=\; K^*(n)\, I_2 \;=\; I_1$$

i.e., the effects of stepup and stepdown transformers cancel each other and the load current is equal to the generator current. The load voltage is

$$V_3 \;=\; Z_{\text{load}} I_3 \;=\; Z_{\text{load}} I_1 \;=\; V_1 \cdot \frac{Z_{\text{load}}}{2Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}}}$$

Hence V_3 relates to V_1 according to the voltage-divider rule where V_1 is the voltage drop across the series of impedances $2Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}}$ and V_3 is the voltage drop across Z_{load}. The complex power delivered to the load is

$$V_3 I_3^* \;=\; Z_{\text{load}} \cdot \left| \frac{V_1}{2Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}}} \right|^2 \;=\; Z_{\text{load}} \cdot \frac{|V_{\text{line}}|^2}{3\left|2Z_l + \frac{Z_{\text{line}}}{|K(n)|^2} + Z_{\text{load}}\right|^2}$$

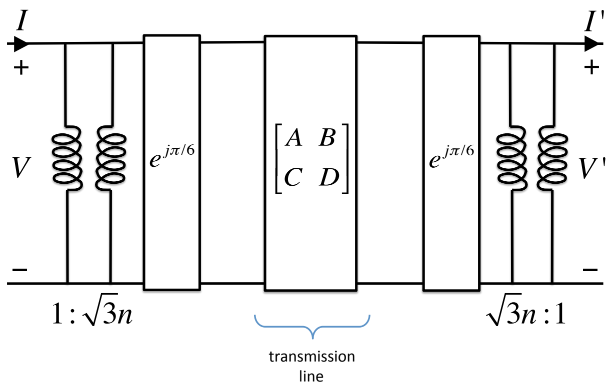
$\square$

Simplified per-phase diagram for external behavior. In Example 3.6, only the transmission line current I_2 that is in between the pair of transformers depends on the *connection-induced phase shift* $e^{j\pi/6}$ in the complex transformer gain $K(n)$. Outside the pair of transformers, the driving point impedance V_1/I_1, the generator current I_1, the load current I_3, the load voltage V_3, and the power delivered to the load do not. They depend only on $|K(n)|^2$. This is the case even if we use the more detailed Π-model of the transmission line instead of the short-line model used here. Indeed, suppose the series impedance Z_{line} in Figure 3.26(b) is replaced by the matrix given by (2.9) or (2.13)(2.14) as in Figure 3.27(a). Then the voltage and current (V, I) on the left is related to the voltage and current (V', I') by

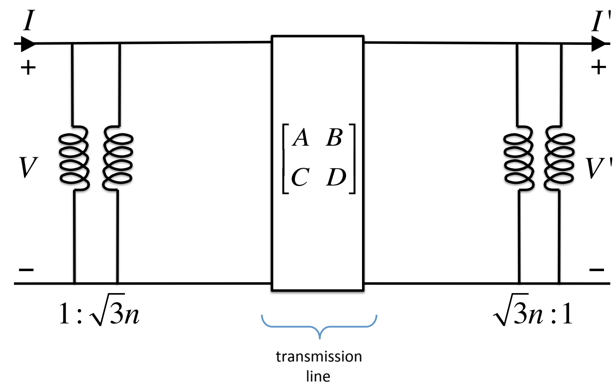
$$\begin{bmatrix} V\,|K(n)|\,e^{j\pi/6} \\ I\,|K(n)|^{-1}\,e^{j\pi/6} \end{bmatrix} \;=\; \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} V'\,|K(n)|\,e^{j\pi/6} \\ I'\,|K(n)|^{-1}\,e^{j\pi/6} \end{bmatrix}$$

$$\begin{bmatrix} V\,|K(n)| \\ I\,|K(n)|^{-1} \end{bmatrix} \;=\; \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} V'\,|K(n)| \\ I'\,|K(n)|^{-1} \end{bmatrix}$$

Therefore the external behavior is as if the connection-induced phase shift $e^{j\pi/6}$ is absent, as shown in Figure 3.27(b). This motivates a simplified per-phase diagram for external behavior that ignores all the connection-induced phase shifts of transformers as long as every path contains stepup and stepdown transforms in pairs and wired in opposite directions. This is generally true for radial networks in practice where no transmission lines nor transformers are in parallel. Radial networks are a special case of a normal system that we discuss next.



(a) Transmission line Π-model



(b) Equivalent circuit without connection-induced phase shift

Figure 3.27: Π-model of transmission line in place of the series impedance Z_{line} model in Figure 3.26(b).

3.4.2 Normal system

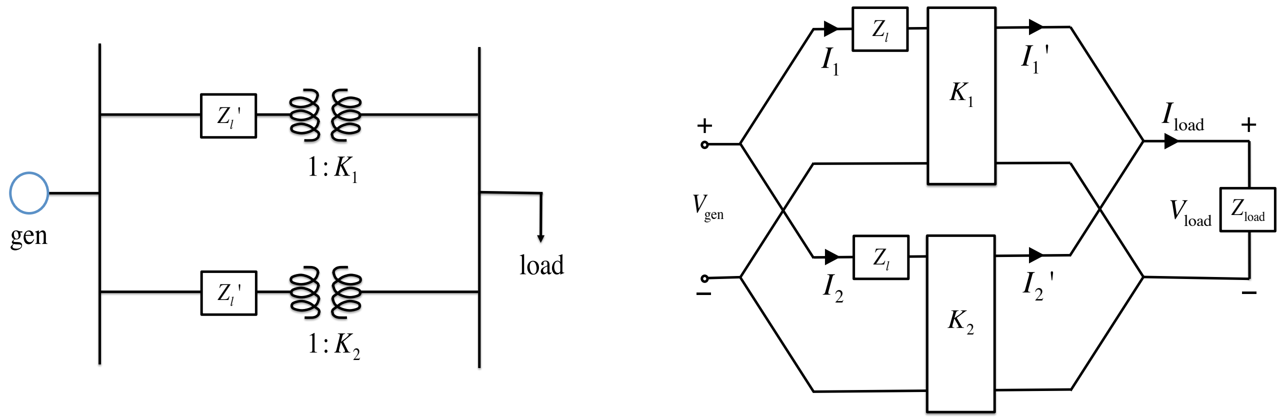
A system is called *normal* if, in the per-phase equivalent circuit, the product of the *complex ideal* transformer gains around every loop is 1. Equivalently, on each parallel path,

1. the product of ideal transformer gain magnitudes is the same, and
2. the sum of ideal transformer phase shifts is the same.

Normal systems have a normalization that greatly simplifies analysis which we will discuss in Chapter 3.5. The following example motivates such a system.

Example 3.7 (Loop flows)**.** Consider a generator and a load connected by two three-phase transformer banks in parallel forming a loop as shown in Figure 3.28(a). The transformer in the upper path is characterized by a series impedance and a complex gain K_1. The transformer in the lower path is characterized by the same series impedance and a possibly different complex gain K_2. Suppose line-to-neutral voltage of the generator bus is V_{gen}, the series impedance Z_l of the transformer and the load impedance Z_{load} in the per-phase equivalent circuit are given, as shown in Figure 3.28(b). Derive the currents I_{load}, I_1', I_2' in terms of $V_{\text{gen}}, Z_l, Z_{\text{load}}$. Discuss the implications when

1. $K_2 = K_1$. This is the case if both transformer banks are YY-configured.
2. $K_2 = K_1\, e^{j\theta}$. This is the case if the upper transformer bank is YY-configure with a voltage gain of n but the lower transformer bank is ΔY-configured with a voltage gain of $n/\sqrt{3}$ and $\theta = \pi/6$.
3. $K_2 = k \cdot K_1$, $k > 0$. This is the case if both transformer banks are YY-configured but with different turns ratios.



(a) Transmission line Π-model (b) Equivalent circuit

Figure 3.28: Two buses connected in a loop with two parallel transformers.

Solution. We cannot directly apply the shortcut (3.14) and (3.15) to refer the impedances Z_{load} and Z_l to the primary side because of the parallel paths, and must analyze the per-phase circuit using Kirchhoff's and Ohm's laws.

We have five unknowns currents $I_{\text{load}}, I_1', I_2', I_1, I_2$. The five equations that relate them are

$$\begin{aligned} I_{\text{load}} &= I_1' + I_2' \\ Z_{\text{load}} I_{\text{load}} &= K_1 \cdot \left(V_{\text{gen}} - Z_l I_1\right) \\ Z_{\text{load}} I_{\text{load}} &= K_2 \cdot \left(V_{\text{gen}} - Z_l I_2\right) \\ I_j' &= \frac{I_j}{K_j^*}, \qquad j = 1, 2 \end{aligned}$$

where the first equation expresses KCL, the second and third equations express the load voltage seen on the upper and lower paths, respectively, and follow from the transformer equation and KVL, and the last equations express current gains of the transformers. Eliminating $I_{\text{load}}, I_1', I_2'$ we have

$$\begin{aligned} Z_{\text{load}} \left(\frac{I_1}{K_1^*} + \frac{I_2}{K_2^*} \right) &= K_1 \cdot \left(V_{\text{gen}} - Z_l I_1\right) \\ Z_{\text{load}} \left(\frac{I_1}{K_1^*} + \frac{I_2}{K_2^*} \right) &= K_2 \cdot \left(V_{\text{gen}} - Z_l I_2\right) \end{aligned}$$

or

$$\begin{bmatrix} Z_l + Z_{\text{load}} |K_1|^{-2} & Z_{\text{load}} (K_1 K_2^*)^{-1} \\ Z_{\text{load}} (K_1^* K_2)^{-1} & Z_l + Z_{\text{load}} |K_1|^{-2} \end{bmatrix} \cdot \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} V_{\text{gen}} \\ V_{\text{gen}} \end{bmatrix}$$

Inverting the matrix, we obtain

$$\begin{aligned} I_1 &= \frac{V_{\text{gen}}}{Z_l + Z_{\text{load}} \left(|K_1|^{-2} + |K_2|^{-2}\right)} \cdot \alpha_1 \\ I_2 &= \frac{V_{\text{gen}}}{Z_l + Z_{\text{load}} \left(|K_1|^{-2} + |K_2|^{-2}\right)} \cdot \alpha_2 \end{aligned}$$

where

$$\alpha_1 \;=\; 1 \,+\, \frac{Z_{\text{load}}}{Z_l} \cdot \frac{K_1 - K_2}{K_1\,|K_2|^2}$$
$$\alpha_2 \;=\; 1 \,+\, \frac{Z_{\text{load}}}{Z_l} \cdot \frac{K_2 - K_1}{|K_1|^2\,K_2}$$

Hence

$$I_1' \;=\; \frac{I_1}{K_1^*} \;=\; \frac{V_{\text{gen}}}{Z_l \,+\, Z_{\text{load}}\,(|K_1|^{-2} + |K_2|^{-2})} \cdot \frac{\alpha_1}{K_1^*}$$
$$I_2' \;=\; \frac{I_2}{K_2^*} \;=\; \frac{V_{\text{gen}}}{Z_l \,+\, Z_{\text{load}}\,(|K_1|^{-2} + |K_2|^{-2})} \cdot \frac{\alpha_2}{K_2^*}$$

and

$$I_{\text{load}} \;=\; I_1' + I_2' \;=\; \frac{V_{\text{gen}}}{Z_l \,+\, Z_{\text{load}}\,(|K_1|^{-2} + |K_2|^{-2})} \cdot \left(\frac{1}{K_1^*} \,+\, \frac{1}{K_2^*} \right)$$

where we have used

$$\frac{\alpha_1}{K_1^*} + \frac{\alpha_2}{K_2^*} \;=\; \left(\frac{1}{K_1^*} \,+\, \frac{Z_{\text{load}}}{Z_l} \cdot \frac{K_1 - K_2}{|K_1|^2\,|K_2|^2} \right) + \left(\frac{1}{K_2^*} \,+\, \frac{Z_{\text{load}}}{Z_l} \cdot \frac{K_2 - K_1}{|K_1|^2\,|K_2|^2} \right) \;=\; \frac{1}{K_1^*} \,+\, \frac{1}{K_2^*}$$

1. When $K_2 = K_1$, then $\alpha_1 = \alpha_2 = 1$ and

$$I_1' \;=\; I_2' \;=\; \frac{V_{\text{gen}}}{Z_l \,+\, Z_{\text{load}}\,(2\,|K_1|^{-2})} \cdot \frac{\alpha_1}{K_1^*} \;=\; \frac{K_1\,V_{\text{gen}}}{|K_1|^2 Z_l \,+\, 2\,Z_{\text{load}}}$$

   and

$$I_{\text{load}} \;=\; \underbrace{\frac{V_{\text{gen}}}{|K_1|^2 Z_l \,+\, 2\,Z_{\text{load}}}}_{I_0} \cdot 2K_1 \;=\; I_0 \cdot 2K_1 \tag{3.18}$$

2. When $K_2 = K_1\,e^{j\theta}$, then, for $i = 1,2$,

$$I_i' \;=\; \frac{V_{\text{gen}}}{Z_l \,+\, Z_{\text{load}}\,(2\,|K_1|^{-2})} \cdot \frac{\alpha_i}{K_i^*} \;=\; \frac{V_{\text{gen}}}{|K_1|^2 Z_l \,+\, 2\,Z_{\text{load}}} \cdot (\alpha_i\,K_i)$$

   Since $\alpha_1\,K_1 \,+\, \alpha_2\,K_2 \;=\; K_1 + K_2 \;=\; K_1\,(1 + e^{j\theta})$ and $|K_1| = |K_2|$, we have

$$I_{\text{load}} \;=\; \frac{V_{\text{gen}}}{|K_1|^2 Z_l \,+\, 2\,Z_{\text{load}}} \cdot \left(1 + e^{j\theta}\right) K_1 \;=\; I_0 \left(1 + e^{j\theta}\right) K_1$$

   Hence I_{load} reduces to the load current in (3.18) when the transformer gains are equal with $\theta = 0$. When the transformer gains K_1 and K_2 are not in phase, $\left(1 + e^{j\theta}\right)$ can be much smaller than 2 and the current $|I_{\text{load}}|$ that enters the load can be much smaller than the currents $|I_i'|$, $i = 1,2$. In particular

$$\frac{|I_{\text{load}}|}{|I_1'|} \;=\; \frac{|1 + e^{j\theta}|}{|\alpha_1|} \quad \text{and} \quad \frac{|I_{\text{load}}|}{|I_2'|} \;=\; \frac{|1 + e^{j\theta}|}{|\alpha_2|}$$

To appreciate the issue, take $K_1 = 10$, $K_2 = 10\,e^{j\pi/6}$, $V_{\text{gen}} = 8$ kV, $Z_l = j0.05\,\Omega$, $Z_{\text{load}} = 800\angle 0°\Omega$. Then

$$\begin{aligned}
I_1' &= 3{,}754.99 \angle -164.85 \text{ A} \\
I_2' &= 4{,}527.24 \angle 14.88 \text{ A} \\
I_{\text{load}} &= I_1' + I_2' \;=\; 772.50 \angle 13.57 \text{ A} \\
\frac{|I_{\text{load}}|}{|I_1'|} &= 20.57\%, \qquad \frac{|I_{\text{load}}|}{|I_2'|} \;=\; 17.06\%
\end{aligned}$$

Hence $|I_1'|$ and $|I_2'|$ are much larger than $|I_{\text{load}}|$. The interpretation is that most of the current loops between the two transformer banks without entering the load. This is undesirable because the circulating current serves no purpose and heats up the transformers. The problem arises because the connection-induced phase shifts in the two parallel paths are different. In practice we will not parallelize these transformers.

The complex generation power and load power are respectively

$$\begin{aligned}
S_{\text{gen}} &:= V_{\text{get}}(I_1 + I_2)^* \;=\; 182.98 \angle 70.97° \text{ MVA} \\
S_{\text{load}} &:= Z_{\text{load}}|I_{\text{load}}|^2 \;=\; 59.68 \angle 0° \text{ MVA}
\end{aligned}$$

Again the apparent load power is a small fraction of the apparent generation power. However, since the transformers have zero resistance, their real powers are the same:

$$P_{\text{gen}} = P_{\text{load}} = 59.68 \text{ MW}$$

3. When $K_2 = k \cdot K_1$, we have

$$\begin{aligned}
I_1' &= \frac{K_1\, V_{\text{gen}}}{|K_1|^2 Z_l \;+\; (1+k^{-2})\, Z_{\text{load}}} \cdot \alpha_1 \\
I_2' &= \frac{K_1\, V_{\text{gen}}}{|K_1|^2 Z_l \;+\; (1+k^{-2})\, Z_{\text{load}}} \cdot \frac{\alpha_2}{k} \\
I_{\text{load}} &= \frac{V_{\text{gen}}}{|K_1|^2 Z_l \;+\; (1+k^{-2})\, Z_{\text{load}}} \cdot \left(1 + \frac{1}{k}\right) K_1
\end{aligned}$$

Hence

$$\frac{|I_{\text{load}}|}{|I_1'|} = \frac{1+k^{-1}}{|\alpha_1|} \quad \text{and} \quad \frac{|I_{\text{load}}|}{|I_2'|} = \frac{1+k}{|\alpha_2|}$$

If we take $K_1 = 10$, $K_2 = 20$, $V_{\text{gen}} = 8$ kV, $Z_l = j0.05\ \Omega$, $Z_{\text{load}} = 800\angle 0°\ \Omega$. Then

$$\begin{aligned}
I_1' &= 3{,}260.76 \angle 76.40 \text{ A} \\
I_2' &= 3{,}213.39 \angle -86.58 \text{ A} \\
I_{\text{load}} &= I_1' + I_2' \;=\; 959.23 \angle -2.29 \text{ A} \\
\frac{|I_{\text{load}}|}{|I_1'|} &= 29.42\%, \qquad \frac{|I_{\text{load}}|}{|I_2'|} \;=\; 29.85\%
\end{aligned}$$

Again $|I_1'|$ and $|I_2'|$ are much larger than $|I_{\text{load}}|$ and there is a large loop flow between the transformer banks. This time the problem arises because the voltage gains in the two parallel paths are different. In practice we will not parallelize these transformers.

□

3.5 Per-unit normalization

In this section we describe a normalization method that will simplify the analysis of balanced three-phase systems. For a normal system where all connection-induced phase shifts of transformers can be ignored in the per-phase equivalent circuit, the system after normalization will contain no transformers if there is no off-nominal transformer in the original system. For general systems, normalization will typically still simplify the equivalent circuit and per-phase analysis, but the system after normalization may contain ideal transformers with real or complex voltage gains. We are usually interested in four types of generally complex quantities: power S, voltages V, currents I, and impedances Z and functions of these quantities. We will choose *base values* for these quantities and define the quantities in per unit as:

$$\text{quantity in p.u.} \quad := \quad \frac{\text{actual quantity}}{\text{base value of quantity}}$$

The base values are chosen to be real positive values and have the same units as the corresponding actual quantities. For example a power base S_B will be in unit VA when it serves as the base value for complex power, W for real power, var for reactive power. Hence the per-unit quantities generally have different magnitudes from, but always the same phase as, the corresponding actual quantities. Furthermore they are dimensionless. The base values are chosen so that the per-unit quantities behave exactly as the actual quantities do, as we now explain.

Consider a power network that consists of multiple areas connected by transformers. It represents either a single-phase system or the per-phase equivalent circuit of a balanced three-phase system. The nominal voltage magnitudes are the same within each area and those in neighboring areas are related by transformer turns ratios. It is common to choose the power base value S_{1B} for the entire network and the voltage base value V_{1B} for one of the areas, say, area 1. For example the base value V_{1B} can be chosen to be the nominal voltage magnitude for area 1 and the base value S_B can be the rated apparent power of one of the transformers in area 1, so that its rated voltage is 1 pu and the rated power is 1 pu. The base values for all other quantities in the entire network are then calculated from these two values (S_B, V_{1B}) so that these base values satisfy:

- Kirchhoff's laws within each area;
- ideal transformer gains across areas;
- three-phase relations.

We derive in Chapter 3.5.1 the base values within area 1 and in Chapter 3.5.2 the base values of other areas connected by transformers to area 1. In Chapter 3.5.3 we describe the normalization of off-nominal transformers. In Chapter 3.5.4 we describe how to calculate base values of three-phase quantities in a balanced three-phase system. In Chapter 3.5.5 we summarize the procedure for per-unit per-phase analysis.

3.5.1 Kirchhoff's and Ohm's laws

Consider a single-phase system or the per-phase equivalent circuit of a three-phase system. Start with area 1 for which we have the power base S_B in VA (or W or var for real and reactive powers respectively) for the entire network, and the voltage base V_{1B} in V. The base values I_{1B}, Z_{1B} of currents and impedances respectively are calculated as:

$$I_{1B} := \frac{S_B}{V_{1B}} \ A, \qquad Z_{1B} := \frac{V_B^2}{S_B} \ \Omega \tag{3.19}$$

so that the base values satisfy the Kirchhoff's laws:

$$V_{1B} = Z_{1B} I_{1B} \ V, \qquad S_B = V_{1B} I_{1B} \ VA$$

Since

$$\frac{V_1}{V_{1B}} = \frac{Z_1 I_1}{Z_{1B} I_{1B}}, \qquad \frac{S_1}{S_B} = \frac{V_1 I_1^*}{V_{1B} I_{1B}}$$

the per-unit quantities satisfy Kirchhoff's laws as the actual quantities do:

$$V_{1\text{pu}} = Z_{1\text{pu}} I_{1\text{pu}}, \qquad S_{1\text{pu}} = V_{1\text{pu}} I_{1\text{pu}}^*$$

We can therefore perform circuit analysis using the per-unit quantities instead of the actual quantities. We can convert the result of the analysis back to the original quantities by multiplying the per-unit quantities by their base values.

Extensions to other related quantities are straightforward. For example S_B is also the base value for real power in W and reactive power in var so that

$$P_{1\text{pu}} := \frac{P_1}{S_B}, \qquad Q_{1\text{pu}} := \frac{Q_1}{S_B}$$

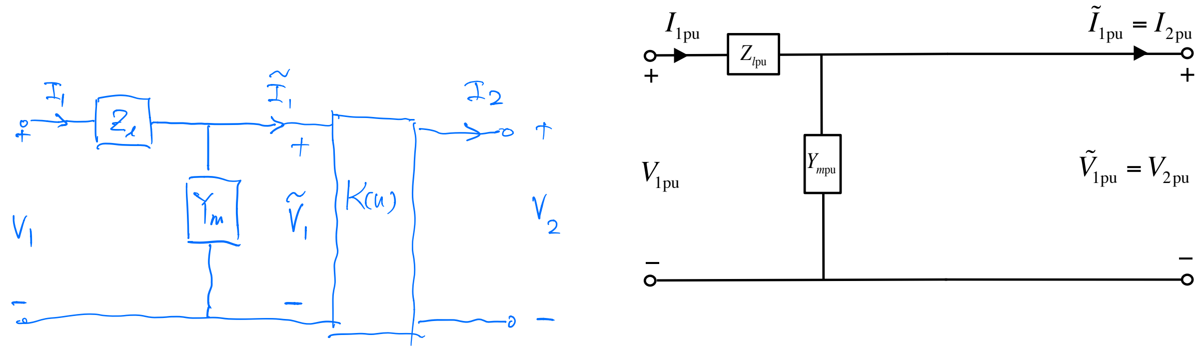
and $S_{1\text{pu}} = P_{1\text{pu}} + jQ_{1\text{pu}}$. Z_B is the base value for resistances and reactances so that

$$R_{1\text{pu}} := \frac{R_1}{Z_{1B}}, \qquad X_{1\text{pu}} := \frac{X_1}{Z_{1B}}$$

and $Z_{1\text{pu}} = R_{1\text{pu}} + jX_{1\text{pu}}$. Similarly $Y_{1B} := 1/Z_{1B}$ in Ω^{-1} is the base value for admittances $Y_1 := 1/Z_1 = G - jB$ in Ω^{-1} as well as conductances G and susceptances B also in Ω^{-1}.

3.5.2 Across ideal transformer

Consider now a neighboring area, say, area 2 that is connected to area 1 through a transformer. We choose the bases for different sides of the transformer in a way that respects the transformer gains. Consider the circuit in Figure 3.29(a) where areas 1 and 2 are connected through a transformer with a voltage gain $K(n)$. If it is a single-phase system then $K(n) = n$, the reciprocal of the turns ratio. If it is the per-phase equivalent of a balanced three-phase system then $K(n)$ may be complex if the transformer is not in YY or



(a) In standard unit (b) In per unit

Figure 3.29: Per-phase equivalent circuit of balanced three-phase transformers with gain $K(n)$.

$\Delta\Delta$ configuration. Given the bases $(S_B, V_{1B}, I_{1B}, Z_{1B})$ for area 1 calculated in Chapter 3.5.1, the bases for the other side of the transformer are calculated according to:

$$V_{2B} \;:=\; |K(n)|\, V_{1B} \;\; V, \qquad I_{2B} \;:=\; \frac{I_{1B}}{|K(n)|} \;\; A, \qquad Z_{2B} \;:=\; |K(n)|^2\, Z_{1B} \;\; \Omega \tag{3.20}$$

The base power value remains $S_B = V_{1B}I_{1B} = V_{2B}I_{2B}$ for all areas since the power gain across an ideal transformer is 1. Even though $K(n)$ may be complex all base values remain real positive numbers.

Referring to Figure 3.29(a), the per-unit quantities $(\tilde{V}_{1\text{pu}}, \tilde{I}_{1\text{pu}})$ at the input and the per-unit quantities $(V_{2\text{pu}}, I_{2\text{pu}})$ at the output of the *ideal* transformer satisfy $(a := 1/n)$

$$\begin{aligned}
\tilde{V}_{1\text{pu}} &= \frac{\tilde{V}_1}{V_{1B}} = \frac{V_2}{K(n)}\frac{|K(n)|}{V_{2B}} = V_{2\text{pu}}\, e^{-j\angle K(n)} \\
\tilde{I}_{1\text{pu}} &= \frac{\tilde{I}_1}{\tilde{I}_{1B}} = \frac{K^*(n) I_2}{|K(n)| I_{2B}} = I_{2\text{pu}}\, e^{-j\angle K(n)}
\end{aligned}$$

This also implies that the per-unit power $\tilde{S}_{1\text{pu}} := \tilde{V}_{1\text{pu}}\tilde{I}^*_{1\text{pu}} = V_{2\text{pu}}I^*_{2\text{pu}} = S_{2\text{pu}}$. If $\angle K(n)$ can be taken as zero then on the input side of the transformer, $(\tilde{V}_{1\text{pu}}, \tilde{I}_{1\text{pu}}, \tilde{S}_{1\text{pu}})$ can be replaced by $(V_{2\text{pu}}, I_{2\text{pu}}, S_{2\text{pu}})$, i.e., the voltages, currents, and power remain the same, in per unit, when crossing an *ideal* transformer. Within each side of the ideal transformer the per-unit quantities $(S_{i\text{pu}}, V_{i\text{pu}}, I_{i\text{pu}}, Z_{i\text{pu}})$ satisfy the Kirchhoff's laws as explained in Chapter 3.5.1. Hence the per-phase equivalent circuit can be simplified into that in Figure 3.29(b) where the ideal transformer has disappeared. The voltage gain angle $\angle K(n) = 0$ if (i) the system is single phased, or (ii) it is balanced three phased with transformers in YY or $\Delta\Delta$ configuration, or (iii) it is a normal system where the connection induced phase shift $\angle K(n)$ can be ignored for external behavior. Hence ideal transformers and connection-induced phase shifts can be omitted in a normal per-phase system if we use the simplified per-phase diagram and the per-unit normalization. This simplified per-phase per-unit diagram is called an *impedance diagram*. Otherwise the per-unit circuit will contain a phase-shifting transformer with voltage gain $e^{j\angle K(n)}$; see Example 3.9.

We proceed in a similar manner to calculate the base values $(S_B, V_{iB}, I_{iB}, Z_{iB})$ in each neighboring area i, until all connected areas are covered. It can be easily checked that the per-unit quantities in each area satisfy the Kirchhoff's laws, as long as the per-unit quantities in area 1 satisfy the Kirchhoff's laws and those in other areas respect transformer gains. This is where system normality is important: on each parallel path in its per-phase equivalent circuit, (i) the product of ideal transformer gain magnitudes is the same, and (ii) the sum of ideal transformer phase shifts is the same. As discussed above these properties prevent loop flows between transformers, as illustrated in Example 3.7. Note that in Figure 3.28(b) of that example, the secondary-side voltages of the two *ideal* transformers are the same but their primary-side voltages are different when $K_2 = K_1 e^{j\theta}$ with $\theta \neq 0$. The first property also ensures that the calculation (3.20) of base values across areas is consistent, i.e., does not depend on the order in which the areas are chosen for calculation; see Exercise 3.13.

Example 3.8 (Single-phase system)**.** Consider the single-phase system in Figure 3.30 where the voltage source has a nameplate rated voltage magnitude of v V and a nameplate rated power of s VA. Calculate

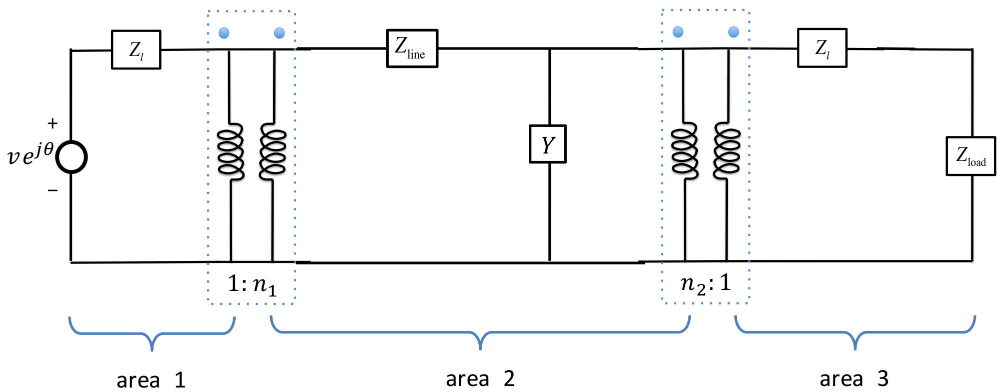


Figure 3.30: Single-phase system for Example 3.8 with a rated voltage magnitude of v in V and a rated apparent power of s in VA.

the base values for the system.

Solution. Let the base value for power be $S_B := s$ in VA for the entire system and the base value for voltage in area 1 (where the voltage source is) be $V_{1B} := v$ in V. Then the base values for currents and impedances in area 1 are respectively:

$$I_{1B} \; := \; \frac{s}{v} \; A \qquad \text{and} \qquad Z_{1B} \; := \; \frac{v^2}{s} \; \Omega$$

The base values in area 2 connected by the first transformer with a voltage gain n_1 are:

$$\begin{aligned}
V_{2B} \;&:=\; n_1 V_{1B} \;=\; n_1 v \;\; V \\
I_{2B} \;&:=\; \frac{I_{1B}}{n_1} \;=\; \frac{s}{n_1 v} \;\; A \\
Z_{2B} \;&:=\; n_1^2 Z_{1B} \;=\; \frac{(v_1 v)^2}{s} \;\; \Omega, \qquad Y_{2B} \;:=\; \frac{1}{Z_{2B}} \;=\; \frac{s}{(v_1 v)^2} \;\; \Omega^{-1}
\end{aligned}$$

The base values in area 3 connected by the second transformer are:

$$\begin{aligned} V_{3B} &:= \frac{V_{2B}}{n_2} = \frac{n_1}{n_2} v \ V \\ I_{3B} &:= n_2 I_{2B} = \frac{n_2}{n_1}\frac{s}{v} \ A \\ Z_{3B} &:= \frac{1}{n_2^2} Z_{2B} = \frac{n_1^2}{n_2^2}\frac{v^2}{s} \ \Omega, \qquad Y_{3B} := \frac{1}{Z_{3B}} = \frac{n_2^2}{n_1^2}\frac{s}{v^2} \ \Omega^{-1} \end{aligned}$$

□

3.5.3 Off-nominal transformer

Power systems employ two types of regulating transformers. The first type regulates voltage magnitudes, e.g., through variable taps on some of its windings that control the number of turns and hence the voltage gain. Such a transformer is usually connected at the end of a line to regulate the voltage magnitude at a node. Its turns ratio may be variable and different from the ratio of the voltage bases in its primary and secondary areas. The second type regulates phase angle displacement between two nodes. Their voltage gains may be complex $K(n) = \rho\angle\phi$ where ϕ may be variable and cannot be omitted in normalization. These transformers are said to be *off-nominal*. They will not disappear under per-unit normalization but will appear as a transformer with a different (normalized) voltage gain, as we now explain.

Consider an ideal transformer with a possibly complex voltage gain $\frac{V_2}{V_1} =: K(n)$ as shown in Figure 3.31(a). Suppose the ratio of the voltage base in area 2 to that in area 1 is $\frac{V_{2B}}{V_{1B}} =: \rho$. Since

$$V_2 = K(n)V_1 = \frac{K(n)}{\rho}\cdot\rho V_1$$

the transformer is equivalent to two ideal transformers in series with voltage gains ρ and $K(n)/\rho$ respectively as shown in Figure 3.31(b). Since the first transformer has an voltage gain of ρ, it disappears in per-unit normalization and hence the per-unit equivalent circuit of the original transformer has a gain reduced by ρ as shown in Figure 3.31(c). For instance for a phase shifting transformer with voltage gain $K(n) = \rho\angle\phi$ its voltage gain in the per-unit circuit will be $1\angle\phi$.

Example 3.9 (Normalization with connection-induced phase shifts)**.** Consider a balanced three-phase ideal transformer in ΔY or $Y\Delta$ configuration with a complex voltage gain $K(n)$. Let the bases for one side of the transformer be $(S_B, V_{1B}, I_{1B}, Z_{1B})$. Choose the bases for the other side according to (3.20). Suppose we cannot ignore the connection-induced phase shift. Then the per-unit equivalent circuit of the ideal transformer will be an off-nominal phase shifting transformer with a gain $\frac{K(n)}{|K(n)|} = \angle K(n)$ as shown in Figure 3.32. □

As we will see in Chapter 4.2 a *nonideal* transformer, whether in standard unit or per unit, can be represented by a phase impedance matrix for power flow analysis.

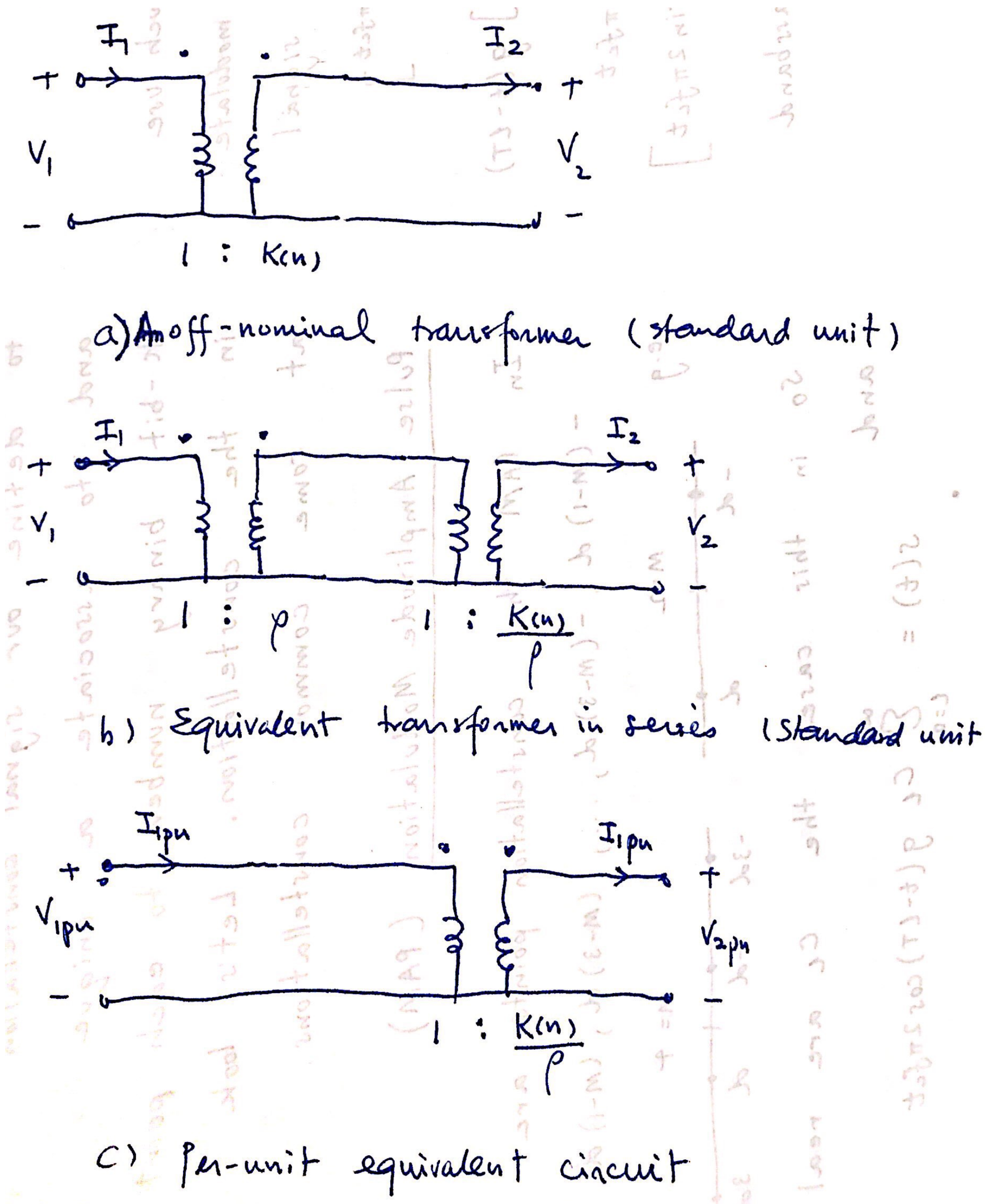


Figure 3.31: Normalization of an off-nominal transformer.

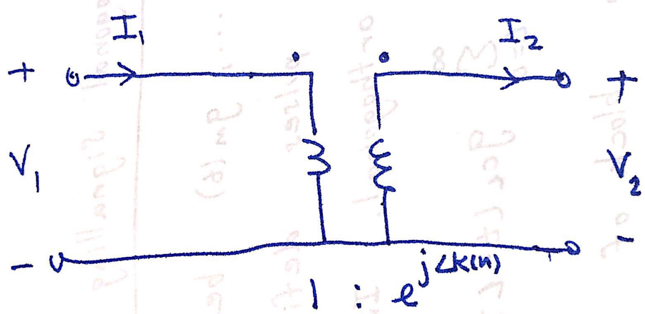


Figure 3.32: Normalization when connection-induced phase shifts cannot be ignored.

3.5.4 Three-phase quantities

In Chapters 3.5.1–3.5.3 we explain how to choose bases for a single-phase system. They are also applicable to the per-phase equivalent of a three-phase system where the voltages and currents are line-to-neutral voltages and line-to-neutral currents. Suppose the base values $(S_B^{1\phi}, V_B^{1\phi}, I_B^{1\phi}, Z_B^{1\phi})$ for a single-phase system are given. When single-phase devices (sources, loads, impedances, transformers) are connected to form a balanced three-phase system, three-phase quantities are created for which base values need to be defined. For instance the ratings of a three-phase transformer are always specified in terms of three-phase power and line-to-line voltages. In this subsection we will derive these base values, in terms of $(S_B^{1\phi}, V_B^{1\phi}, I_B^{1\phi}, Z_B^{1\phi})$, in a way that respects three-phase relations. The main issue is to define the meaning of these base values and the relation they intend to capture in Y and in Δ configurations.

Let $(S^{1\phi}, V^{1\phi}, I^{1\phi}, Z^{1\phi})$ denote respectively the power generated or consumed by a single-phase device, the voltage across and current through the device, and the impedance of the device. We are interested in the following three-phase quantities. The three-phase power $S^{3\phi}$ is defined to be the sum of power generated or consumed by each device in either Y or Δ configuration. The line-to-line voltages V^{ll} and terminal (line) currents $I^{3\phi}$ are external quantities. In an Y configured three-phase device, a line-to-neutral voltage V^{ln} and a three-phase impedance $Z^{3\phi}$ are equal to the voltage $V^{1\phi}$ and impedance $Z^{1\phi}$ respectively associated with each single-phase device. For a Δ configured three-phase device V^{ln} and $Z^{3\phi}$ are defined to be the line-to-neutral voltage and the impedance respectively in its *Y equivalent circuit*. As explained in Chapter 1 these quantities are related to the corresponding single-phase quantities according to:[5]

$$S^{3\phi} = 3S^{1\phi}, \qquad V^{\text{ll}} = \sqrt{3}e^{\mathbf{i}\pi/6}V^{\text{ln}} \tag{3.21a}$$

$$I^{3\phi} = \begin{cases} I_{an} = I^{1\phi} & \text{for } Y \text{ configuration} \\ I_{ab} - I_{ca} = \sqrt{3}e^{-j\pi/6}I^{1\phi} & \text{for } \Delta \text{ configuration} \end{cases} \tag{3.21b}$$

$$V^{\text{ln}} = \begin{cases} V^{1\phi} & \text{for } Y \text{ configuration} \\ \left(\sqrt{3}e^{j\pi/6}\right)^{-1} V^{1\phi} & \text{for } \Delta \text{ configuration} \end{cases} \tag{3.21c}$$

$$Z^{3\phi} = \begin{cases} Z^{1\phi} & \text{for } Y \text{ configuration} \\ Z^{1\phi}/3 & \text{for } \Delta \text{ configuration} \end{cases} \tag{3.21d}$$

Motivated by the three-phase relations (3.21) we define the base values $(S_B^{3\phi}, V_B^{\text{ll}}, I_B^{3\phi}, V_B^{\text{ln}}, Z_B^{3\phi})$ for the three-phase quantities $(S^{3\phi}, V^{\text{ll}}, I^{3\phi}, V^{\text{ln}}, Z^{3\phi})$ in terms of the single-phase base values $(S_B^{1\phi}, V_B^{1\phi}, I_B^{1\phi}, Z_B^{1\phi})$ as follows:

$$S_B^{3\phi} := 3S_B^{1\phi}, \qquad V_B^{\text{ll}} := \sqrt{3}V_B^{\text{ln}} \tag{3.22a}$$

$$I_B^{3\phi} := \begin{cases} I_B^{1\phi} & \text{for } Y \text{ configuration} \\ \sqrt{3}I_B^{1\phi} & \text{for } \Delta \text{ configuration} \end{cases} \tag{3.22b}$$

$$V_B^{\text{ln}} := \begin{cases} V_B^{1\phi} & \text{for } Y \text{ configuration} \\ \left(\sqrt{3}\right)^{-1} V_B^{1\phi} & \text{for } \Delta \text{ configuration} \end{cases} \tag{3.22c}$$

$$Z_B^{3\phi} := \begin{cases} Z_B^{1\phi} & \text{for } Y \text{ configuration} \\ Z_B^{1\phi}/3 & \text{for } \Delta \text{ configuration} \end{cases} \tag{3.22d}$$

5

In light of (3.19) we could also have defined the base values $I_B^{3\phi}$ and $Z_B^{3\phi}$ in terms of $S_B^{3\phi}$ and V_B^{ll} as (see Exercise 3.14):

$$I_B^{3\phi} \ := \ \frac{S_B^{3\phi}}{\sqrt{3}V_B^{\text{ll}}}, \qquad Z_B^{3\phi} \ := \ \frac{\left(V_B^{\text{ll}}\right)^2}{S_B^{3\phi}} \tag{3.22e}$$

These definitions replace (3.22b) and (3.22d) and are applicable for both Y and Δ configurations (note that V_B^{ll} are different functions of $V_B^{1\phi}$ for Y and Δ configurations).

With these base values the per-unit quantities satisfy the following relations (see Exercise 3.15):

$$S_{\text{pu}}^{3\phi} \ = \ S_{\text{pu}}^{1\phi}, \qquad V_{\text{pu}}^{\text{ll}} \ = \ V_{\text{pu}}^{\text{ln}}, \qquad Z_{\text{pu}}^{3\phi} \ = \ Z_{\text{pu}}^{1\phi} \tag{3.23a}$$

$$\left|I_{\text{pu}}^{3\phi}\right| \ = \ \left|I_{\text{pu}}^{1\phi}\right|, \qquad \left|V_{\text{pu}}^{\text{ln}}\right| \ = \ \left|V_{\text{pu}}^{1\phi}\right| \tag{3.23b}$$

Therefore in per unit, the three-phase power, voltage, current and impedance equal their per-phase quantities (at least in magnitude). In particular when one says that the voltage magnitude is 1 pu, it means that the line-to-line voltage magnitude is 1 pu (i.e., equal to its base value V_B^{ll} which is $\sqrt{3}V_B^{1\phi}$ for Y configuration and $V_B^{1\phi}$ for Δ configuration), and the phase voltage magnitude is 1 pu (i.e., equal to its base value V_B^{ln} which is $V_B^{1\phi}$ for Y configuration and $\left(\sqrt{3}\right)^{-1} V_B^{1\phi}$). We sometimes need not specify whether a per-unit voltage is line-to-line or line-to-neutral, or whether a per-unit power is single-phase or three-phase. In Δ configuration the line-to-neutral voltage $V_{\text{pu}}^{\text{ln}}$ is related to single-phase voltage $V_{\text{pu}}^{1\phi}$ according to

$$V_{\text{pu}}^{\text{ln}} := \frac{V^{\text{ln}}}{V_B^{\text{ln}}} = \frac{\left(\sqrt{3}e^{i\pi/6}\right)^{-1} V^{1\phi}}{\left(\sqrt{3}\right)^{-1} V_B^{1\phi}} = e^{-i\pi/6} V_{\text{pu}}^{1\phi}$$

Similarly for line currents $I_{\text{pu}}^{3\phi}$ and $I_{\text{pu}}^{1\phi}$.

The next example illustrates the calculation of three-phase bases from single-phase bases. It shows in particular that impedances, including transformer parameters, will have the same per-unit values in single-phase or three-phase circuits and regardless of Y or Δ configuration.

Example 3.10 (Three-phase system)**.** Consider a single-phase distribution transformer with nameplate ratings of

- Power rating (1ϕ): 50 kVA;
- Voltage ratio: 408 V – 120 V;
- Transformer parameter: $X_l = 0.1$ pu, $X_m = 100$ pu (referred to the primary).

They are used to build three-phase transformer banks in YY, $\Delta\Delta$, ΔY or $Y\Delta$ configurations. Find the per-unit normalization "induced" by the nameplate ratings and the impedance diagram of the per-phase circuit in per unit.

Solution. The nameplate-induced base for the *single-phase* transformer is such that the power rating is 1pu and voltage rating is 1pu. Hence

$$S_B^{1\phi} := 50\,\text{kVA}, \qquad V_{1B}^{1\phi} := 408\,\text{V}, \qquad V_{2B}^{1\phi} := 120\,\text{V}$$

Therefore the current bases are

$$I_{1B}^{1\phi} := \frac{S_B^{1\phi}}{V_{1B}^{1\phi}} = \frac{50\,\text{kVA}}{408\,\text{V}} = 122.55\,\text{A}, \qquad I_{2B}^{1\phi} := \frac{S_B^{1\phi}}{V_{2B}^{1\phi}} = \frac{50\,\text{kVA}}{120\,\text{V}} = 416.67\,\text{A}$$

Since $S = |V|^2/Z$, the impedance base for the single-phase transformer induced by the nameplate ratings is:

$$Z_{1B}^{1\phi} = \frac{\left(V_{1B}^{1\phi}\right)^2}{S_B^{1\phi}} = \frac{(408\,\text{V})^2}{50\,\text{kVA}} = 3.33\,\Omega, \qquad Z_{2B}^{1\phi} = \frac{\left(V_{2B}^{1\phi}\right)^2}{S_B^{1\phi}} = \frac{(120\,\text{V})^2}{50\,\text{kVA}} = 0.29\,\Omega$$

Hence the actual transformer reactances X_l and X_m in Ω in the single-phase system are:

$$X_l = (0.1)\,Z_{1B}^{1\phi} = 0.333\,\Omega, \qquad X_m = (100)\,Z_{1B}^{1\phi} = 333\,\Omega$$

Consider now a three-phase transformer bank obtained from connecting three of these single-phase transformers. We consider first the base values for the primary side; the base values for the secondary side can be similarly chosen. What we will find is that if we choose our bases $(S_B^{3\phi}, V_B^{\text{ll}}, I_B^{3\phi}, Z_B^{3\phi})$ according to (3.22), then the impedance diagram of the per-phase equivalent circuit is independent of Y or Δ configuration.

Case 1: primary side in Y configuration. From (3.22), the base values of the three-phase power and line-to-line voltage induced by the nameplate ratings are

$$\begin{aligned} S_B^{3\phi} &:= 3\,S_B^{1\phi} = 3(50) = 150\,\text{kVA} \\ V_{1B}^{\text{ll}} &:= \sqrt{3}\,V_B^{1\phi} = \sqrt{3}\,(408) = 706.68\,\text{V} \end{aligned}$$

These three-phase quantities are used as the power and voltage ratings on the three-phase transformer nameplate. Hence a line voltage of 1 pu corresponds to the rated primary voltage (706.68 V) on the nameplate. The base values for the terminal currents and impedances are:

$$I_{1B}^{3\phi Y} := I_{1B}^{1\phi} = 122.55\,A, \qquad Z_{1B}^{3\phi Y} := Z_{1B}^{1\phi} = 3.33\,\Omega$$

It can be checked that $(S_B^{3\phi}, V_B^{\text{ll}}, I_B^{3\phi}, Z_B^{3\phi})$ as defined indeed satisfy three-phase relations:

$$I_{1B}^{3\phi Y} = \frac{S_B^{3\phi}}{\sqrt{3}\,V_{1B}^{\text{ll}}}, \qquad Z_{1B}^{3\phi Y} = \frac{\left(V_B^{\text{ll}}\right)^2}{S_B^{3\phi}}$$

Since $Z_{1B}^{3\phi Y} = Z_{1B}^{1\phi}$, $X_l = 0.1$pu and $X_m = 100$pu as before for the three-phase transformer.

Case 2: primary side in Δ configuration. From (3.22), the base values of the three-phase power and line-to-line voltage induced by the nameplate ratings are

$$S_B^{3\phi} \;:=\; 3\,S_B^{1\phi} \;=\; 3(50) \;=\; 150\,\text{kVA}, \qquad V_{1B}^{\text{ll}} \;:=\; V_B^{1\phi} \;=\; 408\,\text{V}$$

The terminal current and the impedance bases are:

$$I_{1B}^{3\phi Y} \;:=\; \sqrt{3}\,I_{1B}^{1\phi} \;=\; \sqrt{3}(122.55) \;=\; 212.26\,A, \qquad Z_{1B}^{3\phi\Delta} \;=\; \frac{Z_B^{1\phi}}{3} \;=\; \frac{3.33}{3} \;=\; 1.11\,\Omega$$

To convert the transformer circuit model in Δ configuration to its equivalent Y configuration, the transformer reactances are reduced by a factor of 3, i.e., $X_l^Y = X_l/3$ and $X_m^Y = X_m/3$. Hence the transformer reactances in pu are:

$$\begin{aligned}
X_{l\text{pu}}^Y &:= \frac{X_l^Y}{Z_{1B}^{3\phi}} = \frac{X_l/3}{Z_{1B}^{1\phi}/3} = \frac{X_l}{Z_{1B}^{1\phi}} = 0.1\text{ pu} \\
X_{m\text{pu}}^Y &:= \frac{X_m^Y}{Z_{1B}^{3\phi}} = \frac{X_m/3}{Z_{1B}^{1\phi}/3} = \frac{X_m}{Z_{1B}^{1\phi}} = 100\text{ pu}
\end{aligned}$$

as expected.

In summary, with the three-phase base values defined in (3.22), the transformer reactances X_l and X_m remain the same in pu regardless of how the single-phase transformers are connected into a three-phase transformer bank. The impedance diagram of its per-phase circuit is shown in Figure 3.33. □

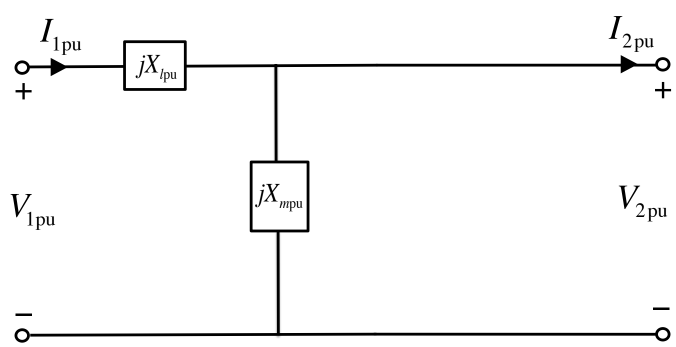


Figure 3.33: Impedance diagram of a three-phase transformer bank.

3.5.5 Per-unit per-phase analysis

Consider a balanced three-phase normal system. Recall that the nameplate ratings of three-phase transformers are specified in terms of their three-phase power and line-to-line voltages. The procedure for per-unit per-phase analysis is summarized as follows:

1. For a single-phase system, pick a power base $S_B^{1\phi}$ for the *entire* system and a voltage base V_{1B}^{ln} in *one* of the areas, e.g., induced by the nameplate ratings of one of the single-phase transformers.

2. For a balanced three-phase system, pick a three-phase power base $S_B^{3\phi}$ and line-to-line voltage base V_{1B}^{ll} induced by the nameplate ratings of one of the three-phase transformers in area 1 (choose either the primary or secondary circuit as area 1). Then choose the power and voltage bases for the per-phase equivalent circuit of the balanced three-phase system according to (3.22a):

$$S_B^{1\phi} := \frac{S_B^{3\phi}}{3} \quad \text{and} \quad V_{1B}^{1\phi} := \frac{V_{1B}^{\text{ll}}}{\sqrt{3}}$$

   $S_B^{1\phi}$ will be the power base for the entire per-phase circuit.

3. Calculate the current and impedance bases in that area by:

$$I_{1B} := \frac{S_B^{1\phi}}{V_{1B}^{1\phi}} \quad \text{and} \quad Z_{1B} := \frac{\left(V_{1B}^{1\phi}\right)^2}{S_B^{1\phi}}$$

4. Calculate the base values for voltages, currents, and impedances in areas i connected to area 1 by the magnitudes n_i of the transformer gains (assuming area 1 is the primary side of the transformers):

$$V_{iB}^{1\phi} := n_i V_{1B}^{1\phi}, \quad V_{iB}^{\text{ll}} := n_i V_{1B}^{\text{ll}} \quad I_{iB} := \frac{1}{n_i} I_{1B}, \quad Z_{iB} := n_i^2 Z_{1B}$$

   Continue this process to calculate the voltage, current, and impedance base values for all areas.

5. For real, reactive, apparent power in the entire system, use $S_B^{1\phi}$ as the base value. For resistances and reactances, use Z_{iB} as the base value in area i. For admittances, conductances, and susceptances, use $Y_{iB} := 1/Z_{iB}$ as the base value in area i.

6. Draw the impedance diagram of the entire system, and solve for the desired per unit quantities.

7. Convert back to actual quantities if desired.

3.6 Bibliographical notes

There are many excellent textbooks on basic power system concepts and many materials in this chapter follow [1]. Some of the materials on per-unit normalization, e.g., off-nominal regulating transformer in Chapter 3.5.3, follow [2]. [8] describes a rigorous approach that treats per-unit normalization as a similarity transformation of a dynamical system in the time domain. The per-unit normalization presented in this chapter represents the steady-state of the per-unit dynamical system of [8].

3.7 Problems

Chapter 3.1.

Exercise 3.1 (T model of transformer)**.** For the T equivalent circuit of transformer in Figure 3.34, show that the transmission matrix T is given by

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \underbrace{\begin{bmatrix} a\left(1+z_p y_m\right) & a z_s\left(1+z_p y_m\right)+n z_p \\ a y_m & n+a z_s y_m \end{bmatrix}}_{T} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{3.24}$$

Hence if $y_m = 0$ then

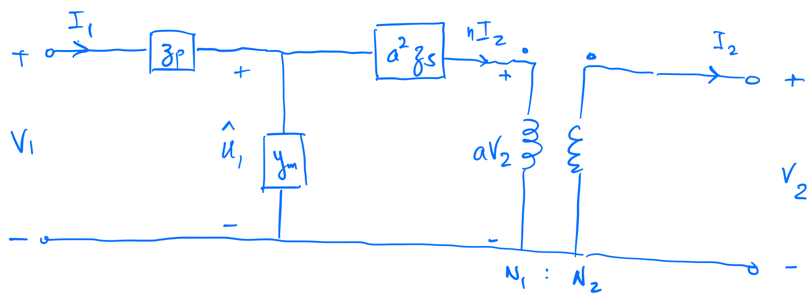


Figure 3.34: Example 3.1: T equivalent circuit of transformer with $n := N_2/N_1$ and $a := N_1/N_2$.

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} a & n\left(z_p + a^2 z_s\right) \\ 0 & n \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

which is the same as the transmission matrix $\hat{T}$ in (3.7).

Exercise 3.2 (T model of transformer)**.** Given the primary voltages and primary currents (V_{sc}, I_{sc}) and (V_{oc}, I_{oc}) of a short-circuit and open-circuit tests respectively, derive (3.6), reproduced here:

$$V_{sc} = \left(z_p + \left(y_m + \frac{1}{a^2 z_s}\right)^{-1}\right) I_{sc}, \qquad V_{oc} = \left(z_p + \frac{1}{y_m}\right) I_{oc} \tag{3.25}$$

from (3.4), reproduced here:

$$\text{Nonideal elements:} \quad V_1 = z_p I_1 + \hat{V}_1, \quad \hat{I}_m = y_m \hat{V}_1, \quad \hat{V}_2 = z_s I_2 + V_2 \tag{3.26a}$$

$$\text{Ideal transformer:} \quad \hat{V}_2 = \frac{N_2}{N_1}\hat{V}_1, \quad I_2 = \frac{N_1}{N_2}\left(I_1 - \hat{I}_m\right) \tag{3.26b}$$

where the series impedances

Exercise 3.3 (Simplified model)**.** Consider the transformer model in Figure 3.5 with $z_l = z_p + a^2 z_s$ and its transmission matrix $\hat{T}$ in (3.7), reproduced here

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \underbrace{\begin{bmatrix} a\left(1+z_l y_m\right) & n z_l \\ a y_m & n \end{bmatrix}}_{\hat{T}} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{3.27}$$

This question shows that when the shunt admittance matrix y_m is small compared with the series admittances z_s, $\hat{T}$ is a good approximation of T, the transmission matrix in (3.24). Let $\varepsilon := a^2 z_s y_m$.

1. Show that their difference is

$$\hat{T} - T \;=\; \varepsilon \begin{bmatrix} a & -nz_p \\ 0 & -n \end{bmatrix}$$

2. Suppose $z_p = \eta z_s = \eta(r_s + \mathbf{i}x_s)$ for some real number $\eta > 0$ with $r_s > 0$ and $x_s > 0$, $y_m = -\mathbf{i}b_m$ with $b_m > 0$, and $|\varepsilon| \ll 1$. Show that

$$\frac{\|\hat{T} - T\|}{\|T\|} \;<\; |\varepsilon| \;\ll\; 1$$

where $\|A\|$ denotes the sum norm $\|A\| := \sum_{i,j} |A_{ij}|$.

Exercise 3.4 (Unitary voltage network)**.** Show that the T equivalent circuit described by

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} a(1+z_p y_m) & az_s(1+z_p y_m) + nz_p \\ ay_m & n + az_s y_m \end{bmatrix}}_{T} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix} \tag{3.28}$$

is equivalent to the transformer model $I = (MY_{\text{uvn}}M)V$ given by (3.11).

Exercise 3.5 (Unitary voltage network)**.** Show that, instead of the numbers N_1, N_2 of turns of the primary and secondary windings respectively, the admittance matrix $MY_{\text{uvn}}M$ in (3.11) can equivalently be written in terms of the turns ratio $a := N_1/N_2$:

$$MY_{\text{uvn}}M \;=\; \frac{y_p y_s}{a^2 y_m + a^2 y_p + y_s} \begin{bmatrix} 1 + a^2 y_m / y_s & -a \\ -a & a^2(1 + y_m/y_p) \end{bmatrix}$$

Chapter 3.2.

Exercise 3.6 ($Y\Delta$ configuration)**.** Consider the three-phase transformer with Y-configured primary side on the left and Δ-configured secondary side on the right as shown in Figure 3.35 with a voltage gain n. Show that the voltage gain $V_{\text{secondary}}/V_{\text{primary}}$ is $K(n) := \frac{n}{\sqrt{3}} e^{-\mathbf{i}\pi/6}$.

Note that the voltage gain $K(n) := \frac{n}{\sqrt{3}} e^{-\mathbf{i}\pi/6}$ is equal to the inverse of the voltage gain $K_{\Delta Y}(1/n)$ for a ΔY-configured transformer, i.e.,

$$K(n) \;=\; \frac{1}{K_{\Delta Y}(1/n)}$$

This means that this transformer is identical to a ΔY transformer with its primary and secondary sides switched. It is unlike the voltage gain of the $Y\Delta$-configured transformer with connectivity shown in Figure 3.13(b).

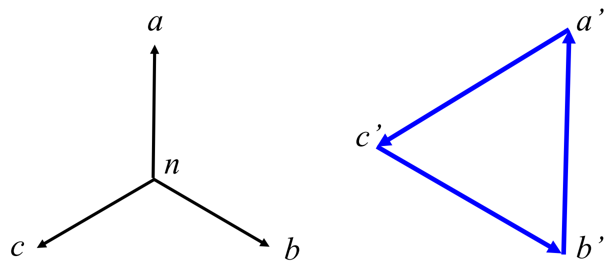
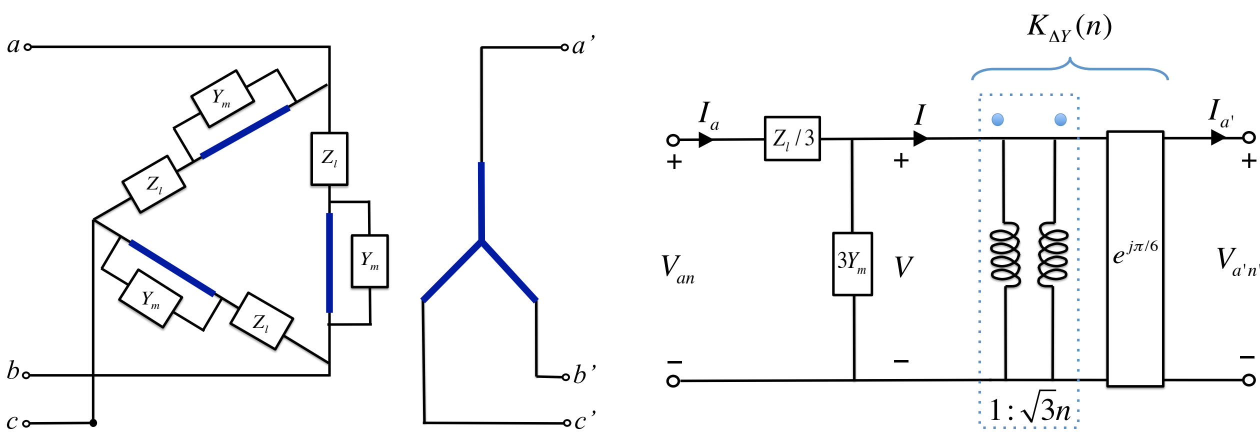


Figure 3.35: Exercise 3.6: Three-phase transformers in $Y\Delta$ configuration.



(a) ΔY configuration

(b) Per-phase circuit

Figure 3.36: Model of three-phase transformers in ΔY configuration and its per-phase equivalent circuit.

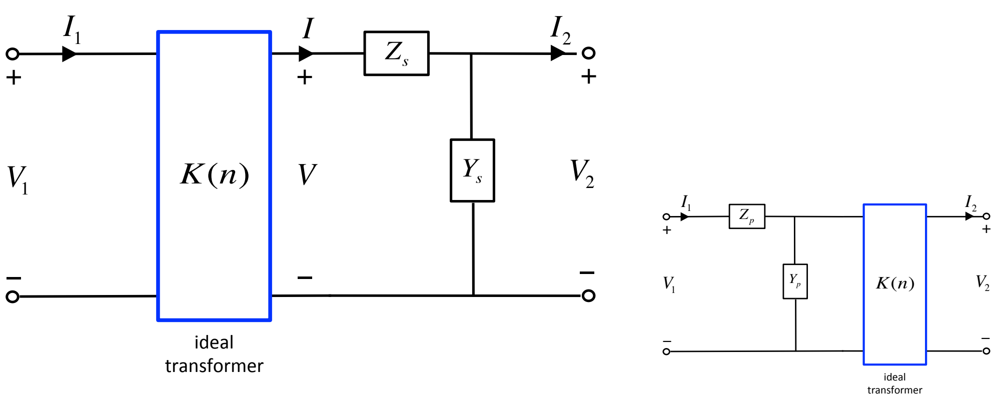
Exercise 3.7. Figure 3.36 shows a model of balanced three-phase transformers in ΔY configuration and its per-phase equivalent circuit. Show that the mapping from $(V_{a'n'}, I_{a'})$ to (V_{an}, I_{an}) is given by

$$\begin{bmatrix} V_{an} \\ I_a \end{bmatrix} = \begin{bmatrix} K_{\Delta Y}^{-1}(n)\,(1+Z_l Y_m) & K_{\Delta Y}^{*}(n)\frac{Z_l}{3} \\ K_{\Delta Y}^{-1}(n)\,(3Y_m) & K_{\Delta Y}^{*}(n) \end{bmatrix} \begin{bmatrix} V_{a'n'} \\ I_{a'} \end{bmatrix}$$

where $K_{\Delta Y}(n) := \sqrt{3}n\, e^{j\pi/6}$ and $Y_m := -j/X_m$.

Exercise 3.8 (Referring shunt admittance in one side to the other)**.** Show that the transmission matrix for the circuit in Figure 3.20(a) is the same as that in Figure 3.20(b) provided that the relation (3.14b) between shunt admittances Y_p and Y_s holds.

Exercise 3.9 (Transmission matrix)**.** Consider a balanced three-phase ideal transformer with a complex gain $K(n)$ connected to a balanced three-phase series impedance Z_s and a balanced three-phase shunt admittance Y_s on the secondary side. The per-phase equivalent circuit is shown in Figure 3.37(a). Show



(a) (Z_s, Y_s) in the secondary (b) (Z_p, Y_p) in the primary

Figure 3.37: Referring (Z_s, Y_s) on the secondary to the primary for an ideal transformer with a complex gain $K(n)$.

directly that transmission matrix of the circuit in Figure 3.37(a) is the same as that in Figure 3.37(b) provided the relation (3.14) between impedances/admittances (Z_p, Y_p) and (Z_s, Y_s) holds.

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Exercise 3.10 (Driving-point impedance)**.** Refer to Figure 3.23.

1. Show that the driving-point impedance V_1/I_1 on the primary side is the same in both circuits in Figure 3.23(a).

2. Show that the driving-point impedance V_2/I_2 on the secondary side is the same in both circuits in Figure 3.23(b).

Exercise 3.11 (Driving-point impedance on primary side)**.** Suppose the secondary sides of the (equivalent) circuits in Figure 3.37 are connected to an identical load Z_{load} so that $V_2 = Z_{\text{load}} I_2$ in both circuits.

1. Show that the driving-point impedances on the primary side of the circuit in Figure 3.37(a) is:

$$\frac{V_1}{I_1} = \frac{1}{|K(n)|^2}\left(Z_s + \frac{1}{Y_s + Z_{\text{load}}^{-1}}\right) \tag{3.29a}$$

   The term in the bracket is the Thévain equivalent impedance in the secondary circuit, seen from the output of the ideal transformer.

2. Show that the driving-point impedances on the primary side of the circuit in Figure 3.37(b) is:

$$\frac{V_1}{I_1} = Z_p + \frac{1}{Y_p + |K(n)|^2 Z_{\text{load}}^{-1}} \tag{3.29b}$$

3. Show that (3.29a) and (3.29b) are equivalent provided that (Z_p, Y_p) and (Z_s, Y_s) satisfy (3.14).

Exercise 3.12. Consider the balanced three phase system in Figure 3.38 where the line-to-line voltage of the three-phase generator in Δ configuration is V_{gen}. The 3ϕ transformer consists of single-phase transformers in ΔY configuration. Each single-phase transformer is modeled by a series impedance Z_l (and negligible shunt admittance) on the primary side followed by an ideal transformer with turn ratio n. The transmission line is modeled by a Π-model with a series impedance Z_s and a shunt admittance $Y_m/2$ at each end of the line. The transmission line is connected to a balanced 3ϕ impedance load in Y configuration with an impedance Z_{load} in each phase.

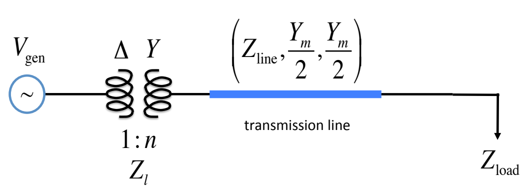


Figure 3.38: A three-phase generator in Δ configuration connected to a three-phase ΔY transformer and then to a three-phase load in Y configuration through a three-phase AC transmission line.

1. Draw the equivalent per-phase circuit.

2. Derive the complex power delivered to the load Z_{load} in each phase.

Exercise 3.13 (Bases across transformers)**.** For a normal system, on each parallel path in its per-phase equivalent circuit, the product of ideal transformer gain magnitudes is the same. Show that this property allows us to consistently define base values between two neighboring areas using (3.20). (Hint: Show that around any loop, (3.20) holds only if the product of voltage gain magnitudes around the loop is 1.)

Exercise 3.14 (Terminal current and three-phase impedance bases)**.** Show the definition (3.22b) (3.22d) for base values $I_B^{3\phi}$ and $Z_B^{3\phi}$ respectively are equivalent to definition (3.22e).

Exercise 3.15 (Per unit properties)**.** Prove the per-unit properties (3.23).

Exercise 3.16 (Caltech ACN: transformers)**.** Figure 3.39 shows the layout of the Adaptive Charging Network (ACN) for electric vehicles (EVs) in a Caltech garage. The Caltech ACN consists of two three-phase

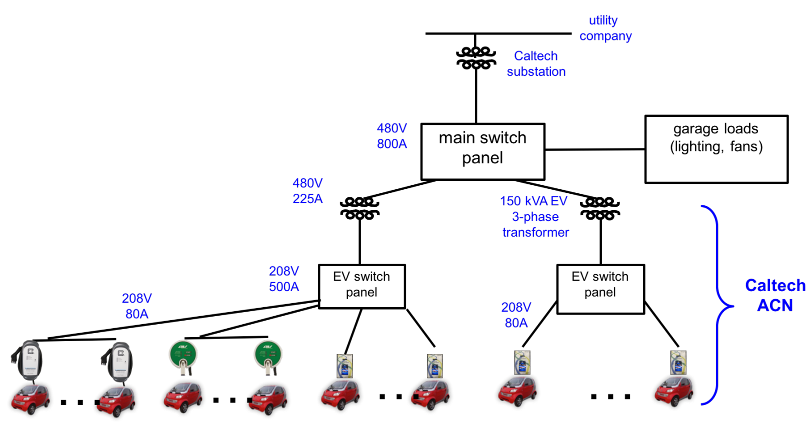
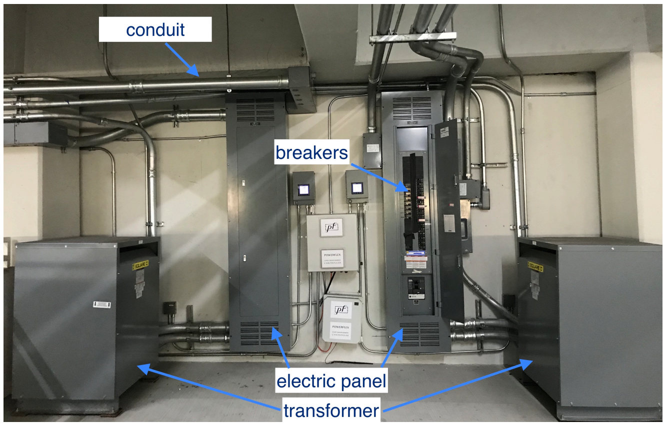


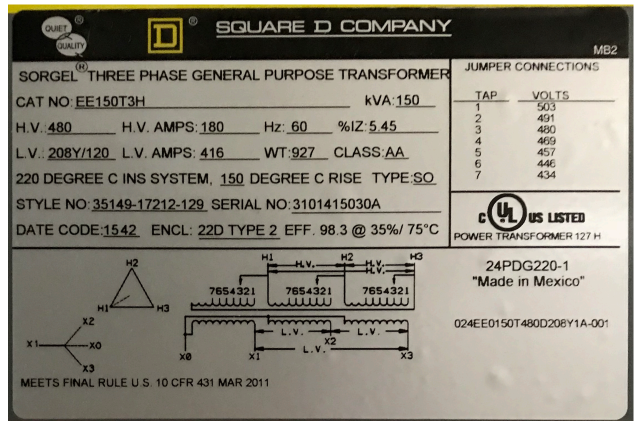
Figure 3.39: Caltech Adaptive Charging Network (ACN) layout.

stepdown transformers in ΔY configuration with Δ on the primary side. Each of these transformers is connected to an electric panel, to which charging stations and subpanels are connected. Figure 3.40(a) shows the two three-phase transformers and the two electric panels. Figure 3.40(b) shows the ratings of each of the three-phase transformers:

- Power rating 150 kVA (three-phase).
- Primary (high voltage) side: 480V in Delta configuration with rated line current of 180A.
- Secondary (low voltage) side: 208Y/120V in Wye configuration with rated line current of 416A.
- Impedance voltage (percentage impedance): $\beta = 5.45\%$ on the primary side (the shunt admittance is negligible).



(a) Transformers and panels



(b) Transformer ratings

Figure 3.40: (a) The two 150 kVA transformers and two electric panels in Caltech ACN to which charging stations and electric subpanels are connected. (b) The transformer ratings.

The impedance voltage is the voltage drop across the series impedance Z_l on the primary side of the transformer in a short-circuit test, as a percentage of the rated primary voltage. In a short-circuit test the secondary side is short-circuited. The β specification means that the voltage needed on the primary side to produce a rated primary current is β times the rated primary voltage.

Verify that the rated line currents on the primary and secondary sides are consistent with the power rating and voltage ratings. Determine the magnitude $|Z_l|$ of the series impedance of the transformer and draw the circuit model of the three-phase transformer.

Exercise 3.17 (Caltech ACN: estimating distribution line impedances)**.** Suppose the transformer in Exercise 3.16 is connected to a three-phase voltage source with a line voltage of $|V_{\text{line}}| = 480V$ on the primary side through a three-phase distribution line modeled by a series impedance $Z_{\text{line},1}$, and to a three-phase load on the secondary side through another three-phase distribution line modeled by a series impedance $Z_{\text{line},2}$, as shown in Figure 3.41. Suppose the system is balanced. The load is a three-phase constant-current load

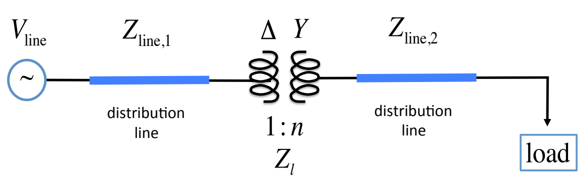


Figure 3.41: The three-phase transformer is connected to a three-phase voltage source and a three-phase load through two three-phase lines.

in Δ configuration with a known current I_{load} from phase a to phase b. The voltage is measured to be V_2 across the load between phase a and phase b. The phase a voltage on the secondary side of the transformer (before the distribution line) is measured to be V_{an}.

Determine the distribution line impedances $Z_{\text{line},1}$ and $Z_{\text{line},2}$ in terms of the line voltage $|V_{\text{line}}|$, the series impedance Z_l of the transformer, and the complex gain $K(n)$ of the ideal ΔY transformer, as well as the measured voltages V_2, V_{an} and current I_{load}. Assume without loss of generality that the voltage source has $V_{ab} = |V_{\text{line}}|\angle 0^\circ$ and the sources are in positive sequence.

Exercise 3.18 (Caltech ACN: network design)**.** This problem considers the deployment costs of different network designs for ACN. Referring to Figure 3.40(a), the output (secondary side) of each of the 150 KVA transformers is connected to the input of one of the two electric panels. A wire connects a circuit breaker in the panel to an electric vehicle (EV) charger or a subpanel and these wires are housed in conduits. We consider the network that connects all the EV chargers to one of the two panels in Figure 3.40(a). In this network, the main components are wires, conduits, and subpanels and the types and sizes of these hardware determine the deployment costs, both parts and labor. The types and sizes depend on the current limit (ampacity) of each wire segment required to carry the current to chargers it supplies and the distance of that wire segment. Consider an *idealized* layout in Figure 3.42 where the network connects a total of nk EV chargers to the electric panel. These chargers are clustered into n groups. Each group i is associated

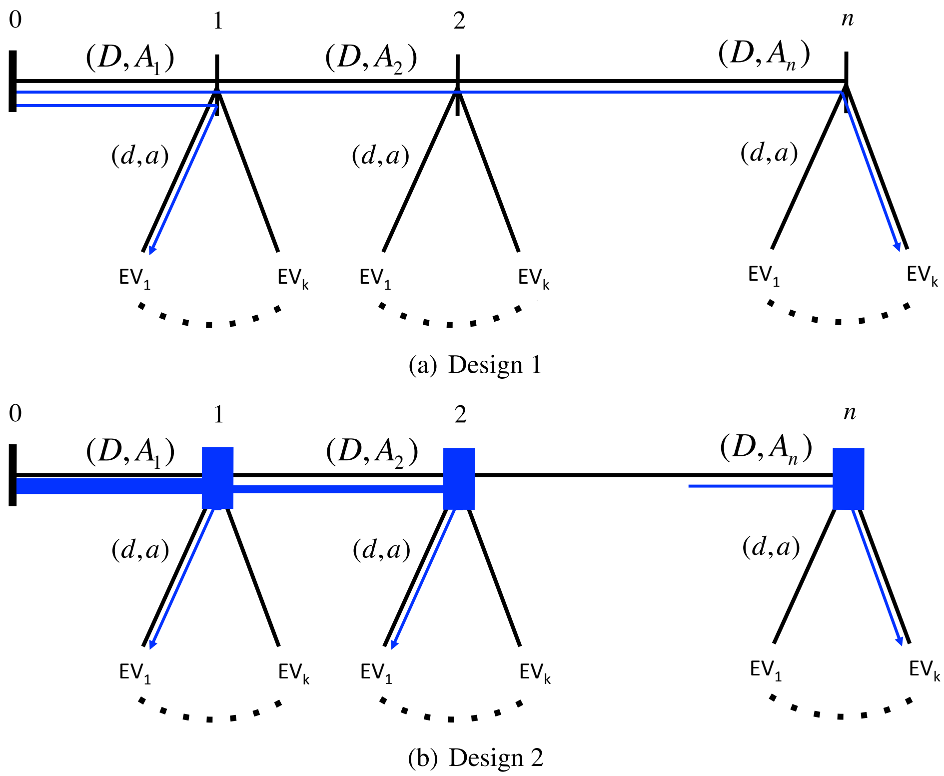


Figure 3.42: Caltech ACN network design.

with a junction $i = 1, \ldots, n$ as shown in the figure. Every group consists of k identical chargers labeled by $EV_1, \ldots, EV_k$. Each charger can draw a maximum current of I (in A).

Design 1. The first design runs a wire from the electric panel at junction 0 directly to each charger following the path labeled in black in Figure 3.42(a). Let (D, A_i) denote the distance and the cross-sectional area of the wire between each junction $i-1$ to i. Let (d, a) denote the distance and the cross-sectional area of the wire from a junction to every EV in its group. The cross-sectional area of a wire depends on the maximum current it needs to supply. We assume the maximum current that can be drawn by any charger is the same, and therefore the wires from a junction to any EV in its group all have the same size a. The wire size A_i between junctions $i-1$ and i depends on the layout. In design 1, $A_i = a$ for all i. This will be different in design 2 (see below).

For example, the wire connecting EV_1 in group 1 goes from junction 0 (electric panel) to junction 1 to the charger, as shown in blue, and has a total length of $D+d$ and size a. The wire connecting EV_k in group n goes from junction 0 to junctions $1,\dots,n$, to the charger, and has a total length of $nD+d$ and size a.

Design 2. In this design a single wire of length D and size A_1 connects the electric panel at junction 0 to an electric subpanel at junction 1; see Figure 3.42(b). Then k wires each of length d and size a connects the k chargers in group 1 to the subpanel. A single wire of length D and size $A_2 < A_1$ connects the subpanel at junction 1 to a subpanel at junction 2, and k wires each of (d,a) then connects the k chargers in group 2, and so on.

For both design 1 and design 2, the cross-sectional area of the wire used for any segment of the layout depends on the maximum current (called the *ampacity* of the wire in ampere) that it needs to carry. That is, the wire sizes a, A_i above are functions $\alpha(x)$ where x is the ampacity. See below for an example of $\alpha(x)$.

Deployment costs. The total deployment cost (parts and labor) involve mainly three types of hardware.

1. *Wire.* The cost of deploying a wire of length λ and cross-sectional area α is denoted by the function $C_w(\lambda,\alpha)$.

2. *Conduit.* The cost $C_c(\lambda,\alpha)$ of deploying a conduit of length λ that carries wires with a *total* cross-sectional areas α has two components:

$$C_c(\lambda,\alpha) \ := \ C_{c1}(\lambda,\alpha) + C_{c2}(\alpha)$$

   The first component $C_{c1}(\lambda,\alpha)$ depends on the length λ and total wire size α, the longer and larger the conduit, the higher the cost. The second component $C_{c2}(\alpha)$ depends only on the total wire size α and is usually a step function: when the total wire size exceeds a threshold, a special machine is needed to deploy the conduit at an extra cost. In Design 1, all wires that share the same segment (say) between junctions $i-1$ to i will be housed in the same conduit. For example, the conduit between junction 1 and junction 2 will carry $(n-1)k$ wires. We assume that if a conduit carries wires of areas $\alpha_1,\dots,\alpha_m$, then the total wire size is simply its sum $\alpha := \sum_{i=1}^m \alpha_i$.

3. *Subpanel.* For simplicity we assume every subpanel (in design 2) has the same cost c_s.

Assumptions on cost functions. Assume the cost functions take the following form:

$$C_w(\lambda,\alpha) \ := \ c_w\,\lambda\,\alpha, \qquad C_{c1}(\lambda,\alpha) \ := \ c_c\,\lambda\,\alpha, \qquad C_{c2}(\alpha) = \beta\,\mathbf{1}(\alpha\geq\tau) \tag{3.30a}$$

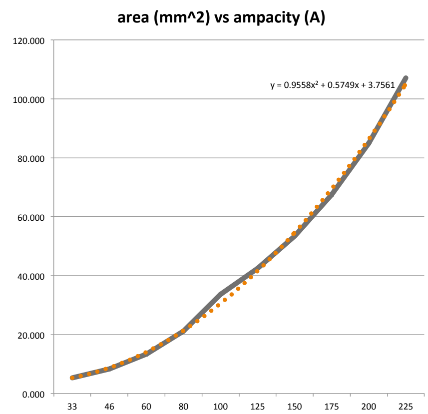
Figure 3.43(a) shows the wire size dependence $\alpha(x)$ on ampacity x from (a version of) the American Wire Gauge (AWG) standard. Based on the data, Figure 3.43(b) shows that $\alpha(x)$ can be well approximated by a quadratic function

$$\alpha(x) \quad := \quad x^2 + 0.6x + 4 \tag{3.30b}$$

with x in A and $\alpha(x)$ in mm^2. The quadratic term represents the fact that the thermal power loss due

AWG #	area (mm^2)	ampacity (enclosed, A)
10	5.269	33
8	8.347	46
6	13.332	60
4	21.156	80
2	33.593	100
1	42.429	125
0	53.456	150
00	67.491	175
000	84.949	200
0000	107.146	225

(a) AWG table



(b) AWG plot

Figure 3.43: (a) American Wire Gauge (AWG) standard: dependence of wire cross-sectional area $\alpha(x)$ on ampacity x. (b) The data for $\alpha(x)$ in the table can be approximated by the quadratic function in (3.30b). The black solid line is the plot of the data and the orange dashed line is the quadratic fit.

to a current I_0 through a wire with resistance r is roughly rI_0^2. Doubling the current means that the resistance must be scaled down by a factor of 4 in order to maintain the same heat loss. Since r is inversely proportional to the cross-sectional area of the wire, this requires a wire with 4 times the area.

1. Evaluate the total cost of network design 1 and design 2.
2. Prove that design 1 is always less expensive than design 2 as long as the maximum current I that can be drawn by a charger is at least 2A.[6]

Exercise 3.19 (Caltech ACN: network design)**.** This problem generalizes problem 3.18 to show that design 2 is more expensive even for very general cost functions and wire size dependency. Suppose the cost functions $C_w(\lambda, \alpha), C_{c1}(\lambda, \alpha), C_{c2}(\alpha)$ and the dependency of wire size $\alpha(x)$ on its ampacity satisfy the following conditions:

C1: For any fixed α, $C_w(\lambda, \alpha)$ is linear in λ. For any fixed λ, $C_w(\lambda, \alpha)$ linear and increasing in α.

C2: $C_{c1}(\lambda, \alpha)$ is increasing in α for any fixed λ. $C_{c2}(\alpha)$ is increasing in α.

C3: There is an ampacity set X such that for all $x \in X$, $\alpha(ix) \geq i\alpha(x)$ for any integer $i \geq 1$.

[6]Currently a level-2 EV charger typically has a current limit of 32A or higher.

Prove that design 2 is more expensive for any ampacity $x \in X$.

It can be easily verified that the cost functions and $\alpha(x)$ in (3.30) satisfy these conditions. In particular the ampacity set X in condition C3 is $X = \{x \geq 2\text{A}\}$. Therefore the conditions C1–C3 allow a much larger set of cost functions and $\alpha(x)$ than (3.30).

We now interpret these conditions to illustrate that they are realistic. Condition C1 says that the total deployment cost (parts and labor) grows linearly in wire length λ and in wire size α. If either one doubles, the cost exactly doubles. Condition C2 says that regardless of its length, both the first and second cost components of the conduit increase as the cross-sectional area of the conduit increases. Finally condition C3 implies in particular that, for any ampacity x in X, doubling the ampacity more than doubles the cost. As explained immediately after (3.30b), since thermal loss is quadratic in ampacity, the required wire size satisfies this condition. The proof reveals that this is the key condition that makes design 2 more expensive than design 1, i.e., it is always cheaper to use more and longer small wires because the wire size *grows faster than linearly in ampacity*.

Exercise 3.20 (Caltech ACN: network design)**.** Problem 3.19 shows that, under very general and realistic conditions, design 2 is always more expensive than design 1. This assumes that, in design 2, the ampacity of the wire between junction $i-1$ and i must be the sum of the ampacities of all the downstream wires supplying groups $i, i+1, \ldots, n$. In practice however it is unlikely all the EV chargers in these groups will draw maximum currents simultaneously and therefore it is reasonable to install a smaller ampacity between junction $i-1$ and i, i.e., each subpanel can be over-subscribed. Discuss over-subscription conditions under which design 2 is less expensive than design 1 (open-ended problem).

Chapter 4

Bus injection models

In previous chapters we introduce mathematical models of main power system components. In this and the next chapters we use these models to describe a power network consisting of an interconnection of basic components such as generators, loads, transmission lines and transformers. In Chapter 4.1 we summarize the component models from previous chapters. In Chapter 4.2 we explain how to model a power network by a matrix that linearly relates current injections to voltages at each node of the network. In Chapter 4.3 we present power flow equations that relate power injections and voltages at each node. In Chapter 4.5 we discuss classical solution methods.

4.1 Component models

4.1.1 Single-phase sources and impedance

In Chapters 1.1.2 and 1.3.1 we describe circuit models of single-phase single-terminal devices as summarized here. They are also per-phase models of balanced three-phase devices. Associated with each device j is its terminal voltage, current, and power $(V_j, I_j, s_j) \in \mathbb{C}^3$. There is an arbitrary *reference point* with respect to which all voltages are defined. If the common reference point is taken to be the ground then voltage V_j is the voltage drop between terminal j and the ground. The current from terminal j flows from the terminal to the reference point; see Figure 4.1. Such a single-terminal device is characterized by relations between the terminal variables (V_j, I_j, s_j).

1. *Voltage source* (E_j, z_j). This is a device with a constant internal voltage E_j in series with an impedance z_j as shown in Figure 1.1.2(a). Its external model is the relation $V_j = E_j - z_j I_j$ between its terminal voltage and current (V_j, I_j). This yields a relation $s_j = V_j I_j^{\mathsf{H}} = V_j \left(E_j - V_j\right)^{\mathsf{H}} / z_j^{\mathsf{H}}$ between the terminal variables (V_j, s_j).

2. *Current source* (J_j, y_j). This is a device with a constant internal current J_j in parallel with an admittance y_j as shown in Figure 1.1.2(b). Its external model is the relation $I_j = J_j - y_j V_j$ between its terminal voltage and current (V_j, I_j). This yields a relation $s_j = V_j I_j^{\mathsf{H}} = V_j \left(J_j - y_j V_j\right)^{\mathsf{H}}$ between

the terminal variables $\left(V_j, s_j\right)$.

3. *Power source* $\left(\sigma_j, z_j\right)$. This is a device with a constant internal power σ_j in series with an impedance z_j. Its external model is the relation $\sigma_j = \left(V_j - z_j I_j\right) I_j^{\mathsf{H}}$ between (V_j, I_j). Its terminal power is $s_j = V_j I_j^{\mathsf{H}} = \sigma_j + z_j I_j I_j^{\mathsf{H}}$.

4. *Impedance* z_j. The external (and internal) model is $V_j = z_j I_j$ and $s_j = \left|V_j\right|^2 / z_j^{\mathsf{H}}$.

4.1.2 Single-phase line

In Chapter 2 we describe the Π circuit model of a single-phase line. It is also a per-phase model of balanced three-phase lines. A line has two terminals (j,k) and is specified by a three-tuple $(y^s_{jk}, y^m_{jk}, y^m_{kj}) \in \mathbb{C}^3$ where $y^s_{jk} = y^s_{kj}$ is the series admittance of the line, y^m_{jk} is the shunt admittance of the line at terminal j, and y^m_{kj} is the shunt admittance of the line at terminal k; see Figure 4.1. Recall that if (j,k) represents a transmission

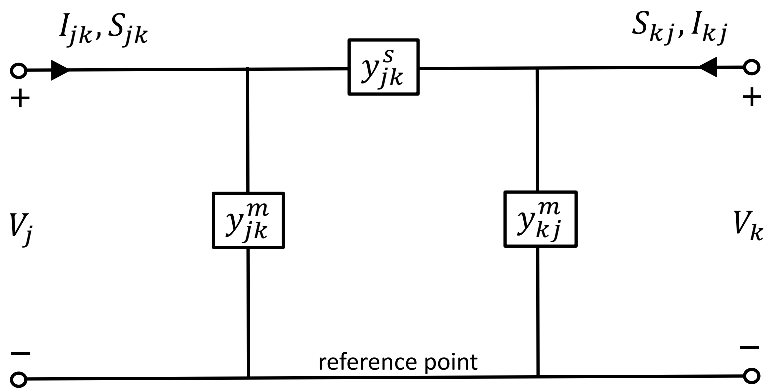


Figure 4.1: Π circuit model of a single-phase line.

line then (y^m_{jk}, y^m_{kj}) models the line capacitance, called *line charging* or *shunt admittances of line* (j,k), and the currents through these shunt admittances model the current supplied to the line capacitance called the charging current.

Associated with terminal j is the terminal voltage V_j, and the sending-end line current I_{jk} and power S_{jk} from j to k. Similarly, associated with terminal k is $(V_k, I_{kj}, S_{kj}) \in \mathbb{C}^3$. Unlike in Chapter 2.2.2 we have defined here I_{kj} to be the current injected from the right terminal into the line. A line is characterized by the relation between the terminal voltages (V_j, V_k) and line currents $\left(I_{jk}, I_{kj}\right)$ or that between (V_j, V_k) and line powers $\left(S_{jk}, S_{kj}\right)$, which we now explain.

***VI* relation.** The terminal voltages with respect to, and the sending-end currents flowing from the terminals to, the reference point are related by

$$I_{jk} \; = \; y^s_{jk}(V_j - V_k) \; + \; y^m_{jk} V_j, \qquad\qquad I_{kj} \; = \; y^s_{kj}(V_k - V_j) \; + \; y^m_{kj} V_k \tag{4.1a}$$

This defines a matrix Y_{line} for a line that maps terminal voltages to sending-end currents:

$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} = \underbrace{\begin{bmatrix} y^s_{jk} + y^m_{jk} & -y^s_{jk} \\ -y^s_{jk} & y^s_{jk} + y^m_{kj} \end{bmatrix}}_{Y_{\text{line}}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{4.1b}$$

where we have used the fact that $y^s_{jk} = y^s_{kj}$ to obtain a symmetric Y_{line}. The off-diagonal entries of Y_{line} are the negatives of the series admittances while the diagonal entries are the sum of series and shunt admittances. As we will see this structure holds for general networks.

In general the sending-end currents $\left(I_{jk}, I_{kj}\right)$ are not negative of each other when the shunt admittances are nonzero. Since $y^s_{jk} = y^s_{kj}$, their sum represents the total current loss along the line due to shunt admittances:

$$I_{jk} + I_{kj} = y^m_{jk} V_j + y^m_{kj} V_k \neq 0$$

Thermal limits on branch current flows should be imposed on both $|I_{jk}|$ and $|I_{kj}|$:

$$\begin{aligned} |I_{jk}| &= \left| y^s_{jk}(V_j - V_k) + y^m_{jk} V_j \right| \leq I^{\max}_{jk} \\ |I_{kj}| &= \left| y^s_{kj}(V_k - V_j) + y^m_{kj} V_k \right| \leq I^{\max}_{kj} \end{aligned}$$

not just on $\left| y^s_{jk}(V_j - V_k) \right|$ unless the shunt admittances are zero.

Vs **relation.** The sending-end line power flows from terminals j to k and that from terminals k to j are respectively (using (4.1a)):

$$S_{jk} := V_j I^{\mathsf{H}}_{jk} = \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V^{\mathsf{H}}_k\right) + \left(y^m_{jk}\right)^{\mathsf{H}} |V_j|^2 \tag{4.2a}$$

$$S_{kj} := V_k I^{\mathsf{H}}_{kj} = \left(y^s_{kj}\right)^{\mathsf{H}} \left(|V_k|^2 - V_k V^{\mathsf{H}}_j\right) + \left(y^m_{kj}\right)^{\mathsf{H}} |V_k|^2 \tag{4.2b}$$

They are not negatives of each other because of power loss along the line. Since $y^s_{jk} = y^s_{kj}$, the total complex power loss is:

$$S_{jk} + S_{kj} = \left(y^s_{jk}\right)^{\mathsf{H}} \left|V_j - V_k\right|^2 + \left(y^m_{jk}\right)^{\mathsf{H}} |V_j|^2 + \left(y^m_{kj}\right)^{\mathsf{H}} |V_k|^2 \tag{4.3}$$

The first term on the right-hand side is loss due to series impedance and the last two terms are losses due to shunt admittances of the line. Thermal limits on branch power flows should be imposed on both $|S_{jk}|$ and $|S_{kj}|$:

$$\begin{aligned} |S_{jk}| &= \left| \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V^{\mathsf{H}}_k\right) + \left(y^m_{jk}\right)^{\mathsf{H}} |V_j|^2 \right| \leq S^{\max}_{jk} \\ |S_{kj}| &= \left| \left(y^s_{kj}\right)^{\mathsf{H}} \left(|V_k|^2 - V_k V^{\mathsf{H}}_j\right) + \left(y^m_{kj}\right)^{\mathsf{H}} |V_k|^2 \right| \leq S^{\max}_{kj} \end{aligned}$$

not just on $\left|\left(y^s_{jk}\right)^{\mathsf{H}}\left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right)\right|$ and $\left|\left(y^s_{kj}\right)^{\mathsf{H}}\left(|V_k|^2 - V_k V_j^{\mathsf{H}}\right)\right|$ unless the shunt admittances are zero.

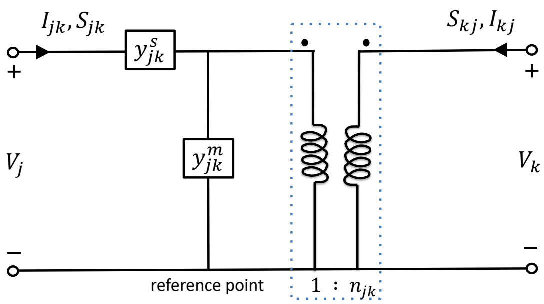
If the shunt admittances y^m_{jk} and y^m_{kj} of the line are zero then the power loss has a simple relation with line currents. Setting $y^m_{jk} = y^m_{kj} = 0$ in (4.3) and (4.1a) and using $y^s_{jk} = y^s_{kj}$, we have

$$S_{jk} + S_{kj} \quad = \quad z^s_{jk} \cdot \left|y^s_{jk}\right|^2 \left|V_j - V_k\right|^2 \quad = \quad z^s_{jk} \left|I_{jk}\right|^2$$

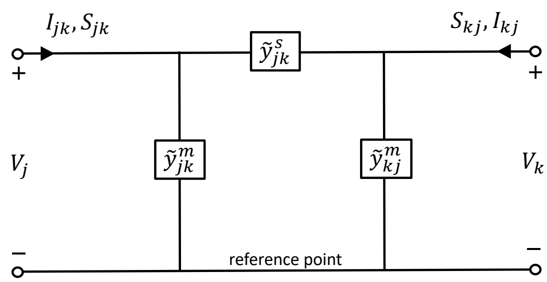
because $I_{jk} = y^s_{jk}(V_j - V_k) = -I_{kj}$ when the shunt elements are zero and $y^s_{jk} = y^s_{kj}$. This is not the case otherwise.

4.1.3 Single-phase transformer

In Chapter 3 we describe circuit models of a single-phase transformer. They are also per-phase models of balanced three-phase transformers. A transformer has two terminals (j,k) and is specified by its voltage gain n_{jk} which is the reciprocal of the turns ratio $a_{jk} := 1/n_{jk}$. If the single-phase transformer is the per-phase model of a balanced three-phase transformer, then the voltage gain $K(n_{jk})$ can be complex, e.g., $K(n_{jk}) = \sqrt{3} n_{jk} e^{\mathbf{i}\pi/6}$ for ΔY configuration. In addition to the voltage gain n_{jk}, a single-phase transformer also has series resistance and leakage inductance and shunt admittance due to the primary and secondary magnetizing currents. These effects can be modeled by a series admittance y^s_{jk} and shunt admittance y^m_{jk} in the primary circuit, as shown in Figure 4.2(a).



(a) Non-ideal transformer



(b) Equivalent Π circuit model (real n_{jk})

Figure 4.2: Single-phase transformer.

As for a line model, associated with terminals j and k are the terminal voltages, sending-end line currents and sending-end line power flows $(V_j, I_{jk}, S_{jk}) \in \mathbb{C}^3$ and $(V_k, I_{kj}, S_{kj}) \in \mathbb{C}^3$ respectively. Notice that the direction of I_{kj} at terminal k is opposite to that in Chapter 3. The behavior of the (possibly off-nominal) transformer in Figure 4.2 is characterized by the relation between the terminal voltages (V_j, V_k) and line currents $\left(I_{jk}, I_{kj}\right)$ or that between (V_j, V_k) and line powers $\left(S_{jk}, S_{kj}\right)$, which we now explain.

Real voltage gain n_{jk}. Using Kirchhoff's and Ohm's laws and transformer gains we have

$$I_{jk} \quad = \quad y^s_{jk}\left(V_j - a_{jk} V_k\right), \qquad\qquad I_{jk} \quad = \quad y^m_{jk} a_{jk} V_k + n_{jk}(-I_{kj}) \tag{4.4a}$$

This defines a matrix $Y_{\text{transformer}}$ for the single-phase transformer:

$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} y^s_{jk} & -a_{jk}\, y^s_{jk} \\ -a_{jk}\, y^s_{jk} & a^2_{jk}\left(y^s_{jk} + y^m_{jk}\right) \end{bmatrix}}_{Y_{\text{transformer}}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{4.4b}$$

If transformer gains are real then their terminal behavior can be modeled by a Π circuit, the same way a transmission line is. Specifically $Y_{\text{transformer}}$ can be rewritten in terms of admittances $\left(\tilde{y}^s_{jk}, \tilde{y}^m_{jk}, \tilde{y}^m_{kj}\right)$ of a Π circuit:

$$Y_{\text{transformer}} \;:=\; \begin{bmatrix} \tilde{y}^s_{jk} + \tilde{y}^m_{jk} & -\tilde{y}^s_{jk} \\ -\tilde{y}^s_{jk} & \tilde{y}^s_{jk} + y^m_{kj} \end{bmatrix}$$

where

$$\tilde{y}^s_{jk} \;:=\; a_{jk}\, y^s_{jk}, \qquad \tilde{y}^m_{jk} \;:=\; (1 - a_{jk}) y^s_{jk}, \qquad \tilde{y}^m_{kj} \;:=\; a_{jk}(a_{jk} - 1) y^s_{jk} + a^2_{jk}\, y^m_{jk}$$

as illustrated in Figure 4.2(b). In particular the shunt admittances $\tilde{y}^m_{jk}$ and $\tilde{y}^m_{kj}$ of the Π circuit model are different unless $(1 - a^2) y^s_{jk} = a^2 y^m_{jk}$.

Complex voltage gain $K(n)$. A physical transformer always has a real voltage gain n. The per-phase model of three-phase transformer in a balanced setting however can have a complex voltage gain $K(n)$ as we have seen in Chapter 3.2. In that case $-nI_{kj}$ in the above derivation should be replaced by $K^{\mathsf{H}}_{jk}(n) I_{kj}$, and we have instead:

$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} y^s_{jk} & -y^s_{jk}/K_{jk}(n) \\ -y^s_{jk}/K^{\mathsf{H}}_{jk}(n) & \left(y^s_{jk} + y^m_{jk}\right)/|K_{jk}(n)|^2 \end{bmatrix}}_{Y_{\text{transformer}}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{4.5}$$

In this case the matrix $Y_{\text{transformer}}$ is *not* symmetric. This means that the terminal behavior of the transformer does not have an equivalent Π circuit model, unless we can ignore connection-induced phase shifts so that $K(n)$ and $K^*(n)$ are both taken as $|K(n)|$. For transformers with complex gains we have to use (4.5) for power flow analysis. See Chapter 8.2.3 for a more general discussion in the context of unbalanced three-phase systems.

4.2 Network model: VI relation

In this section we explain how to use the component models of Chapter 4.1 to model a single-phase network consisting of generators and loads connected by a network of transmission lines and transformers. We will construct an equivalent circuit consisting of *ideal* voltage and current sources connected by a network of series and shunt admittances. The nodal current injections and linearly related to nodal voltages through a matrix called an *admittance matrix*. An admittance matrix is also called a network admittance matrix or a bus admittance matrix. The relation represents the Kirchhoff's laws and the Ohm's law. In this section we derive the admittance matrix.

We start in Chapter 4.2.1 with a few examples, present in Chapter 4.2.2 our line model, and define in Chapter 4.2.3 the admittance matrix Y for general networks. In Chapter 4.2.6 we explain Kron reduction of an admittance matrix Y. In Chapter 4.2.4 we present a common numerical method for solving the current balance equation $I = YV$. Finally in Chapter 4.2.5 we discuss sufficient conditions for the invertibility of Y.

4.2.1 Examples

In this subsection we derive the admittance matrix Y of a single-phase network shown in Figure 4.3 where:

1. The generator on the left end is modeled as a current source with parameters (I_1, y_1).
2. The non-ideal single-phase transformer has a real voltage gain n, a series admittance y^l and shunt admittance y^m in the primary circuit.
3. The transmission line is modeled as a series admittance y (and zero shunt admittances).
4. The motor load on the right end is modeled as another current source (I_2, y_2).

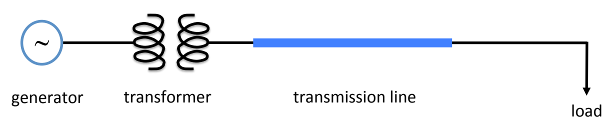


Figure 4.3: One-line diagram of a generator supplying a load through a transformer and a transmission line.

We will derive the admittance matrix Y for the overall system in two steps.

Example 4.1 (Non-ideal transformer and transmission line)**.** Figure 4.4 shows the circuit model of the non-ideal transformer in series with the transmission line. To determine the admittance matrix that relates

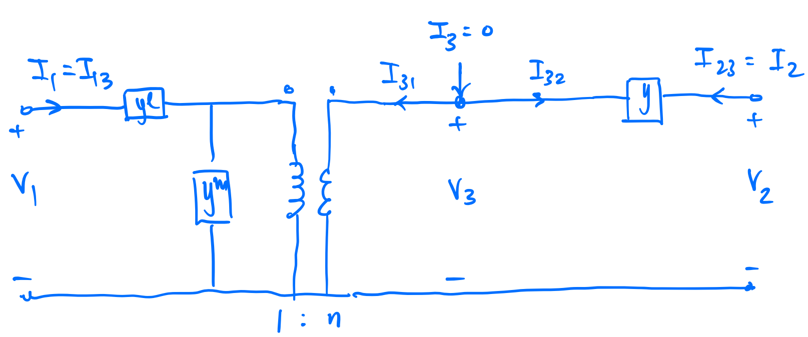


Figure 4.4: A non-ideal transformer in series with a transmission line.

(I_1, I_2) to (V_1, V_2), we introduce an additional network node in between the two series admittances y^l and ny with auxiliary voltage V_3 and an auxiliary injection current I_3 at node 3, as shown in the figure.

Since the voltage gain n is real, use the transformer model (4.4) and the line model (4.1) to get

$$\begin{bmatrix} I_{13} \\ I_{31} \end{bmatrix} = \begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2\left(y^l + y^m\right) \end{bmatrix} \begin{bmatrix} V_1 \\ V_3 \end{bmatrix}, \qquad \begin{bmatrix} I_{32} \\ I_{23} \end{bmatrix} = \begin{bmatrix} y & -y \\ -y & y \end{bmatrix} \begin{bmatrix} V_3 \\ V_2 \end{bmatrix}$$

Kirchhoff's current law at each node gives:

$$I_1 = I_{13}, \qquad 0 = I_3 = I_{31} + I_{32}, \qquad I_2 = I_{23}$$

where $I_3 = 0$ because node 3 is internal to the non-ideal transformer. Eliminating branch currents relates nodal currents (I_1, I_2, I_3) to nodal voltages (V_1, V_2, V_3) through matrix Y_1:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \underbrace{\begin{bmatrix} y^l & 0 & -ay^l \\ 0 & y & -y \\ -ay^l & -y & y + a^2\left(y^l + y^m\right) \end{bmatrix}}_{Y_1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \tag{4.6}$$

The matrix Y_1 is complex symmetric and is therefore an admittance matrix that can be represented as a Π circuit as shown in Figure 4.5 where $\tilde{y}^s_{13} := ay^l$, $\tilde{y}^m_{13} := (1-a)y^l$ and $\tilde{y}^m_{31} := a(a-1)y^l + a^2 y^m$. $\square$

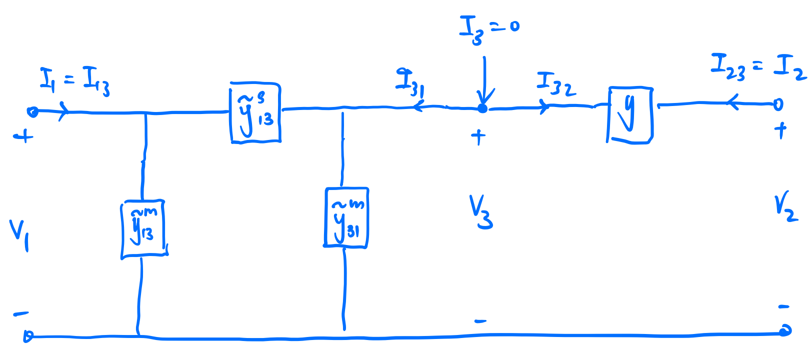
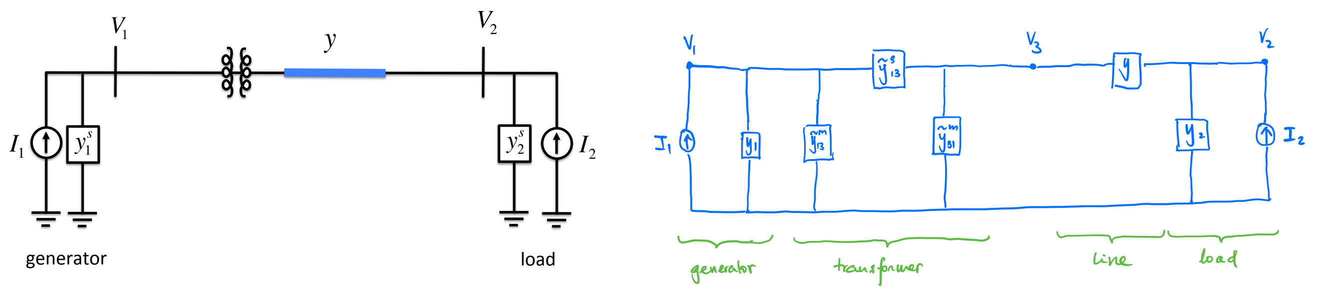


Figure 4.5: Π circuit model of the system in Figure 4.4.

Example 4.2 (Overall system)**.** Finally the circuit model of the overall system that includes the two current sources that model the generator and the load is shown in Figure 4.6(a). The only changes to the admittance matrix, compared with the admittance matrix Y_1 in (4.6), are the additional shunt admittances y_1, y_2 at nodes 1 and 2 respectively. They should be added to the first two diagonal entries of Y_1. The overall network can therefore be modeled by an admittance matrix Y that relates nodal current injections and nodal voltages (setting $I_3 = 0$):

$$\begin{bmatrix} I_1 \\ I_2 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} y^l + y_1 & 0 & -ay^l \\ 0 & y + y_2 & -y \\ -ay^l & -y & y + a^2\left(y^l + y^m\right) \end{bmatrix}}_{Y_1} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

The terminal behavior can be modeled by a 2×2 admittance matrix that relates (I_1, I_2) and (V_1, V_2) which can be obtained from Y through Kron reduction making use of the fact that the internal injection $I_3 = 0$; see Chapter 4.2.6. $\square$



(a) One-line diagram (b) Equivalent circuit model

Figure 4.6: Generator, transformer, transmission line and load.

4.2.2 Line model

In general we model a power network by a connected undirected graph $G = (\overline{N}, E)$ where $\overline{N} := \{0\} \cup N$, $N := \{1, 2, \ldots, N\}$ and $E \subseteq \overline{N} \times \overline{N}$. Each node j in $\overline{N}$ may represent a bus and each edge (j,k) in E may represent a transmission or distribution line or transformer. We also write $j \sim k$ instead of $(j,k) \in E$. We use "bus, node, terminal" interchangeably and "line, branch, link, edge" interchangeably.

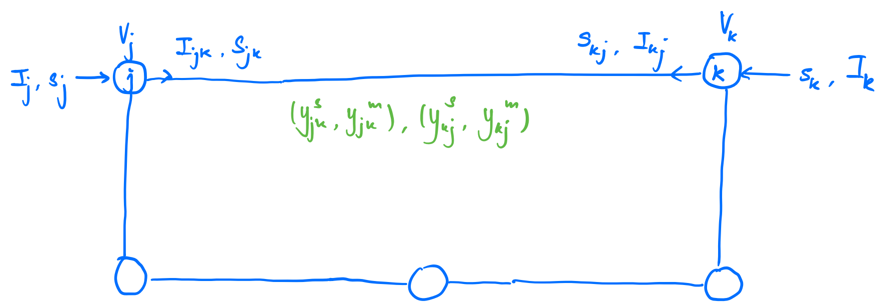


Figure 4.7: Network graph and notations.

For each line $(j,k) \in E$ let (V_j, V_k) denote the terminal voltages at each end of the line. Let I_{jk} denote the sending-end line current from j to k and I_{kj} the sending-end line current from k to j. Each line $(j,k) \in E$ is characterized by four admittances $\left(y_{jk}^s, y_{jk}^m\right) \in \mathbb{C}^2$ from j to k and $\left(y_{kj}^s, y_{kj}^m\right) \in \mathbb{C}^2$ from k to j; see Figure 4.7. We call $\left(y_{jk}^s, y_{jk}^s\right)$ the *series admittances* and $\left(y_{jk}^m, y_{jk}^m\right)$ the *shunt admittances* of line (j,k). They define the relation between (V_j, V_k) and (I_{jk}, I_{kj}):

$$I_{jk} \;=\; y_{jk}^s(V_j - V_k) \;+\; y_{jk}^m V_j, \qquad\qquad I_{kj} \;=\; y_{kj}^s(V_k - V_j) \;+\; y_{kj}^m V_k \tag{4.7a}$$

or in matrix form:

$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} y_{jk}^s + y_{jk}^m & -y_{jk}^s \\ -y_{kj}^s & y_{kj}^s + y_{kj}^m \end{bmatrix}}_{Y_{jk}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{4.7b}$$

We emphasize that the series admittances y_{jk}^s and y_{kj}^s may be different and therefore this general model may not have a Π circuit representation. It can model single-phase transmission or distribution lines, single-phase transformers, and per-phase models of balanced three-phase transformers with real or complex voltage gains, as derived in Chapters 4.1.2 and 4.1.3. Moreover, as we have seen in Example 4.2, a line (j,k) in the graph G, the matrix Y_{jk} may also contain generator and load impedances. In general the shunt admittances y^m_{jk} and y^m_{kj} are different.

We will often restrict ourselves to the special case where the series admittances are equal $y^s_{jk} = y^s_{kj}$. Then (4.7) reduces to

$$I_{jk} \ = \ y^s_{jk}(V_j - V_k) \ + \ y^m_{jk}V_j, \qquad\qquad I_{kj} \ = \ y^s_{jk}(V_k - V_j) \ + \ y^m_{kj}V_k \tag{4.8a}$$

or in terms of Y_{jk}:

$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} \ = \ \underbrace{\begin{bmatrix} y^s_{jk} + y^m_{jk} & -y^s_{jk} \\ -y^s_{jk} & y^s_{jk} + y^m_{kj} \end{bmatrix}}_{Y_{jk}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{4.8b}$$

Since Y_{jk} is symmetric, it has a Π circuit representation and behaves like a transmission or distribution line (though with generally different y^m_{jk} and y^m_{kj}). We characterize such a line by three admittances $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$. This model cannot be used as the per-phase model of a balanced three-phase transformer in ΔY or $Y\Delta$ configuration that has a complex voltage gain $K(n)$. It is however widely applicable, e.g., when the network does not contain transformers with complex voltage gains or when used in per unit systems where (nominal) transformers disappear. We therefore often adopt this model and will explicitly state it as assumption C4.1 when we use it.

4.2.3 Admittance matrix Y

In bus injection models we are interested in nodal variables $(V_j, I_j, s_j) \in \mathbb{C}^3$, $j \in \overline{N}$, where V_j is the complex voltage at bus j with respect to an arbitrary but fixed common reference point, e.g., the ground. Here I_j and s_j are the complex nodal current and power injections respectively into the network at bus j. As mentioned above they can be interpreted as flowing from terminal j to the common reference point in the circuit model. Bus 0 is the *slack bus*. Its voltage is fixed and we sometimes assume that $V_0 = 1\angle 0°$ per unit (pu), i.e., the voltage drop between bus 0 and the reference point is $1\angle 0°$. A bus $j \in \overline{N}$ can have a generator, a load, both or neither and $\left(I_j, s_j\right)$ are the net current and power injections (generation minus load) at bus j. We often use s_j to denote both the complex number $p_j + \mathbf{i}q_j \in \mathbb{C}$ and the real pair $(p_j, q_j) \in \mathbb{R}^2$ depending on the context. The nodal quantities are related by $s_j = V_j I_j^{\mathsf{H}}$ for each bus $j \in \overline{N}$.

The nodal current injections $I := (I_j, j \in \overline{N})$ and voltages $V := (V_j, j \in \overline{N})$ are linearly related. The admittance matrix Y relates, not the line currents, but the *net* nodal current injections I to nodal voltages V. Applying (4.7) to KCL $I_j = \sum_{k:j\sim k} I_{jk}$ at each node j, we have[1]

$$I_j \ = \ \sum_{k:j\sim k} I_{jk} \ = \ \left(\sum_{k:j\sim k} y^s_{jk} \ + \ y^m_{jj}\right) V_j \ - \ \sum_{k:j\sim k} y^s_{jk} V_k, \qquad j \in \overline{N} \tag{4.9a}$$

[1] If there is a load attached to bus j with shunt admittance y^{sh}_j, then the *net* injection becomes $I_j - y^{\mathrm{sh}}_j V_j = \sum_{k:j\sim k} I_{jk}$ instead of I_j on the left-hand side of (4.9a).

where y^m_{jj} denotes the total shunt admittance of the lines connected to bus j:

$$y^m_{jj} \quad := \quad \sum_{k:j\sim k} y^m_{jk} \tag{4.9b}$$

In vector form, this is $I = YV$ where the matrix Y is given by:

$$Y_{jk} \quad = \quad \begin{cases} -y^s_{jk}, & j \sim k \;\; (j \neq k) \\ \sum_{l:j\sim l} y^s_{jl} \,+\, y^m_{jj}, & j = k \\ 0 & \text{otherwise} \end{cases} \tag{4.9c}$$

We refer to Y that maps nodal voltages to nodal current injections as an *admittance matrix*, or a network admittance matrix or bus admittance matrix. Equation (4.9c) prescribes a way to write down the admittance matrix Y by inspection of the network connectivity and line admittances: its off-diagonal entries are the negatives of series admittances $\left(y^s_{jk}, y^s_{kj}\right)$ in each direction on line (j,k) while its diagonal entries are the sum of the series and shunt admittances incident on the corresponding buses. Note that Y_{jk} and Y_{kj} may not be equal if (j,k) models a transformer.

If we restrict ourselves to the special line model (4.8) where $y^s_{jk} = y^s_{kj}$ for all $(j,k) \in E$, then each line (j,k) has a Π circuit representation, i.e., behaves like a transmission or distribution line. The admittance matrix Y is then complex symmetric. It is not Hermitian unless Y is a real matrix. Since this special line model is widely used, we label the following assumption and will explicitly state it when it is required:

C4.1: The series admittances $y^s_{jk} = y^s_{kj}$ for every line $(j,k) \in E$ so that the single-phase admittance matrix Y is complex symmetric.

Example 4.3. Consider the three-bus network shown in Figure 4.8. Under condition C4.1, each line (j,k)

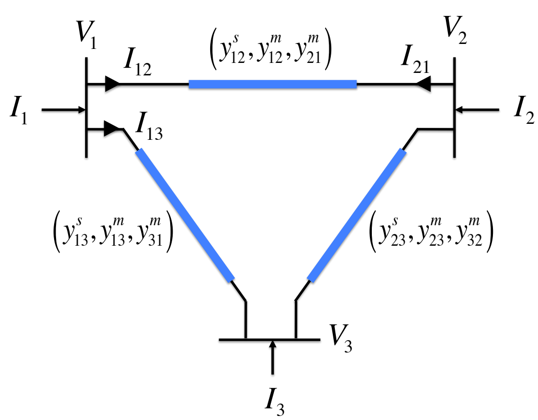


Figure 4.8: Three-bus network of Example 4.3.

is modeled by a Π circuit with series and shunt admittances $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$. The sending-end branch current from bus j to bus k is I_{jk} and that from bus k to bus j is I_{kj}. Applying Kirchhoff's current law and Ohm's law at bus 1 gives

$$\begin{aligned}
I_{12} &= y^s_{12}(V_1 - V_2) + y^m_{12}V_1 \\
I_{13} &= y^s_{13}(V_1 - V_3) + y^m_{13}V_1 \\
\therefore \quad I_1 &= I_{12} + I_{13} \;=\; (y^s_{12} + y^s_{13} + y^m_{12} + y^m_{13})V_1 \;-\; y^s_{12}V_2 \;-\; y^s_{13}V_3
\end{aligned}$$

Similarly applying KCL and Ohm's law at buses 2 and 3 we obtain

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} y^s_{12} + y^s_{13} + y^m_{11} & -y^s_{12} & -y^s_{13} \\ -y^s_{12} & y^s_{12} + y^s_{23} + y^m_{22} & -y^s_{23} \\ -y^s_{13} & -y^s_{23} & y^s_{13} + y^s_{23} + y^m_{33} \end{bmatrix}}_{Y} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

where

$$y^s_{jk} \;=\; y^s_{kj}, \qquad\qquad y^m_{jj} \;:=\; \sum_{k:j\sim k} y^m_{jk}$$

Again the off-diagonal entries of the admittance matrix Y are given by the series admittances on the lines:

$$Y_{jk} \;:=\; \begin{cases} -y^s_{jk} & \text{if } j \sim k \;\; (j \neq k) \\ \quad 0 & \text{otherwise} \end{cases}$$

and the diagonal entries of Y by the sum of series and shunt admittances incident on buses j:

$$Y_{jj} \;:=\; \sum_{k:j\sim k} y^s_{jk} \;+\; y^m_{jj}$$

$\square$

Under Assumption C4.1, the admittance matrix Y given in (4.9) can also be expressed in terms of more elementary matrices. Fix an arbitrary orientation for the graph $G := (\overline{N}, E)$ so that a line $l = i \to j \in E$ is now considered pointing from bus i to bus j. Let $C \in \{-1,0,1\}^{|\overline{N}|\times|E|}$ be the bus-by-line incidence matrix defined by:

$$C_{jl} \;=\; \begin{cases} 1 & \text{if } l = j \to k \text{ for some bus } k \\ -1 & \text{if } l = i \to j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

Let $Y^s := \text{diag}\left(y^s_l,\, l \in E\right)$ be the $|E| \times |E|$ diagonal matrix with the series admittances y^s_l as its diagonal entries. Let $Y^m := \text{diag}\left(y^m_{jj},\, j \in \overline{N}\right)$ be the $|\overline{N}| \times |\overline{N}|$ diagonal matrix with the total shunt admittances y^m_{jj} in (4.9b) as its diagonal entries. Then the admittance matrix in (4.9c) is, when $y^s_{jk} = y^s_{kj}$,

$$Y \;=\; CY^sC^{\mathsf{T}} \;+\; Y^m \tag{4.10}$$

Clearly the matrix $CY^s\overline{C}^{\mathsf{T}}$ has zero row and column sums. It verifies that Y is symmetric but not Hermitian unless Y^s and Y^m are real matrices. This representation can be used to study the inverse of Y; see Exercise 4.7.

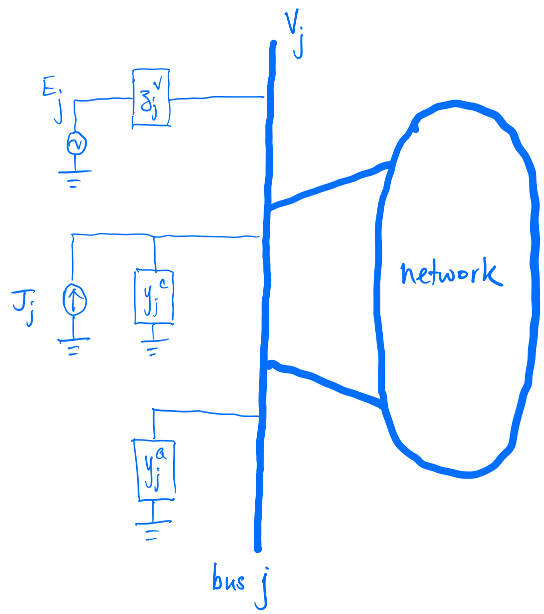


Figure 4.9: Multiple devices connected to the same bus.

Remark 4.1 (Nodal devices)**.** Our notation for current injection I_j suggests that there is a single device at each bus j. This simplifies notation and loses no generality. If there are multiple devices connected to bus j, e.g., a non-ideal voltage source $\left(E_j, z_j^v\right)$, a non-ideal current source $\left(J_j, y_j^c\right)$, and a bus shunt admittance y_j^a or equivalently its impedance $z_j^i = \left(y_j^a\right)^{-1}$, as shown in Figure 4.9. then I_j is the net current injection from bus j to the rest of the network:

$$I_j \quad = \quad \underbrace{\frac{E_j - V_j}{z_j^v}}_{\text{voltage source}} \;+\; \underbrace{\left(J_j - y_j^c V_j\right)}_{\text{current source}} \;-\; \underbrace{y_j^a V_j}_{\text{shunt admittance}}$$

This assumes all voltages are defined with respect to the ground and if a single-phase device is the per-phase model of a three-phase Y configured device, then its neutral is grounded directly. Then (4.9a) becomes

$$\frac{E_j - V_j}{z_j^v} + \left(J_j - y_j^c V_j\right) - y_j^a V_j \;=\; \left(\sum_{k:j\sim k} y_{jk}^s + y_{jj}^m\right) V_j - \sum_{k:j\sim k} y_{jk}^s V_k, \qquad j \in \overline{N}$$

$\square$

4.2.4 Solving $I = YV$

Suppose we are given $I \in \mathbb{C}^{N+1}$ and want to determine $V \in \mathbb{C}^{N+1}$ from $I = YV$. For large networks taking the inverse of Y can be difficult computationally even when it exists. A common method is to compute the LU factorization of Y, i.e., $Y = LU$ where L is a lower triangular matrix with all diagonal entries being 1 and U an upper triangular matrix. Any square matrix $A \in \mathbb{C}^{n\times n}$ has an LU factorization after possibly an appropriate re-ordering of the rows, i.e., there exists a permutation matrix P such that $PA = LU$ for some L, U. If A is invertible then it admits an LU factorization without permutation (i.e., $A = LU$ for some L, U) if and only if all its leading principal minors are nonzero.[2] In that case, the LU factorization is unique. For a singular A, necessary and sufficient conditions for the existence and uniqueness of LU factorization are known but are more involved.

Possibly after an appropriate permutation of Y (such that e.g. $Y_{11} \neq 0$), we can compute the entries of L and U recursively. From

$$\begin{bmatrix} Y_{00} & Y_{01} & Y_{02} & \cdots & Y_{0N} \\ Y_{10} & Y_{11} & Y_{12} & \cdots & Y_{1N} \\ Y_{20} & Y_{21} & Y_{22} & \cdots & Y_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y_{N0} & Y_{N1} & Y_{N2} & \cdots & Y_{NN} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ L_{10} & 1 & 0 & \cdots & 0 \\ L_{20} & L_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{N0} & L_{N1} & L_{N2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} U_{00} & U_{01} & U_{02} & \cdots & U_{0N} \\ 0 & U_{11} & U_{12} & \cdots & U_{1N} \\ 0 & 0 & U_{22} & \cdots & U_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & Y_{NN} \end{bmatrix}$$

we proceed as follows:

1. The 0th row of U is set to the 0th row of Y since $L_{00} = 1$:
$$U_{0j} \;=\; Y_{0j}, \qquad j = 0, \ldots, N$$
2. To compute row-1 entry L_{10} of L, we have
$$Y_{10} \;=\; L_{10}U_{00} \quad \Rightarrow \quad L_{10} \;=\; \frac{Y_{10}}{U_{00}}$$
To compute row-1 entries U_{1j} of U, we have for columns $j = 1, \ldots, N$,
$$Y_{1j} \;=\; L_{10}U_{0j} + U_{1j} \quad \Rightarrow \quad U_{1j} \;=\; Y_{1j} - L_{10}U_{0j}$$
3. In general, to compute row-i entries L_{ij} of L $(i = 2, \ldots, N)$, we have for columns $j = 0, \ldots, i-1$,
$$\begin{aligned} Y_{i0} &= L_{i0}U_{00} & &\Rightarrow & L_{i0} &= \frac{Y_{i0}}{U_{00}} \\ Y_{i1} &= L_{i0}U_{01} + L_{i1}U_{11} & &\Rightarrow & L_{i1} &= \frac{1}{U_{11}}\left(Y_{i1} - L_{i0}U_{01}\right) \\ &\vdots & & & &\vdots \\ Y_{i(i-1)} &= \sum_{j=0}^{i-2} L_{ij}U_{j(i-1)} + L_{i(i-1)}U_{(i-1)(i-1)} & &\Rightarrow & L_{i(i-1)} &= \frac{1}{U_{(i-1)(i-1)}}\left(Y_{i(i-1)} - \sum_{j=0}^{i-2} L_{ij}U_{j(i-1)}\right) \end{aligned}$$

---

[2]Consider a matrix $A \in \mathbb{C}^{n\times n}$. Let $I := \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, $J := \{j_1, \ldots, j_l\} \subseteq \{1, \ldots, n\}$, and A_{IJ} denote the submatrix obtained from deleting rows not in I and columns not in J.

- If $k = l$, i.e., A_{IJ} is square, then the *minor* M_{IJ} of A is the determinant of the submatrix A_{IJ}.
- If $I = J$, then A_{IJ} is called a *principal submatrix* and M_{IJ} a *principal minor* of A.
- If $I = J = \{1, \ldots, k\}$ with $k \leq n$, then A_{IJ} is called a *leading principal submatrix* of order k and M_{IJ} a *leading principal minor* of order k.

To compute row-i entries U_{ij} of U $(i = 2, \dots, N)$, we have for columns $j = i, \dots, N$,

$$
\begin{array}{rclcrcl}
Y_{ii} &=& \sum_{j=0}^{i-1} L_{ij} U_{ji} + U_{ii} & \Rightarrow & U_{ii} &=& Y_{ii} - \sum_{j=0}^{i-1} L_{ij} U_{ji} \\
Y_{i(i+1)} &=& \sum_{j=0}^{i-1} L_{ij} U_{j(i+1)} + U_{i(i+1)} & \Rightarrow & U_{i(i+1)} &=& Y_{i(i+1)} - \sum_{j=0}^{i-1} L_{ij} U_{j(i+1)} \\
\vdots & & & & \vdots & & \\
Y_{iN} &=& \sum_{j=0}^{i-1} L_{ij} U_{jN} + U_{iN} & \Rightarrow & U_{iN} &=& Y_{iN} - \sum_{j=0}^{i-1} L_{ij} U_{jN}
\end{array}
$$

Once the factorization is obtained we have $I = YV = LUV$. Hence, given I, V can be solved in two steps from:

$$
I = L\tilde{V} \tag{4.11}
$$

$$
\tilde{V} = UV \tag{4.12}
$$

In step 1, $\tilde{V}$ is solved using (4.11) by forward substitution (compute $\tilde{V}_1$ then $\tilde{V}_2$ and so on). In step 2, V is solved using (4.12) by backward substitution (compute V_n then V_{n-1} and so on).

Example 4.4. Suppose

$$
Y = \begin{bmatrix} 2(0.5-j)+j0.5 & -0.5+j & -0.5+j \\ -0.5+j & (0.5-j)+j0.1 & 0 \\ -0.5+j & 0 & (0.5-j)+j0.2 \end{bmatrix}
$$

Then

$$
Y = \begin{bmatrix} 1 & 0 & 0 \\ -0.6154+j0.0769 & 1 & 0 \\ -0.6154+j0.0769 & -1.6763+j0.8960 & 1 \end{bmatrix} \begin{bmatrix} 1-j1.5 & -0.5+j & -0.5+j \\ 0 & 0.2692-j0.2462 & -0.2308+j0.6538 \\ 0 & 0 & 0.4682+j1.1566 \end{bmatrix}
$$

Given I, V can be obtained in two steps: solve for $\tilde{V}$ from:

$$
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -0.6154+j0.0769 & 1 & 0 \\ -0.6154+j0.0769 & -1.6763+j0.8960 & 1 \end{bmatrix} \begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{V}_3 \end{bmatrix}
$$

and then solve for V from:

$$
\begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{V}_3 \end{bmatrix} = \begin{bmatrix} 1-j1.5 & -0.5+j & -0.5+j \\ 0 & 0.2692-j0.2462 & -0.2308+j0.6538 \\ 0 & 0 & 0.4682+j1.1566 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}
$$

4.2.5 Properties of Y

In this subsection we collect some analytical properties of the admittance matrix Y, particularly on their invertibility. Invertibility is of interests because given $I \in \mathbb{C}^{N+1}$ we may be interested in inverting Y to obtain $V \in \mathbb{C}^{N+1}$ from $I = YV$ as discussed in Chapter 4.2.4. This is referred to as the $Z_{\text{bus}} := Y^{-1}$ method in the literature. The matrix Z_{bus} is called a *bus impedance matrix* or an *impedance matrix* and is useful for fault analysis (which we will not cover in this book). The admittance matrix Y can be constructed easily by inspection of a network graph or its one-line diagram as specified by (4.9c). It inherits the sparsity structure of the network graph. The impedance matrix Z on the other hand cannot be easily inferred from the one-line diagram and is usually dense even for a sparse network. LU decomposition can be used for both computing Z and solving V from $I = YV$.

We first consider the case where the shunt admittances of lines are negligible, i.e., $y_{jj}^m = 0$ for all $j \in \overline{N}$, so that all row sums of Y are zero. In this case Y is not invertible and we present its pseudo-inverse. We then discuss conditions under which Y with nonzero shunt admittances is invertible. Kron reduction in Chapter 4.2.6 requires that the submatrix Y_{22} be invertible. The invertibility of Y_{22} is studied in Chapter 4.2.7.

We often assume C4.1 holds in this section and will explicitly state it where it is needed.

4.2.5.1 Pseudo-inverse and Takagi decomposition

Suppose $y_{jj}^m = 0$ for all $j \in \overline{N}$ so that Y has zero row (and hence column) sums.[3] Then Y is not invertible. Its pseudo-inverse always exists and can be obtained through singular value decomposition (see Chapter 25.1.5 for singular value decomposition and Chapter 25.1.6 for pseudo-inverse). Let $\bar{Y}$ denote the componentwise complex conjugate of Y, i.e., $[\bar{Y}]_{jk} = \left(Y_{jk}\right)^{\mathsf{H}}$. Then $Y = Y^{\mathsf{T}} = (\bar{Y})^{\mathsf{H}}$. Let the singular value decomposition of Y be

$$Y \;=\; U\Sigma W^{\mathsf{H}}$$

where $\Sigma := \text{diag}\left(\sigma_0, \cdots, \sigma_N\right)$ is a $(N+1)\times(N+1)$ real nonnegative diagonal matrix whose diagonal entries $\sigma_j \geq 0$, called the singular values of Y, are the nonnegative square roots of the eigenvalues of $Y\bar{Y}$, and $U, W \in \mathbb{C}^{(N+1)\times(N+1)}$ are unitary matrices (see discussion after Theorem 25.8 in Chapter 25.1.5 for their derivation). The pseudo-inverse of Y is then

$$Y^{\dagger} \;:=\; W\Sigma^{\dagger}U^{\mathsf{H}}$$

where $\Sigma^{\dagger}$ is the real nonnegative diagonal matrix obtained from Σ by replacing the nonzero singular values σ_j by $1/\sigma_j$.

If null(Y) =span$(\mathbf{1})$ then, for each current vector I with $\mathbf{1}^{\mathsf{T}} I = 0$, there is a subspace of solutions to $I = YV$ given by

$$V \;=\; Y^{\dagger}I + \gamma\mathbf{1}, \qquad \gamma \in \mathbb{C}$$

[3] If Y were real symmetric with zero row sums, then its rank is N and its null space is span$(\mathbf{1})$ when the network is connected. This property generally does not hold when Y is complex symmetric; see Exercise 4.3 for a sufficient condition for this property.

parametrized by γ. Hence V is unique up to an arbitrary reference voltage. For example the solution $V = Y^{\dagger}I$ corresponds to a solution with $\gamma = 0$. Alternatively γ can be chosen so that $V_0 = 1\angle 0^\circ$ at bus 0. If null$(Y) \supset$span(1) then I needs to be orthogonal to all vectors in null(Y) for $I = YV$ to have a solution for V.

Under assumption C4.1, Y is symmetric. Since it is generally not Hermitian, it may not be *unitarily* diagonalizable. A matrix is unitarily diagonalizable if and only if it is normal (Theorem 25.10 in Appendix 25.1.5). Y may or may not be normal. See Exercise 4.2 for sufficient conditions under which Y is normal and hence unitarily diagonalizable. Even when Y is not normal, it can still be diagonalized but the unitary matrix U may consist of neither the singular vectors nor the eigenvectors of Y, according to Theorem 25.14 in Appendix 25.1.5.

Theorem 4.1 (Takagi decomposition of Y)**.** Suppose $y^m_{jj} = 0$ for all $j \in \overline{N}$ and condition C4.1 holds. There exists a unitary matrix $U \in \mathbb{C}^{(N+1)\times(N+1)}$ and a real nonnegative diagonal matrix $\Sigma :=$diag$(\sigma_1, \ldots, \sigma_{N+1})$ such that $Y = U\Sigma U^{\mathsf{T}}$ where the diagonal entries $\sigma_j \geq 0$ of Σ are the singular values of Y. □

Since $U^{\mathsf{T}} \neq U^{\mathsf{H}}$ in general, the Takagi decomposition is generally different from the singular decomposition of Y and therefore $Y^{\dagger}$ is generally not equal to $U\Sigma^{\dagger}U^{\mathsf{T}}$.

4.2.5.2 Inverse of Y

In this subsection we derive the inverse of Y, assuming it is invertible, in terms of its real and imaginary parts when either is invertible. We will study conditions under which Y is invertible in Chapter 4.2.5.3.

Let $Y =: G + \mathbf{i}B$ with $G, B \in \mathbb{R}^{(N+1)\times(N+1)}$. Let $Z := R + \mathbf{i}X$ with $R, X \in \mathbb{R}^{(N+1)\times(N+1)}$. By definition Y^{-1} exists and is equal to Z if and only if there exist unique R, X such that $ZY = YZ = I$, the identity matrix. Consider

$$YZ \;=\; (G+\mathbf{i}B)(R+\mathbf{i}X) \;=\; (GR-BX)+\mathbf{i}(BR+GX) \;=\; I$$

or

$$\underbrace{\begin{bmatrix} G & -B \\ B & G \end{bmatrix}}_{M} \begin{bmatrix} R \\ X \end{bmatrix} \;=\; \begin{bmatrix} I \\ 0 \end{bmatrix} \tag{4.13a}$$

Therefore Y^{-1} exists if and only if the matrix $M := \begin{bmatrix} G & -B \\ B & G \end{bmatrix}$ is nonsingular. Suppose G is nonsingular. According to Theorem 25.1 in Appendix 25.1.2.1, M is nonsingular if and only if the Schur complement $M/G := G + BG^{-1}B$ of G is nonsingular (given that G is nonsingular). Moreover the inverse of M is

$$M^{-1} \;=\; \begin{bmatrix} (M/G)^{-1} & (M/G)^{-1}BG^{-1} \\ -G^{-1}B(M/G)^{-1} & G^{-1} - G^{-1}B(M/G)^{-1}BG^{-1} \end{bmatrix}$$

Hence if both G and M/G are nonsingular, then Y is nonsingular and, from (4.13a), its inverse $Z := R + \mathbf{i}X$ is given by

$$\begin{bmatrix} R \\ X \end{bmatrix} \;=\; \begin{bmatrix} (M/G)^{-1} \\ -G^{-1}B(M/G)^{-1} \end{bmatrix} \;=\; \begin{bmatrix} (G+BG^{-1}B)^{-1} \\ -G^{-1}B(G+BG^{-1}B)^{-1} \end{bmatrix} \tag{4.13b}$$

Suppose B is nonsingular. Then (4.13a) can be written equivalently as

$$\underbrace{\begin{bmatrix} B & G \\ G & -B \end{bmatrix}}_{M'} \begin{bmatrix} R \\ X \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{4.14a}$$

Applying again Theorem 25.1 in Appendix 25.1.2.1, M' is nonsingular if and only if the Schur complement $M'/B := -B - GB^{-1}G$ of B is nonsingular (given that B is nonsingular). Moreover the inverse of M' is

$$M'^{-1} = \begin{bmatrix} B^{-1} + B^{-1}G(M'/B)^{-1}GB^{-1} & -B^{-1}G(M'/B)^{-1} \\ -(M'/B)^{-1}GB^{-1} & (M'/B)^{-1} \end{bmatrix}$$

Hence if both B and M'/B are nonsingular, then Y is nonsingular and, from (4.14a), its inverse $Z := R + \mathbf{i}X$ is given by

$$\begin{bmatrix} R \\ X \end{bmatrix} = \begin{bmatrix} -B^{-1}G(M'/B)^{-1} \\ (M'/B)^{-1} \end{bmatrix} = \begin{bmatrix} B^{-1}G(B + GB^{-1}G)^{-1} \\ -(B + GB^{-1}G)^{-1} \end{bmatrix} \tag{4.14b}$$

To recap, Y is invertible when both G and M/G are invertible or when both B and M'/B are invertible. When neither G nor B is invertible, $Y = G + \mathbf{i}B$ may still be invertible though its inverse $Z := R + \mathbf{i}X$ is not given by (4.13b) or (4.14b) (Exercise 4.4).

4.2.5.3 Invertibility of Y

Nonzero shunt admittances do not guarantee the invertibility of Y. A strictly diagonally dominant matrix is invertible (Theorem 25.5 in Appendix 25.1.2). Shunt admittances however does not guarantee strict diagonal dominance, i.e., $|Y_{ii}| > \sum_{j:j\neq i} |Y_{ij}|$ may not hold for some i. This can be the case for a transmission line since the susceptances of line charging admittances and those of series admittances are typically of different signs. Strict diagonal dominance is however only sufficient for invertibility and a network of transmission lines typically has an invertible Y (see Remark 4.3). We now discuss two sufficient conditions for Y to be invertible.

The first sufficient condition builds on (4.13) and (4.14). It ensures both G and M/G are nonsingular, or both B and M'/B are nonsingular. Recall that a real matrix A is positive definite, denoted $A \succ 0$, if A is symmetric and $v^{\mathsf{T}}Av > 0$ for all real vectors v (see Remark 25.1 in Appendix 25.1.4). A positive definite matrix is nonsingular since all its eigenvalues are strictly positive. A real matrix A is negative definite, denoted $A \prec 0$, if $-A \succ 0$.

Theorem 4.2. Suppose a complex matrix Y is symmetric (i.e., satisfies condition C4.1).

1. If $\mathrm{Re}(Y) \succ 0$ then Y^{-1} exists and is symmetric. Moreover $\mathrm{Re}(Y^{-1}) \succ 0$.
2. If $\mathrm{Im}(Y) \prec 0$ then Y^{-1} exists and is symmetric. Moreover $\mathrm{Im}(Y^{-1}) \succ 0$.

Proof. For part 1, suppose $\mathrm{Re}(Y) =: G \succ 0$. Then G is nonsingular. The Schur complement M/G of G is, from (4.13a), $M/G := G + BG^{-1}B$. Since $B = B^{\mathsf{T}}$ and G, G^{-1} are positive definite, $M/G := G + BG^{-1}B \succ 0$. This implies that Y is nonsingular according to Theorem 25.1 in Appendix 25.1.2.1. It also implies that $\mathrm{Re}(Y^{-1}) \succ 0$ since, from (4.13b), $\mathrm{Re}(Y^{-1}) = (M/G)^{-1}$ which is positive definite since M/G is.

Finally if $Z := Y^{-1}$ then Z is the unique matrix such that $YZ = ZY = I$ where I is the identity matrix. Then

$$Z^{\mathsf{T}} Y^{\mathsf{T}} \ = \ Y^{\mathsf{T}} Z^{\mathsf{T}} \ = \ Z^{\mathsf{T}} Y \ = \ Y Z^{\mathsf{T}} \ = \ I$$

Hence $Z^{\mathsf{T}} = Y^{-1}$. Since inverse is unique, $Z^{\mathsf{T}} = Z$, i.e., Y^{-1} is (complex) symmetric.

Part 2 follows the same argument and is left as Exercise 4.5. (Also see Exercise 4.6 for an alternative proof of the nonsingularity of Y.) □

Remark 4.2 (Generalization)**.** Theorem 4.2 holds with small modifications as long as either $\mathrm{Re}(Y)$ or $\mathrm{Im}(Y)$ is not indefinite. Specifically if Y is complex symmetric then

1. Y^{-1} exists and is symmetric if (a) $\mathrm{Re}(Y) \succ 0$; or (b) $\mathrm{Re}(Y) \prec 0$; or (c) $\mathrm{Im}(Y) \succ 0$; or (d) $\mathrm{Im}(Y) \prec 0$.
2. (a) If $\mathrm{Re}(Y) \succ 0$ then $\mathrm{Re}(Y^{-1}) \succ 0$; and (b) if $\mathrm{Re}(Y) \prec 0$ then $\mathrm{Re}(Y^{-1}) \prec 0$.
3. (a) If $\mathrm{Im}(Y) \succ 0$ then $\mathrm{Im}(Y^{-1}) \prec 0$; and (b) if $\mathrm{Im}(Y) \prec 0$ then $\mathrm{Im}(Y^{-1}) \succ 0$.

□

The second set of sufficient conditions for the invertibility of Y is in terms of the series admittances y^s_{jk} and shunt admittances y^m_{jk}. These conditions ensure either $\mathrm{Re}(Y)$ or $\mathrm{Im}(Y)$ is either positive or negative definite, and hence Y is nonsingular by Theorem 4.2 and Remark 4.2.

Let $Y = G + \mathbf{i}B$, i.e., for all $(j,k) \in E$,

$$y^s_{jk} \ =: \ g^s_{jk} + \mathbf{i} b^s_{jk}, \qquad\qquad y^m_{jk} \ =: \ g^m_{jk} + \mathbf{i} b^m_{jk}, \qquad\qquad y^m_{kj} \ =: \ g^m_{kj} + \mathbf{i} b^m_{kj}$$

Recall $y^m_{jj} := \sum_{k:j\sim k} y^m_{jk}$ and let $g^m_{jj} := \sum_{k:j\sim k} g^m_{jk}$, $b^m_{jj} := \sum_{k:j\sim k} b^m_{jk}$. Previous discussion implies that, for Y to be invertible, it is necessary to have at least one nonzero shunt element. Additional conditions on $\left(g^s_{jk}, g^m_{jk}, g^m_{kj}\right)$ are needed to guarantee invertibility, as follows.

C4.2: For all lines $(j,k) \in E$, $g^s_{jk}, g^m_{jk}, g^m_{kj}$ are nonnegative.

C4.3a: For all buses $j \in \overline{N}$, $g^m_{jj} := \sum_{k:k\sim j} g^m_{jk} \neq 0$, i.e., for all j, there exists a line $(j,k) \in E$ such that $g^m_{jk} \neq 0$.

C4.3b: For all lines $(j,k) \in E$, $g^s_{jk} \neq 0$. Furthermore there exists a line $(j',k') \in E$ such that $g^m_{j'k'} \neq 0$.

Condition C4.2 can be replaced by: for all lines $(j,k) \in E$, all nonzero $g^s_{jk}, g^m_{jk}, g^m_{kj}$ have the same sign, and the invertibility conditions below will still hold with obvious modifications. Indeed if $g^s_{jk}, g^m_{jk}, g^m_{kj}$ are all nonpositive then the proof below shows that $\mathrm{Re}(Y) \prec 0$ (see Remark 4.2).

Theorem 4.3. Suppose the network is connected and the admittance matrix Y satisfies condition C4.1. If C4.2 and one of C4.3a and C4.3b hold, then

1. $\mathrm{Re}(Y) \succ 0$.
2. Y^{-1} exists and is symmetric. Moreover $\mathrm{Re}(Y^{-1}) \succ 0$.

Proof. Recall that $\mathrm{Re}(Y) =: G \in \mathbb{R}^{(N+1)\times(N+1)}$ is given by $G_{jk} = -g^s_{jk}$ if $j \sim k$, $\sum_{i:j\sim i}(g^s_{ji} + g^m_{ji})$ if $j = k$, and 0 otherwise. Hence for any nonzero vector $\rho \in \mathbb{R}^{N+1}$ we have

$$\begin{aligned}
\rho^{\mathsf{T}} G \rho \;&=\; \sum_j \sum_k \rho_j \rho_k G_{jk} \;=\; \sum_j \left(\sum_{k:j\sim k} -\rho_j \rho_k g^s_{jk} \;+\; \rho_j^2 \sum_{i:j\sim i} (g^s_{ji} + g^m_{ji}) \right) \\
&=\; \sum_{(j,k)\in E} \left(\rho_j^2 - 2\rho_j\rho_k + \rho_k^2\right) g^s_{jk} \;+\; \sum_{j\in\overline{N}} \rho_j^2\, g^m_{jj} \\
&=\; \sum_{(j,k)\in E} \left(\rho_j - \rho_k\right)^2 g^s_{jk} \;+\; \sum_{j\in\overline{N}} \rho_j^2\, g^m_{jj}
\end{aligned}$$

Every summand is nonnegative by C4.2. Moreover if C4.3a holds then the second summation is strictly positive since $\rho \neq 0$. If C4.3b holds then for the first summation to be zero, $\rho_j = \rho_k$. Since the network is connected this implies $\rho_j = \rho_1$ for all j. Then the second summation becomes $\sum_j \rho_j^2\, g^m_{jj} \geq \rho_1^2 g^m_{j'k'} > 0$ since $\rho \neq 0$. Therefore $\mathrm{Re}(Y) = G \succ 0$. Theorem 4.2 then completes the proof. $\square$

See Exercise 4.9 for an alternative proof of Theorem 4.3.

Instead of $\left(g^s_{jk}, g^m_{jk}, g^m_{kj}\right)$ conditions on $\left(b^s_{jk}, b^m_{jk}, b^m_{kj}\right)$ can also ensure the invertibility of Y.

C4.4: For all lines $(j,k) \in E$, $b^s_{jk}, b^m_{jk}, b^m_{kj}$ are nonpositive.

C4.5a: For all buses $j \in \overline{N}$, $b^m_{jj} := \sum_{k:k\sim j} b^m_{jk} \neq 0$, i.e., for all j, there exists a line $(j,k) \in E$ such that $b^m_{jk} \neq 0$.

C4.5b: For all lines $(j,k) \in E$, $b^s_{jk} \neq 0$. Furthermore there exists a line $(j',k') \in E$ such that $b^m_{j'k'} \neq 0$.

As before C4.2 can be replaced by: for all lines $(j,k) \in E$, all nonzero $b^s_{jk}, b^m_{jk}, b^m_{kj}$ have the same sign, and the invertibility conditions below will still hold with obvious modifications.

Theorem 4.4. Suppose the network is connected and the admittance matrix Y satisfies condition C4.1. If C4.4 and one of C4.5a and C4.5b hold, then

1. $\mathrm{Im}(Y) \prec 0$.
2. Y^{-1} exists and is symmetric. Moreover $\mathrm{Im}(Y^{-1}) \succ 0$.

Proof. The proof is similar to that for Theorem 4.3. For $\mathrm{Im}(Y) =: B$, for any nonzero real vector ρ, the same calculation yields

$$\rho^{\mathsf{T}} B \rho \; = \; \sum_{(j,k)\in E} (\rho_j - \rho_k)^2 \, b^s_{jk} \; + \; \sum_{j \in \overline{N}} \rho_j^2 \, b^m_{jj}$$

Every summand is nonpositive by C4.4. Moreover if C4.5a holds then the second summation is strictly negative since $\rho \neq 0$. If C4.5b holds then for the first summation to be zero, $\rho_j = \rho_1$ for all j since the network is connected. Then the second summation becomes $\sum_j \rho_j^2 \, b^m_{jj} \leq \rho_1^2 b^m_{j'k'} < 0$ since $\rho \neq 0$. Therefore $\mathrm{Im}(Y) = B \prec 0$. Theorem 4.2 then completes the proof. □

Remark 4.3 (Transmission line)**.** A transmission line (j,k) typically has nonnegative series conductance $g^s_{jk} \geq 0$ and negative series susceptance $b^s_{jk} < 0$ (inductive line). Its shunt conductances $g^m_{jk} \geq 0$ are usually nonnegative, but shunt susceptances $b^m_{jk} \geq 0$ are usually nonnegative (capacitive).

1. Hence the conditions in Theorem 4.3 are usually satisfied for transmission lines (but not for transformers; see Example 4.5).
2. Since $b^s_{jk} < 0$ but $b^m_{jk} \geq 0$ for a typical transmission line, condition C4.4 in Theorem 4.4 is usually not satisfied.

□

Remark 4.4 (Distribution feeder test systems)**.** 1. It has been verified on a set of test distribution feeders that indeed $\mathrm{Re}(Y^{-1}) \succ 0$ and hence Theorem 4.2 holds....

The conditions in Theorems 4.3 and 4.4 are sufficient but not necessary. The next example shows that, even though Condition C4.2 in Theorem 4.3 is usually not satisfied for a transformer, the admittance matrix may nonetheless be nonsingular.

Example 4.5 (Sufficiency only)**.** Consider Example 4.1. An alternative solution approach is to introduce an internal node 3 on the primary side of the ideal transformer, not the secondary side as in Example 4.1. [4] Then the parameters of lines $(1,3)$ and $(2,3)$ are

$$\begin{aligned}
(y^s_{13}, y^m_{13}, y^m_{31}) \; &:= \; \left(y^l, 0, y^m\right) \\
(y^s_{23}, y^m_{23}, y^m_{32}) \; &:= \; (ny, (1-n)y, n(n-1)y)
\end{aligned}$$

where n is the voltage gain of the transformer, y is the series admittance of the line and (y^l, y^m) are the series and shunt admittances of the transformer. The admittance matrix is therefore

$$Y \;=\; \begin{bmatrix} y^l & 0 & -y^l \\ 0 & y & -ny \\ -y^l & -ny & y^l + y^m + n^2 y \end{bmatrix}$$

Let the admittances be of the form:

$$y \; =: \; g^s + \mathbf{i} b^s, \qquad y^l \; =: \; \frac{1}{r + j\omega L^l} \; = \; g^l + \mathbf{i} b^l, \qquad y^m \; =: \; j\omega L^m \; = \; \mathbf{i} b^m$$

4

with $g^s, g^l \geq 0$, $b^s, b^l \leq 0$, and $b^m \geq 0$. We now show that the admittance matrix Y does not satisfy condition C4.2 in Theorem 4.3, but Y is invertible if and only if $b^m > 0$.

We have

$$\begin{aligned} & & y^s_{13} \;=\; y^s_{31} \;=\; g^l + \mathbf{i}b^l, & \qquad y^s_{23} \;=\; y^s_{32} \;=\; ng^s + \mathbf{i}nb^s \\ y^m_{11} \;=\; 0, & \qquad y^m_{22} \;=\; (1-n)g^s + \mathbf{i}(1-n)b^s, & & \qquad y^m_{33} \;=\; n(n-1)g^s + \mathbf{i}\left(n(n-1)b^s + b^m\right) \end{aligned}$$

Hence condition C4.1 is satisfied but C4.2 is not since $g^m_{22} := (1-n)g^s$ and $g^m_{33} := n(n-1)g^s$ have opposite signs unless $n = 1$. For any nonzero vector $\alpha^{\mathsf{H}} Y \alpha \neq 0$, one can show (Exercise 4.9)

$$\alpha^{\mathsf{H}} Y \alpha \;=\; \left(\sum_{(j,k)\in E} g^s_{jk} \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in\overline{N}} g^m_{jj} \left|\alpha_j\right|^2 \right) + \mathbf{i} \left(\sum_{(j,k)\in E} b^s_{jk} \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in\overline{N}} b^m_{jj} \left|\alpha_j\right|^2 \right)$$

Hence

$$\begin{aligned} \mathrm{Re}\left(\alpha^{\mathsf{H}} Y \alpha\right) &= \left(g^l|\alpha_1 - \alpha_3|^2 + ng^s|\alpha_2 - \alpha_3|^2\right) + \left((1-n)g^s|\alpha_2|^2 + n(n-1)g^s|\alpha_3|^2\right) \\ &= g^l|\alpha_1 - \alpha_3|^2 \;+\; g^s|\alpha_2 - n\alpha_3|^2 \end{aligned}$$

Therefore

$$\mathrm{Re}\left(\alpha^{\mathsf{H}} Y \alpha\right) = 0 \quad \text{if and only if} \quad \alpha_1 = \alpha_3 = \frac{\alpha_2}{n} \tag{4.15}$$

On the other hand

$$\begin{aligned} \mathrm{Im}\left(\alpha^{\mathsf{H}} Y \alpha\right) &= \left(b^l|\alpha_1 - \alpha_3|^2 + nb^s|\alpha_2 - \alpha_3|^2\right) + \left((1-n)b^s|\alpha_2|^2 + \left(n(n-1)b^s + b^m\right)|\alpha_3|^2\right) \\ &= b^l|\alpha_1 - \alpha_3|^2 \;+\; b^s|\alpha_2 - n\alpha_3|^2 \;+\; b^m|\alpha_3|^2 \end{aligned}$$

In light of (4.15), if $b^m > 0$ then $\alpha^{\mathsf{H}} Y \alpha = 0$ if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Hence if $b^m > 0$ then Y is invertible.

If $b^m = 0$ then there exists nonzero $\alpha \in \mathbb{C}^3$ with $\alpha^{\mathsf{H}} Y \alpha = 0$. Exercise 4.8 says that, since Y is complex symmetric (but not Hermitian), this does not necessarily imply $Y\alpha = 0$ and hence may not imply that Y is singular. Using the admittance matrix given above, however, it can be verified that, when $y^m = \mathbf{i}b^m = 0$, $\alpha := \begin{bmatrix} 1 & n & 1 \end{bmatrix}^{\mathsf{T}}$ is indeed an eigenvector of Y corresponding to zero eigenvalue. Hence Y is singular if the (only) shunt element b^m in the model is zero, even when y^m_{22} and y^m_{33}, which originate from the effect of an ideal transformer, are nonzero. □

4.2.6 Kron reduction Y/Y_{22}

In many applications we are interested in the relation between the current injections and voltages at only a subset $N_{\text{red}} \subset \overline{N}$ of the buses. For example we are interested in the external behavior of a system defined by the relationship between currents and voltages of the end devices. Denote the number of buses in N_{red} also by N_{red}. Without loss of generality we can partition the buses such that $I_1 \in \mathbb{C}^{N_{\text{red}}}$ denotes the first N_{red}

current injections and I_2 the remaining $N+1-N_{\text{red}}$ current injections. Similarly partition the voltages into (V_1, V_2) with $V_1 \in \mathbb{C}^{N_{\text{red}}}$, $V_2 \in \mathbb{C}^{N+1-N_{\text{red}}}$. Partition the admittance matrix Y so that

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}}_{Y} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

If Y_{22} is invertible then we can eliminate V_2 by substituting $V_2 = -Y_{22}^{-1}Y_{21}V_1 + Y_{22}^{-1}I_2$ to obtain

$$\left(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}\right) V_1 = I_1 - Y_{12}Y_{22}^{-1}I_2 \tag{4.16}$$

The $N_{\text{red}} \times N_{\text{red}}$ matrix $Y/Y_{22} := Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}$ is the Schur complement of Y_{22} of matrix Y (see Appendix 25.1.2 for its properties). It can be interpreted as the admittance matrix of the reduced network consisting only of buses in N_{red} and describes the effective connectivity and line admittances of the reduced network. The quantity $I_1 - Y_{12}Y_{22}^{-1}I_2$ describes the effective current injections at these buses. This is called a *Kron reduction* of network G. If Y is complex symmetric, its Kron reduced admittance matrix Y/Y_{22} is also complex symmetric and hence satisfies Assumption C4.1. Two buses j and k are adjacent in the Kron-reduced network, i.e., $[Y/Y_{22}]_{jk} \neq 0$, if and only if j and k are adjacent in the original graph (i.e., $Y_{jk} \neq 0$) or if there is a path in the original graph that connects j and k. These properties are studied in Chapter 4.2.7.

Example 4.6 (Kron reduction)**.** Consider the network shown in Figure 4.10(a).

(a) Original network

(b) Kron reduced network

Figure 4.10: Kron reduction: $N_{\text{red}} := \{1, 2, 3\}$ with internal bus 4. While the original network is a tree, the Kron reduced network is fully connected.

Under condition C4.1 its admittance matrix Y is (0 and symmetric entries are omitted for simplicity)

$$Y := \begin{bmatrix} y_{14}^s + y_{11}^m & & & -y_{14}^s \\ & y_{24}^s + y_{22}^m & & -y_{24}^s \\ & & y_{34}^s + y_{33}^m & -y_{34}^s \\ & & & \sum_j y_{j4}^s + y_{44}^m \end{bmatrix}$$

with $Y_{22} := \sum_j y^s_{j4} + y^m_{44}$. The Schur complement Y/Y_{22} of Y_{22} is

$$
\begin{aligned}
& Y_{11} \; - \; Y_{12}Y_{22}^{-1}Y_{21} \\
= & \begin{bmatrix} y^s_{14}+y^m_{11} & & \\ & y^s_{24}+y^m_{22} & \\ & & y^s_{34}+y^m_{33} \end{bmatrix} \; - \; \frac{1}{Y_{22}} \begin{bmatrix} -y^s_{14} \\ -y^s_{24} \\ -y^s_{34} \end{bmatrix} \begin{bmatrix} -y^s_{14} & -y^s_{24} & -y^s_{34} \end{bmatrix} \\
= & \begin{bmatrix} \frac{y^s_{14}}{Y_{22}}\left(y^s_{24}+y^s_{34}\right)+\left(y^m_{11}+\gamma y^s_{14}\right) & \frac{-y^s_{14}y^s_{24}}{Y_{22}} & \frac{-y^s_{14}y^s_{34}}{Y_{22}} \\ & \frac{y^s_{24}}{Y_{22}}\left(y^s_{14}+y^s_{34}\right)+\left(y^m_{22}+\gamma y^s_{24}\right) & \frac{-y^s_{24}y^s_{34}}{Y_{22}} \\ & & \frac{y^s_{34}}{Y_{22}}\left(y^s_{14}+y^s_{24}\right)+\left(y^m_{33}+\gamma y^s_{34}\right) \end{bmatrix}
\end{aligned}
$$

where $\gamma := y^m_{44}/Y_{22} = y^m_{44}/\left(\sum_j y^s_{j4} + y^m_{44}\right)$. The Kron reduced network corresponding to Y/Y_{22} is fully connected as shown in Figure 4.10(b).

The effective current injections in the Kron reduced network are

$$
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \; - \; Y_{12}Y_{22}^{-1}I_3 \;\; = \;\; \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} \; + \; \begin{bmatrix} y^s_{14} \\ y^s_{24} \\ y^s_{34} \end{bmatrix} \frac{I_3}{Y_{22}}
$$

□

An admittance matrix Y has zero row, and hence column, sums if and only if all line charging admittances are zero, $y^m_{jk} = y^m_{kj} = 0$ for $(j,k) \in E$. In that case the Kron-reduced admittance matrix Y/Y_{22} also has zero, and hence column, sums (Exercise 4.10). The converse may not hold.

Given current injections $I = (I_1, I_2)$, we can obtain V_1 in terms of the Schur complement Y/Y_{22} and the effective current injections:

$$
V_1 \;\; = \;\; \left(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}\right)^{-1}\left(I_1 \; - \; Y_{12}Y_{22}^{-1}I_2\right)
$$

If additional information is available that expresses I_2 linearly in terms of I_1, say, $I_2 = AI_1$ for an appropriate $(N+1-N_{\text{red}}) \times N_{\text{red}}$ matrix A, then we can obtain from (4.16) a relationship between I_1 and V_1:

$$
\left(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}\right)V_1 \;\; = \;\; \left(I_{N_{\text{red}}} \; - \; Y_{12}Y_{22}^{-1}A\right)I_1
$$

where $I_{N_{\text{red}}}$ denotes the identity matrix of size N_{red}. In many applications current injections $I_2 = 0$. For example the buses in $\overline{N} \backslash N_{\text{red}}$ represent internal buses without generators or loads (see Example 4.1). Then (4.16) reduces to:

$$
I_1 \;\; = \;\; \underbrace{\left(Y_{11} - Y_{12}Y_{22}^{-1}Y_{21}\right)}_{Y/Y_{22}} V_1
$$

and the reduced network is described by the Schur complement Y/Y_{22} that directly relates V_1 and I_1.

4.2.7 Properties of Y/Y_{22}

In this section we summarize some useful analytical properties of the Kron-reduced admittance matrix Y/Y_{22}.

4.2.7.1 Invertibility of Y_{22}

The principal submatrix Y_{22} may not be strictly diagonal dominant nor invertible.[5] The situation is similar to the invertibility of Y. We now extend Theorems 4.3 and 4.4 to Y_{22}. They ensure the invertibility of Y_{22} and hence the validity of the Kron reduction.

Let $A \subsetneq \overline{N}$ denote the set of buses corresponding to Y_{22} and assume A is a strict subset of $\overline{N}$. For the rest of this subsection denote the (j,k) entry of a matrix M by $M[j,k]$, e.g., $Y[j,k], Y_{22}[j,k]$. Note that the indices j,k of Y_{22} take values in A, e.g., they run from $N-n+2,\ldots,N+1$, not $1,\ldots,n$, if Y_{22} corresponds to the last n buses. The argument is similar to that for the invertibility of Y. By definition Y_{22} is not invertible if and only if zero is an eigenvalue of Y_{22}. [6] If λ is an eigenvalue and $\alpha \in \mathbb{C}^n$ is a corresponding eigenvector then

$$\alpha^{\mathsf{H}} Y_{22} \alpha \;=\; \sum_{j\in A}\sum_{k\in A} Y[j,k]\alpha_j^{\mathsf{H}}\alpha_k \;=\; \lambda ||\alpha||^2 \tag{4.17}$$

where $||\cdot||$ denotes the Euclidean norm. Hence for Y_{22} to be invertible it is sufficient, but not necessary, that $\alpha^{\mathsf{H}} Y_{22}\alpha \neq 0$ for all nonzero vectors $\alpha \in \mathbb{C}^n$ (see Exercise 4.8). We have from (4.9c)

$$Y_{22}[j,j] \;=\; \sum_{k\notin A:(j,k)\in E} y^s_{jk} + \sum_{k\in A:(j,k)\in E} y^s_{jk} \,+\, y^m_{jj}, \qquad j\in A$$

Substituting this and $Y[j,k] = -y^s_{jk}$ for $j \sim k$ into (4.17) we have

$$\begin{aligned}
\alpha^{\mathsf{H}} Y_{22}\alpha \;&=\; \sum_{j\in A}\left(\left(\sum_{k\notin A:(j,k)\in E} y^s_{jk} + \sum_{k\in A:(j,k)\in E} y^s_{jk} \,+\, y^m_{jj}\right)|\alpha_j|^2 \,-\, \sum_{k\in A:(j,k)\in E} y^s_{jk}\,\alpha_j^{\mathsf{H}}\alpha_k\right) \\
&=\; \sum_{j,k\in A:(j,k)\in E}\left(y^s_{jk}|\alpha_j|^2 - y^s_{jk}\alpha_j^{\mathsf{H}}\alpha_k - y^s_{kj}\alpha_k^{\mathsf{H}}\alpha_j + y^s_{kj}|\alpha_k|^2\right) \,+\, \sum_{j\in A}\left(\sum_{k\notin A:(j,k)\in E} y^s_{jk} + y^m_{jj}\right)|\alpha_j|^2 \\
&=\; \sum_{j,k\in A:(j,k)\in E} y^s_{jk}\left|\alpha_j-\alpha_k\right|^2 \,+\, \sum_{j\in A}\left(\sum_{k\notin A:(j,k)\in E} y^s_{jk} + y^m_{jj}\right)|\alpha_j|^2
\end{aligned}$$

where the third equality uses from $y^s_{jk} = y^s_{kj}$ in condition C4.1. The first term sums over links in the subgraph induced by A. The second term sums over links between the subgraph induced by A and that by

[5] This is in contrast to the Laplacian matrix $Y = L$ in the DC power flow model whose a strict principal submatrix is always strictly diagonally dominant and hence invertible. See Chapter **??**.

[6]

$\overline{N} \setminus A$. Recall $y^s_{jk} =: g^s_{jk} + \mathbf{i}b^s_{jk}$ and $y^m_{jj} =: g^m_{jj} + \mathbf{i}b^m_{jj}$. Then

$$\mathrm{Re}\left(\alpha^{\mathsf{H}} Y_{22} \alpha\right) = \sum_{j,k\in A:(j,k)\in E} g^s_{jk} \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in A} \left(\sum_{k\notin A:(j,k)\in E} g^s_{jk} + g^m_{jj}\right) |\alpha_j|^2 \tag{4.18a}$$

$$\mathrm{Im}\left(\alpha^{\mathsf{H}} Y_{22} \alpha\right) = \sum_{j,k\in A:(j,k)\in E} b^s_{jk} \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in A} \left(\sum_{k\notin A:(j,k)\in E} b^s_{jk} + b^m_{jj}\right) |\alpha_j|^2 \tag{4.18b}$$

The subgraph corresponding to Y_{22} may consist of multiple connected components $C_i \subseteq A$. Each connected component C_i is a disjoint set of buses that are connected to each other and to no buses outside C_i such that $\cup_i C_i = A$. Let

$$G_j := \sum_{k\notin A:(j,k)\in E} g^s_{jk} + g^m_{jj}, \qquad B_j := \sum_{k\notin A:(j,k)\in E} b^s_{jk} + b^m_{jj}, \qquad j \in A \tag{4.19a}$$

Then we can rewrite (4.18) in terms of the connected components C_i and G_j, B_j:

$$\mathrm{Re}\left(\alpha^{\mathsf{H}} Y_{22} \alpha\right) = \sum_i \left(\sum_{j,k\in C_i:(j,k)\in E} g^s_{jk} \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in C_i} G_j |\alpha_j|^2\right) \tag{4.19b}$$

$$\mathrm{Im}\left(\alpha^{\mathsf{H}} Y_{22} \alpha\right) = \sum_i \left(\sum_{j,k\in C_i:(j,k)\in E} b^s_{jk} \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in C_i} B_j |\alpha_j|^2\right) \tag{4.19c}$$

Consider the following conditions on the conductances g^s_{jk} and G_j:

C4.6: For all lines $(j,k) \in E$, $g^s_{jk} \geq 0$ and for all buses $j \in \overline{N}$, $G_j \geq 0$.

C4.7a: For all buses $j \in \overline{N}$, $G_j \neq 0$,

C4.7b: For all lines $(j,k) \in E$, $g^s_{jk} \neq 0$. Furthermore on each connected component C_i there exists a bus $j_i \in C_i$ such that $G_{j_i} \neq 0$.

Conditions C4.6 can be changed to g^s_{jk}, G_j having the same sign. Theorem 4.3 extends directly to Y_{22}.

Theorem 4.5. Suppose the admittance matrix Y satisfies condition C4.1. If C4.6 and one of C4.7a and C4.7b hold, then the strict principal submatrix Y_{22} satisfies

1. $\mathrm{Re}(Y_{22}) \succ 0$.
2. Y_{22}^{-1} exists and is symmetric. Moreover $\mathrm{Re}\left(Y_{22}^{-1}\right) \succ 0$.

Proof. The proof is similar to that for Theorem 4.3. Condition C4.6 implies that every summand in (4.19b) is nonnegative. Moreover if C4.7a holds then the second summation is strictly positive if $\alpha \neq 0$. If C4.7b holds then for the first summation to be zero, $\alpha_j = \alpha_k$ for all j,k in each connected component C_i. Then the second summation becomes, on each C_i, $\sum_{j\in C_i} G_j|\alpha_j|^2 \geq G_{j_i}|\alpha_{j_i}|^2 > 0$ unless $\alpha_j = \alpha_{j_i} = 0$ for all $j \in C_i$. Therefore $\mathrm{Re}\left(\alpha^{\mathsf{H}} Y_{22}\alpha\right) > 0$ if $\rho \neq 0$, i.e., $\mathrm{Re}(Y_{22}) \succ 0$. Since Y_{22} is symmetric Theorem 4.2 then completes the proof. $\square$

Consider the following conditions on the susceptances b^s_{jk} and B_j:

C4.8: $b^s_{jk} \le 0$ for all lines $(j,k) \in E$ and $B_j \le 0$ for all buses $j \in \overline{N}$.

C4.9a: For all buses $j \in \overline{N}$, $B_j \neq 0$,

C4.9b: For all lines $(j,k) \in E$, $b^s_{jk} \neq 0$. Furthermore on each connected component C_i there exists a bus $j_i \in C_i$ such that $B_{j_i} \neq 0$.

Conditions C4.8 can be changed to b^s_{jk}, B_j having the same sign respectively. Theorem 4.4 extends directly to Y_{22} in the following result whose proof is omitted.

Theorem 4.6. Suppose the admittance matrix Y satisfies condition C4.1. If C4.8 and one of C4.9a and C4.9b hold, then the strict principal submatrix Y_{22} satisfies

1. $\mathrm{Im}(Y_{22}) \prec 0$.
2. Y_{22}^{-1} exists and is symmetric. Moreover $\mathrm{Im}\left(Y_{22}^{-1}\right) \succ 0$.

The invertibility conditions in Theorems 4.5 and 4.6 for the submatrix Y_{22} are less restrictive than those in Theorems 4.3 and 4.4 for Y, as we explain in Remark 4.5. Therefore if conditions of Theorem 4.3 or 4.4 are satisfied then Y^{-1}, Y_{22}^{-1} and Y/Y_{22} all exist.

Remark 4.5 (Transmission line)**.** As discussed in Remark 4.3, for a transmission line, we usually have $g^s_{jk} \ge 0$, $b^s_{jk} < 0$, $g^m_{jj} \ge 0$ and $b^m_{jj} \ge 0$.

1. If all lines (j,k), $j,k \in E$, have nonzero conductances, then conditions C4.6 and C4.7b are satisfied. This is the case even with zero shunt admittances $y^m_{jk} = y^m_{kj} = 0$ in which case Y has zero row sums and is singular.
2. For C4.8, even though b^s_{jk} and b^m_{jj} have opposite signs, the shunt susceptances b^m_{jk} are typically much smaller than the series susceptances b^s_{jk} such that usually B_j in (4.19a) has the same sign as b^s_{jk}. Hence both C4.8 and C4.9a are likely to be satisfied since b^s_{jk} are usually nonzero for transmission lines.

When shunt admittances $y^m_{jk} = y^m_{kj} = 0$. When $y^m_{jk} = y^m_{kj} = 0$ for all lines $(j,k) \in E$ a symmetric admittance matrix Y has zero row and column sums and is hence singular. In this case G_j and B_j in (4.19a) becomes

$$G_j := \sum_{k \notin A : (j,k) \in E} g^s_{jk}, \qquad B_j := \sum_{k \notin A : (j,k) \in E} b^s_{jk}, \qquad j \in A$$

Hence Theorems 4.5 and 4.6 imply the following simple conditions for the invertibility of a strict principal submtirx Y_{22} of Y.

Corollary 4.7. Suppose the admittance matrix Y satisfies condition C4.1 and $y^m_{jk} = y^m_{kj} = 0$ for all lines $(j,k) \in E$. Consider the strict principal submatrix Y_{22}.

1. If $g^s_{jk} > 0$ for all lines $(j,k) \in E$ then Y_{22}^{-1} exists and is symmetric. Moreover both $\mathrm{Re}(Y_{22}) \succ 0$ and $\mathrm{Re}\left(Y_{22}^{-1}\right) \succ 0$.

2. If $b^s_{jk} < 0$ for all lines $(j,k) \in E$ then then Y_{22}^{-1} exists and is symmetric. Moreover $\mathrm{Im}(Y_{22}) \prec 0$ but $\mathrm{Im}\left(Y_{22}^{-1}\right) \succ 0$.

Even when not all g^s_{jk} are strictly positive and not all b^s_{jk} are strictly negative the admittance matrix Y can still be invertible because they cannot be zero simultaneously, as the following result from [9] shows.

Theorem 4.8. Suppose the admittance matrix Y satisfies condition C4.1 and $y^m_{jk} = y^m_{kj} = 0$ for all lines $(j,k) \in E$. If $g^s_{jk} \geq 0$ and $b^s_{jk} \leq 0$ for all lines $(j,k) \in E$ then the strict principal submatrix Y_{22} satisfies

1. $\mathrm{Re}(Y_{22}) \succeq 0$, $\mathrm{Im}(Y_{22}) \preceq 0$.

2. Moreover $\mathrm{Re}(Y_{22}) - \mathrm{Im}(Y_{22}) \succ 0$.

3. Y_{22}^{-1} exists and is symmetric.

Proof. Write $Y =: G + \mathbf{i}B$ and $Y_{22} =: G_{22} + \mathbf{i}B_{22}$. Denote the (j,k) element of a matrix M by $M[j,k]$, e.g., $Y[j,k]$, $G_{22}[j,k]$, etc. Since $y^m_{jk} = y^m_{kj} = 0$ for all lines $(j,k) \in E$ and hence Y has zero row (and column) sums, each row of G_{22} and B_{22} are diagonally dominant:

$$\begin{aligned}
|G_{22}[j,j]| \ &= \ \left| \sum_{k \notin A:(j,k)\in E} g^s_{jk} + \sum_{k \in A:(j,k)\in E} g^s_{jk} \right| \ \geq \ \sum_{k \in A:(j,k)\in E} g^s_{jk} \ = \ \sum_{k\in A: k\neq j} |G_{22}[j,k]|, \qquad j \in A \\
|B_{22}[j,j]| \ &= \ \left| \sum_{k \notin A:(j,k)\in E} b^s_{jk} + \sum_{k \in A:(j,k)\in E} b^s_{jk} \right| \ \geq \ \sum_{k \in A:(j,k)\in E} -b^s_{jk} \ = \ \sum_{k\in A: k\neq j} |B_{22}[j,k]|, \qquad j \in A
\end{aligned}$$

Since G_{22} and B_{22} are real and symmetric their eigenvalues are all real. The Geršgorin disc theorem states that all eigenvalues of a real matrix $M \in \mathbb{R}^{n\times n}$ lie in the union of n discs

$$\cup_{i=1}^n \left\{ z \in \mathbb{C}^n \ : \ |z - M_{ii}| \leq \sum_{j:j\neq i} |M_{ij}| \right\}$$

Therefore all eigvenvalues of the G_{22} are nonnegative and those of B_{22} are nonpositive, i.e., $G_{22} \succeq 0$ and $B_{22} \preceq 0$. This implies that $G_{22} - B_{22} \succeq 0$.

We now show that, indeed, $G_{22} - B_{22} \succ 0$ because the network is connected and $A \subset \overline{N}$ is a strict subset. Since $G_{22} - B_{22}$ is real symmetric, consider, for any nonzero real vector ρ,

$$\begin{aligned}
\rho^{\mathsf{T}}(G_{22} - B_{22})\rho \ &= \ \sum_{j\in A}\sum_{k\in A} \rho_j (G_{22}[j,k] - B_{22}[j,k])\rho_k \\
&= \ \sum_{j\in A}\sum_{k\in A:(j,k)\in E} \rho_j(-g^s_{jk} + b^s_{jk})\rho_k \ + \ \sum_{j\in A} \rho_j^2 \left(\sum_{k\in A:(j,k)\in E} (g^s_{jk} - b^s_{jk}) + \sum_{k\notin A:(j,k)\in E} (g^s_{jk} - b^s_{jk}) \right) \\
&= \ \sum_{j,k\in A:(j,k)\in E} \left(\rho_j - \rho_k\right)^2 (g^s_{jk} - b^s_{jk}) \ + \ \sum_{j\in A} \rho_j^2 G_j
\end{aligned}$$

where the third equality uses $g^s_{jk} = g^s_{kj}$ and $b^s_{jk} = b^s_{kj}$ from C4.1. Here $G_j := \sum_{k \notin A:(j,k)\in E} (g^s_{jk} - b^s_{jk})$ for $j \in A$ and the summation is not vacuous because the network is connected and $A \subsetneq \overline{N}$. For every line $(j,k) \in E$, $y^s_{jk} \neq 0$ and hence $g^s_{jk} - b^s_{jk} > 0$ since $g^s_{jk} \geq$ and $b^s_{jk} \geq 0$. This implies $G_j > 0$ as well for all $j \in A$. Therefore for $\rho^\mathsf{T}(G_{22} - B_{22})\rho > 0$ for any real vector $\rho \neq 0$, i.e., $G_{22} - B_{22} \succ 0$.

Finally we use $G_{22} - B_{22} \succ 0$ to show that Y_{22} is nonsingular (it is clear that Y_{22}^{-1} is symmetric if it exists). If Y_{22} is singular then it has a nonzero eigenvector $\alpha = \rho + \mathbf{i}\varepsilon$ corresponding to the zero eigenvalue and hence

$$0 \;=\; Y_{22}\alpha \;=\; (G_{22} + \mathbf{i}B_{22})(\rho + \mathbf{i}\varepsilon) \;=\; (G_{22}\rho - B_{22}\varepsilon) \;+\; \mathbf{i}\,(G_{22}\varepsilon + B_{22}\rho)$$

Hence

$$G_{22}\rho - B_{22}\varepsilon \;=\; 0, \qquad B_{22}\rho + G_{22}\varepsilon \;=\; 0$$

To solve for (ρ, ε), subtract the second equation from the first to get $(G_{22} - B_{22})\rho = (G_{22} + B_{22})\varepsilon$. Since $G_{22} - B_{22} \succ 0$ we have $\rho = (G_{22} - B_{22})^{-1}(G_{22} + B_{22})\varepsilon$. Substituting into the first equation we have

$$\begin{aligned} 0 \;&=\; \left(G_{22}(G_{22} - B_{22})^{-1}(G_{22} + B_{22}) - B_{22}\right)\varepsilon \\ &=\; \left(G_{22}(G_{22} - B_{22})^{-1}G_{22} + G_{22}(G_{22} - B_{22})^{-1}B_{22} - B_{22}\right)\varepsilon \end{aligned}$$

But $G_{22}(G_{22} - B_{22})^{-1}B_{22} - B_{22} = (G_{22} - (G_{22} - B_{22}))\,(G_{22} - B_{22})^{-1}B_{22} = B_{22}(G_{22} - B_{22})^{-1}B_{22}$ and hence

$$0 \;=\; \left(G_{22}(G_{22} - B_{22})^{-1}G_{22} + B_{22}(G_{22} - B_{22})^{-1}B_{22}\right)\varepsilon$$

Multiplying on the left by ε^T we have

$$0 \;=\; \varepsilon^\mathsf{T}\left(G_{22}(G_{22} - B_{22})^{-1}G_{22} + B_{22}(G_{22} - B_{22})^{-1}B_{22}\right)\varepsilon$$

which implies $\varepsilon = 0$ since $(G_{22} - B_{22})^{-1} \succ 0$. But then $\rho = (G_{22} - B_{22})^{-1}(G_{22} + B_{22})\varepsilon = 0$ and therefore $\alpha = \rho + \mathbf{i}\varepsilon = 0$, contradicting that the eigenvector α is nonzero. Hence Y_{22} is nonsingular. □

4.2.7.2 Properties of Y/Y_{22}

Theorem 4.2 extends directly to the Schur complement $Y/Y_{22} := Y_{11} - Y_{12}Y_{22}^{-1}Y_{12}^\mathsf{T}$.

Theorem 4.9. Consider a complex symmetric matrix $Y =: \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^\mathsf{T} & Y_{22} \end{bmatrix}$ with symmetric Y_{11}, Y_{22} (i.e., satisfies condition C4.1). Suppose Y_{22} is nonsingular.

1. If $\mathrm{Re}(Y) \succ 0$ then the Schur complement Y/Y_{22} of Y_{22} is invertible and $(Y/Y_{22})^{-1}$ is symmetric. Moreover both $\mathrm{Re}(Y/Y_{22}) \succ 0$ and $\mathrm{Re}\big((Y/Y_{22})^{-1}\big) \succ 0$.

2. If $\mathrm{Im}(Y) \prec 0$ then the Schur complement Y/Y_{22} of Y_{22} is invertible and $(Y/Y_{22})^{-1}$ is symmetric. Moreover $\mathrm{Im}(Y/Y_{22}) \prec 0$ but $\mathrm{Im}\big((Y/Y_{22})^{-1}\big) \succ 0$.

Proof. It is clear that the Schur complement Y/Y_{22} of Y_{22} is symmetric since Y_{11} and Y_{22}, and hence Y_{22}^{-1}, are symmetric. From Theorem 25.1 in Appendix 25.1.2.1, Y is nonsingular if and only if Y/Y_{22} is nonsingular, given that Y_{22} is nonsingular. If $\mathrm{Re}(Y) \succ 0$ or $\mathrm{Im}(Y) \prec 0$, Theorem 4.2 implies that Y^{-1} exists and $\mathrm{Re}(Y^{-1}) \succ 0$ or $\mathrm{Im}(Y^{-1}) \prec 0$ respectively. Hence Y/Y_{22} is nonsingular if $\mathrm{Re}(Y) \succ 0$ or $\mathrm{Im}(Y) \prec 0$.

Write Y^{-1} in terms of the Schur complement Y/Y_{22} (from Theorem 25.1):

$$Y^{-1} \;=\; \begin{bmatrix} (Y/Y_{22})^{-1} & -(Y/Y_{22})^{-1}Y_{12}Y_{22}^{-1} \\ -Y_{22}^{-1}Y_{12}^{\mathsf{T}}(Y/Y_{22})^{-1} & A \end{bmatrix}$$

where $A := Y_{22}^{-1} + Y_{22}^{-1}Y_{12}^{\mathsf{T}}(Y/Y_{22})^{-1}Y_{12}Y_{22}^{-1}$. If $\mathrm{Re}(Y) \succ 0$ then Theorem 4.2 implies that $\mathrm{Re}(Y^{-1}) \succ 0$. Hence all the principal submatrices of $\mathrm{Re}(Y^{-1})$ are (symmetric and) positive definite. In particular $\mathrm{Re}\big((Y/Y_{22})^{-1}\big) \succ 0$. But $(Y/Y_{22})^{-1}$ is symmetric and therefore Theorem 4.2 implies that $\mathrm{Re}(Y/Y_{22}) \succ 0$.

If on the other hand $\mathrm{Im}(Y) \prec 0$, then Theorem 4.2 implies that $\mathrm{Im}(Y^{-1}) \succ 0$. Hence its principal submatrix $\mathrm{Im}\big((Y/Y_{22})^{-1}\big) \succ 0$. But $(Y/Y_{22})^{-1}$ is symmetric and therefore Remark 4.2 implies that $\mathrm{Im}(Y/Y_{22}) \prec 0$. □

Application: admittance matrix Y identification.

4.2.8 Summary

We have explained how to model different network components, such as transmission lines, transformers, generators and loads, as nodes in a graph with links connecting these nodes parameterized by $(y_{jk}^s, y_{jk}^m, y_{kj}^m)$. This can be described by an admittance matrix Y. The equation $I = YV$ expresses nodal current balance due to KCL. Finally we have discussed sufficient conditions for the invertibility of Y and its principal submatrices.

In this setting if all generators and loads can be modeled by constant current sources then, given I, the voltages V on the network can be computed from the equation $I = YV$ as discussed in Chapter 4.2.4. This is simple as it involves linear equations only. The power injection at each node j can then be computed as $s_j = V_j I_j^{\mathsf{H}}$. Other quantities such as power or current flows on the lines or active power loss in the network can all be computed from V. For instance the current over line (j,k) is $I_{jk} = y_{jk}^s(V_j - V_k) + y_{jk}^m V_j$. The complex sending-end power flow over line (j,k) from node j and that from node k are respectively

$$\begin{aligned} S_{jk} \;&:=\; V_j I_{jk}^{\mathsf{H}} \;=\; \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y_{jk}^m\right)^{\mathsf{H}} |V_j|^2 \\ S_{kj} \;&:=\; V_k I_{kj}^{\mathsf{H}} \;=\; \left(y_{kj}^s\right)^{\mathsf{H}} \left(|V_k|^2 - V_k V_j^{\mathsf{H}}\right) \;+\; \left(y_{kj}^m\right)^{\mathsf{H}} |V_k|^2 \end{aligned}$$

Hence the active power loss over line (i,j) is

$$\begin{aligned} P_{jk} + P_{kj} \;&=\; \mathrm{Re}\left(S_{jk} + S_{kj}\right) \\ &=\; \mathrm{Re}\left((y_{jk}^s)^* \left(|V_j|^2 - V_j V_k^*\right) \;+\; (y_{kj}^s)^* \left(|V_k|^2 - V_k V_j^*\right) \;+\; (y_{jk}^m)^*|V_j|^2 \;+\; (y_{kj}^m)^*|V_k|^2\right) \\ &=\; \mathrm{Re}\left(y_{jk}^s\right) \left|V_j - V_k\right|^2 \;+\; \mathrm{Re}\left(y_{jk}^m\right) \left|V_j\right|^2 \;+\; \mathrm{Re}\left(y_{kj}^m\right) |V_k|^2 \end{aligned}$$

where the last equality follows if condition C4.1 holds.

4.3 Network model: Vs relation

In many applications however loads and generators are not specified as constant currents. They may be described instead in terms of power injections or removals. For instance, for electric vehicle charging, the travel need is specified in terms of the number of miles required which translates to the amount of energy in kWh required that must be delivered by a deadline. For example it requires roughly 3 kWh for an electric vehicle to travel 10 miles. Hence a charging facility is often characterized by its power requirement to support a certain electric vehicle charging capacity. In this section we present power flow equations that describe the relation between power injections and voltages on the network. As we will see this involves nonlinear equations which are much more difficult to solve.

4.3.1 Complex form

The *bus injection model* (BIM) in its complex form is defined by power balance $s_j = \sum_{k:j\sim k} S_{jk}$ at each node j where S_{jk} are sending-end line powers from j to its neighbors k given in (4.2). This leads to the power flow equations that relate power injections and voltages through (4.2):

$$s_j \;=\; \sum_{k:j\sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2, \qquad j \in \overline{N} \tag{4.20a}$$

where, from (4.9b), the total shunt admittance $y^m_{jj} := \sum_{k:j\sim k} y^m_{jk}$ associated with bus j is the sum of shunt admittances y^m_{jk} of all lines (j,k) incident on bus j. We can also express (4.20a) in terms of the elements of the admittance matrix Y as

$$s_j \;=\; \sum_{k=0}^{N} Y_{jk}^{\mathsf{H}} V_j V_k^{\mathsf{H}}, \qquad j \in \overline{N} \tag{4.20b}$$

where Y is given by:

$$Y_{jk} \;=\; \begin{cases} -y^s_{jk}, & j \sim k \;\; (j \neq k) \\ \sum_{i:j\sim i} y^s_{ji} \,+\, y^m_{jj} & j = k \\ 0 & \text{otherwise} \end{cases} \tag{4.20c}$$

When the total shunt admittance $y^m_{jj} = 0$, (4.20a) reduces to

$$s_j \;=\; \sum_{k:j\sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right), \qquad j \in \overline{N}$$

For convenience we include V_0 in the vector variable $V := (V_j, j \in \overline{N})$ with the understanding that $V_0 := 1\angle 0^\circ$ is fixed. There are $N+1$ equations in (4.20a) in $2(N+1)$ complex variables $(s_j, V_j, j \in \overline{N})$.

This model does not require assumption C4.1.

Remark 4.6 (Nodal devices)**.** If bus j in Remark 4.1 includes, in addition, a power source with a fixed power injection σ^p_j, then s_j is the net bus injection (assuming all neutrals are grounded and all voltages are defined with respect to the ground):

$$s_j \;=\; \underbrace{-\left(z_j^{v\mathsf{H}}\right)^{-1}\left(|V_j|^2 - V_j E_j^{\mathsf{H}}\right)}_{\text{voltage source}} + \underbrace{V_j\left(J_j - y_j^c V_j\right)^{\mathsf{H}}}_{\text{current source}} - \underbrace{y_j^{a\mathsf{H}}|V_j|^2}_{\text{shunt admittance}} + \underbrace{\sigma_j^p}_{\text{power source}}$$

and (4.20a) becomes:

$$\begin{aligned} & -\left(z_j^{v\mathsf{H}}\right)^{-1}\left(|V_j|^2 - V_j E_j^{\mathsf{H}}\right) + V_j\left(J_j - y_j^c V_j\right)^{\mathsf{H}} - y_j^{a\mathsf{H}}|V_j|^2 + \sigma_j^p \\ = & \sum_{k:j\sim k}\left(y_{jk}^s\right)^{\mathsf{H}}\left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) + \left(y_{jj}^m\right)^{\mathsf{H}}|V_j|^2, \qquad j \in \overline{N} \end{aligned}$$

$\square$

4.3.2 Polar form

We may alternatively treat (4.20) as $2(N+1)$ equations in $4(N+1)$ real variables $(p_j, q_j, |V_j|, \theta_j, j \in \overline{N})$ where $s_j := p_j + \mathbf{i}q_j$ are the complex injections and $V_j := |V_j|\,e^{\mathbf{i}\theta_j}$ are the complex voltages. Let $y_{jk}^s =: g_{jk}^s + \mathbf{i}b_{jk}^s$ denote the series admittance of line (j,k). Similarly let $y_{jk}^m =: g_{jk}^m + \mathbf{i}b_{jk}^m$ denote the shunt admittance line line (j,k) associated with bus j and $y_{jj}^m =: g_{jj}^m + \mathbf{i}b_{jj}^m$ the total shunt admittance associated with bus j. As discussed in Remark 4.5, usually $g_{jk}^s \geq 0$, $b_{jk}^s < 0$ (inductive line), $g_{jk}^m \geq 0$, but $b_{jk}^m \geq 0$ (capacitive shunt). Substituting

$$Y_{jk} \;=\; \begin{cases} -(g_{jk}^s + \mathbf{i}b_{jk}^s), & j \sim k \;\; (j \neq k) \\ \sum_{i:j\sim i}\left(g_{ji}^s + g_{ji}^m\right) + \mathbf{i}\sum_{i:j\sim i}\left(b_{ji}^s + b_{ji}^m\right) & j = k \\ 0 & \text{otherwise} \end{cases}$$

into (4.20b) we have

$$s_j \;=\; \sum_{k=0}^{N}\left(g_{jk} - \mathbf{i}b_{jk}\right)\left|V_j\right|^2 - \sum_{k=0,k\neq j}^{N}\left(g_{jk} - \mathbf{i}b_{jk}\right)\left|V_j\right|\,|V_k|\,e^{\mathbf{i}\theta_{jk}} \qquad j \in \overline{N}$$

where $\theta_{jk} := \theta_j - \theta_k$ is the voltage phase angle difference across each line $(j,k) \in E$, and

$$g_{jk} \;:=\; \begin{cases} \sum_{i:j\sim i}\left(g_{ji}^s + g_{ji}^m\right) & \text{if} \quad j = k \\ g_{jk}^s & \text{if} \quad j \neq k,\ (j,k) \in E \\ 0 & \text{if} \quad j \neq k,\ (j,k) \notin E \end{cases} \tag{4.21a}$$

$$b_{jk} \;:=\; \begin{cases} \sum_{i:j\sim i}\left(b_{ji}^s + b_{ji}^m\right) & \text{if} \quad j = k \\ b_{jk}^s & \text{if} \quad j \neq k,\ (j,k) \in E \\ 0 & \text{if} \quad j \neq k,\ (j,k) \notin E \end{cases} \tag{4.21b}$$

Then we can write (4.20a) in the polar form:

$$
\begin{aligned}
p_j &= \left(\sum_{k=0}^{N} g_{jk}\right)|V_j|^2 - \sum_{k\neq j}|V_j||V_k|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right), \qquad j\in\overline{N} && \text{(4.22a)}\\
q_j &= -\left(\sum_{k=0}^{N} b_{jk}\right)|V_j|^2 - \sum_{k\neq j}|V_j||V_k|\left(g_{jk}\sin\theta_{jk} - b_{jk}\cos\theta_{jk}\right), \qquad j\in\overline{N} && \text{(4.22b)}
\end{aligned}
$$

where (g_{jk}, b_{jk}) are defined in (4.21).

This model does not require assumption C4.1.

4.3.3 Cartesian form

The power flow equations (4.20) or (4.22) can also be reformulated in the real domain by writing V_j in terms of its real and reactive components (e_j, f_j), i.e., $V_j =: e_j + \mathbf{i} f_j$. Then (4.22) becomes (using $e_j = |V_j|\cos\theta_j$ and $f_j = |V_j|\sin\theta_j$)

$$
\begin{aligned}
p_j &= \left(\sum_{k} g_{jk}\right)\left(e_j^2 + f_j^2\right) - \sum_{k\neq j}\left(g_{jk}(e_je_k + f_jf_k) + b_{jk}(f_je_k - e_jf_k)\right) && \text{(4.23a)}\\
q_j &= -\left(\sum_{k} b_{jk}\right)\left(e_j^2 + f_j^2\right) - \sum_{k\neq j}\left(g_{jk}(f_je_k - e_jf_k) - b_{jk}(e_je_k + f_jf_k)\right) && \text{(4.23b)}
\end{aligned}
$$

where (g_{jk}, b_{jk}) are defined in (4.21). These are $2(N+1)$ quadratic equations in $4(N+1)$ variables $(p_j, q_j, e_j, f_j,\ j\in\overline{N})$.

This model does not require assumption C4.1.

4.3.4 Types of buses

Each set of power flow equations (4.20)(4.22)(4.23) is a set of $2(N+1)$ nonlinear real equations in $4(N+1)$ real variables corresponding to real and imaginary parts of power injections s and voltage phasors V. Given any $2(N+1)$ of these real variables, these equations can be used to solve for the remaining $2(N+1)$ real variables. There can be zero, unique or multiple solutions. Solving for these solutions is the *power flow* or *load flow* problem (Chapter 4.5).

A popular formulation of the power flow problem uses the polar form where each bus j is classified into one of three types based on which two of the four real variables $(p_j, q_j, |V_j|, \theta_j)$ are specified:

- *PV bus.* This is a bus where the real power injection p_j and the voltage magnitude $|V_j|$ are specified and the reactive power injection q_j and voltage angle θ_j are to be determined. It usually models a bus with a conventional generator.

- *PQ bus.* This is a constant-power bus where the injection (p_j, q_j) is specified and the complex voltage $|V_j| e^{j\theta_j}$ is to be determined. It usually models a load but can also model a renewable generator with undispatchable generation.

- *Slack bus.* Bus 0 is taken as a slack bus where $V_0 = |V_0| \angle 0^\circ$ is specified and the injection $s_0 = (p_0, q_0)$ is to be determined. This is usually used for mathematical convenience to avoid an ill specified power flow problem that has no solution.

A slack bus (or a set of slack buses) is needed because power needs to be balanced over the network. For example if the resistance of every line is zero then $\sum_j p_j$ must be zero. If all buses are *PV* or *PQ* buses then all active powers p_j are specified; if the specified values do not satisfy power balance then the set of power flow equations will have no solution. This is resolved by taking an arbitrary bus (denoted by bus 0 here) as a slack bus with its power injection s_0 unspecified in order to balance power. For instance a distribution system with a substation at bus 0 and N constant power loads or generations can be modeled by a slack bus and N PQ buses with V_0 and (p_j, q_j) specified. The power flow problem solves (4.22) for the N complex voltages V_j, $j \neq 0$, and the power injection s_0.

For optimal power flow problems p_j and $|V_j|$ on generator buses or s_j on load buses can be variables as well. For instance economic dispatch optimizes real power generations p_j at generator buses; demand response optimizes demands s_j at load buses; and volt/var control optimizes reactive powers q_j at capacitor banks, tap changers, or inverters. We will discuss optimal power flow problems in Part III of the book.

4.3.5 Real power loss

For each line $(j,k) \in E$, let its series and shunt admittances be $y^s_{jk} = g^s_{jk} + \mathbf{i} b^s_{jk}$ and $y^m_{jk} = g^m_{jk} + \mathbf{i} b^m_{jk}$. Suppose $y^s_{jk} = y^s_{kj}$ and $g^s_{jk} \geq 0$, $g^m_{jk} \geq 0$ (these conditions are satisfied if (j,k) models a transmission line). Define the total real power loss as the total real power injections as functions of voltages V:

$$C_0(V) \;:=\; \sum_j \mathrm{Re}\left(s_j(V)\right)$$

Then the complex power flow equation (4.20a) implies

$$C_0(V) \;=\; \sum_j \mathrm{Re}\left(\sum_{k: j \sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2 \right)$$

It can be shown (Exercise 13.6) that $C_0(V)$ is a quadratic form $C_0(V) = V^{\mathsf{H}} C_0 V$ where the cost matrix $C_0 := \mathrm{Re}(Y)$ is the real part of the admittance matrix Y. Moreover C_0 is a positive definite matrix when $g^m_{jk} + g^m_{kj} > 0$ for at least one line $(j,k) \in E$.

Suppose $y^m_{jk} = y^m_{kj} = 0$. Define the total thermal loss as:

$$C_0(V) \;:=\; \sum_{(j,k) \in E} r^s_{jk} |I_{jk}(V)|^2 \;=\; \sum_{(j,k) \in E} r^s_{jk} \left| y^s_{jk}(V_j - V_k) \right|^2$$

where $z^s_{jk} = r^s_{jk} + \mathbf{i} x^s_{jk} := 1/y^s_{jk}$. Then it can be shown (Exercise 13.6) that $C_0(V)$ is a quadratic form $C_0(V) = V^{\mathsf{H}} C_0 V$ where the cost matrix $C_0 = \mathrm{Re}(Y) =: G^s$ when $y^m_{jk} = y^m_{kj} = 0$. Therefore the total real power loss reduces to total thermal loss when $y^m_{jk} = y^m_{kj} = 0$. The matrix G^s has zero row sum and is a positive semidefinite matrix.

4.4 Properties of power flow solutions

Example 4.7 (Two-bus network)**.** Consider two buses 1 and 2 connected by a line with admittance $y = g + \mathbf{i}b$ with $g > 0, b < 0$. Assume zero charging admittances, and we ignore reactive powers. Assume $V_1 := 1\angle 0^\circ$ and $V_2 = e^{\mathbf{i}\theta}$, i.e., voltage magnitudes are fixed at 1 pu. Then the real power injections (p_1, p_2) depend on θ according to the power flow equations in polar form are:

$$p_1 \ := \ p_1(\theta) \ := \ g - g\cos\theta - b\sin\theta \tag{4.24a}$$

$$p_2 \ := \ p_2(\theta) \ := \ g - g\cos\theta + b\sin\theta \tag{4.24b}$$

or in vector form

$$P - g\mathbf{1} \ = \ A\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \tag{4.25}$$

where $\mathbf{1} := [1\ 1]^{\mathsf{T}}$ and A is an invertible (indeed negative definite) matrix:

$$A \ := \ \begin{bmatrix}-g & -b \\ -g & b\end{bmatrix}$$

Show that, as θ ranges from 0 to 2π, $(p_1(\theta), p_2(\theta))$ traces out an ellipse.

4.5 Computation methods

Suppose we are given a set of power flow equations in the bus injection model. Suppose $2(N+1)$ of the $4(N+1)$ real variables are specified and we are interested in solving for the remaining variables. We now present three solution methods. These methods do not require assumption C4.1.

4.5.1 Gauss-Seidel algorithm

Consider the power flow equations (4.20a) in the complex form. To illustrate the basic idea consider first the case with a slack bus and load buses only.

Case 1: Given V_0 and $(s_1, \ldots, s_N)$, determine s_0 and $(V_1, \ldots, V_N)$. The power flow equations are:

$$s_0 \ = \ \sum_k Y^{\mathsf{H}}_{0k} V_0 V^{\mathsf{H}}_k \tag{4.26a}$$

$$s_j \ = \ \sum_k Y^{\mathsf{H}}_{jk} V_j V^{\mathsf{H}}_k, \qquad j \in N \tag{4.26b}$$

Once we have computed $(V_1,\ldots,V_N)$, s_0 can be evaluated using (4.26a). Hence the main task is to compute $(V_1,\ldots,V_N)$ from (4.26b). We have from (4.26b):

$$\frac{s_j^{\mathsf{H}}}{V_j^{\mathsf{H}}} \;=\; Y_{jj}V_j + \sum_{\substack{k=0\\k\neq j}}^{N} Y_{jk}V_k, \qquad j\in N$$

Rearrange to obtain

$$V_j \;=\; \frac{1}{Y_{jj}}\left(\frac{s_j^{\mathsf{H}}}{V_j^{\mathsf{H}}} - \sum_{\substack{k=0\\k\neq j}}^{N} Y_{jk}V_k\right) \;=:\; f_j\left(V_1,\ldots,V_N\right), \qquad j\in N$$

Hence a power flow solution $V := (V_1,\ldots,V_N)$ is a fixed point of $f := (f_1,\ldots,f_N)$ with

$$V \;=\; f(V)$$

The Gauss algorithm is the standard fixed point iteration $V(t+1) = f(V(t))$, or

$$\begin{aligned}
V_1(t+1) &= f_1\left(V_1(t),\ldots,V_N(t)\right)\\
V_2(t+1) &= f_2\left(V_1(t),\ldots,V_N(t)\right)\\
&\vdots\\
V_N(t+1) &= f_n\left(V_1(t),\ldots,V_N(t)\right)
\end{aligned}$$

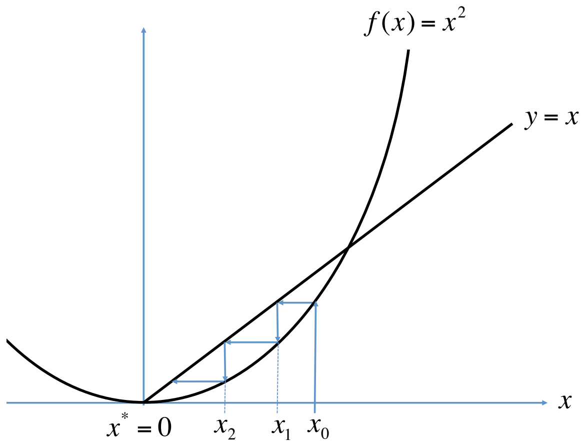
Starting from an initial vector $V(0)$ (e.g., $V_j(0) = 1\angle 0^\circ$ pu for all j), the Gauss algorithm produces a sequence $V(1),V(2),\ldots$. If the sequence converges to a limit V^{lim} then V^{lim} is a fixed point of f and a power flow solution.

When this iteration is carried out sequentially then when $V_2(t+1)$ is computed, $V_1(t+1)$ is already known and can be used in the computation of $V_2(t+1)$, and so on. This is the Gauss-Seidel algorithm where the latest value of $V_i(t+1)$ is used to compute $V_{j+1}(t+1)$, $j>i$:

$$\begin{aligned}
V_1(t+1) &= f_1\left(V_1(t),V_2(t),\ldots,V_N(t)\right)\\
V_2(t+1) &= f_2\left(V_1(t+1),V_2(t),\ldots,V_N(t)\right)\\
&\vdots\\
V_N(t+1) &= f_N\left(V_1(t+1),\ldots,V_{N-1}(t+1),V_N(t)\right)
\end{aligned}$$

Case 2: Given $(V_0,V_1,\ldots,V_m)$ *and* $(s_{m+1},\ldots,s_N)$*, determine* $(s_0,s_1,\ldots,s_m)$ *and* $(V_{m+1},\ldots,V_N)$. In this case, first determine $(V_{m+1},\ldots,V_N)$ from the reduced set of power flow equations (4.26b) for $j=m+1,\ldots,N$, using the same algorithm. Then determine $(s_0,s_1,\ldots,s_m)$ given $(V_0,\ldots,V_N)$.

The Gauss-Seidel algorithm is simple and does not require the evaluation of any derivatives. If the function f is a contraction mapping then it has a unique fixed point V^{lim} and the Gauss or Gauss-Seidel algorithm is guaranteed to converge to V^{lim}. See Exercise 4.11. Otherwise there is no guarantee that



(a) Convergence (b) Divergence

Figure 4.11: The fixed point iteration $x(t+1) = f(x(t)) := x^2(t)$ is not a contraction mapping and its convergence depends on the initial point $x(0) = x_0$.

the algorithms will converge, but if it does, it produces a fixed point which is a power flow solution $V^{\lim}$. Whether it converges can depend on the choice of the initial vector $V(0)$. Take for an example $x = f(x) := x^2$ for $x \in \mathbb{R}$ as shown in Figure 4.11. It has two fixed points $x^{\lim} = 0$ or 1. The fixed point iteration $x(t+1) = f(x(t)) = x^2(t)$ converges to $x^{\lim} = 0$ if the initial point $x(0) \in (-1,1)$ and diverges to positive infinity if $|x(0)| > 1$. The fixed point $x^{\lim} = 0$ is stable in the sense that the iterate $x(t)$ converges back to the origin after a small perturbation. The fixed point $x^{\lim} = 1$ is unstable in the sense that $x(t)$ leaves and will not return after a small perturbation in the positive direction.

4.5.2 Newton-Raphson algorithm

The Newton-Raphson algorithm is popular for iteratively solving the equation

$$f(x) \quad = \quad 0$$

where $x \in \mathbb{R}^n$ and f is a vector-valued function $f : \mathbb{R}^n \to \mathbb{R}^n$. The iteration is motivated by the Taylor series expansion of f. Suppose we have computed $x(t)$ and wish to determine the next iterate $x(t+1) =: x(t) + \Delta x(t)$. The Taylor series of f around $x(t)$ is

$$f(x(t) + \Delta x(t)) \quad = \quad f(x(t)) \; + \; J(x(t))\Delta x(t) \; + \; \text{higher-order terms}$$

where $J(x(t))$ is the Jacobian of f evaluated at $x(t)$:

$$J(x) \quad := \quad \frac{\partial f}{\partial x}(x) \quad = \quad \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1}(x) & \cdots & \frac{\partial f_n}{\partial x_n}(x) \end{bmatrix}$$

If we ignore the higher-order terms in the Taylor expansion and set $f(x(t+1)) = 0$ then we have

$$J(x(t))\Delta x(t) \;=\; -f(x(t)) \tag{4.27}$$

This is illustrated in Figure 4.12. If $J(x(t))$ is invertible then $\Delta x(t) \;=\; -J^{-1}(x(t))\,f(x(t))$, yielding the

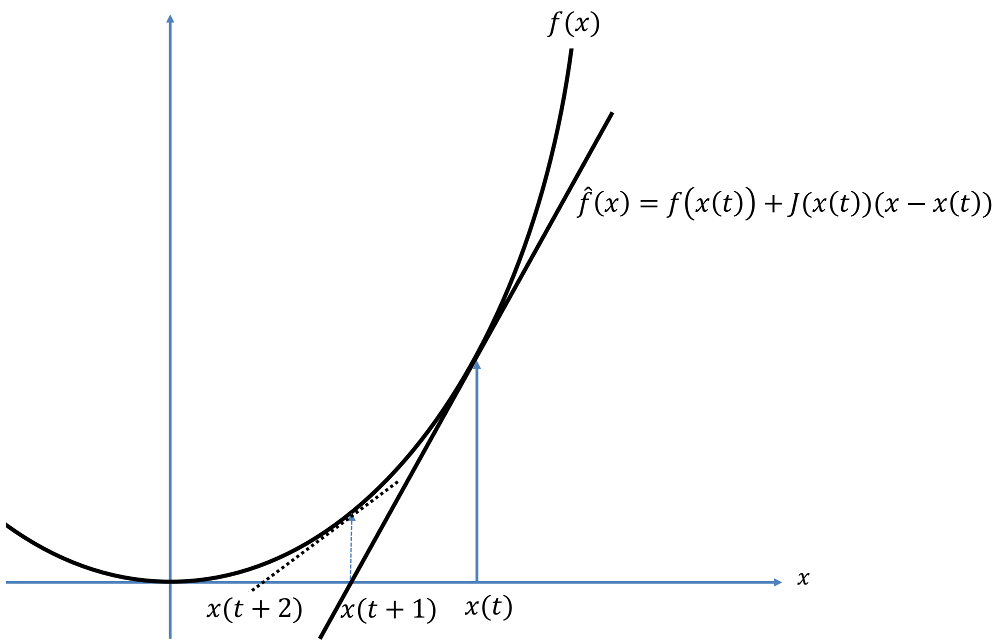


Figure 4.12: Newton-Raphson algorithm: The next iterate $x(t+1)$ is obtained by approximating f by its linear approximation at $x(t)$ and setting the linear approximation $\hat{f}(x) = 0$.

Newton-Raphson iteration:

$$x(t+1) \;=\; x(t) \;-\; J^{-1}(x(t))\,f(x(t)) \tag{4.28}$$

In practice we usually do not evaluate the inverse $J^{-1}(x(t))$ except for very small systems. Instead we solve the linear equation (4.27) for $\Delta x(t)$. The next iterate is then $x(t+1) = x(t) + \Delta x(t)$.

We now apply this method to solve the power flow equations in the polar form. To illustrate the idea we consider the case where every bus in the network is either a slack bus (with V_0 specified and s_0 unknown), a *PV* bus (with $(p_j, |V_j|)$ specified and (q_j, θ_j) unknown), or a *PQ* bus (with (p_j, q_j) specified and $(\theta_j, |V_j|)$ unknown). The idea can be extended to more general cases. As mentioned before, (p_j, q_j) can be evaluated directly from the power flow equations once all $(\theta_j, |V_j|)$ are determined. Hence the main task is to solve for those $(\theta_j, |V_j|)$ that are not specified.

Let $N_{pq} \subseteq N$ be the set of *PQ* buses where $|V_j|$ (as well as θ_j) are unknown. We abuse notation and use N_{pq} to also denote the number $|N_{pq}|$ of buses in N_{pq}. Let

$$\begin{aligned} \theta &:= (\theta_j,\ j \in N) \\ |V| &:= (|V_j|,\ j \in N_{pq}) \end{aligned}$$

i.e., θ collects all unknown phase angles and $|V|$ collects all unknown voltage magnitudes. Rewrite (4.22) as (right-hand sides are given constants):

$$
\begin{aligned}
p_j(\theta, |V|) &= p_j, \qquad j \in N \\
q_j(\theta, |V|) &= q_j, \qquad j \in N_{pq}
\end{aligned}
$$

where we have abused notation to use (p_j, q_j) to denote both power injections and as functions of $(\theta, |V|)$ given by:

$$
p_j(\theta, |V|) := \left(\sum_{k=0}^{N} g_{jk}\right)|V_j|^2 - \sum_{k \neq j} |V_j||V_k| \left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right), \qquad j \in N \tag{4.29a}
$$

$$
q_j(\theta, |V|) := -\left(\sum_{k=0}^{N} b_{jk}\right)|V_j|^2 - \sum_{k \neq j} |V_j||V_k| \left(g_{jk}\sin\theta_{jk} - b_{jk}\cos\theta_{jk}\right), \qquad j \in N_{pq} \tag{4.29b}
$$

Here $\theta_{jk} := \theta_j - \theta_k$ and (g_{jk}, b_{jk}) are defined in (4.21). Define the function $f : \mathbb{R}^{N+N_{pq}} \to \mathbb{R}^{N+N_{pq}}$ by

$$
f(\theta, |V|) := \begin{bmatrix} \Delta p(\theta, |V|) \\ \Delta q(\theta, |V|) \end{bmatrix} := \begin{bmatrix} p(\theta, |V|) - p \\ q(\theta, |V|) - q \end{bmatrix} \tag{4.30}
$$

where $p := (p_j, j \in N)$, $q := (q_j, j \in N_{pq})$ are constants and

$$
p(\theta, |V|) := \begin{bmatrix} p_1(\theta, |V|) \\ \vdots \\ p_N(\theta, |V|) \end{bmatrix}, \qquad q(\theta, |V|) := \begin{bmatrix} q_1(\theta, |V|) \\ \vdots \\ q_N(\theta, |V|) \end{bmatrix}
$$

Our goal is to compute a root of $f(\theta, |V|) = 0$ iteratively. The Jacobian of f is the $(N + N_{pq}) \times (N + N_{pq})$ matrix

$$
J(\theta, |V|) := \begin{bmatrix} \frac{\partial p}{\partial \theta} & \frac{\partial p}{\partial |V|} \\ \frac{\partial q}{\partial \theta} & \frac{\partial q}{\partial |V|} \end{bmatrix} \tag{4.31}
$$

Hence the Newton-Raphson algorithm is:

1. Choose an initial point $(\theta(0), |V|(0))$.
2. Iterate until converge (or the maximum number of iterations has been reached):
   (a) Solve $(\Delta\theta(t), \Delta|V|(t))$ from

$$
J(\theta(t), |V|(t)) \begin{bmatrix} \Delta\theta(t) \\ \Delta|V|(t) \end{bmatrix} = -\begin{bmatrix} \Delta p(\theta(t), |V|(t)) \\ \Delta q(\theta(t), |V|(t)) \end{bmatrix} \tag{4.32}
$$

   (b) Set

$$
\begin{bmatrix} \theta(t+1) \\ |V|(t+1) \end{bmatrix} := \begin{bmatrix} \theta(t) \\ |V|(t) \end{bmatrix} + \begin{bmatrix} \Delta\theta(t) \\ \Delta|V|(t) \end{bmatrix}
$$

The right-hand side of (4.32) is defined in (4.30) and represents the mismatch in injections at iteration t. This mismatch is used to compute the increment $(\Delta\theta(t), \Delta|V|(t))$ that updates the current iterate $(\theta(t), |V|(t))$.

The Newton-Raphson algorithm is widely used in industry to compute power flow solution and solve optimal power flow problems. It converges, typically quadratically, to a solution if it starts close to a solution; see Kantorovich Theorem in Exercise 4.13. Like the Gauss-Seidel algorithm, it may not converge if the initial point is far away from a solution.

Remark 4.7. Usually the injection q_j at a *PV* bus j must be constrained within a range. After solving for $(\theta, |V|)$ and evaluating the resulting q_j at bus j, if it hits or exceeds its limit then q_j is set to the limit and bus j is re-classified as a *PQ* bus with $|V_j|$ (as well as θ_j) to be determined. The updated power flow equations are then re-solved for the remaining unknown quantities.

4.5.3 Fast decoupled algorithm

We now take a closer look at the Jacobian J in (4.31). Using (4.29) it can be shown that for the diagonal blocks (see Exercise 4.14):

$$\frac{\partial p_j}{\partial \theta_k} = \begin{cases} -|V_j||V_k|\left(g_{jk}\sin\theta_{jk} - b_{jk}\cos\theta_{jk}\right), & j \neq k,\ j,k \in N \\ -q_j(\theta,|V|) - \left(\sum_i b_{ji}\right)|V_j|^2, & j = k,\ j \in N \end{cases} \tag{4.33a}$$

$$\frac{\partial q_j}{\partial |V_k|} = \begin{cases} -|V_j|\left(g_{jk}\sin\theta_{jk} - b_{jk}\cos\theta_{jk}\right), & j \neq k,\ j,k \in N_{pq} \\ \frac{q_j(\theta,|V|)}{|V_j|} - \left(\sum_i b_{ji}\right)|V_j|, & j = k,\ j \in N_{pq} \end{cases} \tag{4.33b}$$

and for the off-diagonal blocks:

$$\frac{\partial p_j}{\partial |V_k|} = \begin{cases} -|V_j|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right), & j \neq k,\ j \in N, k \in N_{pq} \\ \frac{p_j(\theta,|V|)}{|V_j|} + \left(\sum_i g_{ji}\right)|V_j|, & j = k,\ j,k \in N_{pq} \end{cases} \tag{4.33c}$$

$$\frac{\partial q_j}{\partial \theta_k} = \begin{cases} |V_j||V_k|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right), & j \neq k,\ j \in N_{pq}, k \in N \\ p_j(\theta,|V|) - \left(\sum_i g_{ji}\right)|V_j|^2, & j = k,\ j \in N_{pq} \end{cases} \tag{4.33d}$$

From (4.21), $g_{jk} = b_{jk} = 0$ if buses j and k are not connected. Hence the corresponding off-diagonal entries of all the submatrices of the Jacobian J are zero. This means that the sparsity of the network graph induces a sparse Jacobian matrix J.

Moreover if line losses and angle differences θ_{jk} are small then it is reasonable to approximate $g_{jk} = 0$ and $\sin\theta_{jk} = 0$. In this case it can be verified that the off-diagonal blocks are approximately zero (see

Exercise 4.14), i.e.,

$$\frac{\partial p_j}{\partial |V_k|} \approx 0 \quad \text{and} \quad \frac{\partial q_j}{\partial \theta_k} \approx 0, \qquad \forall j,k$$

This means that the voltage magnitudes and the real power injections (at the same or different buses) are approximately decoupled, and the voltage angles and the reactive power injections are approximately decoupled. This motivates a fast decoupled algorithm where an approximate Jacobian $\hat{J}$ matrix with the off-diagonal blocks of J set to zero is used in place of J in the Newton-Raphson's algorithm (step 2):

$$\hat{J}(\theta, |V|) \;:=\; \begin{bmatrix} \frac{\partial p}{\partial \theta} & 0 \\ 0 & \frac{\partial q}{\partial |V|} \end{bmatrix}$$

Then equation (4.32) to compute the increments in the Newton-Raphson algorithm is replaced by the following equations that decouple active and reactive power:

$$\frac{\partial p}{\partial \theta}(\theta(t), |V|(t))\, \Delta\theta(t) \;=\; -\Delta p(\theta(t), |V|(t)) \tag{4.34a}$$

$$\frac{\partial q}{\partial |V|}(\theta(t), |V|(t))\, \Delta|V|(t) \;=\; -\Delta q(\theta(t), |V|(t)) \tag{4.34b}$$

There are other properties of J one can exploit to obtain symmetric matrices that saves storage and computation in executing the exact Newton-Raphson algorithm; see [1, p. 350–351]. The fast decoupled algorithm (4.34) can be further simplified with more approximations; see [1, p. 353–354].

4.6 Bibliographical notes

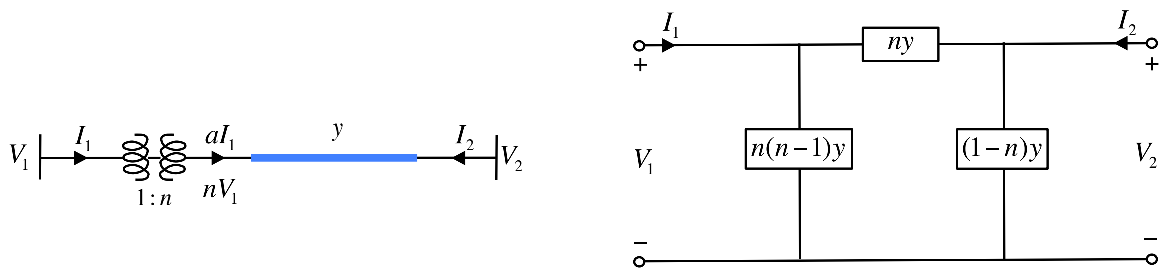
The description of LU decomposition to solve $I = YV$ and algorithms to compute power flow solutions are adapted from [1]. For properties of complex symmetric matrices such as the admittance matrix Y, see [10, Chapter 4.4]. For invertibility of Y, the first part of Theorem 4.2 is from [11, Lemma 1] though we have used properties of Schur complement to simplify its proof. See also [12, **?**].

The use of Newton-Raphson algorithm for solving power flow problems is first proposed in [13]. An implementation at BPA is reported in [14] with major improvements, especially a heuristic to optimize the order of Gaussian elimination of the Jacobian matrix in solving $J(x(t))\Delta x(t) = -f(x(t)$. A method is introduced in [15] that computes a new voltage solution $V' = V + \sum_l i_{j_l k_l}(e_{j_l} - e_{k_l})$ to $I = Y'V'$ in terms of the old voltage solution V to $I = YV$ when the admittance matrix changes from Y to Y' (line changes). The quantities $i_{j_l k_l}$ are called compensation currents and are computed from using the old admittance matrix Y. This method, well explained in [16], has the advantage of not having to factorize new matrix Y' into its LU decomposition when relatively few number of lines are changed. The Fast Decoupled algorithm is proposed in [17].

4.7 Problems

Chapter 4.2

Exercise 4.1 (Ideal transformer and transmission line)**.** Consider the cascade in the one-line diagram of Figure 4.13(a) of an *ideal* transformer with voltage gain n and a transmission line modeled by a series admittance y (and zero shunt admittances). Show that its external behavior is equivalent to that of the Π



(a) One-line diagram (b) Equivalent Π circuit model

Figure 4.13: An ideal transformer with turns ratio $a = n^{-1}$ followed by a transmission line modeled by a series admittance y.

circuit model in Figure 4.13(b).

Exercise 4.2 (Unitary diagonalizability of Y)**.** Suppose condition C4.1 holds. Let the bus admittance matrix $Y := G + \mathbf{i}B$ where G and B are real matrices (whose rows may not sum to zero).

1. Show that Y is normal (i.e., $YY^H = Y^HY$) and hence unitarily diagonalizable if and only if G and B commute, or if and only if BG is symmetric.

2. Suppose all lines have the same RX ratio, i.e., for some real α, $b^s_{jk} = \alpha g^s_{jk}$ for all $(j,k) \in E$ and $b^m_{jj} = \alpha g^m_{jj}$ for all $j \in \overline{N}$ (or all shunt elements are zero). Show that Y is normal. (Hint: Use part 1.)

Exercise 4.3 (Real Laplacian matrix)**.** Suppose the $n \times n$ admittance matrix Y of a connected graph is real symmetric with zero row sums (e.g., Y is the admittance matrix of a DC network), i.e., $Y_{jk} = Y_{kj} \le 0$ for all $j \ne k$ and $Y_{jj} = -\sum_{k:j\ne k} Y_{jk}$ for all j.

1. Show that rank $Y = n - 1$ and hence Y is not invertible and null(Y) = span$(\mathbf{1})$.

2. Show that the $(n-1) \times (n-1)$ matrix Y' obtained from Y by removing the jth row and column, for any j, has rank $n - 1$ and is hence invertible.

3. Give a counter-example to part 1 if Y is real symmetric but Y_{jk} have different signs for $j \ne k$.

Exercise 4.4 (Inverse of Y)**.** Consider a complex matrix $A =: G + \mathbf{i}B$ where $G, B \in \mathbb{R}^{n\times n}$. Show that, even if both G and B are singular, its inverse $A^{-1} =: R + \mathbf{i}X$ may exist though not given by the formulae (4.13b) or (4.14b). This is the case even if G and B are symmetric.

Exercise 4.5 (Invertibility of Y)**.** Prove part 2 of Theorem 4.2: For a complex symmetric matrix Y, if $\mathrm{Im}(Y) \prec 0$ then Y^{-1} exists and is symmetric. Moreover $\mathrm{Im}(Y^{-1}) \succ 0$.

Exercise 4.6 (Invertibility of Y, [11])**.** This is an alternative proof from [11, Lemma 1] of (part of) Theorem 4.2: a complex symmetric matrix Y is nonsingular if $\mathrm{Re}(Y) \succ 0$ or if $\mathrm{Im}(Y) \prec 0$. Prove the claim by showing that there exists no nonzero vector α such that $Y\alpha = 0$.

The next problem expresses the invertibility of the admittance matrix Y in terms of the invertibility of another matrix $\hat{Y}^s$ defined below. If Y is invertible then

$$Y^{-1} \;=\; (Y^m)^{-1} - (Y^m)^{-1}\left(C\left(\hat{Y}^s\right)^{-1}C^T\right)(Y^m)^{-1}$$

This is known as the *matrix inversion lemma*.

Exercise 4.7 (Invert Y using matrix inversion lemma)**.** Recall that, under condition C4.1, the admittance matrix Y can be written in terms of the incidence matrix C as (from (4.10)):

$$Y \;=\; CY^sC^T \;+\; Y^m$$

where $Y^s := \mathrm{diag}\left(y_l^s, l \in E\right)$ and $Y^m := \mathrm{diag}\left(y_{jj}^m, j \in \overline{N}\right)$. Suppose $y_l^s \neq 0$ for all l and $y_{jj}^m \neq 0$ for all j so that the diagonal matrices Y^s and Y^m are invertible. Show that Y is invertible if and only if the $M \times M$ matrix

$$\hat{Y}^s \;:=\; (Y^s)^{-1} + C^T (Y^m)^{-1} C$$

is invertible. (Hint: Use the property that a matrix is nonsingular if and only if a principal submatrix and its Schur complement are both nonsingular, according to Theorem 25.1 in Appendix 25.1.2.)

Exercise 4.8 (Invertibility of Y)**.** For any matrix $A \in \mathbb{C}^{n\times n}$, prove the following.

1. A is invertible if $v^HAv \neq 0$ for all nonzero $v \in \mathbb{C}^n$.
2. Show that the converse is not true by providing a counter-example A that is Hermitian (including real symmetric) and a counter-example A that is complex symmetric. (Hint: Consider 2×2 diagonal matrices.)
3. Suppose A is (Hermitian and) positive semidefinite. Then the following are equivalent:
   - A is invertible
   - $v^HAv \neq 0$ for all nonzero $v \in \mathbb{C}^n$.
   - A is positive definite.

Exercise 4.9 (Alternative proof of Theorem 4.3)**.** Consider the complex symmetric admittance matrix $Y \in \mathbb{C}^{(N+1)\times(N+1)}$. Let λ be an eigenvalue of Y and $\alpha \in \mathbb{C}^{N+1}$ a corresponding eigenvector. Then $\alpha^{\mathsf{H}} Y \alpha = \lambda ||\alpha||^2$ where $||\cdot||$ denotes the Euclidean norm. A sufficient (but not necessary) condition for Y to be invertible is that $\alpha^{\mathsf{H}} Y \alpha \neq 0$ for all nonzero vectors $\alpha \in \mathbb{C}^{N+1}$. Let $y^s_{jk} =: g^s_{jk} + \mathbf{i} b^s_{jk}$, $y^m_{jj} =: g^m_{jj} + \mathbf{i} b^m_{jj}$.

1. Suppose condition C4.1 holds. Show that

$$\alpha^{\mathsf{H}} Y \alpha \;=\; \left(\sum_{(j,k)\in E} g^s_{jk} \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in\overline{N}} g^m_{jj} |\alpha_j|^2 \right) + \mathbf{i} \left(\sum_{(j,k)\in E} b^s_{jk} \left|\alpha_j - \alpha_k\right|^2 + \sum_{j\in\overline{N}} b^m_{jj} |\alpha_j|^2 \right)$$

2. Show that the conditions in Theorem 4.3 imply that $\alpha^{\mathsf{H}} Y \alpha \neq 0$ for all nonzero vectors $\alpha \in \mathbb{C}^{N+1}$.

Exercise 4.10 (Kron reduction)**.** Given an admittance matrix Y and its Kron-reduction Y/Y_{22} (assume Y_{22} is invertible):

$$Y \;=:\; \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathsf{T}} & Y_{22} \end{bmatrix}, \qquad\qquad Y/Y_{22} \;:=\; Y_{11} \,-\, Y_{12} Y_{22}^{-1} Y_{12}^{\mathsf{T}}$$

1. Show that if Y has zero row (and hence column) sums, i.e., $y^m_{jk} = y^m_{kj} = 0$ for $(j,k) \in E$, so does Y/Y_{22}.

2. Show that the converse does not necessarily hold.

Chapter 4.5

Exercise 4.11 (Gauss algorithm)**.** Consider solving for the roots of

$$g(x) \;=\; ax^2 - x \tag{4.35}$$

i.e., finding x such that $g(x) = 0$. An x is a root of g if and only if it is a fixed point of $f(x) := ax^2$, i.e., if and only if $x = f(x)$. The Gauss algorithm computes a fixed point of $f(x)$ by performing the fixed-point iteration:

$$x(t+1) \;:= f(x(t)) \tag{4.36}$$

Let $X \subseteq \mathbb{R}$ be closed and convex and suppose f maps X into X. We say f is a *contraction mapping* on X if there exists an $\alpha \in [0,1)$ such that

$$|f(y) - f(x)| \;\le\; \alpha\,|y-x|, \qquad \text{for all } x,y \in X \tag{4.37}$$

If f is a contraction mapping on X then there is a unique fixed point $x^* \in X$ and the fixed-point iteration (4.36) always converges to x^*, starting from any initial point $x(0) \in X$.

1. What are the roots of g in (4.35)?
2. Whenever $|a| < 1$, f maps $X := [-1,1]$ into X. Show that f is a contraction mapping on X if and only if $|a| < 1/2$. In that case, what is the root of g that (4.36) computes?
3. Show that (4.36) converges to $x^* = 0$ if and only if $x(0)$ satisfies $|ax(0)| < 1$.
4. Is it necessary for f to be a contraction mapping for the Gauss algorithm (4.36) to compute a root of g? What is the advantage, if any, if f is indeed a contraction mapping?

Exercise 4.12 (Newton algorithm)**.** The Newton algorithm solves iteratively for $x \in \mathbb{R}^n$ such that $g(x) = 0$ where $g : \mathbb{R}^n \to \mathbb{R}^n$. In each iteration, it approximates g by its linearization at the current iterate $x(t)$ and moves to $x(t+1)$ where the linearization vanishes. Show that if g is linear, $g(x) = Ax + b$ where A is invertible, then the Newton algorithm solves $g(x) = 0$ in one step wherever it starts.

Exercise 4.13 (Kantorovich Theorem)**.** The Newton algorithm converges if the initial point is close to a solution. This is made precise by the Kantorovich Theorem. Consider $g : D \to \mathbb{R}^n$ where $D \subseteq \mathbb{R}^n$ is an open convex set. Suppose g is differentiable on D and ∇g is Lipschitz on D, i.e., there is an L such that

$$\|\nabla g(y) - \nabla g(x)\| \;\; \leq \;\; L\|y - x\|, \qquad \text{for all } x, y \in D$$

where $[\nabla g(x)]_{ij} := \frac{\partial g_i}{\partial x_j}(x)$. Suppose $x_0 \in D$ and that $\nabla g(x_0)$ is invertible. Let

$$\beta \;\geq\; \left\| (\nabla g(x_0))^{-1} \right\|, \qquad \eta \;\geq\; \left\| (\nabla g(x_0))^{-1}\, g(x_0) \right\|$$

$$h \;:=\; \beta \eta L, \qquad r \;:=\; \frac{1 - \sqrt{1 - 2h}}{h}\, \eta$$

The Kantorovich Theorem says that if the closed ball $B_r(x_0) \subseteq D$ and $h \leq 1/2$ then the Newton iteration

$$x(t+1) \;\; := \;\; x(t) \; - \; (\nabla g(x(t)))^{-1}\, g(x(t))$$

converges to a solution x^* of $g(x) = 0$ in the closed ball $B_r(x_0)$.

1. Apply the Kantorovich Theorem to $g(x) := ax^2 - x$ to prove that the Newton iterates converge to a root of g if the initial point x_0 satisfies either of the following conditions, assuming $a > 0$:

   $$x_0 \;\leq\; \frac{1}{2a}\left(1 - \frac{1}{\sqrt{2}}\right) \quad \text{or} \quad x_0 \;\geq\; \frac{1}{2a}\left(1 + \frac{1}{\sqrt{2}}\right)$$

   Which root will the Newton iteration compute in each case?
2. The Kantorovich Theorem provides only a sufficient condition for convergence of the Newton iterates. Show that, for $g(x) := ax^2 - x$, as long as $x_0 \neq (2a)^{-1} = \min_x g(x)$, the Newton iterates will converge. (Hint: use part 1.)

Exercise 4.14 (Fast decoupled algorithm). 1. Use (4.29) to prove (4.33) reproduced here:

$$\frac{\partial p_j}{\partial \theta_k} = \begin{cases} -|V_j||V_k|\left(g_{jk}\sin\theta_{jk} - b_{jk}\cos\theta_{jk}\right), & j \neq k,\ j,k \in N \\ -q_j(\theta,|V|)\ -\ \left(\sum_i b_{ji}\right)|V_j|^2, & j = k,\ j \in N \end{cases}$$

$$\frac{\partial q_j}{\partial |V_k|} = \begin{cases} -|V_j|\left(g_{jk}\sin\theta_{jk} - b_{jk}\cos\theta_{jk}\right), & j \neq k,\ j,k \in N_{pq} \\ \frac{q_j(\theta,|V|)}{|V_j|}\ -\ \left(\sum_i b_{ji}\right)|V_j|, & j = k,\ j \in N_{pq} \end{cases}$$

$$\frac{\partial p_j}{\partial |V_k|} = \begin{cases} -|V_j|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right), & j \neq k,\ j \in N, k \in N_{pq} \\ \frac{p_j(\theta,|V|)}{|V_j|}\ +\ \left(\sum_i g_{ji}\right)|V_j|, & j = k,\ j,k \in N_{pq} \end{cases}$$

$$\frac{\partial q_j}{\partial \theta_k} = \begin{cases} |V_j||V_k|\left(g_{jk}\cos\theta_{jk} + b_{jk}\sin\theta_{jk}\right), & j \neq k,\ j \in N_{pq}, k \in N \\ p_j(\theta,|V|)\ -\ \left(\sum_i g_{ji}\right)|V_j|^2, & j = k,\ j \in N_{pq} \end{cases}$$

2. Show that if $g_{jk} = 0$ and $\sin\theta_{jk} = 0$ then the Jacobian reduces to the approximating block-diagonal matrix:

$$\hat{J}(\theta,|V|)\ :=\ \begin{bmatrix} \frac{\partial p}{\partial \theta} & 0 \\ 0 & \frac{\partial q}{\partial |V|} \end{bmatrix}$$

Chapter 5

Branch flow models

In this chapter we introduce several forms of the branch flow model. Whereas a bus injection model consists of only nodal variables (power and current injections and voltages), a branch flow model involves also branch power flows and branch currents. We present branch flow models in complex form, real form, for general networks in Chapter 5.1 and radial networks in Chapter 5.2, with and without shunt admittances of the lines. We prove in Chapter 5.3 the equivalence of this set of models and the bus injection model of Chapter 4. For radial networks we describe in Chapter 5.4 a fast iterative algorithm, the backward forward sweep, to compute a power flow solution. Finally we present in Chapter 5.5 a linearized model for radial networks and illustrate its application to volt/var control.

Branch flow models were originally proposed for radial networks and have been extended to general networks with cycles. These models have two important advantages when specialized to radial networks: the backward forward sweep for power flow computation and a linearized model that admits an explicit solution and bounds on nonlinear branch powers and voltage magnitudes.

5.1 General network

5.1.1 Line model

As in Chapter 4 we model a power network with $N+1$ buses and M lines as a connected undirected graph $G=(\overline{N},E)$ where $\overline{N}:=\{0\}\cup N$, $N:=\{1,2,\ldots,N\}$ and $E\subseteq\overline{N}\times\overline{N}$; see Figure 4.7. For each bus $j\in\overline{N}$, let V_j its voltage phasor and s_j its complex power injection. For each line $(j,k)\in E$, let (I_{jk},I_{kj}) denote the *sending-end* line currents from buses j to k and buses k to j respectively. Similarly let (S_{jk},S_{kj}) denote the *sending-end* line power flows in each direction. Let $V:=(V_j,\,j\in\overline{N})$, $s:=(s_j,\,j\in\overline{N})$, $I:=(I_{jk},I_{kj},(j,k)\in E)$, and $S:=(S_{jk},S_{kj},(j,k)\in E)$.

Each line $(j,k)\in E$ is characterized by two series admittances and two shunt admittances, $\left(y^s_{jk},y^m_{jk}\right)\in\mathbb{C}^2$ from j to k and $\left(y^s_{kj},y^m_{kj}\right)\in\mathbb{C}^2$ from k to j. They define the relation between $\left(V_j,V_k\right)$ and $\left(I_{jk},I_{kj}\right)$ (see (5.1c)(5.1d) below). A line may model a transmission or distribution line, a single-phase transformer, the per-phase model of a three-phase transformer in balanced setting, and may contain admittances of sources and loads. When it models a transformer with real voltage gain, y^m_{jk} and y^m_{kj} are in general different. When the voltage gain is complex, y^s_{jk} and y^s_{kj} may also be different. (See Chapter 4.2.2 for more details.) Despite the appearance in Figure 5.1, this general line model does not have a Π circuit representation when $y^s_{jk} \neq y^s_{kj}$. Let $z^s_{jk} := \left(y^s_{jk}\right)^{-1}$ and $z^s_{kj} := \left(y^s_{kj}\right)^{-1}$.

We will often restrict ourselves to the special case where the series admittances are equal $y^s_{jk} = y^s_{kj}$, and characterize a line by three admittances $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$. This model can be represented as a Π circuit and behaves like a transmission or distribution line though with generally different y^m_{jk} and y^m_{kj}; see Figure 5.1. It cannot be used as the per-phase model of a balanced three-phase transformer in ΔY or $Y\Delta$ configuration

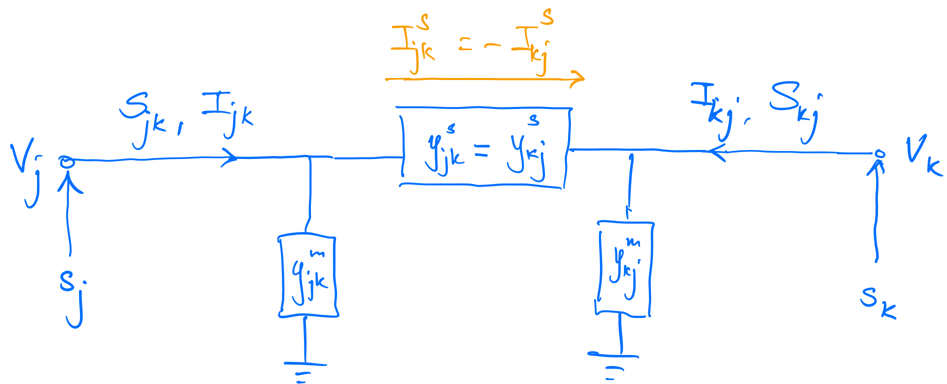


Figure 5.1: Line model under assumption C5.1.

that has a complex voltage gain $K(n)$, but is still widely applicable.

As in Chapter 4 we label the following assumption and will explicitly state it when it is required:

C5.1: The series admittances $y^s_{jk} = y^s_{kj}$ or equivalently the series impedances $z^s_{jk} = y^s_{kj}$ for every line $(j,k) \in E$.

5.1.2 Complex form

The branch flow model (BFM) in the complex form is defined by the following power flow equations in the variables $(s, V, I, S) \in \mathbb{C}^{2(N+1)+4M}$ (from (4.1)(4.2)):

$$s_j = \sum_{k: j \sim k} S_{jk}, \qquad j \in \overline{N} \tag{5.1a}$$

$$S_{jk} = V_j I^{\mathsf{H}}_{jk}, \quad S_{kj} = V_k I^{\mathsf{H}}_{kj}, \qquad (j,k) \in E \tag{5.1b}$$

$$I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk} V_j, \qquad (j,k) \in E \tag{5.1c}$$

$$I_{kj} = y^s_{kj}(V_k - V_j) + y^m_{kj} V_k, \qquad (j,k) \in E \tag{5.1d}$$

where (5.1a) imposes power balance at each bus, (5.1b) defines branch power in terms of the associated voltage and current, and (5.1c)(5.1d) describes Kirchhoff's and Ohm's laws. For convenience we include

V_0 in the vector variable $V := (V_j, j \in \overline{N})$ with the understanding that $V_0 := 1\angle 0°$ is fixed. This model does not require assumption C5.1.

As we will see in Chapter 5.3 this model serves as a bridge between the bus injection model of Chapter 4 in complex form and the branch flow models in real domain in the rest of this chapter.

5.1.3 Real form

A branch flow model, called the DistFlow equations, is proposed in [18, 19] for radial networks. Its key feature is that it does not involve phase angles of voltage and current phasors. For each bus j let

- $s_j := (p_j, q_j)$ and $s_j := (p_j + \mathbf{i}q_j)$ represent the real and reactive power injections at bus j;[1]
- v_j represent the squared voltage magnitude at bus j.

For each line (j,k) let

- $S_{jk} = (P_{jk}, Q_{jk})$ and $S_{jk} = P_{jk} + \mathbf{i}Q_{jk}$ represent the *sending-end* real and reactive branch power flow from bus j to bus k, and S_{kj} represent the sending-end power from k to j;
- ℓ_{jk} represent the squared magnitude of the *sending-end* current from bus j to bus k, and ℓ_{kj} represent the squared current magnitude from k to j.

The variables $v := (v_j, j \in \overline{N})$ and $\ell := (\ell_{jk}, \ell_{kj}, (j,k) \in E)$ will replace the phasors V and I in the model (5.1). The power flow equations below therefore are in terms of a real vector $x := (s, v, \ell, S) \in \mathbb{R}^{3(N+1)+6M}$ that does not involve voltage and phase angles as variables.

The angle information is however embedded in x. Define for each $(j,k) \in E$

$$\begin{aligned} z^s_{jk} &:= \left(y^s_{jk}\right)^{-1} =: z^s_{kj} \\ \alpha_{jk} &:= 1 + z^s_{jk} y^m_{jk}, \qquad \alpha_{kj} := 1 + z^s_{kj} y^m_{kj} \end{aligned}$$

Note that $\alpha_{jk} = \alpha_{kj}$ if and only if $y^m_{jk} = y^m_{kj}$ and $\alpha_{jk} = \alpha_{kj} = 1$ if and only if $y^m_{jk} = y^m_{kj} = 0$ since $|z^s_{jk}| \neq 0$. Given any x define the vector $\beta(x) \in \mathbb{R}^{2M}$ of line angles as a function of x by

$$\beta_{jk}(x) := \angle\left(\alpha^{\mathsf{H}}_{jk} v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right), \qquad (j,k) \in E \tag{5.2a}$$

$$\beta_{kj}(x) := \angle\left(\alpha^{\mathsf{H}}_{kj} v_k - \left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right), \qquad (j,k) \in E \tag{5.2b}$$

Using (5.1b)(5.1c)(5.1d), it can be shown that, if x is a power flow solution, then $(\beta_{jk}(x), \beta_{kj}(x))$ are voltage angle differences across line (j,k) (Exercise 5.1), i.e.,

$$\beta_{jk}(x) = \angle V_j - \angle V_k, \qquad \beta_{kj}(x) = \angle V_k - \angle V_j, \qquad (j,k) \in E$$

[1] We abuse notation and use s to denote both the complex power injection $s = (p + \mathbf{i}q)$ and the real pair $s = (p,q)$, depending on the context. Similarly for $S = (P + \mathbf{i}Q)$ and $S = (P,Q)$, and for $z = (r + \mathbf{i}x)$ and $z = (r,x)$.

This implies in particular that $\beta_{jk}(x) = -\beta_{kj}(x)$, even in the absence of assumption C5.1.

The following branch flow model relaxes the angles of voltages and currents and are applicable to general networks:

$$\begin{aligned}
s_j &= \sum_{k:j\sim k} S_{jk}, \qquad j \in \overline{N} && (5.3a)\\
\left|S_{jk}\right|^2 &= v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 \;=\; v_k \ell_{kj}, \qquad (j,k) \in E && (5.3b)\\
\left|\alpha_{jk}\right|^2 v_j - v_k &= 2\,\mathrm{Re}\left(\alpha_{jk}\left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right) - \left|z^s_{jk}\right|^2 \ell_{jk}, \qquad (j,k) \in E && (5.3c)\\
\left|\alpha_{kj}\right|^2 v_k - v_j &= 2\,\mathrm{Re}\left(\alpha_{kj}\left(z^s_{kj}\right)^{\mathsf{H}} S_{kj}\right) - \left|z^s_{kj}\right|^2 \ell_{kj}, \qquad (j,k) \in E && (5.3d)\\
\text{there exists } \theta \in \mathbb{R}^{N+1} \text{ s.t. } \beta_{jk}(x) &= \theta_j - \theta_k, \qquad (j,k) \in E && (5.3e)\\
\beta_{kj}(x) &= \theta_k - \theta_j, \qquad (j,k) \in E && (5.3f)
\end{aligned}$$

where $\beta_{jk}(x)$ and $\beta_{kj}(x)$ are defined in (5.2). Equation (5.3a) expresses power balance at each bus and is a shorthand for the real equations:

$$p_j \;=\; \sum_{k:j\sim k} P_{jk}, \qquad q_j \;=\; \sum_{k:j\sim k} Q_{jk}, \qquad\qquad j \in \overline{N}$$

Equation (5.3b) defines apparent branch power and follows from (5.1b). The relationship (5.3c)(5.3d) originates from KCL and Ohm's law in (5.1c)(5.1d); see (5.17) in the proof of Theorem 5.2. We call (5.3e)(5.3f) the *cycle condition* and it ensures that the line angles implied by x can indeed be realized by nodal voltage angles. This means that the model (5.3), which does not include phase angles, is consistent with the model (5.1), which does. The cycle condition can also be expressed in vector form in terms of the bus-by-line incidence matrix C as:

$$\beta(x) \;=\; \begin{bmatrix} C^{\mathsf{T}} \\ -C^{\mathsf{T}} \end{bmatrix} \theta, \qquad \text{for some } \theta \in \mathbb{R}^{N+1}$$

A vector x is called a *power flow solution* if it satisfies (5.3) with $v \geq 0$ and $\ell \geq 0$. Given a power flow solution x we can recover the voltage and current phasors; see (5.11) in Chapter 5.2.

We emphasize that, despite the complex notation, (5.3) is a set of $2(N+1)+6M$ real equations in $3(N+1)+6M$ real variables $x := (s, v, \ell, S) = (p_j, q_j, v_j, \ell_{jk}, \ell_{kj}, P_{jk}, P_{kj}, Q_{jk}, Q_{kj},\ j \in \overline{N},\ (j,k) \in E)$. The power flow problem is: given $N+1$ of these variables, determine the remaining $2(N+1)+6M$ variables from these power flow equations. Equations (5.3b) are quadratic, the cycle condition is nonlinear, and the rest are linear in x. This model does not require assumption C5.1.

Remark 5.1. Branch flow models have been most useful for radial networks, which is the focus of the rest of this chapter. These models for radial networks are all special cases of the general model (5.3). Even though the branch flow models (5.3)(5.1) and the bus injection model (4.20a) are defined by different sets of equations in terms of their own variables, all of them are models of Kirchhoff's and Ohm's laws. We will show in Chapter 5.3 that these models are indeed equivalent in a precise sense. □

5.2 Radial network

In this section we assume the network graph G is a tree. The cycle condition (5.3e)(5.3f) for general networks is highly nonlinear in the variable x. When the network graph is a tree, the cycle condition can be replaced by a linear condition on x. When shunts admittances are assumed zero then the cycle condition becomes vacuous.

5.2.1 With shunt admittances

The major simplification for radial networks is the replacement of the nonlinear cycle condition (5.3e)(5.3f) by the following linear *cycle condition* on a power flow solution x:

$$\alpha_{jk}^{\mathsf{H}} v_j \;-\; \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} \;\;=\;\; \left(\alpha_{kj}^{\mathsf{H}} v_k \;-\; \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}, \qquad (j,k) \in E$$

This leads to the following branch flow model for radial networks that generalizes the original DistFlow equations of [18, 19] to allow shunt admittances of lines:

$$\begin{aligned}
s_j \;&=\; \sum_{k:j\sim k} S_{jk}, \qquad j \in \overline{N} && \text{(5.4a)}\\
\left|S_{jk}\right|^2 \;&=\; v_j \ell_{jk}, \quad \left|S_{kj}\right|^2 \;=\; v_k \ell_{kj}, \qquad (j,k) \in E && \text{(5.4b)}\\
\left|\alpha_{jk}\right|^2 v_j - v_k \;&=\; 2\,\mathrm{Re}\left(\alpha_{jk}\left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}\right) - \left|z_{jk}^s\right|^2 \ell_{jk}, \qquad (j,k) \in E && \text{(5.4c)}\\
\left|\alpha_{kj}\right|^2 v_k - v_j \;&=\; 2\,\mathrm{Re}\left(\alpha_{kj}\left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right) - \left|z_{kj}^s\right|^2 \ell_{kj}, \qquad (j,k) \in E && \text{(5.4d)}\\
\alpha_{jk}^{\mathsf{H}} v_j \;-\; \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk} \;&=\; \left(\alpha_{kj}^{\mathsf{H}} v_k \;-\; \left(z_{kj}^s\right)^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}, \qquad (j,k) \in E && \text{(5.4e)}
\end{aligned}$$

where we recall that v_j represents the squared voltage magnitude at bus j. We will show in Theorem 5.2 below that they are equivalent when the network is radial, i.e., an x satisfies the nonlinear cycle condition (5.3e)(5.3f) if and only if it satisfies the linear cycle condition (5.4e). The model (5.4) is a set of $2(N+1)+6M$ real equations in the vector x of $3(N+1)+6M$ real variables ($M=N$ since G is a tree). All equations are linear in x except (5.4b) which are quadratic. This model does not require assumption C5.1.

5.2.2 Without shunt admittances

Consider a radial network where lines have zero shunt admittances and hence $\alpha_{jk}=\alpha_{kj}=1$. Moreover we suppose assumption C5.1 holds. A consequence of substituting $z_{jk}^s = z_{kj}^s$ and $y_{jk}^m = y_{kj}^m = 0$ into (5.4) for all lines $(j,k) \in E$ is the relation between the sending-end power flows S_{jk} and S_{kj} (see Exercise 5.3):

$$S_{jk} \;+\; S_{kj} \;\;=\;\; z_{jk}^s \ell_{jk} \;\;=\;\; z_{jk}^s \ell_{kj} \tag{5.5}$$

It says that the sum of sending-end power flows is equal to the complex line loss across the series impedance z_{jk}^s. We can use this relation to express $\ell_{kj}=\ell_{jk}$ and $S_{kj} = z_{jk}^s \ell_{jk} - S_{jk}$ in terms of (ℓ_{jk}, S_{jk})

and eliminate branch variables (ℓ_{kj}, S_{kj}) in the opposite direction from (5.4). This leads to a simpler set of equations based on a directed, rather than undirected, graph G, as we now explain. In particular the linear cycle condition (5.4e) becomes vacuous.

In this subsection we assume $G = (\overline{N}, E)$ is directed. We denote a line in E from bus j to bus k either by $(j,k) \in E$ or $j \to k \in E$. Associated with each line $j \to k \in E$ are branch variables (ℓ_{jk}, S_{jk}). It is important to remember that, unlike models in the previous sections, (ℓ_{kj}, S_{kj}) in the opposite direction are not defined in the models in this subsection, unless otherwise specified. Let $(s,v) := (s_j, v_j, j \in \overline{N})$ and $(\ell, S) := (\ell_{jk}, S_{jk}, j \to k \in E)$. Let $x := (s, v, \ell, S)$ in $\mathbb{R}^{3(N+1+M)}$ with $M = N$ since G is a tree. Without loss of generality we take bus 0 as the root of the tree. Even though the graph orientation can be arbitrary we discuss two particularly convenient graph orientations: one where every line points *away from* bus 0 and the other where every line points *towards* bus 0; see Figure 5.2. For every bus j there is a unique node i

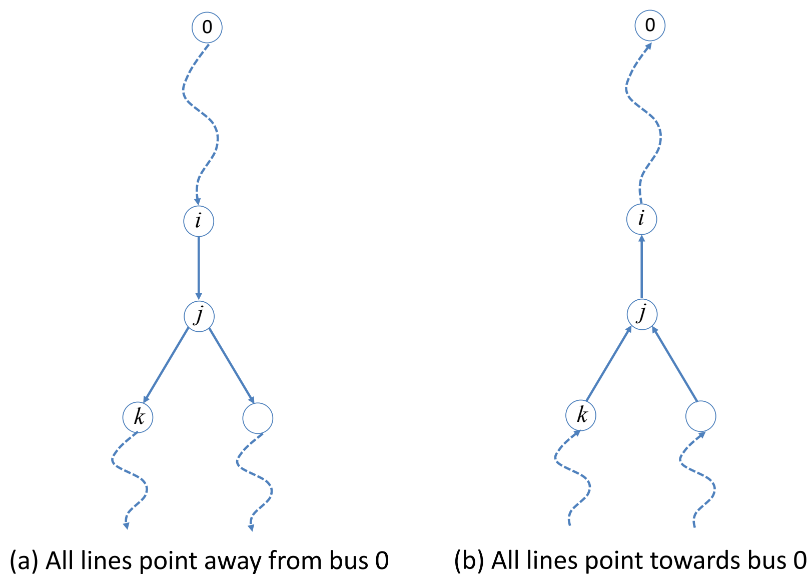


Figure 5.2: Notation for BFM for radial networks.

that is adjacent to j on the path from bus 0 to bus j. We present two sets of power flow equations, one for each graph orientation. These two models are equivalent in the sense that there is a bijection F that maps x to $\hat{x} = F(x)$ such that x is a solution to the first set of equations if and only if $\hat{x}$ is a solution to the second set of equations. A power flow solution x contains no voltage and current angles, but we will explain below how to recover these angles from x using (5.11). To simplify notation we omit the superscript and write $z_{jk} = (r_{jk}, x_{jk}) = \left(y^s_{jk}\right)^{-1}$ as the series impedance of line (j,k).

Down orientation: lines point away from bus 0. Under assumption C5.1, we can substitute $\ell_{kj} = \ell_{jk}$ and $S_{kj} = z^s_{jk}\ell_{jk} - S_{jk}$ into (5.4) to eliminate branch variables (ℓ_{kj}, S_{kj}) and obtain a simpler set of equations for BFM for a radial network without line shunts:

$$\begin{aligned} \sum_{k:j\to k} S_{jk} &= S_{ij} - z_{ij}\ell_{ij} + s_j, \qquad j \in \overline{N} && (5.6a)\\ v_j - v_k &= 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right) - |z_{jk}|^2 \ell_{jk}, \qquad j \to k \in E && (5.6b)\\ v_j \ell_{jk} &= |S_{jk}|^2, \qquad j \to k \in E && (5.6c) \end{aligned}$$

where, in (5.6a), bus $i := i(j)$ denotes the unique adjacent node of j on the path from node 0 to node j, with the understanding that when $j = 0$ then $S_{i0} = 0$ and $\ell_{i0} = 0$. When j is a leaf node[2], all $S_{jk} = 0$ in (5.6a). Here we recall that v_j represents the squared voltage magnitude at bus j. Again this model adopts a directed graph where branch variables (ℓ_{jk}, S_{jk}) are defined only for the direction of the lines $j \to k \in E$, not for the opposite directions. The vector v includes v_0 and s includes s_0. This model is first proposed in [18, 19] for radial networks and is called the DistFlow equations.

Despite the complex notation, (5.6) is a set of $2(N+1+M)$ real equations in $3(N+1+M)$ real variables $x = (p_i, q_i, v_i, \ell_{jk}, P_{jk}, Q_{jk})$ and a shorthand for:

$$\begin{aligned} \sum_{k:j\to k} P_{jk} &= \left(P_{ij} - r_{ij}\ell_{ij}\right) + p_j, \qquad j \in \overline{N}\\ \sum_{k:j\to k} Q_{jk} &= \left(Q_{ij} - x_{ij}\ell_{ij}\right) + q_j, \qquad j \in \overline{N}\\ v_j - v_k &= 2\left(r_{jk}P_{jk} + x_{jk}Q_{jk}\right) - (r_{jk}^2 + x_{jk}^2)\ell_{jk}, \qquad j \to k \in E\\ v_j \ell_{jk} &= P_{jk}^2 + Q_{jk}^2, \qquad j \to k \in E \end{aligned}$$

Since $M = N$, there are $(4N+2)$ equations in $(6N+3)$ real variables. Given $(2N+1)$ of these variables (e.g., given $v_0 = 1$ and non-slack bus injections (p_j, q_j), $j \in N$), the power flow problem is to determine the remaining $4N+2$ variables from these equations. There can be zero, one or more than one solutions.

Up orientation: lines point towards bus 0. When the graph orientation is opposite to that in Case 1, BFM is specified by the following equations in $\overline{x} := (\overline{s}, \overline{v}, \overline{\ell}, \overline{S})$:

$$\begin{aligned} \overline{S}_{ji} &= \sum_{k:k\to j} \left(\overline{S}_{kj} - z_{kj}\overline{\ell}_{kj}\right) + \overline{s}_j, \qquad j \in \overline{N} && (5.7a)\\ \overline{v}_k - \overline{v}_j &= 2\,\mathrm{Re}\left(z_{kj}^{\mathsf{H}} \overline{S}_{kj}\right) - |z_{kj}|^2 \overline{\ell}_{kj}, \qquad k \to j \in E && (5.7b)\\ \overline{v}_k \overline{\ell}_{kj} &= |\overline{S}_{kj}|^2, \qquad k \to j \in E && (5.7c) \end{aligned}$$

where $i := i(j)$ in (5.7a) denotes the node adjacent to j on the unique path between node 0 and node j. The boundary condition is defined by $\overline{S}_{ji} = 0$ in (5.7a) when $j = 0$ and $\overline{S}_{kj} = 0, \overline{\ell}_{kj} = 0$ in (5.7a) when j is a leaf node. For an advantage of this orientation see Remark 5.2. As for the down orientation, (5.7) is a valid model only under assumption C5.1.

[2] A node j is a *leaf* node if there exists no k such that $j \to k \in E$.

General orientation. For general graph orientation, the power flow equations in x are:

$$\sum_{k:j\to k} S_{jk} = \sum_{i:i\to j} \left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j, \qquad j \in \overline{N} \tag{5.8a}$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right) - |z_{jk}|^2 \ell_{jk}, \qquad j \to k \in E \tag{5.8b}$$

$$v_j \ell_{jk} = |S_{jk}|^2, \qquad j \to k \in E \tag{5.8c}$$

The power flow equations (5.8) for different graph orientations are the same, but their boundary conditions may be different. As explained at the beginning of this section, (5.8) is a valid model only under assumption C5.1.

Intuitively nodal injections and voltages (s, v) should not depend on the orientation of the graph while branch currents and power (ℓ, S) do, since branch variables are defined only in the direction of the lines, not in the opposite direction. We can formally relate the power flow solutions defined for opposite graph orientations. Specifically, consider the opposite orientation where the direction of every line is reversed from that in (5.8). The resulting power flow equations are:

$$\sum_{k:j\to k} \hat{S}_{jk} = \sum_{i:i\to j} \left(\hat{S}_{ij} - z_{ij}\hat{\ell}_{ij}\right) + \hat{s}_j, \qquad j \in \overline{N} \tag{5.9a}$$

$$\hat{v}_k - \hat{v}_j = 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} \hat{S}_{kj}\right) - |z_{jk}|^2 \hat{\ell}_{kj}, \qquad k \to j \in E \tag{5.9b}$$

$$\hat{v}_k \hat{\ell}_{kj} = |\hat{S}_{kj}|^2, \qquad k \to j \in E \tag{5.9c}$$

An example is the down and up orientations above. Then it can be shown that there is a bijection g such that x is a power flow solution of (5.8) if and only if $\hat{x} := g(x)$ is a power flow solution of (5.9) (Exercise 5.4). Indeed $\hat{x} = g(x)$ is given by:

$$\hat{s}_j := s_j, \qquad \hat{v}_j := v_j, \qquad \hat{\ell}_{kj} := \ell_{jk}, \qquad \hat{S}_{kj} := -\left(S_{jk} - z_{jk}\ell_{jk}\right) \tag{5.10}$$

Angle recovery, cycle condition. We first show that, for a radial network, the linear cycle condition (5.4e) becomes vacuous when the shunt admittances $y_{jk}^m = y_{kj}^m = 0$ and C5.1 holds. Then we explain how to use this fact to recover phase angles for voltage and current phasors (V, I) as well as line flows S.

As explained above C5.1 allows us to adopt a directed network graph $G := (\overline{N}, E)$ with an arbitrary orientation. Without loss of generality we assume G is connected and hence $M = N$. Let $C \in \{-1, 0, 1\}^{(N+1)\times N}$ denote the bus-by-line incidence matrix defined by:

$$C_{jl} = \begin{cases} 1 & \text{if } l = j \to k \text{ for some bus } k \\ -1 & \text{if } l = i \to j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

where $j \to k = (j, k)$ denotes a directed line in G from bus j to bus k. Let $\beta(x) := \left(\beta_{jk}(x), j \to k \in E\right)$, i.e., $\beta(x)$ contains only $\beta_{jk}(x)$ for all directed lines $j \to k$, but not $\beta_{kj}(x) = -\beta_{jk}(x)$ in the opposite direction. It is proved in Theorem 5.2 below that the cycle condition (5.4e) is equivalent to (5.3e)

$$\beta(x) = C^{\mathsf{T}}\theta \qquad \text{for some } \theta \in \mathbb{R}^{N+1} \tag{5.11a}$$

where the entries $\beta_{jk}(x)$ of $\beta(x)$ are given by (5.2a), reproduced here:

$$\beta_{jk}(x) \;:=\; \angle\left(v_j - \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk}\right), \qquad j \to k \in E$$

The $N \times (N+1)$ matrix C^{T} has rank $N = M$ for a (connected) radial network. The null space of C^{T} is span($\mathbf{1}$) and its pseudo-inverse $\left(C^{\mathsf{T}}\right)^{\dagger} = C\left(C^{\mathsf{T}}C\right)^{-1}$ (Exercise 5.2 shows that C^{T} has full row rank and its pseudo-inverse is therefore given by Corollary 25.17.2 of Appendix 25.1.6). Given a power flow solution x, a solution to (5.11a) is

$$\theta \;=\; C\left(C^{\mathsf{T}}C\right)^{-1}\beta(x) \;+\; \phi\mathbf{1} \tag{5.11b}$$

for an arbitrary angle $\phi \in \mathbb{R}$. The angle ϕ is fixed by the assumption that $\theta_0 := 0$. Hence we have shown that a power flow solution x that satisfies the DistFlow equation (5.6) or (5.7) or (5.8) also satisfies (5.11a) which is equivalent to the cycle condition (5.4e). Hence (5.4e) is vacuous.

To recover the voltage and current phasors, pick any solution $\theta(x)$ in (5.11b), and without loss of generality, we can project it to $\theta_j(x) \in (-\pi, \pi]$. The voltage and current phasors (V, I) can then be obtained in terms of x as:

$$V_j \;:=\; \sqrt{v_j}\, e^{\mathbf{i}\theta_j(x)}, \qquad I_{jk} \;:=\; \sqrt{\ell_{jk}}\, e^{\mathbf{i}\left(\theta_j(x) - \angle S_{jk}\right)} \tag{5.11c}$$

where $\angle S_{jk} := \tan^{-1}(Q_{jk}/P_{jk})$ is the power factor angle.

5.2.3 Power flow solutions

Example 5.1 (Two-bus network)**.** Consider two buses 0 and 1 connected by a line with series impedance $z = r + \mathbf{i}x$ with $r > 0$, $x > 0$ and zero shunt admittances. The power balance at bus 0 (noting that $S_{0k} := 0$) and the other DistFlow equations over line $1 \to 0$ are given by:

$$p_0 - r\ell \;=\; -p_1, \qquad q_0 - x\ell \;=\; -q_1 \tag{5.12a}$$
$$v_1 - v_0 \;=\; 2\left(rp_1 + xq_1\right) - (r^2 + x^2)\ell \tag{5.12b}$$
$$p_1^2 + q_1^2 \;=\; v_1\ell \tag{5.12c}$$

where the voltage v_0 and the injections p_1, q_1 are given. Suppose $q_1 = 0$ and $v_0 = r = x = 1$ pu.

1. Show that power flow solutions (p_0, q_0, v_1, ℓ) exist if and only if

$$\frac{1}{2}\left(1 - \sqrt{2}\right) \;\le\; p_1 \;\le\; \frac{1}{2}\left(1 + \sqrt{2}\right)$$

2. Show that for each injection value p_1 that satisfies the condition in part 1, there are two voltage solutions v_1 given by

$$v_1 \;=\; \frac{1}{2}\left(1 + 2p_1 \mp \sqrt{\Delta}\right) \;=\; \frac{1}{2}\left(1 + 2p_1 \mp \sqrt{4p_1(1 - p_1) + 1}\right)$$

where

$$\Delta \;:=\; 4\left(\left(rp_1 + xq_1\right) - \left(rq_1 - xp_1\right)^2\right) + 1$$

3. Show that the locus (v_1, p_1) that satisfies (15.44) is a (rotated) ellipse. Plot the two solutions for v_1 in Part 2 as functions of p_1. These two curves form the ellipse.

4. Show that the lowest voltage solution is $v_1 = 0$ pu attained at $p_1 = 0$ pu and the highest voltage solution is $v_1 = 2$ pu attained at $p_1 = 1$ pu.

$\square$

For the two-bus network in Example 5.1 power flow solutions, when projected in the (v_1, p_1) coordinate, form an ellipse without the interior. This feature of hollow solution set is generally true for the DistFlow model (5.6), (5.7), (5.8), as the following result shows. Let

$$\mathbb{T} \; := \; \{x : (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies (5.8) under assumption C5.1}\}$$

Theorem 5.1 (Hollow solution set)**.** If $\hat{x}$ and $\tilde{x}$ are distinct power flow solutions in $\mathbb{T}$ with the same voltage $\hat{v}_0 = \tilde{v}_0$ at the root bus 0, then no convex combination of $\hat{x}$ and $\tilde{x}$ can be in $\mathbb{T}$. In particular $\mathbb{T}$ is nonconvex.

Proof. Suppose $\hat{x} \neq \tilde{x}$ are distinct power flow solutions in $\mathbb{T}$. Fix any $a \in [0,1]$ and consider $x := a\hat{x} + (1-a)\tilde{x}$. We now show that if $x \in \mathbb{T}$ then $\hat{x} = \tilde{x}$, contradicting that $\hat{x}$ and $\tilde{x}$ are distinct.

Suppose $x \in \mathbb{T}$. In particular $v_j \ell_{jk} = |S_{jk}|^2$ by (5.8c). Substituting $x := (\hat{x} + \tilde{x})/2$, we have

$$\frac{1}{4}(\hat{v}_j + \tilde{v}_j)(\hat{\ell}_{jk} + \tilde{\ell}_{jk}) \;\; = \;\; \frac{1}{4}\left|\hat{S}_{jk} + \tilde{S}_{jk}\right|^2, \qquad j \to k \in E$$

Substituting $\hat{v}_j \hat{\ell}_{jk} = |\hat{S}_{jk}|^2$ and $\tilde{v}_j \tilde{\ell}_{jk} = |\tilde{S}_{jk}|^2$ yeilds

$$\hat{v}_j \tilde{\ell}_{jk} + \tilde{v}_j \hat{\ell}_{jk} \;\; = \;\; 2\,\mathrm{Re}\left(\hat{S}_{jk}^{\mathsf{H}} \tilde{S}_{jk}\right) \tag{5.13a}$$

The right-hand side satisfies

$$2\,\mathrm{Re}\left(\hat{S}_{jk}^{\mathsf{H}} \tilde{S}_{jk}\right) \;\; \leq \;\; 2\,|\tilde{S}_{jk}||\hat{S}_{jk}| \tag{5.13b}$$

with equality if and only if $\angle \hat{S}_{jk} = \angle \tilde{S}_{jk}$ (mod 2π). The left-hand side of (5.13a) is

$$\hat{v}_j \tilde{\ell}_{jk} + \tilde{v}_j \hat{\ell}_{jk} \;\; = \;\; \eta_j\,|\tilde{S}_{jk}|^2 + \eta_j^{-1}\,|\hat{S}_{jk}|^2 \;\; \geq \;\; 2\,|\tilde{S}_{jk}||\hat{S}_{jk}| \tag{5.13c}$$

with equality if and only if $\eta_j |\tilde{S}_{jk}| = |\hat{S}_{jk}|$, where for $j \in \overline{N}$, $\eta_j := \hat{v}_j / \tilde{v}_j$. But (5.13) implies that equalities are attained in both (5.13b) and (5.13c), and hence

$$\eta_j \tilde{S}_{jk} \;=\; \hat{S}_{jk} \quad \text{and} \quad \eta_j \tilde{\ell}_{jk} \;=\; \hat{\ell}_{jk}, \qquad j \in N \tag{5.14}$$

Define $\eta_0 := \hat{v}_0 / \tilde{v}_0 = 1$. Then for each line $j \to k \in E$ we have, using (5.8b),

$$\begin{aligned}
\eta_k \;\; &= \;\; \frac{\hat{v}_k}{\tilde{v}_k} \;=\; \frac{\hat{v}_j - 2\,\mathrm{Re}(z_{jk}^{\mathsf{H}} \hat{S}_{jk}) + |z_{jk}|^2 \hat{\ell}_{jk}}{\tilde{v}_j - 2\,\mathrm{Re}(z_{jk}^{\mathsf{H}} \tilde{S}_{jk}) + |z_{jk}|^2 \tilde{\ell}_{jk}} \\
&= \;\; \frac{\eta_j \left(\tilde{v}_j - 2\,\mathrm{Re}(z_{jk}^{\mathsf{H}} \tilde{S}_{jk}) + |z_{jk}|^2 \tilde{\ell}_{jk}\right)}{\tilde{v}_j - 2\,\mathrm{Re}(z_{jk}^{\mathsf{H}} \tilde{S}_{jk}) + |z_{jk}|^2 \tilde{\ell}_{jk}} \;=\; \eta_j
\end{aligned}$$

where the third equality follows from (5.14). This implies, since the network graph G is connected, that $\eta_j = \eta_0 = 1$ for all $j \in \overline{N}$, i.e. $\hat{v}_j = \tilde{v}_j$, $j \in \overline{N}$.

We have thus shown that $\hat{S} = \tilde{S}$, $\hat{\ell} = \tilde{\ell}$, $\hat{v} = \tilde{v}$, and hence, by (5.8a), $\hat{s} = \tilde{s}$, i.e., $\hat{x} = \tilde{x}$. This completes the proof. $\square$

This property of the power flow solution set is illustrated vividly in several numerical examples in [20, 21, 22, 23].

5.3 Equivalence

Equivalence. As presented in Chapters 5.1 and 5.2 the branch flow models are defined by different sets of power flow equations:

1. For general networks: (5.1) or (5.3);
2. For radial networks with shunt admittances: (5.4);
3. For radial networks without shunt admittances (DistFlow equations): (5.6), (5.7), (5.8) when assumption C5.1 holds.

BFMs are usually used for modeling radial networks commonly found in distribution systems. The DistFlow equations (5.6), (5.7) are most commonly used in the literature. The model (5.4) is useful in applications where shunt admittances of the lines are important.

The models (5.1) and (5.3) for general networks are used mostly as a bridge to relate BFM models for radial networks to the bus injection models for general networks studied in Chapter 4.3. Specifically even though BFMs and the bus injection model (4.20a), reproduced here:

$$s_j \;=\; \sum_{k:j\sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2, \qquad j \in \overline{N} \tag{5.15}$$

are defined by different sets of equations in terms of their own variables, all of them are models of Kirchhoff's and Ohm's laws. We now clarify the precise sense in which these mathematical models are equivalent.

Let the set of solutions (s,V) of BIM be:

$$\mathbb{V} \;:=\; \{(s,V) \in \mathbb{C}^{2(N+1)} \mid (s,V) \text{ satisfies (5.15)}\} \tag{5.16a}$$

Let the sets of solutions of BFM be:

$$\begin{aligned}
\tilde{\mathbb{X}} \;&:=\; \{\tilde{x} : (s,V,I,S) \in \mathbb{C}^{2(N+1)+4M} \mid \tilde{x} \text{ satisfies (5.1)}\} && \text{(5.16b)}\\
\mathbb{X}_{\text{meshed}} \;&:=\; \{x : (s,v,\ell,S) \in \mathbb{R}^{3(N+1)+6M)} \mid x \text{ satisfies (5.3)}\} && \text{(5.16c)}\\
\mathbb{X}_{\text{tree}} \;&:=\; \{x : (s,v,\ell,S) \in \mathbb{R}^{9N+3} \mid x \text{ satisfies (5.4)}\} && \text{(5.16d)}\\
\mathbb{T}_0 \;&:=\; \{x : (s,v,\ell,S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies (5.6) under assumption C5.1}\} && \text{(5.16e)}\\
\overline{\mathbb{T}}_0 \;&:=\; \{x : (\overline{s},\overline{v},\overline{\ell},\overline{S}) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies (5.7) under assumption C5.1}\} && \text{(5.16f)}
\end{aligned}$$

We say two sets A and B are *equivalent*, denoted by $A \equiv B$, if there is a bijection between them. The equivalence of these power flow models is clarified in the following theorem and illustrated in Figure 5.3.

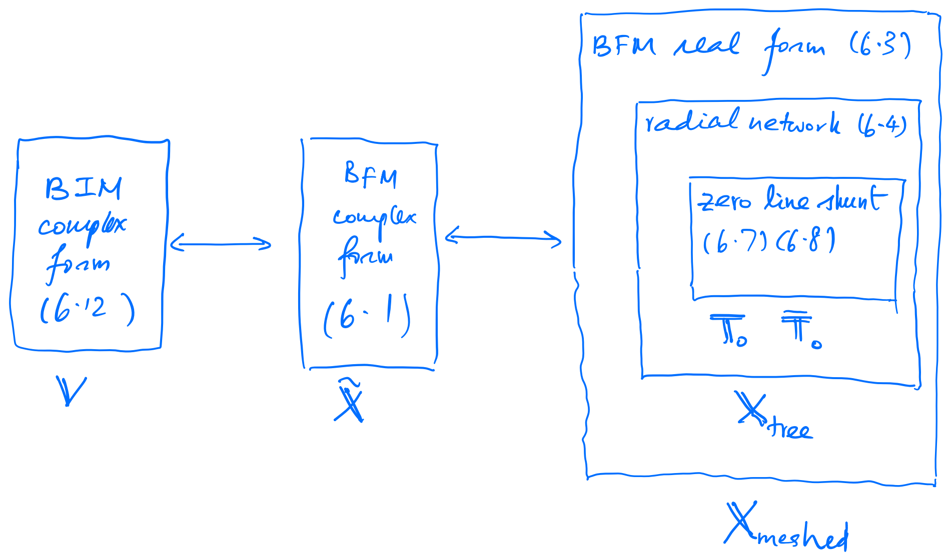


Figure 5.3: Equivalence of BFM and BIM (the model $\mathbb{T}_0$ defined by (5.6) is the original DistFlow model).

Theorem 5.2. Suppose the network G is connected.

1. $\mathbb{V} \equiv \tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$.
2. If G is a tree then $\mathbb{X}_{\text{meshed}} \equiv \mathbb{X}_{\text{tree}}$.
3. Suppose $y^s_{jk} = y^s_{kj}$ (assumption C5.1) and $y^m_{jk} = y^m_{kj} = 0$ for all lines (j,k). If G is a tree then $\mathbb{X}_{\text{tree}} \equiv \mathbb{T}_0 \equiv \overline{\mathbb{T}}_0$.

Proof. Part 1: $\mathbb{V} \equiv \tilde{\mathbb{X}} \equiv \mathbb{X}_{meshed}$. It is obvious $\mathbb{V} \equiv \tilde{\mathbb{X}}$ since, given $(s,V) \in \mathbb{V}$, define I by (5.1c)(5.1d) and S by (5.1b) and the resulting $(s,V,I,S) \in \tilde{\mathbb{X}}$. Conversely given $(s,V,I,S) \in \tilde{\mathbb{X}}$, substituting (5.1b)(5.1c)(5.1d) into (5.1a) shows $(s,V) \in \mathbb{V}$.

To show $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$, fix an $\tilde{x} := (s,V,I,S) \in \tilde{\mathbb{X}}$. Define (v,ℓ) by:

$$v_j \ := \ |V_j|^2, \qquad \ell_{jk} \ := \ |I_{jk}|^2, \qquad \ell_{kj} \ := \ |I_{kj}|^2$$

We now show that $x := (s,v,\ell,S) \in \mathbb{X}_{\text{meshed}}$. That x satisfies (5.3a) follows from (5.1a). Taking the squared magnitude on both sides of (5.1b) gives (5.3b). For (5.3c) rewrite (5.1c) as

$$V_k \ = \ \alpha_{jk} V_j \ - \ z^s_{jk} \left(\frac{S_{jk}}{V_j} \right)^{\mathsf{H}} \tag{5.17}$$

where we have substituted $I_{jk} := S^{\mathsf{H}}_{jk}/V^{\mathsf{H}}_j$ from (5.1b). Taking the squared magnitude on both sides gives

$$v_k \ = \ \left|\alpha_{jk}\right|^2 v_j \ + \ \left|z^s_{jk}\right|^2 \ell_{jk} \ - \ 2\,\mathrm{Re}\left(\alpha_{jk} \left(z^s_{jk}\right)^{\mathsf{H}} S_{jk} \right)$$

which is (5.3c). Similarly (5.3d) can be derived from (5.1d). From (5.1b)(5.1c) we have

$$V_j V_k^{\mathsf{H}} \;=\; \alpha_{jk}^{\mathsf{H}} |V_j|^2 - \left(z_{jk}^s\right)^{\mathsf{H}} S_{jk}, \qquad V_k V_j^{\mathsf{H}} \;=\; \alpha_{kj}^{\mathsf{H}} |V_k|^2 - \left(z_{jk}^s\right)^{\mathsf{H}} S_{kj}$$

The definitions of $\beta_{jk}(x)$ and $\beta_{kj}(x)$ in (5.2) then imply that $\beta_{jk}(x) = \angle V_j - \angle V_k = -\beta_{kj}(x)$ and hence (5.3e)(5.3f) hold with $\theta_j := \angle V_j$. This shows $x \in \mathbb{X}_{\text{meshed}}$.

Conversely fix an $x := (s, v, \ell, S) \in \mathbb{X}_{\text{meshed}}$. Since $\beta_{jk}(x)$ defined in (5.2) satisfy (5.3e), they satisfy (5.11a). Hence we can construct (V, I) from x using (5.11b)(5.11c) as:

$$V_j \;:=\; \sqrt{v_j}\, e^{\mathbf{i}\theta_j}, \qquad\qquad I_{jk} \;:=\; \sqrt{\ell_{jk}}\, e^{\mathbf{i}\left(\theta_j - \angle S_{jk}\right)}$$

We now verify that $\tilde{x} := (s, V, I, S)$ satisfies (5.1). Clearly (5.1a) is (5.3a). For (5.1b), we have from (5.3b) and the construction (5.11c) of (V, I) that

$$|S_{jk}| \;=\; \left|V_j I_{jk}^{\mathsf{H}}\right|, \qquad\qquad \angle S_{jk} \;=\; \angle V_j - \angle I_{jk}$$

Hence $S_{jk} = V_j I_{jk}^{\mathsf{H}}$. Similarly $S_{kj} = V_k I_{kj}^{\mathsf{H}}$. We next show that (5.1c) follows from (5.3c). First note that (5.1c) is equivalent to $z_{jk}^s \left(S_{jk}/V_j\right)^{\mathsf{H}} = \alpha_{jk} V_j \,-\, V_k$ which is equivalent to

$$V_j V_k^{\mathsf{H}} \;=\; \alpha_{jk}^{\mathsf{H}} v_j \,-\, z_{jk}^{s\mathsf{H}} S_{jk} \tag{5.18}$$

We now show that (5.3c) implies that the quantities on both sides of (5.18) have equal magnitudes and angles, thus establishing their equality. For their angles, the definition of $\beta_{jk}(x)$ in (5.2) implies

$$\angle\left(\alpha_{jk}^{\mathsf{H}} v_j \,-\, z_{jk}^{s\mathsf{H}} S_{jk}\right) \;=\; \beta_{jk}(x) \;=\; \theta_j - \theta_k \;=\; \angle\left(V_j V_k^{\mathsf{H}}\right)$$

where the last two equalities follow from the construction of V_j, V_k. The squared magnitude of the right-hand side of (5.18) is

$$\begin{aligned}\left|\alpha_{jk}^{\mathsf{H}} v_j \,-\, z_{jk}^{s\mathsf{H}} S_{jk}\right|^2 \;&=\; \left|\alpha_{jk}\right|^2 v_j^2 \,-\, 2 v_j \operatorname{Re}\left(\alpha_{jk} z_{jk}^{s\mathsf{H}} S_{jk}\right) \,+\, \left|z_{jk}\right|^2 \left|S_{jk}\right|^2 \\ &=\; v_j \left(\left|\alpha_{jk}\right|^2 v_j \,-\, 2\operatorname{Re}\left(\alpha_{jk} z_{jk}^{s\mathsf{H}} S_{jk}\right) \,+\, \left|z_{jk}\right|^2 \ell_{jk}\right) \;=\; v_j v_k\end{aligned}$$

which is the squared magnitude of the quantity on the left-hand side of (5.18) by α_{jk}. The second equality above follows from $|S_{jk}|^2 = v_j \ell_{jk}$ from (5.3b) and the last equality follows from (5.3c). Hence (5.1c) follows from (5.3c). Similarly (5.1d) follows from (5.3d). This proves $\tilde{x} \in \tilde{\mathbb{X}}$. We hence conclude $\tilde{\mathbb{X}} \equiv \mathbb{X}_{\text{meshed}}$.

Part 2: $\mathbb{X}_{meshed} \equiv \mathbb{X}_{tree}$. Suppose G is a tree. We will show that $x := (s, v, \ell, S)$ satisfies (5.3) if and only if it satisfies (5.4). It suffices to show that x satisfies (5.3e)(5.3f) if and only if it satisfies (5.4e). Suppose x satisfies (5.3e)(5.3f) which implies that $\beta_{jk}(x) = -\beta_{kj}(x)$. Using (5.2) we have

$$\angle\left(\alpha_{jk}^{\mathsf{H}} v_j \,-\, z_{jk}^{\mathsf{H}} S_{jk}\right) \;=\; \beta_{jk}(x) \;=\; -\beta_{kj}(x) \;=\; -\angle\left(\alpha_{kj}^{\mathsf{H}} v_k \,-\, z_{kj}^{\mathsf{H}} S_{kj}\right)$$

i.e., the quantities on both sides of (5.4e) have equal angles. We now show that they have equal magnitudes as well. Indeed

$$\left|\alpha_{jk}^{\mathsf{H}} v_j \; - \; z_{jk}^{\mathsf{H}} S_{jk}\right|^2 \;\; = \;\; \left|\alpha_{jk}\right|^2 v_j^2 + \left|z_{jk}\right|^2 |S_{jk}|^2 - 2\,\mathrm{Re}\left(\alpha_{jk} z_{jk}^{\mathsf{H}} v_j S_{jk}\right) \;\; = \;\; v_j v_k$$

where the last equality follows from multiplying both sides of (5.4c) by v_j and then substituting (5.4b). Similarly

$$\left|\alpha_{kj}^{\mathsf{H}} v_k \; - \; z_{jk}^{\mathsf{H}} S_{kj}\right|^2 \;\; = \;\; v_j v_k \;\; = \;\; \left|\alpha_{jk}^{\mathsf{H}} v_j \; - \; z_{jk}^{\mathsf{H}} S_{jk}\right|^2$$

This shows that $\alpha_{jk}^{\mathsf{H}} v_j \, - \, z_{jk}^{\mathsf{H}} S_{jk} = \left(\alpha_{kj}^{\mathsf{H}} v_k \, - \, z_{kj}^{\mathsf{H}} S_{kj}\right)^{\mathsf{H}}$. Hence x satisfies (5.4e). Conversely suppose x satisfies (5.4e). Adopt an arbitrary orientation of the network graph and define $\beta_{jk}(x) := \angle\left(\alpha_{jk}^{\mathsf{H}} v_j - z_{jk}^{\mathsf{H}} S_{jk}\right)$ for each directed line $j \rightarrow k$ (only). Then θ given by (5.11b) is a solution to (5.3e). The condition (5.4e) implies that $\beta_{kj}(x) = -\beta_{jk}(x)$. Hence the θ determined also satisfies (5.3f). This shows that $\mathbb{X}_{\text{meshed}} \equiv \mathbb{X}_{\text{tree}}$.

Part 3: $\mathbb{X}_{tree} \equiv \mathbb{T}_0 \equiv \overline{\mathbb{T}}_0$. The proof for part 3 under assumption C5.1 is left as Exercise 5.5. □

Given the bijection between the solution sets of BIM and BFM, any result in one model is in principle derivable in the other. Some results however are much easier to state or derive in one model than the other. For instance BIM, which is widely used in transmission network problems, allows a much cleaner formulation of semidefinite program (SDP) relaxation. BFM for radial networks has a convenient recursive structure that allows a more efficient computation of power flows and leads to a useful linear approximation; see Chapters 5.4 and 5.5. The sufficient condition for exact relaxation in Chapter **??** provides intricate insights on power flows that are hard to formulate or prove in BIM. BFM for radial networks seems to be much more stable numerically than BIM as the network size scales up. Finally, since BFM directly models branch flows S_{jk} and currents I_{jk}, it is easier to use for some applications. One should freely use either model depending on which is more convenient for the problem at hand.

5.4 Backward forward sweep for radial network

The tree topology induces a spatially recursive structure in power flow equations. This structure can be used to develop an efficient method to compute a power flow solution, called a backward forward sweep (BFS). The Newton-Raphson algorithm of Chapter 4.5.2 works for general networks but needs to compute Jacobian or solve a linear system in each iteration, a significant computational burden for large networks. The Fast Decoupled Algorithm of Chapter 4.5.3 works well when line losses are small, which is a good approximation for high-voltage transmission networks but not distribution systems. In contrast BFS is simple, accurate, and tends to converge quickly in practice.

An outline of BFS is as follows. A power flow solution is partitioned into two groups of variables x and y. Starting from an initial vector y, the components x_i can be recursively computed starting from leaf nodes and working towards the root (backward sweep). Given the newly updated vector x, the components y_i are then updated recursively starting from the root and working towards the leaf nodes (forward sweep).

A BFS method iterates on a backward sweep followed by a forward sweep, until convergence. It can be interpreted as a special Gauss-Siedel algorithm.

Different BFS algorithms mostly differ in their choice of variables x and y and the associated power flow equations. In the following we first provide in Chapter 5.4.1 a general formulation of BFS and then illustrate in Chapters 5.4.2 and 5.4.3 BFM algorithms using the complex form BFM and the DistFlow model. Finally in Chapter 5.4.4 we present convergence analysis.

5.4.1 General BFS

Overview. The method of backward forward sweep can be interpreted as a Gauss-Siedel algorithm discussed in Chapter 4.5.1, with two special structures that exploit that tree topology of the network. First it partitions a power flow variable into two vectors $x \in A^{n_1}$ and $y \in A^{n_2}$ where A is either $\mathbb{C}$ or $\mathbb{R}$. It iteratively computes a fixed point:

$$x(t) \ = \ F(x(t), y(t-1)), \qquad\qquad y(t) \ = \ G(x(t), y(t)) \tag{5.19}$$

where $F : A^{n_1+n_2} \to A^{n_1}$ and $G : A^{n_1+n_2} \to A^{n_2}$. By this notation we mean that each (outer) iteration in (5.19) is computed iteratively in an inner loop that always uses the latest available values to update each component $(x_i(t), y_i(t))$ in turn, i.e.,

$$\begin{aligned}
x_i(t) \ &= \ F_i(x_1(t), \dots, x_{i-1}(t), x_{i+1}(t-1), \dots, x_{n_1}(t-1), y(t-1)), \qquad & i \ &= \ 1, \dots, n_1 \\
y_i(t) \ &= \ G_i(x(t), y_1(t), \dots, y_{i-1}(t), y_{i+1}(t-1), \dots, y_{n_2}(t-1)), \qquad & i \ &= \ 1, \dots, n_2
\end{aligned}$$

Second the inner loop makes use of a spatially recursive structure enabled by the tree topology. Specifically the partition x and y are chosen so that, given a vector y, each component x_i depends on the other components $x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n_1})$ recursively, i.e., the update function F_i depends on x_{-i} only through $(x_1, \dots, x_{i-1})$. This means that, starting from $x_j(t)$ at leaf nodes j and propagating towards the root of the tree, $x_i(t)$ at nodes at successive layers are computed in a Gauss-Siedel manner (backward sweep):

$$x_i(t) \ = \ F_i(x_1(t), \dots, x_{i-1}(t), y(t-1)), \qquad\qquad i \ = \ 1, \dots, n_1$$

Similarly, given $x(t)$, each component $y_i(t)$ depends on $y_{-i}(t)$ recursively and is computed by propagating from the root towards the leaf nodes (forward sweep). The design of BFS involves the choice of power flow equations and variables (x, y) based in part on what information is given in a power flow problem. These choices are not unique and may have different convergence properties.

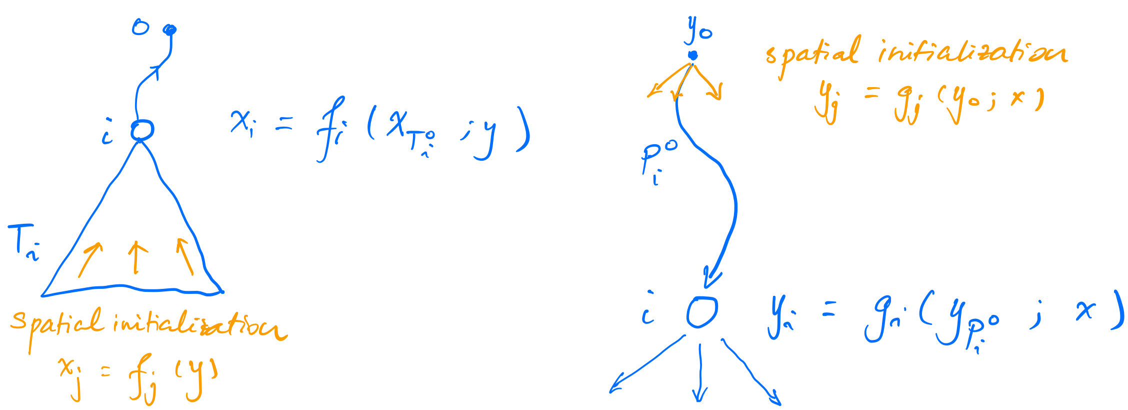
Most BFS algorithms compute line currents or power flows in the backward sweep and voltages in the forward sweep. Typically the voltage at the substation (the root of the tree) is specified and that the line current or power out of a leaf node is zero. These two boundary conditions mean that the computation of line currents or powers must start from the leaf nodes and propagate backward, while that of voltages must start from the root and propagate forward.

Algorithm. Fix any graph orientation. It is natural to use either the down orientation (all lines point away from the root) or the up orientation (all lines point towards the root). Often it is convenient to use the up orientation when designing the backward sweep. The variables associated with a line $j \rightarrow k$ such as line currents I_{jk} or power flows S_{jk}, can be consistently identified by the from node j. In the following we assume such identification has been done and we will identify variables x_i, y_i by nodes i even though they may represent line variables. Let the root of the tree be bus $i = 0$.

Typically x_i depends only on x_j at its child nodes j (i.e., j is adjacent to i and farther away from the root than i). More generally let T_i° denote the set of buses in the subtree rooted at bus i, not including i. Let $x_{\mathsf{T}_i^\circ} := \left(x_j, j \in \mathsf{T}_i^\circ\right)$ denote the variables x_j in the subtree T_i°. We say that x satisfies a *spatially recursive structure* if, given y, x_i depends on the other variables x_{-i} only through $x_{\mathsf{T}_i^\circ} \subseteq x_{-i}$, in the form:

$$x_i \quad = \quad f_i\left(x_{\mathsf{T}_i^\circ};\, y\right), \qquad i \in N$$

Each x_i can be a vector and f_i a vector-valued function. This means that, starting from the leaf nodes and working towards the root (bus 0) in the reverse breadth-first search order, x_i can be recursively updated given a vector y. The boundary condition for the recursion is that, if i is a leaf node, then $\mathsf{T}_i^\circ := \emptyset$ and $x_i = f_i(\emptyset, y) =: f_i(y)$. This relation starts the backward sweep working from the leaf nodes towards the root, as illustrated in Figure 5.4(a).



(a) Backward sweep (b) Forward sweep

Figure 5.4: General backward forward sweep

Similarly x and y are chosen so that, given x, the components y_i depends on the other components y_{-i} only through variables y_j in the path from the root to node i. Specifically let P_i° denote the set of buses in the unique path from the root to bus i, including bus 0 but not including i. Let $y_{\mathsf{P}_i^\circ} := \left(y_j, j \in \mathsf{P}_i^\circ\right)$. The variable y satisfies a *spatially recursive structure* if, given x, y_i depends on y_{-i} only through $y_{\mathsf{P}_i^\circ} \subseteq y_{-i}$, in the form:

$$y_i \quad = \quad g_i\left(y_{\mathsf{P}_i^\circ};\, x\right), \qquad i \in N$$

Each y_i can be a vector and g_i a vector-valued function. The boundary condition for the recursion is that, if i is a child of the root bus 0, then $\mathsf{P}_i^\circ := \{0\}$ and y_0 at bus 0 is given and hence $y_i = g_i(y_0;\, x)$. This relation starts the forward sweep to recursively update y_i, working from the root towards the leaf nodes in the breadth-first search order; see Figure 5.4(b).

In summary let $x := (x_i, i \in N)$ and $y := (y_i(t), i \in N)$. A pair (x, y) is a power flow solution if it satisfies

$$
\begin{aligned}
x_i &= f_i\left(x_{\mathsf{T}_i^\circ};\, y\right), \quad i \in N, & y_i &= g_i\left(y_{\mathsf{P}_i^\circ};\, x\right), \quad i \in N && \text{(5.20a)}\\
\mathsf{T}_i^\circ &= \emptyset \;\text{ for all leaf nodes } i & y_0 & \;\text{ given} && \text{(5.20b)}
\end{aligned}
$$

Let the update functions be $f := (f_i, i = 1, \ldots, N)$ and $g := (g_i, i = 1, \ldots, N)$. A BFS algorithm is a special Gauss-Seidel algorithm that computes a fixed point of (5.20a) starting from the initial conditions in (5.20b). It is defined by the update functions (f, g) and described in Algorithm 1. If it converges and (f, g) are continuous then the limit point is a fixed point and therefore a power flow solution. An advantage of BFS is that it does not need to calculate derivatives of power flow equations and tends to converge quickly in practice.

Algorithm 1: Backward forward sweep

Input: $(f_i, \mathsf{T}_i^\circ, i \in \overline{N})$, $(g_i, \mathsf{P}_i^\circ, i \in N)$, $(y_0, y_{-0}) \in F^{n_2}$ with $y_{-0} := (y_i, i \in N)$.
Output: a solution (x, y) of (5.20).

1. **Initiatization**:
   - $\mathsf{T}_i^\circ := \emptyset$ for all leaf notes i.
   - $y_0(t) \leftarrow y_0$ for $t = 0, 1, \ldots;\; y(0) \leftarrow (y_0, y_{-0})$.
   - $t \leftarrow 0$.
2. **while** *stopping criterion not met* **do**
   (a) $t \leftarrow t + 1$;
   (b) *Backward sweep*: **for** *i starting from the leaf nodes and iterating towards bus 0* **do**
   $$x_i(t) \;\; \leftarrow \;\; f_i\left(x_{\mathsf{T}_i^\circ}(t);\, y(t-1)\right), \qquad i \in N$$
   (c) *Forward sweep*: **for** *i starting from bus 0 and iterating towards the leaf nodes* **do**
   $$y_i(t) \;\; \leftarrow \;\; g_i\left(y_{\mathsf{P}_i^\circ}(t);\, x(t)\right), \qquad i \in N$$
3. **Return**: $x := x(t)$, $y := y(t)$.

5.4.2 Complex form BFM

Without loss of generality we consider G as a directed graph where each line points *away* from the root bus 0. As explained at the beginning of Chapter 5.2.2, we can adopt directed graph and involve line variables

(I_{jk}, S_{jk}) in the direction of the line $j \to k$, but not variables (I_{kj}, S_{kj}) in the opposite direction, only under assumption C5.1 guaranteeing $z^s_{jk} = z^s_{kj}$.

We hence assume C5.1 holds in this subsection. Suppose V_0 and injections s_j at all non-reference buses $j \neq 0$ are given. We will compute the voltage phasor V_j for every bus $j \neq 0$ and the current I^s_{jk} through the series admittance y^s_{jk} for every line $j \to k$ in the direction of the line (see Figure 5.1):

$$I^s_{jk} \;\; := \;\; I_{jk} \; - \; y^m_{jk} V_j$$

All other variables, such as injection s_0 and branch flows S_{jk} in (5.1), can be computed once (V_j, I^s_{jk}) for all $j \in \overline{N}$ and all $j \to k \in E$ are determined (Exercise 5.6).[3]

For each bus j, let $i(j)$ denote the unique parent bus between the buses 0 and j. By Ohm's law we have $V_j - V_k = z^s_{jk} I^s_{jk}$ where $z^s_{jk} := 1/y^s_{jk}$ is the series impedance of line (j,k). The sending-end power from buses j to k is $S_{jk} = V_j I^{\mathsf{H}}_{jk} = V_j \left(I^s_{jk} + y^m_{jk} V_j\right)^{\mathsf{H}}$ and the received power at bus j from $i := i(j)$ is $V_i \left(I^s_{ij} - y^m_{ji} V_j\right)^{\mathsf{H}}$. The BFS algorithm of [24] starts with the power balance equation at each bus j:[4]

$$s_j + V_j \left(I^s_{ij} - y^m_{ji} V_j\right)^{\mathsf{H}} \;=\; \sum_{k: j \to k} V_j \left(I^s_{jk} + y^m_{jk} V_j\right)^{\mathsf{H}} \tag{5.21}$$

where $i := i(j)$ denote the unique parent of j; see Figure 5.5. We therefore have, for all non-reference

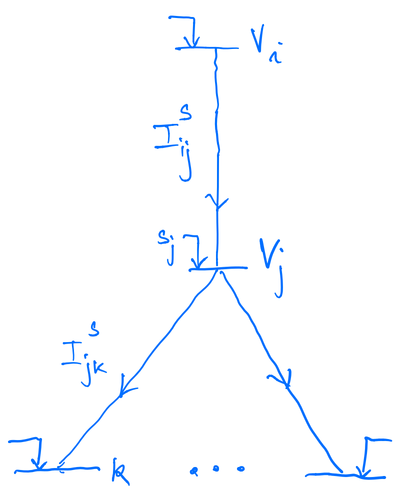


Figure 5.5: Recursive structure of power flow equations (5.22).

buses (dividing both sides of (5.21) by V_j and rearranging),

$$I^s_{ij} \;=\; \sum_{k: j \to k} I^s_{jk} - \left(\left(\frac{s_j}{V_j}\right)^{\mathsf{H}} - y^m_{jj} V_j \right), \qquad j \in N \tag{5.22a}$$

$$V_j \;=\; V_i \; - \; z^s_{ij} I^s_{ij}, \qquad j \in N \tag{5.22b}$$

[3] Instead of I^s_{jk}, we can also compute the branch current (see Exercise 5.7) but the choice of I^s_{jk} over I_{jk} has an advantage of the simple relation $I^s_{jk} = -I^s_{kj}$.

[4] An implicit assumption in (5.21) is that the receiving-end current at bus j from i is $I^s_{ij} - y^m_{ji} V_j$. This holds only under assumption C5.1.

where $y^m_{jj} := y^m_{ji} + \sum_{k:j\to k} y^m_{jk}$ is the total shunt admittance incident on bus j. The boundary conditions are

$$I^s_{jk} \;:=\; 0 \;\text{ for all leaf nodes } j, \qquad V_0 \;\text{ is given }, \qquad V_j(0) := V_0,\; j \in \overline{N} \tag{5.22c}$$

This is spatially recursive in that, given voltages $\left(V_j, j \in N\right)$, propagating (5.22a) backward from the leaf nodes towards the root (bus 0) in the reverse breadth-first search order, the current I^s_{ij} can be updated once all the currents I^s_{jk} in the previous level are determined; see Figure 5.5. In the forward direction, given currents $I^s := \left(I^s_{ij}, (i,j) \in E\right)$, propagating (5.22b) from the root towards the leaf nodes in the breadth-first search order, the voltage V_j can be updated recursively once its parent V_i is determined.

Since the injections s_j at all non-reference buses j are given (i.e., all buses j are PQ buses), (5.22) defines a BFS algorithm where:

- $x_j := I^s_{ij}$ for $j \in N$ are the complex line currents across the series impedance z^s_{ij} from buses i to j. The backward sweep functions f_j in (5.20a) are given by (5.22a).
- $y_j := V_j$ for $j \in N$ are the complex voltage at buses j. The forward sweep functions g_j in (5.20a) are given by (5.22b).
- The initialization in (5.20b) is given by (5.22c).

The update function f is linear in x given y, but not jointly linear in (x,y). The function g is linear in (x,y).

The algorithm proceeds as follows.

Input: voltage V_0 pu and injections $(s_i, i \in N)$.
Output: currents $(I^s_{jk}, j \to k \in E)$ and voltages $(V_i, i \in N)$.

1. *Initialization.*
   - $I^s_{jk}(t) := 0$ for all leaf nodes j for all iterations $t = 1, 2, \ldots$.
   - $V_0(t) := V_0$ at bus $j = 0$ for all $t = 0, 1, \ldots$.
   - $V_j(0) := V_0$ at all buses $j \in N$.
2. *Backward forward sweep.* Iterate for $t = 1, 2, \ldots$ until a stopping criterion (see below) is satisfied:
   (a) *Backward sweep.* Starting from the leaf nodes and iterating towards bus 0, compute

$$I^s_{ij}(t) \;\;\leftarrow\;\; \sum_{k:j\to k} I^s_{jk}(t) \;-\; \left(\left(\frac{s_j}{V_j(t-1)}\right)^{\mathsf{H}} - y^m_{jj} V_j(t-1)\right), \quad i \to j \in E \tag{5.23a}$$

   where $y^m_{jj} := y^m_{ji} + \sum_{k:j\sim k} y^m_{jk}$.

   (b) *Forward sweep.* Starting from bus 0 and iterating towards the leaf nodes, compute

$$V_j(t) \;\;=\;\; V_i(t) \;-\; z^s_{ij} I^s_{ij}(t), \qquad j \in N \tag{5.23b}$$

   where $z^s_{ij} := \left(y^s_{ij}\right)^{-1}$.

3. *Output*: $I^s_{jk} := I^s_{jk}(t)$ and $V_i := V_i(t)$.

A stopping criterion can be based on the discrepancy between the given injections s_j and the injections $s_j(t)$ implied by $I^s(t) := (I^s_{jk}(t), j \to k \in E)$ and $V(t) := (V_j(t), j \in N)$ at the end of each iteration t. Specifically let

$$s_j(t) \;:=\; -V_j(t) I^{s\mathsf{H}}_{ij}(t) \;+\; y^{m\mathsf{H}}_{jj} \left|V_j(t)\right|^2 \;+\; \sum_{k:j\to k} V_j(t) I^{s\mathsf{H}}_{jk}(t)$$

Then a stopping criterion can be

$$\|s(t) - s\| \;:=\; \sum_{j\in N} \left(s_j(t) - s_j\right)^2 \;<\; \varepsilon$$

for a given tolerance $\varepsilon > 0$.

5.4.3 DistFlow model

The BFS algorithm defined by (5.22) assumes all power injections s_j at non-reference buses j are given and computes I^s_{jk} in the backward sweep. If some buses have their voltage magnitudes $|V_j|$ and real power p_j given instead (i.e., these are PV buses), we can develop BFS algorithms based on the DistFlow model of Chapter 5.2.2. The advantage of the DistFlow model is that the BFS algorithms need not compute the voltage angles θ_j. Phase angles can be recovered using (5.11) after BFS has produced a solution.

We will present two algorithms, one where V_0 and $(s_j, j \in N)$ are given, as in Chapter 5.4.2, and the other where $(V_0, v_j, j \in N)$ and $(p_j, j \in N)$ are given. In both cases only v_0 is needed in BFS but the angle $\angle V_0$ is needed in (5.11) to ensure a unique angle vector θ from the solution of BFS.

It will be convenient to adopt a graph orientation where every line $k \to j$ points *towards* the root bus 0. Assume for simplicity zero shunt admittances, $y^m_{jk} = y^m_{kj} = 0$. As explained at the beginning of Chapter 5.2.2, DistFlow model is valid only under assumption C5.1 guaranteeing $z^s_{jk} = z^s_{kj}$. Hence we assume C5.1 holds in this subsection.

Example 5.2 (Given (V_0, s_j))**.** Suppose the complex voltage V_0 and $(s_j, j \in N)$ for all non-reference buses j are given. We will use the DistFlow equations to compute $\left(S_{jk}, \ell_{jk}, j \to k \in E\right)$ and $(v_j, j \in \overline{N})$.

The DistFlow equations for the up orientation are (5.7). The equations (5.7a) and (5.7c) lead to the following backward sweep to compute $\left(S_{kj}, \ell_{kj}, k \to j\right)$:

$$S_{ji} \;=\; s_j \;+\; \sum_{k:k\to j} \left(S_{kj} - z^s_{kj}\ell_{kj}\right), \qquad j \in N \tag{5.24a}$$

$$\ell_{ji} \;=\; \frac{|S_{ji}|^2}{v_j}, \qquad j \in N \tag{5.24b}$$

where $i := i(j)$ in (5.24a) denotes the parent node of j on the unique path between node 0 and node j. The equation (5.7b) leads to a forward sweep to compute $(v_j, j \in N)$:

$$v_j \;=\; v_i \;+\; 2\,\mathrm{Re}\left(z^{s\mathsf{H}}_{ji} S_{ji}\right) - |z^s_{ji}|^2 \ell_{ji}, \qquad j \in N \tag{5.24c}$$

The boundary conditions are

$$S_{0i} \;:=\; 0, \quad S_{kj} \;:=\; 0,\; \ell_{kj} \;:=\; 0 \;\text{ for leaf nodes } j, \quad V_0 \;\text{ is given }, \quad v_j(0) := |V_0|^2,\; j \in \overline{N} \tag{5.24d}$$

This defines a BFS algorithm where:

- $x_j := \left(S_{ji}, \ell_{ji}\right)$ for $j \in N$. The backward sweep functions f_j in (5.20a) are given by (5.24a)(5.24b).
- $y_j := v_j$ for $j \in N$. The forward sweep functions g_j in (5.20a) are given by (5.24c).
- The initialization in (5.20b) is given by (5.24d).

The function f is nonlinear in x given y due to (5.24b). The function g is jointly linear in (x,y).

$\square$

Example 5.3 (Given (v_j, p_j))**.** Suppose the complex voltage V_0, squared voltage magnitudes $(v_j, j \in N)$ and real power injections $(p_j, j \in N)$ for all non-reference buses j are given. We will compute the reactive power injections $(q_j, j \in N)$ as well as the line flows $(S_{jk}, j \to k \in E)$. All other variables, including (p_0, q_0), can then be determined.

Eliminating ℓ_{kj} from (5.24a)(5.24b) we can compute $S_{ji} := \left(P_{ji}, Q_{ji}\right)$ in a backward sweep and q_j in a forward sweep:

$$S_{ji} \;=\; s_j \;+\; \sum_{k:k\to j} \left(S_{kj} - z_{kj} \frac{|S_{kj}|^2}{v_k} \right), \qquad j \in N \tag{5.25a}$$

$$q_j \;=\; Q_{ji} \;-\; \sum_{k:k\to j} \left(Q_{kj} - x_{kj} \frac{|S_{kj}|^2}{v_k} \right), \qquad j \in N \tag{5.25b}$$

The boundary conditions are

$$S_{0i} \;:=\; 0, \qquad S_{kj} \;:=\; 0 \;\text{ for leaf nodes } j, \qquad v_j \;\text{ are given for }\; j \in \overline{N} \tag{5.25c}$$

This defines a BFS algorithm where:

- $x_j := S_{ji}$ for $j \in N$. The backward sweep functions f_j in (5.20a) are given by (5.25a).
- $y_j := q_j$ for $j \in N$. The forward sweep functions g_j in (5.20a) are given by (5.25b).
- The initialization in (5.20b) is given by (5.25c).

The function f is nonlinear in x given y. The function g is linear (constant) in y given x. $\square$

5.4.4 Convergence analysis

Apparently the only convergence analysis is [25]. See [26, Chapter 3] for contraction mapping applied to nonlinear iterative algorithms such as Gauss-Siedel algorithm [26, Proposition 1.4] and Newton algorithms [26, Chapters 3.1.3, 3.2].

5.5 Linearized model for radial network

We now present a linear approximation of BFM for radial networks when the line losses $z_{jk}^s \ell_{jk}$ are small compared with the line flows S_{jk}. In this case it is also reasonable to ignore line charging effects since shunt admittances (y_{jk}^m, y_{kj}^m) are typically much smaller than series admittances y_{jk}^s. Hence the linear model in this section is a reasonable approximation, not only to (5.6)(5.7), but also to (5.4) with nonzero shunt admittances.

It has two advantages. Given injections s, the voltages v_i^{lin} and line flows S_{jk}^{lin} of the linearized model can be solved explicitly in terms of s. Moreover the linear solution $(v^{\text{lin}}, S^{\text{lin}})$ provides bounds on line flows S and voltages v of nonlinear branch flow models (5.6)(5.7).

Since (5.6)(5.7) are valid models only under assumption C5.1, we assume C5.1 in this section in order to derive their linearizations.

5.5.1 Linear DistFlow equations

Consider a radial network. It has $N+1$ nodes and $M := N$ lines since it is a tree. Fix an arbitrary graph orientation. The linearized model from [19] is:

$$\sum_{k: j\rightarrow k} S_{jk} \;=\; \sum_{i: i\rightarrow j} S_{ij} \,+\, s_j, \qquad j \in \overline{N} \tag{5.26a}$$

$$v_j - v_k \;=\; 2\,\text{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right), \qquad j \rightarrow k \in E \tag{5.26b}$$

It can be justified by setting $\ell_{jk} := 0$ in the DistFlow equations (5.8) of Chapter 5.2.2, when line losses $|z_{jk}^s \ell_{jk}|$ are negligible compared with $|S_{jk}|$, and ignoring equation (5.8c). The linearized model can be more compactly represented in vector form in terms of the bus-by-line incidence matrix $C \in \{-1,0,1\}^{(N+1)\times N}$ of the radial network, defined by:

$$C_{jl} \;=\; \begin{cases} 1 & \text{if } l = j \rightarrow k \text{ for some bus } k \\ -1 & \text{if } l = i \rightarrow j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

Let $D_r := \text{diag}(r_l, l \in E)$ and $D_x := \text{diag}(x_l, l \in E)$ be the $N \times N$ diagonal matrices of line resistances and reactances. Let $s := (s_j, j \in \overline{N})$, $v := (v_j, j \in \overline{N})$, and $S := (S_l, l \in E)$. Then (5.26) in vector form is:

$$s \;=\; CS, \qquad\qquad C^{\mathsf{T}} v \;=\; 2\left(D_r P \,+\, D_x Q\right)$$

The matrix C is of rank N since the graph is connected, i.e., its columns are linearly independent. The null space of C^{T} is span($\mathbf{1}$). Any $N \times N$ submatrix of C obtained by removing any row of C is invertible (Theorem 25.31 of Appendix 25.2).

Given reference bus voltage v_0 and injections $\hat{s} := \left(s_j, j \in N\right)$ at other buses, we can derive an explicit solution for non-reference bus voltages $\hat{v} := \left(v_j, j \in N\right)$ and line flows S, from which s_0 can also be determined. Denote by c_0^{T} the first row of C corresponding to bus 0 and by $\hat{C}$ the $N \times N$ submatrix consisting of the remaining rows of C so that

$$C \;=:\; \begin{bmatrix} - & c_0^{\mathsf{T}} & - \\ & \hat{C} & \end{bmatrix}$$

Then the linearized model (5.26) is $s_0 = c_0^{\mathsf{T}} S$ and

$$\hat{s} \;=\; \hat{C} S \tag{5.27a}$$
$$v_0 c_0 \,+\, \hat{C}^{\mathsf{T}} \hat{v} \;=\; 2\left(D_r P \,+\, D_x Q\right) \tag{5.27b}$$

To derive an explicit solution for $(\hat{v}, S)$, let P_j denote the unique path from bus 0 to bus j, including both buses 0 and j. We use "$l \in \mathsf{P}_j$" to refer to a directed line l in the path P_j that points away from bus 0 and "$-l \in \mathsf{P}_j$" to refer to a directed line l in P_j that points towards bus 0. It can be verified direclty that the inverse $\hat{C}^{-1}$ of $\hat{C}$ is given by (Exercise 5.9)

$$\left[\hat{C}^{-1}\right]_{lj} \;=\; \begin{cases} -1 & l \in \mathsf{P}_j \\ 1 & -l \in \mathsf{P}_j \\ 0 & \text{otherwise} \end{cases}$$

and $\hat{C}^{-T} c_0 = -\mathbf{1}$. Then the solution to (5.27) is

$$S \;=\; \hat{C}^{-1} \hat{s} \tag{5.28a}$$
$$\hat{v} \;=\; v_0 \mathbf{1} \;+\; 2\left(R\hat{p} \,+\, X\hat{q}\right) \tag{5.28b}$$

where $R := \hat{C}^{-T} D_r \hat{C}^{-1}$, $X := \hat{C}^{-T} D_x \hat{C}^{-1}$ and $\hat{s} =: (\hat{p}, \hat{q})$.[5] As will be shown in Theorem 5.3, the matrices R and X have a simple interpretation where their (j,m) entries are the total resistance and reactance respectively in the common segment of the paths from bus 0 to buses j and m.

To gain insight on the power flow solution (v, S) given by (5.28) we consider two special graph orientation.

Down orientation: lines point away from bus 0. By setting $\ell_{jk} = 0$ in (5.6) the linearized model (5.26) reduces to:

$$\sum_{k:\, j \to k} S_{jk}^{\text{lin}} \;=\; S_{ij}^{\text{lin}} + s_j, \quad j \in \overline{N} \tag{5.29a}$$
$$v_j^{\text{lin}} - v_k^{\text{lin}} \;=\; 2\,\text{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}^{\text{lin}}\right), \;\; j \to k \in E \tag{5.29b}$$

where bus $i := i(j)$ in (5.29a) denotes the bus adjacent to j on the unique path from bus 0 to bus j. The boundary condition is: $S_{i0}^{\text{lin}} := 0$ in (5.29a) when $j = 0$ and $S_{jk}^{\text{lin}} = 0$ in (5.29a) when j is a leaf node. The set of equations (5.29) is called the *simplified DistFlow equations* in [19].

[5] We can interpret $\hat{L} := \hat{C}(D_r^{-1})\hat{C}^{\mathsf{T}}$ as a reduced Laplacian matrix. Then $R = \hat{L}^{-1}$.

Up orientation: lines point towards bus 0. A linear approximation of (5.7) is (setting $\overline{\ell}_{kj} = 0$):

$$\overline{S}_{ji}^{\text{lin}} = \sum_{k:k \to j} \overline{S}_{kj}^{\text{lin}} + s_j, \quad j \in \overline{N} \tag{5.30a}$$

$$\overline{v}_k^{\text{lin}} - \overline{v}_j^{\text{lin}} = 2\,\text{Re}\left(z_{kj}^{\mathsf{H}} \overline{S}_{kj}^{\text{lin}}\right), \; k \to j \in E \tag{5.30b}$$

where $i := i(j)$ in (5.30a) denotes the node adjacent to j on the unique path between node 0 and node j. The boundary condition is defined by $\overline{S}_{ji}^{\text{lin}} = 0$ in (5.30a) when $j = 0$ and $\overline{S}_{kj}^{\text{lin}} = 0, \ell_{kj} = 0$ in (5.30a) when j is a leaf node.

5.5.2 Analytical properties

Denote by T_j the subtree rooted at bus j, including j. We write "$k \in \mathsf{T}_j$" to mean bus k of T_j and "$(k,l) \in \mathsf{T}_j$" to mean line (k,l) of T_j. Denote by P_k the set of lines on the unique path from bus 0 to bus k. Recursing on the linear power flow equations (5.29) and (5.30) leads to explicit linear solutions given in the following theorem (see Exercise 5.10 for a proof).

Theorem 5.3 (Linear solutions)**.** Suppose assumption C5.1 holds. Fix any v_0 and $\hat{s} = (\hat{p}, \hat{q}) \in \mathbb{R}^{2N}$. Let $(v^{\text{lin}}, S^{\text{lin}}) \in \mathbb{R}^{N+2M}$ be the solution of (5.29) and $(\overline{v}^{\text{lin}}, \overline{S}^{\text{lin}}) \in \mathbb{R}^{N+2M}$ the solution of (5.30).

1. For any graph orientation let $\hat{C}$ denote the corresponding reduced incidence matrix. Then the unique solution of the linear model (5.27) is

$$\begin{aligned} S &= \hat{C}^{-1} \hat{s} \\ v &= v_0 \mathbf{1} \; + \; 2\left(R\hat{p} \; + \; X\hat{q}\right) \end{aligned}$$

   where $R := \hat{C}^{-T} D_r \hat{C}^{-1}$ and $X := \hat{C}^{-T} D_x \hat{C}^{-1}$. Moreover R and X are positive definite matrices.

2. For $(i,j) \in E$

$$\begin{aligned} S_{ij}^{\text{lin}} &= -\sum_{k \in \mathsf{T}_j} s_k, \qquad i \to j \\ \overline{S}_{ji}^{\text{lin}} &= \sum_{k \in \mathsf{T}_j} s_k, \qquad j \to i \end{aligned}$$

   Hence $S_{ij}^{\text{lin}} = -\overline{S}_{ji}^{\text{lin}}$.

3. For $j \in \overline{N}$

$$v_j^{\text{lin}} = \overline{v}_j^{\text{lin}} = v_0 \; + \; 2\sum_m \left(R_{jm} p_m \; + \; X_{jm} q_m\right)$$

   where

$$R_{jm} = \sum_{(i,k) \in \mathsf{P}_j \cap \mathsf{P}_m} r_{ik}, \qquad X_{jm} = \sum_{(i,k) \in \mathsf{P}_j \cap \mathsf{P}_m} x_{ik}$$

Theorem 5.3 says that, on each line $(i,j) \in E$, the power flow S_{ij} from i to j, or the power flow $\overline{S}_{ji}$ in the opposite direction, equals the total load $-\sum_{k\in\mathsf{T}_j} s_k$ in the subtree rooted at node j. These linear line flows neglect line losses and underestimate the required power to supply these loads. With zero line loss, we have $S_{ij}^{\text{lin}} = -\overline{S}_{ji}^{\text{lin}}$. The matrices R and X have a simple interpretation where their (j,m) entries are the total resistance and reactance respectively in the common segment of the paths from bus 0 to buses j and m.

This is not the case for solutions of nonlinear power flow equations (5.6) or (5.7). Indeed fix any v_0 and injections $\hat{s} \in \mathbb{R}^{2N}$ at non-reference buses in N. We can recurse on the power flow equations (5.6), starting from the leaf nodes, to show that any solution (v,ℓ,S) of (5.6) must satisfy (Exercise 5.12):

$$S_{ij} \;=\; -\sum_{k\in\mathsf{T}_j} s_k + \left(z_{ij}\ell_{ij} + \sum_{(k,l)\in\mathsf{T}_j} z_{kl}\ell_{kl} \right) \tag{5.31a}$$

$$v_j \;=\; v_0 - \sum_{(i,k)\in\mathsf{P}_j} \left(2\,\mathrm{Re}\left(z_{ik}^{\mathsf{H}} S_{ik} \right) - |z_{ik}|^2 \ell_{ik} \right) \tag{5.31b}$$

Similarly we can recurse on (5.7) to show that

$$\overline{S}_{ji} \;=\; \sum_{k\in\mathsf{T}_j} s_k - \sum_{(k,l)\in\mathsf{T}_j} z_{kl}\overline{\ell}_{kl} \tag{5.31c}$$

$$\overline{v}_j \;=\; v_0 + \sum_{(i,k)\in\mathsf{P}_j} \left(2\,\mathrm{Re}\left(z_{ik}^{\mathsf{H}} \overline{S}_{ik} \right) - |z_{ik}|^2 \overline{\ell}_{ik} \right) \tag{5.31d}$$

Summing (5.31a) and (5.31c) shows that

$$S_{ij} + \overline{S}_{ji} \;=\; z_{ij}\ell_{ij}$$

as we saw earlier in (5.5). Note that given v_0 and $s \in \mathbb{R}^{2N}$, Theorem 5.3 provides the unique solution $(v^{\text{lin}}, S^{\text{lin}})$ to (5.29) (or unique solution $(\overline{v}^{\text{lin}}, \overline{S}^{\text{lin}})$ to (5.30)). For nonlinear model (5.6) or (5.7), the solutions (v,ℓ,S) or $(\overline{v},\overline{s},\overline{S})$ may not be unique. Any nonlinear solution however must satisfy (5.31).

It is proved in the solution of Exercise 5.10 that, for $j \in N$, the linear solutions satisfy:

$$v_j^{\text{lin}} \;=\; v_0 - \sum_{(i,k)\in\mathsf{P}_j} 2\,\mathrm{Re}\left(z_{ik}^{\mathsf{H}} S_{ik}^{\text{lin}} \right) \tag{5.32a}$$

$$\overline{v}_j^{\text{lin}} \;=\; v_0 + \sum_{(i,k)\in\mathsf{P}_j} 2\,\mathrm{Re}\left(z_{ik}^{\mathsf{H}} \overline{S}_{ik}^{\text{lin}} \right) \tag{5.32b}$$

Comparing these relations and (5.31) leads to bounds on the nonlinear solutions in the following corollary (see Exercise 5.13 for a proof). Recall that, by definition, x is a power flow solution only if $v \geq 0$ and $\ell \geq 0$ componentwise (assuming $z_{ij} = (r_{ij}, x_{ij}) \geq 0$).

Corollary 5.4 (Bounds on nonlinear solutions)**.** Suppose assumption C5.1 holds. Fix any v_0 and $\hat{s} \in \mathbb{R}^{2N}$. Let (v,ℓ,S) and $(\overline{v},\overline{\ell},\overline{S})$ in $\mathbb{R}^{N+3M}$ be any (possibly nonunique) solutions of (5.6) and (5.7) respectively. Let $(v^{\text{lin}}, S^{\text{lin}})$ and $(\overline{v}^{\text{lin}}, \overline{S}^{\text{lin}})$ in R^{N+2M} be the unique solutions of their linearizations (5.29) and (5.30) respectively. Then

1. For $i \to j \in E$, $S_{ij} \geq S_{ij}^{\text{lin}}$ with equality if only if ℓ_{ij} and all ℓ_{kl} in T_j are zero.
2. For $j \to i \in E$, $\overline{S}_{ji} \leq \overline{S}_{ji}^{\text{lin}}$ with equality if and only if all ℓ_{kl} in T_j are zero.
3. For $j \in \overline{N}$, $v_j = \overline{v}_j \leq \overline{v}_j^{\text{lin}} = v_j^{\text{lin}}$.

Remark 5.2. While it is easy to prove $\overline{v}_j \leq \overline{v}_j^{\text{lin}}$ from (5.31) and Theorem 5.3, it does not seem easy to prove $v_j \leq v_j^{\text{lin}}$ directly, except by relating the variables (v_j, v_j^{lin}) to $(\overline{v}_j, \overline{v}_j^{\text{lin}})$ in the opposite direction. This is an advantage of the models (5.7) and (5.30) in the up orientation. □

Remark 5.3. *Bounds for SOCP relaxation.* The bounds in Corollary 5.4 do not depend on the quadratic equalities (5.6c) and (5.7c) as long as $\ell_{jk} \geq 0$. In particular the bounds hold if the equalities are relaxed to inequalities $v_j \ell_{jk} \geq |S_{jk}|^2$. These bounds are used in Chapter **??** to prove a sufficient condition for exact SOCP relaxation of optimal power flow problems for radial networks. □

Remark 5.4. *Linear approximations.* For radial networks the linear approximations (5.29) and (5.30) of BFM have two advantages over the (linear) DC approximation of BIM. First they have a simple recursive structure that leads to simple bounds on power flow quantities. Second DC approximation assumes $r_{jk} = 0$, fixes voltage magnitudes, and ignores reactive power, whereas (5.29) and (5.30) do not. This is important for distribution systems where r_{jk} are not negligible, voltages can fluctuate significantly and reactive powers are used to regulate them. On the other hand (5.29) and (5.30) are applicable only to radial networks whereas DC approximation applies to meshed networks as well. □

5.5.3 Example application: local volt/var control

To illustrate application of the linear model (5.26) or (5.27), we present a method to design a local algorithm to stabilize voltages on a distribution network.

Model. We assume the network is radial and can be modeled by the set of single-phase linear power flow equations (5.26) or (5.27). We assume on each bus j there is a fixed and given active and reactive load $s_j^0 := \left(p_j^0, q_j^0\right)$. In addition there is possibly an inverter on bus j with a fixed active power injection p_j and an adaptable reactive power injection q_j. For example, p_j may represent solar generation. The problem of volt/var control is to adapt the reactive outputs q_j in order to stabilize voltages on the network. In this subsection we will use for notational simplicity $s = (p, q) \in \mathbb{R}^{2N}$, $v \in \mathbb{R}^N$ to denote variables at non-reference buses, instead of $\hat{s}, \hat{v}$. Using (5.28b) the voltages v depend on power injections according to

$$v \;=\; v_0 \mathbf{1} \;+\; 2\left(R(p - p^0) \;+\; X(q - q^0)\right)$$

We write $v := v(q)$ explicitly as a function of the control q as

$$v(q) \;=\; 2Xq \;+\; \tilde{v} \tag{5.33}$$

where $\tilde{v} := v_0 \mathbf{1} + 2R\left(p - p^0\right) - 2Xq^0$ does not depend on q.

A common model of inverters constrains the reactive power q_j to the sector $\{q_j : p_j^2 + q_j^2 \leq \sigma^2\}$ with a power factor limit $-\theta_j \leq \tan^{-1}(q_j/p_j) \leq \theta_j \leq \pi/2$. Equivalently the control q_j is constrained to the sector U_j determined by the given active power $\tilde{p}_j$:

$$U_j \;\; := \;\; \left\{ q_j \; : \; \underline{q}_j \leq q_j \leq \overline{q}_j \right\}, \qquad j = 1, \ldots, N \tag{5.34}$$

where $\overline{q}_j := \min\left\{ p_j \tan\theta_j, \; \sqrt{\sigma^2 - p_j^2} \right\}$ and $\underline{q}_j := \max\left\{ -p_j \tan\theta_j, \; -\sqrt{\sigma^2 - p_j^2} \right\}$. Let $U := U_1 \times \cdots \times U_N$. If the reactive power q_j of the inverter at bus j is fixed and not controllable, this can be modeled by setting $\underline{q}_j = q_j = \overline{q}_j$. If there is no inverter at bus j, then we set $p_j = \underline{q}_j = \overline{q}_j := 0$.

Local memoryless control. Let v^{ref} be a given vector of reference voltages. Our goal is to choose control $q \in U$ to drive voltages towards v^{ref}. We require our control to be local, i.e., $q_j(t+1)$ depends only on voltage $v_j(t)$ at bus j not other buses, and memoryless, i.e., $q(t+1)$ depends only on $v(t)$ but not $v(s), s < t$. In particular, q_j is a function only of voltage discrepancy $v_j(t) - v_j^{\text{ref}}$, of the form

$$q_j(t+1) \;\; = \;\; \left[u_j \left(v_j(t) - v_j^{\text{ref}} \right) \right]_{U_j}, \qquad j = 1, \ldots, N$$

where $v_j(t)$ is the measured local voltage, $u_j : \mathbb{R} \to \mathbb{R}$ is a control function that maps a voltage deviation $v_j(t) - v_j^{\text{ref}}$ into a potential reactive power setting, $[a]_{U_j} := \max\left\{ \underline{q}_j, \min\left\{ a, \overline{q}_j \right\} \right\}$ is the projection onto U_j. Such a local memoryless control is simple to implement as it requires no communications among controllers at different buses.

The local volt/var control problem in our formulation boils down to the design of the control function u_j. Many functions u_j have been proposed and analyzed in the literature. We now present such a control from [27, 28]. From Theorem 5.3.3,

$$\frac{\partial v_j}{\partial q_j} \;\; = \;\; 2X_{jj} \;\; = \;\; \sum_{(i,k) \in \mathsf{P}_j} x_{ik} \;\; > \;\; 0$$

Therefore it is natural to choose a control function u_j that is nonincreasing in voltage discrepancy $v_j(t) - v_j^{\text{ref}}$. An example u_j is shown in Figure 5.6(a).

Closed-loop behavior. Consider the closed-loop system under a local control u_j. Suppose the voltages evolve according to Linear DistFlow model (5.33), i.e., suppose the measured voltages at time t is $v_j(t) = v_j(q(t))$. Then the closed-loop system is a discrete-time dynamical system defined by the control function $u_j : \mathbb{R} \to \mathbb{R}$ followed by a projection onto U_j:

$$q_j(t+1) \;\; = \;\; \left[u_j \left(v_j(q(t)) - v_j^{\text{ref}} \right) \right]_{U_j}, \qquad j = 1, \ldots, N \tag{5.35}$$

where $v_j(q)$ is given by (5.33). If $q^* = \left[u\left(v(q^*) - v^{\text{ref}} \right) \right]_U$ then q^* is called a fixed point, or an equilibrium point, of (5.35).

We now analyze the convergence and optimality of a class of u_j in (5.35) that satisfy the following assumptions:

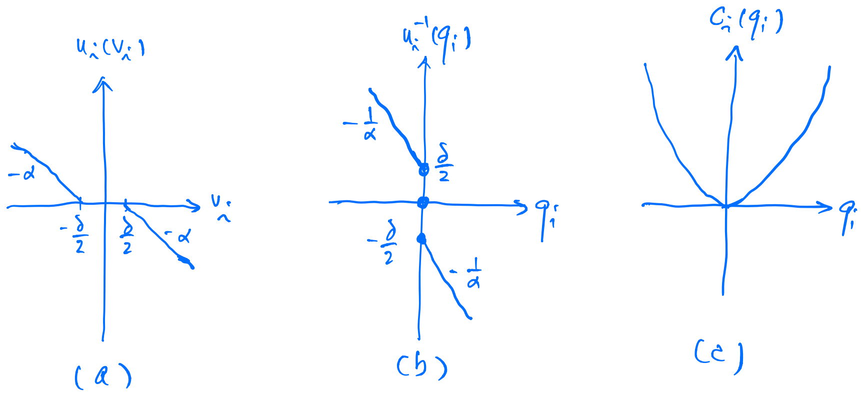


Figure 5.6: (a) The piecewise linear control function $u_j(v_j)$, (b) its inverse $u_j^{-1}(q_j)$, and (c) the implied cost function $c_j(q_j)$.

> C5.2: The control functions u_j are differentiable on $\mathbb{R}$ and there exist α_j such that $\left|u_j'(v_j)\right| \leq \alpha_j$ for all $v_j \in \mathbb{R}$.
>
> C5.3: The control functions u_i are strictly decreasing on $\mathbb{R}$.

The differentiability assumption in C5.2 can be relaxed to allow control functions with a deadband and saturation as shown in Figure 5.6(a) (see [28]). Let $A := \mathrm{diag}(\alpha_j, j \in N)$.

Theorem 5.5 (Convergence)**.** Suppose assumptions C5.1 and C5.2 hold. If the largest singular value $\sigma_{\max}(AX) < 1/2$ then there exists a unique equilibrium point $q^* \in U$ and the volt/var control (5.35) converges to q^* geometrically, i.e.,

$$\|q(t) - q^*\| \quad \leq \quad \beta^t \, \|q(0) - q^*\| \quad \rightarrow \quad 0$$

for some $\beta \in [0, 1)$.

Proof. Applying the mean value theorem to the control function $u_j(v_j)$ we have

$$u_j(v_j) - u_j(\hat{v}_j) \quad = \quad u_j'(w)(v_j - \hat{v}_j)$$

where $w = \lambda v_j + (1-\lambda)\hat{v}_j$ for some $\lambda \in [0, 1]$. Therefore

$$\|u(v) - u(\hat{v})\|_2^2 \quad = \quad \sum_j \left|u_j(v_j) - u_j(\hat{v}_j)\right|^2 \quad \leq \quad \sum_j \left|\alpha_j(v_j - \hat{v}_j)\right|^2 \quad = \quad \|A(v - \hat{v})\|_2^2$$

where the inequality follows from the mean value theorem and assumption C5.2. Hence $\|u(v) - u(\hat{v})\|_2 \leq \|A(v - \hat{v})\|_2$. Applying the chain rule to $Av = Av(q)$ as a vector-valued function of q we have

$$\frac{\partial Av}{\partial q}(q) \quad = \quad A\frac{\partial v}{\partial q} \quad = \quad 2AX$$

Therefore

$$\left\| u\left(v(q) - v^{\mathrm{ref}}\right) \; - \; u\left(v(\hat{q}) - v^{\mathrm{ref}}\right) \right\|_2 \quad \leq \quad \|Av(q) - Av(\hat{q})\|_2 \quad \leq \quad \|2AX\|_2 \, \|q - \hat{q}\|_2$$

where the first inequality follows from $\|u(v)-u(\hat{v})\|_2 \le \|A(v-\hat{v})\|_2$ and the second inequality follows from the mean value Theorem 25.29 for vectored-valued functions in Appendix 25.1.8. Since the induced matrix norm $\|M\|_2 = \sigma_{\max}(M)$ (Exercise 5.14) we have

$$\left\| u\left(v(q) - v^{\text{ref}}\right) \; - \; u\left(v(\hat{q}) - v^{\text{ref}}\right) \right\|_2 \;\; \le \;\; 2\,\sigma_{\max}(AX)\; \|q-\hat{q}\|_2$$

Therefore the control function $u\left(v(q)-v^{\text{ref}}\right)$ as a function of q is a contraction. Since projection onto U is nonexpansive by the Projection Theorem 11.9 of Appendix **??**, the function on the right-hand side of (5.35), as a function of q, is a contraction. The theorem then follows from the Contraction Mapping Theorem 11.11. □

We next show that the equilibrium point q^* implicitly optimizes a cost function implied by the control function u. Under assumption C5.3, the inverse functions u_j^{-1} exist and are strictly decreasing on $\mathbb{R}$. We hence can define $c_j : \mathbb{R} \to \mathbb{R}$ by

$$c_j(q_j) \;\; := \;\; -\int_0^{q_j} u_j^{-1}(\hat{q}_j)\, d\hat{q}_j, \qquad j \in N$$

Moreover c_j is strictly convex since $c_j''(q_j) = -1/u_j'(q_j) > 0$ under assumptions C5.2 and C5.3. Consider the optimization problem

$$\min_{q\in U} \quad \sum_j c_j(q_j) \; + \; q^{\mathsf{T}} X q \; + \; q^{\mathsf{T}} \Delta\tilde{v} \tag{5.36}$$

where $\Delta\tilde{v} := \tilde{v} - v^{\text{ref}}$.

Theorem 5.6 (Optimality)**.** Suppose assumptions C5.1, as well as C5.2 and C5.3 hold. Then the unique equilibrium point $q^* \in U$ of (5.35) is the unique minimizer of (5.36).

Proof. Let $C(q) := \sum_j c_j(q_j) + q^{\mathsf{T}} X q + q^{\mathsf{T}}\Delta\tilde{v}$ denote the objective function of (5.36). Since X is positive definite and c_j are strictly convex, $C(q)$ is strongly convex (and hence also continuous on $\mathbb{R}^N$). This implies, in particular, that if a minimizer of (5.36) exists (e.g., if U is bounded), then it is unique. It therefore suffices to show that q^* is an equilibrium point of (5.35) if and only if it is a minimizer of (5.36).

Since (5.36) is a convex problem, $q^* \in U$ is optimal if and only if

$$\left(\nabla C(q^*)\right)^{\mathsf{T}} (q - q^*) \;\; \ge \;\; 0 \qquad \forall q \in U$$

Since each U_j in (5.34) is a box constraint, this means the optimal $q^* \in U$ is optimal if and only if (Exercise 5.15)

$$q_j^* \;\in\; (\underline{q}_j, \overline{q}_j) \qquad \text{only if} \qquad [\nabla C(q^*)]_j = 0 \tag{5.37a}$$

$$q_j^* \;=\; \underline{q}_j \qquad \text{if} \qquad [\nabla C(q^*)]_j > 0 \tag{5.37b}$$

$$q_j^* \;=\; \overline{q}_j \qquad \text{if} \qquad [\nabla C(q^*)]_j < 0 \tag{5.37c}$$

We have from (5.33) and (5.35)

$$\nabla C(q^*) \;\; = \;\; \nabla c(q^*) \, + \, 2Xq^* \, + \, \Delta\tilde{v} \;\; = \;\; \nabla c(q^*) \, + \, \left(v(q^*) \, - \, v^{\text{ref}}\right)$$

where $\nabla c(q^*) = (c'_j(q^*_j) = -u_j^{-1}(q^*_j),\, i \in N)$. Therefore

$$[\nabla C(q^*)]_j \;=\; -u_j^{-1}(q^*_j) \,+\, \left(v_j(q^*_j) \,-\, v_j^{\text{ref}}\right)$$

Since $u_j(v_j)$ is strictly decreasing in v_j we have

$$\begin{aligned}
[\nabla C(q^*)]_j = 0 \quad &\Longleftrightarrow \quad u_j\left(v_j(q^*_j) - v_j^{\text{ref}}\right) = q^*_j \\
[\nabla C(q^*)]_j > 0 \quad &\Longleftrightarrow \quad u_j\left(v_j(q^*_j) - v_j^{\text{ref}}\right) < q^*_j \\
[\nabla C(q^*)]_j < 0 \quad &\Longleftrightarrow \quad u_j\left(v_j(q^*_j) - v_j^{\text{ref}}\right) > q^*_j
\end{aligned}$$

Substituting this into (5.37) shows that $q^* = \left[u\left(v(q^*) - v^{\text{ref}}\right)\right]_U$, i.e., q^* is the unique equilibrium point of (5.35). This shows that q^* is an equilibrium point of (5.35) if and only if it is a minimizer of (5.36). □

Remark 5.5. Theorem 5.6 shows that the control function in (5.35) implies an objective function $C(q)$ in (5.36) that an equilibrium implicitly optimizes. This is often referred to as reverse engineering. One can also start by designing an objective function $C(q)$ and deriving a control function as an iterative algorithm to solve the optimization problem (5.36). This is referred to as forward engineering; see e.g. [27, 28]. Often these algorithms require some communications among controllers at different buses but are guaranteed to converge under less stringent requirement than that in Theorem 5.5. For instance, since X is positive definite and $c(q)$ is convex, the objective function in (5.36) is strongly convex. Theorem 11.14 in Appendix **??** implies that a gradient projection algorithm is a contraction and converges geometrically to the unique optimal solution.

The formulation here imposes limits $[\underline{q}, \overline{q}]$ on the control q. It is pointed out in [29] that local memoryless control such as (5.35) may not be able to stabilize the equilibrium voltages $v(q^*)$ to within an apriori range $[\underline{v}, \overline{v}]$ (see Exercise 5.17). Alternative formulation imposes apriori limits $[\underline{v}, \overline{v}]$ on equilibrium voltages $v(q^*)$ but relaxes limits on the control q using control laws with internal state, see e.g. [29] □

5.6 Bibliographical notes

A branch flow model, called the DistFlow equations, is proposed in [18, 19] for radial networks. Its key feature is that it does not involve phase angles of voltage and current phasors. This is extended to general meshed network in [30] by introducing a cycle condition. All of these models assume zero shunt admittances on the lines. Shunt admittances of the lines are added to the branch flow model in [31]. The main difference of the model (5.4) from the model in [18, 19, 30] is the use of undirected rather than directed graph when shunt elements are included so that line currents and power flows are defined in both directions. The equivalence of BFM and bus injection model (BIM) is proved in [32]. The equivalence of DistFlow to BFM in complex form and hence equivalent to BIM follows from [30, Theorems 2, 4]. Theorem 5.1 is from [33]. For BFM and SOCP relaxations when a radial network contain ideal transformers and multiple lines between two buses, see [34].

The linearized model (5.26) is first proposed in [19] and called the Simplified DistFlow equations. The paper also states an explicit solution for the squared voltage magnitude v_i as an affine function of the injections s_j whose coefficients ξ_{ij} are the total impedances on the common paths P_i° and P_j° from the root (bus 0) to buses i and j respectively. This is the same solution as that in Theorem 5.3. The properties in Theorem 5.3 and Corollary 5.4 of the linear model appear in [35, 27, 28, 36] where $v_i - v_j$ is sometimes approximated by $2(|V_i| - |V_j|)$ since $|V_i| \approx 1$ pu. Our discussion on the local volt/var control algorithm follows [27, 28].

Backward forward sweep. Power flow solutions for general networks are mostly based on Newton-Raphson and its variants, or more recently, interior-point methods. Another approach has been developed for radial networks, both single-phase and three-phase networks, that exploits their tree structure. The idea of backward forward sweep (BFS) is first proposed in [37] for three-phase distribution systems. Early examples of BFS algorithms for three-phase radial networks are designed in [38][39, Chapter 10.1.3]. The BFS method for single-phase networks described in Chapter 5.4.2 is from [24]. It is extended in [40] to allow PV buses by computing line power flows S_{jk} instead of currents I^s_{jk}. Both algorithms (with extensions for meshed networks) were developed for weakly meshed transmission systems as well as distribution systems. Another variant of BFS, proposed in [**?**], calculates voltages in both forward and backward iterations in linear feeders with voltage-dependent loads. The BFS algorithm in [24] is extended in [41] from single-phase to three-phase networks, and in [**?**] to four-wire neutral-grounded networks. In [**?**], three-phase voltages and line currents are calculated with generalized line models that incorporate transformers and constant impedance loads. Transformers of different configurations have been included in BFS through modified augmented nodal analysis [**?**]. Some of these works are briefly discussed in [42]. BFS algorithms tend to have better convergence properties than general algorithms such as Newton-Raphson. Simulation results in [43] suggest however that Newton-Raphson converges in a smaller number of iterations.

The solution approach in the original DistFlow paper [19] uses one-time backward sweep to express all variables in terms of the power injections at the feeder head and all branch points followed by a Newton-Raphson algorithm to solve for these injections. The existence and uniqueness of solutions are studied in [44]. By exploiting the approximate sparsity of the Jacobian matrix in [19], approximate fast decoupled methods are developed and their convergence properties analyzed in [45]. These methods are extended to three-phase radial networks in [43]. The existence and uniqueness of power flow solutions of three-phase DistFlow model is analyzed in [46].

5.7 Problems

Chapter 5.1.

Exercise 5.1 (Line angles $\beta(x)$). Justify the definition of line angles in (5.2) using (5.1b)(5.1c)(5.1d).

Chapter 5.2.

Exercise 5.2 (Incidence matrix C)**.** Consider the $(N+1)\times M$ incidence matrix C of a (connected) radial network defined by:

$$C_{jl} \quad = \quad \begin{cases} 1 & \text{if } l = j \to k \text{ for some bus } k \\ -1 & \text{if } l = i \to j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

Show that C has rank $N = M$, the null space of C^{T} is span$(\mathbf{1})$ and its pseudo-inverse $\left(C^{\mathsf{T}}\right)^{\dagger} = C\left(C^{\mathsf{T}}C\right)^{-1}$

Exercise 5.3 (Line loss)**.** Consider a radial network where lines have zero shunt admittances. Show that, under assumption C5.1, substituting $y^m_{jk} = y^m_{kj} = 0$ into (5.4) for all lines $(j,k) \in E$ leads to:

$$S_{jk} \; + \; S_{kj} \quad = \quad z^s_{jk}\ell_{jk} \quad = \quad z^s_{jk}\ell_{kj}$$

Exercise 5.4 (Graph orientation)**.** Prove (5.10) under assumption C5.1, i.e., x satisfies (5.8) if and only if $\hat{x} := g(x)$ satisfies (5.9)

Exercise 5.5 (DistFlow equations)**.** Suppose assumption C5.1 holds so that (5.6) and (5.7) are valid models. Show that $\mathbb{X}_{\text{tree}} \equiv \mathbb{T}_0 \equiv \overline{\mathbb{T}}_0$ where these sets are defined in (5.16) when shunt admittances $y^m_{jk} = y^m_{kj} = 0$ for all lines (j,k).

Chapter 5.4

Exercise 5.6 (Backward forward sweep)**.** Suppose assumption C5.1 holds. For the backward forward sweep algorithm in Chapter 5.4.2 based on the branch flow model (5.1) in complex form, show that all other variables, such as injection s_0 and branch flows S_{jk} in (5.1), can be computed once (V_j, I^s_{jk}) for all $j \in \overline{N}$ and all $j \to k \in E$ are determined.

Exercise 5.7 (Backward forward sweep)**.** This exercise solves the same power flow equations as the BFS described in Chapter 5.4.2, under assumption C5.1, except that here we are to derive a BFS algorithm to compute the sending-end current I_{jk} for every line $j \to k$, instead of I^s_{jk} over the series impedance, as well as the voltage V_j for every bus j.

Exercise 5.8 (Backward forward sweep)**.** Consider a 2-bus network and prove a sufficient condition for BFS to converge under assumption C5.1.

Chapter 5.5. Assumption C5.1 is assumed for Chapter 5.5 for linearized DistFlow models and hence the problems in this section.

Exercise 5.9 (Inverse of reduced incidence matrix $\hat{C}$)**.** Consider the $(N+1) \times M$ incidence matrix C of a (connected) radial network defined in Exercise 5.2. Denote by c_0^{T} the first row of C corresponding to bus 0 and by $\hat{C}$ the $N \times N$ submatrix consisting of the remaining rows of C so that

$$C \ =: \ \begin{bmatrix} - \ c_0^{\mathsf{T}} \ - \\ \hat{C} \end{bmatrix}$$

Show that:

1. The inverse $\hat{C}^{-1}$ of $\hat{C}$ is given by

$$\left[\hat{C}^{-1}\right]_{lj} \ = \ \begin{cases} -1 & l \in \mathsf{P}_j \\ 1 & -l \in \mathsf{P}_j \\ 0 & \text{otherwise} \end{cases}$$

   Here "$l \in \mathsf{P}_j$" means a directed line l that points away from bus 0 and is in the unique path P_j from bus 0 to bus j, and "$-l \in \mathsf{P}_j$" means a directed line l in P_j that points towards bus 0.

2. $\hat{C}^{-T} c_0 = -\mathbf{1}$.

Exercise 5.10 (Linear solutions)**.** Prove Theorem 5.3.

Exercise 5.11 (Sensitivity matrix $\hat{L}^{-1}$)**.** Consider the bus injection model for a radial network:

$$I_j \ = \ \sum_{k: j \to k} \tilde{I}_{jk}, \qquad\qquad \tilde{I}_{jk} \ = \ y^s_{jk}(V_j - V_k)$$

where V_j, I_j are nodal voltages and current injections respectively at bus j, and $\tilde{I}_{jk}$ is the (sending-end) line current from buses j to k. We assume $y^m_{jk} = y^m_{kj} = 0$. Let $\hat{L}$ denote the $N \times N$ reduced incidence matrix, which is invertible since the network has a tree topology. Let $\hat{V} := (V_j, j \neq 0)$ and $\hat{I} := (I_j, j \neq 0)$.

1. Show that $\hat{V} = \hat{L}^{-1}\hat{I}$, i.e., the matrix $\hat{L}^{-1}$ is the sensitivity of nodal voltages to current injections.
2. Show that $\left[\hat{L}^{-1}\right]_{jm} = \sum_{(i,k) \in \mathsf{P}_j \cap \mathsf{P}_m} 1/y_{ik}$.

Exercise 5.12 (Nonlinear recursion)**.** Derive (5.31) from the DistFlow equations (5.6) and (5.7).

Exercise 5.13 (Bounds)**.** Prove Corollary 5.4.

Exercise 5.14 (Induced matrix norm)**.** For any $n \times n$ matrix M show that the induced norm

$$\|A\|_2 \; := \; \max_{\|x\|_2 = 1} \|Ax\|_2 \; = \; \sigma_{\max}(A)$$

where $\sigma_{\max}(A)$ is the largest singular value of A.

Exercise 5.15. [Local volt/var control] Let $U_j := \{x_j : \underline{x}_j \leq x_j \leq \overline{x}_j\}$, $j = 1, \ldots, n$, and $U := U_1 \times \cdots \times U_n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$. Show that

$$x^* \in U, \qquad f^{\mathsf{T}}(x^*)\,(x - x^*) \;\geq\; 0 \quad \forall x \in U \tag{5.38}$$

if and only if

$$\begin{aligned} x_j^* &\in (\underline{x}_j, \overline{x}_j) && \text{only if} && f_j(x^*) = 0 && \text{(5.39a)} \\ x_j^* &= \underline{x}_j && \text{if} && f_j(x^*) > 0 && \text{(5.39b)} \\ x_j^* &= \overline{x}_j && \text{if} && f_j(x^*) < 0 && \text{(5.39c)} \end{aligned}$$

Exercise 5.16. [Local volt/var control] Let the control function in (5.35) be $u_j(v_j) = -\gamma_j v_j$ with $\gamma_j > 0$. Derive the condition for convergence and the resulting cost function $C(q)$.

Exercise 5.17. [Local volt/var control] Suppose it is desirable to asymptotically stabilize the voltages v to within a certain bounds $[\underline{v}, \overline{v}]$ while maintaining the limits $[\underline{q}, \overline{q}]$ on the reactive power.

1. Show that there exists $\tilde{v}$ such that no equilibrium point of (5.35) can lie in $[\underline{v}, \overline{v}]$.
2. Fix $\tilde{v}$. For each bus j, find the maximum $\underline{v}_j$ and minimum $\overline{v}_j$ for which it is possible to asymptotically stabilize v_j to within $[\underline{v}_j, \overline{v}_j]$. Note that it may not be possible for v_j to attain $\underline{v}_j$ (or $\overline{v}_j$) *smultaneously for all j.*

Chapter 6

Linear models

Part II

Unbalanced multiphase networks

Chapter 7

Component models, I: devices

Single-phase models are a good approximation of the reality for many transmission network applications where lines are symmetric and loads are balanced. In that case, a similarity transformation produces three networks in a sequence coordinate, called zero, positive, and negative-sequence networks, that are decoupled. Each network can be analyzed using a single-phase model studied in previous chapters. These sequence networks are coupled when lines are not transposed or equally spaced, e.g., as in distribution systems, or when loads are unbalanced or nonlinear, e.g., AC furnaces, high-speed trains, power electronics, or single or two-phase laterals in distribution networks. In that case single-phase analysis can produce incorrect power flow solutions. In this and next chapters we extend single-phase models to unbalanced three-phase models.

We first provide in Chapter 7.1 an overview of models for three-phase devices, lines and transformers, and how to use these component models to compose an overall network model. We summarize in Chapter 7.2 mathematical properties that underly the behavior of three-phase systems. Finally we derive in Chapter 7.3 the models of three-phase voltage sources, current sources, power sources, and impedances in Y and Δ configurations. In Chapter 8 we derive models for three-phase lines and transformers. We will use these component models in Chapters 9 and 10 to construct network models and study unbalanced three-phase analysis.

7.1 Overview

Figure 7.1 shows a simple example of a three-phase system with three components, two devices connected by a line. For example the single-terminal device on the left can model a three-phase generator and the other single-terminal device can be a three-phase load. Each terminal has three wires (or ports or conductors) indexed by its phases a, b, c, and possibly a neutral wire indexed by n. Internally, it can be in Y or Δ configuration, and the Y configuration may have a neutral wire that may be grounded. A three-phase line has two terminals, each terminal with three or four wires, and it connects two single-terminal devices, one at each end of the line. The line may model a transmission or distribution line or a transformer. The distribution line can be underground or overhead with a neutral wire that may be grounded in regular spacing along the line.

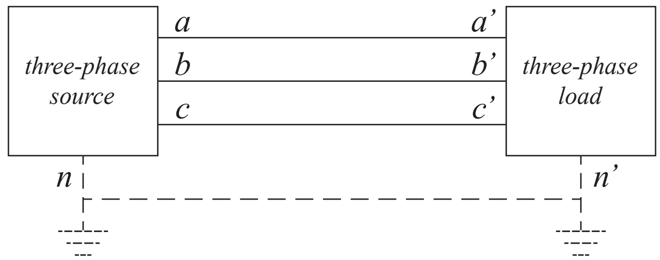
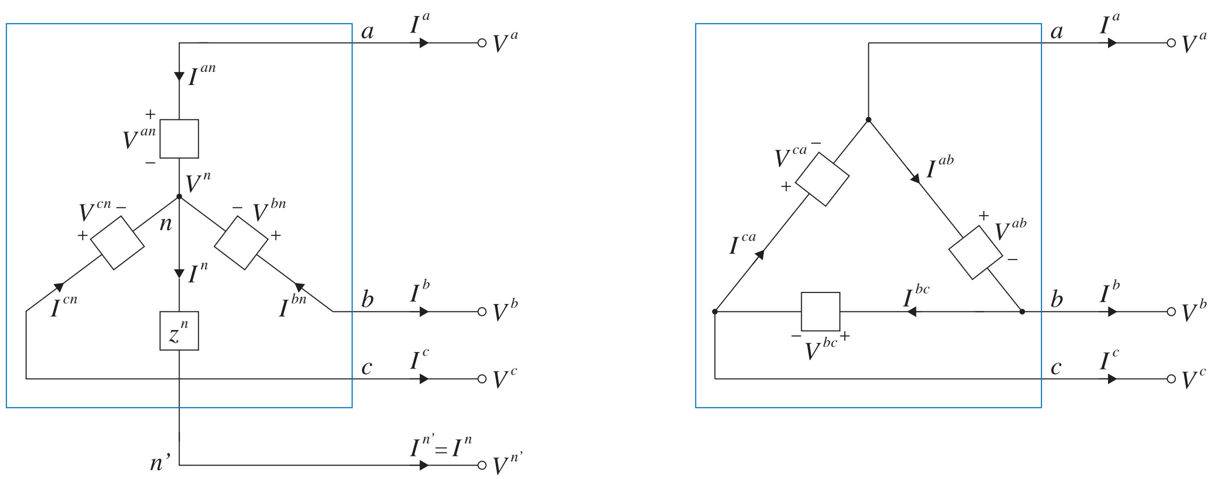


Figure 7.1: A simple model of a three-phase system consisting of a source connected through a line to a load.

The basic idea in modeling a three-phase component is to explicitly separate its model into an *internal model* that specifies the characteristics of the constituent single-phase components in terms of internal variables, and a *conversion rule* that maps its internal variables to its terminal variables. The internal model depends only on the type of components (non-ideal voltage sources, ZIP loads, or different single-phase transformer models) regardless of their configurations. The conversion rule depends only on their configurations regardless of the type of components. They determines an *external model* which is a relation between the terminal variables, obtained by eliminating the internal variables from the set of equations describing the internal model and the conversion rule. We next describe this procedure in detail.

7.1.1 Internal and terminal variables



(a) Y configuration

(b) Δ configuration

Figure 7.2: Internal and external variables associated with a single-terminal device in Y and Δ configurations.

The *internal variables* of a generic single-terminal device are shown in Figure 7.2 and defined as follows:

- $V^Y := \left(V^{an}, V^{bn}, V^{cn}\right) \in \mathbb{C}^3$, $I^Y := \left(I^{an}, I^{bn}, I^{cn}\right) \in \mathbb{C}^3$, $s^Y := \left(s^{an}, s^{bn}, s^{cn}\right) \in \mathbb{C}^3$, $(V^n, I^n, s^n) \in \mathbb{C}^3$:

line-to-neutral voltages, currents, and power across the single-phase devices in Y configuration, as well as the voltage, current, and power across the neutral impedance z^n, respectively. By definition $s^{an} := V^{an} \left(I^{an}\right)^{\mathsf{H}}$ is the power across the phase-a device, etc. The neutral voltage V^n, with respect to a common reference point, is generally nonzero. A Y-configured device may or may not have a neutral line which may or may not be grounded and the grounding impedance z^n may or may not be zero. When present, the current on the neutral line is denoted by I^n in the direction coming out of the device. The Kirchhoff current law dictates that $I^n = \sum_\phi I^{\phi n}$. The internal power across the neutral impedance is $s^n := \left(V^n - V^{n'}\right)\bar{I}^n$ where $\bar{I}^n$ denotes the complex conjugate of I^n. The term $V^n \bar{I}^Y$, in contrast, is the vector power delivered across the neutral and the common reference point (e.g., the ground).

- $V^\Delta := \left(V^{ab}, V^{bc}, V^{ca}\right) \in \mathbb{C}^3$, $I^\Delta := \left(I^{ab}, I^{bc}, I^{ca}\right) \in \mathbb{C}^3$, $s^\Delta := \left(s^{ab}, s^{bc}, s^{ca}\right) \in \mathbb{C}^3$: line-to-line voltages, currents, and power across the single-phase devices respectively in Δ configuration. By definition $s^{ab} := V^{ab} \left(I^{ab}\right)^{\mathsf{H}}$ is the power across the phase-a device, etc.

Note that the direction of the internal power s^{an} or s^{ab} across a single-phase device is defined in the direction of the current across the device. The neutral line, when present, is often assumed grounded, i.e., $V^{n'} = 0$, and the voltage reference point is the ground. In this case $s^n = V^n I^{n\mathsf{H}}$.

The *terminal variables* of the single-terminal device in Figure 7.2 are defined as follows:

- $V := \left(V^a, V^b, V^c\right) \in \mathbb{C}^3$, $I := \left(I^a, I^b, I^c\right) \in \mathbb{C}^3$, $s := \left(s^a, s^b, s^c\right) \in \mathbb{C}^3$, $\left(V^{n'}, I^{n'}, s^{n'}\right) \in \mathbb{C}^3$: terminal voltages, currents, and power respectively. The terminal voltage V is defined with respect to an arbitrary but common reference point, e.g., the ground. The terminal current I is defined in the direction coming out of the device, i.e., I is defined to be the current injection from the device to the rest of the network when it is connected to a bus bar, regardless of whether it generates or consumes power. By definition $s^a := V^a \left(I^a\right)^{\mathsf{H}}$ is the power across the terminal a and the common reference point. When there is a neutral wire its terminal voltage (with respect to the common reference point), current and power are denoted by $\left(V^{n'}, I^{n'}, s^{n'}\right)$ with $I^{n'} = I^n$ and $s^{n'} := V^{n'} I^{n'\mathsf{H}} = V^{n'} I^{n\mathsf{H}}$.

The internal and external variables of a three-phase device are summarized in Table 7.1.

	Voltage	Current	Power	Neutral line
Internal variable	$V^{Y/\Delta}$	$I^{Y/\Delta}$	$s^{Y/\Delta}$	(V^n, I^n, s^n)
External variable	V	I	s	$\left(V^{n'}, I^{n'}, s^{n'}\right)$

Table 7.1: Internal and external variables of single-terminal three-phase devices. The notation $x^{Y/\Delta}$ is a shorthand for the pair (x^Y, x^Δ).

7.1.2 Three-phase device models

An *internal model* of a three-phase device is a relation between the internal variables $\left(V^Y, I^Y, s^Y\right)$ or between $\left(V^\Delta, I^\Delta, s^\Delta\right)$. It describes the behavior of the single-phase devices, and does *not* depend on their Y or Δ configuration nor the absence or presence of a neutral line. For example the internal model of an ideal voltage source specified by its internal voltage $E^{Y/\Delta} \in \mathbb{C}^3$ is

$$V^{Y/\Delta} \;=\; E^{Y/\Delta}, \qquad\qquad s^{Y/\Delta} \;=\; \text{diag}\left(E^{Y/\Delta}\left(I^{Y/\Delta}\right)^{\mathsf{H}}\right)$$

where the notation $x^{Y/\Delta}$ is a shorthand for the pair (x^Y, x^Δ). The internal model of an impedance specified by a complex matrix $z^{Y/\Delta} \in \mathbb{C}^{3\times 3}$ is

$$V^{Y/\Delta} \;=\; z^{Y/\Delta} I^{Y/\Delta}, \qquad\qquad s^{Y/\Delta} \;=\; \text{diag}\left(V^{Y/\Delta}\left(I^{Y/\Delta}\right)^{\mathsf{H}}\right)$$

Denote the internal model of a general three-phase device by

$$f^{\text{int}}\left(V^{Y/\Delta}, I^{Y/\Delta}\right) \;=\; 0, \qquad\qquad s^{Y/\Delta} \;=\; \text{diag}\left(V^{Y/\Delta} I^{Y/\Delta\mathsf{H}}\right) \tag{7.1}$$

The *external model* of a device is the relation between its terminal variables (V, I, s) and possibly $\left(V^{n'}, I^{n'}, s^{n'}\right)$. It describes the externally observable behavior of the device and depends on both the internal model of the single-phase devices and their configuration. How the Y or Δ configuration determines its external model is described by conversion rules that map internal variables to terminal variables. While the internal model depends only on the type of *single-phase* devices, the conversion rules depend only on the configuration, but not on the device type. This will be explained in detailed in Chapter 7.3. Denote the external model by

$$f^{\text{ext}}(V, I) \;=\; 0, \qquad\qquad s \;=\; \text{diag}\left(V I^{\mathsf{H}}\right) \tag{7.2}$$

The importance of the external model is that devices interact over a network only through their terminal variables. The external model of each three-phase device imposes local constraints on its own terminal variables while network equations, to be studied in Chapters 9 and 10, impose global constraints on the terminal variables across devices.

Though not explicit, the functions in (7.1) and (7.2) may be augmented with the internal and terminal variables (V^n, I^n, s^n) and $\left(V^{n'}, I^{n'}, s^{n'}\right)$ respectively associated with the neutral in a Y configuration. The functions f^{int} and f^{ext} are linear for voltage sources, current sources and impedances, but quadratic for power sources; see Chapter 7.3.

A three-phase device can therefore be modeled in two equivalent ways:

1. An internal model (7.1) that describes the relation between its internal variables $\left(V^{Y/\Delta}, I^{Y/\Delta}, s^{Y/\Delta}\right)$ and the conversion rules, (7.8) (7.9) (7.10) below, that map internal variables to external variables.

2. An external model (7.2) that describes the relation between its terminal variables. The external model is obtained by applying the conversion rules to the internal model (7.1) to eliminate the internal variables.

The first model is useful when the application under study needs to determine or optimize some of the internal variables such as the power $s_j^{Y/\Delta}$ generated or consumed by each of the single-phase devices connected at a bus j. Otherwise the external model (7.2) can be used if the application involves only the terminal variables.

Remark 7.1. One should be careful with the direction in which currents and powers are defined when relating internal and external powers (see Chapter7.3). For instance V^{an} is the voltage drop between terminal a and the neutral n and I^{an} is the current from a to n. The power s^{an} is therefore the power delivered to the device in the direction of the current I^{an}. If the device models a generator then the power it generates is $-s^{an} = V^{an}\left(-I^{an}\right)^{\mathsf{H}}$.

7.1.3 Three-phase line and transformer models

Let the terminals of a three-phase line or transformer be indexed by j and k. Let $V_j := \left(V_j^a, V_j^b, V_j^c\right) \in \mathbb{C}^3$ and $V_k := \left(V_k^a, V_k^b, V_k^c\right) \in \mathbb{C}^3$ denote the voltages at terminals j and k respectively with respect to an arbitrary but common reference point. Let $I_{jk} := \left(I_{jk}^a, I_{jk}^b, I_{jk}^c\right) \in \mathbb{C}^3$ denote the sending-end current from terminal j to terminal k along the line or transformer, and I_{kj} denote the sending-end current in the opposite direction. The external behavior of a three-phase line or transformer is described by a linear relation between $\left(V_j, V_k, I_{jk}, I_{kj}\right) \in \mathbb{C}^{12}$ of the form

$$g\left(V_j, V_k, I_{jk}, I_{kj}\right) \quad = \quad 0 \tag{7.3a}$$

where g is defined by 3×3 matrix parameters of the line (j,k).

Let $S_{jk} := \left(S_{jk}^a, S_{jk}^b, S_{jk}^c\right) \in \mathbb{C}^3$ denote the sending-end power from terminal j to terminal k along the line or transformer, and S_{kj} denote the sending-end power in the opposite direction. For each phase $\phi = a,b,c$, $S_{jk}^\phi := V_j^\phi \left(I_{jk}^\phi\right)^{\mathsf{H}}$. In vector form this is

$$S_{jk} \;:=\; \operatorname{diag}\left(V_j I_{jk}^{\mathsf{H}}\right), \qquad\qquad S_{kj} \;:=\; \operatorname{diag}\left(V_k I_{kj}^{\mathsf{H}}\right) \tag{7.3b}$$

When there is a neutral wire between terminals j and k, their voltages are V_j^n and V_k^n. The current in the neutral wire is denoted by $\left(I_{jg}^n, I_{kg}^n\right)$ if the neutral is grounded or $\left(I_{jk}^n, I_{kj}^n\right)$ otherwise. The function g (7.3a) includes neutral voltages and currents and is defined by 4×4 matrix parameters of the line. The power flow equation (7.3b) is modified accordingly.

The equations (7.3) describe the end-to-end behavior of a three-phase line or transformer. We reiterate that they depend on the three-phase devices connected to its terminals only through their external variables.

7.1.4 Three-phase network models

A network of three-phase devices connected by three-phase lines and transformers can be composed from the component models (7.2) and (7.3) for these components through the flow balance equations that relate nodal current and power (s_j, I_j) to line currents and power $\left(I_{jk}, S_{jk}\right)$ connected to the same bus bar j:

$$I_j = \sum_{k:j\sim k} I_{jk}, \qquad \forall j \tag{7.4a}$$

$$s_j = \sum_{k:j\sim k} S_{jk}, \qquad \forall j \tag{7.4b}$$

Depending on the application, what information is available and what quantities are controllable, we can model the network in two ways:

1. *VI model:* We can model the network using the relation $f^{\text{ext}}(V,I) = 0$ in (7.2) and (7.3a) (7.4a) between bus voltage and current vectors (V,I). This model is linear. Once nodal voltages $V_j \in \mathbb{C}^3$ and currents $I_j \in \mathbb{C}^3$ are determined, nodal powers $s_j := \text{diag}\left(V_j I_j^{\mathsf{H}}\right)$ can be calculated.
2. *Vs model:* We can model the network using the device model (7.2) and the power flow equations (7.3b) (7.4b) between bus voltages and power injections (V,s). This model is generally nonlinear.

The linear VI model can always be used if the system contains no power sources. Otherwise either the VI model or the Vs model can be used to describe the network but, since the device model (7.2) is nonlinear, the overall model will always be nonlinear. Network models are studied in Chapter 9 for bus injection models and Chapter 10 for branch flow models.

In summary a complete network model consists of

1. (7.3) (7.4) $+$ (7.1) and (7.8) (7.9) (7.10): involves the internal variables of three-phase devices.
2. (7.3) (7.4) $+$ (7.2): does not involve internal variables of the three-phase devices.

7.1.5 Balanced operation

If the following conditions are satisfied throughout the network:

1. all lines have symmetric geometry;
2. zero total current: $i^a(t) + i^b(t) + i^c(t) = 0$ at all times t;
3. zero total charge: $q^a(t) + q^b(t) + q^c(t) = 0$ at all times t;

then the system is balanced and its phases are decoupled. This means that (7.2) reduces to

$$f^{\text{ext},\phi}(V^\phi, I^\phi) = 0, \qquad s^\phi = V^\phi I^{\phi \mathsf{H}}, \qquad \phi = a,b,c$$

and similarly for equations (7.3)(7.4). For example the line current I_{jk}^a in phase a depends only on voltages $\left(V_j^a, V_k^a\right)$ in phase a, but not on voltages in other phases. This allows per-phase analysis, as we have done in earlier chapters. These decoupling conditions can be satisfied if the terminal voltages of all three-phase sources are balanced (i.e., they have equal magnitudes and are separated by 120° in phase), all three-phase loads consist of identical impedances, and all three-phase lines has symmetric geometry (e.g. through transposition). In that case the magnetic coupling across phases can be modeled by self-impedance alone, i.e., a three-phase line behaves *as if* its mutual inductances and capacitances across phases are zero and self inductances and capacitances are equal in each phase, as shown in Chapter 2.1.4. A general formulation of per-phase analysis of a balanced network and its formal justification is provided in Chapter 9.3. The underlying mathematical property is explained in Corollary 1.4 and Theorem 7.2.

Otherwise, self-impedance alone is not sufficient to model the coupling across phases of a line and per-phase analysis becomes inaccurate. A unbalanced three-phase model is necessary for power flow analysis. The overview of such a model is illustrated in Figure 7.3.

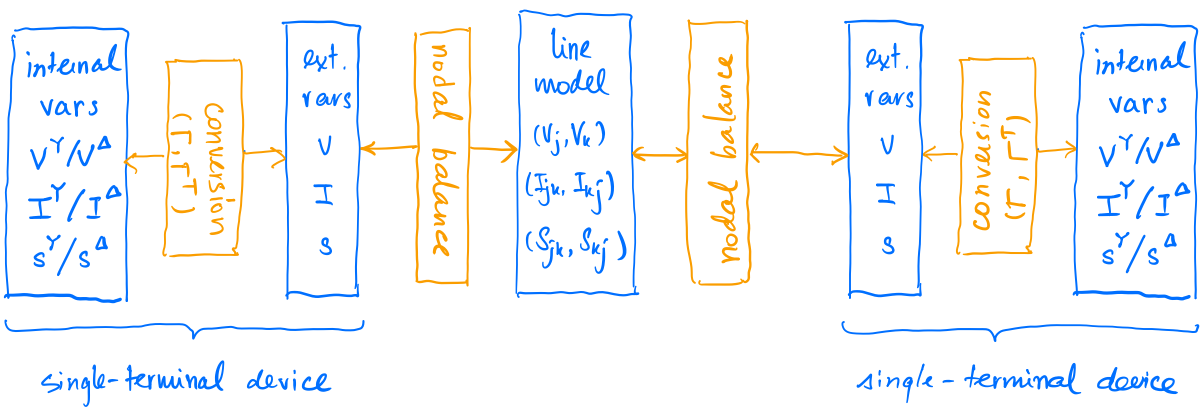


Figure 7.3: Overall network model of the system in Figure 7.1.

Before deriving in detail the internal and external models of these components we first describe some mathematical tools that are important for our derivation.

7.2 Mathematical properties of three-phase network

In this section we collect several mathematical properties that are used in the rest of this chapter, often without explicit references. These properties underlie much of the behavior of three-phase systems. Specifically we use the spectral properties of the conversion matrices Γ and Γ^{T} defined in Chapter 1.2.2 to derive in Chapter 7.2.1 their pseudo inverses. The eigenvectors of Γ are orthogonal and can serve as a basis of $\mathbb{C}^3$. In Chapter 7.2.2 we use this basis to transform voltages and currents to a sequence coordinate in which an unbalanced network may become decoupled.

7.2.1 Pseudo-inverses of conversion matrices $\Gamma, \Gamma^{\mathsf{T}}$.

The main characters of three-phase networks arise from the spectral properties of the conversion matrices Γ and Γ^{T}, defined in (1.12) of Chapter 1.2.2 and reproduced here:

$$\Gamma \; := \; \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad \Gamma^{\mathsf{T}} \; := \; \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{7.5}$$

We have seen in Chapter 1.2.4 that these conversion matrices play an important role in relating the internal and external behaviors of a balanced three-phase system. In such a system, positive-sequence voltages and currents are in span(α_+) and α_+ is an eigenvector of Γ and Γ^{T}. This means that the transformation of balanced voltages and currents under $\Gamma, \Gamma^{\mathsf{T}}$ reduces to a scaling of these variables by their eigenvalues $1-\alpha$ and $1-\alpha^2$ respectively (Corollary 1.4). The voltage and current at every point in a network can be written as linear combinations of transformed source voltages and source currents, transformed by $\left(\Gamma, \Gamma^T\right)$ and line admittance matrices. Therefore if the source voltages and source currents are balanced positive-sequence sets and lines are identical and phase-decoupled, then the transformed voltages and currents remain in span(α_+) and hence are balanced positive-sequence sets. This is the key property that enables balanced sources to induce balanced voltages and currents throughout a balanced network, allowing per-phase analysis of three-phase systems. A formal statement and proof of this property for general three-phase networks is provided in Chapter 9.3.

For unbalanced systems where voltages and currents are not necessarily in span(α_+), Corollary 1.4 is not applicable and we need the concept of pseudo inverses of $\Gamma, \Gamma^{\mathsf{T}}$ in order to convert between terminal variables and line-to-line variables internal to a Δ configuration. Even though Γ and Γ^{T} are not invertible, their pseudo inverses $\Gamma^\dagger$ and $\Gamma^{\mathsf{T}\dagger}$ respectively always exist. The pseudo inverse $M^\dagger$ of a matrix $M \in \mathbb{C}^{n\times n}$ maps the null space of M^{H} to zero. The orthogonal complement of the null space of M^{H} is the range space of M. $M^\dagger$ restricted to the range space acts like an inverse of M in that it maps each vector v in the range space of M to the unique vector $u := M^\dagger v$ in the range space of M^{H}. The vector u is the one in $\mathbb{C}^n$ with the minimum norm such that $Mu = v$. See Appendix 25.1.6 for more properties of pseudo-inverse. The facts relevant to us is summarized in the following lemma (from Theorem 25.10, Theorem 25.16 and Remark 25.2.)

Lemma 7.1. Let $M \in \mathbb{C}^{n\times n}$ be a normal matrix, i.e., $MM^{\mathsf{H}} = M^{\mathsf{H}}M$.

1. *Unitary diagonalization.* There exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a diagonal matrix $\Lambda \in \mathbb{C}^{n\times n}$ with

$$M \;=\; U\Lambda U^{\mathsf{H}} \;=\; \sum_{i=1}^{n} \lambda_i u_i u_i^{\mathsf{H}}$$

where

(a) $\Lambda =$diag$(\lambda_1, \ldots, \lambda_n)$ consists of the eigenvalues of A;

(b) the columns of U are the associated eigenvectors of A.

2. *Pseudo inverse.* The pseudo-inverse of M is given by $M^\dagger = U\Lambda^\dagger U^{\mathsf{H}}$ where $\Lambda^\dagger :=$ diag $\left(\lambda_1^{-1}, \ldots, \lambda_n^{-1}\right)$ with $\lambda_j^{-1} := 0$ if $\lambda_j = 0$.

3. Consider $Mx = b$. A solution x exists if and only if b is orthogonal to $\text{null}\left(M^{\mathsf{H}}\right)$ in which case

$$x \;=\; M^{\dagger} b + w, \qquad\qquad w \in \text{null}(M)$$

   Moreover $M^{\dagger}b$ is the unique solution to $Mx = y$ with the minimum Euclidean norm $\|x\|_2 = \|M^{\dagger}b\|_2 + \|w\|_2$, $w \in \text{null}(M)$.

Theorem 1.3 shows that Γ and Γ^{T} are normal matrices and their spectral decompositions are

$$\Gamma \;=\; F\Lambda\overline{F}, \qquad\qquad \Gamma^{\mathsf{T}} \;=\; \overline{F}\Lambda F \tag{7.6a}$$

where Λ is a diagonal matrix and F is a unitary matrix defined in (1.18), reproduced here:

$$\Lambda \;:=\; \begin{bmatrix} 0 & & \\ & 1-\alpha & \\ & & 1-\alpha^2 \end{bmatrix}, \qquad F \;:=\; \frac{1}{\sqrt{3}}\begin{bmatrix} \mathbf{1} & \alpha_+ & \alpha_- \end{bmatrix} \;:=\; \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \tag{7.6b}$$

with $\alpha := e^{-\mathbf{i}2\pi/3}$ and α_+ and α_- being the standard positive and negative sequence vectors respectively:

$$\alpha_+ \;:=\; \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}, \qquad\qquad \alpha_- \;:=\; \begin{bmatrix} 1 \\ \alpha^2 \\ \alpha \end{bmatrix}$$

Here $\overline{F}$ is the complex conjugate of F componentwise. Since F is symmetric (Theorem 1.3), Lemma 7.1 implies that the pseudo inverses of $\Gamma, \Gamma^{\mathsf{T}}$ are

$$\Gamma^{\dagger} \;=\; F\Lambda^{\dagger}\overline{F}, \qquad\qquad \Gamma^{\mathsf{T}\dagger} \;=\; \overline{F}\Lambda^{\dagger}F \tag{7.6c}$$

where $\Lambda^{\dagger} := \text{diag}\left(0, (1-\alpha)^{-1}, (1-\alpha^2)^{-1}\right)$. This yields the following simple expressions for these pseudo inverses. The proof of the theorem is left as Exercise 7.1.

Theorem 7.2 (Pseudo inverses of $\Gamma, \Gamma^{\mathsf{T}}$)**.** 1. The null spaces of Γ and Γ^{T} are both $\text{span}(1, 1, 1)$.

2. Their pseudo-inverses are

$$\Gamma^{\dagger} \;=\; \frac{1}{3}\Gamma^{\mathsf{T}}, \qquad\qquad \Gamma^{\mathsf{T}\dagger} \;=\; \frac{1}{3}\Gamma$$

3. Consider $\Gamma x = b$ where $b, x \in \mathbb{C}^3$. Solutions x exist if and only if $\mathbf{1}^{\mathsf{T}}b = 0$, in which case the solutions x are given by

$$x \;=\; \frac{1}{3}\Gamma^{\mathsf{T}}b + \gamma\mathbf{1}, \qquad\qquad \gamma \in \mathbb{C}$$

4. Consider $\Gamma^{\mathsf{T}}x = b$ where $b, x \in \mathbb{C}^3$. Solutions x exist if and only if $\mathbf{1}^{\mathsf{T}}b = 0$, in which case the solutions x are given by

$$x \;=\; \frac{1}{3}\Gamma b + \gamma\mathbf{1}, \qquad\qquad \gamma \in \mathbb{C}$$

5. $\Gamma\Gamma^{\dagger} = \Gamma^{\dagger}\Gamma = \frac{1}{3}\Gamma\Gamma^{\mathsf{T}} \;=\; \frac{1}{3}\Gamma^{\mathsf{T}}\Gamma \;=\; \mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^{\mathsf{T}}$ where $\mathbb{I}$ is the identity matrix of size 3.

Recall that $\Gamma\Gamma^{\mathsf{T}} = \Gamma^{\mathsf{T}}\Gamma$ are complex symmetric Laplacian matrices of the graphs in Figure 1.9. This theorem underlies much of the materials in this chapter.

7.2.2 Similarity transformation and symmetrical components

Fortescue transformation. Since Γ and Γ^{T} are normal matrices, they have orthonormal eigenvectors $(\mathbf{1}, \alpha_+, \alpha_-)$ which are the columns of F defined in (7.6b). We can therefore use F to define a similarity transformation (see Appendix 25.1.3 for discussions on similarity transformation). This idea is due to Fortescue [47] and F is sometimes called a (normalized) Fortescue matrix. It simplifies the analysis of an unbalanced three-phase system when the network has a certain symmetry, as explained in Chapter 9.4.

Consider a vector x that may represent a voltage or current. Recall that F is unitary and complex symmetric (Theorem 1.3) and therefore its inverse is:

$$F^{-1} \;=\; F^{\mathsf{H}} \;=\; \overline{F} \;=\; \frac{1}{\sqrt{3}}\begin{bmatrix}\mathbf{1} & \overline{\alpha}_+ & \overline{\alpha}_-\end{bmatrix} \;=\; \frac{1}{\sqrt{3}}\begin{bmatrix}\mathbf{1}^{\mathsf{T}} \\ \alpha_-^{\mathsf{T}} \\ \alpha_+^{\mathsf{T}}\end{bmatrix} \tag{7.7}$$

(Note that $\overline{\alpha}_+ = \alpha_-$, $\overline{\alpha}_- = \alpha_+$; more properties of α are studied in Exercise 1.4). The matrix F defines the transformation:

$$x \;=\; F\tilde{x}, \qquad\qquad \tilde{x} \;:=\; F^{-1}x \;=\; \overline{F}x$$

The vector $\tilde{x}$ is called the *sequence variable* of x. Its components

$$\tilde{x}_0 \;:=\; \frac{1}{\sqrt{3}}\mathbf{1}^{\mathsf{H}}x, \qquad\qquad \tilde{x}_+ \;:=\; \frac{1}{\sqrt{3}}\alpha_+^{\mathsf{H}}x, \qquad\qquad \tilde{x}_- \;:=\; \frac{1}{\sqrt{3}}\alpha_-^{\mathsf{H}}x$$

are called the *zero-sequence, positive-sequence*, and *negative-sequence* components of x. They are also called *symmetrical components* of x. We will sometimes refer to x as a *phase variable* to differentiate it from the sequence variable $\tilde{x}$. The relation $x = F\tilde{x}$ expresses the phase variable in terms of its sequence components:

$$x \;=\; \frac{1}{\sqrt{3}}\left(\tilde{x}_0\mathbf{1} + \tilde{x}_+\alpha_+ + \tilde{x}_-\alpha_-\right) \;=\; \frac{1}{3}\left(\left(\mathbf{1}^{\mathsf{H}}x\right)\mathbf{1} + \left(\alpha_+^{\mathsf{H}}x\right)\alpha_+ + \left(\alpha_-^{\mathsf{H}}x\right)\alpha_-\right)$$

Sequence voltage, current, power. Applying this similarity transformation to phase voltage V and current I, we obtain their sequence variables:

$$\tilde{V} \;=\; \overline{F}V, \qquad\qquad \tilde{I} \;=\; \overline{F}I,$$

The vector of power in the phase coordinate is $s := \text{diag}\left(VI^{\mathsf{H}}\right)$ and that in the sequence coordinate is $\tilde{s} := \text{diag}\left(\tilde{V}\tilde{I}^{\mathsf{H}}\right)$. They are related through the outer product of voltage and current in their respective coordinates according to:

$$\begin{aligned}
\tilde{s} \;&:=\; \text{diag}\left(\tilde{V}\tilde{I}^{\mathsf{H}}\right) \;=\; \text{diag}\left(\overline{F}VI^{\mathsf{H}}\overline{F}^{\mathsf{H}}\right) \;=\; \text{diag}\left(\overline{F}VI^{\mathsf{H}}F\right) \\
s \;&:=\; \text{diag}\left(VI^{\mathsf{H}}\right) \;=\; \text{diag}\left(F\tilde{V}\tilde{I}^{\mathsf{H}}F^{\mathsf{H}}\right) \;=\; \text{diag}\left(F\tilde{V}\tilde{I}^{\mathsf{H}}\overline{F}\right)
\end{aligned}$$

The total powers $\mathbf{1}^\mathsf{T}\tilde{s} = \mathbf{1}^\mathsf{T}s$ however are equal in both coordinates:

$$\mathbf{1}^\mathsf{T}\tilde{s} \;=\; \tilde{I}^\mathsf{H}\tilde{V} \;=\; \left(I^\mathsf{H}\overline{F}^\mathsf{H}\right)\left(\overline{F}V\right) \;=\; I^\mathsf{H}V \;=\; \mathbf{1}^\mathsf{T}s$$

since $\overline{F}^\mathsf{H}\overline{F} = F\overline{F} = \mathbb{I}$. This is sometimes referred to as power invariance property of the similarity transformation F. In Chapter 9.4 we will apply sequence variables to the external models of Chapter 7.3 to define sequence networks.

In Definition 1.1, we call x a balanced vector if its zero-sequence component $\tilde{x}_0 = 0$ and exactly one of $\tilde{x}_+$ and $\tilde{x}_-$ is nonzero. In particular a balanced positive-sequence vector is in $\text{span}(\alpha_+)$. To simplify exposition in this chapter it is convenient to generalize the definition of balanced vector to include a zero-sequence component.

Definition 7.1 (Generalized balanced vector)**.** A vector $\hat{x} := (\hat{x}_1, \hat{x}_2, \hat{x}_3) \in \mathbb{C}^3$ is called a *generalized balanced vector* if $\hat{x} = x + \gamma\mathbf{1}$, for some $\gamma \in \mathbb{C}$, such that x is balanced according to Definition 1.1.

Hence a generalized balanced vector $\hat{x}$ may contain a nontrivial zero-sequence component $\tilde{x}_0$ and exactly one of $\tilde{x}_+$ and $\tilde{x}_-$. We will often refer to a generalized balanced vector $\hat{x}$ simply as *balanced* if there is no risk of confusion or if the differentiation is not important, even if $\gamma \neq 0$. The key property Corollary 1.4 for balanced networks holds for generalized balanced vectors, i.e., $\Gamma(x + \gamma\mathbf{1}) = (1-\alpha)x$ and $\Gamma^\mathsf{T}(x + \gamma\mathbf{1}) = (1-\alpha^2)x$ if x is a balanced positive-sequence vector.

Park transformation. Besides Foretescue transformation F, several other similarity transformations have been proposed that have different advantages and disadvantages for steady-state fault analysis; see [48] that explains their relation. Park's transformation [49] is applicable not only to steady-state voltage and current phasors, but also to instantaneous voltages, currents, and flux linkages. It is originally proposed for analyzing synchronous machines and is defined by the following real orthonormal matrix (which is the normalized version of Park's original matrix; we follow [1]):

$$P \;:=\; \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\theta & \sin\theta \\ \frac{1}{\sqrt{2}} & \cos(\theta - 120^\circ) & \sin(\theta - 120^\circ) \\ \frac{1}{\sqrt{2}} & \cos(\theta + 120^\circ) & \sin(\theta + 120^\circ) \end{bmatrix}$$

It can be verified that P is orthonormal so that $P^{-1} = P^\mathsf{T}$. The matrix can be used to transform instantaneous phase voltages, currents and flux linkages. For example, for instantaneous voltages we have

$$v \;=\; \begin{bmatrix} v^a \\ v^b \\ v^c \end{bmatrix} \;=\; \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \cos\theta & \sin\theta \\ \frac{1}{\sqrt{2}} & \cos(\theta - 120^\circ) & \sin(\theta - 120^\circ) \\ \frac{1}{\sqrt{2}} & \cos(\theta + 120^\circ) & \sin(\theta + 120^\circ) \end{bmatrix}\begin{bmatrix} v^0 \\ v^d \\ v^q \end{bmatrix} \;=\; P\tilde{v}$$

$$\tilde{v} \;=\; \begin{bmatrix} v^0 \\ v^d \\ v^q \end{bmatrix} \;=\; \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \cos\theta & \cos(\theta - 120^\circ) & \cos(\theta + 120^\circ) \\ \sin\theta & \sin(\theta - 120^\circ) & \sin(\theta + 120^\circ) \end{bmatrix}\begin{bmatrix} v^a \\ v^b \\ v^c \end{bmatrix} \;=\; P^\mathsf{T}v$$

The transformed coordinate is called the $0dq$, or zero-direct-quadrature, or rotor coordinate. The *abc* variables are stator-based quantities and the $0dq$ variables are rotor-based quantities. Similarly we can transform abc currents and flux linkages into the $0dq$ coordinate with $\tilde{i} = P^{\mathsf{T}} i$ and $\tilde{\lambda} = P^{\mathsf{T}} \lambda$. The model of a synchronous machine becomes simpler in the rotor coordinate. For example the inductance matrix L in the abc coordinate that relates currents and flux linkages, $\lambda = Li$, becomes diagonal in the rotor coordinate, i.e., $\tilde{\lambda} = \tilde{L}\tilde{i}$ for a diagonal $\tilde{L}$.

7.3 Three-phase device models

In this section we develop the external models (7.2)(7.3) of three-phase devices in terms of their internal specifications. The models of three-phase devices developed in Chapter 1.2 and the phase-decoupled line model of Chapter 2 are special cases of the models in this section.

We start by describing in Chapter 7.3.1 the conversion rules (7.8) and (7.9)(7.10) that maps internal variables $\left(V^{Y/\Delta}, I^{Y/\Delta}, s^{Y/\Delta}\right)$ to external variables (V, I, s) for devices in Y and Δ configurations respectively. These conversion rules depend only on the configuration and are applicable to any types of devices. In Chapters 7.3.2 and 7.3.3 we present the internal models of four types of devices in Y and Δ configuration respectively and apply the conversion rules to these internal models to derive their external models. In Chapter 7.3.4 we explain how to derive the Y equivalent of an ideal Δ-configured voltage or current source in an unbalanced setting.

7.3.1 Conversion rules

Conversion in Y configuration. Consider a generic three-phase device in Y configuration with internal and terminal variables defined as in Figure 7.2(a). Its terminal voltage, current, and power (V, I, s) are related to its internal variables (V^Y, I^Y, s^Y) by:

$$V \ = \ V^Y + V^n \mathbf{1}, \qquad I \ = \ -I^Y, \qquad -\mathbf{1}^{\mathsf{T}} I \ = \ I^n, \qquad s \ = \ -\left(s^Y + V^n \bar{I}^Y\right) \tag{7.8}$$

where $\bar{I}^Y$ denotes the componentwise complex conjugate of the vector $I^Y \in \mathbb{C}^3$. The negative sign on the current and power conversions is due to the definition of $\left(I^Y, s^Y\right)$ as internal current and power delivered to the single-phase devices whereas (I, s) is defined as the terminal current and power injections out of the three-phase device; see Remark 7.1. The property $-\mathbf{1}^{\mathsf{T}} I = I^n$ follows from the KCL at the neutral.

Here $s^Y := \text{diag}\left(V^Y I^{Y\mathsf{H}}\right)$ is the internal power delivered across the single-phase devices, or equivalently, $-s^Y$ is the power generated internally by these devices. The term $V^n \bar{I}^Y$ is the vector power delivered across the neutral and the common reference point (e.g., the ground). The terminal power $s := \text{diag}\left(V I^{\mathsf{H}}\right)$ is power delivered from the device across the phase lines and the common reference point. Hence $-s^Y = s + V^n \bar{I}^Y$ says that the power generated by the device is equal to that delivered to the neutral impedance and the rest of the network. This follows from the conversion between voltages and currents:

$$s \ := \ \text{diag}\left(V I^{\mathsf{H}}\right) \ = \ \text{diag}\left(V^Y \left(-I^Y\right)^{\mathsf{H}}\right) \ + \ V^n \, \text{diag}\left(\mathbf{1} \left(-I^Y\right)^{\mathsf{H}}\right) \ = \ -\left(s^Y \ + \ V^n \bar{I}^Y\right)$$

The conversion rule (7.8) holds whether or not there is a neutral line and whether or not the neutral is grounded with zero or nonzero neutral impedance z^n. If there is not a neutral line then $I^n := 0$ and we have $\mathbf{1}^\mathsf{T} I = \mathbf{1}^\mathsf{T} I^Y = 0$. If the neutral is grounded, then I^n is the current from the neutral to the ground and $V^n = z^n I^n = -z^n \mathbf{1}^\mathsf{T} I$ whether or not $z^n = 0$. If the neutral is ungrounded but connected to the neutral of a 4-wire line, then I^n is the current on the neutral line leaving the neutral of the device. Its value will depend on network interaction; see Example 9.5 and Exercise 9.7.

Remark 7.2 (Neutral voltage V^n)**.** In general the neutral voltage V^n with respect to a common reference point is nonzero whether or not there is a neutral line and whether or not the neutral is grounded. If the neutral is grounded with zero neutral impedance and voltages are defined with respect to the ground, then $V^n = 0$, and hence $V = V^Y$ and $s = -s^Y$. It is important to explicitly include V^n in a network model because not every device in a network may be grounded or grounded with zero neutral impedance. □

Remark 7.3 (Total power)**.** The total terminal power is

$$\mathbf{1}^\mathsf{T} s \;=\; -\mathbf{1}^\mathsf{T} s^Y \;-\; V^n \left(\mathbf{1}^T \bar{I}^Y\right)$$

The first term $\mathbf{1}^\mathsf{T} s^Y$ on the right-hand side is the total power delivered across the single-phase devices. The expression says that the total terminal power injection is equal to the total power $-\mathbf{1}^\mathsf{T} s^Y$ generated internally net of power consumed by the neutral impedance.

If the neutral is ungrounded then $\mathbf{1}^T I^Y = 0$ by KCL and $\mathbf{1}^\mathsf{T} s = -\mathbf{1}^\mathsf{T} s^Y$. If the neutral is grounded (i.e., $V^{n'} = 0$) through an impedance then $V^n \left(\mathbf{1}^T \bar{I}^Y\right)$ is the power delivered to the neutral impedance. In general the internal power delivered to the neutral impedance is $s^n := \left(V^n - V^{n'}\right) \bar{I}^n$ □

Conversion in Δ configuration. Consider a generic three-phase device in Δ configuration with internal and terminal variables defined as in Figure 7.2(b). We now apply Theorem 7.2 to convert between internal and external variables in Δ configuration.

Voltage and current conversion. The relation between terminal voltage and current (V, I) and internal voltage and current $\left(V^\Delta, I^\Delta\right)$ is:

$$\begin{bmatrix} V^{ab} \\ V^{bc} \\ V^{ca} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}_{\Gamma} \begin{bmatrix} V^a \\ V^b \\ V^c \end{bmatrix}, \qquad \begin{bmatrix} I^a \\ I^b \\ I^c \end{bmatrix} = -\underbrace{\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\Gamma^\mathsf{T}} \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix}$$

or in vector form

$$V^\Delta \;=\; \Gamma V, \qquad I \;=\; -\Gamma^\mathsf{T} I^\Delta \tag{7.9a}$$

where $\Gamma, \Gamma^\mathsf{T}$ are given in (7.5). Given appropriate vectors V^Δ and I, solutions V and I^Δ to (7.9a) is provided by Theorem 7.2.

1. Given V^Δ, there is a solution V to (7.9a) if and only if V^Δ is orthogonal to $\mathbf{1}$, i.e.,

$$V^{ab} + V^{bc} + V^{ca} \;=\; 0$$

which expresses Kirchhoff's voltage law. In that case, there is a subspace of solutions V given by

$$V \;=\; \Gamma^{\dagger} V^{\Delta} + \gamma \mathbf{1} \;=\; \frac{1}{3}\Gamma^{\mathsf{T}} V^{\Delta} + \gamma \mathbf{1}, \qquad \gamma \in \mathbb{C} \tag{7.9b}$$

This amounts to an arbitrary reference voltage for V. The quantity $\gamma := \frac{1}{3}\mathbf{1}^T V$ is the (scaled) zero-sequence voltage of V. In most applications we are given a reference voltage (e.g., $V_0 := \alpha_+$ at the reference bus 0) which will fix the constant γ for every Δ-configured device (different devices may have different zero-sequence voltages γ).

2. Given I, there is a solution I^{Δ} to (7.9a) if and only if I is orthogonal to $\mathbf{1}$, i.e.,

$$I^a + I^b + I^c \;=\; 0$$

which expresses Kirchhoff's current law. In that case, there is a subspace of I^{Δ} that satisfy (7.9a), given by

$$I^{\Delta} \;=\; -\Gamma^{\mathsf{T}\dagger} I + \beta \mathbf{1} \;=\; -\frac{1}{3}\Gamma I + \beta \mathbf{1}, \qquad \beta \in \mathbb{C} \tag{7.9c}$$

where β specifies the amount of loop flow in I^{Δ} and does not affect the terminal current I since $\Gamma^{\mathsf{T}} I^{\Delta} = 0$. The quantity $\beta := \frac{1}{3}\mathbf{1}^T I^{\Delta}$ is the (scaled) zero-sequence current of I^{Δ}.

We make two remarks regarding the solutions (V, I^{Δ}). First the minimum-norm solution

$$V \;:=\; \frac{1}{3}\Gamma^{\mathsf{T}} V^{\Delta} \;=\; \frac{1}{3}\begin{bmatrix} V^{ab} - V^{ca} \\ V^{bc} - V^{ab} \\ V^{ca} - V^{bc} \end{bmatrix}$$

sets $\gamma = 0$ such that $\mathbf{1}^{\mathsf{T}} V = 3\gamma = 0$. Note that this solution does not set one of (V^a, V^b, V^c) to zero. A consequence of the arbitrary reference voltage is that, given the internal voltage and current $\left(V^{\Delta}, I^{\Delta}\right)$ with $\mathbf{1}^{\mathsf{T}} V^{\Delta} = 0$ of a Δ-configured device, its terminal power vector s depends on the arbitrary constant γ (similar to the effect of the neutral voltage V^n on s for a Y-configured device); see Remark 7.4. To fix V to be the minimum-norm solution (7.9b) with $\gamma = 0$, it is important to include explicitly the condition $\mathbf{1}^{\mathsf{T}} V = 0$ together with $V^{\Delta} = \Gamma V$, i.e., the minimum-norm solution with $\gamma = 0$ is the unique solution to the system of equations:

$$V^{\Delta} \;=\; \Gamma V, \qquad \mathbf{1}^{\mathsf{T}} V \;=\; 0, \qquad (\text{given } V^{\Delta} \text{ that satisfies } \mathbf{1}^{\mathsf{T}} V^{\Delta} = 0)$$

Second the minimum-norm solution sets $\beta = 0$ and is

$$I^{\Delta} \;=\; -\frac{1}{3}\Gamma I \;=\; -\frac{1}{3}\begin{bmatrix} I^a - I^b \\ I^b - I^c \\ I^c - I^a \end{bmatrix}$$

It contains zero loop flow, i.e., $\mathbf{1}^{\mathsf{T}} I^{\Delta} = 3\beta = 0$. Analogous to the case above, a consequence of an arbitrary β is that, given the terminal voltage and current (V, I) of a Δ-configured device, its internal power vector

s^Δ depends on the zero-sequence current β; see Remark 7.4. To fix I to be the minimum-norm solution (7.9c) with $\beta = 0$, it is important to include explicitly the condition $\mathbf{1}^\mathsf{T} I^\Delta := 0$ together with $I = -\Gamma^\mathsf{T} I^\Delta$, i.e., the minimum-norm solution with $\beta = 0$ is the unique solution to the system of equations:

$$I \;=\; -\Gamma^\mathsf{T} I^\Delta, \qquad\qquad \mathbf{1}^\mathsf{T} I^\Delta \;=\; 0 \qquad\qquad (\text{given } I \text{ that satisfies } \mathbf{1}^\mathsf{T} I = 0)$$

Power conversion. The terminal power injection from the device is $s := \mathrm{diag}\left(VI^\mathsf{H}\right)$ and the internal power delivered across the single-phase devices in the direction ab, bc, ca is $s^\Delta := \mathrm{diag}\left(V^\Delta I^{\Delta\mathsf{H}}\right)$. Unlike a Y-configured power source for which the terminal power s is related directly to the internal power s^Y (see (7.8)), for a Δ-configured power source, the relation between s and s^Δ is indirect through $\left(V^\Delta, I^\Delta\right)$, through (V, I), or through $\left(V, I^\Delta\right)$. We now derive these relations using the voltage and current conversion (7.9).

Specifically, given internal voltage and current $\left(V^\Delta, I^\Delta\right)$ with $\mathbf{1}^\mathsf{T} V^\Delta = 0$, the internal power is $s^\Delta := \mathrm{diag}\left(V^\Delta I^{\Delta\mathsf{H}}\right)$. To express the terminal power s in terms of $\left(V^\Delta, I^\Delta\right)$, we use (7.9a) (7.9b) to write the terminal voltage and current as

$$V \;=\; \Gamma^\dagger V^\Delta + \gamma \mathbf{1}, \quad \gamma \in \mathbb{C}, \qquad\qquad I \;=\; -\Gamma^\mathsf{T} I^\Delta$$

where different γ correspond to different reference voltages. Therefore

$$VI^\mathsf{H} \;=\; \left(\Gamma^\dagger V^\Delta + \gamma \mathbf{1}\right)\left(-\Gamma^\mathsf{T} I^\Delta\right)^\mathsf{H} \;=\; -\Gamma^\dagger \left(V^\Delta I^{\Delta\mathsf{H}}\right)\Gamma \;+\; \gamma\left(\mathbf{1} I^\mathsf{H}\right)$$

Hence the terminal power s can be expressed in terms of the internal voltage and current $\left(V^\Delta, I^\Delta\right)$ as

$$s \;:=\; \mathrm{diag}\left(VI^\mathsf{H}\right) \;=\; -\mathrm{diag}\left(\Gamma^\dagger \left(V^\Delta I^{\Delta\mathsf{H}}\right)\Gamma\right) \;+\; \gamma \bar{I}, \qquad\qquad \mathbf{1}^\mathsf{T} V^\Delta \;=\; 0 \tag{7.10a}$$

where $\bar{I}$ is the componentwise complex conjugate of the terminal current $I = -\Gamma^\mathsf{T} I^\Delta$ and $\gamma \in \mathbb{C}$ is determined by a reference voltage.

Example 7.1. Given internal voltage and current $\left(V^\Delta, I^\Delta\right)$ with $\mathbf{1}^\mathsf{T} V^\Delta = 0$, evaluate the terminal power $s := \mathrm{diag}\left(VI^\mathsf{H}\right)$ directly using the solution (7.9b) with $\gamma := 0$.

Solution. We have

$$I \;=\; -\Gamma^\mathsf{T} I^\Delta \;=\; -\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} I^{ab} \\ I^{bc} \\ I^{ca} \end{bmatrix} \;=\; -\begin{bmatrix} I^{ab} - I^{ca} \\ I^{bc} - I^{ab} \\ I^{ca} - I^{bc} \end{bmatrix}$$

Combine with (7.9b) with $\gamma = 0$ to evaluate $\mathrm{diag}\left(VI^\mathsf{H}\right)$:

$$s \;:=\; -\frac{1}{3}\begin{bmatrix} \left(V^{ab} - V^{ca}\right)\left(I^{ab} - I^{ca}\right)^\mathsf{H} \\ \left(V^{bc} - V^{ab}\right)\left(I^{bc} - I^{ab}\right)^\mathsf{H} \\ \left(V^{ca} - V^{bc}\right)\left(I^{ca} - I^{ab}\right)^\mathsf{H} \end{bmatrix} \;=\; -\frac{1}{3}\left(\begin{bmatrix} s^{ab} + s^{ca} \\ s^{bc} + s^{ab} \\ s^{ca} + s^{bc} \end{bmatrix} + \begin{bmatrix} V^{ca} & 0 & V^{ab} \\ V^{bc} & V^{ab} & 0 \\ 0 & V^{ca} & V^{bc} \end{bmatrix}\begin{bmatrix} \bar{I}^{ab} \\ \bar{I}^{bc} \\ \bar{I}^{ca} \end{bmatrix}\right)$$

This is (7.10a) with $\gamma = 0$. □

We next relate s and s^Δ in terms of terminal voltage and current (V, I). Given (V, I) with $\mathbf{1}^\mathsf{T} I = 0$, $s := \text{diag}\left(VI^\mathsf{H}\right)$. To express s^Δ in terms of (V, I), use (7.9a) (7.9c) to write the internal voltage and current as

$$V^\Delta \;=\; \Gamma V, \qquad\qquad I^\Delta \;=\; -\Gamma^{\mathsf{T}\dagger} I \,+\, \beta\mathbf{1}, \quad \beta \in \mathbb{C}$$

where different β correspond to different loop flows in the Δ configuration. Therefore

$$V^\Delta I^{\Delta\mathsf{H}} \;=\; \Gamma V\left(-\Gamma^{\mathsf{T}\dagger} I + \beta\mathbf{1}\right)^\mathsf{H} \;=\; -\Gamma\left(VI^\mathsf{H}\right)\Gamma^\dagger \,+\, \overline{\beta}\left(V^\Delta\mathbf{1}^\mathsf{T}\right)$$

Hence the internal power $s^\Delta := \text{diag}\left(V^\Delta I^{\Delta\mathsf{H}}\right)$ can be expressed in terms of the terminal voltage and current (V, I) as

$$s^\Delta \;:=\; \text{diag}\left(V^\Delta I^{\Delta\mathsf{H}}\right) \;=\; -\text{diag}\left(\Gamma\left(VI^\mathsf{H}\right)\Gamma^\dagger\right) \,+\, \overline{\beta}V^\Delta, \qquad \mathbf{1}^\mathsf{T} I \;=\; 0 \tag{7.10b}$$

where $V^\Delta = \Gamma V$ and $\beta \in \mathbb{C}$ is determined by the amount of loop flow in I^Δ.

Even though (7.10a) and (7.10a) contain the zero-sequence voltage and current (γ, β), the total powers $\mathbf{1}^\mathsf{T} s$ and $\mathbf{1}^\mathsf{T} s^\Delta$ do not.

Remark 7.4 (Total power)**.**

1. Given an internal voltage and current $\left(V^\Delta, I^\Delta\right)$, the terminal power vector s in (7.10a) does not depend on the zero-sequence current $\beta := \frac{1}{3}\mathbf{1}^\mathsf{T} I^\Delta$ but does depend on the zero-sequence voltage $\gamma := \frac{1}{3}\mathbf{1}^\mathsf{T} V$. Since $I = -\Gamma^\mathsf{T} I^\Delta$ and hence $\mathbf{1}^\mathsf{T} I = 0$, the total terminal power however is independent of γ:

$$\mathbf{1}^\mathsf{T} s \;=\; -\mathbf{1}^\mathsf{T}\text{diag}\left(\Gamma^\dagger\left(V^\Delta I^{\Delta\mathsf{H}}\right)\Gamma\right)$$

This is the same as the effect of neutral voltage V^n on terminal power s and its aggregate $\mathbf{1}^\mathsf{T} s$ in Y configuration when the neutral is ungrounded so that $\mathbf{1}^\mathsf{T} I^Y = 0$ by KCL.

2. Analogously, from (7.10b), the internal power vector s^Δ depends on zero-sequence current β. Since $V^\Delta = \Gamma V$ and hence $\mathbf{1}^\mathsf{T} V^\Delta = 0$, the total internal power however is independent of the loop flow:

$$\mathbf{1}^\mathsf{T} s^\Delta \;=\; -\mathbf{1}^\mathsf{T}\text{diag}\left(\Gamma\left(VI^\mathsf{H}\right)\Gamma^\dagger\right)$$

$\square$

Finally we can relate s and s^Δ through the terminal voltage and internal current $\left(V, I^\Delta\right)$. Indeed both s and s^Δ can be expressed in terms of $\left(V, I^\Delta\right)$ using (7.9a):

$$s \;:=\; \text{diag}\left(VI^\mathsf{H}\right) \;=\; -\text{diag}\left(VI^{\Delta\mathsf{H}}\Gamma\right), \qquad s^\Delta \;:=\; \text{diag}\left(V^\Delta I^{\Delta\mathsf{H}}\right) \;=\; \text{diag}\left(\Gamma V I^{\Delta\mathsf{H}}\right) \tag{7.10c}$$

An important advantage of (7.10c) is that $\left(V, I^\Delta\right)$ contains implicitly both the zero-sequence voltage $\gamma := \frac{1}{3}\mathbf{1}^\mathsf{T} V$ and the zero-sequence current $\beta := \frac{1}{3}\mathbf{1}^\mathsf{T} I^\Delta$. This is often a more computationally convenient model than (7.10a) and (7.10b).

In summary:

- Given internal voltage and current (V^Δ, I^Δ) with $\mathbf{1}^\mathsf{T} V^\Delta = 0$, the terminal power s as a function of (V^Δ, I^Δ) is given by (7.10a).
- Given terminal voltage and current (V, I) with $\mathbf{1}^\mathsf{T} I = 0$, the internal power s^Δ as a function of (V, I) is given by (7.10b).
- Given terminal voltage and internal current $\left(V, I^\Delta\right)$, the terminal power s and the internal power s^Δ are given by (7.10c).

These expressions are used to derive the external model a constant-power source in Δ configuration; see Chapter 7.3.3.

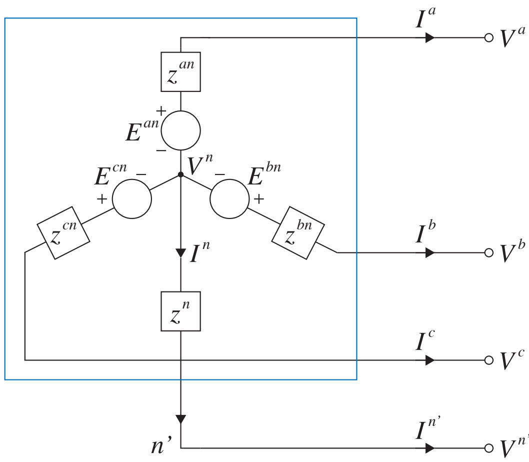
Finally, note that unlike the relation $I = -\Gamma^\mathsf{T} I^\Delta$ which expresses KCL, it is *not* true that $s = -\Gamma^\mathsf{T} s^\Delta$. The relation between terminal power and internal power is given *only indirectly* by (7.10).

7.3.2 Devices in Y configuration

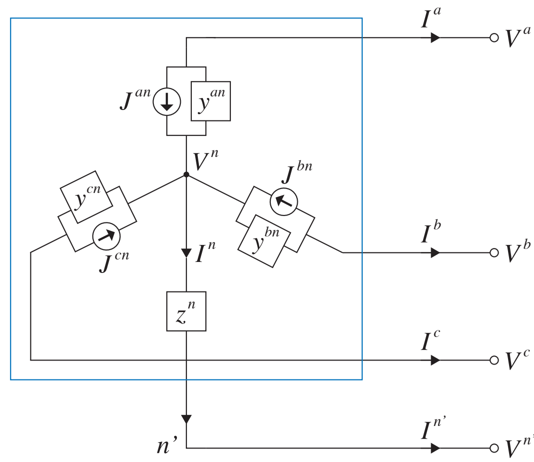
In this subsection we first present parameters of a voltage source, current source, power source, and impedance in Y configuration. For each device we then specify its internal model. Finally we apply the conversion rule (7.8) to the internal model of each device to derive its external model.

Device specification. The devices we study are shown in Figure 7.4.

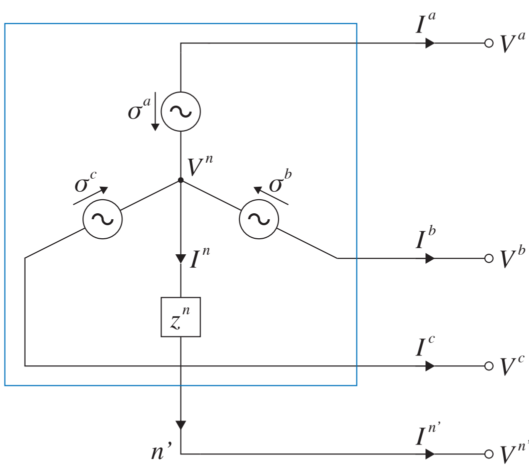
1. *Voltage source* $\left(E^Y, z^Y, z^n\right)$. A voltage source is a single-terminal three or four-wire device. When the configuration is Y, as shown in Figure 7.4(a), it is specified by three parameters. Its internal voltage is fixed at $E^Y := (E^{an}, E^{bn}, E^{cn})$ and its series impedance matrix is $z^Y := \text{diag}\left(z^{an}, z^{bn}, z^{cn}\right)$. If there is a neutral wire then its impedance is a scalar z^n which may or may not be zero whether or not the neutral is grounded. An ideal voltage source is one with $z^Y = 0$ and $z^n = 0$. A voltage source can serve as a Thévenin equivalent circuit of a synchronous generator for which the internal voltage E^Y is typically balanced. It can also model the primary or secondary side of a transformer, or a grid-forming inverter.
2. *Current source* $\left(J^Y, y^Y, z^n\right)$. A current source is a single-terminal three or four-wire device. When the configuration is Y, as shown in Figure 7.4(b), it is specified by three parameters. Its internal current is fixed at $J^Y := (J^{an}, J^{bn}, J^{cn})$ and its shunt admittance matrix is $y^Y := \text{diag}\left(y^{an}, y^{bn}, y^{cn}\right)$. If there is a neutral wire then its impedance is a scalar z^n which may or may not be zero whether or not the neutral is grounded. An ideal current source is one with $y^Y = 0$ and $z^n = 0$. A current source can serve as a Norton equivalent circuit of a synchronous generator. It can also model a load such as an electric vehicle charger, or a grid-following inverter.
3. *Power source* $\left(\sigma^Y, z^n\right)$. A single-terminal three or four-wire power source in Y configuration is shown in Figure 7.4(c) and specified by two parameters. It consumes a constant power $\sigma^Y := \left(\sigma^{an}, \sigma^{bn}, \sigma^{cn}\right)$ or injects a constant power $-\sigma^Y$. If there is a neutral wire then its impedance is a scalar z^n which may or may not be zero whether or not the neutral is grounded. An ideal power



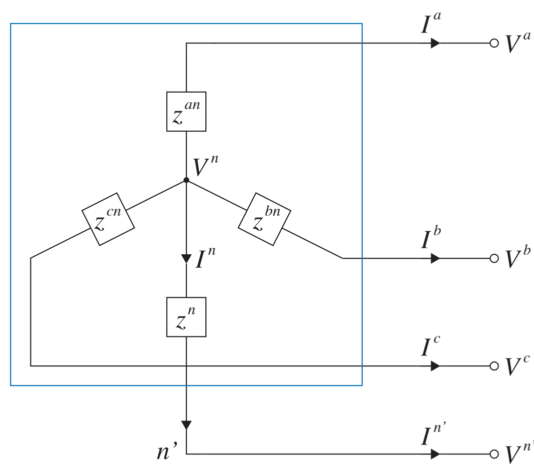
(a) Voltage source



(b) Current source



(c) Power source



(d) Impedance

Figure 7.4: Three-phase devices in Y configuration. (a) A voltage source. (b) A current source. (c) A power source. (d) An impedance. Note that the direction of J^Y and σ^Y is terminal-to-neutral.

source is one with $z^n = 0$. A power source can model a load, a generator, or the primary or secondary side of a transformer.

4. *Impedance* $\left(z^Y, z^n\right)$. A single-terminal three or four-wire impedance in Y configuration as shown in Figure 7.4(d) is specified by an impedance matrix $z^Y := \text{diag}\left(z^{an}, z^{bn}, z^{cn}\right)$. If there is a neutral wire then its impedance is a scalar z^n which may or may not be zero whether or not the neutral is grounded. An impedance can model a load.

Note that the direction of J^Y and σ^Y is defined to be terminal-to-neutral, opposite to that of the terminal current I.

The list above only specifies the internal parameters of a Y-configured device. When it is connected to a network, its neutral voltage V^n will need to be either specified or computed in order to translate between its internal voltage V^Y and external voltage $V = V^Y + V^n\mathbf{1}$ (from (7.8)) and determine voltages, currents, and powers at other parts of the network. We will discuss in Chapter 9.2, for each device in a typical three-phase analysis problem, what quantities are parameters that should be specified and what are variables to be computed through network equations. An assumption that is often made, sometimes implicitly, is:

> C7.1: All neutrals are grounded either through an impedance z^n or directly $(z^n = 0)$ and all voltages are defined with respect to the ground.

This assumption is often satisfied in practice. Under this assumption, $V^{n'} = 0$ (see Figure 7.4). Moreover the internal neutral voltage V^n is not independently specified but is determined by the current through the neutral impedance z^n:

$$V^n \;=\; z^n\left(\mathbf{1}^\top I^Y\right) \;=\; -z^n\left(\mathbf{1}^\top I\right) \tag{7.11}$$

If the neutral is directly grounded, i.e., $z^n = 0$, then $V^n = 0$. Without C7.1 or for an ungrounded voltage source, knowing the internal voltage and current $\left(V^Y, I^Y\right)$ alone may not be sufficient to determine the external voltage V. We will be explicit when we assume C7.1.

Voltage source (E^Y, z^Y, z^n)**.** *Internal model.* Referring to Figure 7.4(a) the internal model of a voltage source is

$$V^Y \;=\; E^Y + z^Y I^Y, \qquad V^n - V^{n'} \;=\; z^n\left(\mathbf{1}^\top I^Y\right), \qquad I^n \;=\; \mathbf{1}^\top I^Y \tag{7.12a}$$

This yields an internal power $s^Y := \text{diag}\left(V^Y I^{Y\mathsf{H}}\right)$ across the non-ideal voltage source and an internal power $s^n := \left(V^n - V^{n'}\right) I^{n\mathsf{H}}$ across the impedance z^n on the neutral line, given by:

$$s^Y \;=\; \text{diag}\left(E^Y I^{Y\mathsf{H}}\right) + \text{diag}\left(z^Y I^Y I^{Y\mathsf{H}}\right) \;=\; \underbrace{\begin{bmatrix} E^{an} I^{an\mathsf{H}} \\ E^{bn} I^{bn\mathsf{H}} \\ E^{cn} I^{cn\mathsf{H}} \end{bmatrix}}_{s^Y_{\text{ideal}}} + \underbrace{\begin{bmatrix} z^{an}\left|I^{an}\right|^2 \\ z^{bn}\left|I^{bn}\right|^2 \\ z^{cn}\left|I^{cn}\right|^2 \end{bmatrix}}_{s_{\text{imp}}} \tag{7.12b}$$

$$s^n \;=\; z^n\left|\mathbf{1}^\top I^Y\right|^2 \tag{7.12c}$$

External model. To derive an external model, apply the conversion rule (7.8), reproduced here:

$$V = V^Y + V^n \mathbf{1}, \qquad I = -I^Y, \qquad -\mathbf{1}^\mathsf{T} I = I^n, \qquad s = -\left(s^Y + V^n \bar{I}^Y\right)$$

to the internal model (7.12) to eliminate the internal variables (here, $\bar{I}^Y$ is the complex conjugate of vector I^Y componentwise). This yields a relation between its terminal variables (V, I, s):

$$V = E^Y + V^n \mathbf{1} - z^Y I, \qquad \mathbf{1}^\mathsf{T} I = -I^n, \qquad s = \text{diag}\left(E^Y I^\mathsf{H}\right) + V^n \bar{I} - \text{diag}\left(z^Y I I^\mathsf{H}\right) \tag{7.13a}$$

The model (7.13a) holds whether there is a neutral line or whether the neutral line is grounded or ungrounded but connected to another device over a four-wire line. As discussed before, $I^n = 0$ if the neutral is ungrounded.

Suppose assumption C7.1 holds so that $V^{n'} = 0$ and $V^n = -z^n \left(\mathbf{1}^\mathsf{T} I\right)$. Then (7.13a) yields the external model:

$$V = E^Y - Z^Y I \tag{7.13b}$$

where

$$Z^Y := z^Y + z^n \mathbf{1}\mathbf{1}^\mathsf{T} = \begin{bmatrix} z^{an} + z^n & z^n & z^n \\ z^n & z^{bn} + z^n & z^n \\ z^n & z^n & z^{cn} + z^n \end{bmatrix}$$

This has the same form as that of a single-phase voltage source discussed in Chapter **??**. The neutral impedance z^n couples the phases. Substituting (7.13b) into $s = \text{diag}\left(V I^\mathsf{H}\right)$ expresses the terminal power s as a quadratic function of V:

$$s = \text{diag}\left(V \left(E^Y - V\right)^\mathsf{H} \left(\left(Z^Y\right)^{-1}\right)^\mathsf{H}\right) \tag{7.13c}$$

assuming Z^Y is invertible. The inverse of Z^Y is calculated in Exercise 7.6.

The linear I-V relation and the nonlinear V-s or I-s relation in (7.2) takes the form of (7.13) for a voltage source.

If $z^n = 0$ then $Z^Y = z^Y$. From (7.13b) the phases are decoupled, i.e., $V^a = E^{an} - z^{an} I^a$, whether or not the current I and the voltage V are balanced. For an ideal voltage source where both $z^n = 0$ and $z^Y = 0$, the internal and external models (7.12) (7.13) here reduce to, under assumption C7.1,

$$V = V = E^Y, \qquad s = s^Y = \text{diag}(E^Y I^\mathsf{H})$$

Example 7.2. Unlike for an ideal voltage source, s^Y in (7.12b) includes both the power $s^Y_{\text{ideal}} := \text{diag}\left(E^Y I^{Y\mathsf{H}}\right)$ across the ideal voltage source and the power $s_{\text{imp}} := \text{diag}\left(z^Y I^Y I^{Y\mathsf{H}}\right)$ delivered to the series impedance z^Y. Hence the net power injection is

$$s = -\left(s^Y_{\text{ideal}} + s_{\text{imp}} + V^n \bar{I}^Y\right)$$

Summing across phases a, b, c shows that the total power generated is equal to the total power injection and total power consumed by the internal impedances of the voltage source:

$$-\mathbf{1}^\mathsf{T} s^Y_{\text{ideal}} = \mathbf{1}^\mathsf{T} s + \mathbf{1}^\mathsf{T} s_{\text{imp}} + s^n$$

where s^n given by (7.12c) is the power delivered to the impedance z^n on the neutral wire.

Current source (J^Y, y^Y, z^n)**.** *Internal model.* Referring to Figure 7.4(b) the internal model of a current source is given by

$$I^Y \;=\; J^Y + y^Y V^Y, \qquad V^n - V^{n'} \;=\; z^n\left(\mathbf{1}^\mathsf{T} I^Y\right), \qquad I^n \;=\; \mathbf{1}^\mathsf{T} I^Y \tag{7.14a}$$

This yields an internal power $s^Y := \text{diag}\left(V^Y I^{Y\mathsf{H}}\right)$ across the non-ideal current source and an internal power $s^n := V^n I^{n\mathsf{H}}$ across the impedance z^n on the neutral line, given by (Exercise 7.7):

$$s^Y \;=\; \text{diag}\left(V^Y J^{Y\mathsf{H}}\right) + \text{diag}\left(V^Y V^{Y\mathsf{H}} y^{Y\mathsf{H}}\right) \;=\; \underbrace{\begin{bmatrix} V^{an} J^{an\mathsf{H}} \\ V^{bn} J^{bn\mathsf{H}} \\ V^{cn} J^{cn\mathsf{H}} \end{bmatrix}}_{s^Y_{\text{ideal}}} + \underbrace{\begin{bmatrix} y^{an\mathsf{H}} \left|V^{an}\right|^2 \\ y^{bn\mathsf{H}} \left|V^{bn}\right|^2 \\ y^{cn\mathsf{H}} \left|V^{cn}\right|^2 \end{bmatrix}}_{s_{\text{adm}}} \tag{7.14b}$$

$$s^n \;:=\; V^n I^{n\mathsf{H}} \;=\; z^n \left|\mathbf{1}^\mathsf{T} J^Y + \text{diag}\left(y^Y\right)^\mathsf{T} V^Y\right|^2 \tag{7.14c}$$

External model. The derivation here is analogous to that for a voltage source above. Applying the conversion rule (7.8) to the internal model (7.14a) yields an external model of a current source that relates its terminal variables:

$$I \;=\; -J^Y - y^Y\left(V - V^n\mathbf{1}\right), \quad \mathbf{1}^\mathsf{T} I \;=\; -I^n, \quad s \;=\; -\text{diag}\left(V J^{Y\mathsf{H}}\right) - \text{diag}\left(V\left(V - V^n\mathbf{1}\right)^\mathsf{H} y^{Y\mathsf{H}}\right) \tag{7.15a}$$

As discussed earlier, $I^n = 0$ if the neutral is ungrounded.

Suppose assumption C7.1 holds so that $V^n = -z^n\left(\mathbf{1}^\mathsf{T} I\right)$. Then (7.15a) yields (Exercise 7.8):

$$V \;=\; -\left(z^Y J^Y + Z^Y I\right), \qquad I \;=\; -A\left(J^Y + y^Y V\right) \tag{7.15b}$$

where, assuming Z^Y is invertible,

$$z^Y \;:=\; \left(y^Y\right)^{-1}, \qquad Z^Y \;:=\; z^Y + z^n\,\mathbf{1}\mathbf{1}^\mathsf{T}, \qquad A \;:=\; \mathbb{I} - \frac{z^n}{1 + z^n\left(\mathbf{1}^\mathsf{T} y^Y \mathbf{1}\right)}\, y^Y \mathbf{1}\mathbf{1}^\mathsf{T}$$

and $\mathbb{I}$ denotes the identity matrix of size 3. The effective impedance matrix Z^Y is the same matrix in (7.13b) for a voltage source. Substituting (7.15b) into $s = \text{diag}\left(V I^\mathsf{H}\right)$ expresses the terminal power s as a quadratic function of V:

$$s \;=\; -\text{diag}\left(V\left(J^{Y\mathsf{H}} + V^\mathsf{H} y^{Y\mathsf{H}}\right) A^\mathsf{H}\right) \tag{7.15c}$$

The linear I-V relation and the nonlinear V-s or I-s relation in (7.2) takes the form of (7.15) for a current source.

Analogous to a voltage source, the phases are decoupled if $z^n = 0$. An ideal current source with $y^Y = 0$ and $z^n = 0$ has $I = -I^Y = -J^Y$ and $s = -\text{diag}\left(V J^{Y\mathsf{H}}\right)$.

Power source $\left(\sigma^Y, z^n\right)$**.** *Internal model:* By definition the power delivered to a constant-power source and the power delivered to the impedance z^n on the neutral line are respectively (Figure 7.4(c))

$$s^Y \ := \ \text{diag}\left(V^Y I^{Y\mathsf{H}}\right) \ = \ \sigma^Y, \qquad s^n \ := \ \left(V^n - V^{n'}\right) I^{n\mathsf{H}} \ = \ z^n \left|\mathbf{1}^\mathsf{T} I^Y\right|^2 \tag{7.16}$$

External model: Apply the conversion rule to the internal model (7.16) yields an external model that relates the terminal variables:

$$\sigma^Y \ = \ \text{diag}\left(I^{Y\mathsf{H}}\right) V^Y \ = \ -\text{diag}\left(I^\mathsf{H}\right)(V - V^n\mathbf{1}), \qquad s \ = \ -\sigma^Y + V^n \bar{I}, \qquad \mathbf{1}^\mathsf{T} I \ = \ -I^n \tag{7.17a}$$

Suppose assumption C7.1 holds so that $V^{n'} = 0$ and $V^n = -z^n\left(\mathbf{1}^\mathsf{T} I\right)$. We can then rewrite the vector $V^n \bar{I}^Y$ as

$$V^n \bar{I} \ = \ -z^n\left(\mathbf{1}^\mathsf{T} I\right)\bar{I} \ = \ -z^n\left(\bar{I} I^\mathsf{T}\right)\mathbf{1}$$

This yields a quadratic relation between V and I (Exercise 7.9):

$$V \ = \ -\left(\text{diag}\bar{I}\right)^{-1}\sigma^Y \ - \ z^n\left(\mathbf{1}\mathbf{1}^\mathsf{T}\right) I \tag{7.17b}$$

and between s and I:

$$s \ = \ -\left(\sigma^Y \ + \ z^n\left(\bar{I} I^\mathsf{T}\right)\mathbf{1}\right) \tag{7.17c}$$

It is generally not possible to solve (7.17b) for I in closed form and hence there is generally not an explicit V-s model for a power source. From (7.17c) the total power $-\mathbf{1}^\mathsf{T}\sigma^Y$ generated by the constant-power source is equal to the total power injection and the power delivered to the impedance on the neutral line:

$$-\mathbf{1}^\mathsf{T}\sigma^Y \ = \ \mathbf{1}^\mathsf{T} s \ + \ \underbrace{z^n\left(\mathbf{1}^\mathsf{T} I^Y\right)}_{-V^n}\underbrace{\left(\mathbf{1}^\mathsf{T}\bar{I}^Y\right)}_{-I^{n\mathsf{H}}} \ = \ \mathbf{1}^\mathsf{T} s \ + \ s^n$$

Clearly $s = -\sigma^Y$ if $z^n = 0$.

Impedance $\left(z^Y, z^n\right)$**.** *Internal model:* Referring to Figure 7.4(d) the internal model of an impedance is

$$V^Y \ = \ z^Y I^Y, \qquad s^Y \ := \ V^Y I^{Y\mathsf{H}}, \qquad s^n \ := \ \left(V^n - V^{n'}\right) I^{n\mathsf{H}} \ = \ z^n \left|\mathbf{1}^\mathsf{T} I^Y\right|^2 \tag{7.18}$$

External model: Application of the conversion rule (7.8) to the internal model (7.18) yields an external model that relates the terminal variables:

$$V \ = \ -z^Y I \ + \ V^n\mathbf{1}, \qquad -\mathbf{1}^\mathsf{T} I \ = \ I^n \tag{7.19a}$$

If assumption C7.1 holds so that $V^{n'} = 0$ and $V^n = -z^n\left(\mathbf{1}^\mathsf{T} I\right)$, then the external model reduces to:

$$V \ = \ -Z^Y I \tag{7.19b}$$

where $Z^Y := z^Y + z^n \mathbf{1}\mathbf{1}^\mathsf{T}$ is the same effective impedance Z^Y in (7.13b) for a voltage source. Substituting (7.19b) into $s = \text{diag}\left(VI^\mathsf{H}\right)$ expresses s as a quadratic function of V:

$$s = -\text{diag}\left(VV^\mathsf{H}\left((Z^Y)^{-1}\right)^\mathsf{H}\right) \tag{7.19c}$$

assuming Z^Y is invertible. If $z^n = 0$ then $Z^Y = z^Y$ is diagonal.

Balanced impedance. When $z^n \neq 0$ but z^Y is balanced, i.e., $z^{an} = z^{bn} = z^{cn}$, then $Z^Y = z^{an}\mathbb{I} + z^n \mathbf{1}\mathbf{1}^\mathsf{T}$ and its off-diagonal entries will couple voltages and currents in different phases. One can perform a similarity transformation using the unitary matrix F to what is called the *sequence coordinate* as explained in Chapter 7.2.2. In the sequence coordinate, the transformed impedance $\tilde{Z}^Y$, called the sequence impedance, is diagonal:

$$\tilde{Z}^Y = \begin{bmatrix} z^{an} + 3z^n & 0 & 0 \\ 0 & z^{an} & 0 \\ 0 & 0 & z^{an} \end{bmatrix}$$

This leads to decoupled voltages and currents in the sequence coordinate called symmetrical components. The decoupled relation between the sequence voltages, currents and impedances can be interpreted as defining separate sequence networks that can be analyzed independently. This is explained in Chapter 9.4.1.

Remark 7.5 (Phase decoupling)**.** The matrix $Z^Y := z^Y + z^n \mathbf{1}\mathbf{1}^\mathsf{T}$ in (7.13) (7.15) (7.19) is called the *phase impedance matrix* or the *impedance matrix*.

1. If $z^n = 0$ in these four devices, i.e., the neutrals are directly grounded, then the phases are decoupled. This is because, for a power source, $s = -\sigma^Y$, and for the other devices, the impedance matrix $Z^Y = z^Y$ becomes diagonal and hence $V = z^Y I$.

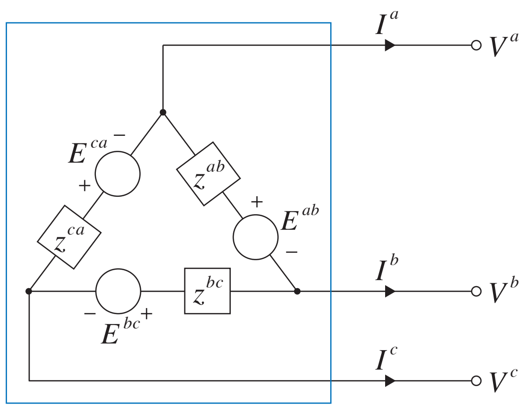
2. If $z^n \neq 0$ but the currents are balanced, i.e., $I^a + I^b + I^c = 0$ then $I^n = 0$ and $V^{ng} = 0$. In this case the phases are also decoupled. If the voltage V is balanced and $z^{an} = z^{an} = z^{cn}$ then I^n will indeed be zero and the phases will be decoupled (Exercise 7.10).

3. In unbalanced operation, however, the neutral current I^n may be nonzero and Z^Y generally has nonzero off-diagonal entries that couple voltages and currents in different phases. As mentioned above, if $z^{an} = z^{an} = z^{cn}$ then the sequence impedance $\tilde{Z}^Y$ is diagonal and hence decoupled in the sequence domain (Chapter 9.4).

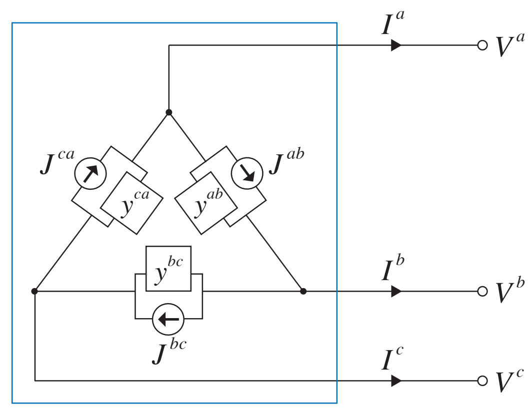
□

7.3.3 Devices in Δ configuration

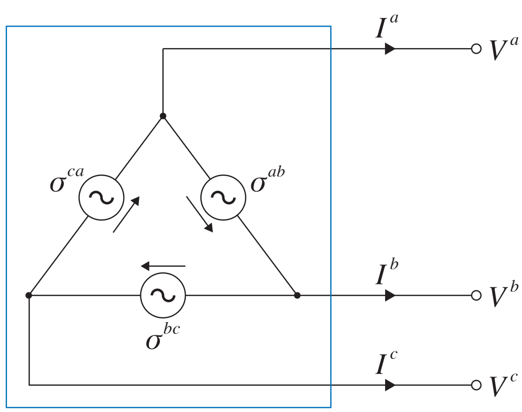
In this subsection we first present parameters of the same single-phase devices studied in Chapter 7.3.2, but arranged in Δ rather than Y configuration. For each device we then specify its internal model. Finally we apply the conversion rule (7.9) (7.10) to the internal model of each device to derive its external models.



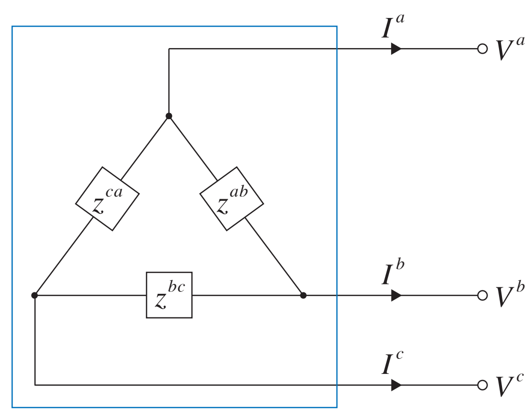
(a) Voltage source



(b) Current source



(c) Power source



(d) Impedance

Figure 7.5: Three-phase devices in Δ configuration. (a) A voltage source. (b) A current source. (c) A power load. (d) An impedance. Note the direction of J^{Δ} and σ^{Δ}.

Internal specification. The three-phase devices we study are shown in Figure 7.5.

1. *Voltage source* $\left(E^\Delta, z^\Delta\right)$. A three-wire voltage source in Δ configuration as shown in Figure 7.5(a) is specified by its internal line-to-line voltage $E^\Delta := (E^{ab}, E^{bc}, E^{ca})$ and series impedance matrix $z^\Delta := \text{diag}\left(z^{ab}, z^{bc}, z^{ca}\right)$. We assume that $z^{ab} + z^{bc} + z^{ca} \neq 0$. An ideal voltage source is one with $z^\Delta = 0$.

2. *Current source* $\left(J^\Delta, y^\Delta\right)$. A three-wire current source in Δ configuration as shown in Figure 7.5(b) is specified by its internal line-to-line current $J^\Delta := (J^{ab}, J^{bc}, J^{ca})$ and shunt admittance matrix $y^\Delta := \text{diag}\left(y^{ab}, y^{bc}, y^{ca}\right)$. An ideal current source is one with $y^\Delta = 0$.

3. *Power source* σ^Δ. A three-wire power source in Δ configuration as shown in Figure 7.5(c) consumes a constant power $\sigma^\Delta := \left(\sigma^{ab}, \sigma^{bc}, \sigma^{ca}\right)$ or injects a constant power $-\sigma^\Delta$.

4. *Impedance* z^Δ. A three-wire impedance in Δ configuration as shown in Figure 7.5(d) is specified by an impedance matrix $z^\Delta := \text{diag}\left(z^{ab}, z^{bc}, z^{ca}\right)$. We assume that $z^{ab} + z^{bc} + z^{ca} \neq 0$.

Voltage source $\left(E^\Delta, z^\Delta\right)$**.** *Internal model.* Referring to Figure 7.5(a) the internal model of a voltage source in Δ configuration is

$$V^\Delta \;=\; E^\Delta + z^\Delta I^\Delta, \qquad s^\Delta \;:=\; \text{diag}\left(V^\Delta I^{\Delta H}\right) \;=\; \text{diag}\left(E^\Delta I^{\Delta \mathsf{H}}\right) + \text{diag}\left(z^\Delta I^\Delta I^{\Delta \mathsf{H}}\right) \tag{7.20}$$

External model. The terminal voltage and current (V, I) are related to the internal voltage and current $\left(V^\Delta, I^\Delta\right)$ according to the conversion rule (7.9a) for Δ-configured devices, reproduced here

$$V^\Delta \;=\; \Gamma V, \qquad I \;=\; -\Gamma^\mathsf{T} I^\Delta$$

We will derive two equivalent relations between the terminal (V, I). Given V, the first relation uniquely determines I in terms of V. Given I, the second relation however determines V in terms of I only up to an arbitrary zero-sequence voltage γ. The asymmetry between these two cases is because V contains more information ($\gamma := \frac{1}{3}\mathbf{1}^\mathsf{T} V$) than I and uniquely determines the internal voltage V^Δ and hence I^Δ (from (7.20)) and I. In contrast I contains no information about the zero-sequence current $\beta := \frac{1}{3}\mathbf{1}^\mathsf{T} I^\Delta$ and hence does not uniquely determine the internal current I^Δ.

For the first relation that maps V to I, define $y^\Delta := \left(z^\Delta\right)^{-1}$ and write from (7.20)

$$I^\Delta \;=\; y^\Delta \left(V^\Delta - E^\Delta\right)$$

Multiplying both sides by $-\Gamma^\mathsf{T}$ and substituting the conversion rule we have

$$I \;=\; \left(\Gamma^\mathsf{T} y^\Delta\right) E^\Delta \;-\; Y^\Delta V \tag{7.21a}$$

where Y_Δ is a complex symmetric Laplacian matrix of the graph in Figure 1.9:[1]

$$Y^\Delta \;:=\; \Gamma^\mathsf{T} y^\Delta \Gamma \;=\; \begin{bmatrix} y^{ab}+y^{ca} & -y^{ab} & -y^{ca} \\ -y^{ab} & y^{bc}+y^{ab} & -y^{bc} \\ -y^{ca} & -y^{bc} & y^{ca}+y^{bc} \end{bmatrix}$$

Note that the terminal current I given by (7.21a) satisfies $\mathbf{1}^\mathsf{T} I = 0$.

For the second relation that maps I to V, substitute the conversion rule into the internal model (7.20) to eliminate the internal variable $\left(V^\Delta, I^\Delta\right)$:

$$\Gamma V \;=\; E^\Delta \;+\; z^\Delta \left(-\Gamma^{\mathsf{T}\dagger} I + \beta \mathbf{1}\right)$$

where we have used $I^\Delta \;=\; -\Gamma^{\mathsf{T}\dagger} I + \beta\mathbf{1}$ from (7.9c) and this is valid if and only if we require

$$\mathbf{1}^\mathsf{T} I \;=\; 0$$

Here $\beta \in \mathbb{C}$ is not arbitrary but depends on E^Δ and I.[2] Multiplying both sides by $\mathbf{1}^\mathsf{T}$ gives

$$0 \;=\; \mathbf{1}^\mathsf{T}\Gamma V \;=\; \mathbf{1}^\mathsf{T} E^\Delta \;-\; \underbrace{\mathbf{1}^\mathsf{T} z^\Delta}_{\tilde{z}^{\Delta\mathsf{T}}} \Gamma^{\mathsf{T}\dagger} I \;+\; \beta \underbrace{\left(\mathbf{1}^\mathsf{T} z^\Delta \mathbf{1}\right)}_{\zeta}$$

Define the column vector $\tilde{z}^\Delta := z^\Delta \mathbf{1} = \left(z^{ab}, z^{bc}, z^{ca}\right)$ and the scalar $\zeta := \mathbf{1}^\mathsf{T} z^\Delta \mathbf{1} = z^{ab} + z^{bc} + z^{ca}$. Then

$$\beta \;=\; \frac{1}{\zeta}\left(\tilde{z}^{\Delta\mathsf{T}} \Gamma^{\mathsf{T}\dagger} I \;-\; \mathbf{1}^\mathsf{T} E^\Delta\right)$$

Note that $\mathbf{1}^\mathsf{T} E^\Delta$ is the zero-sequence internal voltage and $\tilde{z}^\Delta$ is the vector of internal impedances. Both are zero, and hence $\beta = 0$, if the internal voltage E^Δ and impedances $\tilde{z}^\Delta$ are balanced. Therefore

$$\begin{aligned} \Gamma V \;&=\; E^\Delta \;-\; z^\Delta \Gamma^{\mathsf{T}\dagger} I \;+\; \frac{1}{\zeta} z^\Delta \mathbf{1}\left(\tilde{z}^{\Delta\mathsf{T}}\Gamma^{\mathsf{T}\dagger} I \;-\; \mathbf{1}^\mathsf{T} E^\Delta\right) \\ &=\; \left(\mathbb{I} - \frac{1}{\zeta}\tilde{z}^\Delta \mathbf{1}^\mathsf{T}\right) E^\Delta \;-\; z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma^{\mathsf{T}\dagger} I \end{aligned}$$

or

$$V \;=\; \hat{\Gamma} E^\Delta \;-\; Z^\Delta I \;+\; \gamma\mathbf{1}, \qquad\qquad \mathbf{1}^\mathsf{T} I \;=\; 0 \tag{7.21b}$$

where (using Theorem 7.2)

$$\hat{\Gamma} \;:=\; \frac{1}{3}\Gamma^\mathsf{T}\left(\mathbb{I} - \frac{1}{\zeta}\tilde{z}^\Delta \mathbf{1}^\mathsf{T}\right), \qquad\qquad Z^\Delta \;:=\; \frac{1}{9}\Gamma^\mathsf{T} z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma$$

and γ is fixed by a given reference voltage. This is similar to (7.13b) for the Y-configured voltage source.

The two external models (7.21a) and (7.21b) are equivalent in the following sense.

[1] Note however that y^Δ is a complex matrix and therefore Y^Δ is complex symmetric, not Hermitian. Therefore span($\mathbf{1}$) is a subset of the null space of Y^Δ. For a sufficient condition for the null space of Y^Δ to be span($\mathbf{1}$), see Exercise 4.3.

[2] To gain intuition, imagine the voltage source is connected to a constant-voltage device that fixes the terminal voltage V of the voltage source, and hence its internal voltage $V^\Delta = \Gamma V$. Therefore, on each phase line, say, line ab, we have $V^{ab} - E^{ab} = z^{ab} I^{ab}$. Hence I^Δ is uniquely determined which fixes both I and $\beta := \frac{1}{3}\mathbf{1}^\mathsf{T} I^\Delta$.

Theorem 7.3. Given the conversion rules $V^\Delta = \Gamma V$ and $I = -\Gamma^\mathsf{T} I^\Delta$ between the terminal and internal voltages and currents, the following are equivalent:

1. Internal model: $V^\Delta \; = \; E^\Delta + z^\Delta I^\Delta$ and $\mathbf{1}^\mathsf{T}\left(E^\Delta + z^\Delta I^\Delta\right) = 0$.
2. External model: $I \; = \; \left(\Gamma^\mathsf{T} y^\Delta\right) E^\Delta - Y^\Delta V$ where $Y^\Delta := \Gamma^\mathsf{T} y^\Delta \Gamma$.
3. External model: $V = \hat{\Gamma} E^\Delta - Z^\Delta I + \gamma \mathbf{1}$, $\mathbf{1}^\mathsf{T} I = 0$ for some $\gamma \in \mathbb{C}$ where

$$\hat{\Gamma} \;\; := \;\; \frac{1}{3}\Gamma^\mathsf{T}\left(\mathbb{I} - \frac{1}{\zeta}\tilde{z}^\Delta \mathbf{1}^\mathsf{T}\right), \qquad\qquad Z^\Delta \;\; := \;\; \frac{1}{9}\Gamma^\mathsf{T} z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma$$

$\square$

The proof of the theorem is similar to that of Theorem 7.4 and left as Exercise 7.13.

Hence given V, I is uniquely determined by (7.21a) and given I, V is determined by (7.21b) up to a reference voltage specified by γ. These equations allow us to relate terminal power injection s to V or to I as:

$$s \;\; = \;\; \mathrm{diag}\left(VI^\mathsf{H}\right) \;\; = \;\; \mathrm{diag}\left(V\left(\Gamma^\mathsf{T} y^\Delta E^\Delta \; - \; Y^\Delta V\right)^\mathsf{H}\right) \tag{7.21c}$$

$$s \;\; = \;\; \mathrm{diag}\left(VI^\mathsf{H}\right) \;\; = \;\; \mathrm{diag}\left(\left(\hat{\Gamma} E^\Delta \; - \; Z^\Delta I\right) I^\mathsf{H}\right) \; + \; \gamma \bar{I} \tag{7.21d}$$

For an ideal voltage source where $z^\Delta = 0$ we have $\hat{\Gamma} := \frac{1}{3}\Gamma^\mathsf{T}$ and $Z^\Delta = 0$. The external model is, provided $\mathbf{1}^\mathsf{T} E^\Delta = 0$,

$$V \;\; = \;\; \frac{1}{3}\Gamma^\mathsf{T} E^\Delta \; + \; \gamma \mathbf{1}, \qquad\qquad \mathbf{1}^\mathsf{T} I \;\; = \;\; 0, \qquad\qquad s \;\; = \;\; \frac{1}{3}\mathrm{diag}\left(\Gamma^\mathsf{T} E^\Delta I^\mathsf{H}\right) \; + \; \gamma \bar{I}$$

where γ is fixed by a reference voltage.

Current source $\left(J^\Delta, y^\Delta\right)$. *Internal model.* Referring to Figure 7.5(b) the internal model of a current source in Δ configuration is

$$I^\Delta \;\; = \;\; J^\Delta \; + \; y^\Delta V^\Delta, \qquad s^\Delta \;\; := \;\; \mathrm{diag}\left(V^\Delta I^{\Delta H}\right) \;\; = \;\; \mathrm{diag}\left(V^\Delta J^{\Delta\mathsf{H}}\right) \; + \; \mathrm{diag}\left(V^\Delta V^{\Delta\mathsf{H}} y^{\Delta\mathsf{H}}\right) \tag{7.22}$$

External model. Multiplying both sides of $I^\Delta = J^\Delta + y^\Delta V^\Delta$ by $-\Gamma^\mathsf{T}$ and substituting the general conversion rule

$$V^\Delta \;\; = \;\; \Gamma V, \qquad\qquad I \;\; = \;\; -\Gamma^\mathsf{T} I^\Delta$$

for Δ-configured devices, we have

$$I \;\; = \;\; -\left(\Gamma^\mathsf{T} J^\Delta \; + \; Y^\Delta V\right) \tag{7.23a}$$

where $Y^{\Delta} := \Gamma^{\mathsf{T}} y^{\Delta} \Gamma$ is the matrix in (7.21a). The power injection is

$$s \;=\; \text{diag}\left(VI^{\mathsf{H}}\right) \;=\; -\text{diag}\left(VJ^{\Delta\mathsf{H}}\Gamma \,+\, VV^{\mathsf{H}}Y^{\Delta\mathsf{H}}\right) \tag{7.23b}$$

For an ideal current source where $y^{\Delta} = 0$ we have $I = -\Gamma^{\mathsf{T}} J^{\Delta}$ and $s = -\text{diag}(VJ^{\Delta\mathsf{H}}\Gamma)$.

Remark 7.6 (Voltage and current sources)**.** A Δ-configured current source specifies its internal current J^{Δ} which then uniquely determines its terminal current I through the conversion rule (7.9a), as well as its zero-sequence current $\beta := \frac{1}{3}\mathbf{1}^{\mathsf{T}} J^{\Delta}$, whereas a voltage source specifies its internal voltage E^{Δ} which does not uniquely determine its terminal voltage V. This is why the external voltage source model (7.21b) determines V only up to an arbitrary zero-sequence voltage γ and requires $\mathbf{1}^{\mathsf{T}} I = 0$ while both (7.21a) and (7.23a) are valid without any extra condition as their derivation does not involve pseudo-inverse of conversion matrices.

Power source σ^{Δ}. *Internal model.* Referring to Figure 7.5(c) the internal model of a constant-power source is

$$s^{\Delta} \;:=\; \text{diag}\left(V^{\Delta} I^{\Delta\mathsf{H}}\right) \;=\; \sigma^{\Delta} \tag{7.24}$$

This specifies the powers $(\sigma^{ab}, \sigma^{bc}, \sigma^{ca})$ delivered to these single-phase devices.

External model. Applying the power conversion rule (7.10b) to the internal model $s^{\Delta} = \sigma^{\Delta}$ yields an external model of a constant-power source that relates its terminal voltage and current (V, I):

$$\sigma^{\Delta} \;=\; -\frac{1}{3}\,\text{diag}\left(\Gamma\left(VI^{\mathsf{H}}\right)\Gamma^{\mathsf{T}}\right) \,+\, \overline{\beta}\Gamma V, \qquad\qquad \mathbf{1}^{\mathsf{T}} I \;=\; 0 \tag{7.25a}$$

where the first equality follows because $\left(\Gamma^{\mathsf{T}\dagger}\right)^{\mathsf{H}} = \frac{1}{3}\Gamma^{\mathsf{H}} = \frac{1}{3}\Gamma^{\mathsf{T}}$ from Theorem 7.2. Here β represents the amount of loop flow in the internal current I^{Δ}. All three quantities (V, I, β) are variables to be determined by the interaction with other devices through the network; see Chapter 9.1. Here (V, I) are terminal variables but, unlike the external models of other devices, β is a quantity internal to the Δ configuration.

An alternative model of a constant-power source is (7.10c) that relates its terminal voltage V with its *internal* current I^{Δ}:

$$\sigma^{\Delta} \;:=\; \text{diag}\left(V^{\Delta} I^{\Delta\mathsf{H}}\right) \;=\; \text{diag}\left(\Gamma V I^{\Delta\mathsf{H}}\right) \tag{7.25b}$$

An advantage of this model is that it contains implicitly both the zero-sequence terminal voltage $\gamma := \frac{1}{3}\mathbf{1}^{\mathsf{T}} V$ and zero-sequence internal current $\beta := \frac{1}{3}\mathbf{1}^{\mathsf{T}} I^{\Delta}$.

We now study the connection between the two equivalent models (7.25a) and (7.25b) of a constant-power source that relate (V, I) and $\left(V, I^{\Delta}\right)$ respectively. Expand the first equation in (7.25a) to get

$$\sigma^{\Delta} \;=\; -\frac{1}{3}\begin{bmatrix} \left(I^a - I^b\right)^{\mathsf{H}}\left(V^a - V^b\right) \\ \left(I^b - I^c\right)^{\mathsf{H}}\left(V^b - V^c\right) \\ \left(I^c - I^a\right)^{\mathsf{H}}\left(V^c - V^a\right) \end{bmatrix} \,+\, \overline{\beta}\begin{bmatrix} V^a - V^b \\ V^b - V^c \\ V^c - V^a \end{bmatrix} \;=\; \underbrace{\left(\text{diag}\left(\left(-\Gamma^{\mathsf{T}\dagger} I\right)^{H}\right) \,+\, \overline{\beta}\mathbb{I}\right)}_{\text{diag}\left(I^{\Delta\mathsf{H}}\right)}(\Gamma V)$$

which is equivalent to (7.25b). Given a terminal voltage V, the currents I and I^Δ can be uniquely determined in these models (7.25a) and (7.25b) respectively. Given a current I or I^Δ in (7.25a) and (7.25b) respectively, however, V cannot be uniquely determined.

Specifically, given a terminal voltage V, the model (7.25b) provides three linear equations in three unknowns I^Δ, which determines I^Δ uniquely. Both the terminal current I and β are then determined uniquely. Conversely, given I^Δ (and hence β), (7.25b) provides three linear equations in three unknowns V but only $\left(V^a - V^b, V^b - V^c, V^c - V^a\right)$, i.e., $V^\Delta = \Gamma V$, can be uniquely determined. The terminal voltage V (or equivalently, its zero-sequence voltage γ) needs to be determined through network equations or from a reference voltage.

Similarly for the model (7.25b), given a terminal voltage V, (7.25a) provides four linear equations in four unknowns $I := (I^a, I^b, I^c)$ and β which determine (I, β) uniquely (Exercise 7.14). Intuitively, the given terminal voltage V fixes the internal voltage V^Δ which then fixes the internal current I^Δ since $\text{diag}\left(V^\Delta I^{\Delta \mathsf{H}}\right) = \sigma^\Delta$. This then produces a unique terminal current I and the zero-sequence current $\beta := \frac{1}{3}\mathbf{1}^\mathsf{T} I^\Delta$. On the other hand, consider the situation where the terminal current I with $\mathbf{1}^\mathsf{T} I = 0$ is given, instead of I^Δ as for the model (7.25b) above. In this case (7.25a) also does not uniquely determine the terminal voltage V because (7.25a) provides three quadratic equations in four unknowns (V, β), quadratic due to the term $\beta \Gamma V$. Moreover since I contains less information than I^Δ, there is ambiguity in β in addition to γ; see Exercise 7.15. As for the model (7.25b) the terminal voltage V (hence γ) and β will be determined through network equations or from a reference voltage.

For a balanced system however the loop flow β and the internal voltages V^Δ are uniquely determined by σ^Δ and a terminal current I, as the next example illustrates.

Example 7.3 (Balanced systems)**.** Consider a constant-power source with a given σ^Δ whose external behavior is described by (7.25a). Given a terminal current $I = i\alpha_+$ which is a positive-sequence balanced vector with $\mathbf{1}^\mathsf{T} I = 0$:

1. Show that the given σ^Δ and I must satisfy
$$\sigma^\Delta \;\; \in \;\; \text{span}\left(-\frac{1-\alpha}{3}\, i\mathbf{1} \, + \, \overline{\beta}\alpha_+\right)$$
for some $\beta \in \mathbb{C}$. Note that the internal power σ^Δ is different in each phase (with different phase angles separated by 120°) if and only if the loop flow $\beta \neq 0$.
2. Show that the loop flow β and the internal voltage V^Δ are uniquely determined by σ^Δ and I, and that the terminal voltage V is unique only up to an arbitrary reference voltage.

Assume that the internal voltage V^Δ is also a positive-sequence balanced vector.

Solution. By Corollary 1.4 we have for any balanced vector $x \in \mathbb{C}^3$ in positive sequence
$$\Gamma x \;=\; (1-\alpha)x, \qquad\qquad \Gamma^\mathsf{T} x \;=\; (1-\alpha^2)x$$
Hence the internal current is
$$I^\Delta \;=\; -\Gamma^{\mathsf{T}\dagger} I \,+\, \beta\mathbf{1} \;=\; -\frac{1}{3}\Gamma I \,+\, \beta\mathbf{1} \;=\; -\frac{1-\alpha}{3}\, i\alpha_+ \,+\, \beta\mathbf{1}$$

where the second equality follows from Theorem 7.2. By assumption V^Δ is a positive-sequence balanced vector, i.e., $V^\Delta = v\alpha_+$ where $v \in \mathbb{C}$ is a scalar to be determined. Then

$$\begin{aligned}
\sigma^\Delta &= \operatorname{diag}\left(V^\Delta I^{\Delta\mathsf{H}}\right) \;=\; v\operatorname{diag}\left(\alpha_+\left(-\frac{(1-\alpha)i}{3}\alpha_+ \,+\, \beta\mathbf{1}\right)^{\mathsf{H}}\right) \\
&= v\left(-\frac{(1-\alpha)i}{3}\operatorname{diag}\left(\alpha_+\alpha_+^{\mathsf{H}}\right) \,+\, \overline{\beta}\operatorname{diag}\left(\alpha_+\mathbf{1}^{\mathsf{T}}\right)\right) \\
&= v\left(-\frac{(1-\alpha)i}{3}\mathbf{1} \,+\, \overline{\beta}\alpha_+\right)
\end{aligned}$$

i.e., σ^Δ lies in span$\left(-\frac{(1-\alpha)i}{3}\mathbf{1} \,+\, \overline{\beta}\alpha_+\right)$ for some β. To determine v, multiplying both sides by $\mathbf{1}^{\mathsf{T}}$ to get

$$v \;=\; \frac{-\mathbf{1}^{\mathsf{T}}\sigma^\Delta}{(1-\alpha)i}$$

Then $V^\Delta = v\alpha_+$. The terminal voltage V is given by

$$V \;=\; \Gamma^\dagger V^\Delta \,+\, \gamma\mathbf{1} \;=\; \frac{v}{3}\Gamma^{\mathsf{T}}\alpha_+ \,+\, \gamma\mathbf{1} \;=\; \frac{-\mathbf{1}^{\mathsf{T}}\sigma^\Delta(1+\alpha)}{3i}\alpha_+ \,+\, \gamma\mathbf{1}, \qquad \gamma\in\mathbb{C}$$

which is unique up to an arbitrary reference voltage specified by $\gamma \in \mathbb{C}$.

Note that neither V^Δ nor V depends on β, even though from the expression above for σ^Δ in part 1, the internal powers $\sigma^\Delta := \left(\sigma^{ab}, \sigma^{bc}, \sigma^{ca}\right)$ depend on the loop flow specified by β. Moreover the expression uniquely determines β:

$$\sigma^{ab} \;=\; v\left(-\frac{(1-\alpha)i}{3} + \overline{\beta}\right), \quad \sigma^{bc} \;=\; v\left(-\frac{(1-\alpha)i}{3} + \alpha\overline{\beta}\right) \quad\Longrightarrow\quad \overline{\beta} \;=\; \frac{\sigma^{bc}-\sigma^{ab}}{\sigma^{ab}+\sigma^{bc}+\sigma^{ca}}\,i$$

□

Whereas (7.25a) relates the internal power σ^Δ to the external voltage and current (V, I), we can also use the conversion rule (7.10a) to relate the external power s to the internal voltage and current $\left(V^\Delta, I^\Delta\right)$. Specifically, the internal voltage and current (V^Δ, I^Δ) and the terminal power s of a constant-power source must satisfy:

$$s \;=\; -\frac{1}{3}\operatorname{diag}\left(\Gamma^{\mathsf{T}}\left(V^\Delta I^{\Delta\mathsf{H}}\right)\Gamma\right) \,-\, \gamma\Gamma^{\mathsf{T}}\overline{I}^\Delta, \qquad \sigma^\Delta \;=\; \operatorname{diag}\left(V^\Delta I^{\Delta\mathsf{H}}\right), \qquad \mathbf{1}^{\mathsf{T}}V^\Delta \;=\; 0 \tag{7.25c}$$

where γ is fixed by a reference voltage. An equivalent model in terms of $\left(V, I^\Delta\right)$ is (using (7.10c))

$$s \;=\; -\operatorname{diag}\left(VI^{\Delta\mathsf{H}}\Gamma\right), \qquad\qquad \sigma^\Delta \;=\; \operatorname{diag}\left(\Gamma VI^{\Delta\mathsf{H}}\right) \tag{7.25d}$$

The choice of different models in (7.25) for three-phase analysis depends on the specification of the problem. See Example 9.11 in Chapter 9.2.1.

Remark 7.7 (Total power)**.** Since σ^Δ is the power delivered to the single-phase devices while s is the power injected from the three-phase power source to the network it is connected to, (7.25) implies that (the negative of) its total internal power is equal to its total terminal power, i.e., $\mathbf{1}^{\mathsf{T}}s = -\mathbf{1}^{\mathsf{T}}\sigma^\Delta$ (Exercise 7.16). In particular the total terminal power $\mathbf{1}^{\mathsf{T}}s$ is independent of the loop-flow β and zero-sequence voltage γ even when s does.

Impedance z^Δ. *Internal model.* Referring to Figure 7.5(d) the internal model of an impedance z^Δ in Δ configuration is

$$V^\Delta \;=\; z^\Delta I^\Delta, \qquad\qquad s^\Delta \;=\; \text{diag}\left(V^\Delta I^{\Delta \mathsf{H}}\right) \;:=\; \text{diag}\left(z^\Delta I^\Delta I^{\Delta \mathsf{H}}\right) \tag{7.26}$$

External model. The external model can be derived in a similar way to that for a voltage source, by applying the conversion rule $V^\Delta = \Gamma V$, $I = -\Gamma^\mathsf{T} I^\Delta$ to the internal model (7.26). We will derive first a relation that maps a terminal voltage V (which also determines its zero-sequence component γ) uniquely to a terminal current I and then a converse relation that maps I to V up to an arbitrary γ.

Define the admittance matrix $y^\Delta := \left(z^\Delta\right)^{-1}$. Substituting into (7.26), multiplying both sides by $-\Gamma^\mathsf{T}$ and applying the conversion rule, we get

$$I \;=\; -Y^\Delta V \tag{7.27a}$$

where $Y^\Delta := \Gamma^\mathsf{T} y^\Delta \Gamma$ is the same complex symmetric Laplacian matrix in (7.21a) for a voltage source. Note that the terminal current I given by (7.27a) satisfies $\mathbf{1}^\mathsf{T} I = 0$.

For the converse relation, given any terminal current I that satisfies $\mathbf{1}^\mathsf{T} I = 0$, substitute the conversion rule into the internal model (7.26) to eliminate $\left(V^\Delta, I^\Delta\right)$:

$$\Gamma V \;=\; z^\Delta \left(-\Gamma^{\mathsf{T}\dagger} I + \beta \mathbf{1}\right)$$

where $\beta \in \mathbb{C}$ is not arbitrary but depends on I. Multiplying both sides by $\mathbf{1}^\mathsf{T}$ gives

$$0 \;=\; \mathbf{1}^\mathsf{T}\Gamma V \;=\; -\underbrace{\mathbf{1}^\mathsf{T} z^\Delta}_{\tilde{z}^{\Delta\mathsf{T}}} \Gamma^{\mathsf{T}\dagger} I \;+\; \beta \underbrace{\left(\mathbf{1}^\mathsf{T} z^\Delta \mathbf{1}\right)}_{\zeta}$$

where $\tilde{z}^\Delta := z^\Delta \mathbf{1}$ and $\zeta := z^{ab} + z^{bc} + z^{ca}$. Hence

$$\beta \;=\; \frac{1}{\zeta}\left(\tilde{z}^{\Delta\mathsf{T}} \Gamma^{\mathsf{T}\dagger}\right) I$$

Therefore

$$\Gamma V \;=\; -z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\,\tilde{z}^{\Delta\mathsf{T}}\right) \Gamma^{\mathsf{T}\dagger} I$$

or

$$V \;=\; -Z^\Delta I \;+\; \gamma \mathbf{1}, \qquad\qquad \mathbf{1}^\mathsf{T} I \;=\; 0 \tag{7.27b}$$

where γ is a variable to be determined together with V and (using Theorem 7.2)

$$Z^\Delta \;:=\; \frac{1}{9}\Gamma^\mathsf{T} z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\,\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma$$

is the same matrix in (7.21b).

Remark 7.8. Note that (7.27b) is a system of at most 4 linearly independent equations in 7 variables (V, I, γ). We can also eliminate the variable $\gamma := \frac{1}{3}\mathbf{1}^\mathsf{T} V$ and write (7.27b) equivalently in terms of only (V, I):

$$\left(\mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^\mathsf{T}\right) V \;=\; -Z^\Delta I, \qquad\qquad \mathbf{1}^\mathsf{T} I \;=\; 0$$

Since the matrices on both sides of the first equation are singular, this is a system of at most 3 linearly independent equations in 6 variables. It is often more convenient to use (7.27b) in analysis as it expresses V explicitly in terms of I despite the additional variable γ; see Example 9.8. □

As for a voltage source, the two external models (7.27a) and (7.27b) of an impedance are equivalent in the following sense. The theorem also implies that Z^Δ and Y^Δ are pseudo-inverses of each other.

Theorem 7.4. Given the conversion rules $V^\Delta = \Gamma V$ and $I = -\Gamma^\mathsf{T} I^\Delta$ between the terminal and internal voltages and currents, the following are equivalent:

1. Internal model: $V^\Delta \;=\; z^\Delta I^\Delta$ and hence $\mathbf{1}^\mathsf{T} z^\Delta I^\Delta = 0$.
2. External model: $I \;=\; -Y^\Delta V$ where $Y^\Delta := \Gamma^\mathsf{T} y^\Delta \Gamma$.
3. External model: $V = -Z^\Delta I + \gamma\mathbf{1}$, $\mathbf{1}^\mathsf{T} I = 0$ for some $\gamma \in \mathbb{C}$ where

$$Z^\Delta \;:=\; \frac{1}{9}\Gamma^\mathsf{T} z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma$$

Proof. The derivation above of the two external models (7.27a) and (7.27b) shows that 1 ⇒ 2 and 3. For the converse we will show that 2 ⇒ 1 and 3 ⇒ 1.

Suppose $I = -Y^\Delta V = -\left(\Gamma^\mathsf{T} y^\Delta \Gamma\right) V$. Substitute the conversion rules to get

$$\Gamma^\mathsf{T}\left(y^\Delta V^\Delta - I^\Delta\right) \;=\; 0$$

i.e., $y^\Delta V^\Delta - I^\Delta$ is in the null space of Γ^T, or $y^\Delta V^\Delta - I^\Delta = \beta\mathbf{1}$ for some $\beta \in \mathbb{C}$. Therefore

$$V^\Delta \;=\; z^\Delta I^\Delta + \beta\, z^\Delta \mathbf{1}$$

It is important to note that this expression is not of the form $V^\Delta = z'^\Delta I^\Delta + \beta'\mathbf{1}$ for some *diagonal* matrix $z'^\Delta \in \mathbb{C}^3$ and *scalar* $\beta' \in \mathbb{C}$. Since $\mathbf{1}^\mathsf{T} V^\Delta = 0$ because of the conversion rule, multiplying both sides by $\mathbf{1}^\mathsf{T}$ yields

$$\beta \;=\; -\frac{1}{\zeta}\tilde{z}^{\Delta\mathsf{T}} I^\Delta$$

where $\tilde{z}^\Delta := z^\Delta\mathbf{1}$ and $\zeta := z^{ab} + z^{bc} + z^{ca}$. Hence

$$V^\Delta \;=\; z^\Delta I^\Delta - \frac{1}{\zeta}\tilde{z}^{\Delta\mathsf{T}} I^\Delta z^\Delta \mathbf{1} \;=\; \underbrace{z^\Delta\left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\mathbf{1}^\mathsf{T} z^\Delta\right)}_{z'^\Delta} I^\Delta$$

For z'^{Δ} to be a valid three-phase impedance, it must be a diagonal matrix. This is the case if and only if $z^{\Delta}\mathbf{1}\left(\mathbf{1}^{\mathsf{T}}z^{\Delta}I^{\Delta}\right) = 0$ in which case $V^{\Delta} = z^{\Delta}I^{\Delta}$, as desired.

Suppose $V = -Z^{\Delta}I + \gamma\mathbf{1}$, $\mathbf{1}^{\mathsf{T}}I = 0$ for some $\gamma \in \mathbb{C}$. Then †

$$V^{\Delta} \;=\; \Gamma V \;=\; -\frac{1}{3}\Gamma\Gamma^{\dagger}z^{\Delta}\left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma I$$

Since $\mathbf{1}^{\mathsf{T}}I = 0$, there exists I^{Δ} such that $I = -\Gamma^{\mathsf{T}}I^{\Delta}$. Hence

$$V^{\Delta} \;=\; \Gamma\Gamma^{\dagger}z^{\Delta}\left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma\Gamma^{\mathsf{T}}I^{\Delta} \;=\; \underbrace{z^{\Delta}\left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\mathbf{1}^{\mathsf{T}}z^{\Delta}\right)}_{z'^{\Delta}}I^{\Delta}$$

where we have used $\Gamma\Gamma^{\dagger} = \mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^{\mathsf{T}}$ from Theorem 7.2. As before, z'^{Δ} must be a diagonal matrix to be a valid three-phase impedance. This is the case if and only if $z^{\Delta}\mathbf{1}\left(\mathbf{1}^{\mathsf{T}}z^{\Delta}I^{\Delta}\right) = 0$ in which case $V^{\Delta} = z^{\Delta}I^{\Delta}$, as desired. □

Hence given a V, I is uniquely determined by (7.27a) and given an I with $\mathbf{1}^{\mathsf{T}}I = 0$, V is determined by (7.27b) up to a reference voltage specified by γ. These equations allow us to relate terminal power injection s to V or to I as:

$$s \;=\; \mathrm{diag}\left(VI^{\mathsf{H}}\right) \;=\; -\mathrm{diag}\left(VV^{\mathsf{H}}Y^{\Delta\mathsf{H}}\right) \tag{7.27c}$$

$$s \;=\; \mathrm{diag}\left(VI^{\mathsf{H}}\right) \;=\; -\mathrm{diag}\left(Z^{\Delta}II^{\mathsf{H}}\right) \;+\; \gamma\bar{I} \tag{7.27d}$$

Balanced impedance. When the impedance is balanced, i.e., $z^{ab} = z^{bc} = z^{ca}$ then (Exercise 7.17)

$$Z^{\Delta} \;=\; \frac{z^{ab}}{3}\left(\mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^{\mathsf{T}}\right)$$

i.e., Z^{Δ} is not diagonal and the off-diagonal entries will couple voltages and currents in different phases. As we will see in Chapter 9.4.1, in this case, one can perform a similarity transformation using the unitary matrix F to what is called the *sequence coordinate* as explained in Chapter 7.2.2. In the sequence coordinate, the transformed impedance $\tilde{Z}^{\Delta}$, called the sequence impedance, is diagonal:

$$\tilde{Z}^{\Delta} \;=\; \frac{z^{ab}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This leads to decoupled voltages and currents in the sequence coordinate called symmetrical components. The zero-sequence component (first row and column of $\tilde{Z}^{\Delta}$) is zero, reflecting the fact that $I^a + I^b + I^c = 0$ in a Δ configuration since there is no neutral line. The decoupled relation between the sequence voltages, currents and impedances can be interpreted as defining separate sequence networks that can be analyzed independently.

Remark 7.9 (Phase decoupling)**.** Determine conditions under which phases become decoupled (Exercise 7.18). □

7.3.4 Δ-Y transformation

Ideal voltage source $\left(E^\Delta, \gamma\right)$. The terminal voltage of an ideal Δ-configured voltage source $\left(E^\Delta, \gamma\right)$ with zero internal impedance $z^\Delta = 0$ is, from (7.21b):

$$V \;=\; \frac{1}{3}\Gamma^\mathsf{T} E^\Delta \;+\; \gamma\mathbf{1}, \qquad\qquad \mathbf{1}^\mathsf{T} I \;=\; 0$$

where γ is fixed by a given reference voltage. The terminal voltage of an ideal Y-configured voltage source $\left(E^Y, V^n\right)$ with zero internal impedance $z^Y = 0$ is, from (7.13a):

$$V \;=\; E^Y \;+\; V^n\mathbf{1}, \qquad\qquad \mathbf{1}^\mathsf{T} I \;=\; -I^n$$

Hence the Y equivalent of an ideal voltage source $\left(E^\Delta, \gamma\right)$, not necessarily balanced, is given by

$$E^Y \;:=\; \frac{1}{3}\Gamma^\mathsf{T} E^\Delta, \qquad V^n \;:=\; \gamma, \qquad \text{no neutral line so that } \; I^n \;:=\; 0$$

Note that this does not satisfy assumption C7.1 since the neutral is not grounded unless $\gamma = 0$. If E^Δ is balanced then $\Gamma^\mathsf{T} E^\Delta = (1-\alpha^2)E^\Delta = \sqrt{3}\, e^{-\mathbf{i}\pi/6} E^\Delta$ (by Corollary 1.4) and E^Y reduces to the expression (1.32a) derived in Chapter 1.2.4 for balanced systems:

$$E^Y \;=\; \frac{1}{\sqrt{3}\, e^{\mathbf{i}\pi/6}} E^\Delta, \qquad V^n \;:=\; \gamma, \qquad \text{no neutral line so that } \; I^n \;:=\; 0$$

For a non-ideal Δ-configured voltage source $\left(E^\Delta, z^\Delta, \gamma\right)$, its terminal voltage is, from (7.21b):

$$V \;=\; \hat{\Gamma}^\mathsf{T} E^\Delta \;-\; Z^\Delta I \;+\; \gamma\mathbf{1}$$

where

$$\hat{\Gamma} \;:=\; \frac{1}{3}\Gamma^\mathsf{T}\left(\mathbb{I} - \frac{1}{\zeta}\tilde{z}^\Delta \mathbf{1}^\mathsf{T}\right), \qquad\qquad Z^\Delta \;:=\; \frac{1}{9}\Gamma^\mathsf{T} z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma$$

It generally does not have a Y equivalent. Indeed, since the Y equivalent needs to be ungrounded so that $\mathbf{1}^\mathsf{T} I = 0$, its external model is $V \;=\; E^Y - z^Y I + V^n\mathbf{1}$ from (7.13a). In general the effective impedance Z^Δ is not diagonal and hence may not be interpreted as an internal series impedance matrix z^Y of an Y-configured source, even if the impedance is balanced $z^\Delta := z^{ab}\mathbb{I}$ (in which case $Z^\Delta = \frac{z^{ab}}{3}\left(\mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^\mathsf{T}\right)$).

Remark 7.10 (Y-equivalent with equal line-to-line voltage)**.** Given a general Δ-configured device with internal voltage V^Δ, its equivalent line-to-neutral voltage is defined in [39, p.204] to be

$$V^Y \;:=\; \frac{1}{3}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} V^\Delta \tag{7.28}$$

This definition is the same as the Y-equivalent of an ideal voltage source V^Δ derived above with a particular choice of the neutral voltage:

$$V^Y \;:=\; \frac{1}{3}\Gamma^\mathsf{T} V^\Delta, \qquad\qquad V^n \;:=\; \gamma \;=\; 0$$

in the sense that they have the same line-to-line voltages.

To see this, recall that the line-to-line voltage $\tilde{V}^Y$ (not the terminal voltage) of a Y-configured device with internal voltage V^Y is $\tilde{V}^Y = \Gamma V^Y$. If it is equivalent to the given V^Δ then $V^\Delta = \tilde{V}^Y = \Gamma V^Y$. Theorem 7.2 then implies

$$V^Y \;=\; \frac{1}{3}\Gamma^\mathsf{T} V^\Delta + \gamma \mathbf{1} \qquad\qquad \text{for any } \; \gamma \in \mathbb{C}$$

Here γ being arbitrary means that the Δ-configured device has an arbitrary zero-sequence terminal voltage and its Y-equivalent has an arbitrary neutral voltage. Take $\gamma := 0$. Since $\mathbf{1}^\mathsf{T} V^\Delta = \mathbf{1}^\mathsf{T}\left(\Gamma V^Y\right) = 0$ we can add $\frac{1}{3}\mathbf{1}^\mathsf{T} V^\Delta$ to Y^Y to get

$$V^Y \;=\; \frac{1}{3}\left(\Gamma^\mathsf{T} + \mathbf{1}\mathbf{1}^\mathsf{T}\right)V^\Delta \;=\; \frac{1}{3}\left(\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right) V^\Delta \;=\; \frac{1}{3}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} V^\Delta$$

The model (7.28) is applicable only if the zero-sequence voltage $\gamma := \frac{1}{3}\mathbf{1}^\mathsf{T} V$ of the given Δ-configured device is zero. Otherwise its Y-equivalent must have a nonzero neutral voltage $V^j = \gamma$. □

Ideal current source J^Δ. An ideal Δ-configured current source J^Δ has an external model of $I = -\Gamma^\mathsf{T} J^\Delta$. Note that $\mathbf{1}^\mathsf{T} I = 0$. The external model of a Y-configured current source is $I = -J^Y$, $\mathbf{1}^\mathsf{T} I = -I^n$. Hence the Y equivalent is

$$J^Y \;=\; \Gamma^\mathsf{T} J^\Delta, \qquad\qquad \text{no neutral line so that } \; I^n \;:=\; 0$$

If J^Δ is balanced then Corollary 1.4 implies

$$J^Y \;=\; (1-\alpha^2) J^\Delta \;=\; \frac{\sqrt{3}}{e^{\mathrm{i}\pi/6}} J^\Delta$$

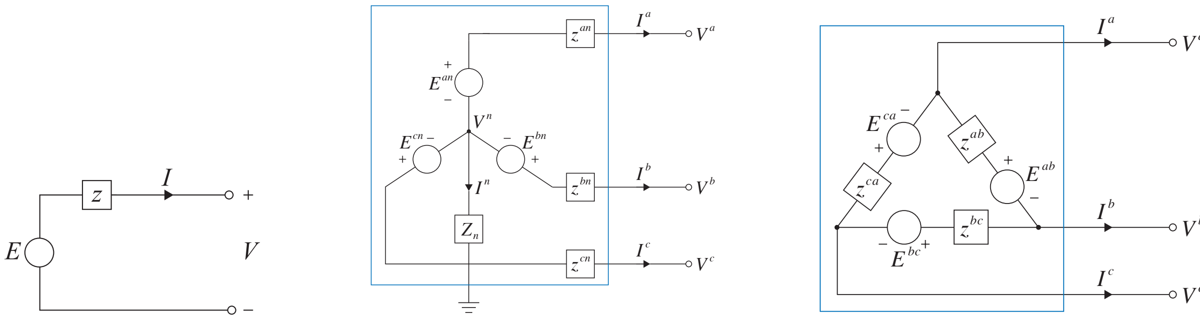
the same expression (1.32a) for balanced systems.

7.3.5 Comparison with single-phase devices

Assume C7.1 holds, i.e., neutrals are grounded and voltages are defined with respect to the ground. We compare the external models of three-phase devices to those of their single-phase counterparts. As we will see they are structurally the same, except for the Δ-configured power source.

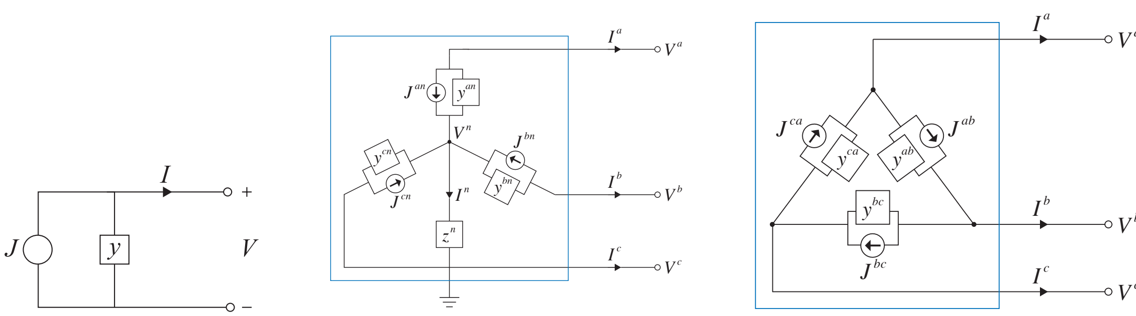
Voltage source. Figure 7.6 shows a single-phase voltage source specified by an internal voltage E and a series impedance z and the three-phase voltage sources in Y and Δ configurations studied in this section. Their external models are, from (7.13b) and (7.21b):

$$\begin{aligned} &\text{single-phase:} && V \;=\; E - zI && \\ &Y\text{-configuration:} && V \;=\; E^Y \;-\; Z^Y I, && Z^Y \;:=\; z^Y + z^n \mathbf{1}\mathbf{1}^\mathsf{T} \\ &\Delta\text{-configuration:} && V \;=\; \hat{\Gamma} E^\Delta - Z^\Delta I + \gamma \mathbf{1}, && \mathbf{1}^\mathsf{T} I \;=\; 0 \end{aligned}$$



(a) Single-phase (b) Y configuration (grounded) (c) Δ configuration

Figure 7.6: Comparison of single-phase and three-phase voltage sources.

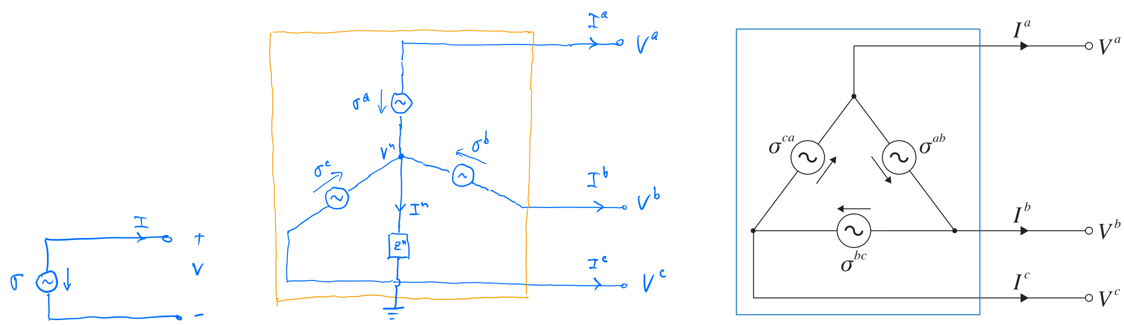


(a) Single-phase (b) Y configuration (grounded) (c) Δ configuration

Figure 7.7: Comparison of single-phase and three-phase current sources.

Current source. Figure 7.7 shows a single-phase current source specified by an internal current J and a shunt admittance y and the three-phase current sources in Y and Δ configurations studied in this section. Their external models are, from (7.15b) and (7.23a):

$$
\begin{aligned}
&\text{single-phase:} && I \;=\; -(J+yV) \\
&Y\text{-configuration:} && I \;=\; -A\left(J^Y + y^Y V\right), && A \;:=\; \mathbb{I} - \frac{z^n}{1+z^n\left(\mathbf{1}^{\mathsf{T}} y^Y \mathbf{1}\right)} y^Y \mathbf{1}\mathbf{1}^{\mathsf{T}} \\
&\Delta\text{-configuration:} && I \;=\; -\left(\Gamma^{\mathsf{T}} J^{\Delta} \,+\, Y^{\Delta} V\right), && Y^{\Delta} \;:=\; \Gamma^{\mathsf{T}} y^{\Delta} \Gamma
\end{aligned}
$$



(a) Single-phase (b) Y configuration (c) Δ configuration

Figure 7.8: Comparison of single-phase and three-phase power sources.

Power source. Figure 7.8 shows a single-phase power source specified by an internal power σ and the three-phase power sources in Y and Δ configurations studied in this section. Their external models are, from (7.17c) and (7.25d):

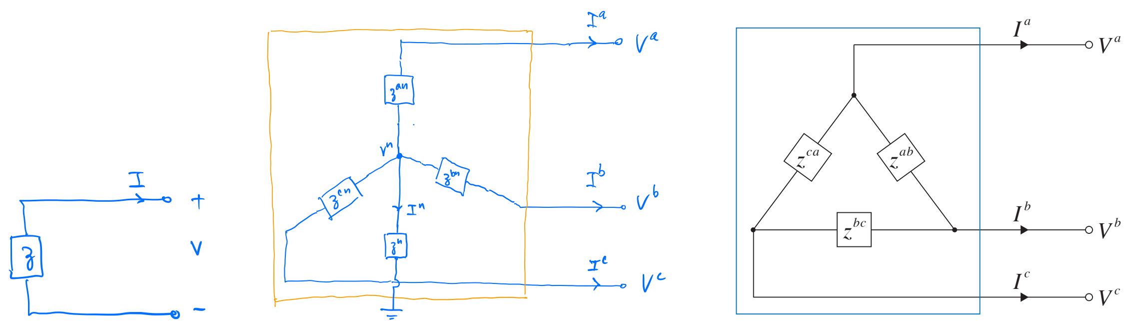
$$
\begin{aligned}
&\text{single-phase:} && s \;=\; -\sigma \\
&Y\text{-configuration:} && s \;=\; -\left(\sigma^Y + z^n\left(\bar{I} I^{\mathsf{T}}\right)\mathbf{1}\right) \\
&\Delta\text{-configuration:} && s \;=\; -\text{diag}\left(V I^{\Delta\mathsf{H}} \Gamma\right), && \sigma^{\Delta} \;=\; \text{diag}\left(\Gamma V I^{\Delta\mathsf{H}}\right)
\end{aligned}
$$

Impedance. Figure 7.9 shows a single-phase impedance specified by z and the three-phase power sources in Y and Δ configurations studied in this section. Their external models are, from (7.19b) and (7.27a):

$$
\begin{aligned}
&\text{single-phase:} && V \;=\; -zI \\
&Y\text{-configuration:} && V \;=\; -Z^Y I, && Z^Y \;:=\; z^Y \,+\, z^n \mathbf{1}\mathbf{1}^{\mathsf{T}} \\
&\Delta\text{-configuration:} && I \;=\; -Y^{\Delta} V, && Y^{\Delta} \;:=\; \Gamma^{\mathsf{T}} y^{\Delta} \Gamma
\end{aligned}
$$

7.3.6 Summary

The external models of three-phase devices are summarized in Table 7.2 and will be used to compose network models in Chapters 9 and 10.



(a) Single-phase (b) Y configuration (c) Δ configuration

Figure 7.9: Comparison of single-phase and three-phase impedances.

Device	Y configuration			Δ configuration		
	Specification	Internal	External	Specification	Internal	External
Voltage source	$\left(E^Y, z^Y, z^n\right)$	(7.12)	(7.13)	$\left(E^\Delta, z^\Delta\right)$	(7.20)	(7.21)
Current source	$\left(J^Y, y^Y, z^n\right)$	(7.14)	(7.15)	$\left(J^\Delta, y^\Delta\right)$	(7.22)	(7.23)
Power source	$\left(\sigma^Y, z^n\right)$	(7.16)	(7.17)	σ^Δ	(7.24)	(7.25)
Impedance	$\left(z^Y, z^n\right)$	(7.18)	(7.19)	z^Δ	(7.26)	(7.27)
Line (3-wire model)	(8.4)					

Table 7.2: Specification, internal and external models of three-phase devices.

When the devices are ideal these models reduce to a simpler form summarized in Tables 7.3 and 7.4. The internal models of ideal devices are:

1. Ideal voltage source $E^{Y/\Delta}$:

$$V^{Y/\Delta} = E^{Y/\Delta}, \qquad s^{Y/\Delta} = \operatorname{diag}\left(E^{Y/\Delta}\left(I^{Y/\Delta}\right)^{\mathsf{H}}\right) \tag{7.29a}$$

2. Ideal current source $J^{Y/\Delta}$:

$$I^{Y/\Delta} = J^{Y/\Delta}, \qquad s^{Y/\Delta} = \operatorname{diag}\left(V^{Y/\Delta}\left(J^{Y/\Delta}\right)^{\mathsf{H}}\right) \tag{7.29b}$$

3. Ideal power source $\sigma^{Y/\Delta}$:

$$s^{Y/\Delta} = \sigma^{Y/\Delta}, \qquad \sigma^{Y/\Delta} = \operatorname{diag}\left(V^{Y/\Delta}\left(I^{Y/\Delta}\right)^{\mathsf{H}}\right) \tag{7.29c}$$

4. Impedance $z^{Y/\Delta}$:

$$V^{Y/\Delta} = z^{Y/\Delta} I^{Y/\Delta}, \qquad s^{Y/\Delta} = \operatorname{diag}\left(V^{Y/\Delta}\left(I^{Y/\Delta}\right)^{\mathsf{H}}\right) \tag{7.29d}$$

Device	Assumption	Y configuration	
Voltage source	$z^n = 0$, $z^Y = 0$	$V = E^Y + \gamma\mathbf{1}$	$s = \text{diag}\left(E^Y I^{\mathsf{H}}\right) + \gamma\bar{I}$
Current source	$z^n = 0$, $y^Y = 0$	$I = -J^Y$	$s = -\text{diag}\left(VJ^{Y\mathsf{H}}\right)$
Power source	$z^n = 0$	$\text{diag}\left(I^{\mathsf{H}}\right)(V - \gamma\mathbf{1}) = -\sigma$	$s = -\sigma^Y + \gamma\bar{I}$
Impedance	$z^n = 0$	$V = -z^Y I + \gamma\mathbf{1}$	$s = -\text{diag}\left(V(V - \gamma\mathbf{1})^{\mathsf{H}} y^{Y\mathsf{H}}\right)$

Table 7.3: External models of ideal single-terminal devices in Y configuration. The quantity $\gamma := V^n$ is the neutral voltage. If all neutrals are directly grounded and voltages are defined with respect to the ground, then $\gamma := V^n = 0$ for all Y-configured devices.

Device	Assumption	Δ configuration	
Voltage source	$z^\Delta = 0$, $\mathbf{1}^{\mathsf{T}} E^\Delta = 0$	$V = \Gamma^\dagger E^\Delta + \gamma\mathbf{1}$, $\mathbf{1}^{\mathsf{T}} I = 0$	$s = \text{diag}\left(\Gamma^\dagger E^\Delta I^{\mathsf{H}}\right) + \gamma\bar{I}$
Current source	$y^\Delta = 0$	$I = -\Gamma^{\mathsf{T}} J^\Delta$	$s = -\text{diag}\left(VJ^{\Delta\mathsf{H}}\Gamma\right)$
Power source			$\sigma^\Delta = \text{diag}\left(\Gamma V I^{\Delta\mathsf{H}}\right)$
		$\mathbf{1}^{\mathsf{T}} I = 0$	$\sigma^\Delta = -\text{diag}\left(\Gamma^{\mathsf{T}\dagger}\left(VI^{\mathsf{H}}\right)\Gamma^{\mathsf{T}}\right) + \overline{\beta}\,\Gamma V$
			$s = \text{diag}\left(VI^{\Delta\mathsf{H}}\Gamma\right)$
		$\mathbf{1}^{\mathsf{T}} V^\Delta = 0$	$s = -\text{diag}\left(\Gamma^\dagger\left(V^\Delta I^{\Delta\mathsf{H}}\right)\Gamma\right) - \gamma\Gamma^{\mathsf{T}}\bar{I}^\Delta$
Impedance		$I = -Y^\Delta V$	$s = -\text{diag}\left(VV^{\mathsf{H}} Y^{\Delta\mathsf{H}}\right)$
		$V = -Z^\Delta I + \gamma\mathbf{1}$, $\mathbf{1}^{\mathsf{T}} I = 0$	$s = -\text{diag}\left(Z^\Delta I I^{\mathsf{H}}\right) + \gamma\bar{I}$

Table 7.4: External models of ideal single-terminal devices in Δ configuration. The quantity $\gamma := \frac{1}{3}\mathbf{1}^{\mathsf{T}} V$ is the zero-sequence voltage of V and $\beta := \frac{1}{3}\mathbf{1}^{\mathsf{T}} I^\Delta$ is the zero-sequence current of I^Δ.

In each case the internal specification of the three-phase device fixes one of the terminal variables (V, I, s) and the relation between the remaining variables characterizes its external behavior. In the rest of this book we often assume sources are ideal and characterized by Tables 7.3 and 7.4 (see Chapter 8.1.4 for a justification).

Consider a network of three-phase voltage sources, current sources, power sources, and impedances connected by three-phase lines and transformers. A power flow problem typically specifies a set of these devices and the objective is to determine other voltages, currents, and powers on the network. The specification of these devices include not only internal voltages, currents, or powers, but also some of the zero-sequence quantities (γ, β). We will clarify in Chapter 9.2 the parameters that should be specified versus variables to be computed of the external models in Tables 7.3 and 7.4.

7.4 Voltage regulators

7.5 Bibliographical notes

The concept of symmetrical component is described in another seminal paper [47] by C. L. Fortescue to simplify the analysis of unbalanced operation of a multiphase system. The use of symmetrical components for fault current analysis is explained in e.g. [7] which also proposes a different transformation called $(\alpha, \beta, 0)$ components. The paper [48] explains that Fortescue's transformation matrix as a particular choice of orthogonal basis for three-dimensional vectors over the complex field (the similarity transformation matrix F in Chapter 7.2.2 is the normalized version of Fortescue's original matrix so that the basis are orthonormal). It shows that other well-known transformations such as those of Clarke, Concordia, Kimbark, and Park can be obtained from Forescue's matrix through elementary row and column transformations and have different advantages and disadvantages mostly for fault analysis. Park transformation [49] is applicable not only to steady state voltage and current phasors, but also to instantaneous voltages, currents, and flux linkages in modeling synchronous machines.

As we will see in Chapter 9 a three-phase network has a single-phase equivalent circuit where the network equations have the form as a single-phase network. The main difference with a single-phase network is the models of three-phase devices in the equivalent circuit, such as models for constant-power devices [50, Chapter 11], loads and voltage regulars [39], as we have studied in Chapter 7.3, as well as three-phase lines and transformers, to be studied in Chapter 8. See also [51, Chapter 3] for comprehensive models of three-phase components including distribution lines, transformers and switches.

7.6 Problems

Chapter 7.2.

Exercise 7.1 (Proof of Theorem 7.2)**.** Let

$$\Gamma \; := \; \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \qquad \Gamma^{\mathsf{T}} \; := \; \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Prove Theorem 7.2:

1. The null spaces of Γ and Γ^{T} are both $\text{span}(1,1,1)$.
2. Their pseudo-inverses are
$$\Gamma^{\dagger} \; = \; \frac{1}{3}\Gamma^{\mathsf{T}}, \qquad \Gamma^{\mathsf{T}\dagger} \; = \; \frac{1}{3}\Gamma$$
3. Consider $\Gamma x = b$. If $\mathbf{1}^{\mathsf{T}} b = 0$ then the solutions x are given by $x = \Gamma^{\dagger} b + \beta\mathbf{1}$ for all $\beta \in \mathbb{C}^3$.
4. Consider $\Gamma^{\mathsf{T}} x = b$. If $\mathbf{1}^{\mathsf{T}} b = 0$ then the solutions x are given by $x = \Gamma^{\mathsf{T}\dagger} b + \beta\mathbf{1}$ for all $\beta \in \mathbb{C}^3$.
5. $\Gamma\Gamma^{\dagger} = \Gamma^{\dagger}\Gamma = \frac{1}{3}\Gamma\Gamma^{\mathsf{T}} \; = \; \frac{1}{3}\Gamma^{\mathsf{T}}\Gamma \; = \; \mathbb{I} \; - \; \frac{1}{3}\mathbf{1}\mathbf{1}^{\mathsf{T}}$ where $\mathbb{I}$ is the identity matrix of appropriate size.

Exercise 7.2. Use $\Gamma^{\dagger} \; = \; \frac{1}{3}\Gamma^{\mathsf{T}}$ (Theorem 7.2) to verify the four defining properties of pseudo-inverse of Γ:

1. $(\Gamma\Gamma^{\dagger})\Gamma \; = \; \Gamma$.
2. $\Gamma^{\dagger}(\Gamma\Gamma^{\dagger}) \; = \; \Gamma^{\dagger}$.
3. $\Gamma\Gamma^{\dagger}$ is Hermitian.
4. $\Gamma^{\dagger}\Gamma$ is Hermitian.

Exercise 7.3. Suppose $I = -\Gamma^{\mathsf{T}} I^{\Delta}$. Show that $VI^{\mathsf{H}}\left(\Gamma^{\dagger}\Gamma\right) \; = \; VI^{\mathsf{H}}$.

Chapter 7.3.1.

Exercise 7.4 (Terminal power s)**.** Consider the three-phase voltage source serving a three-phase impedance load shown in Figure 7.10. Both the source and the load are grounded. Suppose the terminal voltage V is defined with respect to the ground. The terminal current I^a flows from terminal a of the source to the load and returns from the ground, and $s^a := V^a I^{a\mathsf{H}}$ is the power delivered across terminal a and the ground. Relate the terminal power $\mathbf{1}^{\mathsf{T}} s := V^a I^{a\mathsf{H}} + V^b I^{b\mathsf{H}} + V^c I^{c\mathsf{H}}$ and the internal power $\mathbf{1}^{\mathsf{T}} s^Y$ for both the voltage source and the impedance.

Exercise 7.5 (Terminal power s)**.** Repeat Exercise 7.4 but for the case where the neutrals are not grounded, as shown in Figure 7.11. All voltages are defined with respect to an arbitrary but common reference point, e.g., the ground.

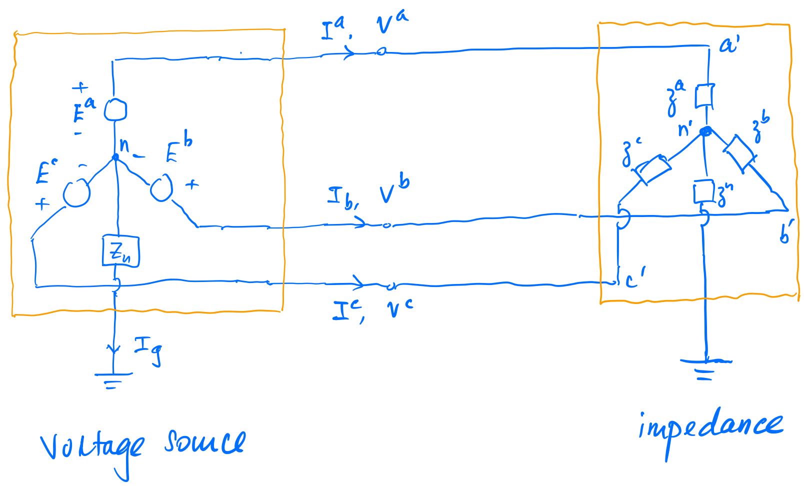


Figure 7.10: Terminal power s and internal power s^Y.

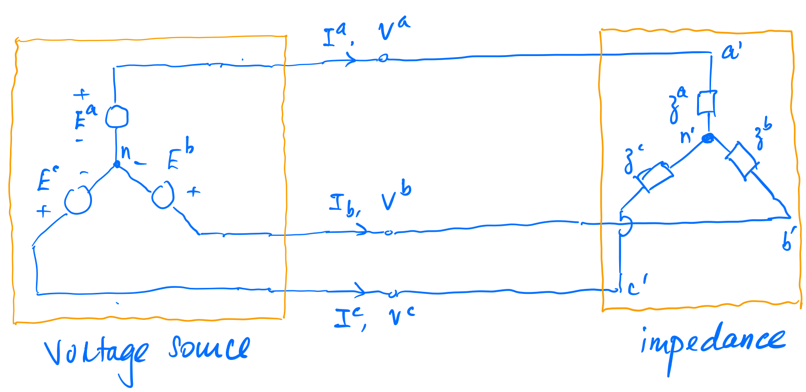


Figure 7.11: Terminal power s and internal power s^Y.

Chapter 7.3.2.

Exercise 7.6 (Y-configured voltage source)**.** Compute the inverse of $Z^Y := z^Y + z_n \mathbf{1}\mathbf{1}^\mathsf{T}$ in (7.13c) using the matrix inversion formula.

Exercise 7.7 (Y-configured current source)**.** Consider the current source in Figure 7.4(b). Derive (7.14) for internal power s^Y and s^n.

Exercise 7.8 (Y-configured current source)**.** Consider the current source in Figure 7.4(b). Suppose assumption C7.1 holds. Derive (7.15b):

$$V \;=\; -\left(z^Y J^Y \,+\, Z^Y I\right), \qquad\qquad I \;=\; -A\left(J^Y + y^Y V\right)$$

where

$$z^Y \;:=\; \left(y^Y\right)^{-1}, \qquad Z^Y \;:=\; z^Y \,+\, z^n \mathbf{1}\mathbf{1}^\mathsf{T}, \qquad A \;:=\; \mathbb{I} - \frac{z^n}{1+z^n\left(\mathbf{1}^\mathsf{T} y^Y \mathbf{1}\right)} y^Y \mathbf{1}\mathbf{1}^\mathsf{T}$$

assuming Z^Y is invertible.

Exercise 7.9 (Y-configured power device)**.** Suppose all voltages are defined with respect to the ground, so that $V^n = -z^n\left(\mathbf{1}^\mathsf{T} I\right)$. Derive (7.17b).

Exercise 7.10 (Y-configured impedance)**.** Consider a three-phase load in Y configuration specified by a series impedance matrix Z^Y:

$$V \;:=\; \begin{bmatrix} V_{ag} \\ V_{bg} \\ V_{cg} \end{bmatrix} \;=\; \begin{bmatrix} z_a + z_n & z_n & z_n \\ z_n & z_b + z_n & z_n \\ z_n & z_n & z_c + z_n \end{bmatrix} \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix}$$

Show that if V is balanced and $z_a = z_b = z_c$ then the neutral current $I_n = 0$ and the phases are decoupled.

Chapter 7.3.3.

Exercise 7.11 (Voltage source in Δ configuration)**.** Consider the voltage source in Figure 7.5(a). Let $V^\Delta = \Gamma V$.

1. Show that $\mathbf{1}^\mathsf{T} I = 0$ implies $\mathbf{1}^\mathsf{T}\left(E^\Delta - z^\Delta \Gamma^{\mathsf{T}\dagger} I\right) = 0$.

2. Show that the converse is not true.

Exercise 7.12 (Voltage source in Δ configuration)**.** Suppose A is a complex symmetric matrix A with zero row sums. Show that its pseudo-inverse $A^\dagger$ is also complex symmetric with zero row sums. (Hint: Use Takagi factorization for complex symmetric matrices in Theorem 25.14 of Appendix 25.1.5.)

Exercise 7.13 (Voltage source in Δ configuration)**.** Prove Theorem 7.3: Given the conversion rules $V^\Delta = \Gamma V$ and $I = -\Gamma^\mathsf{T} I^\Delta$ between the terminal and internal voltages and currents, the following are equivalent:

1. Internal model: $V^\Delta \;=\; E^\Delta + z^\Delta I^\Delta$ and hence $\mathbf{1}^\mathsf{T}\left(E^\Delta + z^\Delta I^\Delta\right) = 0$.
2. External model: $I \;=\; \left(\Gamma^\mathsf{T} y^\Delta\right) E^\Delta - Y^\Delta V$ where $Y^\Delta := \Gamma^\mathsf{T} y^\Delta \Gamma$.
3. External model: $V = \hat{\Gamma} E^\Delta - Z^\Delta I + \gamma \mathbf{1}$, $\mathbf{1}^\mathsf{T} I = 0$ for some $\gamma \in \mathbb{C}$ where

$$\hat{\Gamma} \;:=\; \frac{1}{3}\Gamma^\mathsf{T}\left(\mathbb{I} - \frac{1}{\zeta}\tilde{z}^\Delta \mathbf{1}^\mathsf{T}\right), \qquad\qquad Z^\Delta \;:=\; \frac{1}{9}\Gamma^\mathsf{T} z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma$$

(Hint: See the proof of Theorem 7.4.)

Exercise 7.14 (Voltage source in Δ configuration)**.** Consider (7.25a), reproduced here:

$$\sigma^\Delta \;=\; -\frac{1}{3}\,\mathrm{diag}\left(\Gamma\left(VI^\mathsf{H}\right)\Gamma^\mathsf{T}\right) \;+\; \overline{\beta}\Gamma V, \qquad\qquad \mathbf{1}^\mathsf{T} I \;=\; 0$$

Given any terminal voltage V, show that I and β are uniquely determined in terms of V and σ^Δ.

Exercise 7.15 (Voltage source in Δ configuration)**.** Consider the model of a constant-power source (7.25a), reproduced here:

$$\sigma^\Delta \;=\; -\frac{1}{3}\,\mathrm{diag}\left(\Gamma\left(VI^\mathsf{H}\right)\Gamma^\mathsf{T}\right) \;+\; \overline{\beta}\Gamma V, \qquad\qquad \mathbf{1}^\mathsf{T} I \;=\; 0, \qquad\qquad \beta \;\in\; \mathbb{C}$$

Given a terminal current I with $\mathbf{1}^\mathsf{T} I = 0$, show that the zero-sequence current $\beta := \frac{1}{3}\mathbf{1}^\mathsf{T} I^\Delta$ can take two values.

Exercise 7.16 (Total power in Δ)**.** Consider a power source with internal power $\sigma^\Delta := (\sigma^{ab}, \sigma^{bc}, \sigma^{ca})$ in Δ configuration. Show that (the negative of) its total internal power is equal to its total terminal power, i.e., $\mathbf{1}^\mathsf{T} s = -\mathbf{1}^\mathsf{T}\sigma^\Delta$.

Exercise 7.17 (Balanced impedance z^Δ)**.** Consider a Δ-configured impedance z^Δ whose external equivalent is (from (7.27b)):

$$Z^\Delta \; := \; \frac{1}{9}\Gamma^\mathsf{T} \underbrace{z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)}_{\tilde{z}^\Delta} \Gamma$$

If the impedance is balanced, i.e., $z^{ab} = z^{bc} = z^{ca}$, show that

$$Z^\Delta \; = \; \frac{z^{ab}}{3}\left(\mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^\mathsf{T}\right)$$

Exercise 7.18 (Devices in Δ configuration)**.** Show that the phases are decoupled, i.e., phase a variables (s^a, V^a, I^a) do not depend on variables in phases b and c, if the terminal currents are balanced $I^a + I^b + I^c = 0$ and the terminal voltages $V^a + V^b + V^c = 0$ for the four types of devices in Δ configuration discussed in Chapter 7.3.3.

Chapter 7.3.4.

Exercise 7.19 (Δ-Y transformation)**.** Show that the external behavior of a symmetric non-ideal voltage source $\left(E^\Delta, z^{ab}\mathbb{I}\right)$ with identical series impedance $z^\Delta := z^{ab}\mathbb{I}$ and zero-sequence voltage $\gamma = 0$ is equivalent to a non-ideal Y-configured voltage source $\left(E^Y, z^Y, z^n\right)$ whose neutral is grounded through an impedance z^n with:

$$E^Y \; := \; \frac{1}{3}\Gamma^\mathsf{T} E^\Delta, \qquad z^Y \; := \; \frac{z^{ab}}{3}\mathbb{I}, \qquad z^n \; := \; -\frac{z^{ab}}{9}$$

under assumption C7.1.

Exercise 7.20 (Δ-Y transformation)**.** Consider a symmetric non-ideal current source $\left(J^\Delta, y^{ab}\mathbb{I}\right)$ with identical shunt admittance $y^\Delta := y^{ab}\mathbb{I}$. Show that it cannot be equivalent to a non-ideal Y-configured current source $\left(J^Y, y^Y, z^n\right)$ under assumption C7.1.

Chapter 8

Component models, II: line and transformers

In this chapter we continue the modeling of three-phase components. In Chapter 8.1 we model a three-phase line. In Chapter 8.2 we extend the simplified model of transformers of Chapter 3.1.4 from single-phase to three-phase setting. In Chapter 8.3 we extend the transformer model based on unitary voltage network of Chapter 3.1.5 from single-phase to three-phase setting. In Chapter 8.4 we explain how to identify model parameters from measurements. We will use these component models in Chapters 9 and 10 to construct network models and study unbalanced three-phase analysis.

8.1 Three-phase line models

As explained Chapter 2.1 the electromagnetic interactions among the electric charges in wires of different phases couple the voltages on and currents in these wires. The relation between the voltages and currents in these phases can be modeled by a linear mapping that depends on the line characteristics (resistances, inductances, capacitances).

8.1.1 Review: single-phase model

The linear mapping becomes decoupled when the phases are balanced, leading to a per-phase model of a line as a two-terminal device specified by a Π-equivalent circuit $(y^s_{jk},\, y^m_{jk},\, y^m_{kj})$, as explained in Chapter 2.2.2. The terminal (or bus) voltages (V_j, V_k) and sending-end line currents (I_{jk}, I_{kj}) on this two-terminal device describes the end-to-end behavior of the line. They are linearly related according to Kirchhoff's and Ohm's laws:

$$I_{jk} \;=\; y^s_{jk}\left(V_j - V_k\right) \;+\; y^m_{jk} V_j, \qquad I_{kj} \;=\; y^s_{jk}\left(V_k - V_j\right) \;+\; y^m_{kj} V_k \tag{8.1a}$$

The terms $y^m_{jk} V_j$ and $y^m_{kj} V_k$ assume that the shunt admittances connect the buses j and k both to the common reference point for terminal voltages, e.g., the ground. The sending-end line power $\left(S_{jk}, S_{kj}\right)$ is related to

(V_j, V_k) by

$$S_{jk} = \left(y^s_{jk}\right)^{\mathsf{H}} V_j \left(V_j - V_k\right)^{\mathsf{H}} + \left(y^m_{jk}\right)^{\mathsf{H}} V_j V_j^{\mathsf{H}} \tag{8.1b}$$

$$S_{kj} = \left(y^s_{jk}\right)^{\mathsf{H}} V_k \left(V_k - V_j\right)^{\mathsf{H}} + \left(y^m_{kj}\right)^{\mathsf{H}} V_k V_k^{\mathsf{H}} \tag{8.1c}$$

When the line admittances are zero, i.e., $y^m_{jk} = y^m_{kj} = 0$, then $I_{jk} = -I_{kj}$ and this relation reduces to

$$V_j - V_k = z^s_{jk} I_{jk} \tag{8.1d}$$

where $z^s_{jk} := \left(y^s_{jk}\right)^{-1}$ is the series impedance of the line. We now extend these relations to an unbalanced three-phase line.

8.1.2 Four-wire three-phase model

A three-phase line has three wires one for each phase a,b,c. It may also have a neutral wire which may be grounded at one or both ends if the device connected to that end of the line is in Y configuration. Consider then a four-wire three-phase line where the total current $i^a(t) + i^b(t) + i^c(t)$ and the total charge $q^a(t) + q^b(t) + q^c(t)$ may be nonzero and they flow through the neutral wire (if present) and the earth return. The effect of neutral or earth return on the impedance of a transmission line depends on details such as how many neutral wires are present, whether they are grounded along the lines at regular spacing, etc.

To build intuition we first omit line charging. In this case the three-phase voltages and currents are related by a series impedance matrix, similar to (8.1d) for a single-phase system. We then incorporate the effect of line charging by including shunt admittances to obtain a model that generalizes (8.1a) to a three-phase system.

Without shunt admittances. Consider a four-wire three-phase line with a neutral wire. The voltage between one end of a wire to the other end depends linearly on the current in each of the four wires. Let $\hat{V}_j := \left(V^a_j, V^b_j, V^c_j, V^n_j\right)$ and $\hat{V}_k := \left(V^a_k, V^b_k, V^c_k, V^n_k\right)$ be the *terminal (or nodal or bus)* voltages at terminals j and k respectively of the phase and neutral wire (j,k), with respect to an arbitrary but common reference point, e.g., the ground. Let $\hat{I}_{jk} := \left(I^a_{jk}, I^b_{jk}, I^c_{jk}, I^n_{jk}\right)$ denote the currents in these lines. Then the four-wire three-phase line can be modeled by a *series impedance matrix*[1] $\hat{z}^s_{jk}$ that linearly relates these voltages and currents:

$$\begin{bmatrix} V^a_j \\ V^b_j \\ V^c_j \\ V^n_j \end{bmatrix} - \begin{bmatrix} V^a_k \\ V^b_k \\ V^c_k \\ V^n_k \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{z}^{aa}_{jk} & \hat{z}^{ab}_{jk} & \hat{z}^{ac}_{jk} & \hat{z}^{an}_{jk} \\ \hat{z}^{ba}_{jk} & \hat{z}^{bb}_{jk} & \hat{z}^{bc}_{jk} & \hat{z}^{bn}_{jk} \\ \hat{z}^{ca}_{jk} & \hat{z}^{cb}_{jk} & \hat{z}^{cc}_{jk} & \hat{z}^{cn}_{jk} \\ \hat{z}^{na}_{jk} & \hat{z}^{nb}_{jk} & \hat{z}^{nc}_{jk} & \hat{z}^{nn}_{jk} \end{bmatrix}}_{\hat{z}^s_{jk}} \begin{bmatrix} I^a_{jk} \\ I^b_{jk} \\ I^c_{jk} \\ I^n_{jk} \end{bmatrix} \tag{8.2a}$$

[1] It is sometimes called a series *phase* impedance matrix to differentiate it from a series *sequence* impedance matrix for sequence variables; see Chapter 9.4.

or in vector form

$$\hat{V}_j \ - \ \hat{V}_k \quad = \quad \hat{z}^s_{jk} \hat{I}_{jk} \tag{8.2b}$$

For example, the series impedance matrix $\hat{z}^s_{jk}$ can model an overhead three-phase line with an overhead neutral wire and earth return. Here $\hat{z}^{\phi\phi}_{jk}$ are called the *self-impedances* of phase ϕ wires, including the effect of earth return, and $\hat{z}^{\phi\phi'}_{jk}$ the *mutual impedances* between phase ϕ and phase ϕ' wires, including the effect of earth return. Their values depend on the wire materials, their lengths, distances between them, the operating frequency, and the resistivity of the earth. To relate these impedances to the physical system, suppose a voltage is applied between the phase a terminals and therefore completing the phase a circuit, while circuits of phases b, c, n are open. Then the current I^a_{jk} in the phase a wire is nonzero while all other currents $I^\phi_{jk} = 0$, $\phi \neq a$, so that

$$\begin{bmatrix} V^a_j \\ V^b_j \\ V^c_j \\ V^n_j \end{bmatrix} - \begin{bmatrix} V^a_k \\ V^b_k \\ V^c_k \\ V^n_k \end{bmatrix} = \begin{bmatrix} \hat{z}^{aa}_{jk} & \hat{z}^{ab}_{jk} & \hat{z}^{ac}_{jk} & \hat{z}^{an}_{jk} \\ \hat{z}^{ba}_{jk} & \hat{z}^{bb}_{jk} & \hat{z}^{bc}_{jk} & \hat{z}^{bn}_{jk} \\ \hat{z}^{ca}_{jk} & \hat{z}^{cb}_{jk} & \hat{z}^{cc}_{jk} & \hat{z}^{cn}_{jk} \\ \hat{z}^{na}_{jk} & \hat{z}^{nb}_{jk} & \hat{z}^{nc}_{jk} & \hat{z}^{nn}_{jk} \end{bmatrix} \begin{bmatrix} I^a_{jk} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence the self-impedance

$$\hat{z}^{aa}_{jk} \quad = \quad \frac{V^a_j - V^a_k}{I^a_{jk}}$$

is the ratio of the voltage applied between the phase a terminals to the current in the phase a wire when all other circuits are open. The current I^a_{jk} induces voltages in other phases and the mutual impedance

$$\hat{z}^{ba}_{jk} \quad = \quad \frac{V^b_j - V^b_k}{I^a_{jk}}$$

is the ratio of the voltage induced across the phase b terminals to the phase a current when only the phase a circuit is complete.

With shunt admittances. To incorporate the effect of line charging, let the *series admittance matrix* be $\hat{y}^s_{jk} := \left(\hat{z}^s_{jk}\right)^{-1}$, assuming $\hat{z}^s_{jk}$ is invertible. Let $\left(\hat{y}^m_{jk}, \hat{y}^m_{kj}\right)$ denote the *shunt admittance matrices*. The terminal voltages $\left(V_j, V_k\right) \in \mathbb{C}^8$ and the sending-end currents $\left(I_{jk}, I_{kj}\right) \in \mathbb{C}^8$ respectively are related according to

$$I_{jk} \ = \ \hat{y}^s_{jk}\left(V_j - V_k\right) \ + \ \hat{y}^m_{jk} V_j, \qquad\qquad I_{kj} \ = \ \hat{y}^s_{jk}\left(V_k - V_j\right) \ + \ \hat{y}^m_{kj} V_k \tag{8.3}$$

This model is illustrated in Figure 8.1. It has exactly the same form as (8.1a), except that the variables and admittances are vectors and matrices respectively. It generalizes (8.1a) from a single-phase model to a three-phase model. The terms $y^m_{jk} V_j$ and $y^m_{kj} V_k$ in (8.3) assume that the shunt admittances connect the buses j and k both to the common reference point for terminal voltages, e.g., the ground.

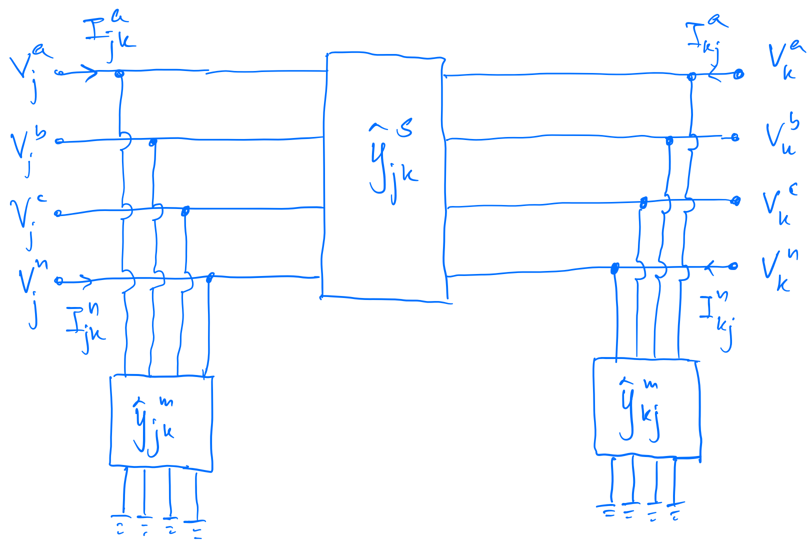


Figure 8.1: A four-wire line characterized by 4×4 series and shunt admittance matrices $\left(\hat{y}^s_{jk}, \hat{y}^m_{jk}, \hat{y}^m_{kj}\right)$.

8.1.3 Three-wire three-phase model

Without shunt admittances. It is typical that either the neutral wire is absent or open circuited, e.g., when connecting devices in Δ configuration, or the neutral is grounded at both the sending and the receiving ends of the line. [2] In both cases, the model (8.2a) can be simplified to involve only phase voltages and currents. In the former case, $I^n_{jk} = 0$ and we can eliminate the last column and row of the impedance matrix $\hat{z}^s_{jk}$ to relate phase voltages to phase currents only. In the latter case, $V^n_j = V^n_k$ and we can eliminate neutral voltages $\left(V^n_j, V^n_k, I^n_{jk}\right)$ by substituting

$$I^n_{jk} \quad = \quad -\frac{1}{\hat{z}^{nn}_{jk}} \left(\hat{z}^{na}_{jk} I^a_{jk} + \hat{z}^{nb}_{jk} I^b_{jk} + \hat{z}^{nc}_{jk} I^c_{jk} \right)$$

into (8.2a). Even though $V^n_j = V^n_k$ across the neutral wire, the current I^n_{jk} in the neutral wire is generally nonzero. Hence when $I^n_{jk} = 0$ or $V^n_j = V^n_k$, we can use a simplified three-wire model without the neutral wire and characterize a three-phase line by a 3×3 series impedance matrix z^s_{jk}:

$$\begin{bmatrix} V^a_j \\ V^b_j \\ V^c_j \end{bmatrix} - \begin{bmatrix} V^a_k \\ V^b_k \\ V^c_k \end{bmatrix} \quad = \quad \underbrace{\begin{bmatrix} z^{aa}_{jk} & z^{ab}_{jk} & z^{ac}_{jk} \\ z^{ba}_{jk} & z^{bb}_{jk} & z^{bc}_{jk} \\ z^{ca}_{jk} & z^{cb}_{jk} & z^{cc}_{jk} \end{bmatrix}}_{y^s_{jk}\ :\ \text{phase impedance matrix}} \begin{bmatrix} I^a_{jk} \\ I^b_{jk} \\ I^c_{jk} \end{bmatrix}$$

[2] The neutral n' of a Y-configured four-wire device may be through a neutral impedance z^n_j to the external terminal n of the device which is then connected to the neutral of the line. The neutral impedance z^n_j of the device may or may not be zero but $V^n_j = V^n_k$.

If we let $V_j := \left(V_j^a, V_j^b, V_j^c\right)$ and $V_k := \left(V_k^a, V_k^b, V_k^c\right)$ denote the terminal phase voltage vectors, and let $I_{jk} := \left(I_{jk}^a, I_{jk}^b, I_{jk}^c\right)$ denote the line phase current vector, then this relation in vector form is

$$V_j \; - \; V_k \quad = \quad z_{jk}^s I_{jk}$$

It is a direct generalization of (8.1d) from a single-phase model to a three-phase model. Even though the three-wire model involves no neutral voltage or current, the 3×3 impedance matrix z_{jk}^s includes the effect of neutral lines and earth return.

With shunt admittances. To incorporate the effect of line charging, let the *series admittance matrix* be $y_{jk}^s := \left(z_{jk}^s\right)^{-1}$, assuming z_{jk}^s is invertible. Let $\left(y_{jk}^m, y_{kj}^m\right)$ denote the *shunt admittance matrices*. The terminal voltages $\left(V_j, V_k\right) \in \mathbb{C}^6$ and the sending-end currents $\left(I_{jk}, I_{kj}\right) \in \mathbb{C}^6$ respectively are related according to

$$I_{jk} \;=\; y_{jk}^s \left(V_j - V_k\right) \,+\, y_{jk}^m V_j, \qquad\qquad I_{kj} \;=\; y_{jk}^s \left(V_k - V_j\right) \,+\, y_{kj}^m V_k \tag{8.4a}$$

This model is the three-wire version of (8.3). It is illustrated in Figure 8.2 which is a three-wire version of Figure 8.1. The terms $y_{jk}^m V_j$ and $y_{kj}^m V_k$ in (8.4a) assume that the shunt admittances connect the buses j and

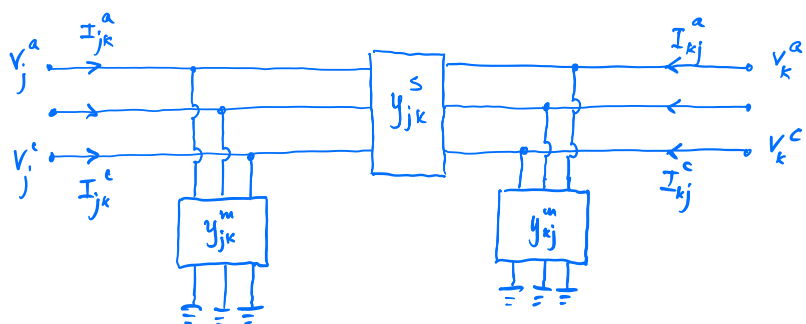


Figure 8.2: A three-wire line characterized by 3×3 series and shunt admittance matrices $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$.

k both to the common reference point for terminal voltages, e.g., the ground.

To describe the relation between the sending-end line power and the voltages $\left(V_j, V_k\right)$, define the matrices $S_{jk}, S_{kj} \in \mathbb{C}^{3\times 3}$ by

$$S_{jk} \;:=\; V_j \left(I_{jk}\right)^{\mathsf{H}} \;=\; V_j \left(V_j - V_k\right)^{\mathsf{H}} \left(y_{jk}^s\right)^{\mathsf{H}} \,+\, V_j V_j^{\mathsf{H}} \left(y_{jk}^m\right)^{\mathsf{H}} \tag{8.4b}$$

$$S_{kj} \;:=\; V_k \left(I_{kj}\right)^{\mathsf{H}} \;=\; V_k \left(V_k - V_j\right)^{\mathsf{H}} \left(y_{jk}^s\right)^{\mathsf{H}} \,+\, V_k V_k^{\mathsf{H}} \left(y_{kj}^m\right)^{\mathsf{H}} \tag{8.4c}$$

The three-phase sending-end line power from terminals j to k along the line is the vector $\operatorname{diag}\left(S_{jk}\right)$ of diagonal entries and that in the opposite direction is the vector $\operatorname{diag}\left(S_{kj}\right)$. The off-diagonal entries of these matrices represent electromagnetic coupling between phases. This generalizes (8.1b)(8.1c) from a single-phase model to a three-phase model.

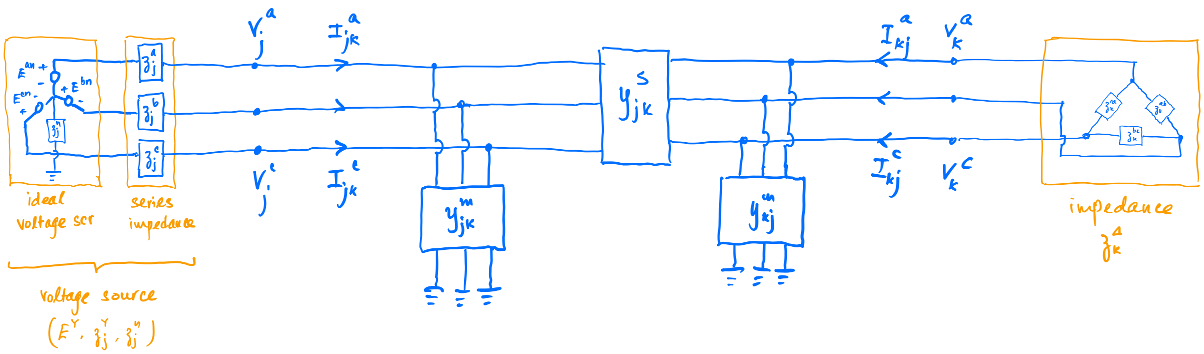


Figure 8.3: A voltage source connected to an impedance load through the line in Figure 8.2.

Example 8.1 (External vs internal variables)**.** Figure 8.3 shows a three-phase voltage source connected to a three-phase impedance load through the line in Figure 8.2. As the figure highlights, the voltages (V_j, V_k) and currents (I_{jk}, I_{kj}) in (8.4a) are terminal voltages and currents regardless of whether the three-phase devices connected to terminals j and k are in Y or Δ configuration. The relation between the terminal variables and internal variables are derived in Chapters 7.3.2 and 7.3.3.

The terminal variable $\left(V_j, I_j, s_j\right)$ at each bus j satisfies both the external device model and the line model (8.4):

$$0 = f_j^{\text{ext}}\left(V_j, I_j\right), \qquad s_j = \text{diag}\left(V_j I_j^{\mathsf{H}}\right)$$

$$I_j = I_{jk}\left(V_j, V_k\right), \qquad s_j = \text{diag}\left(S_{jk}\left(V_j, V_k\right)\right)$$

In particular the nodal balance equation (8.4) relate $\left(V_j, I_j, s_j\right)$ to the terminal voltage V_k at bus k. $\square$

Remark 8.1 (Three-wire model)**.** We will mostly use three-wire line models (8.4) for simplicity, but all analysis extends to four-wire models (including a neutral line) or five-wire models (including a neutral line and the ground return) almost without change with proper definitions that include neutral and ground variables; see Example 9.5 in Chapter 9.2 and Exercise 9.7. $\square$

In most practical situations the series impedance matrix z_{jk}^s is symmetric, i.e., $\left(z_{jk}^s\right)^{\phi\phi'} = \left(z_{jk}^s\right)^{\phi'\phi}$, $\phi, \phi' = a, b, c$, meaning that the coupling between phases ϕ and ϕ' does not depend on direction. It is also common in practice that the shunt admittance matrices y_{jk}^m and y_{kj}^m are symmetric. Formally, we assume throughout this chapter:

C8.1: z_{jk}^s is symmetric and invertible. Moreover $z_{jk}^s = z_{kj}^s$.

C8.2: y_{jk}^m and y_{kj}^m are symmetric matrices.

These matrices are generally complex symmetric, but not Hermitian. By Theorem 4.2, z_{jk}^s is invertible and $\text{Re}(y_{jk}^s) \succ 0$ if $\text{Re}(z_{jk}^s) \succ 0$. Assumption C8.1 implies that y_{jk}^s is symmetric and $y_{jk}^s = y_{kj}^s$ (Exercise 8.1).

Symmetric line. When the line geometry is symmetric (e.g. through transposition) then the series impedance matrix z^s_{jk} has the following important property:

$$z^{aa}_{jk} = z^{bb}_{jk} = z^{cc}_{jk} =: z_{jk} \quad \text{and} \quad z^{ab}_{jk} = z^{ba}_{jk} = z^{bc}_{jk} = z^{cb}_{jk} = z^{ca}_{jk} = z^{ac}_{jk} =: \varepsilon_{jk}$$

so that

$$z^s_{jk} = \begin{bmatrix} z_{jk} & \varepsilon_{jk} & \varepsilon_{jk} \\ \varepsilon_{jk} & z_{jk} & \varepsilon_{jk} \\ \varepsilon_{jk} & \varepsilon_{jk} & z_{jk} \end{bmatrix} = \left(z_{jk} - \varepsilon_{jk}\right) \mathbb{I} + \varepsilon_{jk}\mathbf{1}\mathbf{1}^{\mathsf{T}} \tag{8.5a}$$

Typically $|z_{jk}| > |\varepsilon_{jk}|$. Then the line admittance $y^s_{jk} := \left(z^s_{jk}\right)^{-1}$ has the same structure

$$y^s_{jk} = \begin{bmatrix} y^1_{jk} & y^2_{jk} & y^2_{jk} \\ y^2_{jk} & y^1_{jk} & y^2_{jk} \\ y^2_{jk} & y^2_{jk} & y^1_{jk} \end{bmatrix} = \left(y_{jk} - \delta_{jk}\right) \mathbb{I} + \delta_{jk}\mathbf{1}\mathbf{1}^{\mathsf{T}} \tag{8.5b}$$

where

$$y_{jk} := \frac{z_{jk} + \varepsilon_{jk}}{\left(z_{jk} - \varepsilon_{jk}\right)\left(z_{jk} + 2\varepsilon_{jk}\right)}, \qquad \delta_{jk} := -\frac{\varepsilon_{jk}}{\left(z_{jk} - \varepsilon_{jk}\right)\left(z_{jk} + 2\varepsilon_{jk}\right)} \tag{8.5c}$$

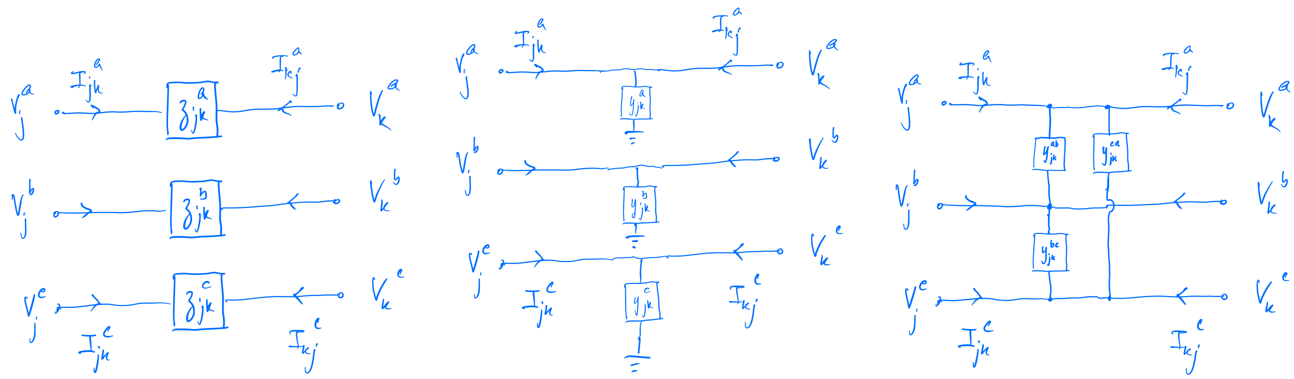
and (8.5c) follows from:

$$\begin{aligned} \mathbb{I} = y^s_{jk} z^s_{jk} &= \left(\left(y_{jk} - \delta_{jk}\right)\mathbb{I} + \delta\mathbf{1}\mathbf{1}^{\mathsf{T}}\right)\left(\left(z_{jk} - \varepsilon_{jk}\right)\mathbb{I} + \varepsilon_{jk}\mathbf{1}\mathbf{1}^{\mathsf{T}}\right) \\ &= \left(y_{jk} - \delta_{jk}\right)\left(z_{jk} - \varepsilon_{jk}\right)\mathbb{I} + \left(\varepsilon_{jk}y_{jk} + z_{jk}\delta_{jk} + \varepsilon_{jk}\delta_{jk}\right)\mathbf{1}\mathbf{1}^{\mathsf{T}} \end{aligned}$$

Typically $|y_{jk}| > |\delta_{jk}|$. If the sources and loads are balanced so that currents sum to zero $i^a(t) + i^b(t) + i^c(t) = 0$ and charges sum to zero $q^a(t) + q^b(t) + q^c(t) = 0$ across phases then $\varepsilon_{jk} = 0$ (see Chapter 2.1.4), i.e., z^s_{jk} is diagonal and the voltages and currents of different phases are decoupled. Otherwise z^s_{jk} is not diagonal and therefore the voltages and currents of different phases are coupled even if the line is symmetric, i.e., even if the series impedance z^s_{jk} satisfies (8.5). As we will see in Chapter 9.4.4, in this case, when shunt admittances are assumed zero, a similarity transformation using the unitary matrix F yields a diagonal impedance matrix $\tilde{z}^s_{jk}$ in the sequence coordinate. This leads to decoupled relation between the sequence voltages and currents across the three-phase line that can be interpreted as defining separate sequence networks.

Example 8.2 (Special lines)**.** The line in (8.4a) is an abstraction that can model a transmission or distribution line, a transformer, or parts of series impedances or shunt admittances of generators or loads. We discuss some degenerate forms of (8.4a) that will be used for this purpose, e.g., for modeling non-ideal voltage and current sources in Chapter 8.1.4. The series impedance z^Y_{jk} in Figure 8.4(a) can be treated as a line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$ with a diagonal series impedance, i.e., $y^m_{jk} = y^m_{kj} = 0$, and

$$y^s_{jk} := \text{diag}^{-1}\left(z^a_{jk}, z^b_{jk}, z^c_{jk}\right), \qquad I_{jk} := y^s_{jk}\left(V_j - V_k\right), \qquad I_{kj} := -I_{jk} \tag{8.6a}$$



(a) Series impedance (b) Shunt admittance in Y (c) Shunt admittance in Δ

Figure 8.4: Special three-wire lines characterized by (8.6).

The Y-configured shunt admittance y^Y_{jk} in Figure 8.4(b) can be treated as a line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$ with a shunt admittance in Y configuration, i.e., $z^s_{jk} = 0$, $y^m_{kj} = 0$, and

$$y^m_{jk} \;:=\; \text{diag}\left(y^a_{jk}, y^b_{jk}, y^c_{jk}\right), \qquad V_j \;=\; V_k, \qquad I_{jk} + I_{kj} \;=\; y^m_{jk} V_j \tag{8.6b}$$

The Δ-configured shunt admittance y^Δ_{jk} in Figure 8.4(c) can be treated as a line $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$ with a shunt admittance in Δ configuration, i.e., $z^s_{jk} = 0$, $y^m_{kj} = 0$, and

$$y^m_{jk} \;:=\; \text{diag}\left(y^{ab}_{jk}, y^{bc}_{jk}, y^{ca}_{jk}\right), \qquad V_j \;=\; V_k, \qquad I^\Delta \;=\; y^m_{jk} \Gamma V_j$$

where $I^\Delta := \left(I^{ab}, I^{bc}, I^{ca}\right)$ are the line-to-line current internal to the Δ configuration. Therefore for any currents I_{jk} and I_{kj} with $\mathbf{1}^\mathsf{T} I_{jk} = \mathbf{1}^\mathsf{T} I_{kj} = 0$, the degenerate line in Figure 8.4(c) is characterized by

$$y^m_{jk} \;:=\; \text{diag}\left(y^{ab}_{jk}, y^{bc}_{jk}, y^{ca}_{jk}\right), \qquad V_j \;=\; V_k, \qquad \Gamma^{\mathsf{T}\dagger}\left(I_{jk} + I_{kj}\right) + \beta \mathbf{1} \;=\; y^m_{jk} \Gamma V_j \tag{8.6c}$$

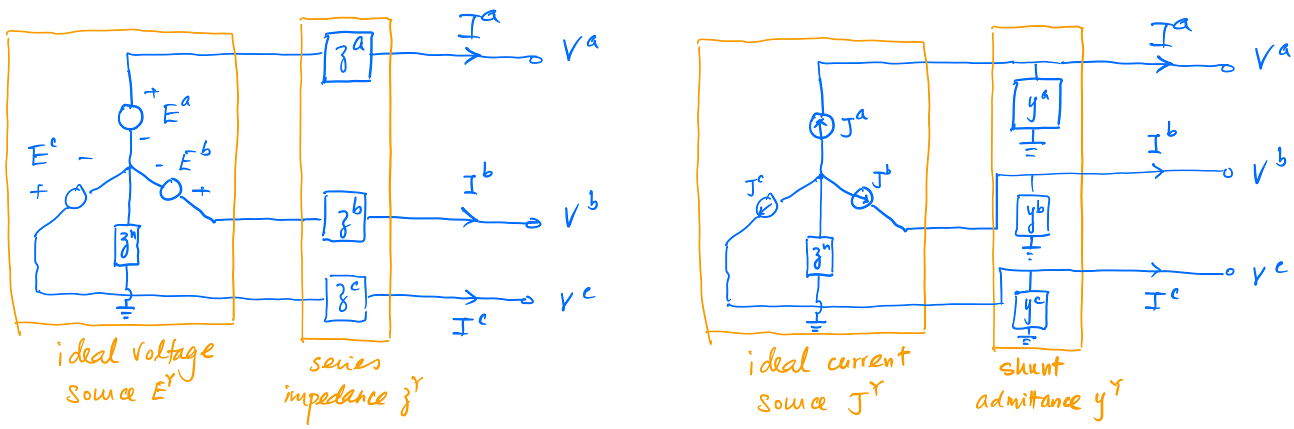
where $\beta \in \mathbb{C}$ depends on the amount of loop flow in the internal current I^Δ.

$\square$

We next use these special lines to simplify models for non-ideal voltage and current sources in Y and Δ configurations.

8.1.4 Ideal voltage and current sources

A voltage or current source in Y configuration may or may not have a neutral line which may or may not be grounded. Figure 8.5 shows the case where the neutral is grounded through an impedance z^n. In this case the voltage source $\left(E^Y, z^Y, z^n\right)$ can be treated as an ideal voltage source $\left(E^Y, z^n\right)$ connected to a (degenerate) three-phase line with a series impedance z^Y characterized by (8.6a). Similarly a grounded

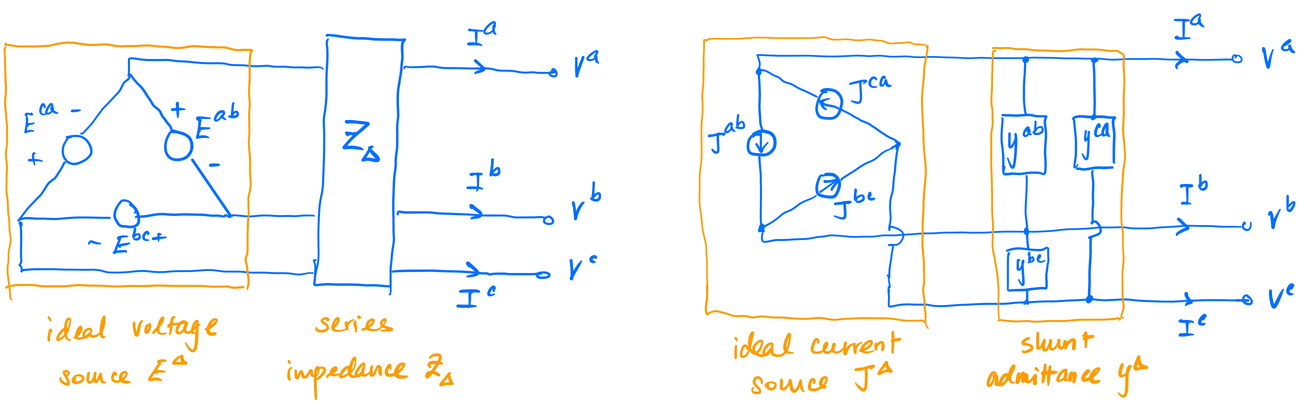


(a) Voltage source (b) Current source

Figure 8.5: Three-wire sources in Y configuration. (a) A voltage source. (b) A current source.

current source $\left(J^Y, y^y, z^n\right)$ in Y configuration, as shown in Figure 8.5(b), can be treated as an ideal current source $\left(J^Y, z^n\right)$ connected to a three-phase line with a shunt admittance y^Y characterized by (8.6b). In both cases the ideal source has no series impedance or shunt admittance. In general the neutral voltage V^n is nonzero whether or not there is a neutral line and whether or not the neutral is grounded.

A voltage source $\left(E^\Delta, z^\Delta\right)$ in Δ configuration, as shown in Figure 8.6(a), can be treated as an ideal voltage source E^Δ in Δ configuration connected to a three-phase line with a series impedance $Z^\Delta :=$ $\frac{1}{9}\Gamma^\mathsf{T} z^\Delta \left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma$ in (7.21b). A current source $\left(J^\Delta, y^\Delta\right)$ in Δ configuration, as shown in Figure 8.6(b),



(a) Voltage source (b) Current source

Figure 8.6: Three-wire sources in Δ configuration. (a) A voltage source. (b) A current source.

can be treated as an ideal current source J^Δ in Δ configuration connected to a three-phase line with a shunt admittance y^Δ in Δ configuration characterized by (8.6c).

Example 8.3 (Ideal sources)**.** Figure 8.7 shows a three-phase voltage source in Y configuration connected to a three-phase current source in Δ configuration through the line in Figure 8.2. The shunt admittance $y_k^\Delta := \text{diag}\left(y_k^{ab}, y_k^{bc}, y_k^{ca}\right)$ of the current source can be absorbed into the shunt admittance matrix y_{kj}^m of the

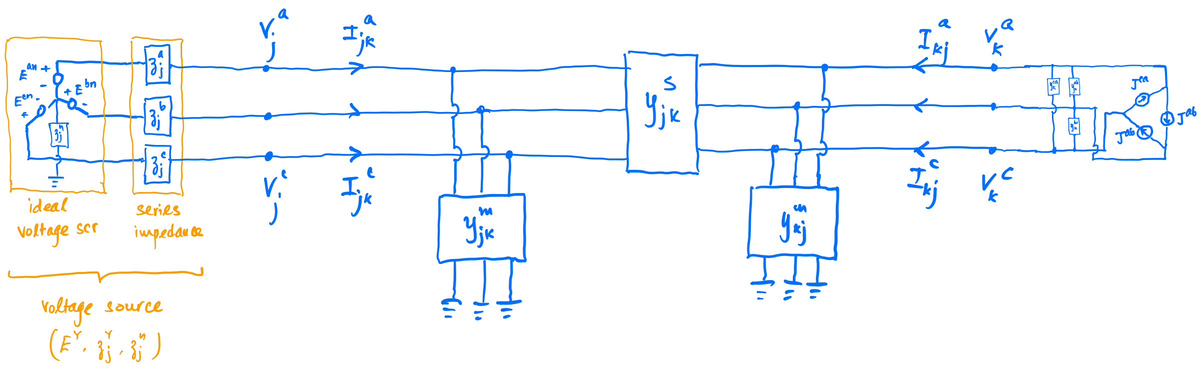


Figure 8.7: A voltage source connected to a current source through the line in Figure 8.2.

line so that the system is equivalent to an ideal current source J_k^Δ connected to terminal k of a line with an equivalent shunt admittance matrix $\tilde{y}_{kj}^m$ given by:

$$\tilde{y}_{kj}^m \quad := \quad \underbrace{\begin{bmatrix} y_{kj}^{aa} & y_{kj}^{ab} & y_{kj}^{ac} \\ y_{kj}^{ba} & y_{kj}^{bb} & y_{kj}^{bc} \\ y_{kj}^{ca} & y_{kj}^{cb} & y_{kj}^{cc} \end{bmatrix}}_{y_{kj}} + \underbrace{\begin{bmatrix} 0 & y_k^{ab} & y_k^{ca} \\ y_k^{ab} & 0 & y_k^{bc} \\ y_k^{ca} & y_k^{bc} & 0 \end{bmatrix}}_{\text{from } y_k^\Delta}$$

Note that in this equivalent model the two shunt admittance matrices y_{jk}^m and $\tilde{y}_{kj}^m$ are generally unequal even if $y_{jk}^m = y_{kj}^m$ originally. Note also that the series impedance matrix z_j^Y of the voltage source cannot be directly absorbed into the line parameters. □

8.2 Three-phase transformer models: simplified circuit

In this section we show that, as for a three-phase line, the external model of a three-phase transformer takes the form of an admittance matrix Y. The general method is similar to that for other three-phase devices: (i) define internal and terminal variables; (ii) derive conversion rules that relate internal and terminal variables; (ii) define internal models that relate these internal variables; and finally (iv) eliminate the internal variables to arrive at the external model. We start by reviewing the single-phase transformer. The notation and the derivation generalize naturally when these transformers are configured into a three-phase transformer.

8.2.1 Review: single-phase transformer

Consider the simplified mode of a single-phase transformer in Figure 3.5 of Chapter 3.1.4, reproduced in Figure 8.8, consisting of an ideal transformer with a voltage gain n, a leakage admittance y^s and a shunt admittance y^m on the primary side. Let the turns ratio be $a := n^{-1}$ (even though a is used to denote both a phase and a turns ratio its meaning should be clear from the context). The currents entering/leaving and the voltages across the ideal transformer are denoted by variables with a hat: $(\hat{V}_j, \hat{I}_j)$, $(\hat{V}_k, \hat{I}_k)$. They are

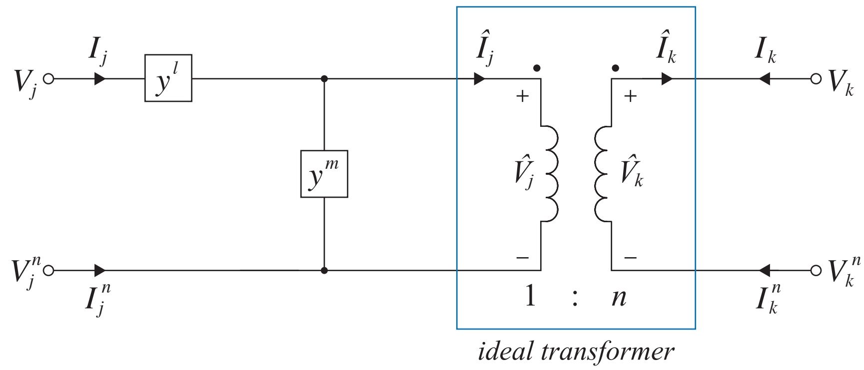


Figure 8.8: Single-phase transformer: simplified model. The internal variables $(\hat{V}_j, \hat{I}_j)$, $(\hat{V}_k, \hat{I}_k)$ and terminal variables (V_j, V_j^n, I_j), (V_k, V_k^n, I_k).

called *internal* variables. The dot notation on the ideal transformer indicates that the internal currents are defined to be positive when $\hat{I}_j$ flows into and $\hat{I}_k$ flows out of the dotted terminals, as indicated in Figure 8.8.

The *terminal voltages* $\left(V_j, V_j^n, V_k, V_k^n\right)$ are defined with respect to an arbitrary but common reference point, e.g., the ground. We emphasize that, while the internal voltages $(\hat{V}_j, \hat{V}_k)$ are defined to be the voltage drops across the ideal transformer windings, the terminal voltages $\left(V_j, V_j^n, V_k, V_k^n\right)$ are defined with respect to a common reference point; in particular the primary and secondary windings are not assumed to be grounded. The *terminal currents* $\left(I_j, I_k\right)$ are defined to be the sending-end currents from buses j and k respectively to the other side, as shown in Figure 8.8. The terminal and internal variables are related by the *conversion rule*:

$$I_j \;=\; y^l\left(V_j - V_j^n - \hat{V}_j\right), \qquad I_j \;=\; y^m \hat{V}_j + \hat{I}_j, \qquad I_j^n \;=\; -I_j \tag{8.7a}$$

$$\hat{V}_k \;=\; V_k \;-\; V_k^n, \qquad \hat{I}_k \;=\; -I_k, \qquad I_k^n \;=\; -I_k \tag{8.7b}$$

where the neutral currents (I_j^n, I_k^n) are injections from the neutral terminals into the ideal transformer and follow from $I_j^n = -(y^m \hat{V}_j + \hat{I}_j) = -I_j$ and $I_k^n = \hat{I}_k = -I_k$ respectively. The *internal model* of the single-phase (ideal) transformer is defined by its transformer gains (n, a):

$$\hat{V}_k \;=\; n\hat{V}_j, \qquad \hat{I}_k \;=\; \frac{1}{n}\hat{I}_j \;=:\; a\hat{I}_j \tag{8.7c}$$

Eliminating the internal variables from (8.7) yields an *external model* that relates the terminal variables:

$$I_j \;=\; y^l\left(\left(V_j - V_j^n\right) - a\left(V_k - V_k^n\right)\right), \qquad I_k \;=\; -a\hat{I}_j \;=\; a y^m\left(V_j - V_j^n\right) - a\left(1 + \frac{y^m}{y^l}\right) I_j$$

or in terms of an admittance matrix Y:

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2(y^l + y^m) \end{bmatrix}}_{Y} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} V_j^n \\ V_k^n \end{bmatrix} \right) \tag{8.8a}$$

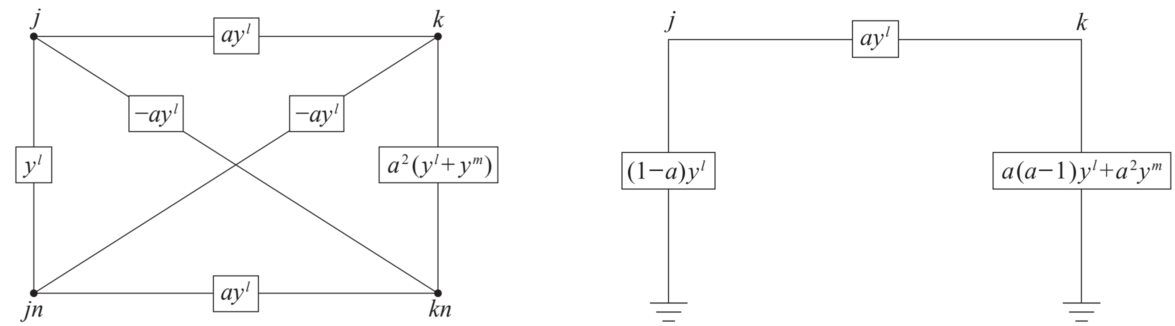
We can add neutral currents from (8.7) to (8.8a):

$$\begin{bmatrix} I_j^n \\ I_k^n \end{bmatrix} = -\begin{bmatrix} I_j \\ I_k \end{bmatrix} = -Y\left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} V_j^n \\ V_k^n \end{bmatrix}\right)$$

to obtain a two-wire model of a single-phase transformer:

$$\begin{bmatrix} I_j \\ I_k \\ I_j^n \\ I_k^n \end{bmatrix} = \underbrace{\begin{bmatrix} Y & -Y \\ -Y & Y \end{bmatrix}}_{Y^{2\text{wire}}} \begin{bmatrix} V_j \\ V_k \\ V_j^n \\ V_k^n \end{bmatrix} \tag{8.8b}$$

Both Y and the 4×4 admittance matrix $Y^{2\text{wire}}$ are complex symmetric. While Y generally has nonzero row and column sums, $Y^{2\text{wire}}$ has zero row and column sums. The admittance matrix $Y^{2\text{wire}}$ is represented by a four-node network in Figure 8.9(a). Since $Y^{2\text{wire}}$ has zero row and column sums, there are no shunt admittances in the four-node network in Figure 8.9(a).



(a) General circuit model. (b) Π circuit model.

Figure 8.9: (a) Circuit model of admittance matrix $Y^{2\text{wire}}$ and (b) when neutrals are grounded with zero grounding impedances, $V_j^n = V_k^n = 0$.

It is often assume implicitly (e.g., in Chapter 3 and Chapter 4.1.3) that neutrals are grounded with zero grounding impedance and voltages are defined with respect to the ground (assumption C7.1). In this case, $V_j^n = V_k^n = 0$ and the model (8.8a) reduces to a Π circuit model:

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = Y \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

The four-node network in Figure 8.9(b) then reduces to a Π circuit in which parallel branches to the ground are combined into shunt admittances, i.e., it can be characterized by series and shunt admittances given by

$$\tilde{y}_{jk}^s := ay^l, \qquad \tilde{y}_{jk}^m := (1-a)y^l, \qquad \tilde{y}_{kj}^m := a(a-1)y^l + a^2y^m \tag{8.8c}$$

like a transmission or distribution line.

We now explain how these relations (8.7)(8.8) extend naturally to three-phase transformers in an unbalanced setting.

8.2.2 General derivation method

The external model of a three-phase transformer depends on the models of its constituent single-phase transformers and their configuration on each side of the three-phase transformer. In particular each of the primary and secondary sides can be in Y or Δ configuration, giving four configurations for a standard three-phase transformer. The external model can be derived in four simple steps, similar to the derivation for a single-phase transformer or other three-phase devices:

1. *Conversion rule*: For the primary side, define the internal variables $(\hat{V}_j, \hat{I}_j)$ and external variables (V_j, V_j^n, I_j) (defined precisely below) and relate them.

2. *Conversion rule*: For the secondary side, define the internal variables $(\hat{V}_k, \hat{I}_k)$ and external variables (V_k, V_k^n, I_k) and relate them.

3. *Internal model*: Couple these relations through the transformer gains (8.7c) on $(\hat{V}_j, \hat{I}_j)$, $(\hat{V}_k, \hat{I}_k)$ for each of the single-phase transformers.

4. *External model*: Derive the external model, a relation between external variables (V_j, I_j) and (V_k, I_k), by eliminating the internal variables.

This method is modular and applicable in a general setting where the single-phase transformers may have different admittances or turns ratios, the neutrals of Y configurations may or may not be connected to the other side, may or may not be grounded, with zero or nonzero grounding impedances. The method can also be generalized to non-standard transformers such as open transformers.

We now describe these steps in more detail.

1. Primary side. Consider the primary circuit of a three-phase transformer in Y or Δ configuration in Figure 8.10. The internal voltages and currents associated with the ideal transformer are denoted by

$$\hat{V}_j^Y := \begin{bmatrix} \hat{V}_j^{an} \\ \hat{V}_j^{bn} \\ \hat{V}_j^{cn} \end{bmatrix}, \qquad \hat{I}_j^Y := \begin{bmatrix} \hat{I}_j^{an} \\ \hat{I}_j^{bn} \\ \hat{I}_j^{cn} \end{bmatrix}, \qquad \hat{V}_j^\Delta := \begin{bmatrix} \hat{V}_j^{ab} \\ \hat{V}_j^{bc} \\ \hat{V}_j^{ca} \end{bmatrix}, \qquad \hat{I}_j^\Delta := \begin{bmatrix} \hat{I}_j^{ab} \\ \hat{I}_j^{bc} \\ \hat{I}_j^{ca} \end{bmatrix}$$

The terminal voltages and currents are denoted by

$$V_j := \begin{bmatrix} V_j^a \\ V_j^b \\ V_j^c \end{bmatrix}, \qquad I_j := \begin{bmatrix} I_j^a \\ I_j^b \\ \hat{I}_j^c \end{bmatrix}$$

regardless of the configuration. For Y configuration the (terminal) neutral voltage and current are denoted by $\left(V_j^n, I_j^n\right)$ in the direction shown in Figure 8.10. As for the single-phase model, these voltages are defined with respect to a common reference point (e.g., the ground); in particular the neutrals are not assumed to be grounded. Note that the internal voltages and currents $\left(\hat{V}_j^{Y/\Delta}, \hat{I}_j^{Y/\Delta}\right)$ are defined across the

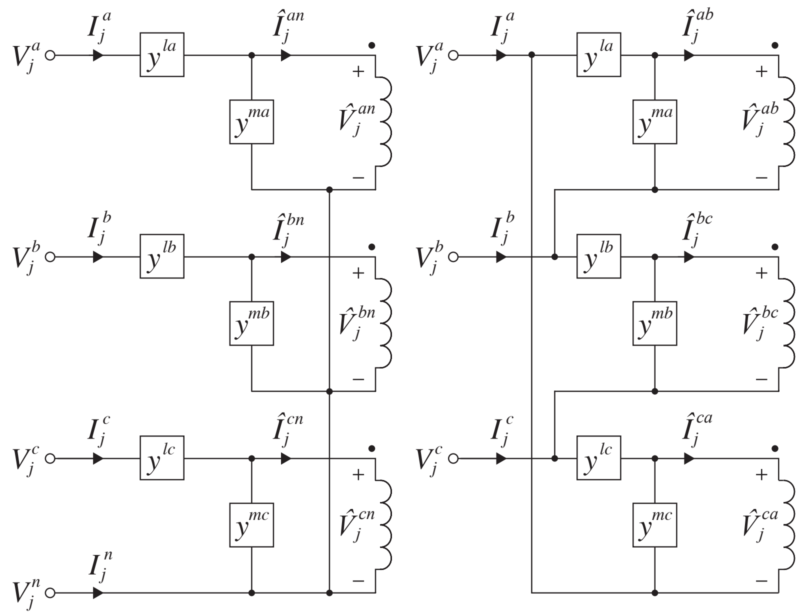


Figure 8.10: Primary side of a three-phase transformer in Y (left) or Δ (right) configuration. (Add shunt admittance y^m; see Figures 8.21, 8.22.)

ideal transformers. In general, $V_j \neq \hat V_j^Y + V_j^n \mathbf{1}$ and $\hat V_j^\Delta \neq \Gamma V_j$. Moreover, $I_j \neq \hat I_j^Y$ and $I_j \neq \Gamma^\mathsf{T} \hat I_j^\Delta$, unless $y^m = 0$.

The leakage admittances of the transformer are denoted by the diagonal matrix $y^l := \text{diag}\left(y^{la}, y^{lb}, y^{lc}\right)$ and the shunt admittances are denoted by $y^m := \text{diag}\left(y^{ma}, y^{mb}, y^{mc}\right)$. From (8.7a) for each single-phase transformer the terminal variables are related to the internal variables according to the *conversion rule*:

$$\begin{aligned} &Y \text{ configuration:} && I_j = y^l\left(V_j - V_j^n\mathbf{1} - \hat V_j^Y\right), \quad I_j = y^m \hat V_j^Y + \hat I_j^Y, \quad I_j^n = -\mathbf{1}^\mathsf{T} I_j && \text{(8.9a)}\\ &\Delta \text{ configuration:} && \hat I_j^\Delta = y^l \Gamma V_j - (y^l + y^m)\hat V_j^\Delta, \quad I_j = \Gamma^\mathsf{T}\left(\hat I_j^\Delta + y^m \hat V_j^\Delta\right) && \text{(8.9b)}\end{aligned}$$

For Y configuration the neutral current I_j^n in (8.9a) follows from $I_j^n = -\mathbf{1}^\mathsf{T}\left(y^m \hat V_j^Y + \hat I_j^Y\right) = -\mathbf{1}^\mathsf{T} I_j$. For Δ configuration $\hat I_j^\Delta$ in (8.9b) follows from $\hat I_j^{ab} + y^{ma}\hat V_j^{ab} = y^{la}\left(V_j^a - V_j^b - \hat V_j^{ab}\right)$. Clearly $\mathbf{1}^\mathsf{T} I_j = 0$ for Δ configuration. Moreover (8.9) implies that the internal and terminal voltages are related according to

$$\begin{aligned} &Y \text{ configuration:} && V_j = \hat V_j^Y + V_j^n\mathbf{1} + z^l I_j && \text{(8.9c)}\\ &\Delta \text{ configuration:} && \hat V_j^\Delta = \Gamma V_j + y^l z^m \Gamma V_j - (z^l + z^m)\hat I_j^\Delta && \text{(8.9d)}\end{aligned}$$

where $z^l := (y^l)^{-1}$ and $z^m := (y^m)^{-1}$.

2. Secondary side. Consider the secondary side of a three-phase transformer in Y or Δ configuration in Figure 8.11. The internal voltages and currents associated with the transformer are denoted by

$$\hat V_k^Y := \begin{bmatrix}\hat V_k^{an}\\ \hat V_k^{bn}\\ \hat V_k^{cn}\end{bmatrix}, \qquad \hat I_k^Y := \begin{bmatrix}\hat I_k^{an}\\ \hat I_k^{bn}\\ \hat I_k^{cn}\end{bmatrix}, \qquad \hat V_k^\Delta := \begin{bmatrix}\hat V_k^{ab}\\ \hat V_k^{bc}\\ \hat V_k^{ca}\end{bmatrix}, \qquad \hat I_k^\Delta := \begin{bmatrix}\hat I_k^{ab}\\ \hat I_k^{bc}\\ \hat I_k^{ca}\end{bmatrix}$$

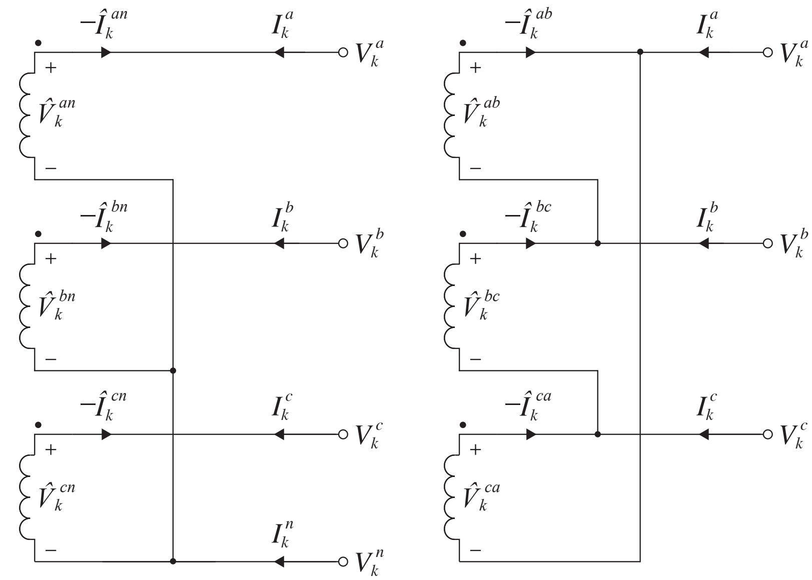


Figure 8.11: Secondary side of a three-phase transformer in Y (left) or Δ (right) configuration.

The terminal voltages and currents are denoted by

$$V_k \; := \; \begin{bmatrix} V_k^a \\ V_k^b \\ V_k^c \end{bmatrix}, \qquad\qquad I_k \; := \; \begin{bmatrix} I_k^a \\ I_k^b \\ \hat{I}_k^c \end{bmatrix}$$

regardless of the configuration. For Y configuration the neutral voltage and current are denoted by $\left(V_k^n, I_k^n\right)$ in the direction shown in Figure 8.11.

From (8.7b) for each single-phase transformer the terminal variables are related to the internal variables according to the *conversion rule*:

$$Y \text{ configuration:} \qquad V_k \; = \; \hat{V}_k^Y \, + \, V_k^n \mathbf{1}, \qquad I_k \; = \; \hat{I}_k^Y, \qquad I_k^n \; = \; -\mathbf{1}^\mathsf{T} \hat{I}_k^Y \; = \; -\mathbf{1}^\mathsf{T} I_k \tag{8.10a}$$

$$\Delta \text{ configuration:} \qquad \hat{V}_k^\Delta \; = \; \Gamma V_k, \qquad I_k \; = \; \Gamma^\mathsf{T} \hat{I}_k^\Delta \tag{8.10b}$$

For Δ configuration, $\mathbf{1}^\mathsf{T} I_k = 0$.

3. Internal model. The voltage and current gains across the ideal transformer define an *internal model* which couples the internal variables in the primary and secondary circuits and connects the relations (8.9) and (8.10). These gains are determined by the turns ratios of the constituent single-phase ideal transformers according to (8.7c), but tailored for different configurations. Denote the voltage gain of the ideal three-phase transformer by a real diagonal matrix $n := \text{diag}\left(n^a, n^b, n^c\right) \in \mathbb{R}^{3\times 3}$ and its turns ratio by

$a := n^{-1} \in \mathbb{R}^{3\times 3}$. Then

$$YY \text{ configuration:} \qquad \hat{V}_k^Y \;=\; n\hat{V}_j^Y, \qquad -\hat{I}_k^Y \;=\; a\hat{I}_j^Y \tag{8.11a}$$

$$\Delta\Delta \text{ configuration:} \qquad \hat{V}_k^\Delta \;=\; n\hat{V}_j^\Delta, \qquad -\hat{I}_k^\Delta \;=\; a\hat{I}_j^\Delta \tag{8.11b}$$

$$\Delta Y \text{ configuration:} \qquad \hat{V}_k^Y \;=\; n\hat{V}_j^\Delta, \qquad -\hat{I}_k^Y \;=\; a\hat{I}_j^\Delta \tag{8.11c}$$

$$Y\Delta \text{ configuration:} \qquad \hat{V}_k^\Delta \;=\; n\hat{V}_j^Y, \qquad -\hat{I}_k^\Delta \;=\; a\hat{I}_j^Y \tag{8.11d}$$

These are internal models of a three-phase (ideal) transformer. The negative signs on $\hat{I}_k^Y$ and $\hat{I}_k^\Delta$ are due to the convention that the transformer current gain is defined for secondary current leaving the dotted terminal of the secondary winding (see Figure 8.11).

4. External model. The *external model* of a three-phase transformer relates the terminal variables (V_j, V_j^n, I_j) and (V_k, V_k^n, I_k) on both sides of the transformer in terms of the leakage admittance y^s, the shunt admittance y^m, and the turns ratio a. It can be derived by eliminating the internal variables $\left(\hat{V}_j^{Y/\Delta}, \hat{I}_j^{Y/\Delta}\right)$ and $\left(\hat{V}_k^{Y/\Delta}, \hat{I}_k^{Y/\Delta}\right)$ from the conversion rules (8.9) (8.10) and the internal model (8.11).

The external models, derived in detail below, turn out to have a striking modular structure. To describe the general form let $V := \left(V_j, V_k\right) \in \mathbb{C}^6$ and $I := \left(I_j, I_k\right) \in \mathbb{C}^6$. Define a 6×6 admittance matrix Y_{YY} and a column vector $\gamma \in \mathbb{C}^6$:

$$Y_{YY} \;:=\; \begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2(y^l + y^m) \end{bmatrix}, \qquad \gamma \;:=\; \begin{bmatrix} V_j^n \mathbf{1} \\ V_k^n \mathbf{1} \end{bmatrix} \tag{8.12a}$$

where $\mathbf{1} := (1,1,1)$. Let D denote a 6×6 block diagonal matrix whose value depends on configuration. As we will explain below Y_{YY} is the admittance matrix of a transformer in YY configuration. It is the same as that in (8.8a) for a single-phase transformer, except that a, y are now 3×3 diagonal matrices rather than scalars. The vector γ is the neutral voltages of a transformer in YY configuration. For $\Delta\Delta$ configuration, $D\gamma = 0 \in \mathbb{C}^6$ in (8.12b), reflecting that a Δ configuration contains no neutral voltage; similarly for ΔY and $Y\Delta$ configurations. The external models of three-phase transformers in YY, $\Delta\Delta$, ΔY and $Y\Delta$ configurations take the form

$$I \;=\; D^{\mathsf{T}} Y_{YY} D\,(V - \gamma) \tag{8.12b}$$

where D is a 6×6 block diagonal matrix that depends on configuration:

$$YY \text{ configuration:} \qquad D \;:=\; \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix} \tag{8.12c}$$

$$\Delta\Delta \text{ configuration:} \qquad D \;:=\; \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \tag{8.12d}$$

$$\Delta Y \text{ configuration:} \qquad D \;:=\; \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix} \tag{8.12e}$$

$$Y\Delta \text{ configuration:} \qquad D \;:=\; \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix} \tag{8.12f}$$

Hence the external models of $\Delta\Delta$, ΔY, $Y\Delta$ configurations can be obtained by pre-multiplying the admittance matrix Y_{YY} of the YY configuration by Γ^{T} and post-multiplying it by Γ for a (primary or secondary) circuit that is in Δ configuration and setting its neutral voltage to zero.

Remark 8.2.

1. Neither the voltage gains $n := \left(n^a, n^b, n^c\right)$ nor the admittances $y^l := \left(y^{la}, y^{lb}, y^{lc}\right)$, $y^m := \left(y^{ma}, y^{mb}, y^{mc}\right)$ may be equal across phases a, b, c. Unless otherwise specified we assume n and a are real matrices. This is the case if they represent voltage gains and turns ratios of constituent single-phase transformers (they can be complex if phase-shifting transformers are involved or if the three-phase transformer is the YY equivalent model of a ΔY-configured transformer in a balanced setting; see Example 8.4).

2. The derivation method is modular. If a different single-phase transformer model is used, e.g., with complex transformer gains, then the relations (8.9) or (8.10) need to be modified but the structure of the derivation remains unchanged.

3. The model (8.12) is a three-wire model that does not include neutral currents. See (8.15c) for a four-wire model that does.

4. The method is also applicable to non-standard transformers such as open transformers. Indeed the external model of an open $\Delta\Delta$ transformer is also given by (8.12b) (8.12d) but with the diagonal matrices y^l, y^m in Y_{YY} in (8.12a) replaced by $\text{diag}\left(y^{la}, y^{lb}, 0\right)$ and $\text{diag}\left(y^{ma}, y^{mb}, 0\right)$ with $y^{lc} = y^{mc} = 0$ on the third leg that has no transformer.

$\square$

We will illustrate this general method by deriving the external models (8.12) of three-phase transformers in YY, $\Delta\Delta$, ΔY and $Y\Delta$ configurations and then show how to adapt the method to non-standard transformers such as open transformers. We start by explaining when a three-phase transformer can be represented by a three-phase Π circuit.

8.2.3 Three-phase Π circuit, block symmetry, symmetry

Refer to the Π circuit model in Figure 8.9(b) for a single-phase transformer where the neutral voltages $V_j^n = V_k^n = 0$. The series and shunt admittances $\left(\tilde{y}_{jk}^s, \tilde{y}_{jk}^m, \tilde{y}_{kj}^m\right)$ of the Π circuit are given by (8.8c). They define a 2×2 admittance matrix Y_{jk} that relates $\left(V_j, V_k\right)$ to $\left(I_{jk}, I_{kj}\right)$ that is complex symmetric. This is because the application of Kirchhoff's laws to this circuit yields

$$I_{jk} \ = \ \tilde{y}_{jk}^s \left(V_j - V_k\right) + \tilde{y}_{jk}^m V_j, \qquad\qquad I_{kj} \ = \ \tilde{y}_{jk}^s \left(V_k - V_j\right) + \tilde{y}_{jk}^m V_k \tag{8.13}$$

Therefore a single-phase transformer always has a Π circuit representation and, in this sense, behaves like a single-phase transmission line.

This is not the case for three-phase transformers. Consider a three-phase transformer and denote by Y_{jk} the 6×6 transmission matrix that maps its voltage vectors $\left(V_j, V_k\right) \in \mathbb{C}^6$ to its current vectors

$(I_{jk}, I_{kj}) \in \mathbb{C}^6$, i.e.,

$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{jk,11} & Y_{jk,12} \\ Y_{jk,21} & Y_{jk,22} \end{bmatrix}}_{Y_{jk}} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

If Y_{jk} can be represented by a *three-phase* Π circuit model, i.e., if it behaves like a three-phase transmission line as shown in Figure 8.2, then (8.13) must also hold but $\left(\tilde{y}^s_{jk}, \tilde{y}^m_{jk}, \tilde{y}^m_{kj}\right)$ are now 3×3 matrices, not scalars. This means that the two off-diagonal submtrices of $Y_{jk} \in \mathbb{C}^6$ must be *equal* $Y_{jk,12} = Y_{jk,21}$ and Y_{jk} must be of the form

$$Y_{jk} = \begin{bmatrix} \tilde{y}^s_{jk} + \tilde{y}^m_{jk} & -\tilde{y}^s_{jk} \\ -\tilde{y}^s_{jk} & \tilde{y}^s_{jk} + \tilde{y}^m_{kj} \end{bmatrix}$$

We call such a matrix *block symmetric* (see Definition 9.1). In contrast, if Y_{jk} is symmetric then $Y^{\mathsf{T}}_{jk,12} = Y_{jk,21}$. As we will see a three-phase transformer may not be block symmetric and hence may not have a three-phase Π circuit representation. For balanced systems, this manifests itself as the per-phase model of a ΔY or $Y\Delta$-configured transformer having no single-phase Π circuit representation because of the its complex voltage gain $K(n)$, as discussed in Chapter 4.1.3. This phenomenon is generalized in the rest of this section for unbalanced systems.

Whether or not Y_{jk} is block symmetric we can always interpret Y_{jk} as the 6×6 admittance matrix of a single-phase network consisting of 6 buses, indexed by $i\phi$, $i = j,k$ and $\phi \in \{a,b,c\}$, as studied in Chapter 4.2. This is referred to as its single-phase equivalent circuit and studied in Chapter 9.1.2.

A matrix can be symmetric but not block symmetric, and vice versa. Symmetry of a matrix is determined only by its off-diagonal entries but its diagonal entries can be arbitrary. Block symmetry is determined only by its off-diagonal blocks but its diagonal blocks can be arbitrary. A symmetric Y_{jk} is block symmetric if $Y^{\mathsf{T}}_{jk,12} = Y_{jk,12}$. A block symmetric Y_{jk} is symmetric if all submatrices $Y_{jk,12}$, $Y_{jk,11}, Y_{jk,22}$ are symmetric. These are reasonable assumptions for modeling a three-phase transmission or distribution line, i.e., Y_{jk} for a transmission or distribution line can be assumed to be both block symmetric and symmetric and therefore has both a three-phase Π circuit representation and a single-phase equivalent circuit. This is not necessarily the case for three-phase transformers.

We will generalize the concepts of block symmetry and single-phase equivalent circuit in Chapter 9.1.2 to a network setting.

8.2.4 YY configuration

Referring to Figure 8.12 and combining the variables defined in Chapter 8.2.2 for each configuration, the internal voltages and currents associated with the ideal transformer are:

$$\hat{V}^Y_j := \begin{bmatrix} \hat{V}^{an}_j \\ \hat{V}^{bn}_j \\ \hat{V}^{cn}_j \end{bmatrix}, \qquad \hat{I}^Y_j := \begin{bmatrix} \hat{I}^{an}_j \\ \hat{I}^{bn}_j \\ \hat{I}^{cn}_j \end{bmatrix}, \qquad \hat{V}^Y_k := \begin{bmatrix} \hat{V}^{an}_k \\ \hat{V}^{bn}_k \\ \hat{V}^{cn}_k \end{bmatrix}, \qquad \hat{I}^Y_k := \begin{bmatrix} \hat{I}^{an}_k \\ \hat{I}^{bn}_k \\ \hat{I}^{cn}_k \end{bmatrix}$$

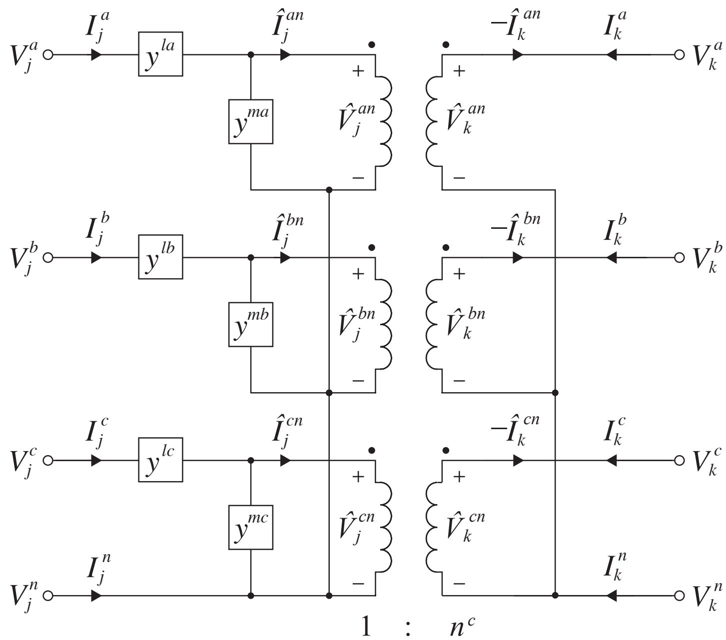


Figure 8.12: YY-configured transformer.

The terminal voltages and currents are:

$$V_j := \begin{bmatrix} V_j^a \\ V_j^b \\ \hat{V}_j^c \end{bmatrix}, \qquad I_j := \begin{bmatrix} I_j^a \\ I_j^b \\ \hat{I}_j^c \end{bmatrix}, \qquad V_k := \begin{bmatrix} V_k^a \\ V_k^b \\ \hat{V}_k^c \end{bmatrix}, \qquad I_k := \begin{bmatrix} I_k^a \\ I_k^b \\ \hat{I}_k^c \end{bmatrix}$$

as well as the the neutral voltages and currents $\left(V_j^n, I_j^n\right)$ and $\left(V_k^n, I_k^n\right)$ as shown in the figure. The relation between the internal and terminal variables is given by (8.9a) and (8.10a) for Y configurations on the primary and secondary sides respectively:

$$I_j = y^l\left(V_j - V_j^n \mathbf{1} - \hat{V}_j^Y\right), \qquad I_j = y^m \hat{V}_j^Y + \hat{I}_j^Y, \qquad I_j^n = -\mathbf{1}^\mathsf{T} I_j \tag{8.14a}$$

$$V_k = \hat{V}_k^Y + V_k^n \mathbf{1}, \qquad I_k = \hat{I}_k^Y, \qquad I_k^n = -\mathbf{1}^\mathsf{T} I_k \tag{8.14b}$$

The transformer gains that relate the internal variables are:

$$\hat{V}_k^Y = n\hat{V}_j^Y, \qquad \hat{I}_k^Y = -a\hat{I}_j^Y \tag{8.14c}$$

Here $y^l := \text{diag}\left(y^{la}, y^{lb}, y^{lc}\right)$ is the leakage admittance matrix, $y^m := \text{diag}\left(y^{ma}, y^{mb}, y^{mc}\right)$ is the shunt admittance matrix, $n := \text{diag}\left(n^a, n^b, n^c\right)$ is the voltage gain matrix and $a := n^{-1}$ is the turns ratio matrix.

We can derive an external model that relates the terminal variables by eliminating the internal variables from (8.14). Specifically we have from (8.14a)(8.14b)

$$\hat{V}_j^Y = (V_j - V_j^n \mathbf{1}) - \left(y^l\right)^{-1} I_j, \qquad \hat{V}_k^Y = V_k - V_k^n \mathbf{1}$$

$$\hat{I}_j^Y = I_j - y^m(V_j - V_j^n \mathbf{1}) + y^m\left(y^l\right)^{-1} I_j, \qquad \hat{I}_k^Y = I_k$$

Substituting it into (8.14c) yields the external model of a three-phase transformer in YY configuration:

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \underbrace{\begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2(y^l+y^m) \end{bmatrix}}_{Y_{YY}} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} V_j^n \mathbf{1} \\ V_k^n \mathbf{1} \end{bmatrix} \right) \tag{8.15a}$$

$$I_j^n = -\mathbf{1}^\top I_j, \qquad I_k^n = -\mathbf{1}^\top I_k \tag{8.15b}$$

where we have used $y^l a = ay^l$ and $a(y^l+y^m)a = a^2(y^l+y^m)$ since they are all diagonal matrices. The expression (8.15a) is the same as the external model (8.8a) for a single-phase transformer, except that, instead of scalars, the variables (V_j, I_j, V_k, I_k) are vectors in $\mathbb{C}^3$ and the parameters a, y^l, y^m are 3×3 matrices. It is the expression (8.12).

We can also express the neutral currents (I_j^n, I_k^n) in terms of the terminal voltages instead of the terminal currents using (8.15a)(8.15b):

$$\begin{bmatrix} I_j^n \\ I_k^n \end{bmatrix} = -\underbrace{\begin{bmatrix} \mathbf{1}^\top & 0 \\ 0 & \mathbf{1}^\top \end{bmatrix} Y_{YY}}_{Y_{YY}^n} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} V_j^n \mathbf{1} \\ V_k^n \mathbf{1} \end{bmatrix} \right)$$

A four-wire model includes the neutral currents. To derive the four-wire model we rewrite this and (8.15a) as

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \underbrace{\begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2(y^l+y^m) \end{bmatrix}}_{Y_{YY}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} - \underbrace{\begin{bmatrix} y^l\mathbf{1} & -ay^l\mathbf{1} \\ -ay^l\mathbf{1} & a^2(y^l+y^m)\mathbf{1} \end{bmatrix}}_{Y_{YY}(\mathbb{I}_2\otimes\mathbf{1})} \begin{bmatrix} V_j^n \\ V_k^n \end{bmatrix}$$

$$\begin{bmatrix} I_j^n \\ I_k^n \end{bmatrix} = -\underbrace{\begin{bmatrix} \mathbf{1}^\top y^l & -\mathbf{1}^\top ay^l \\ -\mathbf{1}^\top ay^l & \mathbf{1}^\top a^2(y^l+y^m) \end{bmatrix}}_{(\mathbb{I}_2\otimes\mathbf{1}^\top)Y_{YY}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{1}^\top y^l\mathbf{1} & -\mathbf{1}^\top ay^l\mathbf{1} \\ -\mathbf{1}^\top ay^l\mathbf{1} & \mathbf{1}^\top a^2(y^l+y^m)\mathbf{1} \end{bmatrix}}_{(\mathbb{I}_2\otimes\mathbf{1}^\top)Y_{YY}(\mathbb{I}_2\otimes\mathbf{1})} \begin{bmatrix} V_j^n \\ V_k^n \end{bmatrix}$$

where $\mathbb{I}_2$ is the identity matrix of size 2, $\mathbf{1}^\top y^l\mathbf{1} = \sum_\phi y^{l\phi}$, $\mathbf{1}^\top ay^l\mathbf{1} = \sum_\phi a^\phi y^{l\phi}$, and $\mathbf{1}^\top a^2(y^l+y^m)\mathbf{1} = \sum_\phi (a^\phi)^2(y^{l\phi}+y^{m\phi})$. Hence the four-wire model of a three-phase transformer in YY configuration is:

$$\begin{bmatrix} I_j \\ I_k \\ I_j^n \\ I_k^n \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{YY} & -Y_{YY}(\mathbb{I}_2\otimes\mathbf{1}) \\ -\left(\mathbb{I}_2\otimes\mathbf{1}^\top\right)Y_{YY} & \left(\mathbb{I}_2\otimes\mathbf{1}^\top\right)Y_{YY}(\mathbb{I}_2\otimes\mathbf{1}) \end{bmatrix}}_{Y_{YY}^{\text{4wire}}} \begin{bmatrix} V_j \\ V_k \\ V_j^n \\ V_k^n \end{bmatrix} \tag{8.15c}$$

This model extends (8.8b) with neutral currents to three-phase transformers. The matrix Y_{YY} in (8.15a) is both symmetric and block symmetric (see Chapter 8.2.3) because a, y^l and y^m are diagonal. This, together with $(A\otimes B)^\top = A^\top\otimes B^\top$, imply that the four-wire admittance matrix Y_{YY}^{4wire} is also symmetric. While the admittance matrix Y_{YY} generally has nonzero row and column sums, Y_{YY}^{4wire} has zero row and column sums.

If both neutrals are grounded with zero impedances and voltages are defined with respect to the ground, then $V_j^n = V_k^n = 0$ and (8.15a) reduces to

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = Y_{YY} \begin{bmatrix} V_j \\ V_k \end{bmatrix} = \begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2(y^l+y^m) \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

which can be represented as a three-phase Π circuit. This means that the external behavior of a YY transformer, when its neutral voltages are zero, has the same structure as that of a three-phase transmission line and can be specified by 3×3 series and shunt admittance matrices $\left(\tilde{y}^s_{jk}, \tilde{y}^m_{jk}, \tilde{y}^m_{kj}\right)$ where

$$\tilde{y}^s_{jk} := ay^l, \qquad \tilde{y}^m_{jk} := (\mathbb{I} - a)\, y^l, \qquad \tilde{y}^m_{kj} := a(a - \mathbb{I})y^l + a^2 y^m \tag{8.15d}$$

This extends the single-phase Π circuit model (8.8c) to the three-phase setting.

8.2.5 $\Delta\Delta$ configuration

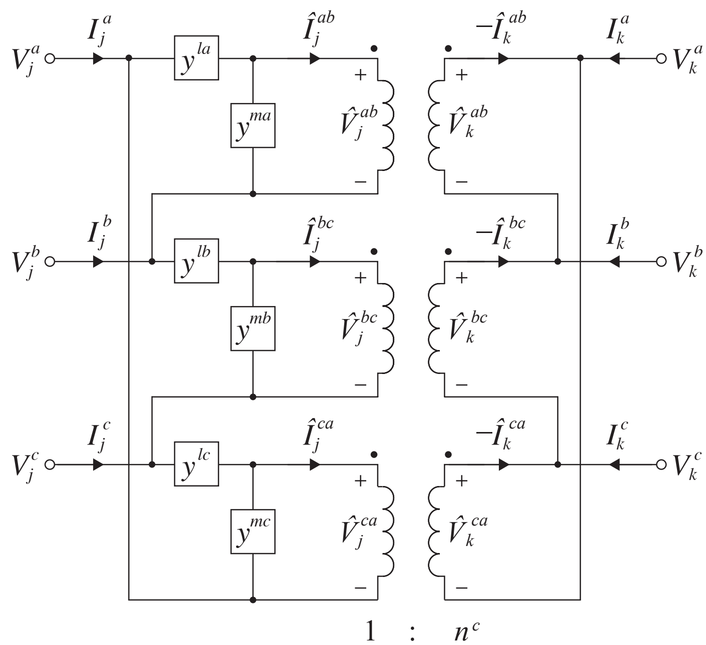


Figure 8.13: $\Delta\Delta$-configured transformer.

Referring to Figure 8.13, and combining the variables defined in Chapter 8.2.2 for each configuration, the internal voltages and currents associated with the ideal transformer are:

$$\hat{V}_j^{\Delta} := \begin{bmatrix} \hat{V}_j^{ab} \\ \hat{V}_j^{bc} \\ \hat{V}_j^{ca} \end{bmatrix}, \qquad \hat{I}_j^{\Delta} := \begin{bmatrix} \hat{I}_j^{ab} \\ \hat{I}_j^{bc} \\ \hat{I}_j^{ca} \end{bmatrix}, \qquad \hat{V}_k^{\Delta} := \begin{bmatrix} \hat{V}_k^{ab} \\ \hat{V}_k^{bc} \\ \hat{V}_k^{ca} \end{bmatrix}, \qquad \hat{I}_k^{\Delta} := \begin{bmatrix} \hat{I}_k^{ab} \\ \hat{I}_k^{bc} \\ \hat{I}_k^{ca} \end{bmatrix}$$

The terminal voltages and currents are denoted by (V_j, I_j), (V_k, I_k), as for a YY-configured transformer. The relation between the internal and terminal variables is given by (8.9b) and (8.10b) for Δ configurations:

$$\hat{I}_j^{\Delta} = y^l \Gamma V_j - (y^l + y^m)\hat{V}_j^{\Delta}, \qquad I_j = \Gamma^{\mathsf{T}} \left(\hat{I}_j^{\Delta} + y^m \hat{V}_j^{\Delta}\right) \tag{8.16a}$$

$$\hat{V}_k^{\Delta} = \Gamma V_k, \qquad I_k = \Gamma^{\mathsf{T}} \hat{I}_k^{\Delta} \tag{8.16b}$$

The transformer gains that relate the internal variables are:

$$\hat{V}_k^{\Delta} \;=\; n\hat{V}_j^{\Delta}, \qquad\qquad \hat{I}_k^{\Delta} \;=\; -a\hat{I}_j^{\Delta} \tag{8.16c}$$

To derive an external model, eliminate the internal variables from (8.16). We obtain from (8.16b)(8.16c):

$$\hat{V}_j^{\Delta} \;=\; n^{-1}\hat{V}_k^{\Delta} \;=\; a\Gamma V_k, \qquad\qquad \Gamma^{\mathsf{T}} a\hat{I}_j^{\Delta} \;=\; -I_k$$

Substitute into the first expression in (8.16a) to eliminate $(\hat{V}_j^{\Delta}, \hat{I}_j^{\Delta})$:

$$I_k \;=\; -\left(\Gamma^{\mathsf{T}} a y^l \Gamma\right) V_j + \left(\Gamma^{\mathsf{T}} a^2 (y^l + y^m)\Gamma\right) V_k$$

Substitute again $\hat{V}_j^{\Delta}$ into the first expression in (8.16a) to obtain $\hat{I}_j^{\Delta} = y^l \Gamma V_j - a(y^l + y^m)\Gamma V_k$. Substitute this and $\hat{V}_j^{\Delta}$ into the second expression in (8.16a) to eliminate $(\hat{V}_j^{\Delta}, \hat{I}_j^{\Delta})$:

$$I_j \;=\; \left(\Gamma^{\mathsf{T}} y^l \Gamma\right) V_j - \left(\Gamma^{\mathsf{T}} a y^l \Gamma\right) V_k$$

The external model of a three-phase transformer in $\Delta\Delta$ configuration is hence

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} \Gamma^{\mathsf{T}} y^l \,\Gamma & -\Gamma^{\mathsf{T}} a y^l \,\Gamma \\ -\Gamma^{\mathsf{T}} a y^l \,\Gamma & \Gamma^{\mathsf{T}} a^2 (y^l + y^m)\,\Gamma \end{bmatrix}}_{Y_{\Delta\Delta}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{8.17a}$$

or in terms of the admittance matrix Y_{YY} in (8.15a) for a YY-configured transformer:

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} \;=\; \begin{bmatrix} \Gamma^{\mathsf{T}} & 0 \\ 0 & \Gamma^{\mathsf{T}} \end{bmatrix} \underbrace{\begin{bmatrix} y^l & -a y^l \\ -a y^l & a^2 (y^l + y^m) \end{bmatrix}}_{Y_{YY}} \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{8.17b}$$

This is the expression (8.12). Unlike Y_{YY} the admittance matrix $Y_{\Delta\Delta}$ is not invertible (it has zero row and column sums). Since $Y_{\Delta\Delta}$ is block symmetric (as well as symmetric) it can be represented as a three-phase Π circuit. This means that its external behavior has the same structure as that of a three-phase transmission line and can be specified by 3×3 series and shunt admittance matrices $\left(\tilde{y}_{jk}^s, \tilde{y}_{jk}^m, \tilde{y}_{kj}^m\right)$ where

$$\tilde{y}_{jk}^s \;:=\; \Gamma^{\mathsf{T}} a y^l \,\Gamma, \qquad \tilde{y}_{jk}^m \;:=\; \Gamma^{\mathsf{T}} (\mathbb{I} - a)\, y^l \,\Gamma, \qquad \tilde{y}_{kj}^m \;:=\; \Gamma^{\mathsf{T}} \left(a(a - \mathbb{I}) y^l + a^2 y^m\right) \Gamma \tag{8.17c}$$

This is the Π circuit model (8.15d) for YY-configured transformer, multiplied on both sides by Γ^{T} and Γ.

The submatrices in (8.17b) are (cf. Y^{Δ} in (7.21a)):

$$\Gamma^{\mathsf{T}} y^l \,\Gamma \;=\; \begin{bmatrix} y^{la} + y^{lc} & -y^{la} & -y^{lc} \\ -y^{la} & y^{lb} + y^{la} & -y^{lb} \\ -y^{lc} & -y^{lb} & y^{lc} + y^{lb} \end{bmatrix}, \qquad \Gamma^{\mathsf{T}} a y^l \,\Gamma \;=\; \begin{bmatrix} \hat{y}^{la} + \hat{y}^{lc} & -\hat{y}^{la} & -\hat{y}^{lc} \\ -\hat{y}^{la} & \hat{y}^{lb} + \hat{y}^{la} & -\hat{y}^{lb} \\ -\hat{y}^{lc} & -\hat{y}^{lb} & \hat{y}^{lc} + \hat{y}^{lb} \end{bmatrix}$$

where $\hat{y}^{l\phi} := a^{\phi} y^{l\phi}$ for $\phi \in \{a, b, c\}$. In the special case where the single-phase transformers are identical, i.e., $y^l = y^{la}\mathbb{I}$ and $a := a^a \mathbb{I}$, these matrices are particularly simple:

$$\left(y^{la}\right) \Gamma^{\mathsf{T}} \Gamma \;=\; y^{la} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \qquad \left(a^a y^{la}\right) \Gamma^{\mathsf{T}} \Gamma \;=\; a^a y^{la} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \tag{8.18}$$

These expressions are often used in simplified models of three-phase transformers.

8.2.6 ΔY configuration

This is a popular configuration for stepdown transformers in distribution systems. ‘Referring to Figure

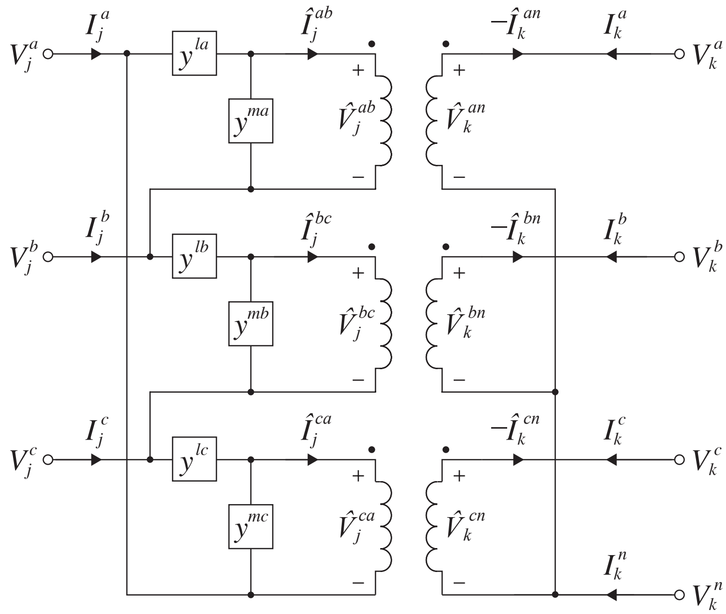


Figure 8.14: ΔY-configured transformer.

8.14, the internal voltages and currents associated with the ideal transformer are:

$$\hat{V}_j^{\Delta} := \begin{bmatrix} \hat{V}_j^{ab} \\ \hat{V}_j^{bc} \\ \hat{V}_j^{ca} \end{bmatrix}, \qquad \hat{I}_j^{\Delta} := \begin{bmatrix} \hat{I}_j^{ab} \\ \hat{I}_j^{bc} \\ \hat{I}_j^{ca} \end{bmatrix}, \qquad \hat{V}_k^{Y} := \begin{bmatrix} \hat{V}_k^{an} \\ \hat{V}_k^{bn} \\ \hat{V}_k^{cn} \end{bmatrix}, \qquad \hat{I}_k^{Y} := \begin{bmatrix} \hat{I}_k^{an} \\ \hat{I}_k^{bn} \\ \hat{I}_k^{cn} \end{bmatrix}$$

The terminal voltages and currents are denoted by (V_j, I_j), (V_k, I_k), as before. The relation between the internal and terminal variables is given by (8.9b) for Δ configuration on the primary side and (8.10a) for Y configuration on the secondary side:

$$\hat{I}_j^{\Delta} = y^l \Gamma V_j - (y^l + y^m)\hat{V}_j^{\Delta}, \qquad I_j = \Gamma^{\mathsf{T}} \left(\hat{I}_j^{\Delta} + y^m \hat{V}_j^{\Delta} \right) \tag{8.19a}$$

$$V_k = \hat{V}_k^{Y} + V_k^n \mathbf{1}, \qquad I_k = \hat{I}_k^{Y}, \qquad I_k^n = -\mathbf{1}^{\mathsf{T}} I_k \tag{8.19b}$$

The transformer gains that relate the internal variables are:

$$\hat{V}_k^{Y} = n \hat{V}_j^{\Delta}, \qquad \hat{I}_k^{Y} = -a \hat{I}_j^{\Delta} \tag{8.19c}$$

Eliminating the internal variables from (8.19), the external model of a three-phase transformer in ΔY configuration is (Exercise 8.2):

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \underbrace{\begin{bmatrix} \Gamma^{\mathsf{T}} y^l \Gamma & -\Gamma^{\mathsf{T}} a y^l \\ -a y^l \Gamma & a^2 (y^l + y^m) \end{bmatrix}}_{Y_{\Delta Y}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} -\Gamma^{\mathsf{T}} a y^l \\ a^2 (y^l + y^m) \end{bmatrix} V_k^n \mathbf{1} \tag{8.20a}$$

or in terms of the admittance matrix Y_{YY} in (8.15a):

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \begin{bmatrix} \Gamma^{\mathsf{T}} & 0 \\ 0 & \mathbb{I} \end{bmatrix} \underbrace{\begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2(y^l+y^m) \end{bmatrix}}_{Y_{YY}} \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} V_j^n \mathbf{1} \\ V_k^n \mathbf{1} \end{bmatrix} \right) \tag{8.20b}$$

It is the expression (8.12). The matrix $Y_{\Delta Y}$ in (8.20a) is not invertible. It is symmetric but not block symmetric. Therefore it cannot be represented as a three-phase Π circuit even if the neutral voltage $V_k^n = 0$.

Even though there is no neutral line on the primary side, the primary current I_j is affected by the neutral voltage V_k^n on the secondary side, unless $a = a^a\mathbb{I}$ and $y = y^a\mathbb{I}$, i.e., the single-phase transformers are identical, in which case $\Gamma^{\mathsf{T}}\mathbf{1} = 0$ and I_j becomes independent of V_k^n.

8.2.7 $Y\Delta$ configuration

Figure 8.15 shows a $Y\Delta$-configured three-phase transformer. Its external model is (Exercise 8.3):

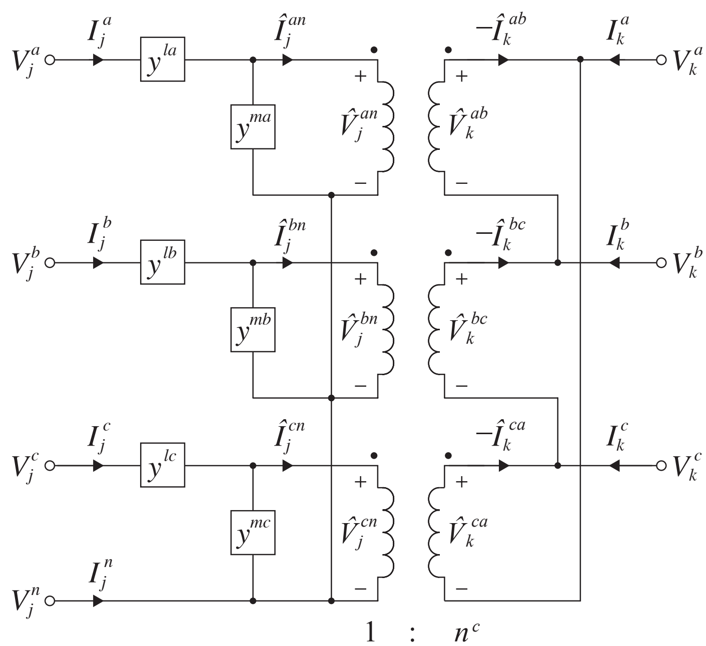


Figure 8.15: $Y\Delta$-configured transformer.

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \underbrace{\begin{bmatrix} y^l & -ay^l\Gamma \\ -\Gamma^{\mathsf{T}}ay^l & \Gamma^{\mathsf{T}}a^2(y^l+y^m)\Gamma \end{bmatrix}}_{Y_{Y\Delta}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} y^l \\ -\Gamma^{\mathsf{T}}ay^l \end{bmatrix} V_j^n \mathbf{1} \tag{8.21a}$$

or in terms of the admittance matrix Y_{YY} in (8.15a):

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma^{\mathsf{T}} \end{bmatrix} \underbrace{\begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2(y^l+y^m) \end{bmatrix}}_{Y_{YY}} \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} V_j^n \mathbf{1} \\ V_k^n \mathbf{1} \end{bmatrix} \right) \tag{8.21b}$$

It is the expression (8.12). The matrix $Y_{Y\Delta}$ is singular, symmetric but not block symmetric. In particular it cannot be represented as a three-phase Π circuit even if the neutral voltage $V_j^n = 0$.

8.2.8 Open transformer

Open transformers where at least one leg of a three-phase transformer is open (not connected) are widely used in distribution systems to connect single-phase loads, e.g., a household. The analysis of a closed transformer can be adapted to that of an open transformer. Indeed their external models are identical, except that the admittance matrices are $\tilde{y}^l = \text{diag}\left(y^{la}, y^{lb}, 0\right)$ and $\tilde{y}^m = \text{diag}\left(y^{ma}, y^{mb}, 0\right)$ for an open transformer without the third leg (compare (8.17) with (8.22) for an open $\Delta\Delta$ transformer). We now derive the external model of an open $\Delta\Delta$ transformer. Other configurations, such as open YY, open ΔY, or open $Y\Delta$, can be analyzed in a similar manner. The analysis proceeds in the same manner as for its closed version, once the voltage gain expression has been modified to represent the open transformer leg where the internal voltages $\hat{V}_j^{ca}$ and $\hat{V}_k^{ca}$ are no longer related by a voltage gain.

Figure 8.16 shows an open $\Delta\Delta$-configured transformer where only two single-phase transformers are used. The leakage admittances of these transformers are $\left(y^a, y^b\right)$ and their voltage gains are $\left(n^a, n^b\right)$. The

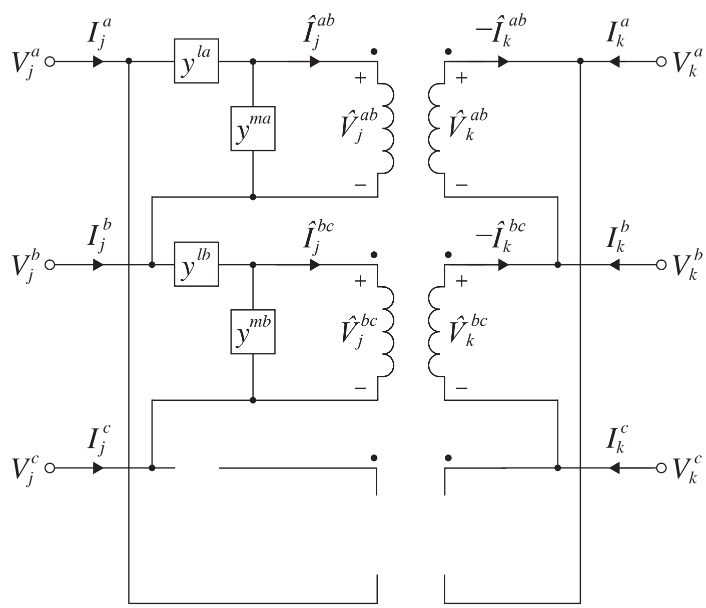


Figure 8.16: Open $\Delta\Delta$-configured transformer.

internal voltages and currents associated with the ideal transformer are:

$$\hat{V}_j^{\Delta} := \begin{bmatrix} \hat{V}_j^{ab} \\ \hat{V}_j^{bc} \\ \hat{V}_j^{ca} \end{bmatrix}, \qquad \hat{I}_j^{\Delta} := \begin{bmatrix} \hat{I}_j^{ab} \\ \hat{I}_j^{bc} \\ \hat{I}_j^{ca} \end{bmatrix}, \qquad \hat{V}_k^{\Delta} := \begin{bmatrix} \hat{V}_k^{ab} \\ \hat{V}_k^{bc} \\ \hat{V}_k^{ca} \end{bmatrix}, \qquad \hat{I}_k^{\Delta} := \begin{bmatrix} \hat{I}_k^{ab} \\ \hat{I}_k^{bc} \\ \hat{I}_k^{ca} \end{bmatrix}$$

The terminal voltages and currents are denoted by $(V_j, I_j) \in \mathbb{C}^6$, $(V_k, I_k) \in \mathbb{C}^6$, as before. We will show that its external model is

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \underbrace{\begin{bmatrix} \Gamma^{\mathsf{T}} \tilde{y}^l \Gamma & -\Gamma^{\mathsf{T}} a \tilde{y}^l \Gamma \\ -\Gamma^{\mathsf{T}} a \tilde{y}^l \Gamma & \Gamma^{\mathsf{T}} a^2 (\tilde{y}^l + \tilde{y}^m) \Gamma \end{bmatrix}}_{Y_{\text{open}\Delta\Delta}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{8.22a}$$

or

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \begin{bmatrix} \Gamma^{\mathsf{T}} & 0 \\ 0 & \Gamma^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} \tilde{y}^l & -a\tilde{y}^l \\ -a\tilde{y}^l & a^2(\tilde{y}^l + \tilde{y}^m) \end{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{8.22b}$$

where

$$\tilde{y}^l := \begin{bmatrix} y^{la} & 0 & 0 \\ 0 & y^{lb} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{y}^m := \begin{bmatrix} y^{ma} & 0 & 0 \\ 0 & y^{mb} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.22c}$$

where $a := \text{diag}\left(a^a, a^b, a^c\right)$. The constant a^c is introduced for notational convenience and can take any arbitrary nonzero finite value, e.g. $a^c = 1$, as its value does not affect the external model. Hence the admittance matrix $Y_{\text{open}\Delta\Delta}$ in (8.22a)(8.22b) are the same as $Y_{\Delta\Delta}$ in (8.17a)(8.17b) for a closed $\Delta\Delta$ transformer, except that $y^{lc} = y^{mc} = 0$ on the third leg that has no transformer. It is also the same as the expression (8.12) with (y^l, y^m) in Y_{YY} replaced by $(\tilde{y}^l, \tilde{y}^m)$. The matrix $Y_{\text{open}\Delta\Delta}$ is block symmetric (as well as symmetric) and therefore has a three-phase Π circuit representation with series and shunt admittance matrices:

$$\tilde{y}^s_{jk} := \Gamma^{\mathsf{T}} a \tilde{y}^l \Gamma, \qquad \tilde{y}^m_{jk} := \Gamma^{\mathsf{T}} (\mathbb{I} - a)\, \tilde{y}^l \Gamma, \qquad \tilde{y}^m_{kj} := \Gamma^{\mathsf{T}} a(a - \mathbb{I})(\tilde{y}^l + \tilde{y}^m)\Gamma \tag{8.22d}$$

which is the same as (8.17c) with (y^l, y^m) replaced by $(\tilde{y}^l, \tilde{y}^m)$.

For notational convenience, we introduce an artificial voltage gain n^c which can take any nonzero finite values, e.g., $n^c := 1$. As before let $n := \text{diag}\left(n^a, n^b, n^c\right)$ and $a := n^{-1}$. As defined above, the leakage and magnetizing admittances are $\tilde{y}^l := \text{diag}\left(y^{la}, y^{lb}, 0\right)$ and $\tilde{y}^m := \text{diag}\left(y^{ma}, y^{mb}, 0\right)$ respectively. The fact that the third leg of the transformer is open requires two adjustments to the derivation of a closed $\Delta\Delta$ transformer. These adjustments modify the internal model (the current and voltage gain on the missing leg) and the derivation then follows the same procedure, as we now explain.

1. The relation between the internal and terminal variables are still given by (8.16a)(8.16b) with the following modifications: replace (y^l, y^m) by $(\tilde{y}^l, \tilde{y}^m)$ and enforce the current on the missing leg on the secondary side to be zero (see Figure 8.16):

$$\tilde{y}^{lc} := 0, \qquad \tilde{y}^{mc} := 0, \qquad \hat{I}^{ca}_k := 0 \tag{8.23a}$$

   This implies that $\hat{I}^{ca}_j = 0$ and $I^c_j = -\hat{I}^{bc}_j$ on the primary side from the last row of (8.16a).

2. For the internal model (8.16c), the current gain $\hat{I}^{\Delta}_k = -a\hat{I}^{\Delta}_j$ remains unchanged (given (8.23a)), but the voltage gain needs modification because the internal voltages $\hat{V}^{ca}_k := V^c_k - V^a_k$ and $\hat{V}^{ca}_j := V^c_j - V^a_j$ are no longer related by the voltage gain n, unlike in a closed transformer.

In order to follow the same derivation we will replace the voltage gain expression $\hat{V}_j^\Delta = a\hat{V}_k^\Delta$ in (8.16c), as follows. In the analysis of a closed $\Delta\Delta$ transformer, the voltage gain is used to relate $\hat{V}_j^\Delta$ to V_k through

$$\hat{V}_j^\Delta \;=\; a\hat{V}_k^\Delta \;=\; a\Gamma V_k$$

For an open $\Delta\Delta$ transformer, the last row of this relation is rewritten as:

$$\hat{V}_j^{ca} \;=\; a^c\hat{V}_k^{ca} + \left(\hat{V}_j^{ca} - a^c\hat{V}_k^{ca}\right)$$

leading to the voltage relation $\hat{V}_j^\Delta = a\hat{V}_k^\Delta + E_3\left(\hat{V}_j^\Delta - a\hat{V}_k^\Delta\right)$ where $E_3 := \text{diag}(0,0,1)$. The right-hand side can then be written in terms of the *terminal* voltage V_j because $\hat{V}_j^{ca} := V_j^c - V_j^a$:

$$\hat{V}_j^\Delta \;=\; E_3\Gamma V_j + (\mathbb{I} - E_3)\,a\hat{V}_k^\Delta \tag{8.23b}$$

which can then be related to V_k using $\hat{V}_k^\Delta = \Gamma V_k$.

In summary, these two modifications (8.23) means that, for open $\Delta\Delta$ transformer, the conversion rules are (8.16a)(8.16b) with (y^l, y^m) replaced by $(\tilde{y}^l, \tilde{y}^m)$:

$$\hat{I}_j^\Delta \;=\; \tilde{y}^l\Gamma V_j - (\tilde{y}^l + \tilde{y}^m)\hat{V}_j^\Delta, \qquad I_j \;=\; \Gamma^\mathsf{T}\left(\hat{I}_j^\Delta + \tilde{y}^m\hat{V}_j^\Delta\right) \tag{8.24a}$$

$$\hat{V}_k^\Delta \;=\; \Gamma V_k, \qquad I_k \;=\; \Gamma^\mathsf{T}\hat{I}_k^\Delta \tag{8.24b}$$

and the internal model (8.16c) is replaced by:

$$\hat{V}_j^\Delta \;=\; E_3\Gamma V_j + (\mathbb{I} - E_3)\,a\hat{V}_k^\Delta, \qquad \hat{I}_k^\Delta \;=\; -a\hat{I}_j^\Delta \tag{8.24c}$$

We then follow the same derivation for the external model. For example we obtain from (8.24b)(8.24c):

$$\hat{V}_j^\Delta \;=\; E_3\Gamma V_j + (\mathbb{I} - E_3)\,a\Gamma V_k, \qquad \Gamma^\mathsf{T} a\hat{I}_j^\Delta \;=\; -I_k$$

Substitute into the first expression in (8.24a) to eliminate $(\hat{V}_j^\Delta, \hat{I}_j^\Delta)$:

$$I_k \;=\; -\left(\Gamma^\mathsf{T} a\tilde{y}^l\Gamma\right)V_j + \left(\Gamma^\mathsf{T} a^2(\tilde{y}^l + \tilde{y}^m)\Gamma\right)V_k$$

where we have used $(\tilde{y}^l + \tilde{y}^m)E_3 = 0$. Similarly we have

$$I_j \;=\; \left(\Gamma^\mathsf{T}\tilde{y}^l\Gamma\right)V_j - \left(\Gamma^\mathsf{T} a\tilde{y}^l\Gamma\right)V_k$$

verifying the external model (8.22). With $y^{lc} = y^{mc} = 0$ the matrices are explicitly:

$$\Gamma^\mathsf{T}\tilde{y}^l\,\Gamma \;=\; \begin{bmatrix} y^{la} & -y^{la} & 0 \\ -y^{la} & y^{lb} + y^{la} & -y^{lb} \\ 0 & -y^{lb} & y^{lb} \end{bmatrix}, \qquad \Gamma^\mathsf{T} a\tilde{y}^l\,\Gamma \;=\; \begin{bmatrix} \hat{y}^{la} & -\hat{y}^{la} & 0 \\ -\hat{y}^{la} & \hat{y}^{lb} + \hat{y}^{la} & -\hat{y}^{lb} \\ 0 & -\hat{y}^{lb} & \hat{y}^{lb} \end{bmatrix}$$

where $\hat{y}^{l\phi} := a^\phi y^{l\phi}$ for $\phi \in \{a, b\}$.

8.2.9 Single-phase equivalent in balanced setting

A three-phase transformer is equivalent to a YY-configured transformer if they have the same external model, i.e., their admittance matrices are equal. In general a three-phase transformer not in YY configuration does not have a YY equivalent, except in a balanced setting. In a balanced setting, not only does a three-phase transformer have a YY equivalent, there is also a single-phase transformer that can be naturally interpreted as the *single-phase equivalent* of the YY equivalent. For simplicity we assume $y^m = 0$.

Consider a $\Delta\Delta$-configured transformer whose external model is determined by the admittance matrix $Y_{\Delta\Delta}$ in (8.17b), reproduced here:

$$Y_{\Delta\Delta} \;:=\; \begin{bmatrix} \Gamma^{\mathsf{T}} & 0 \\ 0 & \Gamma^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2 y^l \end{bmatrix} \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix}$$

Recall from (8.15) that the admittance matrix $\tilde{Y}_{YY}$ of a YY-configured transformer with turns ratio $\tilde{a}$ and leakage admittance $\tilde{y}^l$ is given by

$$\tilde{Y}_{YY} \;:=\; \begin{bmatrix} \tilde{y}^l & -\tilde{a}\tilde{y}^l \\ -\tilde{a}\tilde{y}^l & \tilde{a}^2\tilde{y}^l \end{bmatrix}$$

The $\Delta\Delta$-configured transformer has a YY equivalent if $Y_{\Delta\Delta} = \tilde{Y}_{YY}$ for some $\tilde{Y}_{YY}$. Since the submatrices of $\tilde{Y}_{YY}$ are diagonal while those of $Y_{\Delta\Delta}$ are not, there is generally no YY equivalent, even if the constituent single-phase transformers are identical, i.e., if $y^l = y^{la}\mathbb{I}$ and $a = a^a\mathbb{I}$ (see (8.18)).

The $\Delta\Delta$-configured transformer does have a YY equivalent, however, if the system is balanced, i.e., the single-phase transformers are identical and voltages and currents are positive-sequence sets. This property is used in Chapter 3.4 for per-phase analysis and can be justified using the external models derived here.

Suppose

$$y^l \;:=\; y^{la}\mathbb{I}, \qquad a \;:=\; a^a\mathbb{I}, \qquad V_j \;:=\; v_j\alpha_+, \qquad V_k \;:=\; v_k\alpha_+$$

where we recall that $\alpha_+ := (1, \alpha, \alpha^2)$ is the unit positive-sequence vector and $\alpha := e^{-\mathbf{i}2\pi/3}$. In this case Corollary 1.4 implies

$$\Gamma V_j \;=\; (1-\alpha)V_j, \qquad \Gamma^{\mathsf{T}} V_j \;=\; (1-\alpha^2)V_j$$

The external model (8.17a) of the $\Delta\Delta$-configured transformer then reduces to (with $y^m = 0$):

$$\begin{aligned} I_j \;&=\; \left(\Gamma^{\mathsf{T}} y^l \Gamma\right) V_j \;-\; \left(\Gamma^{\mathsf{T}} a y^l \Gamma\right) V_k \;=\; (1-\alpha)(1-\alpha^2) y^{la} \left(V_j - a^a V_k\right) \\ I_k \;&=\; -\left(\Gamma^{\mathsf{T}} a y^l \Gamma\right) V_j \;+\; \left(\Gamma^{\mathsf{T}} a^2 y^l \Gamma\right) V_k \;=\; (1-\alpha)(1-\alpha^2) y^{la} \left(-a^a V_j + (a^a)^2 V_k\right) \end{aligned}$$

Since $(1-\alpha)(1-\alpha^2) = 3$ we have

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} \tilde{y} & -a\tilde{y} \\ -a\tilde{y} & a^2\tilde{y} \end{bmatrix}}_{\tilde{Y}_{YY}} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

where $\tilde{y}^l = 3y^{la}\mathbb{I}$ and $a = a^a\mathbb{I}$. Hence when the system is balanced a $\Delta\Delta$-configured transformer has a YY equivalent with the same turns ratio a but a leakage admittance $\tilde{y}^l$ three times the original admittance y^l. Since the admittance matrix of the YY equivalent is

$$\tilde{Y}_{YY} := \left(3y^{la} \begin{bmatrix} 1 & -a^a \\ -a^a & (a^a)^2 \end{bmatrix} \right) \otimes \mathbb{I}$$

we can interpret

$$\tilde{Y}_{1\phi} := 3y^{la} \begin{bmatrix} 1 & -a^a \\ -a^a & (a^a)^2 \end{bmatrix}$$

as the admittance matrix of the *single-phase equivalent* of the $\Delta\Delta$ transformer in balanced setting.

In a balanced system a ΔY-configured transformer also has a YY equivalent when $V_k^n = 0$ and hence a single-phase equivalent, but the YY equivalent requires complex, rather than real, turns ratios. This is explained in the next example.

Example 8.4 (Single-phase equivalent of ΔY configuration with $V_k^n = 0$)**.** Consider a ΔY-configured transformer. Suppose, not only is the system balanced, i.e.,

$$y^l := y^{la}\mathbb{I}, \qquad a := a^a\mathbb{I}, \qquad V_j := v_j\alpha_+, \qquad V_k := v_k\alpha_+$$

but the neutral on the secondary side is also grounded with zero grounding impedance, i.e., $V_k^n = 0$. Show that its YY equivalent and single-phase equivalent are respectively

$$\tilde{Y}_{YY} := \tilde{Y}_{1\phi} \otimes \mathbb{I}, \qquad \tilde{Y}_{1\phi} := \tilde{y}^{la} \begin{bmatrix} 1 & -\tilde{a}^a \\ -\tilde{a}^{a\mathsf{H}} & |\tilde{a}^a|^2 \end{bmatrix}$$

where

$$\tilde{y}^{la} := 3y^{la}, \qquad \tilde{a}^a := \frac{a^a}{1-\alpha} = \frac{a^a}{\sqrt{3}e^{\mathbf{i}\pi/6}}$$

Solution. The external model of a ΔY-configured transformer is given by (8.20a). Applying Corollary 1.4 ($\Gamma V_j = (1-\alpha)V_j$, $\Gamma^{\mathsf{T}} V_j = (1-\alpha^2)V_j$), $(1-\alpha)(1-\alpha^2) = 3$ and $\Gamma^{\mathsf{T}}\mathbf{1} = 0$, we have[3]

$$I_j = \left(\Gamma^{\mathsf{T}} y^l \Gamma\right) V_j - \left(\Gamma^{\mathsf{T}} a y^l\right)(V_k - V_k^n\mathbf{1}) = 3y^{la}\left(V_j - \frac{a^a}{1-\alpha}V_k\right)$$

$$I_k = \left(-ay^l\Gamma\right) V_j + \left(a^2 y^l\right)(V_k - V_k^n\mathbf{1}) = 3y^{la}\left(-\frac{a^a}{1-\alpha^2}V_j + \left(\frac{a^a}{\sqrt{3}}\right)^2 (V_k - V_k^n\mathbf{1})\right)$$

Since $a^a \in \mathbb{R}$ we have

$$\left(\frac{a^a}{1-\alpha^2}\right)^{\mathsf{H}} = \frac{a^a}{1-\alpha} = \frac{a^a}{\sqrt{3}e^{\mathbf{i}\pi/6}}$$

[3]To illustrate the effect of V_k^n on YY equivalent we do not substitute $V_k^n = 0$ until the last step.

Define the matrices

$$\tilde{y}^l \;:=\; 3y^{la}\mathbb{I}, \qquad \tilde{a} \;:=\; \frac{a^a}{1-\alpha}\mathbb{I}, \qquad |\tilde{a}|^2 \;:=\; \frac{(a^a)^2}{3}\mathbb{I} \tag{8.25a}$$

The external model of the ΔY-configured transformer is then

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} \;=\; \begin{bmatrix} \tilde{y}^l & -\tilde{a}\tilde{y}^l \\ -\tilde{a}^{\mathsf{H}}\tilde{y}^l & |\tilde{a}|^2\tilde{y}^l \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \;-\; \begin{bmatrix} 0 \\ |\tilde{a}|^2\tilde{y}^l\, V_k^n \mathbf{1} \end{bmatrix} \tag{8.25b}$$

To derive its YY equivalent, consider a YY-configured transformer with a complex voltage gain (matrix) $\hat{n} := \text{diag}\left(\hat{n}^a, \hat{n}^b, \hat{n}^c\right) \in \mathbb{C}^{3\times 3}$ and its turns ratio (matrix) $\hat{a} := \hat{n}^{-1}$. Instead of (8.14c) for real transformer gains, the transformer gains when $\hat{n}$ and $\hat{a}$ are complex are given by

$$\hat{V}_k^Y \;=\; \hat{n}\,\hat{V}_j^Y, \qquad \hat{I}_k^Y \;=\; \hat{a}^{\mathsf{H}}\,\hat{I}_j^Y \tag{8.26a}$$

Let $\hat{y} \in \mathbb{C}^{3\times 3}$ denote its leakage admittance matrix. Then its external model can be shown to be (Exercise 8.5):

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} \hat{y}^l & -\hat{a}\hat{y}^l \\ -\hat{a}^{\mathsf{H}}\hat{y}^l & |\hat{a}|^2\hat{y}^l \end{bmatrix}}_{\tilde{Y}_{YY}} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} \;-\; \begin{bmatrix} V_j^n\mathbf{1} \\ V_k^n\mathbf{1} \end{bmatrix} \right) \tag{8.26b}$$

$$I_j^n \;=\; -\mathbf{1}^{\mathsf{T}} I_j, \qquad I_k^n \;=\; -\mathbf{1}^{\mathsf{T}} I_k \tag{8.26c}$$

where $|\hat{a}|^2$ is the matrix $|\hat{a}|^2 := \text{diag}\left(1/|\hat{n}^a|^2, 1/|\hat{n}^b|^2, 1/|\hat{n}^c|^2\right)$. Note that the matrix $\tilde{Y}_{YY}$ is not complex symmetric and therefore does not have a three-phase Π circuit representation when $\hat{a}$ is complex.

Comparing (8.25b) and (8.26b) we see that, if $V_k^n = 0$, then the ΔY-configured transformer has a YY equivalent whose neutrals are grounded with zero grounding impedances on both sides and whose admittance matrix $\hat{y} = \tilde{y}$ and complex turns ratio matrix $\hat{a} = \tilde{a}$ are given by (8.25a). This completes the proof. □

8.3 Three-phase transformer models: unitary voltage network

In this section we extend the single-phase model in Chapter 3.1.5 with unitary voltage network to three-phase transformers. Multiple copies of the single-phase circuit in Figure 3.8(b) can be connected in Δ or Y configuration on each side of the unitary voltage network, per phase, to create three-phase transformers. The derivation of their external models follows a similar method as that in Chapter 8.2.2: (i) define internal variables for the unitary voltage network in each phase; (ii) derive the internal model that relate these internal variables; (iii) the transformer gains across the two ideal transformers define the conversion between the internal and terminal variables; and finally (iv) eliminate the internal variables to arrive at the external models.

8.3.1 Internal model: UVN per phase

The internal variables on the unitary voltage network in each phase $\phi \in \{a,b,c\}$ are defined in Figure 8.17. Note that the voltages $(\hat{V}_0^\phi, \hat{V}_j^\phi, \hat{V}_k^\phi)$ are defined to be the voltage drops, whether the unitary voltage

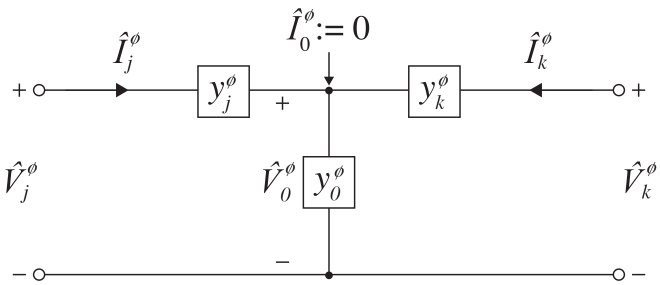


Figure 8.17: Unitary voltage network in each phase ϕ of a three-phase transformer.

network is grounded or not. These variables satisfy (3.10) for each phase ϕ:

$$\hat{I}_j^\phi \;=\; y_j^\phi(\hat{V}_j^\phi - \hat{V}_0^\phi), \qquad \hat{I}_k^\phi \;=\; y_k^\phi(\hat{V}_k^\phi - \hat{V}_0^\phi), \qquad \hat{I}_0^\phi + \hat{I}_j^\phi + \hat{I}_k^\phi \;=\; y_0^\phi \hat{V}_0^\phi, \qquad \phi \in \{a,b,c\} \tag{8.27}$$

Define the internal variables and admittance matrices:

$$\hat{I}_i \;:=\; \begin{bmatrix} \hat{I}_i^a \\ \hat{I}_i^b \\ \hat{I}_i^c \end{bmatrix}, \qquad \hat{V}_i \;:=\; \begin{bmatrix} \hat{V}_i^a \\ \hat{V}_i^b \\ \hat{V}_i^c \end{bmatrix}, \qquad y_i \;:=\; \operatorname{diag}\left(y_i^a, y_i^b, y_i^c\right), \qquad i = 0, j, k$$

Then (8.27) is in vector form:

$$\hat{I}_j \;=\; y_j(\hat{V}_j - \hat{V}_0), \qquad \hat{I}_k \;=\; y_k(\hat{V}_k - \hat{V}_0), \qquad \hat{I}_0 + \hat{I}_j + \hat{I}_k \;=\; y_0 \hat{V}_0$$

or in terms of a 9×9 admittance matrix:

$$\begin{bmatrix} \hat{I}_0 \\ \hat{I}_j \\ \hat{I}_k \end{bmatrix} \;=\; \begin{bmatrix} \sum_i y_i & -y_j & -y_k \\ -y_j & y_j & 0 \\ -y_k & 0 & y_k \end{bmatrix} \begin{bmatrix} \hat{V}_0 \\ \hat{V}_j \\ \hat{V}_k \end{bmatrix} \tag{8.28}$$

where $\sum_i y_i = y_0 + y_j + y_k$ is a diagonal matrix of all admittances. Since $\hat{I}_0 = 0 \in \mathbb{C}^3$ we can eliminate $\hat{V}_0$ and derive the 6×6 Kron-reduced admittance matrix Y_{uvn} that maps $\hat{V} := (\hat{V}_j, \hat{V}_k) \in \mathbb{C}^6$ to $\hat{I} := (\hat{I}_j, \hat{I}_k) \in \mathbb{C}^6$ (Exercise 8.6):

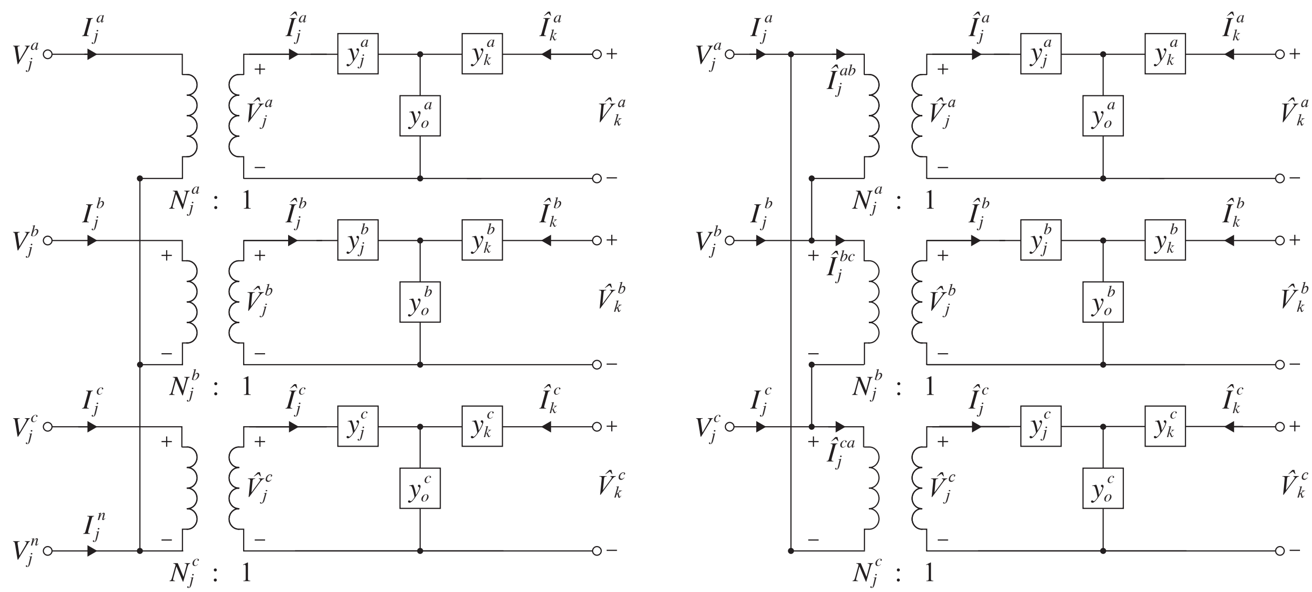
$$\hat{I} \;=\; Y_{\text{uvn}} \hat{V} \qquad \text{where} \qquad Y_{\text{uvn}} \;:=\; \left(\mathbb{I}_2 \otimes \left(\sum_i y_i \right)^{-1} \right) \begin{bmatrix} y_j(y_0 + y_k) & -y_j y_k \\ -y_j y_k & y_k(y_0 + y_j) \end{bmatrix} \tag{8.29}$$

and $\mathbb{I}_2$ is the identity matrix of size 2. This defines the *internal model* that relates $\hat{I}$ and $\hat{V}$. Note that the phases of these internal variables are decoupled in (8.29) since the admittance matrices $y_i \in \mathbb{C}^{3\times 3}$ are diagonal. The phases will be coupled in the terminal variables (V_j, V_k) and (I_j, I_k) through Y or Δ configuration, as we now explain.

8.3.2 Conversion rules

Let the terminal currents of the three-phase transformer be $I_i := (I_i^a, I_i^b, I_i^c)$, its terminal voltages be $V_i := (V_i^a, V_i^b, V_i^c)$, and the terminal neutral voltage of Y configuration be V_i^n, $i = j,k$. The primary side is illustrated in Figure 8.18. These voltages are defined respect to an arbitrary and common reference point, e.g., the ground. Let $M_j := \text{diag}\left(1/N_j^a, 1/N_j^b, 1/N_j^c\right)$ and $M_k := \text{diag}\left(1/N_k^a, 1/N_k^b, 1/N_k^c\right)$ be the transformer gain matrices of the ideal transformers on each side of the unitary voltage network.

To derive the conversion between internal and terminal variables, consider first the primary side where three single-phase ideal transformers are connected to the left end of the unitary voltage network in Figure 8.17. Figure 8.18(a) shows the primary side in Y configuration. The *conversion rule* between the internal



(a) Y configuration (b) Δ configuration

Figure 8.18: Primary side of a three-phase transformer with unitary voltage networks.

variables $(\hat{V}_j, \hat{I}_j)$ and the terminal variables (V_j, I_j, V_j^n) is:

$$Y \text{ configuration:} \qquad \hat{V}_j = M_j\left(V_j - V_j^n \mathbf{1}\right), \qquad \hat{I}_j = M_j^{-1} I_j \tag{8.30a}$$

where $\mathbf{1} := (1,1,1)$. Figure 8.18(b) shows the primary side in Δ configuration. Let $\hat{I}_j^\Delta := (\hat{I}_j^{ab}, \hat{I}_j^{bc}, \hat{I}_j^{ca})$ denote the internal currents entering the primary side of the ideal transformer as indicated in Figure 8.18(b). From (7.9a) the internal variables $(\hat{V}_j, \hat{I}_j, \hat{I}_j^\Delta)$ are related to the terminal variables (V_j, I_j) according to the *conversion rule*:

$$\Delta \text{ configuration:} \qquad \hat{V}_j = M_j \Gamma V_j, \qquad \hat{I}_j = M_j^{-1} \hat{I}_j^\Delta, \qquad I_j = \Gamma^\mathsf{T} \hat{I}_j^\Delta \tag{8.30b}$$

where $\Gamma, \Gamma^\mathsf{T}$ are conversion matrices. Similarly on the secondary side we have the conversion rule (see Figure 8.19):

$$Y \text{ configuration:} \qquad \hat{V}_k = M_k\left(V_k - V_k^n \mathbf{1}\right), \qquad \hat{I}_k = M_k^{-1} I_k \tag{8.30c}$$

$$\Delta \text{ configuration:} \qquad \hat{V}_k = M_k \Gamma V_k, \qquad \hat{I}_k = M_k^{-1} \hat{I}_k^\Delta, \qquad I_k = \Gamma^\mathsf{T} \hat{I}_k^\Delta \tag{8.30d}$$

8.3.3 External model

We can derive an external model by eliminating the internal variables $(\hat{V}, \hat{I}, \hat{I}^{\Delta})$ from the internal model (8.29) and the conversion rules (8.30). Specifically substitute (8.30) into (8.29) to get

$$YY: \quad \begin{bmatrix} M_j^{-1} I_j \\ M_k^{-1} I_k \end{bmatrix} = Y_{\text{uvn}} \begin{bmatrix} M_j(V_j - V_j^n \mathbf{1}) \\ M_k(V_k - V_k^n \mathbf{1}) \end{bmatrix}, \qquad \Delta\Delta: \quad \begin{bmatrix} M_j^{-1} \hat{I}_j^{\Delta} \\ M_k^{-1} \hat{I}_k^{\Delta} \end{bmatrix} = Y_{\text{uvn}} \begin{bmatrix} M_j \Gamma V_j \\ M_k \Gamma V_k \end{bmatrix} \tag{8.31a}$$

$$\Delta Y: \quad \begin{bmatrix} M_j^{-1} \hat{I}_j^{\Delta} \\ M_k^{-1} I_k \end{bmatrix} = Y_{\text{uvn}} \begin{bmatrix} M_j \Gamma V_j \\ M_k(V_k - V_k^n \mathbf{1}) \end{bmatrix}, \qquad Y\Delta: \quad \begin{bmatrix} M_j^{-1} I_j \\ M_k^{-1} \hat{I}_k^{\Delta} \end{bmatrix} = Y_{\text{uvn}} \begin{bmatrix} M_j(V_j - V_j^n \mathbf{1}) \\ M_k \Gamma V_k \end{bmatrix} \tag{8.31b}$$

Let $V := (V_j, V_k) \in \mathbb{C}^6$ and $I := (I_j, I_k) \in \mathbb{C}^6$ denote the vectors of terminal voltages and currents respectively. Let $M := \text{diag}(M_j, M_k) \in \mathbb{R}^{6\times 6}$ be the transformer gain matrices. Then the *external model* of a three-phase transformer is (Exercise 8.7)

$$I = D^{\mathsf{T}} (M Y_{\text{uvn}} M) D (V - \gamma) \tag{8.32a}$$

where Y_{uvn} is defined in (8.29), $D \in \mathbb{C}^{6\times 6}$ and $\gamma \in \mathbb{C}^6$ are defined in (8.12).

We often do not know the numbers N_j^{ϕ}, N_k^{ϕ} of turns of the primary and secondary windings respectively and hence cannot determine the matrices M_j, M_k, but we can always determine the turns ratio matrix $a := M_j^{-1} M_k = \text{diag}\left(N_j^a / N_k^a, N_j^b / N_k^b, N_j^c / N_k^c\right)$ from the specified rated voltages. The 3×3 admittance matrices y_0, y_1, y_2 are assembled from their per-phase admittances and recall from (3.9) (see Figure 3.8):

$$\begin{aligned} y_0 &:= N_j^2 y^m := N_j^2 \,\text{diag}\left(y^{ma}, y^{mb}, y^{mc}\right) \\ y_j &:= N_j^2 y^p := N_j^2 \,\text{diag}\left(y^{pa}, y^{pb}, y^{pc}\right), \qquad & y^{p\phi} &:= \frac{1}{z^{p\phi}}, \quad \phi \in \{a,b,c\} \\ y_k &:= N_k^2 y^s := N_j^2 \,\text{diag}\left(y^{sa}, y^{sb}, y^{sc}\right), \qquad & y^{s\phi} &:= \frac{1}{z^{s\phi}}, \quad \phi \in \{a,b,c\} \end{aligned}$$

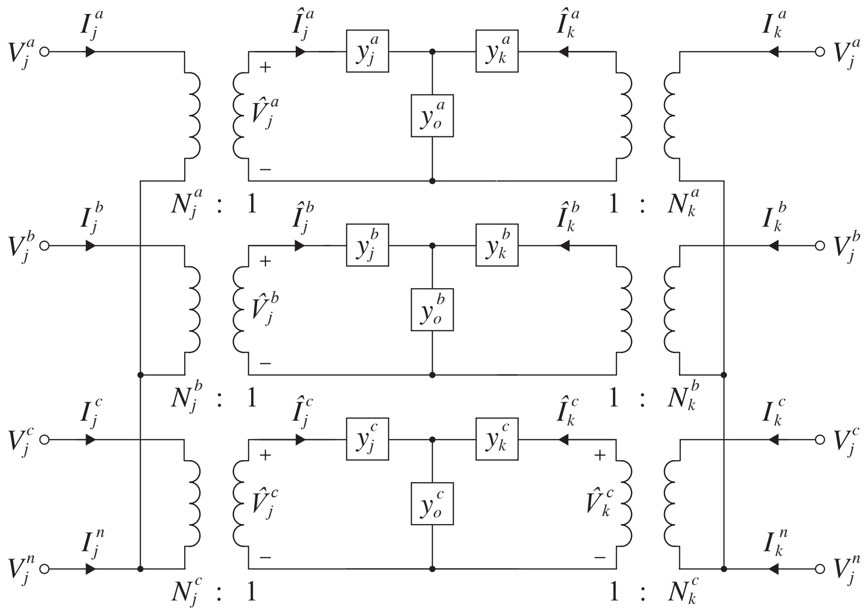
Then the matrix $M Y_{\text{uvn}} M$ in (8.32a) can also be written in terms of the 3×3 turns ratio and admittance matrices a, y^p, y^s, y^m (Exercise 8.8):

$$Y_{YY} := M Y_{\text{uvn}} M = y^p y^s \left(a^2 y^m + a^2 y^p + y^s\right)^{-1} \begin{bmatrix} \mathbb{I} + a^2 y^m (y^s)^{-1} & -a \\ -a & a^2\left(\mathbb{I} + y^m (y^p)^{-1}\right) \end{bmatrix} \tag{8.32b}$$

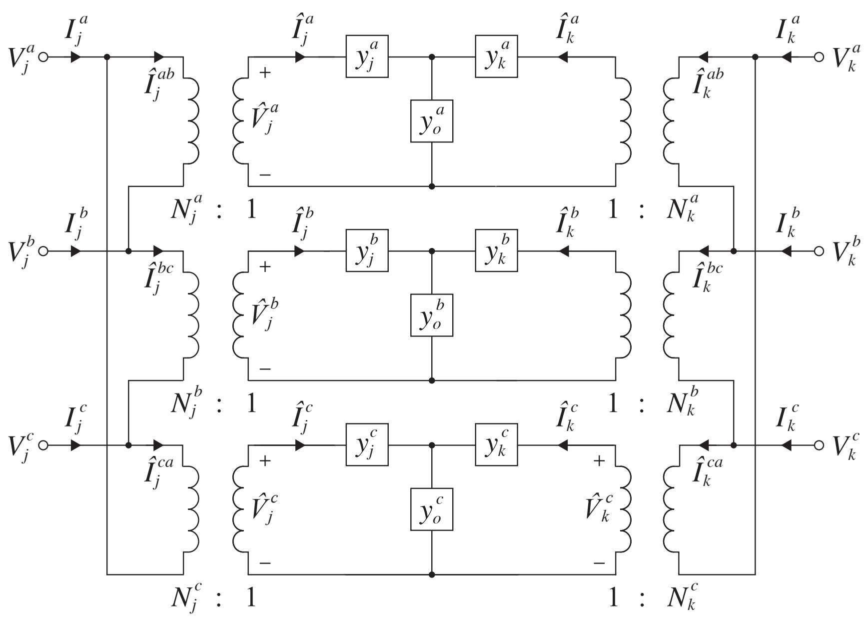
Hence the external model of a standard three-phase transformer is

$$I = D^{\mathsf{T}} Y_{YY} D (V - \gamma) \tag{8.32c}$$

where Y_{YY} is defined in (8.32b), $D \in \mathbb{C}^{6\times 6}$ and $\gamma \in \mathbb{C}^6$ are defined in (8.12), reproduced here: $\gamma := \left(V_j^n \mathbf{1}, V_k^n \mathbf{1}\right)$ are neutral voltages for Y configuration and D is a 6×6 block diagonal matrix that depends



(a) YY configuration



(b) $\Delta\Delta$ configuration

Figure 8.19: Three-phase transformer models with unitary voltage networks.

on configuration:

$$
\begin{aligned}
&YY \text{ configuration:} && D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{bmatrix} \\
&\Delta\Delta \text{ configuration:} && D := \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} \\
&\Delta Y \text{ configuration:} && D := \begin{bmatrix} \Gamma & 0 \\ 0 & \mathbb{I} \end{bmatrix} \\
&Y\Delta \text{ configuration:} && D := \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \Gamma \end{bmatrix}
\end{aligned}
$$

For $\Delta\Delta$ configuration, $D\gamma = 0 \in \mathbb{C}^6$ in (8.32), reflecting that a Δ configuration contains no neutral voltage; similarly for other configurations.

Remark 8.3. 1. As explained in Chapter 3.1.5, the transformer model with unitary voltage networks is equivalent to the T equivalent circuit. This holds in both single-phase and three-phase settings.

2. This model is generally different from the simplified model of Chapter 8.2 which is the three-phase extension of the model in Chapter 3.1.4. From (8.32) and (8.12), these models however have the same structure. They differ only in the admittance matrix Y_{YY} for the YY configuration and the difference is due to different models for single-phase nonideal transformers.

3. When the shunt admittances are assumed zero in both models, i.e., $y_0^\phi = y^{m\phi} = 0$ for $\phi \in \{a,b,c\}$, these two models are equivalent, as in the single-phase case. To see this, recall that per-phase $\phi \in \{a,b,c\}$, the leakage impedances in the simplified model are $z^{l\phi} = z^{p\phi} + (a^\phi)^2 z^{s\phi}$ and hence the leakage admittances per phase are

$$
y^{l\phi} \;=\; \left(z^{la}\right)^{-1} \;=\; \left(1/y^{p\phi} + (a^\phi)^2 y^{s\phi}\right)^{-1} \;=\; \frac{y^{p\phi} y^{s\phi}}{(a^\phi)^2 y^{p\phi} + y^{s\phi}}, \qquad \phi \in \{a,b,c\}
$$

Since all matrices are diagonal we have $y^l = y^p y^s \left(a^2 y^p + y^s\right)^{-1}$. Substituting this and $y^m = 0$ into (8.32b), Y_{YY} for the transformer model based on the unitary voltage network reduces to

$$
Y_{YY} \;=\; y^l \begin{bmatrix} \mathbb{I} & -a \\ -a & a^2 \end{bmatrix}
$$

which is the same as Y_{YY} in (8.12a) for the simplified model. (See Exercise 8.9 for another proof).

4. The model (8.32) generalizes the single-phase model (3.11) in three ways. First the 6×6 admittance matrix $MY_{\text{uvn}}M$ in (8.32) has the same structure as the 2×2 matrix in (3.11). Second the neutrals of the three-phase transformer in Y configuration may not be grounded, i.e., V_j^n, V_k^n may be nonzero whereas V in (3.11) is assumed to be the voltage drop across the windings. Finally the admittance matrix of a three-phase transformer in YY configuration is $Y_{YY} := MY_{\text{unv}}M$, and a Δ configuration in either the primary or the secondary circuit is represented by conversion matrices Γ^{T} and Γ.

8.3.4 Split-phase transformer

8.4 Parameter identification: examples

8.4.1 Simplified circuit

Example 8.5 (Parameter identification)**.** Consider a three-phase transformer in ΔY configuration. Its simplified circuit model is shown in Figure 8.20. Suppose the single-phase transformers are identical, i.e.

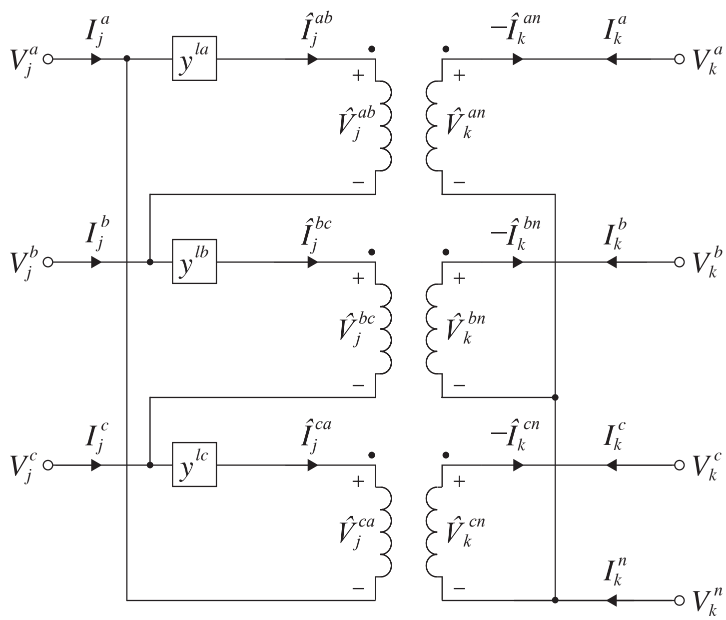


Figure 8.20: ΔY-configured transformer with zero shunt admittances.

their turns ratios $a := a^a \mathbb{I}$ and leakage admittances $y^l := y^{la} \mathbb{I}$ are the same across phases. Suppose the shunt admittances are zero. We discuss parameter identification in two steps.

1. Suppose the following measurements are given:

   - Terminal currents $I_j = i_j \in \mathbb{C}^3$ and $I_k = i_k \in \mathbb{C}^3$.
   - Terminal voltages $V_j = v_j \in \mathbb{C}^3$ (with respect to ground) on the primary (Δ) side.
   - Line-to-line voltages $\Gamma V_k = u_k \in \mathbb{C}^3$ on the secondary (Y) side.
   - The neutral is grounded with zero grounding impedance so that $V_k^n := 0$.

   Assume the measurements are error free and let $x := (i_j, i_k, v_j, u_k)$ be the measurement vector. Calculate:

   - The turns ratio a^a and the leakage admittance y^{la}.
   - The terminal voltage V_k with respect to the ground.

- The internal voltage and current $(\hat{V}_k^Y, \hat{I}_k^Y)$ on the secondary side.
- The internal voltage and current $(\hat{V}_j^\Delta, \hat{I}_j^\Delta)$ on the primary side and hence the loop flow β_j within the Δ configuration.

2. Repeat part 1 when T measurements $(x_1, \dots, x_T)$ are given and measurement errors may be nonzero.

Solution. Under the assumption of zero measurement error, the measurement $x := (i_j, i_k, v_j, u_k) \in \mathbb{C}^{12}$, the parameter $\theta := (a^a, y^{la}) \in \mathbb{C}^2$, and the variable $V_k \in \mathbb{C}$ satisfy (8.20a) with $y^l := y^{la}\mathbb{I}, a := a^a\mathbb{I}$:

$$\begin{bmatrix} i_j \\ i_k \end{bmatrix} = y^{la} \begin{bmatrix} \Gamma^\mathsf{T}\Gamma & -a^a\Gamma^\mathsf{T} \\ -a^a\Gamma & (a^a)^2\mathbb{I} \end{bmatrix} \begin{bmatrix} v_j \\ V_k \end{bmatrix} \tag{8.33}$$

We can obtain $\Gamma^\mathsf{T} V_k$ from the line-to-line voltage measurement $\Gamma V_k = u_k$ by shifting the values of u_k:

$$\Gamma^\mathsf{T} V_k = \begin{bmatrix} V^a - V^c \\ V^b - V^a \\ V^c - V^b \end{bmatrix} = -\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{permutation } P} \begin{bmatrix} V^a - V^b \\ V^b - V^c \\ V^c - V^a \end{bmatrix} = -P\Gamma V_k = -Pu_k$$

Hence the first row of (8.33) becomes

$$i_j = y^{la}\left(\Gamma^\mathsf{T}\Gamma v_j + a^a P u_k\right) \tag{8.34a}$$

where P is the permutation matrix

$$P := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{8.34b}$$

This is a set of 3 quadratic equations in a positive real variable $a^a \in \mathbb{R}_+$ and a complex variables $y^{la} \in \mathbb{C}$. Under appropriate conditions a solution of (8.34) exists and can be computed numerically. Let $\theta := (a^a, y^{la})$ denote such a solution. All other variables can then be derived in terms of the parameter θ and the measurement $x := (i_j, i_k, v_j, u_k)$, as follows.

The terminal voltage V_k can be calculated from the second row of (8.33):

$$V_k = \frac{1}{(a^a)^2 y^{la}} i_k + \frac{1}{a^a}\Gamma v_j \tag{8.35a}$$

On the secondary side the internal voltage and current $(\hat{V}_k^Y, \hat{I}_k^Y)$ are given by the conversion rule in (8.19b) for Y configuration on the secondary side:

$$\hat{V}_k^Y = V_k - V_k^n \mathbf{1} = V_k, \qquad \hat{I}_k^Y = i_k \tag{8.35b}$$

On the primary side the internal voltage $\hat{V}_j^\Delta$ *across the ideal transformers* is given by (8.9d) with $z^m := 0$ (no shunt admittance):

$$\hat{V}_j^\Delta = \Gamma v_j - \frac{1}{y^{la}}\hat{I}_j^\Delta$$

Instead of expressing $\hat{I}_j^\Delta$ in terms of the measurement i_j using $y^m = 0$ and the conversion rule $i_j = \Gamma^\mathsf{T} \hat{I}_j^\Delta$, we will use the transformer current gain in (8.19c) for ΔY configuration to express $\hat{I}_j^\Delta$ in terms of the measurement i_k, yielding

$$\hat{V}_j^\Delta \;=\; \Gamma v_j + \frac{1}{a^a y^{la}} i_k, \qquad \hat{I}_j^\Delta \;=\; -\frac{1}{a^a}\hat{I}_k^Y \;=\; -\frac{1}{a^a} i_k, \qquad \beta_j \;:=\; \frac{1}{3}\mathbf{1}^\mathsf{T}\hat{I}_j^\Delta \;=\; -\frac{1}{3a^a}\mathbf{1}^\mathsf{T} i_k \tag{8.35c}$$

Even though we cannot determine the loop flow β_j from the terminal current i_j, we can from the measurement i_k on the secondary side.

When the measurement error is zero, the measurement vector $x := (i_j, i_k, v_j, u_k)$ and the parameter vector $\theta := (a^a, y^{la})$ satisfy (8.34). This can be represented as $f(x;\theta) = 0$ for some function f. Given T measurements $x := (x_1, \ldots, x_T)$, there may not be any choice of θ such that $f(x_t;\theta) = 0$ for all $t = 1, \ldots, T$ when measurement errors are nonzero. A popular estimate of θ is one that minimizes error subject to certain constraints:

$$\hat{\theta} \;:=\; \arg\min_{\theta} \sum_t \|f(x_t;\theta)\| \qquad \text{s.t.} \qquad g(x_t;\theta) \le 0, \quad t = 1, \ldots, T$$

for some appropriate norm $\|\cdot\|$. Here $g(x_t;\theta) \le 0$ expresses some known relations that must hold, e.g., $a^a \ge 0$ is real. Let $\hat{\theta}$ denote an estimate of the parameter. Then other variables

$$\hat{y}_t \;:=\; (V_k(t), \hat{V}_j^\Delta(t), \hat{I}_j^\Delta(t), \hat{V}_k^Y(t), \hat{I}_k^Y(t)), \qquad\qquad t = 1, \ldots, T$$

can be derived from (8.35) in terms of $\hat{\theta}$ and the measurements x_t.

It is possible that the estimate $\hat{y}_t$ derived in this way may violate some known constraints, e.g., $v_k^{\min} \le \|V_k(t)\|^2 \le v_k^{\max}$ for some t given voltage limits. An alternative identification method is to estimate the parameter θ and the variables $y := (y_1, \ldots, y_T)$ jointly from the measurements $x := (x_1, \ldots, x_T)$, i.e., solve

$$(\hat{\theta}, \hat{y}) \;:=\; \arg\min_{(\theta, y)} \sum_t \|f(x_t, y_t;\theta)\| \qquad \text{s.t.} \qquad g(x_t, y_t;\theta) \le 0, \quad t = 1, \ldots, T$$

where f represents (8.34)(8.35) and $g(x_t, y_t;\theta) \le 0$ express some known constraints on $(\hat{\theta}, \hat{y})$. $\square$

From Figure 8.20 the terminal powers s_j and s_k are powers injected into the transformer at terminals j and k respectively. Hence $\mathbf{1}^\mathsf{T}(s_j + s_k)$ is the total power loss in the three-phase transformer due to the leakage impedance $1/y^l$, as the next example shows.

Example 8.6 (Total power loss)**.** For the three-phase transformer in Example 8.5 show that the total power loss $\mathbf{1}^\mathsf{T}(s_j + s_k)$ in the transformer is equal to (assuming zero measurement error):

$$\mathbf{1}^\mathsf{T}(s_j + s_k) \;=\; \frac{1}{y^{la}} \|n^a i_k\|_2^2$$

where $n^a := 1/a^a$ is the voltage gain. Even though the transformer gain n^a relates the internal currents $(\hat{I}_j^\Delta, \hat{I}_k^Y)$, not terminal currents (I_j, I_k), we can interpret $n^a i_k$ as the "effective" terminal current on the primary side.

Solution. The terminal powers are, from (8.35),

$$\begin{aligned} s_j &:= \operatorname{diag}\left(V_j I_j^{\mathsf{H}}\right) = -n^a \operatorname{diag}\left(v_j\, i_k^{\mathsf{H}} \Gamma\right) \\ s_k &:= \operatorname{diag}\left(V_k I_k^{\mathsf{H}}\right) = n^a \operatorname{diag}\left(\Gamma v_j\, i_k^{\mathsf{H}}\right) + \frac{(n^a)^2}{y^{la}} \operatorname{diag}\left(i_k\, i_k^{\mathsf{H}}\right) \end{aligned}$$

where $n^a := 1/a^a$, the second equality follows from $y^m = 0$ and hence $i_j = \Gamma^{\mathsf{T}} \hat{I}_j^{\Delta} = -n^a \Gamma^{\mathsf{T}} i_k$, and the last equality follows from (8.35a). Hence

$$s_j + s_k = n^a \left(\operatorname{diag}\left(\Gamma v_j\, i_k^{\mathsf{H}}\right) - \operatorname{diag}\left(v_j\, i_k^{\mathsf{H}} \Gamma\right)\right) + \frac{(n^a)^2}{y^{la}} \operatorname{diag}\left(i_k\, i_k^{\mathsf{H}}\right)$$

Now

$$\begin{aligned} \operatorname{diag}\left(\Gamma v_j\, i_k^{\mathsf{H}}\right) - \operatorname{diag}\left(v_j\, i_k^{\mathsf{H}} \Gamma\right) &= \begin{bmatrix} (v_j^a - v_j^b)\bar{i}_k^a \\ (v_j^b - v_j^c)\bar{i}_k^b \\ (v_j^c - v_j^a)\bar{i}_k^c \end{bmatrix} - \begin{bmatrix} v_j^a(\bar{i}_k^a - \bar{i}_k^c) \\ v_j^b(\bar{i}_k^b - \bar{i}_k^a) \\ v_j^c(\bar{i}_k^c - \bar{i}_k^b) \end{bmatrix} = \begin{bmatrix} v_j^a \bar{i}_k^c \\ v_j^b \bar{i}_k^a \\ v_j^c \bar{i}_k^b \end{bmatrix} - \begin{bmatrix} v_j^b \bar{i}_k^a \\ v_j^c \bar{i}_k^b \\ v_j^a \bar{i}_k^c \end{bmatrix} \\ \operatorname{diag}\left(i_k\, i_k^{\mathsf{H}}\right) &= \begin{bmatrix} |i_k^a|^2 \\ |i_k^b|^2 \\ |i_k^c|^2 \end{bmatrix} \end{aligned}$$

where P is the permutation matrix in (8.34b). The total power loss in the three-phase transformer is then

$$\mathbf{1}^{\mathsf{T}}(s_j + s_k) = n^a \left((P i_k)^{\mathsf{H}} v_j - i_k^{\mathsf{H}} (P^{\mathsf{T}} v_j)\right) + \frac{(n^a)^2}{y^{la}} \|i_k\|_2^2 = \frac{1}{y^{la}} \|n^a i_k\|_2^2$$

where the last equality follows from $(P i_k)^{\mathsf{H}} v_j = i_k^{\mathsf{H}} (P^{\mathsf{T}} v_j)$. □

8.4.2 Unitary voltage network

8.5 Bibliographical notes

The modeling of transmission lines with earth return is presented in the seminal paper [52] by J. R. Carson. Circuit models of three-phase line models studied in Chapter 8.1 are developed in e.g. [53, 54, 39]. See e.g. [51, Chapter 3] for comprehensive models of three-phase components including distribution lines, transformers and switches. For the simplified model of Chapter 8.2 see [55, 56, 57, 58] for early work and [39, Ch 8][50, Ch 7.4][59] for recent summary. The idea of decomposing a nonideal transformer into two ideal transformers connected by a unitary voltage network as in Chapter 8.3 is first mentioned, but not explored, in [55]. It is developed in detail in [60] where the unitary network is a Π circuit with a leakage (series) admittance and two shunt admittances. The unitary voltage network in [61] uses a T circuit model, as Chapter 8.3 does. The unitary voltage network that models leakage fluxes and core losses can be quite general e.g. [62, 63].

8.6 Problems

Chapter 8.1.

Exercise 8.1 (Symmetric y_{jk})**.** Let z_{jk} be a phase impedance matrix of a three-phase line (j,k). Assume z_{jk} is symmetric invertible and $z_{jk} = z_{kj}$ (A0). Show that its inverse $y_{jk} := z_{jk}^{-1}$ is symmetric. Moreover $y_{jk} = y_{kj}$.

Chapter 8.2.

Exercise 8.2 (ΔY-configured transformer)**.** Derive the external model (8.20) of the ΔY-configured three-phase transformer in Figure 8.14.

Exercise 8.3 ($Y\Delta$-configured transformer)**.** Derive the external model (8.21) of the $Y\Delta$-configured three-phase transformer in Figure 8.15.

Exercise 8.4 (Open transformers)**.**

Exercise 8.5 (Complex voltage gain)**.** Consider a YY-configured transformer with a *complex* voltage gain (matrix) $n := \text{diag}\left(n^a, n^b, n^c\right) \in \mathbb{C}^{3\times3}$. Let its turns ratio be $a := n^{-1} \in \mathbb{C}^{3\times3}$. Let $y^l \in \mathbb{C}^{3\times3}$ denote its series admittance and assume its shunt admittance $y^m = 0$. Show that its external model is

$$\begin{aligned} \begin{bmatrix} I_j \\ I_k \end{bmatrix} &= \underbrace{\begin{bmatrix} y^l & -ay^l \\ -a^{\mathsf{H}}y^l & |a|^2 y^l \end{bmatrix}}_{Y_{YY}} \left(\begin{bmatrix} V_j \\ V_k \end{bmatrix} - \begin{bmatrix} V_j^n \mathbf{1} \\ V_k^n \mathbf{1} \end{bmatrix} \right) \\ I_j^n &= -\mathbf{1}^{\mathsf{T}} I_j, \qquad I_k^n = -\mathbf{1}^{\mathsf{T}} I_k \end{aligned}$$

where $|a|^2$ is the matrix $|a|^2 := \text{diag}\left(1/|n^a|^2, 1/|n^b|^2, 1/|n^c|^2\right)$.

Exercise 8.6 (Unitary voltage network: 3ϕ transformers)**.** Derive (8.29), reproduced here:

$$\hat{I} = Y_{\text{uvn}} \hat{V}$$

where

$$Y_{\text{uvn}} := \left(\mathbb{I}_2 \otimes \left(\sum_i y_i \right)^{-1} \right) \begin{bmatrix} y_j(y_0 + y_k) & -y_j y_k \\ -y_j y_k & y_k(y_0 + y_j) \end{bmatrix}$$

$\mathbb{I}_2$ is the identity matrix of size 2, and $\sum_i y_i = y_0 + y_j + y_k$ is a diagonal matrix of all admittances.

Exercise 8.7 (Unitary voltage network: 3ϕ transformers)**.** Show that, for the transformer model in Chapter 8.3 with unitary voltage network, the admittance matrices of standard three-phase transformers are given by

$$I \;=\; D^{\mathsf{T}}(MY_{\text{uvn}}M)D\,(V-\gamma)$$

where Y_{uvn} is defined in (8.29), and $D \in \mathbb{C}^{6\times 6}$ and $\gamma \in \mathbb{C}^{6}$ are defined in (8.12).

Exercise 8.8 (Unitary voltage network: turns ratio a)**.** Prove (8.32b): the matrix $MY_{\text{uvn}}M$ in (8.32) can be written in terms of the 3×3 turns ratio and admittance matrices a, y^p, y^s, y^m:

$$Y_{YY} \;:=\; MY_{\text{uvn}}M \;=\; y^p y^s\left(a^2y^m + a^2y^p + y^s\right)^{-1}\begin{bmatrix} \mathbb{I} + a^2y^m(y^s)^{-1} & -a \\ -a & a^2\left(\mathbb{I} + y^m(y^p)^{-1}\right)\end{bmatrix}$$

Exercise 8.9 (3ϕ transformer: $y^m = y_0 = 0$)**.** Suppose shunt admittances $y_0 = y^m = \text{diag}(0,0,0)$. Then the admittance matrices Y_{uvn} defined in (8.29) and Y_{YY} defined in (8.12a) become

$$Y_{\text{uvn}} \;:=\; \left(\mathbb{I}_2 \otimes \left(y_j + y_k\right)^{-1}\right)\right)\begin{bmatrix} y_jy_k & -y_jy_k \\ -y_jy_k & y_jy_k\end{bmatrix}, \qquad\qquad Y_{YY} \;:=\; \begin{bmatrix} y^l & -ay^l \\ -ay^l & a^2y^l\end{bmatrix}$$

Show that $MY_{\text{uvn}}M = Y_{YY}$.

Exercise 8.10 (Split-phase transformer)**.** Consider a split-phase $\Delta\Delta$ transformer in Figure **??**. Suppose $\sum_{\phi\in\{a,b,c\}}\left(I_k^{\phi} + I_k^{\phi'}\right) = 0$. Derive (**??**).

Chapter 9

Bus injection models

In this chapter we use the component models in Chapters 7 and 8 to construct network models and study unbalanced three-phase analysis. In Chapter 9.1 we extend the relation between terminal voltage, current and power (V, I, s) in the single-phase bus injection model of Chapter 4.3 to the unbalanced three-phase setting. In Chapter 9.2 we formulate a general three-phase analysis problem. In Chapter 9.3 we study the analysis problem when the network is balanced. We prove formally that a general balanced network is equivalent to per-phase networks and its analysis can be solved by per-phase analysis. In Chapter 9.4 we explain that, when an unbalanced system has a certain symmetry, we can transform it to a sequence coordinate in which the system becomes decoupled even if the phases are coupled in the original coordinate. Single-phase analysis can then be applied to individual sequence networks.

9.1 Network models

In this section we develop a model for a network of three-phase devices connected by three-phase lines and transformers studied in Chapters 7 and 8. We start in Chapter 9.1.1 with a line model that models a three-phase transmission or distribution line or a three-phase transformer. The line model linearly relates the sending-end line currents $\left(I_{jk}, I_{kj}\right) \in \mathbb{C}^6$ and the nodal voltages $\left(V_j, V_k\right) \in \mathbb{C}^6$ by an admittance matrix Y_{jk} which may or may not have a three-phase Π circuit representation. The line model induces a network model through nodal current balance equations. This is derived in Chapter 9.1.2 and it linearly relates the nodal (terminal) current injections I_j and voltages V_j through a network admittance matrix Y. The admittance matrix Y also implies a single-phase equivalent circuit of the three-phase network. We then use Y to derive in Chapter 9.1.4 nonlinear power flow equations that relate nodal (terminal) power injections s_j and voltages V_j. Finally we explain in Chapter 9.1.5 that the overall model consists of the network equations of Chapters 9.1.2 and 9.1.4 and the three-phase device models of Chapter 7.3. A device model can either be specified as an internal model with conversion rules or an external model relating the terminal variables (V_j, I_j, s_j).

9.1.1 Line model

Consider a network with $N+1$ three-phase devices connected by three-phase lines represented as an undirected graph $G := (\overline{N}, E)$ where every bus $j \in \overline{N}$ and every line $(j,k) \in E$ has 3 phases. A bus is where the terminals of three-phase devices are connected. A line may model a transmission or distribution line, a transformer, or a combination. We will hence refer to $j \in \overline{N}$ interchangeably as a bus, a node, or a terminal, and $(j,k) \in E$ interchangeably as a line, a branch, a link, or an edge. The formulation can be generalized to the case where a bus or a line has a single, two, or three phases.

For simplicity of exposition we assume, by default, we can use three-wire models for these lines and their characterization includes the effects of neutral and earth return on the phase variables. This assumption is reasonable if, e.g., neutral wires are absent, the line connects devices in Δ configuration, or the neutrals are directly grounded with equal spacing along a line and at both ends of the line so that all neutrals have $V_j^n = 0$. Otherwise, the line model in this section needs to be augmented with neutral lines with variables in $\mathbb{C}^4$ instead of $\mathbb{C}^3$ and line admittance matrices in $\mathbb{C}^{4\times 4}$ instead of $\mathbb{C}^{3\times 3}$; see Example 9.5 and Exercise 9.7. As we will see, even though lines are assumed to be three-wired, Y-configured devices such as voltage, current and power sources and impedances do have neutral lines in the our model and their neutral voltages $\gamma_j := V_j^n$ may be nonzero.

For each line $(j,k) \in E$ let $\left(V_j, V_k\right) \in \mathbb{C}^6$ denote the terminal voltages at each end of the line and $\left(I_{jk}, I_{kj}\right) \in \mathbb{C}^6$ denote the sending-end line currents in both directions. In general each line $(j,k) \in E$ is characterized by four 3×3 series and shunt admittance matrices, $\left(y_{jk}^s, y_{jk}^m\right)$ from j to k and $\left(y_{kj}^s, y_{kj}^m\right)$ from k to j. See Figure 9.1. They define the relation between $\left(V_j, V_k\right)$ and $\left(I_{jk}, I_{kj}\right)$:

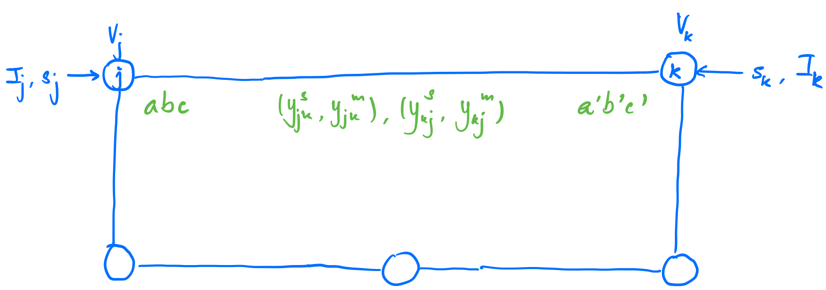


Figure 9.1: A model of three-phase system.

$$I_{jk} \;=\; y_{jk}^s (V_j - V_k) \;+\; y_{jk}^m V_j, \qquad\qquad I_{kj} \;=\; y_{kj}^s (V_k - V_j) \;+\; y_{kj}^m V_k \tag{9.1a}$$

or in matrix form:

$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} = \underbrace{\begin{bmatrix} y_{jk}^s + y_{jk}^m & -y_{jk}^s \\ -y_{kj}^s & y_{kj}^s + y_{kj}^m \end{bmatrix}}_{Y_{jk}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{9.1b}$$

We emphasize that y_{jk}^s and y_{kj}^s may be different matrices and therefore this general model Y_{jk} may not have a three-phase Π circuit representation. It can model, where $y_{jk}^s = y_{kj}^s$:

- A transmission or distribution line where, from (8.4a), $y^s_{jk} = y^s_{kj}$ is its series admittance and $\left(y^m_{jk}, y^m_{kj}\right)$ are its shunt admittances.

- A transformer in YY configuration where neutral voltages are zero, from (8.15d) (recalling that $V^n_j = V^n_k = 0$),

$$y^s_{jk} \;=\; y^s_{kj} \;:=\; ay, \qquad y^m_{jk} \;:=\; (\mathbb{I}-a)\,y, \qquad y^m_{kj} \;:=\; a(a-\mathbb{I})y \tag{9.2a}$$

  with a being the diagonal matrix of its turns ratios and y the diagonal matrix of its leakage admittances.

- A transformer in $\Delta\Delta$ configuration where, from (8.17c),

$$y^s_{jk} \;=\; y^s_{kj} \;:=\; \Gamma^{\mathsf{T}} ay\Gamma, \qquad y^m_{jk} \;:=\; \Gamma^{\mathsf{T}}(\mathbb{I}-a)\,y\Gamma, \qquad y^m_{kj} \;:=\; \Gamma^{\mathsf{T}} a(a-\mathbb{I})y\Gamma \tag{9.2b}$$

  Or a transformer in open $\Delta\Delta$ configuration where, from (8.22a),

$$y^s_{jk} \;=\; y^s_{kj} \;:=\; \Gamma^{\mathsf{T}} a\tilde{y}\Gamma, \qquad y^m_{jk} \;:=\; \Gamma^{\mathsf{T}}(\mathbb{I}-a)\,\tilde{y}\Gamma, \qquad y^m_{kj} \;:=\; \Gamma^{\mathsf{T}} a(a-\mathbb{I})\tilde{y}\Gamma \tag{9.2c}$$

  where $\tilde{y} := \text{diag}\left(y^a, y^b, 0\right)$ is the leakage admittance matrix of the open transformer.

This model can also model transformers in other configurations where $y^s_{jk} \neq y^s_{kj}$:

- A transformer in ΔY configuration with zero neutral voltage where, from (8.20a),

$$y^s_{jk} \;:=\; \Gamma^{\mathsf{T}} ay, \qquad y^s_{kj} \;:=\; ay\Gamma, \qquad y^m_{jk} \;:=\; \Gamma^{\mathsf{T}} y(\Gamma-a), \qquad y^m_{kj} \;:=\; ay(a-\Gamma) \tag{9.3a}$$

- A transformer in $Y\Delta$ configuration with zero neutral voltage where, from (8.21a),

$$y^s_{jk} \;:=\; ay\Gamma, \qquad y^s_{kj} \;:=\; \Gamma^{\mathsf{T}} ay, \qquad y^m_{jk} \;:=\; y(\mathbb{I}-a\Gamma), \qquad y^m_{kj} \;:=\; \Gamma^{\mathsf{T}} ay(a\Gamma-\mathbb{I}) \tag{9.3b}$$

Remark 9.1 (Transformer models)**.**

1. We emphasize that the models (9.2) (9.3) assume that, for three-phase transformers with Y configuration either in the primary or secondary side, their neutrals are directly grounded so the neutral voltages $V^n_j = 0$.

2. While the shunt admittances y^m_{jk} and y^m_{kj} are typically equal for a transmission or distribution line, they are typically different for a transformer.

3. The series and shunt admittance matrices $\left(y^s_{jk}, y^m_{jk}\right)$ and $\left(y^s_{kj}, y^m_{kj}\right)$ in (9.2) are all complex symmetric. None of them are symmetric for series and shunt admittances in (9.3).

For simplicity we often restrict ourselves to the special case where $y^s_{jk} = y^s_{kj}$. In this case we characterize a line (j,k) by three 3×3 series and shunt admittance matrices $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$. With $y^s_{jk} = y^s_{kj}$, (9.1) reduces to

$$I_{jk} \;=\; y^s_{jk}(V_j - V_k) \;+\; y^m_{jk}V_j, \qquad\qquad I_{kj} \;=\; y^s_{jk}(V_k - V_j) \;+\; y^m_{kj}V_k \tag{9.4a}$$

or in terms Y_{jk}:

$$\begin{bmatrix} I_{jk} \\ I_{kj} \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} y^s_{jk} + y^m_{jk} & -y^s_{jk} \\ -y^s_{jk} & y^s_{jk} + y^m_{kj} \end{bmatrix}}_{Y_{jk}} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{9.4b}$$

which is now block symmetric (see Definition 9.1). We say Y_{jk} has a *three-phase* Π *circuit representation* in the sense that its external behavior is the same as the external behavior (8.4a) of a three-phase transmission line; see Figure 8.2.

From (9.3) this more restrictive Y_{jk} cannot be used to model transformers in ΔY and $Y\Delta$ configurations. It is however still widely used. We therefore often adopt this model and will explicitly state it as assumption C9.1 below when we use it.

9.1.2 VI relation

Associated with each bus j are three nodal variables $\left(V_j, I_j, s_j\right) \in \mathbb{C}^9$ representing the nodal voltage, current injection, and power injection respectively at the terminal of the device connected to bus j. To simplify notation we assume, without loss of generality, that at most one single-terminal device (source or load) is connected to a bus but one or more lines can be connected to a bus.[1] The bus current and power injection (I_j, s_j) at bus j therefore refers unambiguously to the injection from the unique device at bus j. As explained in Chapters 7.3.2 and 7.3.3, the external behavior of a three-phase device is described by the relation between (V_j, I_j) or that between (V_j, s_j). We can assume without loss of generality that these three-phase devices are ideal (see Chapter 8.1.4) and their behavior is summarized in Tables 7.3 and 7.4.

Let $(V, I, s) := \left(V_j, I_j, s_j, j \in \overline{N}\right) \in \mathbb{C}^{3(N+1)}$ be nodal variables over the entire network. As for a single-phase network, a three-phase network model is a relation between the *terminal* voltage and current (V, I) or a relation between the *terminal* voltage and power (V, s), independent of the internal Y or Δ configurations of the three-phase devices that are connected by the lines. In this subsection we derive the linear VI relation defined by an admittance matrix Y and show that Y defines a single-phase equivalent circuit of the three-phase network. In the next subsection we derive the Vs relation in the form of nonlinear power flow equations. In both cases the extension of the line model (9.1) to a network is the nodal current or power balance equations:

$$I_j \;=\; \sum_{k: j \sim k} I_{jk}, \qquad\qquad s_j \;=\; \sum_{k: j \sim k} \operatorname{diag}\left(S_{jk}\right), \qquad\qquad j \;\in\; \overline{N}$$

where $S_{jk} := V_j I_{jk}^{\mathsf{H}}$ are matrices defined in (8.4b).

Network admittance matrix Y. Substitute the line currents (9.1) into the current balance equation to get

$$I_j \;=\; \sum_{k: j \sim k} I_{jk} \;=\; \sum_{k: j \sim k} y^s_{jk}(V_j - V_k) \;+\; \left(\sum_{k: j \sim k} y^m_{jk}\right) V_j$$

[1] If K three-phase devices with terminal current injections $I_{j1}, \ldots, I_{jK}$ are connected to bus j then the net bus injection is $I_j := \sum_k I_{jk}$. Unless otherwise specified we assume $K = 1$.

Therefore

$$I_j \;=\; \left(\left(\sum_{k:j\sim k} y^s_{jk}\right) + y^m_{jj}\right)V_j \;-\; \sum_{k:j\sim k} y^s_{jk}V_k, \qquad j\in\overline{N} \tag{9.5a}$$

where

$$y^m_{jj} \;:=\; \sum_{k:j\sim k} y^m_{jk} \tag{9.5b}$$

Note that I_j is the net current injection.[2] In vector form, this relates the bus current vector $I := (I_0,\ldots,I_N)$ to the bus voltage vector $V := (V_0,\ldots,V_N)$:

$$I \;=\; YV \tag{9.6a}$$

through a $3(N+1)\times 3(N+1)$ admittance matrix Y where its 3×3 submatrices $Y_{jk}\in\mathbb{C}^{3\times 3}$ are given by

$$Y_{jk} \;:=\; \begin{cases} -y^s_{jk}, & j\sim k \;\; (j\neq k) \\ \sum_{l:j\sim l} y^s_{jl} + y^m_{jj}, & j=k \\ 0 & \text{otherwise} \end{cases} \tag{9.6b}$$

The submatrices Y_{jk} and Y_{kj} may be different if (j,k) models a three-phase transformer in ΔY or $Y\Delta$ configuration.

Definition 9.1 (Block symmetry and block row sum)**.** Given a matrix $A\in\mathbb{C}^{3n\times 3n}$, partition it into $n\times n$ blocks of 3×3 submatrices. Denote by $A_{jk}\in\mathbb{C}^{3\times 3}$ its jkth submatrix.

1. A is called *block symmetric* if $A_{jk} = A_{kj}$ for all $j,k = 1,\ldots,n$.
2. A is said to have zero *block row sums* if $\sum_k A_{jk} = 0$ for all $j = 1,\ldots,n$.

As discussed in Chapter 8.2.3 a matrix can be symmetric but not block symmetric, and vice versa. Symmetry of a matrix is determined only by its off-diagonal entries but its diagonal entries can be arbitrary. Block symmetry is determined only by its off-diagonal blocks but its diagonal blocks can be arbitrary. A symmetric matrix A is block symmetric if, in addition, all its off-diagonal blocks are themselves symmetric, i.e., $A^{\mathsf{T}}_{jk} = A_{jk}$, for all $j\neq k$. A block symmetric A is symmetric if, in addition, all blocks A_{jk}, including the diagonal blocks, are symmetric (Exercise 9.1). We will remark on zero block row sums below after introducing single-phase equivalent circuit.

In general an admittance matrix Y defined by (9.6) may neither be block symmetric nor symmetric. If the series admittances $y^s_{jk} = y^s_{kj}$ for all lines $(j,k)\in E$ then the admittance matrix Y is block symmetric and hence has a three-phase Π circuit representation. As in Chapter 4 we label the following assumption and will explicitly state it when it is required:

[2] If there is a nodal shunt admittance load y^{sh}_j, e.g., a capacitor bank, in addition to a device whose terminal injection is $\tilde{I}_j$, then the net injection from bus j to the rest of the network is $I_j = \tilde{I}_j - y^{\text{sh}}_j V_j$. This assumes that y^{sh}_j connects bus j to the ground and the terminal voltage V_j is defined with respect to the ground.

C9.1: The series admittance matrices $y^s_{jk} = y^s_{kj}$ for every line $(j,k) \in E$, so that the admittance matrix Y is block symmetric.

If every $(j,k) \in E$ models a transmission or distribution line or a transformer described by (9.2), then Y is block symmetric with a three-phase Π circuit representation. If some $(j,k) \in E$ model transformers described by (9.3), however, then Y is not.

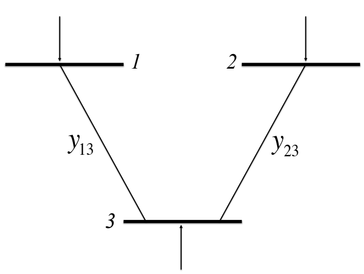
The expression (4.10) for Y for a single-phase network generalizes directly to the three-phase setting. Let $C \in \{-\mathbb{I}, 0, \mathbb{I}\}^{|\overline{N}| \times |E|}$ be the bus-by-line incidence matrix defined by:

$$C_{jl} \quad = \quad \begin{cases} \mathbb{I} & \text{if } l = j \rightarrow k \text{ for some bus } k \\ -\mathbb{I} & \text{if } l = i \rightarrow j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

where $\mathbb{I}$ is the identity matrix of size 3. Let $Y^s := \text{diag}\left(y^s_l,\, l \in E\right)$ be the $3|E| \times 3|E|$ block diagonal matrix with the series admittance matrices $y^s_l \in \mathbb{C}^{3\times 3}$ as its diagonal submatrices. Let $Y^m := \text{diag}\left(y^m_{jj},\, j \in \overline{N}\right)$ be the $|\overline{N}| \times |\overline{N}|$ block diagonal matrix with the total shunt admittances $y^m_{jj} \in \mathbb{C}^{3\times 3}$ in (9.5b) as its diagonal submatrices. Then the admittance matrix in (9.6b) is, when $y^s_{jk} = y^s_{kj}$,

$$Y \quad = \quad C Y^s C^{\mathsf{T}} \; + \; Y^m$$

Example 9.1. The admittance matrix Y for a 3-terminal network with zero shunt admittances is shown in Figure 9.2. $\square$



(a) 3-bus example.

$$Y = \begin{bmatrix} \begin{bmatrix} y_{13}^{aa} & y_{13}^{ab} & y_{13}^{ac} \\ y_{13}^{ba} & y_{13}^{bb} & y_{13}^{bc} \\ y_{13}^{ca} & y_{13}^{cb} & y_{13}^{cc} \end{bmatrix} & 0 & -\begin{bmatrix} y_{13}^{aa} & y_{13}^{ab} & y_{13}^{ac} \\ y_{13}^{ba} & y_{13}^{bb} & y_{13}^{bc} \\ y_{13}^{ca} & y_{13}^{cb} & y_{13}^{cc} \end{bmatrix} \\ 0 & \begin{bmatrix} y_{23}^{aa} & y_{23}^{ab} & y_{23}^{ac} \\ y_{23}^{ba} & y_{23}^{bb} & y_{23}^{bc} \\ y_{23}^{ca} & y_{23}^{cb} & y_{23}^{cc} \end{bmatrix} & -\begin{bmatrix} y_{23}^{aa} & y_{23}^{ab} & y_{23}^{ac} \\ y_{23}^{ba} & y_{23}^{bb} & y_{23}^{bc} \\ y_{23}^{ca} & y_{23}^{cb} & y_{23}^{cc} \end{bmatrix} \\ -\begin{bmatrix} y_{13}^{aa} & y_{13}^{ab} & y_{13}^{ac} \\ y_{13}^{ba} & y_{13}^{bb} & y_{13}^{bc} \\ y_{13}^{ca} & y_{13}^{cb} & y_{13}^{cc} \end{bmatrix} & -\begin{bmatrix} y_{23}^{aa} & y_{23}^{ab} & y_{23}^{ac} \\ y_{23}^{ba} & y_{23}^{bb} & y_{23}^{bc} \\ y_{23}^{ca} & y_{23}^{cb} & y_{23}^{cc} \end{bmatrix} & \begin{bmatrix} y_{13}^{aa}+y_{23}^{aa} & y_{13}^{ab}+y_{23}^{ab} & y_{13}^{ac}+y_{23}^{ac} \\ y_{13}^{ba}+y_{23}^{ba} & y_{13}^{bb}+y_{23}^{bb} & y_{13}^{bc}+y_{23}^{bc} \\ y_{13}^{ca}+y_{23}^{ca} & y_{13}^{cb}+y_{23}^{cb} & y_{13}^{cc}+y_{23}^{cc} \end{bmatrix} \end{bmatrix}$$

(b) Admittance matrix Y.

Figure 9.2: The admittance matrix Y for a 3-terminal network with no shunt admittances.

Single-phase equivalent circuit. The $3(N+1) \times 3(N+1)$ admittance matrix Y in (9.6) defines a single-phase equivalent circuit of the three-phase network. Recall that a three-phase network can be represented by a graph $G := (\overline{N}, E)$ where $\overline{N}$ is a set of $N+1$ three-phase buses and E is a set of three-phase lines. The admittance matrix Y induces a network graph $G^{3\phi} := (\overline{N}^{3\phi}, E^{3\phi})$ where $\overline{N}^{3\phi}$ has $3(N+1)$ buses. Each bus in $\overline{N}^{3\phi}$ is indexed by $j\phi$ with $j \in \overline{N}, \phi \in \{a,b,c\}$ in the original network G. Each line in $E^{3\phi}$ is indexed by $(j\phi, k\phi')$. There is a line between bus $j\phi$ and another distinct bus $k\phi'$ in $G^{3\phi}$ if and only if $Y_{jk}^{\phi\phi'}$ is nonzero. We call this graph $G^{3\phi}$ the *single-phase equivalent* (circuit) of the three-phase network G. All the single-phase modeling and analysis developed in earlier chapters can be directly applied to this single-phase equivalent.

When shunt admittances are assumed zero, $y_{jk}^m = y_{kj}^m = 0$ for all $(j,k) \in E$, the $3(N+1) \times 3(N+1)$ admittance matrix Y has zero block row sums (Definition 9.1), because

$$Y_{jj} \;=\; \sum_{k:(j,k)\in E} y_{jk}^s \;=\; \sum_k -Y_{jk}, \qquad j \in \overline{N}$$

so that $\sum_k Y_{jk} = 0$ for all j. Suppose Y has zero block row sums. Then Y also has zero block column sums if and only if Y is block symmetric. The matrix has zero row sums if $\sum_{k,\phi'} Y_{j\phi,k\phi'} = 0$ for all $j\phi$. This is equivalent to

$$\sum_{k,\phi'} Y_{j\phi,k\phi'} \;=\; \sum_{\phi'\in\{a,b,c\}} y_{jj}^{\phi\phi'} \;-\; \sum_{\substack{k:(j,k)\in E \\ \phi'\in\{a,b,c\}}} y_{jk}^{\phi\phi'} \;=\; 0, \qquad j\phi \in \overline{N} \times \{a,b,c\}$$

i.e., zero row sums requires only that the 3×3 matrix $\sum_k Y_{jk}$ has zero row sums, whereas zero block row sums requires that $\sum_k Y_{jk}$ is a zero matrix. Hence if a matrix has zero block row sums, then all its row sums are zero, but the converse does not necessarily hold.

In general Y is not symmetric (nor block symmetric), i.e., it may not satisfy C4.1 as the admittance matrix of a single-phase network. It is symmetric, and block symmetric, under the following condition:

> C9.2: In addition to C9.1, all series and shunt admittance matrices $y_{jk}^s, y_{jk}^m, y_{kj}^m$ are complex symmetric, so that the admittance matrix Y is both symmetric and block symmetric.

Suppose all transmission and distribution line models satisfy C9.2 (in particular, it satisfies assumptions C8.1 and C8.2). If every $(j,k) \in E$ models a transmission or distribution line or a transformer described by (9.2), then Y is not only block symmetric, but also symmetric (hence satisfying C4.1). Therefore Y has a three-phase Π circuit representation and the admittance matrix of its single-phase equivalent is complex symmetric. If some $(j,k) \in E$ models transformers described by (9.3), however, then Y is neither symmetric nor block symmetric.

Radial network. Even when the multiphase network G is radial (i.e., with tree topology), its single-phase equivalent $G^{3\phi}$ is a meshed network (i.e., has cycles), but in that case, $G^{3\phi}$ has a radial macro-structure in which each line is represented as a clique (complete subgraph). Specifically $G^{3\phi}$ has a maximal clique consisting of the set $\left\{ j\phi, k\phi' \in \overline{N}^{3\phi} \,:\, \phi, \phi' \in \{a,b,c\} \right\}$ of buses if and only if (j,k) is a line in G; see Figure 9.3. The corresponding principal submatrix $Y_{G^{3\phi}}(j,k) \in \mathbb{C}^{6\times 6}$ of Y is:

$$Y_{G^{3\phi}}(j,k) \;=\; \begin{bmatrix} Y_{jj} & Y_{jk} \\ Y_{kj} & Y_{kk} \end{bmatrix}$$

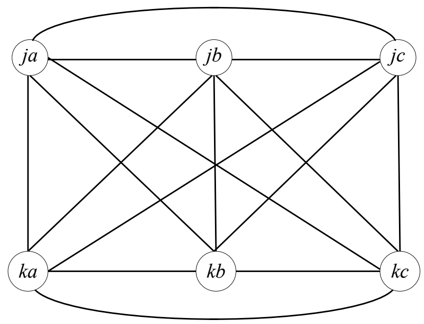


Figure 9.3: A clique of $G^{3\phi}$ corresponding to line (j,k) in G.

We will explain in Chapter **??** that $G^{3\phi}$ is a chordal graph which can be exploited to simplify the semidefinite relaxation of optimal power problems.

9.1.3 Invertibility of Y, Y_{22} and Y/Y_{22}

In this subsection we study the invertibility and properties of Y, Y_{22} and its Schur complement Y/Y_{22}. These results extend those in Chapter 4.2.5 from single-phase to three-phase networks.

Invertibility of Y. Recall that a real matrix G is positive semidefinite (or positive definite), denoted $G \succeq 0$ (or $G \succ 0$), if G is symmetric and $v^\mathsf{T} G v \geq 0$ (or $v^\mathsf{T} G v > 0$) for all real vectors v (see Remark 25.1 in Appendix 25.1.4). Under assumption C9.2 ($y^s_{jk} = y^s_{kj}$, y^m_{jk} and y^m_{kj} are complex symmetric) the admittance matrix $Y \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ is both symmetric and block symmetric. Write admittances in terms of their real and imaginary parts, $y^s_{jk} = g^s_{jk} + \mathbf{i} b^s_{jk}$, $y^m_{jk} = g^m_{jk} + \mathbf{i} b^m_{jk}$, and $y^m_{kj} = g^m_{kj} + \mathbf{i} b^m_{kj}$. Consider the following conditions on the conductances $g^s_{jk}, g^m_{jk}, g^m_{kj} \in \mathbb{R}^{3\times 3}$:

C9.3: For all lines $(j,k) \in E$, $g^s_{jk} \succeq 0$, $g^m_{jk} \succeq 0$, $g^m_{kj} \succeq 0$.

C9.4a: For all buses $j \in \overline{N}$, $g^m_{jj} := \sum_{k:k\sim j} g^m_{jk} \succ 0$, i.e., for all j, there exists a line $(j,k) \in E$ such that $g^m_{jk} \succ 0$

C9.4b: For all lines $(j,k) \in E$, $g^s_{jk} \succ 0$. Furthermore there exists a line $(j',k') \in E$ such that $g^m_{j'k'} \succ 0$.

C9.4c: For all lines $(j,k) \in E$, $g^s_{jk} \succ 0$. Furthermore there exists a line $(j',k') \in E$ such that the intersection of the null spaces of $g^m_{j'k'}$ and $g^m_{k'j'}$ is $\{0\}$.

Condition C9.4b is a special case of C9.4c which does not require positive definiteness of g^m_{jk}. The next result extends Theorems 4.2, 4.3, and 4.9 in Chapter 4.2.5 from single-phase to three-phase networks.

Theorem 9.1. Suppose the network is connected and the admittance matrix $Y \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ satisfies C9.2. If the conductance matrices $g^s_{jk}, g^m_{jk}, g^m_{kj} \in \mathbb{R}^{3\times 3}$ satisfy conditions C9.3 and one of C9.4a, C9.4b, C9.4c, then

1. The admittance matrix $Y^{-1} \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ exists and is symmetric. Moreover both $\mathrm{Re}(Y) \succ 0$ and $\mathrm{Re}(Y^{-1}) \succ 0$.

In addition if $Y =: \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathsf{T}} & Y_{22} \end{bmatrix}$ with invertible Y_{22}, then

2. The Schur complement $Y/Y_{22} := Y_{11} - Y_{12}Y_{22}^{-1}Y_{12}^{\mathsf{T}}$ of Y_{22} is symmetric and invertible. Moreover both $\mathrm{Re}(Y/Y_{22}) \succ 0$ and $\mathrm{Re}\big((Y/Y_{22})^{-1}\big) \succ 0$.

Proof. Let $G := \mathrm{Re}(Y) \in \mathbb{R}^{3(N+1)\times 3(N+1)}$. We will show that $G \succ 0$. The claims then follow from Theorems 4.2 and 4.9.

Fix any real vector $\rho \in \mathbb{R}^{3(N+1)}$ and decompose it into $\rho =: (\rho_j, j \in \overline{N})$ with $\rho_j \in \mathbb{R}^3$. We have using (9.6b) and (9.5b)

$$\rho^{\mathsf{T}} G \rho \;=\; \sum_j \sum_{k:k\sim j} \left(\rho_j^{\mathsf{T}} g_{jk}^s \rho_j - \rho_j^{\mathsf{T}} g_{jk}^s \rho_k\right) \;+\; \sum_{j\in\overline{N}} \rho_j^{\mathsf{T}} g_{jj}^m \rho_j \tag{9.7a}$$

$$= \sum_{(j,k)\in E} \left(\rho_j^{\mathsf{T}} g_{jk}^s \rho_j - \rho_j^{\mathsf{T}} g_{jk}^s \rho_k - \rho_k^{\mathsf{T}} g_{kj}^s \rho_j + \rho_k^{\mathsf{T}} g_{kj}^s \rho_k\right) \;+\; \sum_j \sum_{k:k\sim j} \rho_j^{\mathsf{T}} g_{jk}^m \rho_j \tag{9.7b}$$

$$= \sum_{(j,k)\in E} (\rho_j - \rho_k)^{\mathsf{T}} g_{jk}^s (\rho_j - \rho_k) \;+\; \sum_{(j,k)\in E} \left(\rho_j^{\mathsf{T}} g_{jk}^m \rho_j + \rho_k^{\mathsf{T}} g_{kj}^m \rho_k\right) \tag{9.7c}$$

where the last equality follows because $g_{jk}^s = g_{kj}^s$ for all $(j,k) \in E$ by C9.2. Since $g_{jk}^s, g_{jk}^m, g_{kj}^m \in \mathbb{R}^{3\times 3}$ are positive semidefinite for all lines $(j,k) \in E$ by C9.3, every summand is nonnegative and hence $\rho^{\mathsf{T}} G \rho = 0$ if and only if every summand is zero. We examine each of the three cases:

- *C9.4a holds*: Then for all buses $j \in \overline{N}$, $\rho_j^{\mathsf{T}} g_{jj}^m \rho_j > 0$ unless $\rho_j = 0$. Therefore for the second summation in (9.7a) to be zero we must have $\rho_j = 0$ for all $j \in \overline{N}$. This implies that $G \succ 0$.
- *C9.4b holds*: For the first summation in (9.7c) to be zero we must have $\rho_j = \rho_k$ for all $(j,k) \in E$. Since the network is connected, this implies that $\rho_j = \rho_1$ for all $j \in \overline{N}$. The second summation in (9.7b) then becomes, if $\rho_1 \neq 0$,

  $$\sum_j \sum_{k:k\sim j} \rho_j^{\mathsf{T}} g_{jk}^m \rho_j \;=\; \rho_1^{\mathsf{T}} \left(\sum_j \sum_{k:k\sim j} g_{jk}^m\right) \rho_1 \;\geq\; \rho_1^{\mathsf{T}} g_{j'k'}^m \rho_1 \;>\; 0$$

  Therefore $\rho^{\mathsf{T}} G \rho > 0$ unless $\rho = 0$, i.e., $G \succ 0$.
- *C9.4c holds*: As for the case of C9.4b, we must have $\rho_j = \rho_1$ for all $j \in \overline{N}$. Then the second summation in (9.7c) becomes, if $\rho_1 \neq 0$,

  $$\sum_{(j,k)\in E} \left(\rho_j^{\mathsf{T}} g_{jk}^m \rho_j + \rho_k^{\mathsf{T}} g_{kj}^m \rho_k\right) \;\geq\; \rho_1^{\mathsf{T}} \left(g_{j'k'}^m + g_{k'j'}^m\right) \rho_1 \;>\; 0$$

  where the last inequality follows because $g_{j'k'}^m$ and $g_{k'j'}^m$ are positive semidefinite and their null spaces intersect only at the origin. Therefore $\rho^{\mathsf{T}} G \rho > 0$ unless $\rho = 0$, i.e., $G \succ 0$.

Hence in all three cases G is positive definite. Since Y is complex symmetric and Y_{22} is nonsingular by assumption, Theorems 4.2 and 4.9 complete the proof. □

Consider the following conditions on the conductances $b^s_{jk}, b^m_{jk}, b^m_{kj} \in \mathbb{R}^{3\times 3}$:

C9.5: For all lines $(j,k) \in E$, $b^s_{jk} \preceq 0$, $b^m_{jk} \preceq 0$, $b^m_{kj} \preceq 0$.

C9.6a: For all buses $j \in \overline{N}$, $b^m_{jj} := \sum_{k:k\sim j} b^m_{jk} \prec 0$, i.e., for all j, there exists a line $(j,k) \in E$ such that $b^m_{jk} \prec 0$

C9.6b: For all lines $(j,k) \in E$, $b^s_{jk} \prec 0$. Furthermore there exists a line $(j',k') \in E$ such that $b^m_{j'k'} \prec 0$.

C9.6c: For all lines $(j,k) \in E$, $b^s_{jk} \prec 0$. Furthermore there exists a line $(j',k') \in E$ such that the intersection of the null spaces of $b^m_{j'k'}$ and $b^m_{k'j'}$ is $\{0\}$.

Condition C9.6b is a special case of C9.6c which does not require negative definiteness of b^m_{jk}. The next result extends Theorems 4.2, 4.4, and 4.9 in Chapter 4.2.5 from single-phase to three-phase networks. Its proof is left as Exercise 9.2.

Theorem 9.2. Suppose the network is connected and the admittance matrix $Y \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ satisfies C9.2. If the susceptance matrices $b^s_{jk}, b^m_{jk}, b^m_{kj} \in \mathbb{R}^{3\times 3}$ satisfy conditions C9.5 and one of C9.6a, C9.6b, C9.6c, then

1. The admittance matrix $Y^{-1} \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ exists and is symmetric. Moreover $\mathrm{Im}(Y) \prec 0$ and $\mathrm{Im}(Y^{-1}) \succ 0$.

In addition if $Y =: \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathsf{T}} & Y_{22} \end{bmatrix}$ with invertible Y_{22}, then

2. The Schur complement $Y/Y_{22} := Y_{11} - Y_{12}Y_{22}^{-1}Y_{12}^{\mathsf{T}}$ of Y_{22} is symmetric and invertible. Moreover $\mathrm{Im}(Y/Y_{22}) \prec 0$ but $\mathrm{Im}\big((Y/Y_{22})^{-1}\big) \succ 0$.

□

The conditions in Theorem 9.1 not only ensure $\mathrm{Re}(Y) \succ 0$ and those in Theorem 9.2 not only ensure $\mathrm{Im}(Y) \prec 0$. Each set of conditions also ensures $\alpha^{\mathsf{H}} Y \alpha \neq 0$ for any nonzero $\alpha \in \mathbb{C}^{3(N+1)}$ (Exercise 9.3). Since a necessary condition for Y to be singular is the existence of a nonzero α with $\alpha^{\mathsf{H}} Y \alpha = 0$, these conditions imply the invertibility of Y, as expected, and extend the sufficient conditions in Theorems 4.3 and 4.4 to three-phase networks.

Remark 9.2. The admittance matrix of a three-phase transformer involving Δ configuration is singular (see (8.12) or (8.32)). This causes the admittance matrix Y of a network that contains such transformers to be singular. A proposal in the literature is to add a small shunt admittance (diagonal entries) to the admittance matrix of such a transformer to make it nonsingular. □

Invertibility of Y_{22} when $y^m_{jk} = y^m_{kj} = 0$. Let $A \subsetneq \overline{N}$ and Y_A be the $3|A| \times 3|A|$ principal submatrix of Y consisting of row and column blocks Y_{jk} with $j,k \in A$. Suppose the shunt admittances are zero, $y^m_{jk} = y^m_{kj} = 0$ so that the admittance matrix Y has zero block row sums and is not invertible. The next result provides a set of simple sufficient conditions for a principal submatrix Y_A to be invertible when A is a *strict* subset of $\overline{N}$. Its proof is similar to those of Theorems 4.5 and 4.6 and left as Exercise 9.4.

Theorem 9.3. Suppose the network is connected and the admittance matrix $Y \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ satisfies C9.2. Suppose $y^m_{jk} = y^m_{kj} = 0$ for all lines $(j,k) \in E$. Consider the principal submatrix $Y_A \in \mathbb{C}^{3|A|\times 3|A|}$ for a strict subset $A \subsetneq \overline{N}$.

1. If $g^s_{jk} \succ 0$ for all lines $(j,k) \in E$ then Y_A^{-1} exists and is symmetric. Moreover both $\mathrm{Re}(Y_A) \succ 0$ and $\mathrm{Re}(Y_A^{-1}) \succ 0$.

2. If $b^s_{jk} \prec 0$ for all lines $(j,k) \in E$ then Y_A^{-1} exists and is symmetric. Moreover $\mathrm{Im}(Y_A) \prec 0$ but $\mathrm{Im}(Y_A^{-1}) \succ 0$.

Even when not all g^s_{jk} are positive definite and not all b^s_{jk} are negative definite the admittance matrix Y can still be invertible because they cannot be zero simultaneously. The next result extends Theorem 4.8 from single-phase to three-phase setting.

Theorem 9.4. Suppose the network is connected and the admittance matrix $Y \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ satisfies C9.2. Suppose $y^m_{jk} = y^m_{kj} = 0$ for all lines $(j,k) \in E$. If $g^s_{jk} \succeq 0$ and $b^s_{jk} \preceq 0$ for all lines $(j,k) \in E$ then the principal submatrix $Y_A \in \mathbb{C}^{3|A|\times 3|A|}$ for a strict subset $A \subsetneq \overline{N}$ satisfies:

1. $\mathrm{Re}(Y_A) \succeq 0$, $\mathrm{Im}(Y_A) \preceq 0$.

2. Moreover $\mathrm{Re}(Y_A) - \mathrm{Im}(Y_A) \succ 0$.

3. Y_A^{-1} exists and is symmetric.

Proof. The proof of Theorem 4.8 for single-phase network shows that G_A is diagonally dominant since $g^s_{jk} \in \mathbb{R}$ are nonnegative and hence its eigenvalues are nonnegative by the the Geršgorin disc theorem. In the three-phase case, we cannot use this argument since not every element of the 3×3 conductance matrix g^s_{jk} is nonnegative. We will use the argument in the proof of Theorem 9.1 (see (9.7)): for any real vector $\rho =: (\rho_j, j \in A)$ with $\rho_j \in \mathbb{R}^3$ we have, using $G_A := \mathrm{Re}(Y_A)$,

$$\begin{aligned}
\rho^{\mathsf{T}} G_A \rho &= \sum_j \sum_{k:k\sim j} \left(\rho_j^{\mathsf{T}} g^s_{jk} \rho_j - \rho_j^{\mathsf{T}} g^s_{jk} \rho_k \right) \\
&= \sum_{(j,k)\in E} \left(\rho_j^{\mathsf{T}} g^s_{jk} \rho_j - \rho_j^{\mathsf{T}} g^s_{jk} \rho_k - \rho_k^{\mathsf{T}} g^s_{kj} \rho_j + \rho_k^{\mathsf{T}} g^s_{kj} \rho_k \right) \\
&= \sum_{(j,k)\in E} (\rho_j - \rho_k)^{\mathsf{T}} g^s_{jk} (\rho_j - \rho_k)
\end{aligned}$$

where the last equality has used $g^s_{jk} = g^s_{kj}$ for all $(j,k) \in E$ from C9.2. Since $g^s_{jk} \succeq 0$, $\rho^{\mathsf{T}} G_A \rho \geq 0$ for any ρ, i.e., $G_A \succeq 0$. Similar, using $B_A := \mathrm{Im}(Y_A)$, we have

$$\rho^{\mathsf{T}} B_A \rho \;=\; \sum_j \sum_{k:k\sim j} \left(\rho_j^{\mathsf{T}} b^s_{jk} \rho_j - \rho_j^{\mathsf{T}} b^s_{jk} \rho_k\right) \;=\; \sum_{(j,k)\in E} (\rho_j - \rho_k)^{\mathsf{T}} b^s_{jk} (\rho_j - \rho_k)$$

Therefore $\rho^{\mathsf{T}} B_A \rho \leq 0$ since $b^s_{jk} \preceq 0$, i.e., $B_A \preceq 0$. This implies that $G_A - B_A \succeq 0$.

We now show that, indeed, $G_A - B_A \succ 0$ because the network is connected and $A \subset \overline{N}$ is a strict subset. The argument is the same as that for Theorem 4.8 for single-phase networks. For a $3n \times 3n$ matrix M, let $M[j,k]$ denote the 3×3 submatrix of M consisting of the jth row block and the kth row column. Since $G_A - B_A$ is real symmetric, consider, for any nonzero real vector $\rho \in \mathbb{R}^{3|A|}$,

$$\begin{aligned}
\rho^{\mathsf{T}} (G_A - B_A) \rho \;&=\; \sum_{j\in A} \sum_{k\in A} \rho_j^{\mathsf{T}} (G_A[j,k] - B_A[j,k]) \rho_k \\
&=\; \sum_{j\in A} \sum_{\substack{k\in A:\\ (j,k)\in E}} \rho_j^{\mathsf{T}} (-g^s_{jk} + b^s_{jk}) \rho_k \;+\; \sum_{j\in A} \rho_j^{\mathsf{T}} \left(\sum_{\substack{k\in A:\\ (j,k)\in E}} (g^s_{jk} - b^s_{jk}) + \sum_{\substack{k\notin A:\\ (j,k)\in E}} (g^s_{jk} - b^s_{jk}) \right) \rho_j \\
&=\; \sum_{\substack{j,k\in A:\\ (j,k)\in E}} \left(\rho_j - \rho_k\right)^{\mathsf{T}} (g^s_{jk} - b^s_{jk}) \left(\rho_j - \rho_k\right) \;+\; \sum_{j\in A} \rho_j G_j \rho^{\mathsf{T}}
\end{aligned}$$

where the third equality has used $g^s_{jk} = g^s_{kj}$ for all $(j,k) \in E$ from C9.2. Here $G_j := \sum_{k\notin A:(j,k)\in E} (g^s_{jk} - b^s_{jk})$ for $j \in A$ and the summation is not vacuous because the network is connected and $A \subsetneq \overline{N}$. For every line $(j,k) \in E$, $y^s_{jk} \neq 0$ and hence $g^s_{jk} - b^s_{jk} \succ 0$ since $g^s_{jk} \succeq 0$ and $b^s_{jk} \succeq 0$. This implies $G_j \succ 0$ as well for all $j \in A$. Therefore for $\rho^{\mathsf{T}} (G_A - B_A) \rho > 0$ for any real vector $\rho \neq 0$, i.e., $G_A - B_A \succ 0$.

Finally $G_A - B_A \succ 0$ implies that Y_A is nonsingular (it is clear that Y_A^{-1} is symmetric if it exists). The argument is exactly the same as that for Theorem 4.8 for single-phase networks. $\square$

Application: admittance matrix Y identification.

Uniform lines. Suppose all lines are of the same type specified by an impedance matrix y^{-1} per unit length. These lines differ only in their lengths. We will call y the *unit admittance*.[3] We show that this property is preserved under Schur complement. It means that the effective line admittances of the Kron-reduced admittance matrix Y/Y_A are also specified by the unit admittance y. This assumption makes the iterative construction of the Schur complement particularly simple.

Consider any $3(N+1) \times 3(N+1)$ complex symmetric matrix Y on a graph $G := (N,E)$ where its 3×3 (i,j)th blocks $Y[i,j]$ are given by:

$$Y[i,j] \;=\; \begin{cases} -\mu_{ij}\, y & (i,j) \in E \\ \left(\sum_{k:(i,k)\in E} \mu_{ik}\right) y & i = j \\ 0 & \text{otherwise} \end{cases} \tag{9.8}$$

3

where $y \in \mathbb{C}^{3\times 3}$ is complex symmetric. Suppose $\mathrm{Re}(y) \succ 0$ and $\mu_{ij} > 0$ for all $(i, j) \in E^0$. Then Theorem 4.2 implies that y^{-1} exists, is symmetric, and $\mathrm{Re}(y^{-1}) \succ 0$. Kron reduction preserves this structure.

Theorem 9.5. Suppose $\mathrm{Re}(y) \succ 0$ and $\mu_{ij} > 0$ for all $(i, j) \in E^0$ in the complex symmetric matrix Y defined in (9.8). Let $Y =: \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^{\mathsf{T}} & Y_{22} \end{bmatrix}$ with a $3n \times 3n$ nonsingular submatrix Y_{22}, $1 \le n \le N$.

1. The 3×3 (i, j)th blocks $(Y/Y_{22})[i, j]$ of the Schur complement Y/Y_{22} of Y_{22} of Y are given by

$$(Y/Y)[i,j] \;=\; \begin{cases} -\tilde{\mu}_{ij}\, y & i \rightsquigarrow j \\ \left(\sum_{k: i \rightsquigarrow k} \tilde{\mu}_{ik}\right) y & i = j \\ 0 & \text{otherwise} \end{cases} \tag{9.9}$$

   for some $\tilde{\mu}_{ij} = \tilde{\mu}_{ji} > 0$. Here $i \rightsquigarrow j$ if and only if there is a path in the underlying graph G connecting nodes i and j.

2. If the network is connected and the admittance matrix Y satisfies C9.2, then $(Y/Y_{22})^{-1}$ exists and is symmetric, and both $\mathrm{Re}(Y/Y_{22}) \succ 0$ and $\mathrm{Re}(Y/Y_{22})^{-1} \succ 0$.

Proof. The Schur complement Y/Y_{22} is the admittance matrix describing the effective connectivity between nodes $1, \ldots, N-n+1$ obtained by eliminating interior nodes $N-n+2, \ldots, N+1$ by Kron reduction. We follow the approach of [64] to prove the theorem by induction on the interior nodes to be Kron reduced one by one. Define

$$A^0 := Y, \qquad A^1 := A^0/A^0[n,n], \qquad \cdots \qquad A^n := A^{n-1}/A^{n-1}[N-n+2, N-n+2] \;=\; Y/Y_{22}$$

i.e., A^{l+1} is the admittance matrix for the graph after the last node in A^l has been Kron reduced, and hence $Y/Y_{22} = A^n$. Define the set of lines in the graph underlying $A^0, A^1, \ldots, A^n$ by

$$E^0 := E, \qquad\qquad E^l := \left\{(i,j) : A^l[i,j] \neq 0\right\}, \qquad\qquad l = 1, \ldots, k$$

Hence these sets are well-defined given the matrices A^0, $A^1, \ldots, A^n$. For $0 < l < n$, let the induction hypothesis be

$$A^l[i,j] \;=\; \begin{cases} -\mu_{ij}^l\, y & (i,j) \in E^l \\ \left(\sum_{k:(i,k)\in E^l} \mu_{ik}^l\right) y & i = j \\ 0 & \text{otherwise} \end{cases} \tag{9.10}$$

for some $\mu_{ij}^l = \mu_{ji}^l > 0$. Clearly A^0 satisfies (9.10). Suppose A^l satisfies (9.10). We now prove that $A^{l+1} := A^l/A^l[N-l+1, N-l+1]$ satisfies (9.10).

The 3×3 (i, j)th block $A^{l+1}[i, j]$ is given by

$$A^{l+1}[i,j] \;=\; A^l[i,j] - A^l[i, N-l+1]\left(A^l[N-l+1, N-l+1]\right)^{-1} A^l[j, N-l+1] \tag{9.11}$$

We consider 6 cases by substituting the induction hypothesis (9.10) into (9.11):

1. If $(i,j) \in E^l$ but either $(i, N-l+1) \notin E^l$ or $(j, N-l+1) \notin E^l$ then, substituting the induction hypothesis (9.10) into (9.11), we have $A^{l+1}[i,j] = -\mu_{ij}^{l+1} y$ where $\mu_{ij}^{l+1} := \mu_{ij}^l > 0$.

2. If $(i,j) \notin E^l$ but both $(i, N-l+1) \in E^l$ and $(j, N-l+1) \in E^l$ then

$$A^{l+1}[i,j] \;=\; -\mu_{i(N-l+1)}^l y \left(\sum_{k:(k,N-l+1)\in E^l} \mu_{k(N-l+1)}^l y \right)^{-1} \mu_{j(N-l+1)}^l y \;=\; -\mu_{ij}^{l+1} y$$

   where

$$\mu_{ij}^{l+1} \;:=\; \mu_{i(N-l+1)}^l \mu_{j(N-l+1)}^l \left(\sum_{k:(k,N-l+1)\in E^l} \mu_{k(N-l+1)}^l \right)^{-1} \;>\; 0$$

3. If $(i,j) \in E^l$, $(i, N-l+1) \in E^l$ and $(j, N-l+1) \in E^l$ then

$$\begin{aligned} A^{l+1}[i,j] \;&:=\; -\mu_{ij}^l y - \mu_{i(N-l+1)}^l y \left(\sum_{k:(k,N-l+1)\in E^l} \mu_{k(N-l+1)}^l y \right)^{-1} \mu_{j(N-l+1)}^l y \\ &=\; -\mu_{ij}^{l+1} y \end{aligned}$$

   where

$$\mu_{ij}^{l+1} \;:=\; \mu_{ij}^l + \mu_{i(N-l+1)}^l \mu_{j(N-l+1)}^l \left(\sum_{k:(k,N-l+1)\in E^l} \mu_{k(N-l+1)}^l \right)^{-1} \;>\; 0$$

4. If $i = j$ but $(i, N-l+1) \notin E^l$ then $A^{l+1}[i,i] = \left(\sum_{k:(i,k)\in E^{l+1}} \mu_{ik}^{l+1} \right) y$ where $\mu_{ik}^{l+1} := \mu_{ik}^l > 0$.

5. If $i = j$ and $(i, N-l+1) \in E^l$ then

$$\begin{aligned} A^{l+1}[i,i] \;&:=\; \left(\sum_{k:(i,k)\in E^l} \mu_{ik}^l \right) y - \mu_{i(N-l+1)}^l y \left(\sum_{k:(k,N-l+1)\in E^l} \mu_{k(N-l+1)}^l y \right)^{-1} \mu_{i(N-l+1)}^l y \\ &=\; \left(\sum_{k:(i,k)\in E^{l+1}} \mu_{ik}^{l+1} \right) y \end{aligned}$$

   where $\mu_{ik}^{l+1} := \mu_{ik}^l > 0$ for $(i,k) \in E^l$ and $k = 1, \ldots, N-l+1$, and

$$\mu_{i(N-l+1)}^{l+1} \;:=\; \mu_{i(N-l+1)}^l \left(1 - \frac{\mu_{i(N-l+1)}^l}{\sum_{k:(k,N-l+1)\in E^l} \mu_{k(N-l+1)}^l} \right) \;>\; 0$$

6. Otherwise, $i \neq j$ and $(i,j) \notin E^l$ and $A^{l+1}[i,j] = 0$.

This completes the induction and the proof of part 1. Part 2 follows from $\mathrm{Re}(y) \succ 0$ and Theorem 9.1. $\square$

9.1.4 Vs relation

Power flow equations. The power flow equations that relate bus injections $s := \left(s_j, j \in \overline{N}\right)$ and voltages $V := \left(V_j, j \in \overline{N}\right)$ can be obtained by applying the derivation for single-phase systems to the single-phase equivalent network $G^{3\phi}$. In particular the bus injection model in complex form is defined by the following power flow equation that expresses power balance at each bus $j\phi$ in terms of the elements $Y_{j\phi,k\phi'}$ of the $3(N+1) \times 3(N+1)$ admittance matrix Y defined in (9.6):

$$s_j^{\phi} \;=\; \sum_{\substack{k\in\overline{N} \\ \phi'\in\{a,b,c\}}} Y_{j\phi,k\phi'}^{\mathsf{H}} V_j^{\phi} \left(V_k^{\phi'}\right)^{\mathsf{H}}, \qquad j \in \overline{N},\ \phi \in \{a,b,c\} \tag{9.12a}$$

This directly generalizes (4.20b) from the single-phase setting to the three-phase setting. To generalize (4.20a) to the three-phase setting note that

$$s_j \;=\; \sum_{k:j\sim k} \operatorname{diag}\left(V_j I_{jk}^{\mathsf{H}}\right), \qquad j \in \overline{N}$$

where $s_j, V_j, I_{jk} \in \mathbb{C}^3$ are power injections, voltages, and line currents in all phases. We then have from (8.4)

$$s_j \;=\; \sum_{k:j\sim k} \operatorname{diag}\left(V_j (V_j - V_k)^{\mathsf{H}} \left(y_{jk}^s\right)^{\mathsf{H}} + V_j V_j^{\mathsf{H}} \left(y_{jk}^m\right)^{\mathsf{H}}\right), \qquad j \in \overline{N} \tag{9.12b}$$

Power flow analysis and optimization for unbalanced three-phase networks can be conducted using both forms of the bus injection model (9.12). In particular (9.12b) will be used in Chapter 10.1 to prove the equivalence of the branch flow model and the bus injection model (Theorem 10.1). The model (9.12) does not require condition C9.1 nor C9.2.

9.1.5 Overall model

Most power flow analysis or optimization applications involve three-phase devices, either in Y or Δ configuration, connected by three-phase lines. The lines may not be phase-decoupled and the sources and loads may not be balanced. In this subsection we compose an overall model consisting of the device modes of Chapter 7.3 and the network equations of this section. We use this overall model to formulate a general three-phase analysis problem in the next section.

The overall model consists of:

1. A network model that relates terminal voltage, current, and power (V, I, s). Any equivalent model can be used, whichever is convenient for the problem under study, including:
   - The (linear) current balance equation (9.5)(9.6).
   - The (quadratic) power flow equation that defines the BIM model (9.12).
2. A device model for each three-phase device j. For ideal devices, this can either be:

- Its internal model (7.29) and the conversion rules (7.8) and (7.9)(7.10); or
- Its external model summarized in Tables 7.3 and 7.4 when only terminal quantities are needed.

For non-ideal devices, this can either be:

- Its internal model summarized in Table 7.2 and the conversion rules (7.8) and (7.9)(7.10); or
- Its external model summarized in Table 7.2 when only terminal quantities are needed.

If only voltage sources, current sources and impedances are involved then the overall model is linear, consisting of the nodal current balance equation (9.5)(9.6) and linear device models. If power sources are also involved then, even though (9.5)(9.6) can still be used as the network model, the overall model will be nonlinear because of nonlinear power source models.

9.2 Three-phase analysis

A device model relates its internal and terminal variables. A network equation relates the terminal variables of these devices. A typical three-phase analysis problem is: given a collection of voltage sources, current sources, power sources and impedances connected by three-phase lines, compute a certain set of external and internal variables. We first illustrate this in Chapter 9.2.1 using examples. We then formulate in Chapter 9.2.2 a general three-phase analysis problem and outline in Chapter 9.2.3 a solution strategy based on intuitions from these examples.

9.2.1 Examples

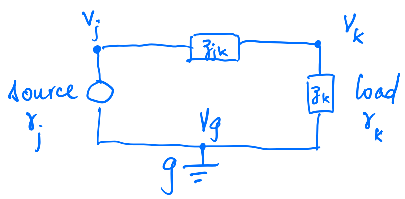
Three-phase analysis or optimization problems in practice are large-scale and can only be solved numerically. The goal of analyzing small examples is to gain intuition on how to specify these problems using the models developed in this chapter and illustrate their structure.

Consider a network of three-phase sources and impedances connected by three-phase lines. Assume without loss of generality that there is exactly one device at each bus j. The quantities of interest include the internal variables $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}, \beta_j\right)$ and the terminal variables $\left(V_j, I_j, s_j, \gamma_j\right)$ at each bus j. The first set of examples is driven by voltage and current sources and the second set by power sources as well. In these examples we specify the parameters of a set of (ideal) devices and our objective is to compute the remaining internal and terminal voltages, currents, and powers.

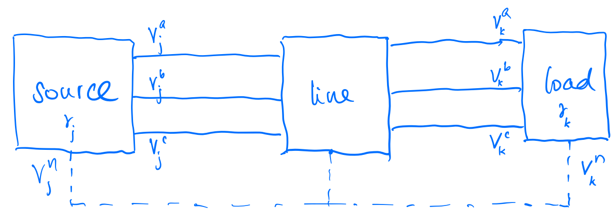
The general analysis problem we formulate in Chapter 9.2.3 will specify γ_j for all voltage sources. The first example shows how γ_j arises in a circuit.

Example 9.2 (Reference voltage and γ_j)**.** We start with a single-phase circuit shown in Figure 9.4(a) where the source can be a voltage, current, or power source, the load is an impedance z_k, and the line is a series impedance z_{jk}. The terminal voltages $\left(V_j, V_k, V_g\right)$ are defined with respect to an arbitrary but fixed reference point. The defining equations are

$$V_j - V_k = z_{jk} I_{jk}, \qquad\qquad V_k - V_g = z_k I_{jk}$$



(a) Single-phase network



(b) Three-phase network

Figure 9.4: Reference voltage and constant γ_j

Suppose the source is a current source with a given J_j from g to terminal j. Then the solution is:

$$I_{jk} = J_j, \qquad V_j = \left(z_{jk}+z_k\right) J_j + V_g \qquad V_k = z_k J_j + V_g$$

The terminal voltages depend on the choice of the reference point through the ground voltage V_g. For this example, $\gamma_j = \gamma_k = V_g$. In particular, if γ_j at the source is specified then γ_k at the load is fixed and the voltages (V_j, V_k) are uniquely determined. If we choose the reference point to be the ground then $\gamma_j = \gamma_k = V_g = 0$.

Consider now a three-phase system shown in Figure 9.4(b) where a device may or may not have a neutral line and the neutrals may or may not be grounded, directly or through an impedance. The voltage conversion rule between internal and terminal voltages for Y and Δ configured devices is (7.8)(7.9), reproduced here:

$$V_j = V_j^Y + \gamma_j \mathbf{1}, \qquad V_j^\Delta = \Gamma V_j \quad \text{or equivalently} \quad V_j = \Gamma^\dagger V_j^\Delta + \gamma_j \mathbf{1}$$

For Y-configured devices, $\gamma_j = V_j^n$, i.e., their neutral voltages with respect to the reference point. In general we need two of $\left(V_j, V_j^Y, \gamma_j\right)$ to determine the third. For Δ-configured devices, γ_j can be determined by specifying one of $\left(V_j^a, V_j^b, V_j^c\right)$ for each device j. Knowing the vector V_j is sufficient to determine both the internal voltage V_j^Δ and γ_j. Knowing V_j^Δ however is not sufficient to determine V_j without γ_j. This is studied in detail in the next few examples. □

Voltage and current sources. For a network driven by constant voltage and current sources without power sources, both the device models and the network equation $I = YV$ are linear. We will therefore focus on linear analysis to compute terminal and internal voltages and currents. Given (V_j, I_j) and $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}\right)$, external and internal powers can be computed. As we will see, the key step in our analysis is to solve for the *internal* currents $I_k^{Y/\Delta}$ of all impedances k, together with other quantities such as the terminal voltages V_j of current sources j, using the network equation, internal models of impedances and the voltage and current conversion rules. All other variables can then be derived. This solution strategy is extended in Chapter 9.2.3 to general three-phase networks.

Example 9.3 (Generator/load in Y configuration)**.** Consider the system in Figure 9.5 where an (ideal) voltage source is connected through a three-phase line to an impedance, both in Y configuration. We assume the neutrals are not grounded and there is not a neutral line. Suppose the following are specified:

- Voltage source $\left(E_j^Y, \gamma_j := V_j^n\right)$.
- Impedance $\left(z_k^Y, \gamma_k := V_k^n\right)$.
- Line parameters $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$. In particular assumption C9.1 is satisfied.

Derive the terminal and internal voltages and currents $\left(V_k, I_k, V_k^Y, I_k^Y\right)$ of the impedance.

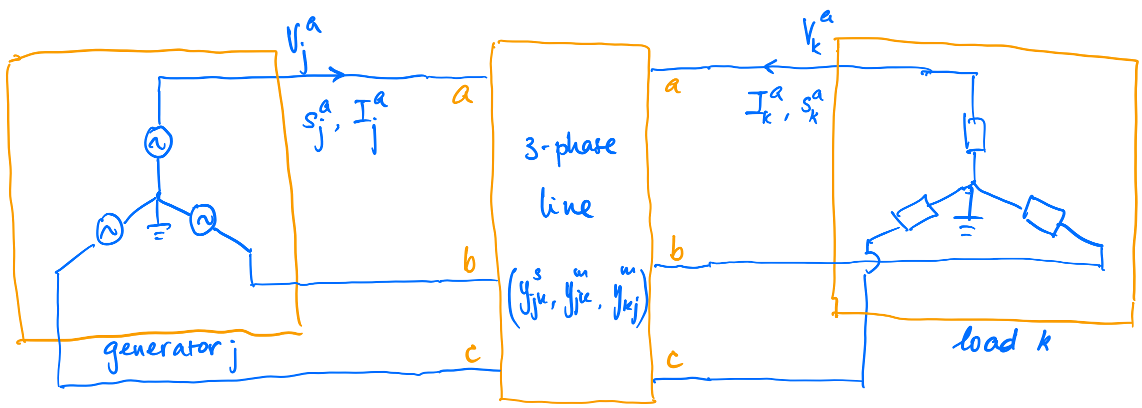


Figure 9.5: Example 9.3: A Y-configured generator connected through a three-phase line to a Y-configured impedance load.

Solution. The terminal voltages (V_j, V_k) and current injections (I_j, I_k) are related according to (9.5):

$$I_j \;=\; y_{jk}^s\left(V_j - V_k\right) \;+\; y_{jk}^m V_j \tag{9.13a}$$

$$I_k \;=\; y_{jk}^s\left(V_k - V_j\right) \;+\; y_{kj}^m V_k \tag{9.13b}$$

From Table 7.3, the external models for the ideal voltage source and impedance in Y configuration are

$$V_j \;=\; E_j^Y + \gamma_j, \qquad\qquad V_k \;=\; -z_k^Y I_k + \gamma_k \mathbf{1} \tag{9.13c}$$

This is a system of 12 linear equations in 12 unknowns $\left(V_j, I_j\right)$ and (V_k, I_k).

Substituting V_j from (9.13c) and the current conversion rule $I_k^Y = -I_k$ into (9.13b) we have

$$-I_k^Y \;=\; -y_{jk}^s\left(E_j^Y + \gamma_j\right) + \left(y_{jk}^s + y_{kj}^m\right) V_k \tag{9.14a}$$

Substituting V_k from (9.13c) we have

$$\left(\left(y_{jk}^s + y_{kj}^m\right)^{-1} + z_k^Y\right) I_k^Y \;=\; \left(y_{jk}^s + y_{kj}^m\right)^{-1} y_{jk}^s V_j \;-\; \gamma_k \mathbf{1} \tag{9.14b}$$

Hence

$$\begin{aligned} I_k^Y \;=\; -I_k \;&=\; \left(\hat{z}_{jk} + z_k^Y\right)^{-1} \hat{z}_{jk} y_{jk}^s V_j \;-\; \gamma_k \left(\hat{z}_{jk} + z_k^Y\right)^{-1} \mathbf{1} \\ &=\; \left(z_k^Y \;+\; z_{jk}^s \;+\; z_{jk}^s y_{kj}^m z_k^Y\right)^{-1} V_j \;-\; \gamma_k \left(\hat{z}_{jk} + z_k^Y\right)^{-1} \mathbf{1} \end{aligned}$$

where $z^s_{jk} := \left(y^s_{jk}\right)^{-1}$, $\hat{z}_{jk} := \left(y^s_{jk} + y^m_{kj}\right)^{-1}$ and $V_j = E^Y_j + \gamma_j$. From (9.13c)

$$V^Y_k \;=\; z^Y_k I^Y_k \;=\; z^Y_k \left(z^Y_k \,+\, z^s_{jk} \,+\, z^s_{jk} y^m_{kj} z^Y_k\right)^{-1} V_j - \gamma_k z^Y_k \left(\hat{z}_{jk} + z^Y_k\right)^{-1} \mathbf{1}$$
$$V_k \;=\; V^Y_k + \gamma_k \mathbf{1} \;=\; z^Y_k \left(z^Y_k \,+\, z^s_{jk} \,+\, z^s_{jk} y^m_{kj} z^Y_k\right)^{-1} V_j + \gamma_k \left(\mathbb{I} - \left(\hat{z}_{jk} + z^Y_k\right)^{-1}\right) \mathbf{1}$$

□

In Example 9.3 the neutral voltages γ_j, γ_k are given explicitly. Often some of them are not explicitly given but additional information is available to indirectly specify them, i.e., to either compute their values, provide additional equations, or eliminate them in terms of other variables. For instance, if a neutral at bus j is grounded with zero grounding impedance and voltages are defined with respect to the ground then $\gamma_j = 0$. The next two examples study this in more detail. In Example 9.4, γ_k of the impedance z^Y_k is not explicitly given, but the additional information shows that its terminal voltage and current satisfy $V_k = -Z^Y_k I_k$; see (9.15). This means that the external model of the impedance is equivalent to that of an impedance with an effective internal impedance Z^Y_k with a known neutral voltage $\gamma_k = 0$. (See also Exercise 9.7 for another four-wire example).

Example 9.4 (Indirect specification of $\gamma_k = V^n_k$)**.** Repeat Example 9.3 with the modification that the impedance is specified only by z^Y_k (i.e., γ_k is not specified), and that the neutral of the impedance is connected through a given impedance z^n_k to the ground and not to the voltage source.

Solution. The equations (9.13) in Example 9.3 is now a system of 4 vector linear equations in 4 vector unknowns $\left(V_j, V_k, I_j, I_k\right)$ and a scalar unknown, the unspecified neutral voltage $\gamma_k := V^n_k$ of the impedance, one more unknown than in Example 9.3. Since the neutral of the impedance is connected only to the ground (and not to the voltage source) through the impedance z^n_k, KCL and Ohm's law provide the additional equation

$$\gamma_k \;:=\; V^n_k \;=\; -z^n_k \left(\mathbf{1}^\mathsf{T} I_k\right)$$

Substituting into γ_k in (9.13c) we have $V_k = -z^Y_k I_k - z^n_k \mathbf{1}\mathbf{1}^\mathsf{T} I_k$. Hence the external device model (9.13c) in Example 9.3 can be replaced by

$$V_j \;=\; E^Y_j + \gamma_j \mathbf{1}, \qquad\qquad V_k \;=\; -\underbrace{\left(z^Y_k I_k + z^n_k \mathbf{1}\mathbf{1}^\mathsf{T}\right)}_{Z^Y_k} I_k \tag{9.15}$$

It says that the external behavior of the impedance z^Y_k when its neutral is grounded through z^n_k is equivalent to an impedance with an effective admittance Z^Y_k that is grounded directly so that $\gamma_k := V^n_k = 0$. The same computation leads to the same solution for (V_k, I_k) with the following replacement:

$$z^Y_k \;\rightarrow\; Z^Y_k, \qquad\qquad \gamma_k \;\rightarrow\; 0$$

□

The next example illustrates the case where the neutrals are not grounded but connected directly to each end of a four-wire line (also see Exercise 9.7). In this case, neither γ_j nor γ_k needs to be explicitly specified and can be determined from the network equation $I = YV$. This is an example where γ_j of a voltage source cannot be specified arbitrarily but is constrained by the network equation, in contrast to the three-wire models of Examples 9.3 and 9.4. This is because, when the neutral of the voltage source j is not grounded nor connected to bus k, the current I_j is determined only by (V_j, V_k) through (9.13a) and γ_j can be arbitrary. With the neutral wire, the additional constraint $I_j^{n'} = \mathbf{1}^{\mathsf{T}} I_j$ determines γ_j uniquely. Similarly for γ_k for the impedance.

Example 9.5 (Four-wire model)**.** Repeat Example 9.3 with the modification that the neutrals of both devices are ungrounded and are connected to the neutral wires at each end of a 4-wire line; see Figure 9.6. Suppose the following are specified:

- Voltage source E_j^Y.
- Impedance z_k^Y.
- Line parameters $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$. In particular assumption C9.1 is satisfied.

Note that neither γ_j nor γ_k is explicitly specified.

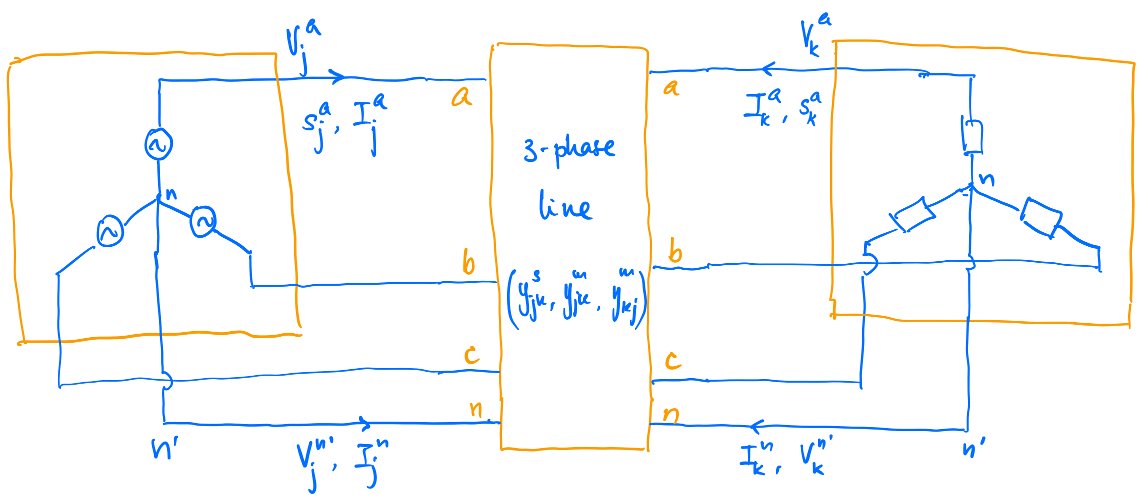


Figure 9.6: Example 9.5: A Y-configured generator connected through a four-wire line to a Y-configured impedance load.

Solution. To indicate the direction of internal currents on the neutral lines, we will use n to denote the internal neutral of a device and n' to denote the external terminal of the neutral line. In this example, $n' = n$ in the sense that $V_j^{n'n} = V_k^{n'n} = 0$. See Exercise 9.7 for the case where the neutrals of the voltage source and the load are connected through internal impedances $\left(z_j^n, z_k^n\right)$ to each end of the four-wire line, so $\left(V_j^{n'n}, V_k^{n'n}\right)$ may not be zero.

Define the terminal voltages (with respect to a common reference point) and currents in $\mathbb{C}^4$:

$$\hat{V}_j \;:=\; \begin{bmatrix} V_j^a \\ V_j^b \\ V_j^c \\ V_j^{n'} \end{bmatrix}, \qquad \hat{V}_k \;:=\; \begin{bmatrix} V_k^a \\ V_k^b \\ V_k^c \\ V_k^{n'} \end{bmatrix}, \qquad \hat{I}_j \;:=\; \begin{bmatrix} I_j^a \\ I_j^b \\ I_j^c \\ I_j^n \end{bmatrix}, \qquad \hat{I}_k \;:=\; \begin{bmatrix} I_k^a \\ I_k^b \\ I_k^c \\ I_k^n \end{bmatrix}$$

As noted above, $z_k^n = 0$ implies that $\gamma_j := V_j^n = V_j^{n'}$ and $\gamma_k := V_k^n = V_k^{n'}$ are variables to be determined. These terminal variables are related by $\hat{I} = \hat{Y}\hat{V}$ as in (9.13a) (9.13b), except that the admittance matrices are replaced by their four-wire counterparts:

$$\hat{I}_j \;=\; \hat{y}_{jk}^s\left(\hat{V}_j - \hat{V}_k\right) \;+\; \hat{y}_{jk}^m \hat{V}_j, \qquad \hat{I}_k \;=\; \hat{y}_{jk}^s\left(\hat{V}_k - \hat{V}_j\right) \;+\; \hat{y}_{kj}^m \hat{V}_k \tag{9.16a}$$

The external model of a four-wire voltage source in Y configuration is, since the neutrals are ungrounded and connected to each other,

$$\hat{V}_j \;=\; \begin{bmatrix} E_j^{an} + V_j^n \\ E_j^{bn} + V_j^n \\ E_j^{cn} + V_j^n \\ V_j^n \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} E_j^Y \\ 0 \end{bmatrix}}_{\hat{E}_j^Y} + \gamma_j \hat{\mathbf{1}} \;=:\; \hat{E}_j^Y \;+\; \gamma_j \hat{\mathbf{1}}, \qquad I_j^n \;=\; \mathbf{1}^{\mathsf{T}} I_j \tag{9.16b}$$

where $\hat{\mathbf{1}}$ is the vector of all 1s of size 4 and $I_j := (I_j^a, I_j^b, I_j^c)$. Similarly the internal model of a four-wire impedance in Y configuration is, since the neutrals are ungrounded and connected to each other,

$$\hat{V}_k \;=\; \begin{bmatrix} z_k^{an} I_k^{an} \\ z_k^{bn} I_k^{bn} \\ z_k^{cn} I_k^{cn} \\ 0 \end{bmatrix} + \gamma_k \hat{\mathbf{1}} \;=\; -\begin{bmatrix} z_k & 0 \\ 0 & 0 \end{bmatrix} \hat{I}_k \;+\; \gamma_k \hat{\mathbf{1}}, \qquad I_k^n \;=\; \mathbf{1}^{\mathsf{T}} I_k \tag{9.16c}$$

This is a set of 18 linear equations in 18 unknowns $\left(\hat{V}_j, \hat{I}_j, \gamma_j\right)$ and $\left(\hat{V}_k, \hat{I}_k, \gamma_k\right)$. It replaces (9.13) when neutrals are ungrounded and unconnected to each other and γ_j, γ_k must be given explicitly. It can be solved as in Example 9.3.

Exercise 9.6 expresses (γ_j, γ_k) in terms of the phase voltages and currents $\left(V_j, V_k, I_j, I_k\right)$. □

The next example considers the setup of Example 9.3 in Δ configuration when the load is supplied by a voltage source. Exercise 9.8 considers the Δ configuration when the load is supplied by a current source. A voltage source $\left(E_j^{\Delta}, \gamma_j, \beta_j\right)$ is fully specified. A current source only needs to specify its internal current J_j^{Δ} if shunt admittances of the line are nonzero. Otherwise its zero-sequence voltage γ_j also needs to be specified (see Exercise 9.8 and Remark 9.8). Neither the zero-sequence voltage nor the zero-sequence current (γ_k, β_k) of the load need to be specified. They will be derived from network equations. A more detailed comparison between Example 9.3 (voltage source) and Exercise 9.8 (current source) is given in Tables 9.1 and 9.2 and in Remark 9.3. We will also explain in Remark 9.6 in Chapter 9.2.2 the asymmetry in the specification of voltage and current sources in Δ configuration.

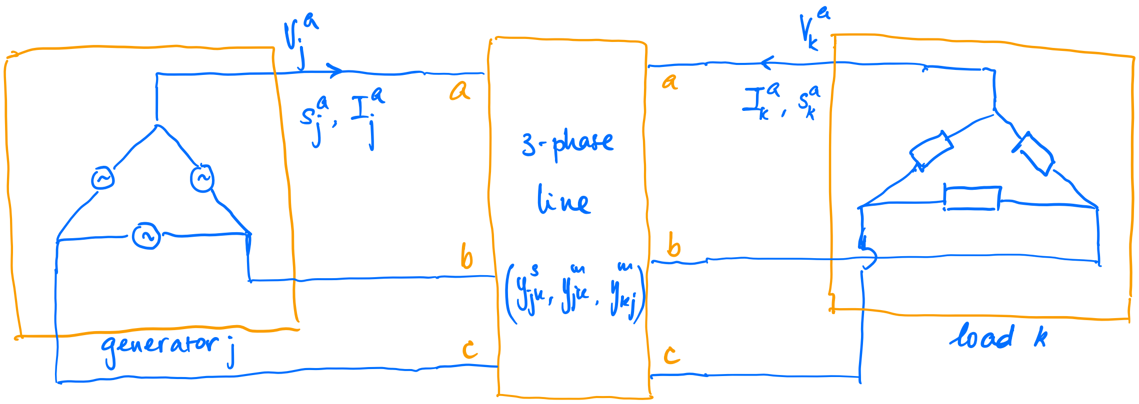


Figure 9.7: Example 9.6: Three-phase generator in Δ configuration connected through a three-phase line to an impedance load in Δ configuration.

Example 9.6 (Generator/load in Δ configuration)**.** Repeat Example 9.3 when the devices are in Δ configuration as shown in Figure 9.7, Suppose the following are specified:

- Voltage source $\left(E_j^\Delta, \gamma_j, \beta_j\right)$.
- Impedance z_k^Δ. (Note that the internal current β_k need not be specified and can be derived.)
- Line admittances $\left(\left(z_{jk}^s\right)^{-1}, y_{jk}^m = y_{kj}^m := 0\right)$. We have assumed assumption C9.1 and that shunt admittances are zero.

1. Compute all the other quantities in Table 9.1. In particular show that the internal voltage and current $\left(V_k^\Delta, I_k^\Delta\right)$ of the impedance depends only on E_j^Δ, but not on $\left(\gamma_j, \beta_j\right)$.
2. Show that $I_j^\Delta - I_k^\Delta = \delta \mathbf{1}$ for some $\delta \in \mathbb{C}$ when $\mathbf{1}\mathbf{1}^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$ is in span$(\mathbf{1})$ where $Z_{\text{Th}} := \Gamma z_{jk}^s \Gamma^\mathsf{T} + z_k^\Delta$.
3. Show that $\gamma_k = \gamma_j$ when the three-phase line is symmetric of the form in (8.5) with $z_{jk}^1 + 2z_{jk}^2 \neq 0$.
4. In deriving the impedance model (7.27b), we have shown that its internal variable β_k and terminal current I_k must satisfy $\beta_k = \frac{1}{\zeta_k}\left(\tilde{z}_k^{\Delta\mathsf{T}} \Gamma^{\mathsf{T}\dagger}\right) I_k$, where $\tilde{z}_k^\Delta := z_k^\Delta \mathbf{1}$ and $\zeta_k := \mathbf{1}^\mathsf{T} z_k^\Delta \mathbf{1}$. Verify this expressions using the answer to part 1.

Solution. We will derive the quantities in the following order: $E_j^\Delta \Rightarrow I_k^\Delta, V_k^\Delta \Rightarrow \beta_k, I_k, I_j$. Then $E_j^\Delta, \gamma_j \Rightarrow V_j, V_k, \gamma_k \Rightarrow I_j^\Delta$.

The current balance equation (9.5) with $y_{jk}^m = y_{kj}^m = 0$ is:

$$V_k \;=\; V_j - z_{jk}^s I_j$$

Multiplying both sides by Γ and substituting the conversion rule $V_k^\Delta = \Gamma V_k$, $E_j^\Delta = \Gamma V_j$, and $I_j = -I_k$, we have

$$V_k^\Delta \;=\; \Gamma V_k \;=\; E_j^\Delta + \Gamma z_{jk}^s I_k \tag{9.17}$$

Substitute the internal model $V_k^\Delta = z_k^\Delta I_k^\Delta$ of impedance and the conversion rule $I_k = -\Gamma^\mathsf{T} I_k^\Delta$ to get

$$\left(\Gamma z_{jk}^s \Gamma^\mathsf{T} + z_k^\Delta\right) I_k^\Delta \;=\; E_j^\Delta \tag{9.18}$$

Hence

$$I_k^\Delta \;=\; Z_{\text{Th}}^{-1} E_j^\Delta, \qquad\qquad V_k^\Delta \;=\; z_k^\Delta Z_{\text{Th}}^{-1} E_j^\Delta$$

where $Z_{\text{Th}} := \Gamma z_{jk}^s \Gamma^\mathsf{T} + z_k^\Delta$ is the Thévenin equivalent of the three-phase line and the three-phase impedance. The expression for V_k^Δ is the three-phase version of the voltage divider rule. Note that the internal variables $\left(V_k^\Delta, I_k^\Delta, \beta_j\right)$ of the impedance does not depend on γ_j.

We now calculate the other variables $\left(V_j, I_j, I_j^\Delta\right)$ and $(V_k, I_k, \gamma_k, \beta_k)$. The zero-sequence current and the terminal current of the impedance are

$$I_k \;=\; -\Gamma^\mathsf{T} I_k^\Delta \;=\; -\Gamma^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta, \qquad\qquad \beta_k \;:=\; \frac{1}{3}\mathbf{1}^\mathsf{T} I_k^\Delta \;=\; \frac{1}{3}\mathbf{1}^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$$

Using the external model of an ideal voltage source from Table 7.4 we have

$$V_j \;=\; \frac{1}{3}\Gamma^\mathsf{T} E_j^\Delta \;+\; \gamma_j \mathbf{1}, \qquad\qquad I_j \;=\; -I_k \;=\; \Gamma^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta \tag{9.19}$$

Hence

$$\begin{aligned} V_k \;&=\; V_j - z_{jk}^s I_j \;=\; \left(\frac{1}{3}\Gamma^\mathsf{T} - z_{jk}^s \Gamma^\mathsf{T} Z_{\text{Th}}^{-1}\right) E_j^\Delta \;+\; \gamma_j \mathbf{1} \\ \gamma_k \;&=\; \frac{1}{3}\mathbf{1}^\mathsf{T} V_k \;=\; \gamma_j \;-\; \frac{1}{3}\left(\mathbf{1}^\mathsf{T} z_{jk}^s \Gamma^\mathsf{T}\right) Z_{\text{Th}}^{-1} E_j^\Delta \end{aligned}$$

Since $-\Gamma^\mathsf{T} I_j^\Delta = I_j = \Gamma^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$ from (9.19) we have $\Gamma^\mathsf{T}\left(I_j^\Delta + Z_{\text{Th}}^{-1} E_j^\Delta\right) = 0$. Therefore (since the null space of Γ^T is $\text{span}(\mathbf{1})$)

$$I_j^\Delta \;=\; -Z_{\text{Th}}^{-1} E_j^\Delta + \beta_j' \mathbf{1}$$

where $\beta_j' \in \mathbb{C}$ is related to the given $\beta_j := \frac{1}{3}\mathbf{1}^\mathsf{T} I_j^\Delta$ by $\beta_j' = \beta_j + \frac{1}{3}\mathbf{1}^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$. Hence[4]

$$I_j^\Delta \;=\; -Z_{\text{Th}}^{-1} E_j^\Delta \;+\; \left(\frac{1}{3}\mathbf{1}^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta + \beta_j\right)\mathbf{1}$$

[4] Alternative derivation is: $-\Gamma^\mathsf{T} I_j^\Delta = \Gamma^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$ implies

$$I_j^\Delta \;=\; -\frac{1}{3}\Gamma\Gamma^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta \;+\; \beta_j \mathbf{1} \;=\; -Z_{\text{Th}}^{-1} E_j^\Delta \;+\; \frac{1}{3}\mathbf{1}\mathbf{1}^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta \;+\; \beta_j \mathbf{1}$$

where the last equality follows from $\frac{1}{3}\Gamma\Gamma^\mathsf{T} = \mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^\mathsf{T}$ by Theorem 7.2.

From the derivation above, $\gamma_k = \gamma_j$ if $\mathbf{1}^\mathsf{T} z^s_{jk} \Gamma^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta = 0$. When the line is symmetric of the form in (8.5) we have

$$\mathbf{1}^\mathsf{T} z^s_{jk} \;=\; \mathbf{1}^\mathsf{T} \begin{bmatrix} z^1_{jk} & z^2_{jk} & z^2_{jk} \\ z^2_{jk} & z^1_{jk} & z^2_{jk} \\ z^2_{jk} & z^2_{jk} & z^1_{jk} \end{bmatrix} \;=\; \left(z^1_{jk} + 2z^2_{jk}\right) \mathbf{1}^\mathsf{T}$$

Hence (since $z^1_{jk} + 2z^2_{jk} \neq 0$)

$$\mathbf{1}^\mathsf{T} z^s_{jk} \Gamma^\mathsf{T} \;=\; \left(z^1_{jk} + 2z^2_{jk}\right) \left(\mathbf{1}^\mathsf{T} \Gamma^\mathsf{T}\right) \;=\; 0$$

Finally we verify that the expressions $\beta_k = \frac{1}{3}\mathbf{1}^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$ and $I_k = -\Gamma^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$ satisfy $\beta_k = \frac{1}{\zeta_k}\left(\tilde{z}_k^{\Delta\mathsf{T}} \Gamma^{\mathsf{T}\dagger}\right) I_k$ where $\tilde{z}_k^\Delta := z_k^\Delta \mathbf{1}$ and $\zeta_k := \mathbf{1}^\mathsf{T} z_k^\Delta \mathbf{1}$. We have

$$\left(\tilde{z}_k^{\Delta\mathsf{T}} \Gamma^{\mathsf{T}\dagger}\right) I_k \;=\; -\tilde{z}_k^{\Delta\mathsf{T}} \left(\Gamma^{\mathsf{T}\dagger} \Gamma^\mathsf{T}\right) Z_{\text{Th}}^{-1} E_j^\Delta \;=\; -\tilde{z}_k^{\Delta\mathsf{T}} \left(\mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^\mathsf{T}\right) Z_{\text{Th}}^{-1} E_j^\Delta \;=\; -\tilde{z}_k^{\Delta\mathsf{T}} Z_{\text{Th}}^{-1} E_j^\Delta + \frac{\zeta_k}{3} \mathbf{1}^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$$

where the second equality follows from Theorem 7.2. But

$$\tilde{z}_k^{\Delta\mathsf{T}} Z_{\text{Th}}^{-1} E_j^\Delta \;=\; \mathbf{1}^\mathsf{T} z_k^\Delta Z_{\text{Th}}^{-1} E_j^\Delta \;=\; \mathbf{1}^\mathsf{T} V_k^\Delta \;=\; 0$$

where the last equality follows from (9.17). Hence $\left(\tilde{z}_k^{\Delta\mathsf{T}} \Gamma^{\mathsf{T}\dagger}\right) I_k = \zeta_k \beta_k$ as desired. □

Voltage source j			
V_j^Δ	given E_j^Δ	V_k^Δ	$z_k^\Delta I_k^\Delta = z_k^\Delta Z_{\text{Th}}^{-1} E_j^\Delta$
I_j^Δ	$-\left(\Gamma\Gamma^\dagger\right) Z_{\text{Th}}^{-1} E_j^\Delta + \beta_j \mathbf{1}$	I_k^Δ	$Z_{\text{Th}}^{-1} E_j^\Delta$
β_j	given	β_k	$\frac{1}{3}\mathbf{1}^\mathsf{T} I_k^\Delta = \frac{1}{3}\mathbf{1}^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$
V_j	$\Gamma^\dagger E_j^\Delta + \gamma_j$	V_k	$\left(\frac{1}{3}\Gamma^\mathsf{T} - z^s_{jk}\Gamma^\mathsf{T} Z_{\text{Th}}^{-1}\right) E_j^\Delta + \gamma_j \mathbf{1}$
I_j	$-I_k = \Gamma^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$	I_k	$-\Gamma^\mathsf{T} I_k^\Delta = -\Gamma^\mathsf{T} Z_{\text{Th}}^{-1} E_j^\Delta$
γ_j	given	γ_k	$\gamma_j - \frac{1}{3}\left(\mathbf{1}^\mathsf{T} z^s_{jk} \Gamma^\mathsf{T}\right) Z_{\text{Th}}^{-1} E_j^\Delta$

Table 9.1: Example 9.6: parameters and variables for a voltage source j where $Z_{\text{Th}} := \Gamma z^s_{jk} \Gamma^\mathsf{T} + z_k^\Delta$.

Remark 9.3 (Comprison: voltage vs current sources)**.** In both Example 9.6 and Exercise 9.8, the key to the derivation is to first calculate the internal current I_k^Δ of the impedance by relating it to the given source parameter E_j^Δ or J_j^Δ. Given I_k^Δ, all other variables can be derived. This insight will be used in Chapter 9.2.3 for analyzing a general three-phase problem.

Compare the results in Table 9.1 from Example 9.6 for the voltage source with the results in Table 9.2 from Exercise 9.8 for the current source.

1. The internal variables $\left(V_k^\Delta, I_k^\Delta, \beta_k\right)$ of the impedance do not depend on $\left(\gamma_j, \beta_j\right)$, but only on E_j^Δ for the voltage source and I_j^Δ for the current source.

Current source j			
V_j^Δ	$V_k^\Delta - \left(\Gamma z_{jk}^s \Gamma^\mathsf{T}\right) I_j^\Delta$	V_k^Δ	$z_k^\Delta I_k^\Delta \ = \ z_k^\Delta A(z_k^\Delta) J_j^\Delta$
I_j^Δ	given J_j^Δ	I_k^Δ	$A(z_k^\Delta) J_j^\Delta$
β_j	$\frac{1}{3}\mathbf{1}^\mathsf{T} J_j^\Delta$	β_k	$\left(\frac{\tilde{z}_k^\Delta}{\zeta_k} - \frac{\mathbf{1}}{3}\right)^\mathsf{T} J_j^\Delta$
V_j	$V_k + z_{jk}^s I_j = V_k - z_{jk}^s \Gamma^\mathsf{T} J_j^\Delta$	V_k	$\frac{1}{3}\Gamma^\mathsf{T} V_k^\Delta + \gamma_k \mathbf{1}$
I_j	$-\Gamma^\mathsf{T} J_j^\Delta$	I_k	$\Gamma^\mathsf{T} J_j^\Delta$
γ_j	given	γ_k	$\gamma_j + \frac{1}{3}\mathbf{1}^\mathsf{T} z_{jk}^s \Gamma^\mathsf{T} J_j^\Delta$

Table 9.2: Exercise 9.8: parameters and variables for a current source j where $\tilde{z}_k^\Delta := z_k^\Delta \mathbf{1}$, $\zeta_k := \mathbf{1}^\mathsf{T} z_k^\Delta \mathbf{1}$, and $A(z_k^\Delta) := \left(\frac{1}{\zeta_k}\mathbf{1}\tilde{z}_k^{\Delta\mathsf{T}} - \mathbb{I}\right)$.

2. For the current source, $I_k^\Delta = A(z_k^\Delta) J_j^\Delta$ depends only on the impedance z_k^Δ but not on the line series admittance y_{jk}^s. This is because of the assumption $z_{jk}^m = z_{kj}^m = 0$. For the voltage source, $I_k^\Delta = Z_{\text{Th}}^{-1} E_j^\Delta$ depends on both z_{jk}^s and z_k^Δ through their Thévenin equivalent. Their values are equal if $E_j^\Delta = Z_{\text{Th}} A\left(z_k^\Delta\right) J_j^\Delta$.
3. For both the voltage and current source, $\gamma_k = \gamma_j$ if z_{jk}^s is symmetric.
4. For the current source, the loop flows β_j and β_k are related as follows (see Exercise 9.8):
   - $\beta_k = -\beta_j$ if and only if $z_k^{ab} J_j^{ab} + z_k^{bc} J_j^{bc} + z_k^{ca} J_j^{ca} = 0$.
   - $\beta_k = 0$ if and only if $z_k^{ab} J_j^{ab} + z_k^{bc} J_j^{bc} + z_k^{ca} J_j^{ca} = \zeta_k \beta_j$.
   - $\beta_k = 0$ if the impedance $z_k^\Delta = \frac{\zeta_k}{3}\mathbb{I}$ is balanced, regardless of whether J_j^Δ is balanced or whether β_j is zero. The converse does not necessarily hold.

□

Example 9.7 (Balanced system)**.** Assume the system in Example 9.6 is a balanced system, i.e., given

- The voltage source parameters $\left(E_j^\Delta, \gamma_j, \beta_j\right)$ with $E_j^\Delta := \lambda_j \alpha_+$ where $\lambda_j \in \mathbb{C}$, $\alpha_+ := (1, \alpha, \alpha^2)$, and $\alpha := e^{-\mathbf{i}2\pi/3}$,
- The impedance $z_k^\Delta := \zeta_k' \mathbb{I}$ where $\zeta_k' \in \mathbb{C}$.
- Line admittances $\left(\left(z_{jk}^s\right)^{-1}, y_{jk}^m = y_{kj}^m := 0\right)$ with $z_{jk}^s = \zeta_{jk}\mathbb{I}$, i.e., the phases are decoupled.

1. Show that $Z_{\text{Th}} = \zeta_k' \mathbb{I} + \zeta_{jk}\Gamma\Gamma^\mathsf{T}$ and $Z_{\text{Th}}^{-1} = a\left(\mathbb{I} - \frac{a\zeta_{jk}}{3a\zeta_{jk}-1}\mathbf{1}\mathbf{1}^\mathsf{T}\right)$ where $a := 1/(\zeta_k' + 3\zeta_{jk})$.
2. Show that all variables $\left(V_j, V_k, I_j, I_k\right)$, $\left(V_k^\Delta, I_k^\Delta\right)$ are balanced positive-sequence sets.

Solution. By definition

$$Z_{\text{Th}} \;:=\; z_k^{\Delta} + \Gamma z_{jk}^{s} \Gamma^{\mathsf{T}} \;=\; \zeta_k' \mathbb{I} + \zeta_{jk} \Gamma\Gamma^{\mathsf{T}}$$

Substituting $\Gamma\Gamma^{\mathsf{T}} = 3\mathbb{I} - \mathbf{1}\mathbf{1}^{\mathsf{T}}$ from Theorem 7.2 we have $Z_{\text{Th}} = (1/a)\left(\mathbb{I} - a\zeta_{jk}\mathbf{1}\mathbf{1}^{\mathsf{T}}\right)$. Apply the matrix inversion formula (25.5) in Appendix 25.1.2: given a scalar $c \in \mathbb{C}$, vectors $b, d \in \mathbb{C}^n$, and the identity matrix $\mathbb{I}_n$ of size n,

$$\left(\mathbb{I}_n + bcd^{\mathsf{T}}\right)^{-1} \;=\; \mathbb{I}_n - b\left(c^{-1} + d^{\mathsf{T}} b\right)^{-1} d^{\mathsf{T}}$$

we therefore have (with $c := -a\zeta_{jk}$, $b = d = \mathbf{1}$)

$$Z_{\text{Th}}^{-1} \;=\; a\left(\mathbb{I} - \frac{a\zeta_{jk}}{3a\zeta_{jk} - 1}\mathbf{1}\mathbf{1}^{\mathsf{T}}\right) \tag{9.20}$$

To show that all voltages and currents are balanced positive-sequence sets, i.e., in span(α_+), the key property that we will use is Corollary 1.4 which states that: For any balanced positive-sequence vector $x + a\mathbf{1} \in \mathbb{C}^3$ with $a \in \mathbb{C}$, we have

$$\Gamma(x + a\mathbf{1}) \;=\; (1-\alpha)x, \qquad\qquad \Gamma^{\mathsf{T}}(x + a\mathbf{1}) \;=\; (1-\alpha^2)x$$

We have from Table 9.1 (substituting $E_j^{\Delta} = \lambda_j \mathbb{I}$ and $z_k^{\Delta} = \zeta_k' \mathbb{I}$)

$$I_k^{\Delta} \;=\; Z_{\text{Th}}^{-1} E_j^{\Delta} \;=\; a\lambda_j \left(\mathbb{I} - \frac{a\zeta_{jk}}{3a\zeta_{jk} - 1}\mathbf{1}\mathbf{1}^{\mathsf{T}}\right)\alpha_+ \;=\; \frac{\lambda_j}{\zeta_k' + 3\zeta_{jk}}\,\alpha_+$$

$$V_k^{\Delta} \;=\; z_k^{\Delta} I_k^{\Delta} \;=\; \frac{\zeta_k'}{\zeta_k' + 3\zeta_{jk}}\,\lambda_j\,\alpha_+, \qquad \beta_k \;:=\; \frac{1}{3}\mathbf{1}^{\mathsf{T}} I_k^{\Delta} \;=\; 0$$

where we have used $\mathbf{1}^{\mathsf{T}}\alpha_+ = 0$. The expression for V_k^{Δ} is the voltage divider rule.

We now calculate the other variables $\left(V_j, I_j, I_j^{\Delta}\right)$ and (V_k, I_k, γ_k). The terminal current of the impedance are

$$I_k \;=\; -\Gamma^{\mathsf{T}} I_k^{\Delta} \;=\; -\frac{\lambda_j}{\zeta_k' + 3\zeta_{jk}}\Gamma^{\mathsf{T}}\alpha_+ \;=\; -\frac{(1-\alpha^2)\lambda_j}{\zeta_k' + 3\zeta_{jk}}\,\alpha_+$$

Using the external model of an ideal voltage source from Table 7.4 we have

$$V_j \;=\; \frac{1}{3}\Gamma^{\mathsf{T}} E_j^{\Delta} \;+\; \gamma_j \mathbf{1} \;=\; \frac{1}{3}(1-\alpha^2)\lambda_j\,\alpha_+ \;+\; \gamma_j\mathbf{1}$$

$$I_j \;=\; -I_k \;=\; \frac{(1-\alpha^2)\lambda_j}{\zeta_k' + 3\zeta_{jk}}\,\alpha_+$$

Hence

$$V_k \;=\; V_j - z_{jk}^{s} I_j \;=\; \frac{(1-\alpha^2)\zeta_k'}{3\left(\zeta_k' + 3\zeta_{jk}\right)}\lambda_j\,\alpha_+ + \gamma_j\mathbf{1}, \qquad \gamma_k \;=\; \frac{1}{3}\mathbf{1}^{\mathsf{T}} V_k \;=\; \gamma_j$$

Finally

$$I_j^{\Delta} \;=\; -\frac{1}{3}\Gamma I_j + \beta_j\mathbf{1} \;=\; -\frac{(1-\alpha)(1-\alpha^2)\lambda_j}{3\left(\zeta_k' + 3\zeta_{jk}\right)}\,\alpha_+ + \beta_j\mathbf{1}$$

□

With power sources. The solution strategy is the same as that for problems without power sources with the addition of quadratic device models of power sources. Specifically we first relate *internal* voltages and currents to power sources $(\sigma_j^\Delta, \gamma_j)$ to obtain a system of quadratic equations that can be solved numerically. Then all other voltages and currents can be obtained analytically in terms of a solution of the quadratic equations. Finally we can calculate internal and external power using $s_j^{Y/\Delta} := \text{diag}\left(V_j^{Y/\Delta} I_j^{Y/\Delta\mathsf{H}}\right)$ and $s_j := \text{diag}\left(V_j I_j^{\mathsf{H}}\right)$ respectively. This solution strategy is extended in Chapter 9.2.3 to general three-phase networks.

Example 9.8 (Power source)**.** Consider the system in Figure 9.7 where, instead of a voltage source, the generator is a three-phase power source. Suppose the following are specified:

- Power source $\left(\sigma_j^\Delta, \gamma_j\right)$.
- Impedance z_k^Δ. (Note that β_k needs not be specified for an impedance and can be derived.)
- Line admittances $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$ with nonzero y_{jk}^m and y_{kj}^m. In particular assumption C9.1 is satisfied.

Find all remaining internal and external variables $\left(V_i^\Delta, I_i^\Delta, s_i^\Delta, \beta_j\right)$ and $(V_i, I_i, s_i, \gamma_k)$, $i = j,k$.

Solution. The current balance equation $I = YV$, the internal models of the power source and impedance, and the conversion rules are:

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} = \begin{bmatrix} y_{jk}^s + y_{jk}^m & -y_{jk}^s \\ -y_{jk}^s & y_{jk}^s + y_{kj}^m \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{9.21a}$$

$$\sigma_j^\Delta = \text{diag}\left(V_j^\Delta I_j^{\Delta\mathsf{H}}\right), \qquad V_k^\Delta = z_k^\Delta I_k^\Delta \tag{9.21b}$$

$$\Gamma V_i = V_i^\Delta, \qquad I_i = -\Gamma^\mathsf{T} I_i^\Delta, \qquad i = j,k \tag{9.21c}$$

Assuming the admittance matrix Y is invertible (e.g., it satisfies the condition in Theorem 4.3), denote its inverse by

$$Y^{-1} := \begin{bmatrix} y_{jk}^s + y_{jk}^m & -y_{jk}^s \\ -y_{jk}^s & y_{jk}^s + y_{kj}^m \end{bmatrix}^{-1} = \begin{bmatrix} z_{jj} & z_{jk} \\ z_{kj} & z_{kk} \end{bmatrix}$$

We can then relate the internal variables $\left(V_i^\Delta, I_i^\Delta\right)$, $i = j,k$, by eliminating the external variables to get

$$\begin{aligned} \begin{bmatrix} V_j^\Delta \\ V_k^\Delta \end{bmatrix} &= -\text{diag}\,(\Gamma, \Gamma) \begin{bmatrix} z_{jj} & z_{jk} \\ z_{kj} & z_{kk} \end{bmatrix} \text{diag}\left(\Gamma^\mathsf{T}, \Gamma^\mathsf{T}\right) \begin{bmatrix} I_j^\Delta \\ I_k^\Delta \end{bmatrix} \\ \sigma_j^\Delta &= \text{diag}\left(V_j^\Delta I_j^{\Delta\mathsf{H}}\right) \end{aligned} \tag{9.22}$$

Eliminating V_k^Δ using $V_k^\Delta = z_k^\Delta I_k^\Delta$ and re-arranging, we get

$$\begin{bmatrix} Z_{jj} & Z_{jk} & \mathbb{I} \\ Z_{kj} & Z_{kk} + z_k^\Delta & 0 \end{bmatrix} \begin{bmatrix} I_j^\Delta \\ I_k^\Delta \\ V_j^\Delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9.23a}$$

$$\text{diag}\left(V_j^\Delta I_j^{\Delta\mathsf{H}}\right) = \sigma_j^\Delta \tag{9.23b}$$

where $Z_{jj} := \Gamma z_{jj} \Gamma^{\mathsf{T}}$ and so on. This is a system of 9 quadratic equations in 9 variables $\left(V_j^{\Delta}, I_j^{\Delta}, I_k^{\Delta}\right)$. It can be solved numerically. All other variables can then be derived analytically in terms of a solution $\left(V_j^{\Delta}, I_j^{\Delta}, I_k^{\Delta}\right)$.

We can further reduce (9.23) by eliminating V_j^{Δ} and I_k^{Δ} to get a quadratic equation in I_j^{Δ}:

$$\operatorname{diag}\left(\left(-Z_{jj} + Z_{jk}\left(Z_{kk} + z_k^{\Delta}\right)^{-1} Z_{kj}\right) I_j^{\Delta} I_j^{\Delta\mathsf{H}}\right) \;=\; \sigma_j^{\Delta}, \qquad j \in \overline{N} \tag{9.24}$$

In summary we can first solve (9.24) numerically to obtain I_j^{Δ} and then derive all other variables, or first solve (9.23) numerically to obtain $\left(V_j^{\Delta}, I_j^{\Delta}, I_k^{\Delta}\right)$ and then all other variables. They are equivalent to solving the original system (9.21) numerically. The decentralized structure of (9.24) is quite striking: the system of power flow equations for the entire network reduces to this quadratic equation separately for each bus j that can be solved in parallel.

We now derive all other variables from I_j^{Δ}, by tracing back the derivation of (9.24). From (9.23a) we have

$$I_k^{\Delta} \;=\; -\left(Z_{kk} + z_k^{\Delta}\right)^{-1} Z_{kj} I_j^{\Delta}, \qquad V_j^{\Delta} \;=\; -Z_{jj} I_j^{\Delta} - Z_{jk} I_k^{\Delta} \;=\; \left(-Z_{jj} + Z_{jk}\left(Z_{kk} + z_k^{\Delta}\right)^{-1} Z_{kj}\right) I_j^{\Delta}$$

From (9.21b) we have

$$V_k^{\Delta} \;=\; z_k^{\Delta} I_k^{\Delta} \;=\; -z_k^{\Delta}\left(Z_{kk} + z_k^{\Delta}\right)^{-1} Z_{kj} I_j^{\Delta},$$

The internal zero-sequence currents are given by

$$\beta_j \;=\; \frac{1}{3}\mathbf{1}^{\mathsf{T}} I_j^{\Delta}, \qquad \beta_k \;=\; \frac{1}{3}\mathbf{1}^{\mathsf{T}} I_k^{\Delta}$$

This completes the derivation of internal voltages and currents.

The terminal currents can be obtained from the conversion rule (9.21c):

$$I_j \;=\; -\Gamma^{\mathsf{T}} I_j^{\Delta}, \qquad I_k \;=\; -\Gamma^{\mathsf{T}} I_k^{\Delta} \;=\; \Gamma^{\mathsf{T}}\left(Z_{kk} + z_k^{\Delta}\right)^{-1} Z_{kj} I_j^{\Delta}$$

Note that $\mathbf{1}^{\mathsf{T}} V_j^{\Delta} = \mathbf{1}^{\mathsf{T}} V_k^{\Delta} = 0$ from (9.22). Hence the conversion rule (9.21c) yields (recall that γ_j is specified)

$$V_j \;=\; \frac{1}{3} V_j^{\Delta} + \gamma_j \mathbf{1} \tag{9.25a}$$

Given the terminal voltage V_j of the power source, (V_k, γ_k) of the impedance can then be determined through the network equation (9.21a):

$$V_k \;=\; \left(y_{jk}^{s} + y_{kj}^{m}\right)^{-1}\left(y_{jk}^{s} V_j + I_k\right), \qquad \gamma_k \;=\; \frac{1}{3}\mathbf{1}^{\mathsf{T}} V_k \tag{9.25b}$$

Notice that the zero-sequence voltage γ_j of the power source uniquely determines γ_k of the impedance. □

The derivation in Example 9.8 relies on the assumption that the admittance matrix Y in (9.21a) is invertible. If the shunt admittances $y^m_{jk} = y^m_{kj} = 0$ then Y has zero block row sums (Definition 9.1), i.e., $\sum_k Y_{jk} = 0$ for all j. This implies that Y has zero row sums, i.e., $\sum_{k,\phi'} Y_{j\phi,k\phi'} = 0$ for all $j\phi$, and is therefore singular. In that case, additional information needs to be specified to obtain a unique solution, as the next example illustrates.

Example 9.9 (Power source)**.** Repeat Example 9.8 but with zero shunt admittances and given zero-sequence currents, i.e., suppose the following are specified:

- Power source $\left(\sigma_j^\Delta, \gamma_j\right)$.
- Impedance z_k^Δ.
- Line admittances $\left(y^s_{jk}, y^m_{jk} = y^m_{kj} = 0\right)$ with nonsingular y^s_{jk}. In particular assumption C9.1 is satisfied.
- $\beta_j + \beta_k := \frac{1}{3}\mathbf{1}^\mathsf{T}\left(I_j^\Delta + I_k^\Delta\right) = \beta'$.

Solution. When $y^m_{jk} = y^m_{kj} = 0$ the network equation (9.21a) reduces to

$$I_j \ = \ -I_k \ = \ y^s_{jk}\left(V_j - V_k\right) \tag{9.26}$$

Hence $\Gamma^\mathsf{T}\left(I_j^\Delta + I_k^\Delta\right) = 0$ from (9.21c), implying that

$$I_j^\Delta + I_k^\Delta \ = \ \left(\beta_j + \beta_k\right)\mathbf{1} \ = \ \beta'\,\mathbf{1} \tag{9.27}$$

with β' a given quantity. We will express V_j^Δ in terms of I_j^Δ in order to write $\sigma_j = \text{diag}\left(V_j^\Delta I_j^{\Delta\mathsf{H}}\right)$ as a quadratic equation in I_j^Δ.

Multiplying both sides of (9.26) by $z^s_{jk} := \left(y^s_{jk}\right)^{-1}$ and using the conversion rule again (9.21b)(9.21c), we have

$$V_j^\Delta \ = \ \left(\Gamma z^s_{jk}\Gamma^\mathsf{T} + z_k^\Delta\right) I_k^\Delta \ = \ Z_{jk}^\Delta\left(-I_j^\Delta + \beta'\mathbf{1}\right) \ = \ -Z_{jk}^\Delta I_j^\Delta + \beta'\tilde{z}_k^\Delta \tag{9.28}$$

where the second equality follows from (9.27), $Z_{jk}^\Delta := \Gamma z^s_{jk}\Gamma^\mathsf{T} + z_k^\Delta$, and $\tilde{z}_k^\Delta := z_k^\Delta\mathbf{1}$. Hence we have

$$\sigma_j^\Delta \ = \ \text{diag}\left(V_j^\Delta I_j^{\Delta\mathsf{H}}\right) \ = \ \text{diag}\left(-Z_{jk}^\Delta I_j^\Delta I_j^{\Delta\mathsf{H}} + \beta'\tilde{z}_k^\Delta I_j^{\Delta\mathsf{H}}\right) \tag{9.29}$$

This is a system of three quadratic equations in three variables $I_j^\Delta \in \mathbb{C}^3$. Assume a solution exists and can be obtained by solving (9.29) numerically.

Given a solution I_j^Δ of (9.29), all other variables can be derived analytically in terms of I_j^Δ by tracing back the derivation of (9.29), similar to the derivation in Example 9.8. Specifically we have $I_j = -\Gamma^\mathsf{T} I_j^\Delta$

and $\beta_j := \frac{1}{3}\mathbf{1}^\mathsf{T} I_j^\Delta$. We obtain V_j^Δ from (9.28), from which we have $V_j = \frac{1}{3}\Gamma^\mathsf{T} V_j^\Delta + \gamma_j \mathbf{1}$. This computes all voltages and currents of the power source j.

The network equation (9.26) then yields $V_k = V_j - z_{jk}^s I_j$ and hence also $\gamma_k := \frac{1}{3}\mathbf{1}^\mathsf{T} V_k$. We also have $I_k = -I_j = \Gamma^\mathsf{T} I_j^\Delta$, $\beta_k = \beta' - \beta_j$, and hence $I_k^\Delta = -\frac{1}{3}\Gamma I_k + \beta_k \mathbf{1}$ and $V_k^\Delta = z_k^\Delta I_k^\Delta$. This computes all voltages and currents of the impedance k. □

The next example shows that if the power source and the impedance are balanced and the line is decoupled and balanced, then all voltages, currents, and powers will be generalized balanced vectors. This will be proved for general networks in Chapter 9.3. Furthermore the given power σ_j^Δ cannot be arbitrary but must be consistent with other parameters of the network such as line and device impedances, e.g., from (9.33), b_j/c must be real. This generalizes the single-phase case where a power source s supplies an impedance load z with a current i. Then $s = z|i|^2$ implying that s/z is a read number. This is because $\angle z = \angle s$ fixes the phase difference between the voltage v and current i across the impedance.

Example 9.10 (Balanced power source)**.** Repeat Example 9.8 when the system is balanced, i.e.,

- Power source $\left(\sigma_j^\Delta, \gamma_j\right)$ with $\sigma_j^\Delta = a_j \alpha_+ + b_j \mathbf{1}$ for given $\left(a_j, b_j\right)$, i.e., a balanced power source must be a generalized balanced vector. Moreover its voltage and current $\left(V_j^\Delta, I_j^\Delta\right)$ are generalized balanced vectors.
- Impedance $z_k^\Delta := \zeta_k^\Delta \mathbb{I}$.
- Line admittances $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right) := \left(\eta_{jk}^s \mathbb{I}, \eta_{jk}^m \mathbb{I}, \eta_{kj}^m \mathbb{I}\right)$ with nonzero η_{jk}^s, η_{jk}^m and η_{kj}^m.

Find all remaining internal and external variables $\left(V_i^\Delta, I_i^\Delta, s_i^\Delta, \beta_i^\Delta\right)$ and $(V_i, I_i, s_i, \gamma_i)$, $i = j, k$. Show that the problem can be solved *analytically* when a reference angle is given, say, $\angle V_j^a := \theta_j^a$.

Solution. Let (recall that $\mathbf{1}^\mathsf{T} V_j^\Delta = 0$)

$$V_j^\Delta \;=:\; v_j^\Delta \alpha_+, \qquad\qquad I_j^\Delta \;=:\; i_j^\Delta \alpha_+ + \beta_j \mathbf{1} \tag{9.30}$$

giving (noting diag $\left(\alpha_+ \alpha_+^\mathsf{H}\right) = \mathbf{1}$)

$$\sigma_j^\Delta \;=\; \text{diag}\left(v_j^\Delta \alpha_+ \left(i_j^\Delta \alpha_+ + \beta_j \mathbf{1}\right)^\mathsf{H}\right) \;=\; \left(v_j^\Delta \overline{\beta}_j\right)\alpha_+ \;+\; \left(v_j^\Delta \bar{i}_j^\Delta\right)\mathbf{1}$$

where $\left(v_j^\Delta, i_j^\Delta, \beta_j \in \mathbb{C}^3\right)$ are to be determined. Recall that $\bar{x}$ denotes the complex conjugate of any $x \in \mathbb{C}$. Therefore, since $\sigma_j^\Delta = a_j \alpha_+ + b_j \mathbf{1}$,

$$v_j^\Delta \overline{\beta}_j \;=\; a_j, \qquad\qquad v_j^\Delta \bar{i}_j^\Delta \;=\; b_j \tag{9.31}$$

which are two quadratic equations in unknowns $\left(v_j^\Delta, i_j^\Delta, \beta_j\right) \in \mathbb{C}^3$. Note that the internal power σ_j^Δ is different in each phase (with different phase angles separated by 120°) if and only if $\beta_j \neq 0$.

We will solve this problem by substituting the given balanced system parameters into the solution of Example 9.8.

Specifically the admittance matrix is

$$Y \;:=\; \begin{bmatrix} y^s_{jk}+y^m_{jk} & -y^s_{jk} \\ -y^s_{jk} & y^s_{jk}+y^m_{kj} \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} \eta^s_{jk}+\eta^m_{jk} & -\eta^s_{jk} \\ -\eta^s_{jk} & \eta^s_{jk}+y^m_{kj} \end{bmatrix}}_{Y^{1\phi}} \otimes \mathbb{I}$$

Assuming the 2×2 admittance matrix $Y^{1\phi}$ is invertible with inverse $\left(Y^{1\phi}\right)^{-1} =: \begin{bmatrix} \zeta_{jj} & \zeta_{jk} \\ \zeta_{kj} & \zeta_{kk} \end{bmatrix}$ we have

$$Y^{-1} \;=\; \left(Y^{1\phi}\right)^{-1} \otimes \mathbb{I} \;=:\; \begin{bmatrix} \zeta_{jj} & \zeta_{jk} \\ \zeta_{kj} & \zeta_{kk} \end{bmatrix} \otimes \mathbb{I}$$

where the first equality follows from $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$ in Lemma 9.6. Then (9.22) becomes

$$\begin{aligned} \begin{bmatrix} V^\Delta_j \\ V^\Delta_k \end{bmatrix} \;&=\; -\mathrm{diag}\,(\Gamma,\Gamma) \left(\begin{bmatrix} \zeta_{jj} & \zeta_{jk} \\ \zeta_{kj} & \zeta_{kk} \end{bmatrix} \otimes \mathbb{I} \right) \mathrm{diag}\left(\Gamma^\mathsf{T},\Gamma^\mathsf{T}\right) \begin{bmatrix} I^\Delta_j \\ I^\Delta_k \end{bmatrix} \\ &=\; \begin{bmatrix} \zeta_{jj} & \zeta_{jk} \\ \zeta_{kj} & \zeta_{kk} \end{bmatrix} \otimes \left(\Gamma\Gamma^\mathsf{T}\right) \begin{bmatrix} I^\Delta_j \\ I^\Delta_k \end{bmatrix} \end{aligned}$$

where $\Gamma\Gamma^\mathsf{T} = 3\mathbb{I} - \mathbf{1}\mathbf{1}^\mathsf{T}$ from Theorem 7.2. Then (9.23) becomes (9.31) together with

$$\begin{bmatrix} \zeta_{jj}\left(\Gamma\Gamma^\mathsf{T}\right) & \zeta_{jk}\left(\Gamma\Gamma^\mathsf{T}\right) & \mathbb{I} \\ \zeta_{kj}\left(\Gamma\Gamma^\mathsf{T}\right) & \zeta_{kk}\left(\Gamma\Gamma^\mathsf{T}\right) + \zeta^\Delta_k\mathbb{I} & 0 \end{bmatrix} \begin{bmatrix} i^\Delta_j\alpha_+ + \beta_j\mathbf{1} \\ I^\Delta_k \\ v^\Delta_j\alpha_+ \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where we have used the specification (9.30). This is a system of 8 (redundant) quadratic equations that can be solved numerically for the 6 unknowns $\left(v^\Delta_j, i^\Delta_j, \beta_j\right) \in \mathbb{C}^3$ and $I^\Delta_k \in \mathbb{C}^3$. It implies that I^Δ_k is a generalized balanced vector of the form $I^\Delta_k = i^\Delta_k\alpha_+ + \beta_k\mathbf{1}$ for some $\left(i^\Delta_k, \beta_k\right)$.

To evaluate (9.24) we have

$$I^\Delta_j I^{\Delta\mathsf{H}}_j \;=\; \left(i^\Delta_j\alpha_+ + \beta_j\mathbf{1}\right)\left(i^\Delta_j\alpha_+ + \beta_j\mathbf{1}\right)^\mathsf{H} \;=\; \left|i^\Delta_j\right|^2 \alpha_+\alpha_+^\mathsf{H} + i^\Delta_j\overline{\beta}_j\,\alpha_+\mathbf{1}^\mathsf{T} + \bar{i}^\Delta_j\beta_j\,\mathbf{1}\alpha_+^\mathsf{H} + \left|\beta_j\right|^2\mathbf{1}\mathbf{1}^\mathsf{T}$$

and therefore

$$\left(\Gamma\Gamma^\mathsf{T}\right) I^\Delta_j I^{\Delta\mathsf{H}}_j \;=\; 3\left(\left|i^\Delta_j\right|^2 \alpha_+\alpha_+^\mathsf{H} + i^\Delta_j\overline{\beta}_j\,\alpha_+\mathbf{1}^\mathsf{T}\right) \tag{9.32a}$$

where we have used $\Gamma\Gamma^\mathsf{T}\alpha_+ = 3\alpha_+$ from Corollary 1.4 and $\Gamma^\mathsf{T}\mathbf{1} = 0$. Furthermore

$$\left(Z_{kk} + z^\Delta_k\right)^{-1} \;=\; \left(\zeta_{kk}\left(\Gamma\Gamma^\mathsf{T}\right) + \zeta^\Delta_k\mathbb{I}\right)^{-1} \;=\; \left(\left(3\zeta_{kk} + \zeta^\Delta_k\right)\mathbb{I} - \zeta_{kk}\mathbf{1}\mathbf{1}^\mathsf{T}\right)^{-1} \;=\; \frac{1}{3\zeta_{kk} + \zeta^\Delta_k}\left(\mathbb{I} - \frac{\zeta_{kk}}{\zeta^\Delta_k}\mathbf{1}\mathbf{1}^\mathsf{T}\right)$$

where the last equality follows from the matrix inversion formula (see Appendix 25.1.2.2)

$$(\mathbb{I}_n + BD)^{-1} \;=\; I_n \,-\, B\,(\mathbb{I}_k + DB)^{-1}\,D$$

when $B, D^{\mathsf{T}} \in \mathbb{C}^{n\times k}$ and $\mathbb{I}_n, \mathbb{I}_k$ denote identity matrices of sizes n,k respectively. Hence

$$Z_{jk}\left(Z_{kk} + z_k^{\Delta}\right)^{-1} Z_{kj} \;=\; \frac{\zeta_{jk}\zeta_{kj}}{3\zeta_{kk}+\zeta_k^{\Delta}}\left(\Gamma\Gamma^{\mathsf{T}}\right)\left(\mathbb{I} - \frac{\zeta_{kk}}{\zeta_k^{\Delta}}\mathbf{1}\mathbf{1}^{\mathsf{T}}\right)\left(\Gamma\Gamma^{\mathsf{T}}\right) \;=\; \frac{3\zeta_{jk}\zeta_{kj}}{3\zeta_{kk}+\zeta_k^{\Delta}}\,\Gamma\Gamma^{\mathsf{T}} \tag{9.32b}$$

Together with $Z_{jj} = \zeta_{jj}\Gamma\Gamma^{\mathsf{T}}$, (9.32) implies that (9.24) is

$$\begin{aligned}\sigma_j \;=\; a_j\alpha_+ + b_j\mathbf{1} \;&=\; \left(-\zeta_{jj} + \frac{3\zeta_{jk}\zeta_{kj}}{3\zeta_{kk}+\zeta_k^{\Delta}}\right)\operatorname{diag}\left(\Gamma\Gamma^{\mathsf{H}} I_j^{\Delta} I_j^{\Delta\mathsf{H}}\right)\\ &=\; \underbrace{3\left(-\zeta_{jj} + \frac{3\zeta_{jk}\zeta_{kj}}{3\zeta_{kk}+\zeta_k^{\Delta}}\right)}_{c}\left(i_j^{\Delta}\overline{\beta}_j\,\alpha_+ + \left|i_j^{\Delta}\right|^2\mathbf{1}\right)\end{aligned}$$

where we have used $\operatorname{diag}\left(\alpha_+\alpha_+^{\mathsf{H}}\right) = \mathbf{1}$. Hence

$$c\,i_j^{\Delta}\overline{\beta}_j \;=\; a_j, \qquad\qquad c\left|i_j^{\Delta}\right|^2 \;=\; b_j \tag{9.33}$$

which is a system of 2 quadratic equations. This yields the magnitude of i_j^{Δ}:

$$\left|i_j^{\Delta}\right|^2 \;=\; \frac{b_j}{c}$$

which in particular means that the specification cannot be arbitrary, e.g., b_j/c must be real.

When the reference angle $\angle V_j^a := \theta_j^a$ is given, let $\phi_j := \angle i_j^{\Delta}$. Given $i_j^{\Delta} := \sqrt{\frac{b_j}{c}}e^{\mathbf{i}\phi_j}$, all the other variables $\left(v_j^{\Delta}, i_j^{\Delta}, \beta_j\right) \in \mathbb{C}^3$ and $I_k^{\Delta} \in \mathbb{C}^3$ can be obtained as in Example 9.8, as a function of ϕ_j which can then be determined from the given reference angle:

$$\angle V_j^a \;=\; \angle\left[\frac{1}{3}\Gamma^{\mathsf{T}} v_j^{\Delta}\alpha_+ + \gamma_j\mathbf{1}\right]^a \;=\; \theta_j^a$$

This also shows that all variables are (generalized) balanced positive-sequence sets. □

Remark 9.4 (Nonuniqueness of specification)**.** Device specification is not unique and depends on the application under study. For Example 9.8, since both internal voltages V_j^{Δ} and V_k^{Δ} are obtained in terms of I_j^{Δ} in (9.25), we can either specify γ_j for the power source and derive γ_k of the impedance through the network equation, as done in Example 9.8, or alternatively, we can specify γ_k and determine γ_j from the network equation instead. While Example 9.8 contains no power sources, the next example illustrates multiple ways to specify and solve the case when both the generator and the load are power sources.

Also see Remark 9.6 for discussions on the asymmetry in device specifications. □

The next example uses the internal model or an external model of power sources, depending on how the power sources are specified. Specifically the solution boils down to a system of quadratic equations that can be solved numerically. All other variables can then be derived analytically in terms of a solution of the quadratic equations. For each of the two power sources, if its zero-sequence voltage γ_i is specified, we will use the internal model for the power source to obtain the system of quadratic equations in the internal currents I_i^Δ. Then the internal voltage V_i^Δ can be derived and, with the given γ_i, the terminal voltages V_i. If its zero-sequence current β_i is specified, on the other hand, we will use an external model to obtain the quadratic equations in the terminal current I_i from which, with the given β_i, the internal current I_i^Δ can then be derived. The network equation is used to express V_i^Δ in terms of I_i^Δ in the first case and express V_i in terms of I_i in the second case in the derivation of the system of quadratic equations.

Example 9.11 (Power sources)**.** Consider the system in Figure 9.7 where both the generator and load are power sources. Suppose the line admittances $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$ are specified with nonzero y_{jk}^m, y_{kj}^m and assumption C9.1, as in Example 9.8.

1. Suppose the power sources are specified as $\left(\sigma_j^\Delta, \gamma_j\right)$ and $\left(\sigma_k^\Delta, \gamma_k\right)$. Determine all variables $\left(V_i^\Delta, I_i^\Delta, \beta_i\right)$ and (V_i, I_i, s_i), $i = j,k$.
2. Suppose the power sources are specified as $\left(\sigma_j^\Delta, \beta_j\right)$ and $\left(\sigma_k^\Delta, \beta_k\right)$. Determine all variables $\left(V_i^\Delta, I_i^\Delta\right)$ and $(V_i, I_i, s_i, \gamma_i)$, $i = j,k$.
3. Suppose the power sources are specified as $\left(\sigma_j^\Delta, \gamma_j\right)$ and $\left(\sigma_k^\Delta, \beta_k\right)$. Determine all variables $\left(V_i^\Delta, I_i^\Delta\right)$ and (V_i, I_i, s_i), $i = j,k$, and β_j, γ_k.

Solution.

1. The internal model of the power sources, the conversion rules, and the current balance equation are

$$\sigma_i^\Delta \;:=\; \text{diag}\left(V_i^\Delta I_i^{\Delta\mathsf{H}}\right), \qquad V_i^\Delta \;=\; \Gamma V_i, \qquad I_i \;=\; -\Gamma^\mathsf{T} I_i^\Delta, \qquad i \;=\; j,k \tag{9.34a}$$

$$\begin{bmatrix} I_j \\ I_k \end{bmatrix} \;=\; \begin{bmatrix} y_{jk}^s + y_{jk}^m & -y_{jk}^s \\ -y_{jk}^s & y_{jk}^s + y_{kj}^m \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix} \tag{9.34b}$$

Assume the admittance matrix Y in (9.34b) is invertible and let $Y^{-1} =: \begin{bmatrix} z_{jj} & z_{jk} \\ z_{kj} & z_{kk} \end{bmatrix}$. Then substituting the conversion rules into the network equation (9.34b) yields

$$\begin{bmatrix} V_j^\Delta \\ V_k^\Delta \end{bmatrix} \;=\; -\underbrace{\text{diag}\left(\Gamma,\Gamma\right) \begin{bmatrix} z_{jj} & z_{jk} \\ z_{kj} & z_{kk} \end{bmatrix} \text{diag}\left(\Gamma^\mathsf{T},\Gamma^\mathsf{T}\right)}_{Z := \begin{bmatrix} Z_{jj} & Z_{jk} \\ Z_{kj} & Z_{kk} \end{bmatrix}} \begin{bmatrix} I_j^\Delta \\ I_k^\Delta \end{bmatrix} \tag{9.35}$$

Substituting V_j^Δ and V_k^Δ into the internal power source models in (9.34a) yelds

$$\sigma_j^\Delta \;:=\; -\text{diag}\left(\left(Z_{jj} I_j^\Delta + Z_{jk} I_k^\Delta\right) I_j^{\Delta\mathsf{H}}\right), \qquad \sigma_k^\Delta \;:=\; -\text{diag}\left(\left(Z_{kj} I_j^\Delta + Z_{kk} I_k^\Delta\right) I_k^{\Delta\mathsf{H}}\right) \tag{9.36}$$

This is a system of 6 quadratic equations that can be solved numerically for $\left(I_j^\Delta, I_k^\Delta\right) \in \mathbb{C}^6$.

All other variables can then be derived in terms of a solution $\left(I_j^\Delta, I_k^\Delta\right)$. Specifically, the internal voltages can be obtained from the internal power source model (9.34a) (or equivalently from (9.35)), $V_i^\Delta = \left(\text{diag}\left(I_i^{\Delta\mathsf{H}}\right)\right)^{-1} \sigma_i^\Delta$, $i = 1,2$. Using γ_i, the terminal voltages are determined by the conversion rule, $V_i = \frac{1}{3}\Gamma^\mathsf{T} V_i^\Delta + \gamma_i \mathbf{1}$, $i = 1,2$. In terms of I_i^Δ we have $\beta_i := \frac{1}{3}\mathbf{1}^\mathsf{T} I_i^\Delta$ and $I_i = -\Gamma^\mathsf{T} I_i^\Delta$, $i = j,k$. The terminal power is $s_i := \text{diag}\left(V_i I_i^\mathsf{H}\right)$, $i = j,k$.

2. When $\left(\gamma_j, \gamma_k\right)$ are given as in part 1, we set up equation (9.36) to solve numerically for $\left(I_j^\Delta, I_k^\Delta\right)$, so that V_i^Δ and then V_i can be derived for $i = j,k$. When $\left(\beta_j, \beta_k\right)$ are given instead, we will solve numerically for (V_j, V_k) by using the external model (7.25a) of a power source, reproduced here:

$$\sigma_i^\Delta \;=\; -\frac{1}{3}\text{diag}\left(\Gamma\left(V_i I_i^\mathsf{H}\right)\Gamma^\mathsf{T}\right) \;+\; \overline{\beta}_i \Gamma V_i, \qquad \mathbf{1}^\mathsf{T} I_i \;=\; 0, \qquad i \;=\; j,k$$

and the network equation (9.34b). Note that all these equations relate terminal voltages and currents.

Specifically, instead of (9.35)), obtain from the network equation (9.34b)

$$\begin{bmatrix} V_j \\ V_k \end{bmatrix} \;=\; \begin{bmatrix} z_{jj} & z_{jk} \\ z_{kj} & z_{kk} \end{bmatrix} \begin{bmatrix} I_j \\ I_k \end{bmatrix}$$

Substituting into the external models of the power sources we have

$$\begin{aligned} \sigma_j^\Delta \;&=\; -\frac{1}{3}\text{diag}\left(\Gamma\left(z_{jj}I_j + z_{jk}I_k\right) I_j^\mathsf{H}\Gamma^\mathsf{T}\right) \;+\; \overline{\beta}_j \Gamma\left(z_{jj}I_j + z_{jk}I_k\right), \qquad \mathbf{1}^\mathsf{T} I_j \;=\; 0 \\ \sigma_k^\Delta \;&=\; -\frac{1}{3}\text{diag}\left(\Gamma\left(z_{kj}I_j + z_{kk}I_k\right) I_k^\mathsf{H}\Gamma^\mathsf{T}\right) \;+\; \overline{\beta}_k \Gamma\left(z_{kj}I_j + z_{kk}I_k\right), \qquad \mathbf{1}^\mathsf{T} I_k \;=\; 0 \end{aligned}$$

This is a system of 8 (redundant) quadratic equations that can be solved numerically for $\left(I_j, I_k\right) \in \mathbb{C}^6$. Given a solution $\left(I_j, I_k\right)$, the internal currents can be determined from the conversion rule and the given $\left(\beta_j, \beta_k\right)$ as $I_i^\Delta = -\frac{1}{3}\Gamma I_i + \beta_i \mathbf{1}$, $i = j,k$. The remaining variables can then be derived as in part 1.

3. This combines the solution approaches of parts 1 and 2. Specifically we use the internal model for power source j, the external model for k:

$$\sigma_j^\Delta \;:=\; \text{diag}\left(V_j^\Delta I_j^{\Delta\mathsf{H}}\right), \qquad V_j^\Delta \;=\; \Gamma V_j, \qquad I_j \;=\; -\Gamma^\mathsf{T} I_j^\Delta \tag{9.37a}$$

$$\sigma_k^\Delta \;=\; -\frac{1}{3}\text{diag}\left(\Gamma\left(V_k I_k^\mathsf{H}\right)\Gamma^\mathsf{T}\right) \;+\; \overline{\beta}_k \Gamma V_k, \qquad \mathbf{1}^\mathsf{T} I_k \;=\; 0 \tag{9.37b}$$

From the network equation (9.34b) we have

$$\begin{bmatrix} V_j^\Delta \\ V_k \end{bmatrix} \;=\; \text{diag}\left(\Gamma, \mathbb{I}\right) \begin{bmatrix} z_{jj} & z_{jk} \\ z_{kj} & z_{kk} \end{bmatrix} \text{diag}\left(-\Gamma^\mathsf{T}, \mathbb{I}\right) \begin{bmatrix} I_j^\Delta \\ I_k \end{bmatrix} \;=\; \begin{bmatrix} -\Gamma z_{jj}\Gamma^\mathsf{T} & \Gamma z_{jk} \\ -z_{kj}\Gamma^\mathsf{T} & z_{kk} \end{bmatrix} \begin{bmatrix} I_j^\Delta \\ I_k \end{bmatrix}$$

Substituting V_j^Δ and V_k into the internal power source models in (9.37) yelds

$$\sigma_j^\Delta \;:=\; \text{diag}\left(\left(-\Gamma z_{jj}\Gamma^\mathsf{T} I_j^\Delta + \Gamma z_{jk} I_k\right) I_j^{\Delta\mathsf{H}}\right)$$
$$\sigma_k^\Delta \;=\; -\frac{1}{3}\text{diag}\left(\Gamma\left(-z_{kj}\Gamma^\mathsf{T} I_j^\Delta + z_{kk} I_k\right) I_k^\mathsf{H}\Gamma^\mathsf{T}\right) \;+\; \overline{\beta}_k\Gamma\left(-z_{kj}\Gamma^\mathsf{T} I_j^\Delta + z_{kk} I_k\right), \quad \mathbf{1}^\mathsf{T} I_k \;=\; 0$$

This is a system of 7 (redundant) quadratic equations that can be solved numerically for $\left(I_j^\Delta, I_k\right) \in \mathbb{C}^6$. All other variables can then be derived analytically in terms of a solution $\left(I_j^\Delta, I_k\right)$ as done in parts 1 and 2.

□

9.2.2 General analysis problem

We now formulate a general three-phase analysis problem. Consider a three-phase network $G := (\overline{N}, E)$ where each line $(j,k) \in E$ is characterized by 3×3 series and shunt admittance matrices $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$. At each bus $j \in \overline{N}$ we assume, without loss of generality, there is a single three-wire device in either Y or Δ configuration. Associated with each device j are its internal variables $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}, \beta_j\right) \in \mathbb{C}^{10}$ (or in $\mathbb{C}^9$ for Y-configured devices j without β_j) and terminal variables $\left(V_j, I_j, s_j, \gamma_j\right) \in \mathbb{C}^{10}$. Some of these variables will be specified in our formulation. The others are to be computed from network equations, device models and the conversion rules.

We start by describing which of these variables are specified for each type of devices using the internal and external device models in Tables 7.3 and 7.4. It is important to keep in mind that device specification is not unique and our formulation here may need to be modified depending on the details of an application, especially for problems involving power sources as discussed in Remark 9.4 and illustrated in Example 9.11. The principle of analysis described here, however, is widely applicable and can be applied to other formulations. For instance, we formulate our analysis problem in a three-wire model. If the neutrals of two Y-configured devices are not grounded and are connected to each other through a four-wire line, then a four-wire model needs to be used; see Example 9.5 and Exercise 9.7. In that case the neutral voltages of these devices may not be arbitrarily specified but must be determined through network equations and device models, even for a voltage source, unlike the formulation here.

Partition $\overline{N}$ into 8 disjoint subsets:

- $N_v^{Y/\Delta}$: buses with ideal voltage sources in Y or Δ configurations. Let $N_v := N_v^Y \cup N_v^\Delta$.
- $N_c^{Y/\Delta}$: buses with ideal current sources in Y or Δ configurations. Let $N_c := N_c^Y \cup N_c^\Delta$.
- $N_i^{Y/\Delta}$: buses with impedances in Y or Δ configurations. Let $N_i := N_i^Y \cup N_i^\Delta$.
- $N_p^{Y/\Delta}$: buses with ideal power sources in Y or Δ configurations. Let $N_p := N_p^Y \cup N_p^\Delta$.

with $\overline{N} = N_v \cup N_c \cup N_i \cup N_p$. These devices are specified as follows.

1. *Voltage source* $\left(E_j^Y, \gamma_j\right)$ *or* $\left(E_j^\Delta, \gamma_j, \beta_j\right)$: It is specified by its internal voltage $E_j^{Y/\Delta}$ and a parameter γ_j where $\gamma_j := V_j^n$ is the neutral voltage for Y configuration and $\gamma_j := \frac{1}{3}\mathbf{1}^\mathsf{T} V_j$ is the zero-sequence terminal voltage for Δ configuration. For Δ configuration, E_j^Δ should satisfy $\mathbf{1}^\mathsf{T} E_j^\Delta = 0$. The zero-sequence internal current $\beta_j := \frac{1}{3}\mathbf{1}^\mathsf{T} I_j^\Delta$ also needs to be specified in order to determine I_j^Δ from the terminal current I_j.

2. *Current source* $\left(J_j^Y, \gamma_j\right)$ *or* J_j^Δ: It is specified by its internal current $J_j^{Y/\Delta}$. For a Y-configured current source, its neutral voltage γ_j is also specified. For a Δ-configured current source, the zero-sequence voltage γ_i generally need not be specified and can be derived in terms of other quantities, but there are exceptions; see Remark 9.8.

3. *Power source* $\left(\sigma^Y, \gamma_j\right)$ *or* $\left(\sigma^\Delta, \gamma_j\right)$: It is specified by its internal power and zero-sequence voltage $\left(\sigma^{Y/\Delta}, \gamma_j\right)$. See Example 9.11 for other power source specifications and their solution methods.

4. *Impedance* $\left(z^Y, \gamma_j\right)$ *or* z^Δ: A Y-configured impedance j is specified by its internal impedance z_j^Y and the neutral voltage γ_j. A Δ-configured impedance j is specified by z_j^Δ. Its zero-sequence voltage and current $\left(\gamma_j, \beta_j\right)$ can generally be derived from network equations as we will see in Chapter 9.2.3.

A three-phase analysis problem is: given devices specified as above connected by lines with given admittance matrices $\left(y_{jk}^s, y_{jk}^m\right)$, $\left(y_{kj}^s, y_{kj}^m\right)$, determine some or all of the internal variables $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}, \beta_j\right)$ and terminal variables $\left(V_j, I_j, s_j, \gamma_j\right)$ at every bus j. This is summarized in Table 9.3. Note that the analysis

Buses j	Specification	Unknowns
N_v^Y	$V_j^Y := E_j^Y,\ \gamma_j$	$\left(I_j^Y, s_j^Y\right)$, $\left(V_j, I_j, s_j\right)$
N_v^Δ	$V_j^\Delta := E_j^\Delta,\ \gamma_j, \beta_j,$	$\left(I_j^\Delta, s_j^\Delta\right)$, $\left(V_j, I_j, s_j\right)$
N_c^Y	$I_j^Y := J_j^Y, \gamma_j$	$\left(V_j^Y, s_j^\Delta\right)$, $\left(V_j, I_j, s_j\right)$
N_c^Δ	$I_j^\Delta := J_j^\Delta$	$\left(V_j^\Delta, s_j^\Delta, \beta_j\right)$, $\left(V_j, I_j, s_j, \gamma_j\right)$
N_i^Y	$z_j^Y,\ \gamma_j$	$\left(V_j^Y, I_j^Y, s_j^Y\right)$, $\left(V_j, I_j, s_j\right)$
N_i^Δ	z_j^Δ	$\left(V_j^\Delta, I_j^\Delta, s_j^\Delta, \beta_j\right)$, $\left(V_j, I_j, s_j, \gamma_j\right)$
N_p^Y	$\sigma_j^Y,\ \gamma_j$	$\left(V_j^Y, I_j^Y\right)$, $\left(V_j, I_j, s_j\right)$
N_p^Δ	$\sigma_j^\Delta,\ \gamma_j$	$\left(V_j^\Delta, I_j^\Delta, \beta_j\right)$, $\left(V_j, I_j, s_j\right)$

Table 9.3: Three-phase analysis problem: given the specification in blue, compute the remaining unknowns in black.

problem does not assume C9.1 and therefore each line (j,k) may model a transmission or distribution line, or a three-phase transformer where its series admittance matrices y_{jk}^s and y_{kj}^s may be different.

We make a few remarks on the voltage γ_j. See Remark 9.3 on how the loop flow β_k of an impedance k may depend on β_j of a current source j.

Remark 9.5 (Voltage γ_j)**.**

1. *Y configuration.* The voltage parameter γ_j needs to be specified for every Y-configured device in our formulation here. It may be specified explicitly, or more likely, indirectly. By that, we mean information additional to generic device models is available to either compute their values, provide additional equations, or eliminate them in terms of other variables. For instance if the neutral of a Y-configured device is grounded and all voltages are defined with respect to the ground, then $\gamma_j = V_j^n = -z_j^n\left(\mathbf{1}^\mathsf{T} I_j\right)$, which allows the elimination of γ_j from the model. If the neutral is grounded directly (i.e., $z_j^n = 0$), then $\gamma_j = 0$. If the neutral is not grounded but the internal voltage V_j^Y is known to satisfy $\mathbf{1}^\mathsf{T} V_j^Y = 0$, then $\gamma_j = \frac{1}{3}\mathbf{1}^\mathsf{T} V_j$. This is studied in detail in Examples 9.3 and 9.4 for a three-wire line model as we have been assuming in almost all of our analysis.

   For a Y-configured current source, γ_j is usually not needed to determine its terminal voltage V_j, but needed to compute its internal voltage $V_j^Y = V_j - \gamma_j\mathbf{1}$ from the terminal voltage V_j.

   As noted above, Example 9.5 and Exercise 9.7 consider a four-wire line model where the neutrals of the voltage source and the impedance are connected to each other. Here the (internal) neutral voltages $\left(\gamma_j, \gamma_k\right)$ of neither device can be arbitrarily specified but must be determined through the network equation and device models.

2. Δ *configuration.* For a Δ-configured voltage source, the zero-sequence voltage $\gamma_j := \frac{1}{3}\mathbf{1}^\mathsf{T} V_j$ needs to be specified, e.g., by specifying one of its terminal voltages, say, V_j^a. For a Δ-configured current source or impedance, γ_j can be determined once its terminal voltage V_j is determined from network equations. For a Δ-configured power source, typically either γ_j or β_j can be specified; see Example 9.11.

3. *Neutral voltage γ_j and zero-sequence voltage.* For any Y-configured device, we have

$$V_j \;=\; V_j^Y \;+\; V_j^n \mathbf{1}$$

   The parameter $\gamma_j := V_j^n$ may or may not equal the zero-sequence voltage $\frac{1}{3}\mathbf{1}^\mathsf{T} V_j$. They are equal if and only if the internal voltages have no zero-sequence component since $\frac{1}{3}\mathbf{1}^\mathsf{T} V_j = \frac{1}{3}\mathbf{1}^\mathsf{T} V_j^Y + V_j^n$.

□

Remark 9.6 (Asymmetry in Δ specification)**.** As summarized in Table 9.3, in our formulation, for Δ configuration, a voltage source needs to specify both $\left(\gamma_j, \beta_j\right)$, but a power source only needs to specify its γ_j, and a current source or impedance needs to specify none. This asymmetry is because internal currents I_j^Δ contain more information (they fix β_j) than internal voltages V_j^Δ (they do not fix γ_j). Device specification and network equation determine $\left(E_j^\Delta, I_j\right)$ for voltage sources, which contains neither β_j nor γ_j. These quantities therefore need to be specified. Device specification and network equation, on the other hand, determine $\left(J_j^\Delta, V_j\right)$ for current sources, which contains both β_j and γ_j. For impedances, as we will see in Chapter 9.2.3, the network equation will determine their internal currents I_j^Δ which contain β_j. When the terminal voltages of all sources, including power sources, are specified or obtained, the terminal voltages V_j of impedances can be determined by the network equation. Therefore both $\left(\gamma_j, \beta_j\right)$ are determined by the network equation in that case. □

9.2.3 Solution strategy

The solution strategy for the problem formulated in Chapter 9.2.2 consists of three steps:

1. Write down a network equation that relates the terminal variables (V, I, s), either the current balance equation (9.5)(9.6) $I = YV$ or the power flow equation (9.12). As discussed in Remark 9.7 we can always use the linear equation $I = YV$.
2. Write down the device models of the given collection of sources and impedances, either their internal models and conversion rules, or their external models.
3. Numerically solve this system of equations for desired variables.

Step 1 specifies, for the entire network, an equation that relates all the terminal variables. For examples, see (9.38) and (9.43) for analysis problems without and with power sources respectively. Step 2 specifies, for each device, equations relating its terminal variables to its internal variables or specified parameters. For examples, see (9.39d)(9.39d) and (9.44a) respectively.

Remark 9.7 (Nonlinearity)**.** Using the nonlinear power flow equations $s_j = \text{diag}\left(V_j(V_j - V_k)^{\mathsf{H}}\left(y^s_{jk}\right)^{\mathsf{H}}\right)$ as the network equation in Step 1 is equivalent to using the linear current balance equation $I = YV$. This is because dividing both sides of the power flow equations by V_j and taking complex conjugate yields $I = YV$. Therefore if no power sources are involved, then the device models of voltage sources, current sources and impedances are linear and therefore the overall model will be linear.

If power sources are involved, then even if we use $I = YV$ as the network equation, the device models of power sources will be quadratic and therefore the overall network will be nonlinear. In this case the power source device model is the only place where nonlinearity appears. □

In the rest of this subsection we first describe in detail Steps 1 and 2 in the general solution strategy outline above to obtain a system of equations that can be solved numerically. In light of Remark 9.7 we will use the current balance equation $I = VY$ as our network equation. Then, motivated by the examples in Chapter 9.2.1, we show how to reduce the entire system of equations obtained from Steps 1 and 2 into a smaller system with possibly much fewer variables, which must be solved numerically. All other variables can then be derived analytically in terms of the solution of the reduced system. (For problems without power sources, this reduces equations (9.38)(9.39) to (9.42).) This simpler solution strategy not only reduces the size of the system that needs numerical solution, but more importantly, it often reveals more clearly the essential structure of the problem. For instance, for problems with power sources, the reduced system is equation (9.47) which consists of a linear equation and a quadratic equation due to power sources.

We first derive the solution for the case without power sources. We then show how to extend the solution to incorporate power sources simply by adding their device models to the systems of equations. We will focus on determining terminal and internal voltages and currents. Once they are determined, internal and external powers can be calculated using $s_j^{Y/\Delta} := \text{diag}\left(V_j^{Y/\Delta} I_j^{Y/\Delta \mathsf{H}}\right)$ and $s_j := \text{diag}\left(V_j I_j^{\mathsf{H}}\right)$ respectively.

Without power sources. Recall that $N_v := N_v^Y \cup V_v^\Delta$, $N_c := N_c^Y \cup V_c^\Delta$, and $N_i := N_i^Y \cup V_i^\Delta$ are the set of buses with, respectively, voltage sources, current sources, and impedances. With a slight abuse of notation define the following (column) vectors of terminal voltages and currents:

$$(V_v, I_v) \;:=\; \left(V_j, I_j,\, j \in N_v\right), \qquad (V_c, I_c) \;:=\; \left(V_j, I_j,\, j \in N_c\right), \qquad (V_i, I_i) \;:=\; \left(V_j, I_j,\, j \in N_i\right)$$

Some of them will be specified and the remaining voltages and currents will be determined from the network equation and device models. Step 1 of the solution strategy is to write the network equation $I = YV$:

$$\begin{bmatrix} I_v \\ I_c \\ I_i \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} Y_{vv} & Y_{vc} & Y_{vi} \\ Y_{cv} & Y_{cc} & Y_{ci} \\ Y_{iv} & Y_{ic} & Y_{ii} \end{bmatrix}}_{Y} \begin{bmatrix} V_v \\ V_c \\ V_i \end{bmatrix} \tag{9.38}$$

where the admittance matrix Y is defined in (9.6).

Step 2 is to describe the device models. The specifications for voltage sources, current sources and impedances are, from Table 9.3:

$$\begin{aligned} \left(E_v^{Y/\Delta}, \gamma_v^{Y/\Delta}, \beta_v^\Delta\right) \;&:=\; \left(E_j^{Y/\Delta}, \gamma_j, j \in N_v^{Y/\Delta};\; \beta_j, j \in N_v^\Delta\right) \\ \left(J_c^{Y/\Delta}, \gamma_c^Y\right) \;&:=\; \left(J_j^{Y/\Delta}, j \in N_c^{Y/\Delta};\; \gamma_j, j \in N_c^Y\right) \\ \left(Z_i^{Y/\Delta}, \gamma_i^Y\right) \;&:=\; \left(\text{diag}\left(z_j^\Delta, j \in N_i^{Y/\Delta}\right);\; \gamma_j, j \in N_i^Y\right) \end{aligned}$$

To unify notation we define the following matrices

$$\begin{aligned} \Gamma_v^{Y\dagger} &:= \mathbb{I}_v^Y \otimes \mathbb{I}, & \Gamma_v^{\Delta\dagger} &:= \mathbb{I}_v^\Delta \otimes \Gamma^\dagger, & \Gamma_v^\dagger &:= \text{diag}\left(\Gamma_v^{Y\dagger}, \Gamma_v^{\Delta\dagger}\right) \\ \Gamma_c^Y &:= \mathbb{I}_c^Y \otimes \mathbb{I}, & \Gamma_c^\Delta &:= \mathbb{I}_c^\Delta \otimes \Gamma, & \Gamma_c &:= \text{diag}\left(\Gamma_c^Y, \Gamma_c^\Delta\right) \\ \Gamma_i^Y &:= \mathbb{I}_i^Y \otimes \mathbb{I}, & \Gamma_i^\Delta &:= \mathbb{I}_i^\Delta \otimes \Gamma, & \Gamma_i &:= \text{diag}\left(\Gamma_i^Y, \Gamma_i^\Delta\right) \end{aligned}$$

where $\mathbb{I}_v^Y, \mathbb{I}_c^Y, \mathbb{I}_i^Y$ are the identity matrices of sizes $|N_v^Y|$, $|N_c^Y|$, $|N_i^Y|$ respectively and $\mathbb{I}_v^\Delta, \mathbb{I}_c^\Delta, \mathbb{I}_i^\Delta$ denote the identity matrices of sizes $|N_v^\Delta|$, $|N_c^\Delta|$, $|N_i^\Delta|$ respectively. Define vectors of specifications

$$E_v \;:=\; \begin{bmatrix} E_v^Y \\ E_v^\Delta \end{bmatrix}, \qquad J_c \;:=\; \begin{bmatrix} J_c^Y \\ J_c^\Delta \end{bmatrix}, \qquad Z_i \;:=\; \text{diag}\left(Z_i^Y, Z_i^\Delta\right) \tag{9.39a}$$

$$\gamma_v \;:=\; \begin{bmatrix} \gamma_v^Y \\ \gamma_v^\Delta \end{bmatrix}, \qquad \gamma_c \;:=\; \begin{bmatrix} \gamma_c^Y \\ 0 \end{bmatrix}, \qquad \gamma_i \;:=\; \begin{bmatrix} \gamma_i^Y \\ 0 \end{bmatrix} \tag{9.39b}$$

so that $\gamma_v \in \mathbb{C}^{|N_v|}$, $\gamma_c \in \mathbb{C}^{|N_c|}$ and $\gamma_i \in \mathbb{C}^{|N_i|}$. Then the terminal voltage and current V_v and I_c in (9.38) are given by

$$V_v \;:=\; \begin{bmatrix} E_v^Y + \gamma_v^Y \otimes \mathbf{1} \\ \Gamma_v^{\Delta\dagger} E_v^\Delta + \gamma_v^\Delta \otimes \mathbf{1} \end{bmatrix} \;=\; \Gamma_v^\dagger E_v \;+\; \gamma_v \otimes \mathbf{1} \tag{9.39c}$$

$$I_c \;:=\; -\begin{bmatrix} J_c^Y \\ \Gamma_c^{\Delta\mathsf{T}} J_c^\Delta \end{bmatrix} \;=\; -\Gamma_c^\mathsf{T} J_c \tag{9.39d}$$

Define the following notations for internal variables of impedances:

$$I_i^Y := \left(I_j^Y,\, j \in N_i^Y\right), \qquad I_i^\Delta := \left(I_j^\Delta,\, j \in N_i^\Delta\right), \qquad I_i^{\text{int}} := \begin{bmatrix} I_i^Y \\ I_i^\Delta \end{bmatrix}$$

$$V_i^Y := \left(V_j^Y,\, j \in N_i^Y\right), \qquad V_i^\Delta := \left(V_j^\Delta,\, j \in N_i^\Delta\right), \qquad V_i^{\text{int}} := \begin{bmatrix} V_i^Y \\ V_i^\Delta \end{bmatrix}$$

The internal model of the impedances in Y and Δ configurations is then

$$V_i^{\text{int}} = Z_i I_i^{\text{int}} \tag{9.39e}$$

where Z_i is defined in (9.39a). The conversion rule for the current and voltage (I_i, V_i) is:

$$I_i = \begin{bmatrix} -I_i^Y \\ -\Gamma_i^{\Delta\mathsf{T}} I_i^\Delta \end{bmatrix} = -\Gamma_i^{\mathsf{T}} I_i^{\text{int}}, \qquad \Gamma_i V_i = \begin{bmatrix} V_i^Y + \gamma_i^Y \otimes \mathbf{1} \\ V_i^\Delta \end{bmatrix} = V_i^{\text{int}} + \gamma_i \otimes \mathbf{1} \tag{9.39f}$$

The analysis problem is: Solve the network equation (9.38) and the device models (9.39) for the unknown external and internal variables. This can be done by numerically solving the system of equations (9.38)(9.39). Note that the analysis problem defined by (9.38)(9.39) does not assume C9.1 and therefore each line (j,k) may model a transmission or distribution line, or a three-phase transformer where its series admittance matrices y_{jk}^s and y_{kj}^s may be different.

The intuition from Example 9.3, Example 9.6 and Exercise 9.8 suggests that, instead of numerically solving (9.38)(9.39), it is possible to reduce it to a smaller system of equations with possibly much fewer variables. Once the reduced system is solved numerically, all other variables can be derived analytically in terms of a solution of the reduced system. The key observation from the examples is to first solve for the internal currents I_i^{int} of all impedances, not their internal voltages V_i^{int} nor other terminal variables (V_i, I_i), using the network equation, the internal device models and the conversion rules. We now explain how to obtain the reduced system of equations in the internal currents I_i^{int} of all impedances and the terminal voltages V_c of all current sources.

Substituting I_i in (9.39f) into (9.38) we have

$$\begin{bmatrix} I_c \\ -\Gamma_i^{\mathsf{T}} I_i^{\text{int}} \end{bmatrix} = \begin{bmatrix} Y_{cv} \\ Y_{iv} \end{bmatrix} V_v + \begin{bmatrix} Y_{cc} & Y_{ci} \\ Y_{ic} & Y_{ii} \end{bmatrix} \begin{bmatrix} V_c \\ V_i \end{bmatrix} \tag{9.40}$$

To express V_i in this equation in terms of I_i^{int}, suppose the inverse

$$\begin{bmatrix} Z_{cc} & Z_{ci} \\ Z_{ic} & Z_{ii} \end{bmatrix} := \begin{bmatrix} Y_{cc} & Y_{ci} \\ Y_{ic} & Y_{ii} \end{bmatrix}^{-1} \tag{9.41}$$

exists and multiplying both sides of (9.40) by this inverse and then by $\text{diag}(\mathbb{I}_c, \Gamma_i)$ we have

$$\text{diag}(\mathbb{I}_c \otimes \mathbb{I}, \Gamma_i) \begin{bmatrix} Z_{cc} & Z_{ci} \\ Z_{ic} & Z_{ii} \end{bmatrix} \begin{bmatrix} I_c \\ -\Gamma_i^{\mathsf{T}} I_i^{\text{int}} \end{bmatrix} = \underbrace{\text{diag}(\mathbb{I}_c \otimes \mathbb{I}, \Gamma_i) \begin{bmatrix} Z_{cc} & Z_{ci} \\ Z_{ic} & Z_{ii} \end{bmatrix} \begin{bmatrix} Y_{cv} \\ Y_{iv} \end{bmatrix}}_{\begin{bmatrix} A_{cv} \\ A_{iv} \end{bmatrix}} V_v + \begin{bmatrix} V_c \\ \Gamma_i V_i \end{bmatrix}$$

where $\mathbb{I}_c$ is the identity matrix of size $|N_c|$. Substituting $\Gamma_i V_i = V_i^{\text{int}} + \gamma_i \otimes \mathbf{1} = Z_i I_i^{\text{int}} + \gamma_i \otimes \mathbf{1}$ from (9.39e) and (9.39f) and re-arranging, we have thus reduced the original system (9.38)(9.39) into the following reduced system in $\left(V_c, I_i^{\text{int}}\right)$:

$$\begin{bmatrix} \mathbb{I}_c \otimes \mathbb{I} & Z_{ci}\Gamma_i^{\mathsf{T}} \\ 0 & \Gamma_i Z_{ii} \Gamma_i^{\mathsf{T}} + Z_i \end{bmatrix} \begin{bmatrix} V_c \\ I_i^{\text{int}} \end{bmatrix} = \begin{bmatrix} Z_{cc} \\ \Gamma_i Z_{ic} \end{bmatrix} I_c - \begin{bmatrix} A_{cv} \\ A_{iv} \end{bmatrix} V_v - \begin{bmatrix} 0 \\ \gamma_i \otimes \mathbf{1} \end{bmatrix} \tag{9.42}$$

Here V_v, I_c, Z_i and γ_i are given by (9.39), the submatrices $Z_{cc}, Z_{ci}, Z_{ic}, Z_{ii}$ are from the inverse in (9.41), and

$$\begin{bmatrix} A_{cv} \\ A_{iv} \end{bmatrix} := \operatorname{diag}(\mathbb{I}_c \otimes \mathbb{I}, \Gamma_i) \begin{bmatrix} Z_{cc} & Z_{ci} \\ Z_{ic} & Z_{ii} \end{bmatrix} \begin{bmatrix} Y_{cv} \\ Y_{iv} \end{bmatrix}$$

All quantities on the right-hand side of (9.42) are known. This is a system of $3(|N_c| + |V_i|)$ linear equations in $3(|N_c| + |V_i|)$ unknowns $\left(V_c, I_i^{\text{int}}\right)$. Assuming the matrix on the left-hand side is invertible, the methods described in Chapter 4.2.4 can be used to compute numerically a solution $\left(V_c, I_i^{\text{int}}\right)$ of (9.42).

We now explain how to derive all the remaining variables.

1. For impedances, with I_i^{int}, the internal voltage $V_i^{\text{int}} = Z_i I_i^{\text{int}}$ and the terminal current $I_i = -\Gamma_i^{\mathsf{T}} I_i^{\text{int}}$ from the internal model (9.39e) and the conversion rule (9.39f). With both I_i^{int} and V_c, we can obtain V_i from (9.40). The zero-sequence voltages and currents $\left(\gamma_j = \frac{1}{3}\mathbf{1}^{\mathsf{T}} V_j,\ \beta_j := \frac{1}{3}\mathbf{1}^{\mathsf{T}} I_j^{\Delta}\right)$ of all Δ-configured impedances $j \in N_i^{\Delta}$ can then be derived from $\left(V_i, I_i^{\text{int}}\right)$. This completes the derivation of all voltages and currents of impedances.
2. For voltage sources, with (V_v, V_c, V_i), the terminal current I_v can be derived from (9.38). For Y-configured voltage sources $j \in N_v^Y$, the internal currents are $I_j^Y = -I_j$. For Δ-configured voltage sources $j \in N_v^{\Delta}$, β_j are given and hence the internal currents are $I_j^{\Delta} = -\frac{1}{3}\Gamma I_j + \beta_j \mathbf{1}$. This completes the derivation of all voltages and currents of power sources.
3. For Y-configured current sources $j \in N_c^Y$, γ_j are given and hence the internal voltages are $V_j^Y = V_j - \gamma_j \mathbf{1}$. For Δ-configured current sources $j \in N_c^{\Delta}$, β_j can be calculated from J_j^{Δ} and $\left(V_j^{\Delta}, \gamma_j\right)$ can be calculated from V_j. This completes the derivation of the voltages and currents of all current sources.

With all voltages and currents determined, the internal and external powers are then $s_j^{Y/\Delta} := \operatorname{diag}\left(V_j^{Y/\Delta} I_j^{Y/\Delta \mathsf{H}}\right)$ and $s_j = \operatorname{diag}\left(V_j I_j^{\mathsf{H}}\right)$, $j \in \overline{N}$, respectively. This completes the derivation of the variables of all devices in the network.

Remark 9.8. The derivation of the reduced system (9.42) depends critically on the assumption that the admittance matrix in (9.40) and the effective impedance matrix $\Gamma_i Z_{ii} \Gamma_i^{\mathsf{T}} + Z_i$ in (9.42) are invertible. When that is not the case, additional information will be needed to uniquely determine all the quantities.

1. If there are voltage sources then the matrix in (9.40) is a strict submatrix of an admittance matrix and therefore will be invertible if the conditions in Theorem 4.5 are satisfied, including the condition $y_{jk}^s = y_{kj}^s$.

In Example 9.3 where a voltage source j supplies an impedance k both in Y configuration over a three-phase line, the equation (9.40) is (9.14a) for which the inverse exists. In Example 9.6 where the devices are in Δ configuration, the equation (9.40) takes the form

$$-\Gamma^{\mathsf{T}} I_k^{\Delta} \ = \ -y_{jk}^s V_j + y_{jk}^s V_k$$

so the inverse $\left(y_{jk}^s\right)^{-1}$ also exists.

2. When only current sources are present, the matrix in (9.40) is the network admittance matrix and is invertible if the conditions in Theorem 4.3 are satisfied, including the condition $y_{jk}^s = y_{kj}^s$. In particular if the shunt admittances of all three-phase lines are assumed zero, then the admittance matrix is not invertible because it will have zero row sums. In that case, additional information needs to be specified to provide an additional equation to (9.40) for solving $\left(V_c, I_i^{\text{int}}\right)$ and V_i.

   In Exercise 9.8 where the voltage source is replaced by a current source j and shunt admittances $\left(y_{jk}^m, y_{kj}^m\right)$ are assumed zero, the equation (9.40) takes the form

$$\begin{bmatrix} I_j \\ -\Gamma^{\mathsf{T}} I_k^{\Delta} \end{bmatrix} \ = \ \begin{bmatrix} y_{jk}^s & -y_{jk}^s \\ -y_{kj}^s & y_{kj}^s \end{bmatrix} \begin{bmatrix} V_j \\ V_k \end{bmatrix}$$

   for which the inverse does not exist. As a result the zero-sequence voltage γ_j of the current source is also specified to provide the additional equation for solving $\left(V_j, I_i^{\text{int}}\right)$. If the shunt admittances $\left(y_{jk}^m, y_{kj}^m\right)$ are nonzero as in Exercise 9.9, γ_j of the current source need not be specified and can be derived because the equation above will be invertible.

3. The reduced system (9.42) generalizes (9.14b) in Example 9.3 and (9.18) in Example 9.6 to general networks and with current sources.

□

With power sources. Analysis problems with power sources can be solved following the same procedure, but with the addition of device models of power sources. Specifically the current balance equation (9.38) is extended to

$$\begin{bmatrix} I_v \\ I_c \\ I_i \\ I_p \end{bmatrix} \ = \ \underbrace{\begin{bmatrix} Y_{vv} & Y_{vc} & Y_{vi} & Y_{vp} \\ Y_{cv} & Y_{cc} & Y_{ci} & V_{cp} \\ Y_{iv} & Y_{ic} & Y_{ii} & Y_{ip} \\ Y_{pv} & Y_{pc} & Y_{pi} & Y_{pp} \end{bmatrix}}_{Y} \begin{bmatrix} V_v \\ V_c \\ V_i \\ V_p \end{bmatrix} \tag{9.43}$$

where $(V_p, I_p) := \left(V_j, I_j,\, j \in N_p\right)$, with $N_p := N_p^Y \cup N_p^{\Delta}$, are the terminal voltages and currents of power sources.

The device model (9.39) also needs to be extended to include power sources. For a Y-configured power source, $\left(s_j^Y := \sigma_j^Y, \gamma_j := V_j^n\right)$ are specified. For a Δ-configured power source, we assume that

$\left(s_j^\Delta := \sigma_j^\Delta, \gamma_j := \frac{1}{3}\mathbf{1}^\mathsf{T} V_j\right)$ are specified. Let $\sigma_p := \begin{bmatrix} \sigma_p^Y \\ \sigma_p^\Delta \end{bmatrix}$. Then the internal models of the power sources in Y and Δ configurations are

$$\sigma_p^Y \;=\; \left(\mathrm{diag}\left(V_j^Y I_j^{Y\mathsf{H}}\right), \, j \in N_p^Y\right), \qquad\qquad \sigma_p^\Delta \;:=\; \left(\mathrm{diag}\left(V_j^\Delta I_j^{\Delta\mathsf{H}}\right), \, j \in N_p^\Delta\right)$$

To simplify notation define the internal currents and voltages for all power sources:

$$\begin{aligned} I_p^Y &:= \left(I_j^Y, \, j \in N_p^Y\right), & I_p^\Delta &:= \left(I_j^\Delta, \, j \in N_p^\Delta\right), & I_p^{\text{int}} &:= \begin{bmatrix} I_p^Y \\ I_p^\Delta \end{bmatrix} \\ V_p^Y &:= \left(V_j^Y, \, j \in N_p^Y\right), & V_p^\Delta &:= \left(V_j^\Delta, \, j \in N_p^\Delta\right), & V_p^{\text{int}} &:= \begin{bmatrix} V_p^Y \\ V_p^\Delta \end{bmatrix} \end{aligned}$$

Then the internal models of the power sources can be written as

$$\sigma_p \;=\; \mathrm{diag}\left(V_p^{\text{int}} I_p^{\text{int}\mathsf{H}}\right) \tag{9.44a}$$

This is a quadratic equation in the unknowns internal voltage and current $\left(V_p^{\text{int}}, I_p^{\text{int}}\right)$.[5] Define

$$\Gamma_p^\Delta \;:=\; \mathbb{I}_p^\Delta \otimes \Gamma, \qquad\qquad \Gamma_p \;:=\; \mathrm{diag}\left(\mathbb{I}_p^Y, \Gamma_p^\Delta\right), \qquad\qquad \gamma_p \;:=\; \begin{bmatrix} \gamma_p^Y \\ 0 \end{bmatrix}$$

where $\mathbb{I}_p^\Delta$ denotes the identity matrix of size $|N_p^\Delta|$, $\mathbb{I}_p^Y$ the identity matrices of size $3|N_p^Y|$, and $\gamma_p^Y := \left(\gamma_j := V_j^n, \, j \in N_p^Y\right)$ are neutral voltages of all Y-configured power sources j. The current and voltage conversion rule is (similar to (9.39f))

$$I_p \;=\; -\Gamma_p I_p^{\text{int}}, \qquad\qquad \Gamma_p V_p \;=\; V_p^{\text{int}} + \gamma_p \otimes \mathbf{1} \tag{9.44b}$$

The analysis problem can be stated as: Solve the network equation (9.43) and the device models (9.39) (9.44) for the unknown external and internal variables. This system of equations (9.43)(9.39)(9.44) can be solved numerically.

We will follow the same procedure to reduce (9.43)(9.39)(9.44) into a smaller system of (nonlinear) equations that involves only $\left(V_c, I_i^{\text{int}}, I_p^{\text{int}}, V_p^{\text{int}}\right)$. All other variables can then be derived from a solution $\left(V_c, I_i^{\text{int}}, I_p^{\text{int}}, V_p^{\text{int}}\right)$.

Substituting I_i, I_p in (9.39f) and (9.44b) respectively into (9.43) we have

$$\mathrm{diag}\left(\mathbb{I}_c, -\Gamma_i^\mathsf{T}, -\Gamma_p^\mathsf{T}\right) \begin{bmatrix} I_c \\ I_i^{\text{int}} \\ I_p^{\text{int}} \end{bmatrix} \;=\; \begin{bmatrix} Y_{cv} \\ Y_{iv} \\ Y_{pv} \end{bmatrix} V_v \;+\; \begin{bmatrix} Y_{cc} & Y_{ci} & Y_{cp} \\ Y_{ic} & Y_{ii} & Y_{ip} \\ Y_{pc} & Y_{pi} & Y_{pp} \end{bmatrix} \begin{bmatrix} V_c \\ V_i \\ V_p \end{bmatrix} \tag{9.45}$$

[5] We can also use the equivalent model $\sigma_p = \mathrm{diag}\left((\Gamma_p V_p - \gamma_p) I_p^{\text{int}\mathsf{H}}\right)$ of power sources in terms of the terminal voltage V_p, the internal current I_p^{int}, and the neutral voltage γ_p^Y. The network equation however will only allow us to solve for $\Gamma_j V_j = V_j^\Delta$ for Δ-configured power sources j. Specifically V_p^{int} in (9.47a) will be replaced by $\Gamma_p V_p$ and (9.47b) by $\sigma_p = \mathrm{diag}\left((\Gamma_p V_p - \gamma_p) I_p^{\text{int}\mathsf{H}}\right)$. Therefore it is simpler to solve for the internal voltage V_p^{int} and then use γ_j to obtain the terminal voltages V_j of Δ-configured power sources j.

Suppose the inverse

$$\begin{bmatrix} Z_{cc} & Z_{ci} & Z_{cp} \\ Z_{ic} & Z_{ii} & Z_{ip} \\ Z_{pc} & Z_{pi} & Z_{pp} \end{bmatrix} := \begin{bmatrix} Y_{cc} & Y_{ci} & Y_{cp} \\ Y_{ic} & Y_{ii} & Y_{ip} \\ Y_{pc} & Y_{pi} & Y_{pp} \end{bmatrix}^{-1} \tag{9.46}$$

exists and multiplying both sides by this inverse and then by $\text{diag}(\mathbb{I}_c, \Gamma_i, \Gamma_p)$ we have

$$\text{diag}(\mathbb{I}_c, \Gamma_i, \Gamma_p) \begin{bmatrix} Z_{cc} & Z_{ci} & Z_{cp} \\ Z_{ic} & Z_{ii} & Z_{ip} \\ Z_{pc} & Z_{pi} & Z_{pp} \end{bmatrix} \text{diag}\left(\mathbb{I}_c, -\Gamma_i^{\mathsf{T}}, -\Gamma_p^{\mathsf{T}}\right) \begin{bmatrix} I_c \\ I_i^{\text{int}} \\ I_p^{\text{int}} \end{bmatrix} = \begin{bmatrix} B_{cv} \\ B_{iv} \\ B_{pv} \end{bmatrix} V_v + \begin{bmatrix} V_c \\ \Gamma_i V_i \\ \Gamma_p V_p \end{bmatrix}$$

where

$$\begin{bmatrix} B_{cv} \\ B_{iv} \\ B_{pv} \end{bmatrix} := \text{diag}(\mathbb{I}_c, \Gamma_i, \Gamma_p) \begin{bmatrix} Z_{cc} & Z_{ci} & Z_{cp} \\ Z_{ic} & Z_{ii} & Z_{ip} \\ Z_{pc} & Z_{pi} & Z_{pp} \end{bmatrix} \begin{bmatrix} Y_{cv} \\ Y_{iv} \\ Y_{pv} \end{bmatrix}$$

Substituting $\Gamma_i V_i = V_i^{\text{int}} + \gamma_i \otimes \mathbf{1} = Z_i I_i^{\text{int}} + \gamma_i \otimes \mathbf{1}$ from (9.39e) and (9.39f), $\Gamma_p V_p = V_p^{\text{int}} + \gamma_p \otimes \mathbf{1}$ from (9.44b), and re-arranging, we have

$$\begin{bmatrix} \mathbb{I}_c & Z_{ci}\Gamma_i^{\mathsf{T}} & Z_{cp}\Gamma_p^{\mathsf{T}} & 0 \\ 0 & \Gamma_i Z_{ii}\Gamma_i^{\mathsf{T}} + Z_i & \Gamma_i Z_{ip}\Gamma_p^{\mathsf{T}} & 0 \\ 0 & \Gamma_p Z_{pi}\Gamma_i^{\mathsf{T}} & \Gamma_p Z_{pp}\Gamma_p^{\mathsf{T}} & \mathbb{I}_p \end{bmatrix} \begin{bmatrix} V_c \\ I_i^{\text{int}} \\ I_p^{\text{int}} \\ V_p^{\text{int}} \end{bmatrix} = \begin{bmatrix} Z_{cc} \\ \Gamma_i Z_{ic} \\ \Gamma_p Z_{pc} \end{bmatrix} I_c - \begin{bmatrix} B_{cv} \\ B_{iv} \\ B_{pv} \end{bmatrix} V_v - \begin{bmatrix} 0 \\ \gamma_i \\ \gamma_p \end{bmatrix} \otimes \mathbf{1} \tag{9.47a}$$

$$\text{diag}\left(V_p^{\text{int}} I_p^{\text{int}\mathsf{H}}\right) = \sigma_p \tag{9.47b}$$

The reduced system of (9.43)(9.39)(9.44) is (9.47) which must be solved numerically. The analysis problem therefore becomes: Solve (9.47) for $\left(V_c, I_i^{\text{int}}, I_p^{\text{int}}, V_p^{\text{int}}\right)$ and derive all other variables analytically (Exercise 9.13). As before, the analysis problem does not assume C9.1 and therefore each line (j,k) may model a transmission or distribution line, or a three-phase transformer where its series admittance matrices y_{jk}^s and y_{kj}^s may be different.

We make three remarks. First, compared with the reduced system (9.42) without power sources, the reduced system (9.47) involves two more variables $\left(V_p^{\text{int}}, V_p^{\text{int}}\right)$ with two additional sets of equations. While (9.42) is linear, (9.47) is quadratic because of the device model (9.47b) of power sources. Even if the inverse in (9.46) exists and the matrix on the left-hand side of (9.47a) is invertible, (9.47) may or may not have a solution which may or may not be unique because of the nonlinearity. Second, these inverses may not exist in which case more information is needed to determine a solution. For example, when there are no voltage sources as in Example 9.9 and the shunt admittances $y_{jk}^m = y_{kj}^m = 0$, the admittance matrix in (9.45) has zero row sums and is singular. In that case additional information $(\beta_j + \beta_k)$ is given, compared with the case in Example 9.8; see also Remark 9.8. Finally, the linearity of (9.47a) is the consequence of using the linear current balance equation $I = YV$ in (9.43), and this is always possible as discussed in Remark 9.7.

9.3 Balanced network

In this section we show that, if the voltage sources, current sources, and impedances are generalized balanced vectors and the lines are decoupled, then the analysis problem in Chapter 9.2 can be solved by analyzing certain simpler per-phase networks. The intuition is that the balanced voltage and current sources render all voltages and currents in the network to be balanced due to Corollary 1.4. To simplify exposition we only consider the case without power sources so that our problem remains linear.

With today's abundant computing power the smaller problem size may not be an important advantage of per-phase analysis. Rather, per-phase analysis clarifies the simple structure underlying a balanced network and enhances our conceptual understanding of three-phase networks in general, balanced or unbalanced.

We start in Chapter 9.3.1 by summarizing properties of Kronecker product which underlies the equivalence of three-phase analysis and per-phase analysis for a balanced network.

9.3.1 Kronecker product

The simple structure that underlies balanced networks depends critically on properties of the Kronecker product. For instance the admittance matrix Y of a balanced three-phase network can be written as the Kronecker product of a per-phase admittance matrix and the identity matrix $\mathbb{I}$ of size 3. This is explained in Chapter 9.3. In particular we will use the following properties in the proof of Theorem **??** there.

Lemma 9.6 (Kronecker product)**.** Let A,B,C,D be complex matrices of appropriate dimensions.

1. $(A+B)\otimes C = (A\otimes C) + (B\otimes C); C\otimes(A+B) = (C\otimes A) + (C\otimes B)$.
2. $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$.
3. $(A\otimes B)^{\mathsf{T}} = A^{\mathsf{T}}\otimes B^{\mathsf{T}}; (A\otimes B)^{\mathsf{H}} = A^{\mathsf{H}}\otimes B^{\mathsf{H}}$.
4. $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1};\ \ (A\otimes B)^{\dagger} = A^{\dagger}\otimes B^{\dagger}$ where $A^{\dagger}$ denotes the pseudo-inverse of A.
5. rank $(A\otimes B) =$ rank $A\cdot$rank B.
6. If $A\in\mathbb{C}^{m\times n}$ is invertible and $X,Y\in\mathbb{C}^{p\times q}$ then
$$A\otimes X = A\otimes Y, \qquad\qquad \Longleftrightarrow \qquad\qquad X = Y$$

The proof of the lemma is left as Exercise **??**

9.3.2 Three-phase analysis

We first explain how the device models and the admittance matrix simplify in a balanced system. We then use that to simplify the three-phase analysis problem in Chapter 9.2. Finally we show that the problem is equivalent to solving per-phase systems.

Balanced devices. When the devices are balanced positive-sequence sets with parameters $\lambda_j, \mu_j, \zeta_j \in \mathbb{C}$:

$$E_j^{Y/\Delta} := \lambda_j \alpha_+, \quad j \in N_v, \qquad J_j^{Y/\Delta} := \mu_j \alpha_+, \quad j \in N_c, \qquad z_j^{Y/\Delta} := \zeta_j \mathbb{I}, \quad j \in N_i$$

their internal models in Table 9.3 reduce to those specified in Table 9.4. In vector form the voltage sources are

$$E_v^Y = \lambda_v^Y \otimes \alpha_+, \qquad E_v^\Delta = \lambda_v^\Delta \otimes \alpha_+, \qquad E_v := \begin{bmatrix} E_v^Y \\ E_v^\Delta \end{bmatrix} = \lambda_v \otimes \alpha_+$$

where $\lambda_v^Y := \left(\lambda_j, j \in N_v^Y\right)$, $\lambda_v^\Delta := \left(\lambda_j, j \in N_v^\Delta\right)$ and $\lambda_v := \left(\lambda_j, j \in N_v\right)$. Defining similar quantities for current sources and impedances, the specification (9.39a)(9.39b) in vector form reduces to

$$E_v := \begin{bmatrix} \lambda_v^Y \\ \lambda_v^\Delta \end{bmatrix} \otimes \alpha_+ = \lambda_v \otimes \alpha_+, \qquad \gamma_v := \begin{bmatrix} \gamma_v^Y \\ \gamma_v^\Delta \end{bmatrix} \tag{9.48a}$$

$$J_c := \begin{bmatrix} \mu_c^Y \\ \mu_c^\Delta \end{bmatrix} \otimes \alpha_+ = \mu_c \otimes \alpha_+, \qquad \gamma_c^0 := \begin{bmatrix} \gamma_c^Y \\ 0 \end{bmatrix} \tag{9.48b}$$

$$Z_i := \text{diag}\left(\zeta_i^Y, \zeta_i^\Delta\right) \otimes \mathbb{I} = \zeta_i \otimes \mathbb{I}, \qquad \gamma_i^0 := \begin{bmatrix} \gamma_i^Y \\ 0 \end{bmatrix} \tag{9.48c}$$

where $\zeta_i^Y := \text{diag}\left(\zeta_j, j \in N_i^Y\right)$, $\zeta_i^\Delta := \text{diag}\left(\zeta_j, j \in N_i^\Delta\right)$, $\zeta_i := \text{diag}\left(\zeta_i^Y, \zeta_i^\Delta\right)$ are diagonal matrices of sizes $\left|V_i^Y\right|$, $\left|V_i^\Delta\right|$, $|V_i|$ respectively.

The external models in Table 9.4 are obtained by substituting these specifications into the external models in Table 9.3 and applying Corollary 1.4 and Theorem 7.2, specifically

$$\Gamma \alpha_+ = (1-\alpha)\alpha_+, \qquad \Gamma^{\mathsf{T}} \alpha_+ = \left(1-\alpha^2\right)\alpha_+, \qquad \Gamma^\dagger = \frac{1}{3}\Gamma^{\mathsf{T}}, \qquad \Gamma^{\mathsf{T}\dagger} = \frac{1}{3}\Gamma$$

The derivation of the impedance model in Table 9.4 in Δ configuration is left as Exercise 9.14. These

Buses j	Specification	External model	Vars	Internal vars
N_v^Y	$E_j^Y = \lambda_j \alpha_+,\ \gamma_j$	$V_j = \lambda_j \alpha_+ + \gamma_j \mathbf{1}$	I_j	$I_j^Y = -I_j$
N_v^Δ	$E_j^\Delta = \lambda_j \alpha_+,\ \gamma_j,\ \beta_j$	$V_j = \frac{1}{3}(1-\alpha^2)\lambda_j \alpha_+ + \gamma_j \mathbf{1}$	I_j	$I_j^\Delta = -\Gamma^{\mathsf{T}\dagger} I_j + \beta_j \mathbf{1}$
N_c^Y	$J_j^Y = \mu_j \alpha_+, \gamma_j$	$I_j = -\mu_j \alpha_+$	V_j	$V_j^Y = V_j - \gamma_j \mathbf{1}$
N_c^Δ	$J_j^\Delta = \mu_j \alpha_+$	$I_j = -(1-\alpha^2)\mu_j \alpha_+$	V_j	$V_j^\Delta = \Gamma V_j,\ \gamma_j := \frac{1}{3}\mathbf{1}^{\mathsf{T}} V_j$ $\beta_j := \frac{1}{3}\mathbf{1}^{\mathsf{T}} I_j^\Delta$
N_i^Y	$z_j^Y = \zeta_j \mathbb{I},\ \gamma_j$	$I_j = -\eta_j \left(V_j - \gamma_j \mathbf{1}\right)$	(V_j, I_j)	$V_j^Y = V_j - \gamma_j \mathbf{1},\ I_j^Y = -I_j$
N_i^Δ	$z_j^\Delta = \zeta_j \mathbb{I},\ \beta_j$	$I_j = -3\eta_j \left(V_j - \gamma_j \mathbf{1}\right)$	(V_j, I_j)	$V_j^\Delta = \Gamma V_j,\ \gamma_j := \frac{1}{3}\mathbf{1}^{\mathsf{T}} V_j$ $I_j^\Delta = -\Gamma^{\mathsf{T}\dagger} I_j + \beta_j \mathbf{1}$

Table 9.4: Internal and external models of balanced positive-sequence sources and impedances with $\eta_j := \zeta_j^{-1}$. The impedance model for N_i^Δ in the table is equivalent to $I_j = -3\eta_j \left(V_j - \left(\frac{1}{3}\mathbf{1}^{\mathsf{T}} V_j\right)\mathbf{1}\right)$ which is the model $I_j = -Y_j^\Delta V_j$ in Table 9.3.

models are special cases of the three-phase devices in Chapters 7.3.2 and 7.3.3. To simplify the notation for the external models of voltage and current sources, define

$$\hat{\alpha}_j \;:=\; \begin{cases} 1 & \text{if} \quad j \in N_v^Y \cup N_c^Y \cup N_i^Y \\ (1-\alpha^2)/3 & \text{if} \quad j \in N_v^\Delta \qquad \text{(voltage sources)} \\ (1-\alpha^2) & \text{if} \quad j \in N_c^\Delta \qquad \text{(current sources)} \\ 3 & \text{if} \quad j \in N_i^\Delta \qquad \text{(admittance)} \end{cases}$$

Then when the voltage and current sources are balanced, their external models (9.39c)(9.39d) reduce to:

$$V_v \;=\; \left(\hat{\alpha}_j \lambda_j \alpha_+ + \gamma_j \mathbf{1}, j \in N_v\right) \;=:\; \hat{\lambda}_v \otimes \alpha_+ \;+\; \gamma_v \otimes \mathbf{1} \tag{9.48d}$$

$$I_c \;=\; \left(-\hat{\alpha}_j \mu_j \alpha_+, j \in N_c\right) \;=:\; -\hat{\mu}_c \otimes \alpha_+ \tag{9.48e}$$

where $\hat{\lambda}_v, \gamma_v \in \mathbb{C}^{|N_v|}$ and $\hat{\mu}_c \in \mathbb{C}^{|N_c|}$.

Remark 9.9 (Δ-Y transformation)**.** The specification (9.48d)(9.48e) corresponds to the first step of per-phase analysis in Chapter 1.2.5 that converts all Δ configured devices to their Y equivalents that have the same external behavior. It generalizes the standard practice of assuming $\gamma_j = 0$ to the case where γ_j may be nonzero, because some Y-configured devices on the network are not grounded, some are grounded through nonzero earthing impedances, and some Δ-configured devices have nonzero zero-sequence voltages. $\square$

The internal models of impedances (9.39e) and the conversion rules (9.39f) become

$$V_i^{\text{int}} \;=\; Z_i I_i^{\text{int}} \;=\; (\zeta_i \otimes \mathbb{I})\, I_i^{\text{int}} \tag{9.48f}$$

$$I_i \;=\; \begin{bmatrix} -I_i^Y \\ -\left(\mathbb{I}_i^\Delta \otimes \Gamma_i^{\Delta\mathsf{T}}\right) I_i^\Delta \end{bmatrix} \;=\; -\Gamma_i^{\mathsf{T}} I_i^{\text{int}} \tag{9.48g}$$

$$\Gamma_i V_i \;=\; \begin{bmatrix} V_i^Y + \gamma_i^Y \otimes \mathbf{1} \\ V_i^\Delta \end{bmatrix} \;=\; V_i^{\text{int}} + \gamma_i^0 \otimes \mathbf{1} \tag{9.48h}$$

where Z_i, ζ_i, γ_i^0 are defined in (9.48c), and $\mathbb{I}_i^Y, \mathbb{I}_i^\Delta$ are the identity matrices of sizes $\left|V_i^Y\right|, \left|V_i^\Delta\right|$ respectively.

Balanced admittance matrix Y. We assume all lines are balanced, i.e.,

$$y_{jk}^s \;=\; \eta_{jk}^s \mathbb{I}, \qquad\qquad y_{jk}^m \;=\; \eta_{jk}^m \mathbb{I}, \qquad\qquad y_{kj}^m \;=\; \eta_{kj}^m \mathbb{I} \tag{9.49a}$$

for some constants $\eta_{jk}^s, \eta_{jk}^m, \eta_{kj}^m \in \mathbb{C}$. The terminal voltages and currents $V := (V_0, \ldots, V_N)$ and $I := (I_0, \ldots, I_N)$ are described by (9.5) which, with balanced lines, reduces to

$$I_j \;=\; \sum_{k: j \sim k} \left(y_{jk}^s \;+\; y_{jk}^m\right) V_j \;-\; \sum_{k: j \sim k} y_{jk}^s V_k \;=\; \sum_{k: j \sim k} \eta_{jk} V_j \;-\; \sum_{k: j \sim k} \eta_{jk}^s V_k, \qquad j \in \overline{N} \tag{9.49b}$$

where $\eta_{jk} := \eta_{jk}^s + \eta_{jk}^m$ and $V_j, I_j \in \mathbb{C}^3$. This in vector form is $I = YV$. The balanced lines in (9.49a) allow us to write the admittance matrix Y using the Kronecker product. This is the key mathematical structure, in addition to the conversion matrices $\Gamma, \Gamma^{\mathsf{T}}$ as described in Corollary 1.4, that underlies the balanced property of all voltages and currents in the network.

Specifically, define the $(N+1)\times(N+1)$ *per-phase admittance matrix* $Y^{1\phi}$ by

$$Y^{1\phi}_{jk} \;:=\; \begin{cases} -\eta^s_{jk}, & (j,k)\in E,\;\;(j\neq k) \\ \sum_{k:j\sim k}\left(\eta^s_{jk} + \eta^m_{jk}\right), & j=k \\ 0 & \text{otherwise} \end{cases} \tag{9.50a}$$

As we will see, this is the bus admittance matrix studied in Chapter 4.2 for the per-phase circuit of a balanced three-phase network where each line is characterized by four complex scalars $\left(\eta^s_{jk},\eta^m_{jk}\right)$, $\left(\eta^s_{kj},\eta^m_{kj}\right)$. In particular Y does not assume C9.1 and hence $Y^{1\phi}$ may not satisfy C4.1. Therefore each line (j,k) may model a transmission or distribution line, or a three-phase transformer where its series admittance matrices y^s_{jk} and y^s_{kj} may be different.

Substituting (9.49a) into the admittance matrix Y in (9.6) for the three-phase network, we can write Y in terms of the per-phase admittance matrix $Y^{1\phi}$ using the Kronecker product:

$$Y \;=\; Y^{1\phi}\otimes\mathbb{I} \tag{9.50b}$$

The relation $I=YV$ for the three-phase network becomes

$$I \;=\; \left(Y^{1\phi}\otimes\mathbb{I}\right)V \tag{9.50c}$$

Three-phase analysis. We are interested in determining the (column) vectors of terminal and internal variables

$$V_{-v} \;:=\; (V_c,V_i) \;:=\; \left(V_j,\, j\in N_c\cup N_i\right), \qquad I_{-c} \;:=\; (I_v,I_i) \;:=\; \left(I_j,\, j\in N_c\cup N_i\right) \tag{9.51a}$$

$$V^{\text{int}}_{-v} \;:=\; \left(V^{\text{int}}_c,V^{\text{int}}_i\right) \;:=\; \left(V^{Y/\Delta}_j,\, j\in N_c\cup N_i\right), \qquad I^{\text{int}}_{-c} \;:=\; \left(I^{\text{int}}_v,I^{\text{int}}_i\right) \;:=\; \left(I^{Y/\Delta}_j,\, j\in N_c\cup N_i\right) \tag{9.51b}$$

$$\gamma^{\Delta}_{-v} \;:=\; \left(\gamma^{\Delta}_c,\gamma^{\Delta}_i\right) \;:=\; \left(\gamma_j,\, j\in N^{\Delta}_c\cup N^{\Delta}_i\right), \qquad \beta^{\Delta}_{-v} \;:=\; \left(\beta^{\Delta}_j,\beta^{\Delta}_j\right) \;:=\; \left(\beta_j,\, j\in N^{\Delta}_c\cup N^{\Delta}_i\right) \tag{9.51c}$$

Let $x:=\left(V_{-v},I_{-c},V^{\text{int}}_{-v},I^{\text{int}}_{-c},\gamma^{\Delta}_{-v},\beta^{\Delta}_{-v}\right)$. When the network is balanced the three-phase analysis problem in Chapter 9.2 reduces to: solve for x given the device specification (9.48) and the network equation (9.50).

9.3.3 Balanced voltages and currents

In this subsection we prove a structural result that says that, when the internal voltages and currents of non-power sources are balanced, so are all other voltages and currents in the network.

Partition the per-phase admittance matrix $Y^{1\phi}$ defined in (9.50) into submatrices (A_{11},A_{21},A_{22}):

$$Y^{1\phi} \;=:\; \left[\begin{array}{c|cc} Y^{1\phi}_{vv} & Y^{1\phi}_{vc} & Y^{1\phi}_{vi} \\ \hline Y^{1\phi}_{cv} & Y^{1\phi}_{cc} & Y^{1\phi}_{ci} \\ Y^{1\phi}_{iv} & Y^{1\phi}_{ic} & Y^{1\phi}_{ii} \end{array}\right] \;=:\; \left[\begin{array}{c|c} A_{11} & A_{21}^{\mathsf{T}} \\ \hline A_{21} & A_{22} \end{array}\right] \tag{9.52}$$

The matrix A_{22} is complex symmetric and therefore a legitimate admittance matrix. We will make two assumptions on the per-phase admittance matrix $Y^{1\phi}$.

C9.7: The submatrix A_{22} is invertible.

Assuming C9.7 (see Chapter 4.2.5 for sufficient conditions for the invertibility of principal submatrices of an admittance matrix), denote the inverse of the submatrix A_{22} by

$$\begin{bmatrix} Z_{cc}^{1\phi} & Z_{ci}^{1\phi} \\ Z_{ic}^{1\phi} & Z_{ii}^{1\phi} \end{bmatrix} \;:=\; \begin{bmatrix} Y_{cc}^{1\phi} & Y_{ci}^{1\phi} \\ Y_{ic}^{1\phi} & Y_{ii}^{1\phi} \end{bmatrix}^{-1} \;=\; A_{22}^{-1} \tag{9.53a}$$

Then the inverse in (9.41) exists and is:

$$\begin{bmatrix} Z_{cc} & Z_{ci} \\ Z_{ic} & Z_{ii} \end{bmatrix} := \begin{bmatrix} Y_{cc} & Y_{ci} \\ Y_{ic} & Y_{ii} \end{bmatrix}^{-1} \;=\; A_{22}^{-1} \otimes \mathbb{I} \tag{9.53b}$$

where we have used $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ (Lemma 9.6). The second assumption is:

C9.8: The impedances $\zeta_j \in \mathbb{C}$ are nonzero for all $j \in N_i$, the submatrix $Z_{ii}^{1\phi}$ in (9.53a) and the matrix

$$\hat{C}_i \;=\; \left(\left(Z_{ii}^{1\phi} \right)^{-1} \otimes \mathbb{I} \right) \;+\; \Gamma_i^{\mathsf{T}} \left(\zeta_i^{-1} \otimes \mathbb{I} \right) \Gamma_i \tag{9.54}$$

are invertible.

Theorem 9.7 (Balanced voltages and currents)**.** Suppose C9.7 and C9.8 hold.

1. Any solution x of (9.48)(9.50) consists of generalized balanced vectors in positive sequence, i.e., any voltage or current x_j in (9.51) at bus j is of the form $x_j = a_j \alpha_+ + b_j \mathbf{1}$ for some $a_j, b_j \in \mathbb{C}$.

2. Moreover all x_j are balanced vectors, i.e., $b_j = 0$, if $\gamma_v = 0$ for all voltage sources and the neutral voltages $\gamma_i^Y = 0$ for all Y configured impedances.

In the rest of this subsection we prove the theorem following the solution strategy in Chapter 9.2.3 to show that any solution $\left(V_c, I_i^{\text{int}}\right)$ of the reduced system (9.42) consists of generalized balanced vectors. All other variables can then be derived analytically in terms of the solution $\left(V_c, I_i^{\text{int}}\right)$ and shown to be generalized balanced vectors (Exercise 9.15).

The variable $\left(V_c, I_i^{\text{int}}\right)$ satisfies (9.42), reproduced here:

$$\begin{bmatrix} \mathbb{I}_c \otimes \mathbb{I} & Z_{ci}\Gamma_i^{\mathsf{T}} \\ 0 & \Gamma_i Z_{ii} \Gamma_i^{\mathsf{T}} + Z_i \end{bmatrix} \begin{bmatrix} V_c \\ I_i^{\text{int}} \end{bmatrix} \;=\; \begin{bmatrix} Z_{cc} \\ \Gamma_i Z_{ic} \end{bmatrix} I_c \;-\; \begin{bmatrix} A_{cv} \\ A_{iv} \end{bmatrix} V_v \;-\; \begin{bmatrix} 0 \\ \gamma_i^0 \otimes \mathbf{1} \end{bmatrix} \tag{9.55}$$

where

$$\begin{bmatrix} A_{cv} \\ A_{iv} \end{bmatrix} \;:=\; \operatorname{diag}(\mathbb{I}_c \otimes \mathbb{I}, \Gamma_i) \begin{bmatrix} Z_{cc} & Z_{ci} \\ Z_{ic} & Z_{ii} \end{bmatrix} \begin{bmatrix} Y_{cv} \\ Y_{iv} \end{bmatrix}$$

We now prove Theorem 9.7 in the following three lemmas. The first lemma simplifies (9.55) using balanced devices (9.48) and balanced lines (9.50).

Lemma 9.8. Suppose C9.7 holds. Balanced devices and lines (9.48)(9.50) reduces (9.55) to

$$\underbrace{\begin{bmatrix} \mathbb{I}_c \otimes \mathbb{I} & \left(Z_{ci}^{1\phi} \otimes \mathbb{I}\right) \Gamma_i^{\mathsf{T}} \\ 0 & \Gamma_i \left(Z_{ii}^{1\phi} \otimes \mathbb{I}\right) \Gamma_i^{\mathsf{T}} + (\zeta_i \otimes \mathbb{I}) \end{bmatrix}}_{M} \begin{bmatrix} V_c \\ I_i^{\text{int}} \end{bmatrix} \;=\; a' \otimes \alpha_+ \,+\, b' \otimes \mathbf{1} \tag{9.56a}$$

where

$$a' \;:=\; -\begin{bmatrix} Z_{cc}^{1\phi}\,\hat{\mu}_c + B_{cv}\,\hat{\lambda}_v \\ Z_{ic}^{1\phi,Y}\,\hat{\mu}_c + B_{iv}^{Y}\hat{\lambda}_v \\ (1-\alpha)\left(Z_{ic}^{1\phi,\Delta}\hat{\mu}_c + B_{iv}^{\Delta}\hat{\lambda}_v\right) \end{bmatrix}, \qquad b' \;:=\; -\begin{bmatrix} B_{cv}\,\gamma_v \\ B_{iv}^{Y}\gamma_v^{Y} + \gamma_i^{Y} \\ 0 \end{bmatrix} \tag{9.56b}$$

for some matrices $B_{cv}, B_{iv}^{Y}, B_{iv}^{\Delta}$.

The second lemma shows that the inverse M^{-1} of the matrix in (9.53a) has a structure that preserve the balanced nature of voltages and currents.

Lemma 9.9. Suppose C9.7 and C9.8 hold.

1. The matrix M in (9.56) is invertible.
2. Each 3×3 block $\left[M^{-1}\right]_{jk}$ of M^{-1} corresponding to phases *abc* is of the form

$$\left[M^{-1}\right]_{jk} \;:=\; v_{jk}\,\mathbb{I} \,+\, w_{jk}\,W_{jk} \tag{9.57}$$

   where $v_{jk}, w_{jk} \in \mathbb{C}$ are scalars and $W_{jk} \in \mathbb{C}^{3\times 3}$ is one of $\mathbb{I}$, Γ, Γ^{T}, $\Gamma\Gamma^{\mathsf{T}}$ and $\Gamma^{\mathsf{T}}\Gamma$.

The structure (9.57) of M^{-1} in Lemma 9.9 is what allows $\left(V_c, V_i^{\text{int}}\right)$ to remain generalized balanced vectors. It requires that $\hat{C}_i$ in C9.8 be invertible. The following lemma is the crucial fact in determining the inverse of $\hat{C}_i$ that appears in M^{-1}. The lemma can be verified directly using $\left(\Gamma^{\mathsf{T}}\Gamma\right)\left(\Gamma^{\mathsf{T}}\Gamma\right) = 3\Gamma^{\mathsf{T}}\Gamma$ (Theorem 7.2). It says that taking the inverse of the sum of a Kronecker product with $\mathbb{I}$ and a Kronecker product with $\Gamma^{\mathsf{T}}\Gamma$ preserves the Kronecker structure.

Lemma 9.10. For any matrix A and B of appropriate sizes, if A and $A+3B$ are invertible then

$$\left(A \otimes \mathbb{I} + B \otimes \Gamma^{\mathsf{T}}\Gamma\right)^{-1} \;=\; \left(A^{-1} \otimes \mathbb{I} - \left((A+3B)^{-1} BA^{-1}\right) \otimes \Gamma^{\mathsf{T}}\Gamma\right)$$

We now prove Lemmas 9.8 and 9.9.

Proof of Lemma 9.8. From (9.53) and (9.48c), the matrix on the left-hand side of (9.55) reduces to

$$\begin{bmatrix} \mathbb{I}_c \otimes \mathbb{I} & \left(Z_{ci}^{1\phi} \otimes \mathbb{I}\right) \Gamma_i^{\mathsf{T}} \\ 0 & \Gamma_i \left(Z_{ii}^{1\phi} \otimes \mathbb{I}\right) \Gamma_i^{\mathsf{T}} + (\zeta_i \otimes \mathbb{I}) \end{bmatrix} \tag{9.58}$$

On the right-hand side partition $Z_{ic}^{1\phi}$ in (9.53) into submatrices corresponding to impedances in Y and Δ configurations:

$$Z_{ic}^{1\phi} \;=:\; \begin{bmatrix} Z_{ic}^{1\phi,YY} & Z_{ic}^{1\phi,Y\Delta} \\ Z_{ic}^{1\phi,\Delta Y} & Z_{ic}^{1\phi,\Delta\Delta} \end{bmatrix} \;=:\; \begin{bmatrix} Z_{ic}^{1\phi,Y} \\ Z_{ic}^{1\phi,\Delta} \end{bmatrix}$$

where $Z_{ic}^{1\phi,Y}$ denotes the first $|N_i^Y|$ rows of $Z_{ic}^{1\phi}$ corresponding to Y configured impedances and $Z_{ic}^{1\phi,\Delta}$ denotes the remaining $|N_i^\Delta|$ rows of $Z_{ic}^{1\phi}$ corresponding to Δ configured impedances. We then have, using $\Gamma_i = \text{diag}\left(\mathbb{I}_i^Y \otimes \mathbb{I}, \mathbb{I}_i^\Delta \otimes \Gamma\right)$,

$$\Gamma_i Z_{ic} \;=\; \Gamma_i\left(Z_{ic}^{1\phi} \otimes \mathbb{I}\right) \;=\; \begin{bmatrix} Z_{ic}^{1\phi,Y} \otimes \mathbb{I} \\ Z_{ic}^{1\phi,\Delta} \otimes \Gamma \end{bmatrix}$$

The important structure is that the conversion matrix Γ appears on the right as "$\otimes\Gamma$" which allows the current I_c transformed by $\Gamma_i Z_{ic}$ to remain in $\text{span}(\alpha_+)$ on the right-hand side (using (9.48e)):

$$\begin{bmatrix} Z_{cc} \\ \Gamma_i Z_{ic} \end{bmatrix} I_c \;=\; -\begin{bmatrix} Z_{cc}^{1\phi} \otimes \mathbb{I} \\ \Gamma_i\left(Z_{ic}^{1\phi} \otimes \mathbb{I}\right) \end{bmatrix} \hat{\mu}_c \otimes \alpha_+ \;=\; -\begin{bmatrix} Z_{cc}^{1\phi}\,\hat{\mu}_c \\ Z_{ic}^{1\phi,Y}\,\hat{\mu}_c \\ (1-\alpha) Z_{ic}^{1\phi,\Delta}\,\hat{\mu}_c \end{bmatrix} \otimes \alpha_+ \tag{9.59a}$$

where we have used $(A \otimes B)(C \otimes D) \;=\; (AC) \otimes (BD)$ (Lemma 9.6) and $\Gamma\alpha_+ = (1-\alpha)\alpha_+$ (Corollary 1.4).

The second term $\begin{bmatrix} A_{cv} \\ A_{iv} \end{bmatrix} V_v$ on the right-hand side of (9.55) can be simplified in a similar manner but with more steps. We have from (9.53)

$$\begin{bmatrix} A_{cv} \\ A_{iv} \end{bmatrix} \;=\; \begin{bmatrix} Z_{cc}^{1\phi} \otimes \mathbb{I} & Z_{ci}^{1\phi} \otimes \mathbb{I} \\ \Gamma_i\left(Z_{ic}^{1\phi} \otimes \mathbb{I}\right) & \Gamma_i\left(Z_{ii}^{1\phi} \otimes \mathbb{I}\right) \end{bmatrix} \begin{bmatrix} Y_{cv}^{1\phi} \otimes \mathbb{I} \\ Y_{iv}^{1\phi} \otimes \mathbb{I} \end{bmatrix}$$

Similarly partition $Z_{ii}^{1\phi}$ into its first $|N_i^Y|$ and the remaining $|N_i^\Delta|$ rows:

$$Z_{ii}^{1\phi} \;=:\; \begin{bmatrix} Z_{ii}^{1\phi,Y} \\ Z_{ii}^{1\phi,\Delta} \end{bmatrix}$$

Then, using (9.48d) and $\Gamma_i = \text{diag}\left(\mathbb{I}_i^Y \otimes \mathbb{I}, \mathbb{I}_i^\Delta \otimes \Gamma\right)$, we have

$$\begin{aligned} \begin{bmatrix} A_{cv} \\ A_{iv} \end{bmatrix} V_v \;&=\; \begin{bmatrix} Z_{cc}^{1\phi} \otimes \mathbb{I} & Z_{ci}^{1\phi} \otimes \mathbb{I} \\ Z_{ic}^{1\phi,Y} \otimes \mathbb{I} & Z_{ii}^{1\phi,Y} \otimes \mathbb{I} \\ Z_{ic}^{1\phi,\Delta} \otimes \Gamma & Z_{ii}^{1\phi,\Delta} \otimes \Gamma \end{bmatrix} \begin{bmatrix} Y_{cv}^{1\phi} \otimes \mathbb{I} \\ Y_{iv}^{1\phi} \otimes \mathbb{I} \end{bmatrix} \left(\hat{\lambda}_v \otimes \alpha_+ + \gamma_v \otimes \mathbf{1}\right) \\ &=:\; \underbrace{\begin{bmatrix} B_{cv}\,\hat{\lambda}_v \\ B_{iv}^Y \hat{\lambda}_v \\ (1-\alpha) B_{iv}^\Delta \hat{\lambda}_v \end{bmatrix}}_{a'} \otimes \alpha_+ \;+\; \underbrace{\begin{bmatrix} B_{cv}\,\gamma_v \\ B_{iv}^Y \gamma_v^Y \\ 0 \end{bmatrix}}_{b'} \otimes \mathbf{1} \end{aligned} \tag{9.59b}$$

where

$$B_{cv} \;:=\; Z_{cc}^{1\phi} Y_{cv}^{1\phi} + Z_{ci}^{1\phi} Y_{iv}^{1\phi}, \qquad B_{iv}^{Y} \;:=\; Z_{ic}^{1\phi,Y} Y_{cv}^{1\phi} + Z_{ii}^{1\phi,Y} Y_{iv}^{1\phi}, \qquad B_{cv}^{\Delta} \;:=\; Z_{ic}^{1\phi,\Delta} Y_{cv}^{1\phi} + Z_{ii}^{1\phi,\Delta} Y_{iv}^{1\phi}$$

The factor $1-\alpha$ in (9.59b) is due to $\Gamma\alpha_+ = (1-\alpha)\alpha_+$ and the 0 entry is due to $\Gamma\mathbf{1} = 0$ and originates from the fact that the internal voltages in a Δ configuration sum to zero, i.e., $\mathbf{1}^{\mathsf{T}} V^{\Delta} = 0$.

Substituting (9.58)(9.59) into (9.55) then yields (9.56) (recall from (9.48c) that $\gamma_i^0 := \left(\gamma_i^Y, 0\right)$). □

Proof of Lemma 9.9. The matrix in (9.56):

$$M \;:=\; \begin{bmatrix} \mathbb{I}_c \otimes \mathbb{I} & \left(Z_{ci}^{1\phi} \otimes \mathbb{I}\right)\Gamma_i^{\mathsf{T}} \\ 0 & \Gamma_i \left(Z_{ii}^{1\phi} \otimes \mathbb{I}\right)\Gamma_i^{\mathsf{T}} + (\zeta_i \otimes \mathbb{I}) \end{bmatrix}$$

is invertible if its submatrix

$$M_{22} \;:=\; (\zeta_i \otimes \mathbb{I}) \,+\, \Gamma_i \left(Z_{ii}^{1\phi} \otimes \mathbb{I}\right)\Gamma_i^{\mathsf{T}} \tag{9.60a}$$

is invertible in which case its inverse is

$$M^{-1} \;:=\; \begin{bmatrix} \mathbb{I}_c \otimes \mathbb{I} & -\left(Z_{ci}^{1\phi} \otimes \mathbb{I}\right)\Gamma_i^{\mathsf{T}} M_{22}^{-1} \\ 0 & M_{22}^{-1} \end{bmatrix} \tag{9.60b}$$

(see Appendix 25.1.2 for discussions on Schur complement for the inverse of general block matrices). To study the invertibility of M_{22} we use the matrix inversion formula (25.5):

$$(A + BCD)^{-1} \;=\; A^{-1} \,-\, A^{-1}\left(B\tilde{C}^{-1}D\right)A^{-1}$$

where $\tilde{C} := C^{-1} + DA^{-1}B$ in Appendix 25.1.2.2. The matrix $A+BCD$ is invertible if A, C and $\tilde{C} := C^{-1} + DA^{-1}B$ are invertible. Therefore M_{22} in (9.60a) is invertible if (i) the impedances $\zeta_j \in \mathbb{C}$ are nonzero for all $j \in N_i$; (ii) $Z_{ii}^{1\phi}$ is invertible; and (iii) the matrix $\hat{C}_i$ in (9.54) is invertible, as claimed in Lemma 9.9.

We now prove (9.57). To apply Lemma 9.10 to determine the inverse of $\hat{C}_i$, use $\Gamma_i = \mathrm{diag}\left(\mathbb{I}_i^Y \otimes \mathbb{I}, \mathbb{I}_i^{\Delta} \otimes \Gamma\right)$ to get

$$\begin{aligned} \Gamma_i^{\mathsf{T}}\left(\zeta_i^{-1} \otimes \mathbb{I}\right)\Gamma_i \;&=\; \mathrm{diag}\left(\mathbb{I}_i^Y \otimes \mathbb{I}, \mathbb{I}_i^{\Delta} \otimes \Gamma^{\mathsf{T}}\right)\mathrm{diag}\left(\left(\zeta_i^Y\right)^{-1} \otimes \mathbb{I}, \left(\zeta_i^{\Delta}\right)^{-1} \otimes \mathbb{I}\right)\mathrm{diag}\left(\mathbb{I}_i^Y \otimes \mathbb{I}, \mathbb{I}_i^{\Delta} \otimes \Gamma\right) \\ &=\; \mathrm{diag}\left(\eta_i^Y \otimes \mathbb{I}, \eta_i^{\Delta} \otimes \Gamma^{\mathsf{T}}\Gamma\right) \end{aligned}$$

where $\eta_i^{Y/\Delta} := \left(\zeta_i^{Y/\Delta}\right)^{-1}$. Partition $\left(Z_{ii}^{1\phi}\right)^{-1}$ into submatrices:

$$\left(Z_{ii}^{1\phi}\right)^{-1} \;=:\; \begin{bmatrix} A^{YY} & A^{Y\Delta} \\ A^{\Delta Y} & A^{\Delta\Delta} \end{bmatrix}$$

Then

$$\hat{C}_i \;:=\; \left(\left(Z_{ii}^{1\phi}\right)^{-1} \otimes \mathbb{I}\right) \,+\, \Gamma_i^{\mathsf{T}}\left(\zeta_i^{-1} \otimes \mathbb{I}\right)\Gamma_i \;=\; \begin{bmatrix} \tilde{A}^{YY} & A^{Y\Delta} \\ A^{\Delta Y} & A^{\Delta\Delta} \end{bmatrix} \otimes \mathbb{I} \,+\, \mathrm{diag}\left(0, \eta_i^{\Delta}\right) \otimes \Gamma^{\mathsf{T}}\Gamma$$

where $\tilde{A}^{YY} := A^{YY} + \eta_i^Y$. We can then apply Lemma 9.10 to get

$$\hat{C}_i^{-1} \;=\; \tilde{A} \otimes \mathbb{I} \;-\; \tilde{B} \otimes \Gamma^\mathsf{T}\Gamma \tag{9.61}$$

where

$$\tilde{A} \;:=\; \begin{bmatrix} A^{YY} + \eta_i^Y & A^{Y\Delta} \\ A^{\Delta Y} & A^{\Delta\Delta} \end{bmatrix}^{-1}, \qquad \tilde{B} \;:=\; \begin{bmatrix} A^{YY} + \eta_i^Y & A^{Y\Delta} \\ A^{\Delta Y} & A^{\Delta\Delta} + 3\eta_i^\Delta \end{bmatrix}^{-1} \operatorname{diag}\left(0, \eta_i^\Delta\right) \tilde{A}$$

Applying the matrix inversion formula with $\hat{C}_i^{-1}$ given by (9.61) we obtain the inverse of M_{22} in (9.60a) as

$$\begin{aligned} M_{22}^{-1} \;&=\; (\eta_i \otimes \mathbb{I}) \;-\; (\eta_i \otimes \mathbb{I})\Gamma_i \left(\tilde{A} \otimes \mathbb{I} \;-\; \tilde{B} \otimes \Gamma^\mathsf{T}\Gamma\right) \Gamma_i^\mathsf{T} (\eta_i \otimes \mathbb{I}) \\ &=\; (\eta_i \otimes \mathbb{I}) \;-\; \left(\begin{bmatrix} \hat{A}^{YY} \otimes \mathbb{I} & \hat{A}^{Y\Delta} \otimes \Gamma^\mathsf{T} \\ \hat{A}^{\Delta Y} \otimes \Gamma & \hat{A}^{\Delta\Delta} \otimes \Gamma\Gamma^\mathsf{T} \end{bmatrix} \;-\; \begin{bmatrix} \hat{B}^{YY} \otimes \Gamma^\mathsf{T}\Gamma & 3\hat{B}^{Y\Delta} \otimes \Gamma^\mathsf{T} \\ 3\hat{B}^{\Delta Y} \otimes \Gamma & 3\hat{B}^{\Delta\Delta} \otimes \Gamma\Gamma^\mathsf{T} \end{bmatrix} \right) \end{aligned} \tag{9.62}$$

where

$$\begin{aligned} [\hat{A}/\hat{B}]^{YY} \;&:=\; \eta_i^Y [\tilde{A}/\tilde{B}]^{YY} \eta_i^Y, \qquad & [\hat{A}/\hat{B}]^{Y\Delta} \;&:=\; \eta_i^Y [\tilde{A}/\tilde{B}]^{Y\Delta} \eta_i^\Delta \\ [\hat{A}/\hat{B}]^{\Delta Y} \;&:=\; \eta_i^\Delta [\tilde{A}/\tilde{B}]^{\Delta Y} \eta_i^Y, \qquad & [\hat{A}/\hat{B}]^{\Delta\Delta} \;&:=\; \eta_i^\Delta [\tilde{A}/\tilde{B}]^{\Delta\Delta} \eta_i^\Delta \end{aligned}$$

and $\eta_i^{Y/\Delta} := \left(\zeta_i^{Y/\Delta}\right)^{-1}$, $\eta_i := \operatorname{diag}\left(\eta_i^Y, \eta_i^\Delta\right)$. Therefore each 3×3 block of M_{22}^{-1} is of the desired form of $v_{jk}\mathbb{I} + w_{jk}W_{jk}$ where $v_{jk}, w_{jk} \in \mathbb{C}$ are scalars and $W_{jk} \in \mathbb{C}^{3\times 3}$ is one of $\mathbb{I}$, Γ, Γ^T, $\Gamma\Gamma^\mathsf{T}$ and $\Gamma^\mathsf{T}\Gamma$.

Finally substituting (9.61) into (9.60b) we see that each 3×3 block of the $3\left(|N_c| + |N_i|\right) \times 3\left(|N_c| + |N_i|\right)$ matrix M^{-1} will also be of the desired form of $w_{jk}W_{jk}$ if this property holds for its off-diagonal submatrix $\left(Z_{ci}^{1\phi} \otimes \mathbb{I}\right)\Gamma_i^\mathsf{T} M_{22}^{-1}$. We now show that this is indeed the case. Partition $Z_{ci}^{1\phi}$ into submatrices corresponding to impedances in Y and Δ configurations:

$$Z_{ci}^{1\phi} \;=:\; \begin{bmatrix} Z_{ci}^{1\phi,YY} & Z_{ci}^{1\phi,Y\Delta} \\ Z_{ci}^{1\phi,\Delta Y} & Z_{ci}^{1\phi,\Delta\Delta} \end{bmatrix}$$

Using $Z_{ci}^{1\phi}$ and $\Gamma_i = \operatorname{diag}\left(\mathbb{I}_i^Y \otimes \mathbb{I}, \mathbb{I}_i^\Delta \otimes \Gamma\right)$ we have

$$\left(Z_{ci}^{1\phi} \otimes \mathbb{I}\right)\Gamma_i^\mathsf{T} M_{22}^{-1} \;=\; \begin{bmatrix} Z_{ci}^{1\phi,YY} \otimes \mathbb{I} & Z_{ci}^{1\phi,Y\Delta} \otimes \Gamma^\mathsf{T} \\ Z_{ci}^{1\phi,\Delta Y} \otimes \mathbb{I} & Z_{ci}^{1\phi,\Delta\Delta} \otimes \Gamma^\mathsf{T} \end{bmatrix} M_{22}^{-1}$$

Substituting M_{22}^{-1} in (9.62) and using

$$\Gamma^\mathsf{T}\Gamma\Gamma^\mathsf{T} \;=\; \left(3\mathbb{I} - \mathbf{1}\mathbf{1}^\mathsf{T}\right)\Gamma^\mathsf{T} \;=\; 3\Gamma^\mathsf{T}$$

we see that each 3×3 block of $\left(Z_{ci}^{1\phi} \otimes \mathbb{I}\right)\Gamma_i^\mathsf{T} M_{22}^{-1}$ is of the desired form of $v_{jk}\mathbb{I} + w_{jk}W_{jk}$ where $v_{jk}, w_{jk} \in \mathbb{C}$ are scalars and $W_{jk} \in \mathbb{C}^{3\times 3}$ is one of $\mathbb{I}$, Γ, Γ^T, $\Gamma\Gamma^\mathsf{T}$ and $\Gamma^\mathsf{T}\Gamma$.

This completes the proof of (9.57). $\square$

Lemmas 9.8 and 9.9 imply Theorem 9.7.

Proof of Theorem 9.7. Multiplying both sides of (9.56) by M^{-1} in (9.57) we see that the jth 3×3 block of $\left(V_c, I_i^{\text{int}}\right)$ is of the form

$$\sum_k \left[M^{-1}\right]_{jk} \left(a'_k \alpha_+ + b'_k \mathbf{1}\right) \;=\; \sum_k a'_k \left(v_{jk}\mathbb{I} + w_{jk} W_{jk}\right) \alpha_+ \;+\; \sum_k b'_k \left(v_{jk}\mathbb{I} + w_{jk} W_{jk}\right) \mathbf{1}$$

Since

$$W_{jk}\alpha_+ \;=\; \begin{cases} \alpha_+ & \text{if} \quad W_{jk} = \mathbb{I} \\ (1-\alpha)\alpha_+ & \text{if} \quad W_{jk} = \Gamma \\ (1-\alpha^2)\alpha_+ & \text{if} \quad W_{jk} = \Gamma^{\mathsf{T}} \\ 3\alpha_+ & \text{if} \quad W_{jk} = \Gamma\Gamma^{\mathsf{T}} \text{ or } \Gamma^{\mathsf{T}}\Gamma \end{cases}$$

and $W_{jk}\mathbf{1} = \mathbf{1}$ if $W_{jk} = \mathbb{I}$ and 0 otherwise, $\left(V_c, I_i^{\text{int}}\right)$ consists of generalized balanced vectors of the form $a_j\alpha_+ + b_j\mathbf{1}$. When $\gamma_v = 0$ for all voltage sources and $\gamma_i^Y = 0$ for all Y configured impedances, then $b' = 0$ in (9.56) and hence $b = 0$. This completes the proof of Theorem 9.7. $\square$

9.3.4 Phase decoupling and per-phase analysis

In this subsection we show that phases in a balanced network are decoupled so that the three-phase analysis problem can be solved by solving two per-phase networks.

Substitute the per-phase admittance matrix (9.52), and the external models of voltage and current sources (9.48d)(9.48e) into the current balance equation $I = YV$ (9.50c) to get

$$\begin{bmatrix} I_v \\ -\hat{\mu}_c \otimes \alpha_+ \\ I_i \end{bmatrix} \;=\; \left(\begin{bmatrix} Y_{vv}^{1\phi} & Y_{vc}^{1\phi} & Y_{vi}^{1\phi} \\ Y_{cv}^{1\phi} & Y_{cc}^{1\phi} & Y_{ci}^{1\phi} \\ Y_{iv}^{1\phi} & Y_{ic}^{1\phi} & Y_{ii}^{1\phi} \end{bmatrix} \otimes \mathbb{I} \right) \begin{bmatrix} \hat{\lambda}_v \otimes \alpha_+ + \gamma_v \otimes \mathbf{1} \\ V_c \\ V_i \end{bmatrix} \tag{9.63}$$

Instead of following the solution strategy of Chapter 9.2.3 to compute the internal impedance current I_i^{int} from the reduced system (9.42) we will compute the terminal voltage V_i, as well as V_c, using (9.63). We can then compute (I_v, I_i) and all other variables such as internal voltages and currents and zero-sequence voltages and currents.

We know from Theorem 9.7 that all voltages and currents consist of generalized balanced vectors of the form $a_j\alpha_+ + b_j\mathbf{1}$. We now describe separately external models for devices in Δ and Y configurations.

Δ configuration. Consider a Δ configured device $j \in N_v^\Delta \cup N_c^\Delta \cup N_i^\Delta$. Let

$$V_j \;=:\; v_j\alpha_+ + \gamma_j\mathbf{1}, \qquad j \in N_c^\Delta \cup N_i^\Delta \tag{9.64a}$$

$$I_j \;=:\; i_j\alpha_+, \qquad j \in N_v^\Delta \cup N_i^\Delta \tag{9.64b}$$

for some (v_j, γ_j) and i_j to be determined. Here $\gamma_j = \frac{1}{3}\mathbf{1}^\mathsf{T} V_j$ is the zero-sequence voltage of V_j. As expected, $\mathbf{1}^\mathsf{T} I_j = 0$ since $I_j = -\Gamma^\mathsf{T} I_j^\Delta$. For an impedance $j \in N_i^\Delta$, we can express its terminal current I_j in terms of its terminal voltage V_j using its external model (from Table 9.4)

$$I_j \;=\; -3\eta_j \left(V_j - \gamma_j \mathbf{1}\right) \;=\; -3\eta_j v_j\, \alpha_+, \qquad j \in N_i^\Delta \tag{9.64c}$$

Hence the variables (v_j, i_j) for an impedance $j \in N_i^\Delta$ satisfies $i_j = -3\eta_j v_j$, the negative sign due to the definition of I_j being injection from the device to the rest of the network.

Y **configuration.** Consider a Y configured device $j \in N_v^Y \cup N_c^Y \cup N_i^Y$. Let its internal voltage and internal current be generalized balanced vectors:

$$\begin{aligned} V_j^Y \;&=:\; v_j^{\text{int}} \alpha_+ + \gamma_j^{\text{int}} \mathbf{1}, && j \in\in N_c^Y \cup N_i^Y \\ I_j^Y \;&=:\; -\left(i_j^{\text{int}} \alpha_+ + \beta_j^{\text{int}} \mathbf{1}\right), && j \in N_v^Y \cup N_i^Y \end{aligned}$$

for some $(v_j^{\text{int}}, \gamma_j^{\text{int}})$ and $\left(i_j^{\text{int}}, \beta_j^{\text{int}}\right)$ to be determined. Here $\gamma_j^{\text{int}} := \frac{1}{3}\mathbf{1}^\mathsf{T} V_j^Y$ is the zero-sequence voltage of the *internal* voltage V_j^Y, not the neutral voltage $\gamma_j := V_j^n$, and $\beta_j^{\text{int}} := \frac{1}{3}\mathbf{1}^\mathsf{T} I_j^Y$ is the zero-sequence current of the *internal* current I_j^Y. Since $V_j = V_j^Y + V_j^n \mathbf{1}$ and $I_j = -I_j^Y$, the terminal voltage and current are:

$$V_j \;=:\; v_j^{\text{int}} \alpha_+ + \left(\gamma_j^{\text{int}} + \gamma_j\right) \mathbf{1}, \qquad j \in N_c^Y \cup N_i^Y \tag{9.65a}$$

$$I_j \;=:\; i_j^{\text{int}} \alpha_+ + \beta_j^{\text{int}} \mathbf{1}, \qquad j \in N_v^Y \cup N_i^Y \tag{9.65b}$$

Recall that the neutral voltages $\gamma_j := V_j^n$ are given for all Y configured devices. The zero-sequence voltage of the terminal voltage V_j is the sum of the zero-sequence voltage γ_j^{int} of the internal voltage V_j^Y and the neutral voltage γ_j. Hence the terminal voltage V_j is balanced if and only if the neutral voltage γ_j is offset by γ_j^{int} so that $\gamma_j^{\text{int}} + \gamma_j = 0$ (see below for a sufficient condition). Moreover $\mathbf{1}^\mathsf{T} I_j = -\mathbf{1}^\mathsf{T} I_j^Y = -I_j^n$ is the negative of the neutral current. Hence $\beta_j^{\text{int}} = \frac{1}{3}\mathbf{1}^\mathsf{T} I_j = 0$ if device j has no neutral line. For an impedance $j \in N_i^Y$, we can express its terminal current I_j in terms of its terminal voltage V_j using the external model (from Table 9.4 and (9.65a))

$$I_j \;=\; -\eta_j \left(V_j - \gamma_j \mathbf{1}\right) \;=\; -\eta_j \left(v_j^{\text{int}} \alpha_+ + \gamma_j^{\text{int}} \mathbf{1}\right), \qquad j \in N_i^Y \tag{9.65c}$$

Hence $i_j^{\text{int}} = -\eta_j v_j^{\text{int}}$ and $\beta_j^{\text{int}} = -\eta_j \gamma_j^{\text{int}}$.

Before substituting (9.64)(9.65) into the network equation (9.63) we unify notations by defining

$$\hat{v}_j \;:=\; \begin{cases} v_j^{\text{int}}, \\ v_j, \end{cases} \qquad \hat{\gamma}_j \;:=\; \begin{cases} \gamma_j^{\text{int}} + \gamma_j, & j \in N_c^Y \cup N_i^Y \\ \gamma_j, & j \in N_c^\Delta \cup N_i^\Delta \end{cases} \tag{9.66a}$$

$$\hat{i}_j \;:=\; \begin{cases} i_j^{\text{int}}, \\ i_j, \end{cases} \qquad \hat{\beta}_j \;:=\; \begin{cases} \beta_j^{\text{int}}, & j \in N_v^Y \cup N_i^Y \\ 0, & j \in N_v^\Delta \cup N_i^\Delta \end{cases} \tag{9.66b}$$

Even though $\gamma_j = V_j^n$ are given for $j \in N_c^Y \cup N_i^Y$, γ_j^{int} (as well as $\gamma_j := \frac{1}{3}\mathbf{1}^\mathsf{T} V_j$ for $j \in N_c^\Delta \cup N_i^\Delta$) are unknown, and hence $\hat{\gamma}_j$ is unknown for $j \in N_c \cup N_i$. Therefore all the quantities in (9.66a) (9.66b) are to be determined. Collect currents and voltages associated with voltage and current sources respectively into

$$\hat{i}_v := \left(\hat{i}_j,\, j \in N_v\right), \quad \hat{\beta}_v := \left(\hat{\beta}_j,\, j \in N_v\right), \quad \hat{v}_c := \left(\hat{v}_j,\, j \in N_c\right), \quad \hat{\gamma}_c := \left(\hat{\gamma}_j,\, j \in N_c\right) \tag{9.66c}$$

Collect currents and voltages associated with impedances into

$$\hat{i}_i := \left(\hat{i}_j,\, j \in N_i\right), \quad \hat{\beta}_i := \left(\hat{\beta}_j,\, j \in N_i\right), \quad \hat{v}_i := \left(\hat{v}_j,\, j \in N_i\right), \quad \hat{\gamma}_i := \left(\hat{\gamma}_j,\, j \in N_i\right) \tag{9.66d}$$

Using the same notation for $\hat{\alpha}_j$ as in (9.48d)(9.48e), we can apply (9.66) to the external impedance models (9.65c) and (9.64c) to relate $\hat{v}_i$ and $\hat{i}_i$:

$$\hat{i}_i \otimes \alpha_+ \,+\, \hat{\beta}_i \otimes \mathbf{1} \;=\; -\left(\hat{\eta}_i \otimes \mathbb{I}\right)\left(\hat{v}_i \otimes \alpha_+ + (\hat{\gamma}_i - \gamma_i) \otimes \mathbf{1}\right) \tag{9.67a}$$

where the diagonal matrix $\hat{\eta}_i \in \mathbb{C}^{|N_i| \times |N_i|}$ and the vector $\gamma_i \in \mathbb{C}^{|N_i|}$ are defined as

$$\hat{\eta}_i := \text{diag}\left(\hat{\alpha}_j \eta_j,\, j \in N_i\right), \qquad \gamma_i := \begin{bmatrix} \gamma_i^Y \\ \gamma_i^\Delta \end{bmatrix} := \begin{bmatrix} \left(\gamma_j := V_j^n,\, j \in N_i^Y\right) \\ \left(\gamma_j := \frac{1}{3}\mathbf{1}^\mathsf{T} V_j,\, j \in N_i^\Delta\right) \end{bmatrix} \tag{9.67b}$$

Hence $\hat{\gamma}_i - \gamma_i = \begin{bmatrix} \gamma_i^{\text{int}} \\ 0 \end{bmatrix}$ with $\gamma_i^{\text{int}} := \left(\gamma_j^{\text{int}},\, j \in N_i^Y\right)$. Note the difference between γ_i defined here and the specification $\gamma_i^0 := \begin{bmatrix} \gamma_i^Y \\ 0 \end{bmatrix}$ defined in (9.48c). Recall that γ_i^Y is given, but γ_i^{int} and hence $\hat{\gamma}_i$ are to be determined.

Substituting (9.66) into (9.63) we have

$$\begin{bmatrix} \hat{i}_v \\ -\hat{\mu}_c \\ \hat{i}_i \end{bmatrix} \otimes \alpha_+ + \begin{bmatrix} \hat{\beta}_v \\ 0 \\ \hat{\beta}_i \end{bmatrix} \otimes \mathbf{1} \;=\; \left(\begin{bmatrix} Y_{vv}^{1\phi} & Y_{vc}^{1\phi} & Y_{vi}^{1\phi} \\ Y_{cv}^{1\phi} & Y_{cc}^{1\phi} & Y_{ci}^{1\phi} \\ Y_{iv}^{1\phi} & Y_{ic}^{1\phi} & Y_{ii}^{1\phi} \end{bmatrix} \otimes \mathbb{I} \right) \left(\begin{bmatrix} \hat{\lambda}_v \\ \hat{v}_c \\ \hat{v}_i \end{bmatrix} \otimes \alpha_+ + \begin{bmatrix} \gamma_v \\ \hat{\gamma}_c \\ \hat{\gamma}_i \end{bmatrix} \otimes \mathbf{1} \right) \tag{9.68}$$

where the voltage sources $\hat{\lambda}_v$, current sources $-\hat{\mu}_c$, as well as $\left(\gamma_v, \gamma_c^0, \gamma_i^0\right)$ are given, and $\left(\hat{v}_{-v}, \hat{\gamma}_{-v}, \hat{i}_{-c}, \hat{\beta}_{-c}\right)$ are variables to be determined. Since α_+ and $\mathbf{1}$ are orthogonal this induces two sets of equations that can be interpreted as two per-phase networks.

Positive-sequence per-phase network. Equating the α_+ coordinates on both sides of (9.68) the per-phase variables must satisfy

$$\begin{bmatrix} \hat{i}_v \\ -\hat{\mu}_c \\ \hat{i}_i \end{bmatrix} = \begin{bmatrix} Y_{vv}^{1\phi} & Y_{vc}^{1\phi} & Y_{vi}^{1\phi} \\ Y_{cv}^{1\phi} & Y_{cc}^{1\phi} & Y_{ci}^{1\phi} \\ Y_{iv}^{1\phi} & Y_{ic}^{1\phi} & Y_{ii}^{1\phi} \end{bmatrix} \begin{bmatrix} \hat{\lambda}_v \\ \hat{v}_c \\ \hat{v}_i \end{bmatrix} \tag{9.69a}$$

This defines the following per-phase network:

- The admittance matrix is $Y^{1\phi}$.
- The voltage sources have given voltages $\hat{\lambda}_v$.
- The current sources have given currents $-\hat{\mu}_c$.
- The impedances are $\hat{\eta}_i$ so that (from (9.67a))

$$\hat{i}_i = -\hat{\eta}_i \hat{v}_i \tag{9.69b}$$

This is a system of 4 sets of equations in 4 sets of variables $\left(\hat{v}_c, \hat{v}_i, \hat{i}_v, \hat{i}_i\right)$. Substituting (9.69b) into (9.69a) we obtain

$$\begin{bmatrix} Y^{1\phi}_{cc} & Y^{1\phi}_{ci} \\ Y^{1\phi}_{ic} & Y^{1\phi}_{ii} + \hat{\eta}_i \end{bmatrix} \begin{bmatrix} \hat{v}_c \\ \hat{v}_i \end{bmatrix} = -\left(\begin{bmatrix} \hat{\mu}_c \\ 0 \end{bmatrix} + \begin{bmatrix} Y^{1\phi}_{cv} \\ Y^{1\phi}_{iv} \end{bmatrix} \hat{\lambda}_v \right) \tag{9.70}$$

If the matrix on the left-hand side is invertible then $(\hat{v}_c, \hat{v}_i)$ can be uniquely determined. The other variables $\left(\hat{i}_v, \hat{i}_i\right)$ can then be derived in terms of a solution $(\hat{v}_c, \hat{v}_i)$.

Zero-sequence per-phase network. Equating the **1** coordinates in (9.68) the per-phase variables must satisfy

$$\begin{bmatrix} \hat{\beta}_v \\ 0 \\ \hat{\beta}_i \end{bmatrix} = \begin{bmatrix} Y^{1\phi}_{vv} & Y^{1\phi}_{vc} & Y^{1\phi}_{vi} \\ Y^{1\phi}_{cv} & Y^{1\phi}_{cc} & Y^{1\phi}_{ci} \\ Y^{1\phi}_{iv} & Y^{1\phi}_{ic} & Y^{1\phi}_{ii} \end{bmatrix} \begin{bmatrix} \gamma_v \\ \hat{\gamma}_c \\ \hat{\gamma}_i \end{bmatrix} \tag{9.71a}$$

This defines the following per-phase network:

- The network is described by the admittance matrix is $Y^{1\phi}$.
- The voltage sources have given voltages γ_v.
- The current sources inject 0 currents, i.e., no device is connected at buses j of the zero-sequence per-phase network where three-phase current sources are connected in the original network.
- The impedances are $\hat{\eta}_i$ so that (from (9.67a))

$$\hat{\beta}_i = -\hat{\eta}_i (\hat{\gamma}_i - \gamma_i) = -\text{diag}\left(\hat{\eta}^Y_i, 0\right) \begin{bmatrix} (\hat{\gamma}^Y_i - \gamma^Y_i) \\ 0 \end{bmatrix} \tag{9.71b}$$

where $\hat{\eta}^Y_i := \text{diag}\left(\eta_j, j \in N^Y_i\right)$, $\hat{\gamma}^Y_i := \left(\hat{\gamma}_j, j \in N^Y_i\right)$ and $\gamma^Y_i := \left(V^n_j, j \in N^Y_i\right)$. Note that γ^Y_i is given and $\hat{\gamma}^Y_i$ is unknown.

This is a system of 4 sets of equations in 4 sets of variables $\left(\hat{\gamma}_c, \hat{\gamma}_i, \hat{\beta}_v, \hat{\beta}_i\right)$. Substituting (9.71b) into (9.71a) we obtain

$$\begin{bmatrix} Y_{cc}^{1\phi} & Y_{ci}^{1\phi} \\ Y_{ic}^{1\phi} & Y_{ii}^{1\phi} + \text{diag}\left(\hat{\eta}_i^Y, 0\right) \end{bmatrix} \begin{bmatrix} \hat{\gamma}_c \\ \hat{\gamma}_i \end{bmatrix} \ = \ - \begin{bmatrix} Y_{cv}^{1\phi} \\ Y_{iv}^{1\phi} \end{bmatrix} \gamma_v \ + \ \begin{bmatrix} 0 \\ \hat{\eta}_i \gamma_i^0 \end{bmatrix} \tag{9.72}$$

where we recall $\hat{\eta}_i$ in (9.67) and the given neutral voltages $\gamma_i^0 := \begin{bmatrix} \gamma_i^Y \\ 0 \end{bmatrix}$. If the matrix on the left-hand side is invertible then $(\hat{\gamma}_c, \hat{\gamma}_i)$ can be uniquely determined. The other variables $\left(\hat{\beta}_v, \hat{\beta}_i\right)$ can then be derived in terms of a solution $(\hat{\gamma}_c, \hat{\gamma}_i)$.

Assume the matrix in (9.72) is invertible. If $\gamma_v = 0$ and $\gamma_i^Y = 0$ as in Theorem 9.7.2, then $\hat{\gamma}_c = 0$ and $\hat{\gamma}_i = 0$ and all voltages consist of balanced vectors. In this case we do not have to compute the zero-sequence network but simply set $\hat{\gamma}_{-v} := 0$ and $\hat{\beta}_{-c} := 0$. Recall from (9.66a)(9.66b) that this means $\gamma_j^{\text{int}} + V_j^n = 0$ and $\beta_j^{\text{int}} = 0$ for Y configured devices and $\gamma_j = 0$ for Δ configured devices.

Note that, even though $\hat{\beta}_{-c}$ is determined from (9.72) (9.71), its components $\hat{\beta}_j = 0$ for $j \in N_v^\Delta \cup N_i^\Delta$ from (9.66b). This is consistent because, for $j \in N_v^\Delta \cup N_i^\Delta$, multiplying both sides of (9.49b) by $\mathbf{1}^\mathsf{T}$ gives, using $\gamma_j := \frac{1}{3}\mathbf{1}^\mathsf{T} V_j$,

$$\sum_{k:j\sim k} \left(y_{jk}^s + y_{jk}^m\right) \gamma_j - \sum_{k:j\sim k} y_{jk}^s \gamma_k \ = \ 0$$

which is (9.71) for rows corresponding to $j \in N_v^\Delta \cup N_i^\Delta$.

Per-phase analysis. Per-phase analysis for solving (9.63) is as follows:

1. Solve the positive-sequence per-phase network (9.70) for $(\hat{v}_c, \hat{v}_i)$ and then derive $\left(\hat{i}_v, \hat{i}_i\right)$.
2. If $\gamma_v = 0$ and $\gamma_i^Y = 0$, set $\hat{\gamma}_{-v} := 0$, $\hat{\beta}_{-c} := 0$, and goto the next step. Otherwise, solve the zero-sequence per-phase network (9.72) for $(\hat{\gamma}_c, \hat{\gamma}_i)$ and then derive $\left(\hat{\beta}_v, \hat{\beta}_i\right)$.
3. Substitute into (9.64)(9.65) to obtain (V_{-v}, I_{-c}).

Example 9.12 ($\gamma^Y = 0$)**.** Explain per-phase analysis in the special case where all neutrals are grounded with zero neutral impedances and voltages are defined with respect to the ground, i.e., $\gamma_j = 0$ for $j \in N_v^Y \cup N_c^Y \cup N_i^Y$. □

9.4 Symmetric network

We have formulated a general three-phase analysis problem in Chapter 9.2.2 and described a solution strategy in Chapter 9.2.3. When the network is balanced, the phases are decoupled and the network decomposes into two independent per-phase networks and the problem can be solved using per-phase analysis as explained in Chapter 9.3.

When the network is not balanced, e.g., the sources are unbalanced or the transmission lines are not phase-decoupled, then we can apply the similarity transformation F defined in Chapter 7.2.2 to transform *terminal* phase voltage and current (V, I) into sequence voltage and current $(\tilde{V}, \tilde{I})$. Even though the phases are coupled, we show in Chapters 9.4.1–9.4.4 that if three-phase lines are symmetric and loads are identical, then their external models are decoupled in the sequence coordinate. They define sequence networks that can be analyzed separately, similar to the per-phase networks of a balanced network studied in Chapter 9.3. The results from analyzing the sequence networks can then be transformed back to the original phase coordinate. We describe in Chapter 9.4.5 how to compose the sequence networks from the sequence models of individual devices and how to solve the three-phase analysis problem using these decoupled sequence networks when the original network is symmetric.

Symmetric components and sequence networks are most useful for fault analysis in a system that is more or less balanced, e.g., a three-phase network that remains balanced until the fault location. Without any symmetry, symmetrical components may not offer much advantage because they do not lead to decoupled sequence networks. Even though we do not study fault analysis in this book, the discussion in this section illustrates the application of various three-phase models developed in this chapter.

9.4.1 Sequence impedances

Y **configuration** $\left(z^Y, z^n\right)$**.** Consider the four-wire three-phase impedance $\left(z^Y, z^n\right)$ in Y configuration shown in Figure 7.4 of Chapter 7.3.2. Under assumption C7.1 (all neutrals are grounded and all voltages are defined with respect to the ground), recall the external model (7.19b) relating the terminal voltage and current (V, I):

$$V \;=\; -Z^Y I \qquad \text{with} \qquad Z^Y \;:=\; z^Y \,+\, z^n\,\mathbf{1}\mathbf{1}^{\mathsf{T}} \;=\; \begin{bmatrix} z^{an}+z^n & z^n & z^n \\ z^n & z^{an}+z^n & z^n \\ z^n & z^n & z^{cn}+z^n \end{bmatrix}$$

Substitute $V = F\tilde{V}$ and $I = F\tilde{I}$ to obtain the external model in the sequence coordinate:

$$\tilde{V} \;=\; -\underbrace{\overline{F} Z^Y F}_{\tilde{Z}^Y}\, \tilde{I} \;=\; -\tilde{Z}^Y \tilde{I}$$

where F from (7.6b) and its inverse $F^{-1} = \overline{F}$ from (7.7) are

$$F \;=\; \frac{1}{\sqrt{3}}\begin{bmatrix}\mathbf{1} & \alpha_+ & \alpha_-\end{bmatrix} \;=\; \frac{1}{\sqrt{3}}\begin{bmatrix}\mathbf{1}^{\mathsf{T}} \\ \alpha_+^{\mathsf{T}} \\ \alpha_-^{\mathsf{T}}\end{bmatrix} \;:=\; \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha\end{bmatrix} \tag{9.73a}$$

$$\overline{F} \;=\; \frac{1}{\sqrt{3}}\begin{bmatrix}\mathbf{1} & \alpha_- & \alpha_+\end{bmatrix} \;=\; \frac{1}{\sqrt{3}}\begin{bmatrix}\mathbf{1}^{\mathsf{T}} \\ \alpha_-^{\mathsf{T}} \\ \alpha_+^{\mathsf{T}}\end{bmatrix} \;:=\; \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2\end{bmatrix} \tag{9.73b}$$

We call $\tilde{Z}^Y$ a *sequence impedance matrix* to differentiate it from the (phase) impedance matrix Z^Y. Substituting $Z^Y = z^Y + z^n \mathbf{1}\mathbf{1}^\mathsf{T}$, F and $\overline{F}$, we have (Exercise 9.18)

$$\tilde{Z}^Y \;=\; \frac{1}{3}\begin{bmatrix} \mathbf{1}^T z & \alpha_+^\mathsf{T} z & \alpha_-^\mathsf{T} z \\ \alpha_-^\mathsf{T} z & \mathbf{1}^T z & \alpha_+^\mathsf{T} z \\ \alpha_+^\mathsf{T} z & \alpha_-^\mathsf{T} z & \mathbf{1}^T z \end{bmatrix} + \begin{bmatrix} 3z^n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $z := \left(z^{an}, z^{bn}, z^{cn}\right)$ is the column vector of phase impedances. Hence the neutral impedance z^n appears only in the zero-sequence impedance.

If the impedance is balanced $z^{an} = z^{bn} = z^{cn}$, then $\mathbf{1}^T z = 3z^{an}$ and $\alpha_+^\mathsf{T} z = \alpha_-^\mathsf{T} z = 0$ and

$$\tilde{Z}^Y \;=\; \begin{bmatrix} z^{an} + 3z^n & 0 & 0 \\ 0 & z^{an} & 0 \\ 0 & 0 & z^{an} \end{bmatrix} \tag{9.74a}$$

Hence the sequence impedance matrix $\tilde{Z}^Y$ is diagonal even though the phase impedance Z^Y is not. This implies that the external model $\tilde{V} = -\tilde{Z}^Y \tilde{I}$ relating the sequence voltage and current in the sequence coordinate is decoupled:

$$\begin{bmatrix} \tilde{V}_0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} \;=\; -\begin{bmatrix} z^{an} + 3z^n & 0 & 0 \\ 0 & z^{an} & 0 \\ 0 & 0 & z^{an} \end{bmatrix} \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix} \tag{9.74b}$$

i.e., the external model consists of three separate impedances:

$$\begin{aligned} &\text{zero-seq impedance:} && \tilde{V}_0 \;=\; -\left(z^{an} + 3z^n\right) \tilde{I}_0 \\ &\text{positive-seq impedance:} && \tilde{V}_+ \;=\; -z^{an} \tilde{I}_+ \\ &\text{negative-seq impedance:} && \tilde{V}_- \;=\; -z^{an} \tilde{I}_- \end{aligned}$$

The interpretation is as follows. When the similarity transformation defined by the unitary matrix F transforms a power network from the *abc* phase coordinate to $0{+}{-}$ sequence coordinate (see Chapter 7.2.2), a balanced impedance with $z^{an} = z^{bn} = z^{cn}$ becomes decoupled in the sequence coordinate. If all devices are decoupled in the sequence coordinate, the entire *sequence networks* are decoupled and the sequence impedances are impedances on these decoupled sequence networks. Each sequence network can be analyzed separately like a single-phase network. We will explain in Chapter 9.4.5 on how to compose the sequence networks from sequence models of individual devices.

Note that if the impedance is not balanced then the relation $\tilde{V} = \tilde{Z}^Y \tilde{I}$ is generally coupled and power flow analysis using the sequence variables may not offer any advantage over using the phase variables.

Δ configuration z^Δ. Consider the three-wire three-phase impedance z^Δ in Δ configuration shown in Figure 7.5 of Chapter 7.3.3. Recall the external model (7.27b) relating the terminal voltage and current (V, I):

$$V \;=\; -Z^\Delta I \,+\, \gamma \mathbf{1}, \qquad\qquad \mathbf{1}^\mathsf{T} I \;=\; 0 \tag{9.75}$$

where the zero-sequence voltage $\gamma := \frac{1}{3}\mathbf{1}^\mathsf{T} V$ is also a variable to be determined in an analysis problem and

$$Z^\Delta \;:=\; \frac{1}{9}\Gamma^\mathsf{T} \underbrace{z^\Delta\left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\,\tilde{z}^{\Delta\mathsf{T}}\right)}_{\hat{z}^\Delta}\Gamma$$

Substitute $V = F\tilde{V}$ and $I = F\tilde{I}$ to obtain the external model in the sequence coordinate:

$$\tilde{V} \;=\; -\underbrace{\left(\overline{F}Z^\Delta F\right)}_{\tilde{Z}^\Delta}\tilde{I} \,+\, \gamma\overline{F}\mathbf{1}, \qquad\qquad \mathbf{1}^\mathsf{T} F\tilde{I} \;=\; 0 \tag{9.76}$$

where F and its inverse $\overline{F}$ is given in (9.73). It can be shown (Exercise 9.19) that

$$\tilde{Z}^\Delta \;:=\; \frac{1}{9}\left(F\Lambda\right)^\mathsf{H}\hat{z}^\Delta\left(F\Lambda\right) \qquad\text{with}\qquad \Lambda \;:=\; \begin{bmatrix} 0 & & \\ & 1-\alpha & \\ & & 1-\alpha^2 \end{bmatrix}$$

Moreover $\gamma\overline{F}\mathbf{1} = \tilde{V}_0\, e_1$ and $\mathbf{1}^\mathsf{T} F\tilde{I} = \sqrt{3}\tilde{I}_0 = 0$.

If the impedance is balanced, i.e., $z^{ab} = z^{bc} = z^{ca}$ then (Exercise 9.19)

$$Z^\Delta \;=\; \frac{z^{ab}}{3}\left(\mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^\mathsf{T}\right), \qquad\qquad \tilde{Z}^\Delta \;=\; \frac{z^{ab}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.77a}$$

and the external model (9.76) of a Δ-configured impedance in the sequence coordinate becomes decoupled:

$$\begin{bmatrix} 0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} \;=\; -\frac{z^{ab}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix}, \qquad\qquad \tilde{I}_0 \;=\; \frac{1}{\sqrt{3}}\left(I_a + I_b + I_c\right) \;=\; 0 \tag{9.77b}$$

For a Δ-configured load, $\tilde{I}_0 = 0$ because there is no neutral wire and therefore KCL dictates that the line currents sum to zero. The model (9.77) defines three separate impedances in the sequence coordinate:

zero-seq impedance:	null $\quad(\tilde{I}_0 \;=\; 0,\ \tilde{Z}_0 \;=\; \infty$, open circuit)
positive-seq impedance:	$\tilde{V}_+ \;=\; -\frac{z^{ab}}{3}\tilde{I}_+$
negative-seq impedance:	$\tilde{V}_- \;=\; -\frac{z^{ab}}{3}\tilde{I}_-$

The interpretation is that a balanced Δ-configured impedance with $z^{ab} = z^{bc} = z^{ca}$ connected to a bus in a power network is transformed into an impedance of $z^{ab}/3$ at that bus (as we have seen in Chapter 1.2.4) in the positive and the negative-sequence networks and no impedance at that bus in the zero-sequence network (i.e., in the circuit model for the zero-sequence network, the connection between this bus and the ground is open; see (**??**) and discussions therein). This does not mean that the voltage $V_{j,0} = 0$ at bus j in the zero-sequence network where the impedance is connected. Rather, it means that there is zero injection at bus j ($\tilde{I}_{j,0} = 0$) and $\tilde{V}_{j,0}$ will be determined by the network equation; see Chapter 9.4.5.

Remark 9.10 (Terminal variables)**.** It is important to remember that the external models derived in this section relate the sequence variables $(\tilde{V}, \tilde{I})$ of the terminal voltage and current (V, I), not the internal voltage and current $\left(V^{Y/\Delta}, I^{Y/\Delta}\right)$. See Example 9.13 on how to use sequence networks to calculate internal currents and powers. □

9.4.2 Sequence voltage sources

Y **configuration** $\left(E^Y, z^Y, z^n\right)$**.** Consider the four-wire three-phase voltage source $\left(E^Y, z^Y, z^n\right)$ in Y configuration shown in Figure 7.4 of Chapter 7.3.2. Under assumption C7.1 (all neutrals are grounded and all voltages are defined with respect to the ground), recall the external model (7.13b) relating the terminal voltage and current (V, I):

$$V \;=\; E^Y \;-\; Z^Y I \qquad\qquad \text{with} \qquad\qquad Z^Y \;:=\; z^Y \;+\; z^n\, \mathbf{1}\mathbf{1}^\mathsf{T}$$

where Z^Y is the same matrix as that for Y-configured impedance. Substitute $V = F\tilde{V}$ and $I = F\tilde{I}$ to obtain the external model in the sequence coordinate:

$$\tilde{V} \;=\; \underbrace{\overline{F}E^Y}_{\tilde{E}^Y} - \underbrace{\overline{F}Z^Y F}_{\tilde{Z}^Y}\tilde{I} \;=:\; \tilde{E}^Y \;-\; \tilde{Z}^Y\tilde{I}$$

The sequence impedance matrix $\tilde{Z}^Y := \overline{F}Z^Y F$ is the same matrix as that for Y-configured impedance and the sequence internal voltage is:

$$\tilde{E}^Y \;:=\; \overline{F}E^Y \;=\; \frac{1}{\sqrt{3}}\begin{bmatrix}\mathbf{1}^\mathsf{H}E^Y\\ \alpha_+^\mathsf{H}E^Y\\ \alpha_-^\mathsf{H}E^Y\end{bmatrix}$$

When the impedance z^Y is balanced, i.e., $z^{an} = z^{bn} = z^{cn}$, even if the internal voltage E^Y is unbalanced, its external model in the sequence coordinate becomes decoupled (using (9.74b)):

$$\begin{bmatrix}\tilde{V}_0\\ \tilde{V}_+\\ \tilde{V}_-\end{bmatrix} \;=\; \begin{bmatrix}\tilde{E}_0^Y\\ \tilde{E}_+^Y\\ \tilde{E}_-^Y\end{bmatrix} \;-\; \begin{bmatrix}z^{an}+3z^n & 0 & 0\\ 0 & z^{an} & 0\\ 0 & 0 & z^{an}\end{bmatrix}\begin{bmatrix}\tilde{I}_0\\ \tilde{I}_+\\ \tilde{I}_-\end{bmatrix} \tag{9.78a}$$

This defines three separate non-ideal voltage sources:

$$\begin{aligned}
\text{zero-seq voltage source:} &\qquad \tilde{V}_0 \;=\; \tilde{E}_0^Y \;-\; (z^{an}+3z^n)\,\tilde{I}_0\\
\text{positive-seq voltage source:} &\qquad \tilde{V}_+ \;=\; \tilde{E}_+^Y \;-\; z^{an}\tilde{I}_+\\
\text{negative-seq voltage source:} &\qquad \tilde{V}_- \;=\; \tilde{E}_-^Y \;-\; z^{an}\tilde{I}_-
\end{aligned}$$

As for a balanced impedance, the voltage source becomes decoupled in the sequence coordinate even if they remain unbalanced.

Furthermore, if $E^Y = E^{an}\alpha_+$ is a balanced positive-sequence set then only the positive-sequence voltage is nonzero:

$$\overline{F}E^Y \;=\; \tilde{E}^Y \;=\; \frac{1}{\sqrt{3}}\begin{bmatrix}\mathbf{1}^\mathsf{T}\\ \alpha_-^\mathsf{T}\\ \alpha_+^\mathsf{T}\end{bmatrix}(E^{an}\alpha_+) \;=\; \frac{E^{an}}{\sqrt{3}}\begin{bmatrix}\mathbf{1}^\mathsf{H}\alpha_+\\ \alpha_+^\mathsf{H}\alpha_+\\ \alpha_-^\mathsf{H}\alpha_+\end{bmatrix} \;=\; \begin{bmatrix}0\\ \sqrt{3}E^{an}\\ 0\end{bmatrix}$$

The external model of a balanced Y-configured voltage source in the sequence coordinate becomes (from (9.78a)):

$$\begin{bmatrix}\tilde V_0\\ \tilde V_+\\ \tilde V_-\end{bmatrix} = \begin{bmatrix}0\\ \sqrt{3}E^{an}\\ 0\end{bmatrix} - \begin{bmatrix}z^{an}+3z^n & 0 & 0\\ 0 & z^{an} & 0\\ 0 & 0 & z^{an}\end{bmatrix}\begin{bmatrix}\tilde I_0\\ \tilde I_+\\ \tilde I_-\end{bmatrix} \tag{9.78b}$$

This defines a voltage source $\left(\sqrt{3}E^{an}, z^{an}\right)$ on the positive-sequence network and impedances on the other sequence networks:

$$\begin{aligned} \text{zero-seq impedance:} \qquad & \tilde V_0 \;=\; -\left(z^{an}+3z^n\right)\tilde I_0 \\ \text{positive-seq voltage source:} \qquad & \tilde V_+ \;=\; \sqrt{3}E^{an} \;-\; z^{an}\tilde I_+ \\ \text{negative-seq impedance:} \qquad & \tilde V_- \;=\; -z^{an}\tilde I_- \end{aligned}$$

They are illustrated in Figure 9.8. [6]

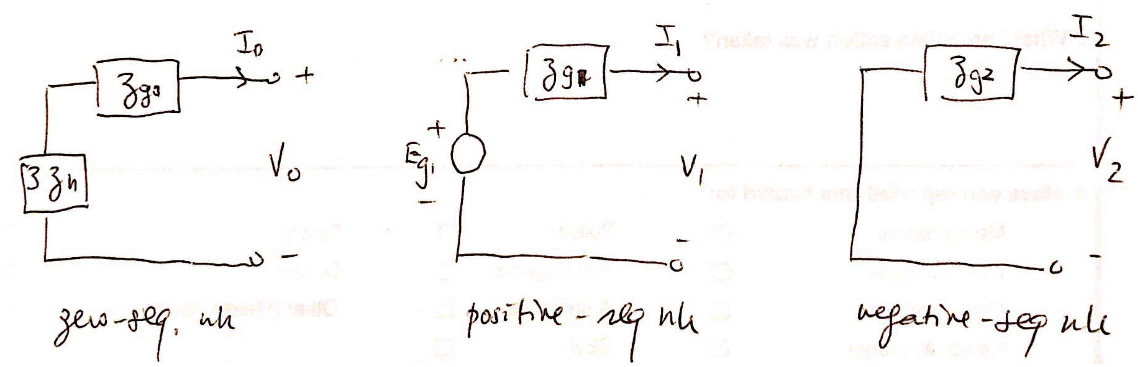


Figure 9.8: The sequence networks of a balanced voltage source $\left(E^Y, z^Y, z^n\right)$ in Y configuration.

Δ configuration $\left(E^\Delta, z^\Delta\right)$. Consider the three-phase voltage source $\left(E^\Delta, z^\Delta\right)$ in Δ configuration shown in Figure 7.5 of Chapter 7.3.3. One of its external models is (7.21b), reproduced here [7]

$$V \;=\; \hat\Gamma E^\Delta \;-\; Z^\Delta I \;+\; \gamma\mathbf{1}, \qquad\qquad \mathbf{1}^\mathsf{T} I \;=\; 0$$

where

$$\hat\Gamma \;:=\; \frac{1}{3}\Gamma^\mathsf{T}\left(\mathbb{I}-\frac{1}{\zeta}\tilde z^\Delta\mathbf{1}^\mathsf{T}\right), \qquad\qquad Z^\Delta \;:=\; \frac{1}{9}\Gamma^\mathsf{T} z^\Delta\left(\mathbb{I}-\frac{1}{\zeta}\mathbf{1}\tilde z^{\Delta\mathsf{T}}\right)\Gamma$$

where $\tilde z^\Delta := z^\Delta\mathbf{1}$ is a column vector and $\zeta := \mathbf{1}^\mathsf{T}\tilde z^\Delta$ is a scalar. This is similar to the model (9.75) of Δ-configured impedance with the extra term $\hat\Gamma E^\Delta$. Substitute $V = F\tilde V$ and $I = F\tilde I$ to obtain the external model in the sequence coordinate:

$$\tilde V \;=\; \underbrace{\overline{F}\hat\Gamma E^\Delta}_{\tilde E^\Delta} - \underbrace{\overline{F}Z^\Delta F}_{\tilde Z^\Delta}\tilde I \;+\; \gamma\overline{F}\mathbf{1} \;=:\; \tilde E^\Delta \;-\; \tilde Z^\Delta\tilde I \;+\; \tilde V_0 e_1, \qquad\qquad \mathbf{1}^\mathsf{T}F\tilde I \;=\; 0$$

[6] The sequence networks of synchronous generators are generally more complicated and their sequence impedances (mostly reactances) are generally unequal unlike the model in (9.78b); see e.g. [55, Section 2.3].

[7]

where $\mathbf{1}^\mathsf{T} F\tilde{I} = \sqrt{3}\tilde{I}_0 = 0$. This is similar to (9.76) with the extra term (Exercise 9.20)

$$\tilde{E}^\Delta \ := \ \overline{F}\hat{\Gamma}E^\Delta \ = \ \Lambda^\dagger \overline{F}\left(\mathbb{I} - \frac{1}{\zeta}\tilde{z}^\Delta \mathbf{1}^\mathsf{T}\right)E^\Delta \qquad \text{with} \qquad \Lambda^\dagger \ := \ \begin{bmatrix} 0 & & \\ & (1-\alpha)^{-1} & \\ & & (1-\alpha^2)^{-1} \end{bmatrix}$$

If the impedance is balanced, i.e., $z^{ab} = z^{bc} = z^{ca}$ then $\tilde{z}^\Delta := z^{ab}\mathbf{1}$, $\zeta := 3z^{ab}$, and (Exercise 9.20 and from (9.77a))

$$\tilde{E}^\Delta \ = \ \begin{bmatrix} 0 \\ (1-\alpha)^{-1}\tilde{E}_+^\Delta \\ (1-\alpha^2)^{-1}\tilde{E}_-^\Delta \end{bmatrix}, \qquad Z^\Delta \ = \ \frac{z^{ab}}{3}\left(\mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^\mathsf{T}\right), \qquad \tilde{Z}^\Delta \ = \ \frac{z^{ab}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where the sequence voltages are $\tilde{E}_+^\Delta := \frac{1}{3}\alpha_+^\mathsf{H} E^\Delta$ and $\tilde{E}_-^\Delta := \frac{1}{3}\alpha_-^\mathsf{H} E^\Delta$. The zero-sequence voltage $\tilde{E}_0^\Delta = 0$ because there is no neutral line in Δ configuration. Hence the external model in the sequence coordinate is

$$\begin{bmatrix} 0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} \ = \ \begin{bmatrix} 0 \\ (1-\alpha)^{-1}\tilde{E}_+^\Delta \\ (1-\alpha^2)^{-1}\tilde{E}_-^\Delta \end{bmatrix} - \frac{z^{ab}}{3}\begin{bmatrix} 0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix}, \qquad \tilde{I}_0 \ = \ \frac{1}{\sqrt{3}}\left(I^a + I^b + I^c\right) \ = \ 0 \tag{9.79a}$$

Hence the voltage sources in the sequence coordinate are unbalanced but decoupled:

$$\begin{aligned} \text{zero-seq voltage source:} & \qquad \text{null} \quad (\tilde{I}_0 \ = \ 0,\ \tilde{Z}_0 \ = \ \infty, \text{ open circuit}) \\ \text{positive-seq voltage source:} & \qquad \tilde{V}_+ \ = \ \frac{E_+^\Delta}{1-\alpha} - \frac{z^{ab}}{3}\tilde{I}_+ \\ \text{negative-seq voltage source:} & \qquad \tilde{V}_- \ = \ \frac{E_-^\Delta}{1-\alpha^2} - \frac{z^{ab}}{3}\tilde{I}_- \end{aligned}$$

As for a Δ-configured impedance, a symmetric voltage source in a power network is transformed into voltage sources in the positive and negative-sequence networks. The equivalent series impedance of the sequence voltage sources is $z^{ab}/3$ as we have seen in Chapter 1.2.4. There is no device (open circuit) in the zero-sequence network, which means that, when the voltage source is connected to bus j, there is zero injection at bus j in the zero-sequence network ($\tilde{I}_{j,0} = 0$) and $\tilde{V}_{j,0}$ will be determined by the network equation; see Chapter 9.4.5.

Furthermore, if $E^\Delta := E^{ab}\alpha_+$ is a balanced positive-sequence set then

$$\tilde{E}_+^\Delta \ = \ \sqrt{3}E^{ab}, \qquad\qquad \tilde{E}_-^\Delta \ = \ 0$$

and

$$\begin{bmatrix} 0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} \ = \ \begin{bmatrix} 0 \\ e^{-\mathbf{i}\pi/6}E^{ab} \\ 0 \end{bmatrix} - \frac{z^{ab}}{3}\begin{bmatrix} 0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix} \tag{9.79b}$$

since $\sqrt{3}/(1-\alpha) = e^{-\mathbf{i}\pi/6}$. This defines a voltage source $\left(e^{-\mathbf{i}\pi/6}E^{ab}, z^{ab}/3\right)$ in the positive-sequence network and an impedance $z^{ab}/3$ in the negative-sequence network:

$$\begin{aligned}
\text{zero-seq voltage source:} & \qquad \text{null} \quad (\tilde{I}_0 \;=\; 0,\, \tilde{Z}_0 \;=\; \infty, \text{ open circuit}) \\
\text{positive-seq voltage source:} & \qquad \tilde{V}_+ \;=\; e^{-\mathbf{i}\pi/6}E^{ab} \;-\; \frac{z^{ab}}{3}\tilde{I}_+ \\
\text{negative-seq voltage source:} & \qquad \tilde{V}_- \;=\; -\,\frac{z^{ab}}{3}\tilde{I}_-
\end{aligned}$$

There is no device (open circuit) in the zero-sequence network.

9.4.3 Sequence current sources

Y **configuration** $\left(J^Y, y^Y, z^n\right)$**.** An external model of a Y-configured current source $\left(J^Y, y^Y, z^n\right)$ is (from (7.15a)):

$$I \;=\; -J^Y \;-\; y^Y\left(V - V^n\mathbf{1}\right)$$

Substitute $V = F\tilde{V}$ and $I = F\tilde{I}$ to obtain the external model in the sequence coordinate:

$$\tilde{I} \;=\; -\underbrace{\overline{F}J^Y}_{\tilde{J}^Y} \;-\; \underbrace{\overline{F}y^YF}_{\tilde{Y}^Y}\tilde{V} \;+\; V^n\overline{F}y^Y\mathbf{1}$$

where $\tilde{J}^Y := \overline{F}J^Y$ and

$$\tilde{Y}^Y \;:=\; \overline{F}y^YF \;=\; \frac{1}{3}\left(y^{an}\mathbf{1}\mathbf{1}^{\mathsf{H}} \;+\; y^{bn}\alpha_-\alpha_-^{\mathsf{H}} \;+\; y^{cn}\alpha_+\alpha_+^{\mathsf{H}}\right) \tag{9.80}$$

If the phase admittance $y^Y := y^{an}\mathbb{I}$ is balanced then the sequence admittance is also balanced:

$$\tilde{Y}^Y \;:=\; \overline{F}y^YF \;=\; y^{an}\mathbb{I}, \qquad\qquad \overline{F}y^Y\mathbf{1} \;=\; y^{an}\overline{F}\mathbf{1} \;=\; y^{an}\begin{bmatrix}\sqrt{3}\\0\\0\end{bmatrix}$$

The current source becomes decoupled in the sequence coordinate even though it is unbalanced:

$$\begin{bmatrix}\tilde{I}_0\\ \tilde{I}_+\\ \tilde{I}_-\end{bmatrix} \;=\; -\begin{bmatrix}\tilde{J}_0^Y\\ \tilde{J}_+^Y\\ \tilde{J}_-^Y\end{bmatrix} \;-\; y^{an}\left(\begin{bmatrix}\tilde{V}_0\\ \tilde{V}_+\\ \tilde{V}_-\end{bmatrix} \;-\; \begin{bmatrix}\sqrt{3}V^n\\0\\0\end{bmatrix}\right)$$

In particular the neutral voltage V^n appears only in the zero-sequence network. If, furthermore, the current source $J^Y := J^{an}\alpha_+$ is in a balanced positive sequence then

$$\tilde{J}^Y \;=\; \overline{F}J^Y \;=\; \frac{J^{an}}{\sqrt{3}}\begin{bmatrix}\mathbf{1}^{\mathsf{H}}\\ \alpha_+^{\mathsf{H}}\\ \alpha_-^{\mathsf{H}}\end{bmatrix}\alpha_+ \;=\; \begin{bmatrix}0\\ \sqrt{3}J^{an}\\ 0\end{bmatrix}$$

The current source in the sequence coordinate becomes a current source $\left(\sqrt{3}J^{an}, y^{an}\right)$ in the positive-sequence network and the impedance $(y^{an})^{-1}$ in each of the other two sequence networks:

$$\begin{aligned} \text{zero-seq impedance:} \qquad & \tilde{I}_0 \;=\; -y^{an}\left(\tilde{V}_0 - \sqrt{3}V^n\right) \\ \text{positive-seq current source:} \qquad & \tilde{I}_+ \;=\; -\sqrt{3}J^{an} \;-\; y^{an}\tilde{V}_+ \\ \text{negative-seq impedance:} \qquad & \tilde{I}_- \;=\; -y^{an}\tilde{V}_- \end{aligned}$$

The interpretation of the zero-sequence impedance is that the voltage drop across the impedance $(y^{an})^{-1}$ is $\tilde{V}_0 - \sqrt{3}V^n$ with one end of the impedance at a potential $\sqrt{3}V^n$ with respect to the common voltage reference point.

When assumption C7.1 holds (the neutral is grounded and voltages are defined with respect to the ground) so that $V^n = -z^n\left(\mathbf{1}^\mathsf{T} I\right)$, we have

$$V^n \;=\; -z^n\left(\mathbf{1}^\mathsf{T} F \tilde{I}\right) \;=\; -\frac{z^n}{\sqrt{3}}\left(\mathbf{1}^\mathsf{T}\begin{bmatrix}\mathbf{1} & \alpha_+ & \alpha_-\end{bmatrix}\tilde{I}\right) \;=\; -\sqrt{3}z^n\,\tilde{I}_0$$

i.e., the neutral voltage depends only on the zero-sequence current $\tilde{I}_0$ (of the terminal current I). Substitute this into expressions above, the sequence voltage and current $\left(\tilde{V}, \tilde{I}\right)$ satisfies, when $y^Y := y^{an}\mathbb{I}$,

$$\begin{bmatrix}(1+3\,y^{an}\,z^n)\,\tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_-\end{bmatrix} \;=\; -\begin{bmatrix}\tilde{J}_0^Y \\ \tilde{J}_+^Y \\ \tilde{J}_-^Y\end{bmatrix} - y^{an}\begin{bmatrix}\tilde{V}_0 \\ \tilde{V}_+ \\ \tilde{V}_-\end{bmatrix} \tag{9.81a}$$

and the current source becomes decoupled in the sequence coordinate even if they remain unbalanced:

$$\begin{aligned} \text{zero-seq current source:} \qquad & \tilde{I}_0 \;=\; -\frac{\tilde{J}_0^Y}{1+3\,y^{an}\,z^n} \;-\; \frac{y^{an}}{1+3\,y^{an}\,z^n}\tilde{V}_0 \\ \text{positive-seq current source:} \qquad & \tilde{I}_+ \;=\; -\tilde{J}_+^Y \;-\; y^{an}\tilde{V}_+ \\ \text{negative-seq current source:} \qquad & \tilde{I}_- \;=\; -\tilde{J}_-^Y \;-\; y^{an}\tilde{V}_- \end{aligned}$$

If, furthermore, the current source $J^Y := J^{an}\alpha_+$ they become:

$$\text{zero-seq admittance:} \qquad \tilde{I}_0 \;=\; -\frac{y^{an}}{1+3\,y^{an}\,z^n}\tilde{V}_0 \tag{9.81b}$$

$$\text{positive-seq current source:} \qquad \tilde{I}_+ \;=\; -\sqrt{3}J^{an} \;-\; y^{an}\tilde{V}_+ \tag{9.81c}$$

$$\text{negative-seq admittance:} \qquad \tilde{I}_- \;=\; -y^{an}\tilde{V}_- \tag{9.81d}$$

Instead of sequence current sources in (9.81), equivalent voltage sources in the sequence domain can also be derived starting from the external model of a current source (from (7.15b)): $V = -\left(z^Y J^Y \;+\; Z^Y I\right)$ where $z^Y := \left(y^Y\right)^{-1}$ and $Z^Y := z^Y \;+\; z^n\,\mathbf{1}\mathbf{1}^\mathsf{T}$; see Exercise 9.22.

Δ configuration $\left(J^\Delta, y^\Delta\right)$. The external model of a Δ-configured current source is (from (7.23a)):

$$I \;=\; -\left(\Gamma^{\mathsf{T}} J^\Delta \,+\, Y^\Delta V\right)$$

where $Y^\Delta := \Gamma^{\mathsf{T}} y^\Delta \Gamma$ is the matrix in (7.21a). Substitute $V = F\tilde{V}$ and $I = F\tilde{I}$ to obtain the external model in the sequence coordinate:

$$\tilde{I} \;=\; -\left(\underbrace{\overline{F}\Gamma^{\mathsf{T}} J^\Delta}_{\tilde{J}^\Delta} + \underbrace{\overline{F} Y^\Delta F}_{\tilde{Y}^\Delta}\tilde{V}\right) \;=:\; -\left(\tilde{J}^\Delta \,+\, \tilde{Y}^\Delta \tilde{V}\right)$$

where

$$\begin{aligned}
\tilde{J}^\Delta &:= \overline{F}\Gamma^{\mathsf{T}} J^\Delta \;=\; 3\Lambda^\dagger \overline{F} J^\Delta \\
\tilde{Y}^\Delta &:= \overline{F}\left(\Gamma^{\mathsf{T}} y^\Delta \Gamma\right) F \;=\; \overline{F}\left(3F\Lambda^\dagger \overline{F}\right) y^\Delta \left(F\Lambda\overline{F}\right) F \;=\; 3\Lambda^\dagger \left(\overline{F} y^\Delta F\right)\Lambda
\end{aligned}$$

where we have used $\Gamma = F\Lambda\overline{F}$ and $\Gamma^{\mathsf{T}} = 3\Gamma^\dagger = 3F\Lambda^\dagger\overline{F}$ from (7.6).

If the phase admittance $y^Y := y^{ab}\mathbb{I}$ is balanced, then the effective phase admittance Y^Δ is not diagonal but its sequence admittance $\tilde{Y}^\Delta$ is unbalanced but diagonal:

$$\begin{aligned}
Y^\Delta &:= y^{ab}\Gamma^{\mathsf{T}}\Gamma \;=\; 3y^{ab}\left(\mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^{\mathsf{T}}\right) \\
\tilde{Y}^\Delta &:= \overline{F} Y^\Delta F \;=\; 3y^{ab}\left(\mathbb{I} \,-\, e_1 e_1^{\mathsf{T}}\right)
\end{aligned}$$

where we have used $\Gamma^{\mathsf{T}}\Gamma = 3\left(\mathbb{I} - \frac{1}{3}\mathbf{1}\mathbf{1}^{\mathsf{T}}\right)$ from Theorem 7.2 and $\overline{F}\mathbf{1} = \sqrt{3}e_1$. Hence the current source is unbalanced but decoupled in the sequence coordinate:

$$\begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix} \;=\; -\begin{bmatrix} \tilde{J}_0^\Delta \\ \tilde{J}_+^\Delta \\ \tilde{J}_-^\Delta \end{bmatrix} - 3y^{ab}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{V}_0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} \;=\; -\begin{bmatrix} \tilde{J}_0^\Delta \\ \tilde{J}_+^\Delta \\ \tilde{J}_-^\Delta \end{bmatrix} - 3y^{ab}\begin{bmatrix} 0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} \tag{9.82a}$$

The zero-sequence network has an ideal current source $\tilde{J}_0^\Delta$ and the other two sequence networks each has a non-ideal current source:

$$\begin{aligned}
&\text{zero-seq current source:} && \tilde{I}_0 \;=\; -\tilde{J}_0^\Delta \\
&\text{positive-seq current source:} && \tilde{I}_+ \;=\; -\tilde{J}_+^\Delta \,-\, 3y^{ab}\tilde{V}_+ \\
&\text{negative-seq current source:} && \tilde{I}_- \;=\; -\tilde{J}_-^\Delta \,-\, 3y^{ab}\tilde{V}_-
\end{aligned}$$

If, furthermore, the current source $J^\Delta := J^{ab}\alpha_+$ is a balanced positive sequence then

$$\tilde{J}^\Delta \;:=\; 3J^{ab}\Lambda^\dagger\overline{F}\alpha_+ \;=\; 3J^{ab}\begin{bmatrix} 0 & & \\ & (1-\alpha)^{-1} & \\ & & \left(1-\alpha^2\right)^{-1} \end{bmatrix}\begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} 0 \\ 3e^{-\mathbf{i}\pi/6}J^{ab} \\ 0 \end{bmatrix}$$

where we have used $\overline{F}\alpha_+ = \sqrt{3}e_2$ and $\sqrt{3}/(1-\alpha) = e^{-\mathbf{i}\pi/6}$. A balanced positive-sequence current source is therefore transformed into a current source $\left(3e^{-\mathbf{i}\pi/6}J^{ab}, 3y^{ab}\right)$ in the positive-sequence network and an admittance $3y^{ab}$ in the negative-sequence network:

$$\text{zero-seq current source:} \qquad \text{null} \quad (\tilde{I}_0 \;=\; 0) \tag{9.82b}$$

$$\text{positive-seq current source:} \qquad \tilde{I}_+ \;=\; -3e^{-\mathbf{i}\pi/6}J^{ab} \;-\; 3y^{ab}\tilde{V}_+ \tag{9.82c}$$

$$\text{negative-seq admittance:} \qquad \tilde{I}_- \;=\; -3y^{ab}\tilde{V}_- \tag{9.82d}$$

There is no device in the zero-sequence network because Δ configuration has no neutral line.

9.4.4 Sequence line model

Consider a three-phase line connecting bus j and bus k that is modeled by only a series phase impedance matrix z^s_{jk}. We omit shunt admittances for simplicity.[8] The terminal voltages and the line current is related by Ohm's law:

$$V_j \;-\; V_k \;\;=\;\; z^s_{jk}I_{jk}$$

Convert to the sequence coordinate by substituting $V_j = F\tilde{V}_j$, $V_k = F\tilde{V}_k$ and $I_{jk} = F\tilde{I}_{jk}$ to get

$$\tilde{V}_j \;-\; \tilde{V}_k \;\;=\;\; \underbrace{\left(\overline{F}z^s_{jk}F\right)}_{\tilde{z}^s_{jk}}\tilde{I}_{jk} \;\;=:\;\; \tilde{z}^s_{jk}\tilde{I}_{jk} \tag{9.83a}$$

where $\tilde{z}^s_{jk} := \overline{F}z^s_{jk}F$ is called the *sequence impedance matrix* of line (j,k). This does not assume C9.1, i.e., z^s_{jk} and z^s_{kj} may be different.

If the phase impedance matrix z^s_{jk} is symmetric of the form in (8.5) then (omitting the subscript jk for simplicity)

$$\tilde{z}^s_{jk} \;\;=\;\; \frac{1}{3}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha^2 & \alpha\\ 1 & \alpha & \alpha^2\end{bmatrix}\begin{bmatrix}z^1 & z^2 & z^2\\ z^2 & z^1 & z^2\\ z^2 & z^2 & z^1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & \alpha & \alpha^2\\ 1 & \alpha^2 & \alpha\end{bmatrix} \;\;=\;\; \begin{bmatrix}z^1+2z^2 & 0 & 0\\ 0 & z^1-z^2 & 0\\ 0 & 0 & z^1-z^2\end{bmatrix} \tag{9.83b}$$

i.e., the sequence impedance matrix of line (j,k) is diagonal. This defines three separate sequence networks:

$$\text{zero-seq impedance:} \qquad \tilde{V}_{j,0} - \tilde{V}_{k,0} \;=\; \left(z^1+2z^2\right)\tilde{I}_{jk,0}$$

$$\text{positive-seq impedance:} \qquad \tilde{V}_{j,+} - \tilde{V}_{k,+} \;=\; \left(z^1-z^2\right)\tilde{I}_{jk,+}$$

$$\text{negative-seq impedance:} \qquad \tilde{V}_{j,-} - \tilde{V}_{k,-} \;=\; \left(z^1-z^2\right)\tilde{I}_{jk,-}$$

---

[8]Shunt admittances can be included using (8.4a): $I_{jk} = y^s_{jk}(V_j - V_k) + y^m_{jk}V_j$ in which case the sequence admittance matrices $\left(\tilde{y}^s_{jk}, \tilde{y}^m_{jk}, \tilde{y}^m_{kj}\right)$ are given by:

$$\tilde{I}_{jk} \;\;=\;\; \underbrace{\left(\overline{F}y^s_{jk}F\right)}_{\tilde{y}^s_{jk}}\left(\tilde{V}_j - \tilde{V}_k\right) \;+\; \underbrace{\left(\overline{F}y^m_{jk}F\right)}_{\tilde{y}^m_{jk}}\tilde{V}_j$$

The phase impedance matrix z^s_{jk} in (8.5) is complex symmetric but not Hermitian. In general a complex symmetric matrix may not be diagonalizable (see Exercise 9.23 for an example). The matrix z^s_{jk} however is normal and hence unitarily diagonalizable through the unitary matrix F (Exercise 9.24).

9.4.5 Three-phase analysis

We now explain how to compose sequence networks from individual device models in the sequence coordinate derived in Chapters 9.4.1–9.4.4. We will show that if a network is unbalanced but symmetric, its sequence networks are decoupled and can be analyzed separately.

Definition 9.2 (Symmetric network)**.** A network $G := \left(\overline{N}, E\right)$ that connects a set of three-phase devices by three-phase lines is called *symmetric* if the following assumptions hold:

C9.9: All impedances are symmetric $z_j^{Y/\Delta} = z_j^{an/ab}\mathbb{I}$.

C9.10: All voltage sources have symmetric series impedances $z_j^{Y/\Delta} = z_j^{an/ab}\mathbb{I}$.

C9.11: All current sources have symmetric shunt admittances $y_j^{Y/\Delta} = y_j^{an/ab}\mathbb{I}$.

C9.12: All three-phase lines (j,k) have series impedances $z^s_{jk} = z^s_{kj}$ that satisfy (8.5) and zero shunt admittances. In particular we assume for simplicity that assumption C9.1 holds.

Suppose we are given a symmetric network with a single three-phase device at each bus. As before, partition the set $\overline{N}$ of buses into 6 disjoint subsets:

- $N_v^{Y/\Delta}$: buses with non-ideal voltage sources in Y or Δ configurations: $\left(E^Y, z^Y, z^n\right)$, $\left(E^\Delta, z^\Delta\right)$.
- $N_c^{Y/\Delta}$: buses with non-ideal current sources in Y or Δ configurations: $\left(J^Y, y^Y, z^n\right)$, $\left(J^\Delta, y^\Delta\right)$.
- $N_i^{Y/\Delta}$: buses with impedances in Y or Δ configurations: $\left(z^Y, z^n\right)$, z^Δ.

Suppose assumption C7.1 holds (i.e., all neutrals are grounded and voltages are defined with respect to the ground). C7.1 and the assumption of a single three-phase device at each bus are made without loss of generality only to simplify presentation (see Example 9.13 for a network where there are two devices connected to a single bus). We will follow the solution strategy of Chapter 9.3.4 that solves

$$\begin{bmatrix} I_v \\ I_c \\ I_i \end{bmatrix} = \underbrace{\begin{bmatrix} Y_{vv} & Y_{vc} & Y_{vi} \\ Y_{cv} & Y_{cc} & Y_{ci} \\ Y_{iv} & Y_{ic} & Y_{ii} \end{bmatrix}}_{Y} \begin{bmatrix} V_v \\ V_c \\ V_i \end{bmatrix} \tag{9.84}$$

for the terminal voltage $V_{-v} := (V_c, V_i)$ and current $I_{-c} := (I_v, I_i)$. All other variables such as internal voltages and currents $\left(V^{Y/\Delta}, I^{Y/\Delta}\right)$ can then be derived in terms of the terminal voltages and currents (V, I).

We now show that (9.84) decomposes into three separate sequence networks so that it can be solved by analyzing three simpler networks. Furthermore, if not only is the network symmetric but all voltage and current sources are also balanced positive-sequence sets, then it is sufficient to analyze only the positive-sequence network. This is because in that case there are only impedances and admittances, but no voltage or current sources, in the zero-sequence and the negative-sequence networks.

Let $\mathbb{I}_{N+1}$ be the identity matrix of size $N+1$ so that $\mathbb{I}_{N+1} \otimes F$ is a matrix of size $3(N+1) \times 3(N+1)$. Convert both sides of (9.84) into the sequence coordinate by substituting

$$I \;=:\; (\mathbb{I}_{N+1} \otimes F)\,\tilde{I}, \qquad\qquad V \;=:\; (\mathbb{I}_{N+1} \otimes F)\,\tilde{V}$$

to obtain

$$\begin{bmatrix} \tilde{I}_v \\ \tilde{I}_c \\ \tilde{I}_i \end{bmatrix} \;=\; \underbrace{\begin{bmatrix} \tilde{Y}_{vv} & \tilde{Y}_{vc} & \tilde{Y}_{vi} \\ \tilde{Y}_{cv} & \tilde{Y}_{cc} & \tilde{Y}_{ci} \\ \tilde{Y}_{iv} & \tilde{Y}_{ic} & \tilde{Y}_{ii} \end{bmatrix}}_{\tilde{Y}} \begin{bmatrix} \tilde{V}_v \\ \tilde{V}_c \\ \tilde{V}_i \end{bmatrix} \qquad \text{where} \qquad \tilde{Y} \;:=\; \left(\mathbb{I}_{N+1} \otimes \overline{F}\right) Y \left(\mathbb{I}_{N+1} \otimes F\right) \tag{9.85a}$$

and we have used $(\mathbb{I}_{N+1} \otimes F)^{-1} = \mathbb{I}_{N+1} \otimes \overline{F}$ from Lemma 9.6. The three rows $(3j+1, 3j+2, 3j+3)$ of (9.85a) corresponding to the sequence current $\tilde{I}_j \in \mathbb{C}^3$ of device $j = 0, \ldots, N$, are:

$$\tilde{I}_j \;=\; \sum_{\substack{j:j\sim k \\ k\in N_v}} \tilde{y}_{jk}\left(\tilde{V}_j - \tilde{V}_k\right) \;+\; \sum_{\substack{j:j\sim k \\ k\in N_c}} \tilde{y}_{jk}\left(\tilde{V}_j - \tilde{V}_k\right) \;+\; \sum_{\substack{j:j\sim k \\ k\in N_i}} \tilde{y}_{jk}\left(\tilde{V}_j - \tilde{V}_k\right), \qquad j \in \overline{N} \tag{9.85b}$$

where $\tilde{y}_{jk} := \left(\tilde{z}_{jk}\right)^{-1} := \left(\overline{F} z^s_{jk} F\right)^{-1}$ are the series admittance matrices of lines (j,k) in the sequence coordinate from (9.83). The network equation (9.85) relates terminal variables. To show that the three-phase network decomposes into decoupled sequence networks we have to show both of the following:

1. The three rows of (9.85b) are decoupled, i.e., the zero-sequence current $\tilde{I}_{j,0}$ depends only on voltages $\tilde{V}_{k,0}$ of its adjacent buses $k \neq j$ in the zero-sequence network but not on voltages $\tilde{V}_{k,s}$ in the other sequence networks $s \in \{+,-\}$. Similarly for the positive and negative-sequence currents $\left(\tilde{I}_{j,+}, \tilde{I}_{j,-}\right)$.

2. At each bus j, the terminal voltage and current $\left(\tilde{V}_j, \tilde{I}_j\right)$ are decoupled, i.e., the zero-sequence voltage $\tilde{V}_{j,0}$ does not depend on the positive or negative-sequence currents $\left(\tilde{I}_{j,+}, \tilde{I}_{j,-}\right)$ at bus j. Similarly for $\tilde{V}_{j,+}$ and $\tilde{V}_{j,-}$.

The first claim follows from C9.12 in Definition 9.2 which implies that $\tilde{y}_{jk}$ is diagonal (from (9.83)). This means that the three rows of (9.85b) are decoupled at all buses $j \in \overline{N}$. We hence only need to prove the second claim that locally at each bus j the sequence voltage $\tilde{V}_{j,s}$, $s \in \{0,+,-\}$, does not couple the sequence currents $\tilde{I}_{j,s'}$, $s' \neq s$. This can be shown using the models derived in Chapters 9.4.1–9.4.3.

Specifically the external models of the three-phase devices are as follows.

1. Voltage source $j \in N_v$ from (9.78a) and (9.79a):

$$\begin{bmatrix} \tilde{V}_{j,0} \\ \tilde{V}_{j,+} \\ \tilde{V}_{j,-} \end{bmatrix} = \begin{bmatrix} \tilde{E}^Y_{j,0} \\ \tilde{E}^Y_{j,+} \\ \tilde{E}^Y_{j,-} \end{bmatrix} - \begin{bmatrix} z^{an}_j + 3z^n_j & & \\ & z^{an}_j & \\ & & z^{an}_j \end{bmatrix} \begin{bmatrix} \tilde{I}_{j,0} \\ \tilde{I}_{j,+} \\ \tilde{I}_{j,-} \end{bmatrix}, \qquad j \in N^Y_v \tag{9.86a}$$

$$\begin{bmatrix} 0 \\ \tilde{V}_{j,+} \\ \tilde{V}_{j,-} \end{bmatrix} = \begin{bmatrix} 0 & & \\ & \frac{1}{1-\alpha} & \\ & & \frac{1}{1-\alpha^2} \end{bmatrix} \begin{bmatrix} \tilde{E}^\Delta_{j,0} \\ \tilde{E}^\Delta_{j,+} \\ \tilde{E}^\Delta_{j,-} \end{bmatrix} - \frac{z^{ab}_j}{3} \begin{bmatrix} \tilde{I}_{j,0} \\ \tilde{I}_{j,+} \\ \tilde{I}_{j,-} \end{bmatrix}, \qquad j \in N^\Delta_v \tag{9.86b}$$

2. Current sources $j \in N_c$ from (9.81a) and (9.82a):

$$\begin{bmatrix} \tilde{I}_{j,0} \\ \tilde{I}_{0,+} \\ \tilde{I}_{j,-} \end{bmatrix} = -\frac{1}{1+3y^{an}z^n} \begin{bmatrix} \tilde{J}^Y_{j,0} \\ \tilde{J}^Y_{0,+} \\ \tilde{J}^Y_{0,-} \end{bmatrix} - \frac{y^{an}}{1+3y^{an}z^n} \begin{bmatrix} \tilde{V}_{j,0} \\ \tilde{V}_{j,+} \\ \tilde{V}_{0,-} \end{bmatrix}, \qquad j \in N^Y_c \tag{9.86c}$$

$$\begin{bmatrix} \tilde{I}_{j,0} \\ \tilde{I}_{0,+} \\ \tilde{I}_{j,-} \end{bmatrix} = -\begin{bmatrix} \tilde{J}^\Delta_{j,0} \\ \tilde{J}^\Delta_{j,+} \\ \tilde{J}^\Delta_{j,-} \end{bmatrix} - 3y^{ab} \begin{bmatrix} 0 \\ \tilde{V}_{j,+} \\ \tilde{V}_{j,-} \end{bmatrix}, \qquad j \in N^\Delta_c \tag{9.86d}$$

3. Impedances $j \in N_i$ from (9.74b) and (9.77b):

$$\begin{bmatrix} \tilde{V}_{j,0} \\ \tilde{V}_{j,+} \\ \tilde{V}_{j,-} \end{bmatrix} = -\begin{bmatrix} z^{an}+3z^n & 0 & 0 \\ 0 & z^{an} & 0 \\ 0 & 0 & z^{an} \end{bmatrix} \begin{bmatrix} \tilde{I}_{j,0} \\ \tilde{I}_{j,+} \\ \tilde{I}_{j,-} \end{bmatrix}, \qquad j \in N^Y_i \tag{9.86e}$$

$$\begin{bmatrix} 0 \\ \tilde{V}_{j,+} \\ \tilde{V}_{j,-} \end{bmatrix} = -\frac{z^{ab}}{3} \begin{bmatrix} \tilde{I}_{j,0} \\ \tilde{I}_{j,+} \\ \tilde{I}_{j,-} \end{bmatrix}, \qquad j \in N^\Delta_i \tag{9.86f}$$

Therefore the terminal voltage and current $\left(\tilde{V}_j, \tilde{I}_j\right)$ at each bus j are decoupled, even if they are unbalanced. The network equation (9.85) and the device models (9.86) thus decompose into separate 0/+/− sequence networks that can be analyzed separately, similar to per-phase analysis for balanced networks.

We illustrate the analysis of sequence networks with an example.

Example 9.13 (Sequence network analysis)**.** Consider the network shown in Figure 9.9 where a voltage source and a current source supply power through two lines to two loads in parallel. Suppose the network is symmetric (Definition 9.2) and C7.1 holds (i.e., all neutrals are grounded and voltages are defined with respect to the ground). Given the Y-configured voltage source $\left(E^Y, z^Y, z^n\right)$, the Δ-configured current source $\left(J^\Delta, y^\Delta\right)$, the balanced impedances $\left(z^Y, z^n\right)$, z^Δ, and the symmetric lines with series impedance matrices (z_{12}, z_{23}), calculate:

1. the terminal load voltages $V_2 := \left(V^a_2, V^b_2, V^c_2\right)$;

2. the internal current $I^Y_2 := \left(I^{an}_2, I^{bn}_2, I^{cn}_2\right)$ and the total complex power $\mathbf{1}^\mathsf{T} s^Y_2$ delivered to the Y-configured load;

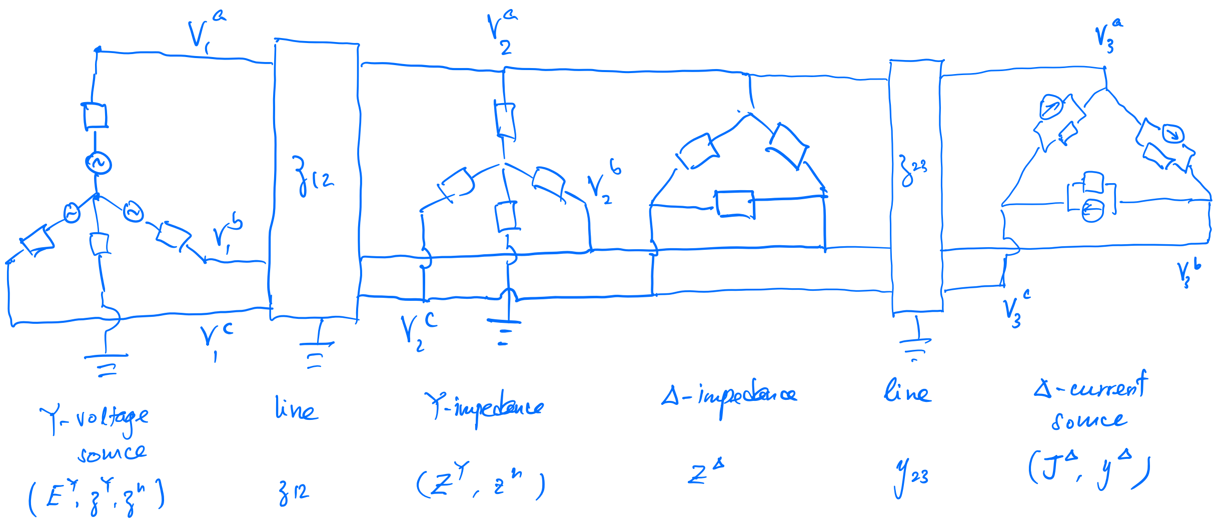


Figure 9.9: Example 9.13: Three-phase unbalanced sources supplies power two balance loads in parallel through symmetric lines.

3. the internal current $I_2^\Delta := \left(I_2^{ab}, I_2^{bc}, I_2^{ca}\right)$ and the total complex power $\mathbf{1}^\mathsf{T} s_2^\Delta$ delivered to the Δ-configured load;

Solution. The network equation (9.85) and the device models (9.86) decompose into separate 0/+/− sequence networks as shown in Figure 9.10. We will first determine the terminal sequence voltage $\tilde{V}_2$ and then the *terminal* sequence currents $\tilde{I}_2^1$ and $\tilde{I}_2^2$ coming out of the Y-configured and Δ-configured impedances respectively. The *terminal* phase variables are then $V_2 = F\tilde{V}_2$, $I_2^1 = F\tilde{I}_2^1$, and $I_2^2 = F\tilde{I}_2^2$. Given these terminal variables we can determine internal currents $\left(I_2^Y, I_2^\Delta\right)$ and powers $\left(s_2^Y, s_2^\Delta\right)$ using the conversion rules.

To determine $\tilde{V}_2$, apply KCL at bus 2 of the zero-sequence networks to get

$$\frac{\tilde{E}_{1,0}^Y - \tilde{V}_{2,0}}{\left(z_1^{an} + 3z_1^n\right) + \left(z_{12}^s + 2z_{12}^m\right)} \;=\; \frac{\tilde{V}_{2,0}}{z_2^{an} + 3z_2^n} + \tilde{J}_{3,0}^\Delta \tag{9.87a}$$

To analyze the positive and negative-sequence networks let the Thévenin equivalent load admittance be

$$\tilde{Y}_2 \;=\; y_2^{an} + 3y_2^{ab}$$

where $y_2^{an} := \left(z_2^{an}\right)^{-1}$ and $y_2^{ab} := \left(z_2^{ab}\right)^{-1}$. KCL at bus 2 of the positive-sequence network gives

$$\frac{\tilde{E}_{1,+}^Y - \tilde{V}_{2,+}}{z_1^{an} + \left(z_{12}^s - z_{12}^m\right)} \;=\; \tilde{Y}_2\tilde{V}_{2,+} + 3y_3^{ab}\tilde{V}_{3,+} + \tilde{J}_{3,+}^\Delta$$

Hence we have, after eliminating $\tilde{V}_{3,+}$

$$\frac{\tilde{E}_{1,+}^Y - \tilde{V}_{2,+}}{z_1^{an} + z_{12}^s - z_{12}^m} \;=\; \left(\tilde{Y}_2 + 3\tilde{\rho}_3 y_3^{ab}\right)\tilde{V}_{2,+} + \left(1 - 3\tilde{\rho}_3 y_3^{ab}\left(z_{23}^s - z_{23}^m\right)\right)\tilde{J}_{3,+}^\Delta \tag{9.87b}$$

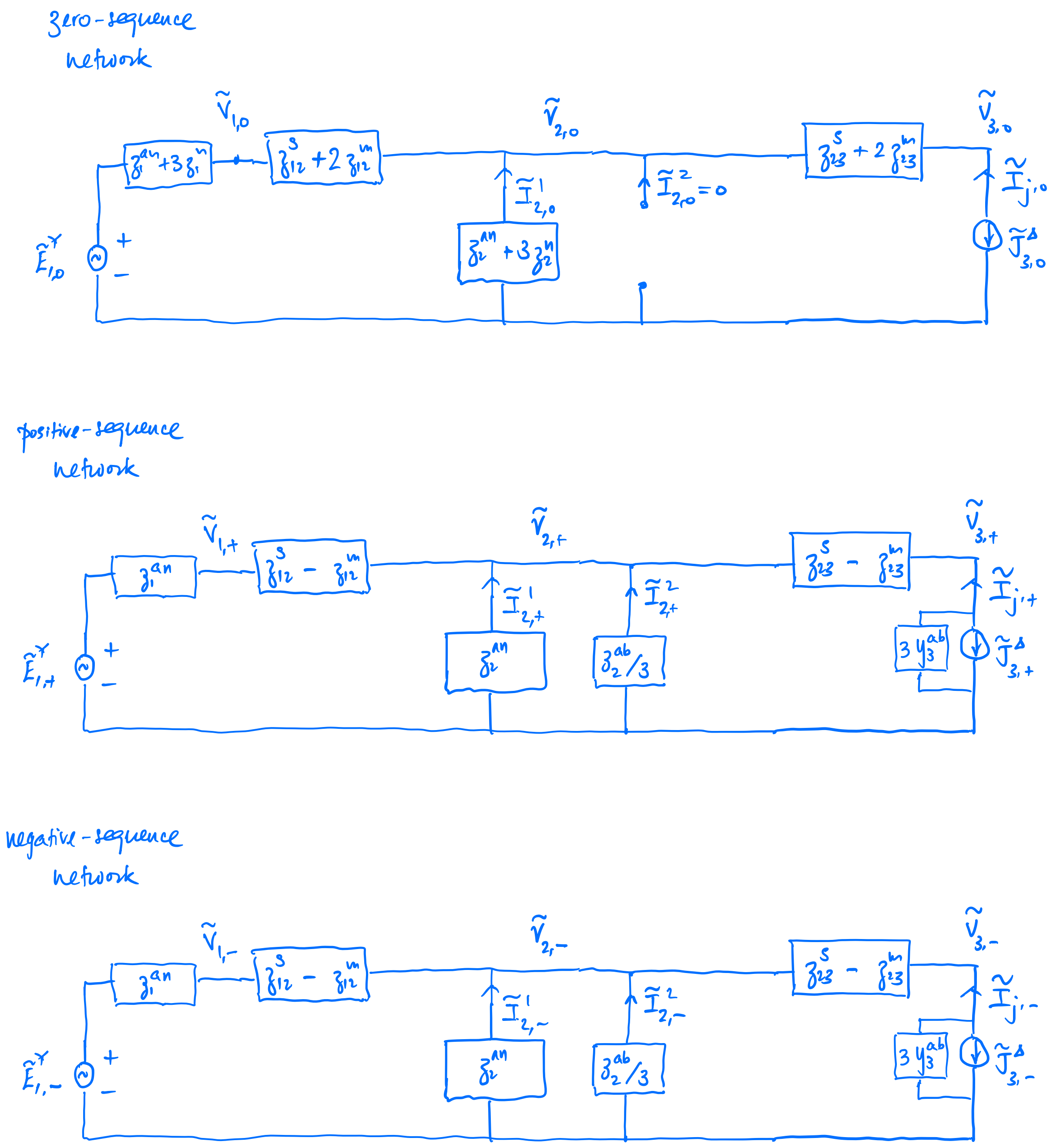


Figure 9.10: Example 9.13: Circuit models of sequence networks.

Similarly, from the negative-sequence network, we get

$$\frac{\tilde{E}^Y_{1,-} - \tilde{V}_{2,-}}{z_1^{an} + z_{12}^s - z_{12}^m} \;=\; \left(\tilde{Y}_2 + 3\tilde{\rho}_3 y_3^{ab}\right)\tilde{V}_{2,-} \;-\; \left(1 \;-\; 3\tilde{\rho}_3 y_3^{ab}\left(z_{23}^s - z_{23}^m\right)\right)\tilde{J}^{\Delta}_{3,-} \tag{9.87c}$$

The terminal sequence voltage $\tilde{V}_2 := \left(\tilde{V}_{2,0}, \tilde{V}_{2,+}, \tilde{V}_{2,-}\right)$ can be obtained from (9.87). From the 0/+/− sequence networks, the terminal sequence load currents are

$$\begin{aligned} \tilde{I}^1_{2,0} &= -\frac{\tilde{V}_{2,0}}{z_2^{an} + 3z_2^n}, & \tilde{I}^1_{2,+} &= -\frac{\tilde{V}_{2,+}}{z_2^{an}}, & \tilde{I}^1_{2,-} &= -\frac{\tilde{V}_{2,-}}{z_2^{an}} \\ \tilde{I}^2_{2,0} &= 0, & \tilde{I}^2_{2,+} &= -\frac{3\tilde{V}_{2,+}}{z_2^{ab}}, & \tilde{I}^2_{2,-} &= -\frac{3\tilde{V}_{2,-}}{z_2^{ab}} \end{aligned}$$

From the terminal sequence variables $\left(\tilde{V}_2, \tilde{I}^1_2, \tilde{I}^2_2\right)$ we can obtain the terminal phase variables

$$V_2 \;=\; F\tilde{V}_2, \qquad I^1_2 \;=\; F\tilde{I}^1_2, \qquad I^2_2 \;=\; F\tilde{I}^2_2$$

To obtain the internal currents I_2^Y and I_2^{Δ}, apply the conversion rules to get

$$I_2^Y \;=\; -I_2^1, \qquad I_2^{\Delta} \;=\; -\Gamma^{\mathsf{T}\dagger} I_2^2 \;+\; \beta_2 \mathbf{1} \;=\; -\frac{1}{3}\Gamma I_2^2 \;+\; \beta_2\mathbf{1}$$

for an arbitrary $\beta \in \mathbb{C}$, where I_2^{Δ} exists because $\tilde{I}^2_{2,0} = 0$ means $\mathbf{1}^{\mathsf{T}} I_2^2 = 0$.

Finally to calculate the internal powers s_2^Y and s_2^{Δ} we first obtain the internal voltages:

$$V_2^Y \;=\; V_2 \;-\; V_2^n \mathbf{1} \;=\; V_2 \;+\; z_2^n\left(\mathbf{1}\mathbf{1}^{\mathsf{T}}\right) I_2^1, \qquad V_2^{\Delta} \;=\; \Gamma V_2$$

where the second equality follows from $V_2^n = -z_2^n\left(\mathbf{1}^{\mathsf{T}} I_2^1\right)$ under C7.1. Hence

$$\begin{aligned} s_2^Y &:= \operatorname{diag}\left(V_2^Y I_2^{Y\mathsf{H}}\right) \;=\; -\operatorname{diag}\left(V_2 I_2^{1\mathsf{H}} \;+\; z_2^n\left(\mathbf{1}\mathbf{1}^{\mathsf{T}}\right) I_2^1 I_2^{1\mathsf{H}}\right) \\ s_2^{\Delta} &:= \operatorname{diag}\left(V_2^{\Delta} I_2^{\Delta\mathsf{H}}\right) \;=\; -\operatorname{diag}\left(\Gamma V_2 I_2^{2\mathsf{H}}\Gamma^{\dagger}\right) \;+\; \overline{\beta}_2 \Gamma V_2 \end{aligned}$$

The total internal powers are $\mathbf{1}^{\mathsf{T}} s_2^Y$ and $\mathbf{1}^{\mathsf{T}} s_2^{\Delta}$ which is independent of β_2. □

9.5 Bibliographical notes

Three-phase load flow solvers have been developed since at least the 1960s, e.g., see [65] for solution in the sequence coordinate and [37, 55] in the phase coordinate. A three-phase network is equivalent to a single-phase circuit where each node in the equivalent circuit is indexed by a (bus, phase) pair [55]. The main difference with a single-phase network is the models of three-phase devices in the equivalent circuit, such as models for generators and loads studied in Chapter 7, and lines and transformers studied in Chapter 8. Single-phase power flow algorithms such as Newton Raphson [66] or Fast Decoupled methods

[67] can be directly applied to the equivalent circuit. See also [50, Chapter 11] for recent algorithms for solving three-phase power flows. A sufficient condition is derived in [68] to ensure a fixed-point iteration of an AC power flow equation converges to a unique power flow solution. Sufficient conditions are also proved in [11] for the invertibility of three-phase admittance matrix which then ensures the validity of Z-bus method for computing power flow solutions. Finally recent studies on three-phase AC optimal power flow problems and their semidefinite relaxations include e.g. [69, 70, 71].

9.6 Problems

Chapter 9.1.

Exercise 9.1 (Symmetry and block symmetry)**.** Consider a $3n \times 3n$ matrix A partitioned as in Definition 9.1.

1. Suppose A is symmetric. Show that it is block symmetric if all its off-diagonal blocks are symmetric, i.e., $A_{jk}^{\mathsf{T}} = A_{jk}$, for all $j \neq k$.
2. Suppose A is block symmetric. Show that it is symmetric if all blocks A_{jk}, including the diagonal blocks, are symmetric.

Exercise 9.2 (Invertibility of Y)**.** Prove Theorem 9.2.

Exercise 9.3 (Invertibility of Y)**.** This exercise shows that the set of conditions in Theorem 9.1 and that in Theorem 9.2 each ensures $\alpha^{\mathsf{H}} Y \alpha \neq 0$ for any nonzero $\alpha \in \mathbb{C}^{3(N+1)}$. Suppose C9.2 is satisfied, i.e., $y_{jk}^s = y_{kj}^s$, y_{jk}^m and y_{kj}^m are complex symmetric, so that the admittance matrix Y is both symmetric and block symmetric. Consider $\alpha^{\mathsf{H}} Y \alpha$ for any $\alpha \in \mathbb{C}^{3(N+1)}$, and write $y_{jk}^s, y_{jj}^m := \sum_{k: j\sim k} y_{jk}^m$ and α_j in terms of their real and imaginary parts:

$$y_{jk}^s \;=:\; g_{jk}^s + \mathbf{i} b_{jk}^s \;\in\; \mathbb{C}^{3\times 3}, \qquad y_{jj}^m \;=:\; g_{jj}^m + \mathbf{i} b_{jj}^m \;\in\; \mathbb{C}^{3\times 3}, \qquad \alpha_j \;=:\; \rho_j + \mathbf{i}\varepsilon_j \;\in\; \mathbb{C}^3$$

1. Show that the real and imaginary parts of $\alpha^{\mathsf{H}} Y \alpha$ are:

$$\mathrm{Re}\left(\alpha^{\mathsf{H}} Y \alpha\right) = \sum_{(j,k)\in E} \left(\begin{bmatrix}\rho_j\\ \varepsilon_j\end{bmatrix} - \begin{bmatrix}\rho_k\\ \varepsilon_k\end{bmatrix}\right)^{\mathsf{T}} \begin{bmatrix} g_{jk}^s & 0\\ 0 & g_{jk}^s\end{bmatrix} \left(\begin{bmatrix}\rho_j\\ \varepsilon_j\end{bmatrix} - \begin{bmatrix}\rho_k\\ \varepsilon_k\end{bmatrix}\right) + \sum_{j\in\overline{N}} \begin{bmatrix}\rho_j^{\mathsf{T}} & \varepsilon_j^{\mathsf{T}}\end{bmatrix} \begin{bmatrix} g_{jj}^m & 0\\ 0 & g_{jj}^m\end{bmatrix} \begin{bmatrix}\rho_j\\ \varepsilon_j\end{bmatrix}$$

$$\mathrm{Im}\left(\alpha^{\mathsf{H}} Y \alpha\right) = \sum_{(j,k)\in E} \left(\begin{bmatrix}\rho_j\\ \varepsilon_j\end{bmatrix} - \begin{bmatrix}\rho_k\\ \varepsilon_k\end{bmatrix}\right)^{\mathsf{T}} \begin{bmatrix} b_{jk}^s & 0\\ 0 & b_{jk}^s\end{bmatrix} \left(\begin{bmatrix}\rho_j\\ \varepsilon_j\end{bmatrix} - \begin{bmatrix}\rho_k\\ \varepsilon_k\end{bmatrix}\right) + \sum_{j\in\overline{N}} \begin{bmatrix}\rho_j^{\mathsf{T}} & \varepsilon_j^{\mathsf{T}}\end{bmatrix} \begin{bmatrix} b_{jj}^m & 0\\ 0 & b_{jj}^m\end{bmatrix} \begin{bmatrix}\rho_j\\ \varepsilon_j\end{bmatrix}$$

2. Show that the conditions in Theorem 9.1 ensure $\alpha^{\mathsf{H}} Y \alpha \neq 0$ for any nonzero $\alpha \in \mathbb{C}^{3(N+1)}$.
3. Show that the conditions in Theorem 9.2 ensure $\alpha^{\mathsf{H}} Y \alpha \neq 0$ for any nonzero $\alpha \in \mathbb{C}^{3(N+1)}$.

Exercise 9.4 (Invertibility of Y_{22})**.** Prove Theorem 9.3.

Exercise 9.5 (Power flow equation)**.** Express the three-phase power injection $s_j \in \mathbb{C}^3$ in terms of the voltage vector $V \in \mathbb{C}^{3(N+1)}$:

$$s_j = \sum_{k:j\sim k} \operatorname{diag}\left(\left(e_j^{\mathsf{T}} \otimes \mathbb{I}\right) VV^{\mathsf{H}} \left((e_j - e_k) \otimes y_{jk}^{s\mathsf{H}}\right) + \left(e_j^{\mathsf{T}} \otimes \mathbb{I}\right) VV^{\mathsf{H}} \left(e_j \otimes y_{jk}^{m\mathsf{H}}\right)\right)$$

Chapter 9.2.

Exercise 9.6 (Four-wire model in Y-configured)**.** For Example 9.3 express the neutral voltages (γ_j, γ_k) in terms of the phase voltages and currents (V_j, V_k, I_j, I_k).

Exercise 9.7 (Four-wire model in Y-configured)**.** Repeat Example 9.5 but for the case where the neutrals n of the voltage source and the impedance are connected through impedances $\left(z_j^{n'n}, z_k^{n'n}\right)$ to their respective external neutral terminals n' which are then connected to the four-wire line. See Figure 9.6.

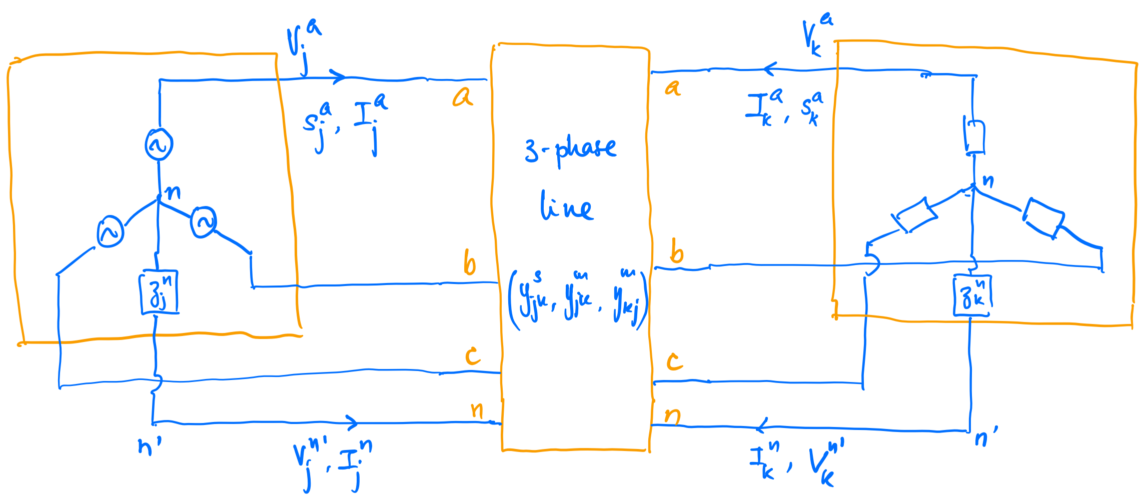


Figure 9.11: Exercise 9.7: A Y-configured generator connected through a four-wire line to a Y-configured impedance load.

Note that V_j^n is the voltage (with respect to a common reference point) at the neutral internal of the device, and $V_j^{n'}$ is the voltage at the terminal of the neutral line of the device, and that $\left(V_j^{n'}, V_k^{n'}\right)$ do not need to be given or grounded.

Exercise 9.8 (Current Source in Δ configuration)**.** Consider Example 9.6 but with an ideal current source instead of the ideal voltage source. Specifically suppose the following are specified:

- Current source $\left(J_j^{\Delta}, \gamma_j\right)$.

- Impedance z_k^Δ. (Note that β_k need not be specified but can be derived.)
- Line admittances $\left(\left(z_{jk}^s\right)^{-1}, y_{jk}^m = y_{kj}^m := 0\right)$. We have assumed for simplicity that shunt admittances are zero.

1. Compute all the other quantities in Table 9.2.
2. Show that if z_{jk}^s is symmetric of the form in (8.5) with $z_{jk}^1 + 2z_{jk}^2 \neq 0$, then $\gamma_k = \gamma_j$.
3. Show the following relation between the loop flows β_j and β_k:
   - $\beta_k = -\beta_j$ if and only if $z_k^{ab} J_j^{ab} + z_k^{bc} J_j^{bc} + z_k^{ca} J_j^{ca} = 0$.
   - $\beta_k = 0$ if and only if $z_k^{ab} J_j^{ab} + z_k^{bc} J_j^{bc} + z_k^{ca} J_j^{ca} = \zeta_k \beta_j$ where $\zeta_k := \mathbf{1}^\top \zeta_k \mathbf{1}$.
   - $\beta_k = 0$ if the impedance $z_k^\Delta = \frac{\zeta_k}{3}\mathbb{I}$ is balanced, regardless of whether J_j^Δ is balanced or whether β_j is zero. The converse does not necessarily hold.

Note that if the shunt admittances $\left(y_{jk}^m, y_{kj}^m\right)$ are nonzero, then γ_j need not be specified and can be derived; see Remark 9.8.

Exercise 9.9 (Y and Δ devices)**.** Consider a Y-configured current source connected to a Δ-configured impedance as shown in Figure 9.12. Suppose the following are specified:

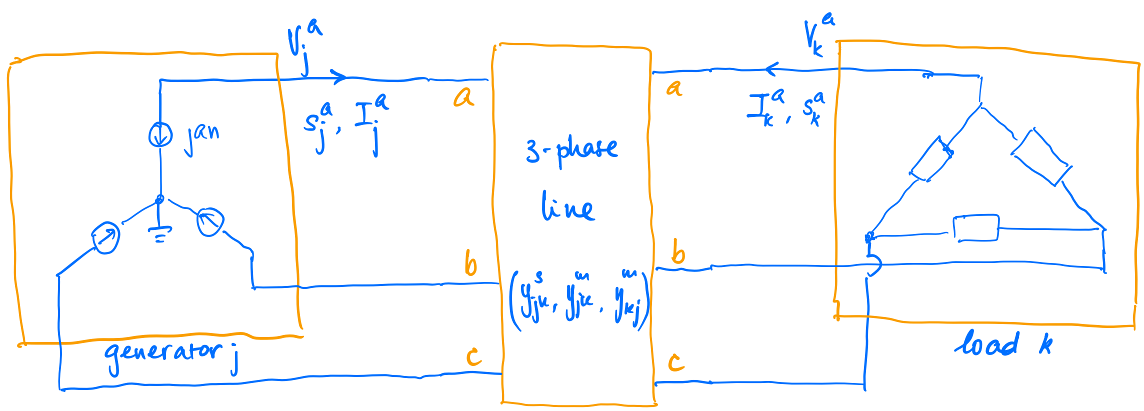


Figure 9.12: Three-phase Y-configured current source connected through a three-phase line to a Δ-configured impedance load.

- Current source J_j^Y.
- Impedance z_k^Δ.

- Line admittances $\left(\left(z^s_{jk}\right)^{-1}, y^m_{jk}, y^m_{kj}\right)$ with at least one of $\left(y^m_{jk}, y^m_{kj}\right)$ being nonzero.

Follow the solution strategy outlined in Chapter 9.2.3 to solve the network. State any invertibility assumptions in your derivation. An alternative approach is that used in Exercise 9.8.

Exercise 9.10 (Balanced power source)**.** Solve Example 9.9 when the system is balanced, i.e.,

- Power source $\left(\sigma^{\Delta}_j, \gamma_j\right)$ with $\sigma^{\Delta}_j = a_j\alpha_+ + b_j\mathbf{1}$ for given $\left(a_j, b_j\right)$. i.e., a balanced power source must be a generalized balanced vector. Moreover its voltage and current $\left(V^{\Delta}_j, I^{\Delta}_j\right)$ are generalized balanced vectors.
- Impedance $z^{\Delta}_k := \zeta^{\Delta}_k \mathbb{I}$.
- Line admittances $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right) := \left(\eta^s_{jk}\mathbb{I}, 0, 0\right)$.
- $\beta_j + \beta_k := \frac{1}{3}\mathbf{1}^\mathsf{T}\left(I^{\Delta}_j + I^{\Delta}_k\right) = \beta'$.

Use the external model (7.27b) of impedance.

Exercise 9.11 (Power sources)**.** Repeat Example 9.11 when the shunt admittances are zero, i.e., the three-phase line is specified as $\left(y^s_{jk}, y^m_{jk} = y^m_{kj} = 0\right)$ with nonsingular y^s_{jk}, as in Example 9.9. Since the admittance matrix is no longer invertible, suppose $\beta_j + \beta_k := \frac{1}{3}\mathbf{1}^\mathsf{T}\left(I^{\Delta}_j + I^{\Delta}_k\right) = \beta'$ is also given.

Exercise 9.12 (Balanced power sources)**.** Consider the system in Figure 9.7 where both the generator and load are power sources and the lines have zero shunt admittances, as in Example 9.9. Suppose the system is balanced and the following are specified:

- Power source $\left(\sigma^{\Delta}_j, \gamma_j\right)$ with $\sigma^{\Delta}_j = a_j\alpha_+ + b_j\mathbf{1}$ for given $\left(a_j, b_j\right)$, with its voltage and current $\left(V^{\Delta}_j, I^{\Delta}_j\right)$ being generalized balanced vectors.
- Power source $\sigma^{\Delta}_k = a_k\alpha_+ + b_k\mathbf{1}$ for given (a_k, b_k), with its voltage and current $\left(V^{\Delta}_j, I^{\Delta}_j\right)$ being generalized balanced vectors. Note that γ_k is not specified.
- Line admittances $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right) := \left(\eta^s_{jk}\mathbb{I}, 0, 0\right)$.
- Suppose a reference voltage $\angle V^a_j := \theta^a_j$ is given.

Show how to derive all variables $\left(V_i^{\Delta}, I_i^{\Delta}, \beta_i\right)$ and $\left(V_i, I_i, \gamma_j\right)$, $i = j,k$, *analytically*. In particular show that $\gamma_j = \gamma_k$.

Exercise 9.13 (Power sources)**.** Given a solution $\left(V_c, I_i^{\text{int}}, I_p^{\text{int}}, V_p^{\text{int}}\right)$ to the reduced system (9.47), derive all the unknown internal variables $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}, \beta_j\right)$ and external variables $\left(V_j, I_j, s_j, \gamma_j\right)$ over the network.

Chapter 9.3

Exercise 9.14 (Balanced network)**.** The two equivalent external models of an impedance z_j^{Δ} in Tables 7.3 and 7.4 are

$$\begin{aligned} V_j &= -Z^{\Delta} I_j + \gamma_j \mathbf{1}, & \mathbf{1}^{\mathsf{T}} I_j &= 0 \\ I_j &= -Y^{\Delta} V_j \end{aligned}$$

where the effective impedance and admittance matrices are $Z_j^{\Delta} := \frac{1}{9}\Gamma^{\mathsf{T}} z_j^{\Delta}\Gamma$ and and $Y_j^{\Delta} := \Gamma^{\mathsf{T}} y_j^{\Delta}\Gamma$. For balanced networks where the impedance $z_j^{\Delta} = \varepsilon_j^{-1}\mathbb{I}$, show that these models reduce to:

$$\begin{aligned} V_j &= -\frac{1}{3\varepsilon_j} I_j + \gamma_j \mathbf{1}, & \mathbf{1}^{\mathsf{T}} I_j &= 0 \\ I_j &= -3\varepsilon_j \left(V_j - \gamma_j \mathbf{1}\right) \end{aligned}$$

Exercise 9.15 (Balanced voltages & currents)**.** Consider the reduced system (9.42) of (9.48)(9.50). We have shown that any solution $\left(V_c, I_i^{\text{int}}\right)$ of (9.42) consists of generalized balanced vectors. Derive all other variables analytically in terms of the solution $\left(V_c, I_i^{\text{int}}\right)$ and show that they are generalized balanced positive-sequence sets.

Exercise 9.16 (Balanced network)**.** Suppose $(A \times \mathbb{I})V = b \otimes \alpha_+ + c \otimes \mathbf{1}$ where $A \in \mathbb{C}^{n\times n}$, $b, c \in \mathbb{C}^n$, $\mathbb{I}$ is the identity matrix of size 3 and $\mathbf{1}$ is the vector of all 1s of size 3. Let $\gamma_j := \frac{1}{3}\mathbf{1}^{\mathsf{T}} V_j$ be the zero-sequence component of $V_j \in \mathbb{C}^3$. Show that $A\gamma = c$.

Chapter 9.4.

Exercise 9.17. Prove that if a vector V of three-phase voltages is a balanced negative sequence then the negative-sequence voltage $\tilde{V}_- = \sqrt{3}V_a$ and the zero-sequence and the positive-sequence voltages are both zero, $\tilde{V}_0 = \tilde{V}_+ = 0$.

Exercise 9.18 (Sequence impedance $\tilde{Z}^Y$)**.** Consider the phase impedance matrix $Z^Y := z^Y + z^n \mathbf{1}\mathbf{1}^\mathsf{T}$ of a Y-configured impedance z^Y. Show that its sequence impedance matrix is

$$\tilde{Z}^Y \;=\; \frac{1}{3}\begin{bmatrix} \mathbf{1}^T z & \alpha_+^\mathsf{T} z & \alpha_-^\mathsf{T} z \\ \alpha_-^\mathsf{T} z & \mathbf{1}^T z & \alpha_+^\mathsf{T} z \\ \alpha_+^\mathsf{T} z & \alpha_-^\mathsf{T} z & \mathbf{1}^T z \end{bmatrix} + \begin{bmatrix} 3z^n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $z^{an} = z^{bn} = z^{cn}$ then

$$\tilde{Z}^Y \;=\; \begin{bmatrix} z^{an} + 3z^n & 0 & 0 \\ 0 & z^{an} & 0 \\ 0 & 0 & z^{an} \end{bmatrix}$$

Exercise 9.19 (Sequence impedance $\tilde{Z}^\Delta$)**.** Consider a Δ-configured impedance z^Δ whose external model is (from (9.75)):

$$V \;=\; -Z^\Delta I \,+\, \gamma\mathbf{1}, \qquad\qquad \mathbf{1}^\mathsf{T} I \;=\; 0 \tag{9.88}$$

where the zero-sequence voltage $\gamma := \frac{1}{3}\mathbf{1}^\mathsf{T} V$ is also a variable to be determined and

$$Z^\Delta \;:=\; \frac{1}{9}\Gamma^\mathsf{T} \underbrace{z^\Delta\left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\,\tilde{z}^{\Delta\mathsf{T}}\right)}_{\hat{z}^\Delta}\Gamma$$

Show that its sequence impedance matrix is

$$\tilde{Z}^\Delta \;:=\; \frac{1}{9}\left(F\Lambda\right)^\mathsf{H} \hat{z}^\Delta \left(F\Lambda\right)$$

where F is given in (**??**) and

$$\Lambda \;:=\; \begin{bmatrix} 0 & & \\ & 1-\alpha & \\ & & 1-\alpha^2 \end{bmatrix}$$

If $z^{ab} = z^{bc} = z^{ca}$ then

$$\tilde{Z}^\Delta \;=\; \frac{z^{ab}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the external model of the Δ-configured impedance in the sequence coordinate is:

$$\begin{bmatrix} 0 \\ \tilde{V}_+ \\ \tilde{V}_- \end{bmatrix} \;=\; -\frac{z^{ab}}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_+ \\ \tilde{I}_- \end{bmatrix}, \qquad\qquad \tilde{I}_0 \;=\; 0$$

Exercise 9.20 (Sequence network: Δ-configured voltage source)**.** One of the external models of a Δ-configured voltage source is (from (7.21b)):

$$V \;=\; \hat{\Gamma} E^{\Delta} \,-\, Z^{\Delta} I \,+\, \gamma \mathbf{1}, \qquad\qquad \mathbf{1}^{\mathsf{T}} I \;=\; 0$$

where

$$\hat{\Gamma} \;:=\; \frac{1}{3}\Gamma^{\mathsf{T}}\left(\mathbb{I} - \frac{1}{\zeta}\tilde{z}^{\Delta}\mathbf{1}^{\mathsf{T}}\right), \qquad\qquad Z^{\Delta} \;:=\; \frac{1}{9}\Gamma^{\mathsf{T}} z^{\Delta}\left(\mathbb{I} - \frac{1}{\zeta}\mathbf{1}\tilde{z}^{\Delta\mathsf{T}}\right)\Gamma$$

where $\tilde{z}^{\Delta} := \text{diag}\left(z^{\Delta}\right)\mathbf{1}$ and $\zeta := \mathbf{1}^{\mathsf{T}}\tilde{z}^{\Delta}$.

1. Show that

Exercise 9.21 (Sequence network: Δ-configured voltage source)**.** Repeat Exercise 9.20 starting with the alternative external models of a Δ-configured voltage source is (from (7.21a)).

Exercise 9.22 (Sequence network: Y-configured current source)**.** Suppose assumption C9.1 holds (all neutrals are grounded and voltages are defined with respect to the ground) so that $V^n = -z^n\left(\mathbf{1}^{\mathsf{T}} I\right)$. Derive the sequence networks for a Y-configured current source (as those in Chapter 9.4.3) starting from the external model in the phase domain (from (7.15b)):

$$V \;=\; -\left(z^Y J^Y \,+\, Z^Y I\right)$$

where $z^Y := \left(y^Y\right)^{-1}$ and $Z^Y := z^Y \,+\, z^n\,\mathbf{1}\mathbf{1}^{\mathsf{T}}$.

Exercise 9.23. Consider the complex symmetric matrix

$$M \;:=\; \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$$

Show that M is not diagonalizable by computing its Jordan form and that:

1. Its eigenvalue $\lambda = 0$ has algebraic multiplicity of 2 and geometric multiplicity of 1.
2. Its eigenvector is $v_1 = (-i, 1)$ and generalized eigenvector is $v_2 = (-2i, 1)$.

Exercise 9.24. Consider the complex symmetric phase impedance matrix

$$z \;:=\; \begin{bmatrix} s & m & m \\ m & s & m \\ m & m & s \end{bmatrix}$$

where $s, m \in \mathbb{C}$.

1. Check directly that $zz^{\mathsf{H}} = z^{\mathsf{H}}z$. Hence, even though z is symmetric but not Hermitian, it is normal.
2. Since z is normal, it is unitarily similar to a diagonal matrix $\tilde{z}$, i.e., there exists a unitary matrix F such that $\tilde{z} = F^{\mathsf{H}}zF$. Find F and $\tilde{z}$.

Exercise 9.25 (Unbalanced currents)**.** Consider a balanced load in (a) Y configuration, or (b) Δ configuration, with one of the loads open-circuited, as shown in Figure 9.13. Find the sequence currents $\tilde{I} := (\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ and the neutral current I_n (for Y configuration) when the terminal phase currents are

$$I = \begin{bmatrix} i_a \\ i_a e^{\mathrm{i}2\pi/3} \\ I_c \end{bmatrix}$$

Why is only the negative-sequence component nonzero even though the loads are unbalanced because of the open circuit?

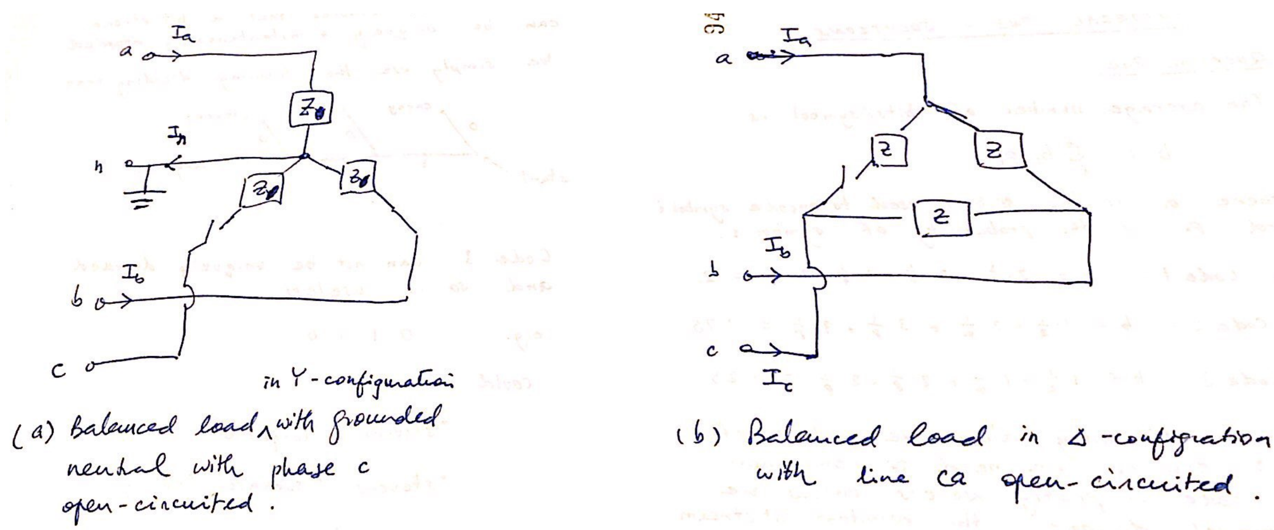


Figure 9.13: Sequence components of unbalanced phase currents.

Exercise 9.26. Repeat Example 9.13 without using symmetrical components and sequence networks.

Exercise 9.27. Repeat Example 9.13 but with the Y and Δ-impedances in series (instead of in parallel) connected by a line with the same series-phase impedance matrix z_{line}, as shown in Figure 9.14.

Exercise 9.28. Repeat Exercise 9.27 without using symmetrical components and sequence networks.

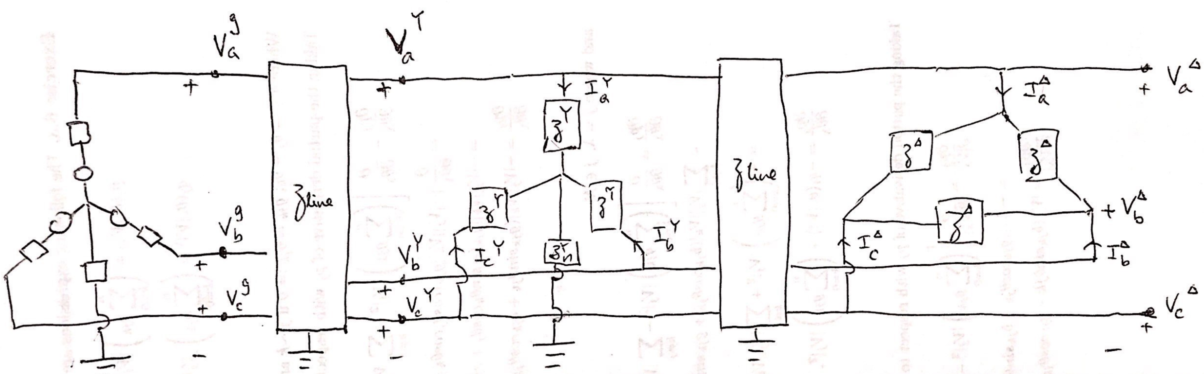


Figure 9.14: Exercise 9.27: A three-phase unbalanced voltage source supplies power two balance loads in series through symmetric lines.

Chapter 10

Branch flow models

In this chapter we extend the single-phase branch flow models of Chapter 5 to unbalanced three-phase networks. We will build on materials in Chapter 9 on unbalanced bus injection models.

10.1 General network

In this section we extend the branch flow model of Chapter 5.1 for single-phase networks to unbalanced three-phase networks, under the following condition throughout this chapter:

C10.1: For every line $(j,k) \in E$, the series impedance matrices $z^s_{jk} = z^s_{kj}$.

This means that the admittance admittance Y is block symmetric and has a three-phase Π circuit representation. If $(j,k) \in E$ models a three-phase transformer then it is in YY or $\Delta\Delta$ configuration.

We often assume the shunt admittances $y^m_{jk} = y^m_{kj} = 0$ as well in which the admittance matrix Y has zero block row sums. This, together with C10.1, allow us to adopt a directed graph for network model.

10.1.1 Three-phase model

Review: single-phase model. Consider the branch flow model (5.1) of Chapter 5.1 for general networks. We assume C5.1 ($z^s_{jk} = z^s_{kj}$ for every line $(j,k) \in E$) and zero shunt admittances $y^m_{jk} = y^m_{kj} = 0$. We therefore often omit the superscript s and write series impedances and admittances as $z_{jk} =: \left(y_{jk}\right)^{-1}$. Then the branch variables in one direction of a line can be easily expressed in terms of those in the other direction:

$$I_{kj} \;=\; -I_{jk}, \qquad\qquad S_{kj} \;=\; -\left(S_{jk} - z_{jk}\left|I_{jk}\right|^2\right)$$

This allows us to adopt a (connected) directed graph $G = (\overline{N}, E)$ with an arbitrary but fixed orientation and reduce the number of variables by defining branch variables only in the direction of the lines. Substituting

$y_{jk}^m = y_{kj}^m = 0$ into (5.1) then simplifies the model to (Exercise 10.1):

$$\begin{aligned}
\sum_{k:j\rightarrow k} S_{jk} &= \sum_{i:i\rightarrow j} \left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j, \quad j \in \overline{N} && \text{(10.1a)}\\
V_j - V_k &= z_{jk} I_{jk}, \quad j \rightarrow k \in E && \text{(10.1b)}\\
S_{jk} &= V_j I_{jk}^{\mathsf{H}}, \quad j \rightarrow k \in E && \text{(10.1c)}\\
\ell_{jk} &= |I_{jk}|^2, \quad j \rightarrow k \in E && \text{(10.1d)}
\end{aligned}$$

Here (10.1a) imposes power balance at each bus, (10.1b) is Ohm's law, and (10.1c) defines branch power in terms of the associated voltage and current. The quantity ℓ_{jk} in (10.1d) is the squared magnitude of the branch current and $z_{ij}\ell_{jk}$ represents line loss so that $S_{ij} - z_{ij}\ell_{ij}$ is the receiving-end complex power at bus j from bus i. For convenience we include V_0 in the vector variable $V := (V_j, j \in \overline{N})$ with the understanding that $V_0 := 1\angle 0^\circ$ is fixed. The BFM (10.1) is applicable to both radial networks and meshed networks.

Three-phase BFM. We now generalize the single-phase BFM (10.1) for general networks to unbalanced three-phase networks under assumption C10.1. We also assume zero shunt admittance matrices $y_{jk}^m = y_{kj}^m = 0$ for all lines $j \rightarrow k \in E$. We therefore often omit the superscript s and write series impedance and admittance matrices as $z_{jk} =: \left(y_{jk}\right)^{-1}$. For notational simplicity we assume that all buses and all lines have three phases a, b, c; the generalization to the case where a bus/line has one, two, or three phases is straightforward.

As explained in Chapter 8.1 a three-phase line is characterized by its series impedance matrix:

$$z_{jk} := \left(y_{jk}^s\right)^{-1} := \begin{bmatrix} z_{jk}^{aa} & z_{jk}^{ab} & z_{jk}^{ac} \\ z_{jk}^{ba} & z_{jk}^{bb} & z_{jk}^{bc} \\ z_{jk}^{ca} & z_{jk}^{cb} & z_{jk}^{cc} \end{bmatrix} \in \mathbb{C}^{3\times 3}$$

For each line $j \rightarrow k$ let $I_{jk} \in \mathbb{C}^3$ be the 3-phase complex line current. For each bus $j \in \overline{N}$ let $V_j \in \mathbb{C}^3$ be the 3-phase complex voltage and $s_j \in \mathbb{C}^3$ be the 3-phase net complex power injection at bus j. A key to generalizing single-phase BFM to the 3-phase setting is the generalization of the quadratic relation in (10.1c) and (10.1d) between $\left(S_{jk}, \ell_{jk}\right)$ and $\left(V_j, I_{jk}\right)$. They are generalized in [70] to the 3-phase setting using outer product:

$$S_{jk} = V_j I_{jk}^{\mathsf{H}}, \qquad \ell_{jk} = I_{jk} I_{jk}^{\mathsf{H}}$$

i.e., S_{jk} and ℓ_{jk} are 3×3 rank-1 matrices. The diagonal terms of S_{jk} are the 3-phase sending-end power on line $(j,k) \in E$ and those of ℓ_{jk} are the squared magnitudes of the 3-phase branch currents. The BFM for unbalanced multiphase networks is then the following set of equations:

$$\begin{aligned}
\sum_{k:j\rightarrow k} \mathrm{diag}(S_{jk}) &= \sum_{i:i\rightarrow j} \mathrm{diag}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j, \quad j \in \overline{N} && \text{(10.2a)}\\
V_j - V_k &= z_{jk} I_{jk}, \quad j \rightarrow k \in E && \text{(10.2b)}\\
S_{jk} &= V_j I_{jk}^{\mathsf{H}}, \quad j \rightarrow k \in E && \text{(10.2c)}\\
\ell_{jk} &= I_{jk} I_{jk}^{\mathsf{H}}, \quad j \rightarrow k \in E && \text{(10.2d)}
\end{aligned}$$

with a given $V_0 \in \mathbb{C}^3$. This model generalizes directly the single-phase model (10.1) where the variables I_{jk}, V_j, s_j are now vectors and S_{jk}, ℓ_{jk} are now 3×3 matrices instead of scalars. The power balance equation (10.2a) constrains only the diagonal terms of S_{jk} and ℓ_{jk}. Their off-diagonal terms are determined by (10.2c)(10.2d). For convenience we assume here the vector V_0, not just V_0^ϕ, $\phi \in \{a,b,c\}$, is given (see angle recovery in Chapter 10.2.1 and a backward forward sweep method in Chapter 10.4.2).

10.1.2 Equivalence

Therefore the bus injection model and the branch flow model differ only in their power flow equations (9.12) and (10.2) respectively. We now show that these models for unbalanced multiphase networks are equivalent in the following sense. Define the solution sets:

$$\begin{aligned} \mathbb{V} &:= \mathbb{V}(V_0) := \left\{ (s,V) \in \mathbb{C}^{6(N+1)} \mid (s,V) \text{ satisfies (9.12) with a given } V_0 \right\} \\ \tilde{\mathbb{X}} &:= \tilde{\mathbb{X}}(V_0) := \left\{ \tilde{x} := (s,V,I,\ell,S) \in \mathbb{C}^{6(N+1)+21M} \mid \tilde{x} \text{ satisfies (10.2) with a given } V_0 \right\} \end{aligned}$$

where $N+1$ is the number of nodes and $M := |E|$ is the number of lines in G.[1] We say that two sets A and B are *equivalent*, denoted by $A \equiv B$, if there is a bijection between them. The following theorem generalizes Theorem 5.2.1 of Chapter 5.3 from single-phase to multiphase networks, assuming the admittance matrix $y_{jk} := (z_{jk})^{-1}$ exists (assumption C8.1).

Theorem 10.1. Suppose assumptions C10.1 and C8.1 hold. Then $\mathbb{V} \equiv \tilde{\mathbb{X}}$.

Proof. Fix any $(s,V) \in \mathbb{V}$. We will construct an $\tilde{x} := (s,V,I,\ell,S) \in \tilde{\mathbb{X}}$. For each line $j \to k \in E$ define $(I_{jk}\ell_{jk}, S_{jk})$ in terms of V by

$$I_{jk} := y_{jk}\left(V_j - V_k\right) \tag{10.3a}$$

$$\ell_{jk} := I_{jk} I_{jk}^{\mathsf{H}} = y_{jk}\left(V_j - V_k\right)\left(V_j - V_k\right)^{\mathsf{H}} y_{jk}^{\mathsf{H}} \tag{10.3b}$$

$$S_{jk} := V_j I_{jk}^{\mathsf{H}} = V_j\left(V_j - V_k\right)^{\mathsf{H}} y_{jk}^{\mathsf{H}} \tag{10.3c}$$

By construction, $\tilde{x}$ satisfies (10.2b)(10.2c) (10.2d). We now show that since (s,V) satisfies (9.12b), $\tilde{x}$ also satisfies (10.2a). We have from (9.12b)

$$s_j = \sum_{k:j\to k} \mathrm{diag}\left(V_j (V_j - V_k)^{\mathsf{H}} \left(y_{jk}^s\right)^{\mathsf{H}} \right) + \sum_{i:i\to j} \mathrm{diag}\left(V_j (V_j - V_i)^{\mathsf{H}} \left(y_{ji}^s\right)^{\mathsf{H}} \right)$$

The core of the first term on the right-hand side equals S_{jk} because of (10.3c), noting $y_{jk} := y_{jk}^s$. We claim that the core of the second term equals $-\left(S_{ij} - z_{ij}\ell_{ij}\right)$ and therefore (10.2a) is satisfied. To see this, use (10.3b) (10.3b) to get

$$-\left(S_{ij} - z_{ij}\ell_{ij}\right) = -V_i\left(V_i - V_j\right)^{\mathsf{H}} y_{ij}^{\mathsf{H}} + z_{ij}\left(y_{ij}\left(V_i - V_j\right)\left(V_i - V_j\right)^{\mathsf{H}} y_{ij}^{\mathsf{H}}\right) = V_j (V_j - V_i)^{\mathsf{H}} \left(y_{ji}^s\right)^{\mathsf{H}}$$

[1] The assumption that V_0 fixed and equal in both $\mathbb{V}$ and $\tilde{\mathbb{X}}$ is not necessary for Theorem 10.1. This condition is used in the proof of Theorem 10.2 on the equivalence of BFM for radial networks and $\tilde{\mathbb{X}}$, and hence is added here as well. Another reason is that a given reference angle, say, $\angle V_0^a = a_0$, is needed for most three-phase analysis problems whose specifications determine a power flow solution only up to an arbitrary reference angle; see Examples 9.8 and 9.11 for BIM and Examples 10.1 and 10.2 for BFM.

This shows that $\tilde{x}$ satisfies (10.2a).

Conversely, if $\tilde{x} := (s, V, I, \ell, S)$ satisfies (10.2a) then the argument above also shows that its component (s, V) satisfies (9.12b), and hence $(s, V) \in \mathbb{V}$. $\square$

10.2 Radial network

10.2.1 Three-phase model

Review: single-phase model. Assume C5.1 ($z^s_{jk} = y^s_{kj}$ for every line $(j,k) \in E$) and zero shunt admittances. When the network is radial we adopt, without loss of generality, the graph orientation in which all lines point away from bus 0. Then the power flow equations are (5.6) in Chapter 5.2, reproduced here:

$$\sum_{k:j\to k} S_{jk} = \left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j, \qquad j \in \overline{N} \tag{10.4a}$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z_{jk}S^{\mathsf{H}}_{jk}\right) - |z_{jk}|^2\ell_{jk}, \qquad j \to k \in E \tag{10.4b}$$

$$v_j\ell_{jk} = |S_{jk}|^2, \qquad j \to k \in E \tag{10.4c}$$

where $V_0 \in \mathbb{C}$ is given. In (10.4a), $i := i(j)$ denotes the unique bus between bus 0 and bus j so that $i \to j \in E$ is a line. Here we omit the superscript in z^s_{jk}.

Three-phase BFM. We now specialize to radial networks and generalize the single-phase model (10.4) to multiphase networks, under assumptions C10.1 and zero shunt admittance matrices $y^m_{jk} = y^m_{kj} = 0 \in \mathbb{C}^{3\times 3}$. Without loss of generality we adopt the graph orientation where all lines point away from the root bus 0. The variables for an unbalanced three-phase network are:

$$s_j \in \mathbb{C}^3, \qquad v_j \in \mathbb{S}^3_+, \qquad j \in \overline{N}$$
$$\ell_{jk} \in \mathbb{S}^3_+, \qquad S_{jk} \in \mathbb{C}^{3\times 3}, \qquad j \to k \in E$$

where $\mathbb{S}^n_+ \subseteq \mathbb{C}^{n\times n}$ is the set of of $n \times n$ complex (Hermitian and) positive semidefinte matrices. Let $s := (s_j, j \in \overline{N}), v := (v_j, j \in \overline{N})$, $\ell := (\ell_{jk}, (j,k) \in E)$, $S := (S_{jk}, (j,k) \in E)$, and let $x := (s, v, \ell, S)$. The following equations are proposed in [70] to generalize the DistFlow model from the single-phase to the three-phase setting:

$$\sum_{k:j\to k} \mathrm{diag}(S_{jk}) = \mathrm{diag}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j, \qquad j \in \overline{N} \tag{10.5a}$$

$$v_j - v_k = \left(z_{jk}S^{\mathsf{H}}_{jk} + S_{jk}z^{\mathsf{H}}_{jk}\right) - z_{jk}\ell_{jk}z^{\mathsf{H}}_{jk}, \qquad j \to k \in E \tag{10.5b}$$

$$\begin{bmatrix} v_j & S_{jk} \\ S^{\mathsf{H}}_{jk} & \ell_{jk} \end{bmatrix} \succeq 0, \qquad j \to k \in E \tag{10.5c}$$

$$\mathrm{rank}\begin{bmatrix} v_j & S_{jk} \\ S^{\mathsf{H}}_{jk} & \ell_{jk} \end{bmatrix} = 1, \qquad j \to k \in E \tag{10.5d}$$

where $V_0 \in \mathbb{C}^3$ is given and bus $i := i(j)$ is the unique parent of bus j in (10.5a). Here, and below, we omit the superscript in z^s_{jk} when shunt admittances $z^m_{jk} = z^m_{kj} = 0$. Even though V_0^ϕ for any $\phi \in \{a,b,c\}$ is sufficient to ensure unique voltage and current angles from a solution x of (10.5), as we explain below, we assume for convenience that the vector V_0 is given because V_0 enables Algorithm 2 below that explicitly constructs a $\tilde{x} := (s,V,I,\ell,S) \in \tilde{\mathbb{X}}$ from $x := (s,v,\ell,S)$. A given V_0 also enables a backward forward sweep method in Chapter 10.4.2. Note however that fixing V_0 may not guarantee the uniqueness of power flow solutions x since (10.5) is nonlinear.

Angle recovery. We now explain how to recover the phase angles for voltage and current phasors (V,I) for a radial network with zero shunt admittance matrices $y^m_{jk} = y^m_{kj} = 0$ and under assumption C5.1, i.e., given a power solution $x = (s,v,\ell,S)$ that satisfies (10.5) we will construct the phasors (V,I).

The BFM (10.5) does not contain the vectors V_j or I_{jk}, but the psd rank-1 constraints (10.5c)(10.5d) ensure that there exist V_j and I_{jk} such that

$$v_j \;=\; V_jV_j^{\mathsf{H}}, \qquad\qquad \ell_{jk} \;=\; I_{jk}I_{jk}^{\mathsf{H}}, \qquad\qquad S_{jk} \;=\; V_jI_{jk}^{\mathsf{H}} \tag{10.6a}$$

or equivalently

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \;=\; \begin{bmatrix} V_j \\ I_{jk} \end{bmatrix} \cdot \begin{bmatrix} V_j^{\mathsf{H}} & I_{jk}^{\mathsf{H}} \end{bmatrix}, \qquad j \to k \in E \tag{10.6b}$$

Given matrices $\left(v_j, \ell_{jk}, S_{jk}\right)$, the vectors $\left(V_j, I_{jk}\right)$ are determined uniquely up to a reference angle. If a reference angle is given, e.g., $\angle V_0^a = 0^\circ$, the power flow equation (10.5) will fix the angles of all variables. See Example 10.1 in Chapter 10.3.

If V_0 is given, not just V_0^ϕ, $\phi \in \{a,b,c\}$, then given a power solution $x := (s,v,\ell,S)$ that satisfies (10.5), an $\tilde{x} := (s,V,I,\ell,S) \in \tilde{\mathbb{X}}$ can be explicitly constructed using the iterative Algorithm 2 from [70] that makes use of the tree topology. The basic idea in Step 5 of the algorithm is to compute the phasors V_i and I_{ij} recursively, starting from bus 0 when V_0 is given: since $S_{ij} = V_iI_{ij}^{\mathsf{H}}$, taking the Hermitian transpose and multiplying both sides by V_i, we have

$$V_iI_{ij}^{\mathsf{H}} \;=\; S_{ij} \quad \Rightarrow \quad I_{ij}\left(V_i^{\mathsf{H}}V_i\right) \;=\; S_{ij}^{\mathsf{H}}V_i \quad \Rightarrow \quad I_{ij} \;=\; \frac{1}{\mathrm{tr}(v_i)}\, S_{ij}^{\mathsf{H}}V_i \tag{10.7}$$

Tree topology and cycle condition. An x satisfying (10.5) is a legitimate power flow solution, i.e., from which a unique (up to an arbitrary reference angle) phasor (V,I) can be constructed as described above, only if the network is radial. To see this, substituting $I_{jk} = y_{jk}\left(V_j - V_k\right)$ into $S_{jk} = V_jI_{jk}^{\mathsf{H}}$ we get

$$V_jV_k^{\mathsf{H}} \;=\; v_j - S_{jk}z_{jk}^{\mathsf{H}}, \qquad j \to k \in E$$

Algorithm 2: Recover $\tilde{x} = (s, V, I, \ell, S)$ from $x = (s, v, \ell, S)$.
Down orientation where all lines point away from root bus 0.

Input: $x = (s, v, \ell, S) \in \mathbb{X}$; $V_0 \in \mathbb{C}^3$.
Output: $\tilde{x} = (\tilde{s}, \tilde{V}, \tilde{I}, \tilde{\ell}, \tilde{S}) \in \tilde{\mathbb{X}}$

1: $\tilde{s} \leftarrow s$; $\tilde{\ell} \leftarrow \ell$; $\tilde{S} \leftarrow S$;
2: $N_{\text{visit}} \leftarrow \{0\}$;
3: **while** $N_{\text{visit}} \neq \overline{N}$ **do**
4: find $i \to j$ such that $i \in N_{\text{visit}}$ and $j \notin N_{\text{visit}}$;
5: compute

$$\begin{aligned} \tilde{I}_{ij} &\leftarrow \frac{1}{\text{tr}\,(v_i)} S_{ij}^{\mathsf{H}} \tilde{V}_i \\ \tilde{V}_j &\leftarrow \tilde{V}_i - z_{ij} \tilde{I}_{ij} \\ N_{\text{visit}} &\leftarrow N_{\text{visit}} \cup \{j\} \end{aligned}$$

6: **end while**

Taking the diagonal vectors on both sides, we conclude that given a solution x of (10.5), voltage phasors V_j exist if and only if there exist $\theta_j := \left(\theta_j^a, \theta_j^b, \theta_j^c\right)$, for all $j \in \overline{N}$, such that

$$\begin{bmatrix} |V_j^a V_k^a|\, e^{\mathbf{i}\left(\theta_j^a - \theta_k^a\right)} \\ |V_j^b V_k^b|\, e^{\mathbf{i}\left(\theta_j^b - \theta_k^b\right)} \\ |V_j^c V_k^c|\, e^{\mathbf{i}\left(\theta_j^c - \theta_k^c\right)} \end{bmatrix} = \begin{bmatrix} |U_{jk}^a|\, e^{\mathbf{i}\beta_{jk}^a} \\ |U_{jk}^b|\, e^{\mathbf{i}\beta_{jk}^b} \\ |U_{jk}^c|\, e^{\mathbf{i}\beta_{jk}^c} \end{bmatrix}, \qquad j \to k \in E$$

where the vectors $\beta_{jk} := \beta_{jk}(x) \in \mathbb{R}^3$ of angles depend on x and are defined by $\beta_{jk}(x) := \angle\text{diag}\left(v_j - S_{jk} z_{jk}^{\mathsf{H}}\right)$. In particular there must exist $\theta := \left(\theta_j \in \mathbb{R}^3,\ j \in \overline{N}\right) \in \mathbb{R}^{3(N+1)}$ such that

$$\beta(x) = \left(C^{\mathsf{T}} \otimes \mathbb{I}\right) \theta \tag{10.8a}$$

where $\beta(x) := \left(\beta_{jk}(x), j \to k \in E\right) \in \mathbb{C}^{3M}$ and C is the $(N+1) \times M$ bus-by-line incidence matrix whose rank is N. See Chapter 25.2 for more properties of C. The condition (10.8a) is the cycle condition that generalizes (5.11a) from single-phase to three-phase networks. We now show that the cycle condition is vacuous for radial networks, i.e., any x satisfying (10.5) also satisfies (10.8a) when the network is radial.

Partition C into its first row c_0^{T} and an $N \times M$ matrix $\hat{C}$ of the remaining rows so that

$$C^{\mathsf{T}} =: \begin{bmatrix} c_0 & \hat{C}^{\mathsf{T}} \end{bmatrix}$$

Similarly partition $\theta =: \left(\theta_0, \hat{\theta}\right) \in \mathbb{R}^{3(N+1)}$. Suppose G is a (connected) tree with $M = N$. Then $\hat{C}^{\mathsf{T}}$ is $N \times N$ and of full rank. Therefore $c_0 = \hat{C}^{\mathsf{T}} \eta$ for some $\eta \in \mathbb{C}^N$. It is proved in Exercise 10.2 that $\left(\hat{C}^{\mathsf{T}} \eta\right) \otimes \mathbb{I} = \left(\hat{C}^{\mathsf{T}} \otimes \mathbb{I}\right)(\eta \otimes \mathbb{I})$. Hence (10.8a) becomes

$$\beta(x) = \left(c_0^{\mathsf{T}} \otimes \mathbb{I}\right) \theta_0 + \left(\hat{C}^{\mathsf{T}} \otimes \mathbb{I}\right) \hat{\theta} = \left(\hat{C}^{\mathsf{T}} \otimes \mathbb{I}\right) \left(\hat{\theta} + (\eta \otimes \theta_0)\right) \tag{10.8b}$$

where we have used $(\eta \otimes \mathbb{I})\,\theta_0 = \eta \otimes \theta_0$. Since $\hat{C}^{\mathsf{T}}$ and hence $\left(\hat{C}^{\mathsf{T}} \otimes \mathbb{I}\right)$ are invertible, for any x satisfying (10.5), there always exists an $\theta = \left(\theta_0, \hat{\theta}\right) \in \mathbb{R}^{3(N+1)}$ that satisfies (10.8b). Indeed the solution θ of (10.8b) is not unique. Given any $\theta_0 \in \mathbb{C}^3$, there is always a (unique) $\hat{\theta} := \left(\left(\hat{C}^{\mathsf{T}}\right)^{-1} \otimes \mathbb{I}\right)\beta(x) - \eta \otimes \theta_0$ that satisfies (10.8b).[2]

If G contains cycles, on the other hand, then $M > N$ and the $3M \times 3(N+1)$ matrix $\left(C^{\mathsf{T}} \otimes \mathbb{I}\right)$ in (10.8a) has a column rank of $3N < 3M$ since rank $(A \otimes B) =$ rank $A \cdot$ rank B from Lemma 9.6. This means that the column space of $\left(C^{\mathsf{T}} \otimes \mathbb{I}\right)$ does not span $\mathbb{R}^{3M}$ and hence there may be $\beta(x)$ for which no θ exists that satisfies (10.8a), regardless of whether θ_0 is given. A power flow model for a meshed network consists of (10.5) augmented with the cycle condition (10.8a).

10.2.2 Equivalence

The two unbalanced three-phase BFM models, (10.2) and (10.5), are defined by different sets of variables that satisfy different sets of equations. In the first model, the variables are $\tilde{x} := (s, V, I, \ell, S)$ where s_j, V_j, I_{jk} are vectors in $\mathbb{C}^3$, and ℓ_{jk}, S_{jk} are matrices in $\mathbb{C}^{3\times 3}$. They satisfy (10.2). In the second model, the variables are $x := (s, v, \ell, S)$ where s_j are vectors in $\mathbb{C}^3$ and v_j, ℓ_{jk}, S_{jk} are matrices in $\mathbb{C}^{3\times 3}$. They satisfy a different set of equations (10.5). Nonetheless, for radial networks, they are equivalent in the following sense. Let

$$\mathbb{X} \;:=\; \mathbb{X}(V_0) \;:=\; \left\{\, x := (s, v, \ell, S) \in \mathbb{C}^{12(N+1)+18M} \mid x \text{ satisfies (10.5) with a given } V_0 \,\right\}$$

See footnote 1 for the need for a reference angle, say, $\angle V_0^a = a_0$, that underlies the condition of a fixed and given V_0. The following theorem generalizes Theorem 5.2 of Chapter 5.3 from single-phase to multiphase radial networks.

Theorem 10.2. Suppose assumptions C10.1 and C8.1 hold. If G is a (connected) tree then $\tilde{\mathbb{X}} \equiv \mathbb{X}$.

Proof. We explicitly construct a bijection between these two sets. Fix any $\tilde{x} := (s, V, I, \ell, S)$ with the given V_0 that satisfies (10.2). The mapping $\tilde{x} \mapsto x$ is defined by $x := (s, v, \ell, S)$ where v is

$$v_j \;:=\; V_j V_j^{\mathsf{H}} \tag{10.9}$$

We first show that x satisfies (10.5). Then we show that the mapping $\tilde{x} \mapsto x$ defined by (10.9) is both injective and surjective, and it is therefore a bijection.

First x clearly satisfies (10.5a). To prove (10.5b), rewrite (10.2b) as

$$V_k \;=\; V_j \;-\; z_{jk} I_{jk}$$

Multiply both sides on the right by its Hermitian transpose to get

$$v_k \;=\; v_j \;+\; z_{jk}\,\ell_{jk}\,z_{jk}^{\mathsf{H}} \;-\; V_j\,I_{jk}^{\mathsf{H}}\,z_{jk}^{\mathsf{H}} \;-\; z_{jk}\,I_{jk}\,V_j^{\mathsf{H}}$$

[2] Here the vector θ_0 can be arbitrary to satisfy (10.8b) whereas a single angle e.g. θ_0^a fixes all other angles in (10.6). This is because (10.6) uses the matrix v_j whereas (10.8) uses only the diagonal entries of v_j.

where we have identified the psd rank-1 matrices $v_j = V_j V_j^{\mathsf{H}}$ and $\ell_{jk} = I_{jk} I_{jk}^{\mathsf{H}}$. Substituting $S_{jk} = V_j I_{jk}^{\mathsf{H}}$ from (10.2c), we have

$$v_k \;\;=\;\; v_j \,+\, z_{jk}\,\ell_{jk}\,z_{jk}^{\mathsf{H}} \,-\, \left(S_{jk}\,z_{jk}^{\mathsf{H}} \,+\, z_{jk}\,S_{jk}^{\mathsf{H}}\right)$$

which is (10.5b). To prove (10.5c)–(10.5d), note that (10.9) and (10.2c)(10.2d) imply

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \;\;=\;\; \begin{bmatrix} V_j \\ I_{jk} \end{bmatrix} \cdot \begin{bmatrix} V_j^{\mathsf{H}} & I_{jk}^{\mathsf{H}} \end{bmatrix}, \qquad j \to k \in E \tag{10.10}$$

i.e., this matrix is psd and rank-1. We hence have constructed a mapping $\tilde{x} \mapsto x$ through (10.9) that maps any $\tilde{x}$ that satisfies (10.2) to an x that satisfies (10.5).

We next show that, when V_0 is fixed, the mapping $\tilde{x} \mapsto x$ is injective. For the sake of contradiction, suppose both $\tilde{x} := (\tilde{s}, \tilde{V}, \tilde{I}, \tilde{\ell}, \tilde{S})$ and $\hat{x} = (\hat{s}, \hat{V}, \hat{I}, \hat{\ell}, \hat{S})$, with $\tilde{V}_0 = \hat{V}_0$, are mapped to $x = (s, v, \ell, S)$ through (10.9). By definition of $\tilde{x} \mapsto x$ we have $\tilde{s} = s = \hat{s}$, $\tilde{\ell} = \ell = \hat{\ell}$, $\tilde{S} = S = \hat{S}$. Moreover $\tilde{V}_j \tilde{V}_j^{\mathsf{H}} = v_j = \hat{V}_j \hat{V}_j^{\mathsf{H}}$ for all $j \in \overline{N}$. We have to show that $\tilde{V} = \hat{V}$ and $\tilde{I} = \hat{I}$. Since the rank-1 decomposition (10.10) is unique up to an arbitrary phase, $\left(\tilde{V}_j, \tilde{I}_{jk}\right)$ and $\left(\hat{V}_j, \hat{I}_{jk}\right)$ can differ only by an arbitrary phase shift φ_{jk}. We argue that φ_{jk} must be the same for all lines $j \to k \in E$ as long as the network is connected. It is convenient to assume, without loss of generality, that all lines point *towards* bus 0 (only) in this proof. Start from a leaf node i and consider a line $i \to j \in E$. Let

$$\hat{V}_i \;=\; \tilde{V}_i e^{\mathbf{i}\varphi_{ij}}, \qquad\qquad \hat{I}_{ij} \;=\; \tilde{I}_{ij} e^{\mathbf{i}\varphi_{ij}} \tag{10.11}$$

Similarly, for all lines $j \to k$ connected to j, we have $\hat{V}_j \;=\; \tilde{V}_j e^{\mathbf{i}\varphi_{jk}}$. Substituting $\hat{V}_i, \hat{V}_j, \hat{I}_{jk}$ into (10.2b) yields $\varphi_{ij} = \varphi_{jk}$ (strictly speaking it is $\varphi_{ij} - \varphi_{jk} = 2\pi$ but we ignore this nonuniqueness issue). On the other hand, if $j = 0$ (i.e., there is no line $j \to k$) then $\hat{V}_j = \tilde{V}_j$ by assumption and substituting $\hat{V}_i, \hat{V}_j, \hat{I}_{jk}$ into (10.2b) yields $\varphi_{ij} = 0$. Propagating towards bus 0 in a reverse breadth-first search order, we conclude that the angles φ_{jk} must be the same on all lines $j \to k$ since the network is connected. Moreover $\varphi_{jk} = 0$. Hence $\tilde{x} = \hat{x}$ and the mapping $\tilde{x} \mapsto x$ is injective.

To show that the mapping $\tilde{x} \mapsto x$ is surjective, we show that for any $x := (s, v, \ell, S)$ that satisfies (10.5) there is a $\tilde{x}$ that satisfies (10.2). Fix such an $x := (s, v, \ell, S)$. As explained above, when G is a tree, if v_j, ℓ_{jk}, S_{jk} satisfy (10.5c)–(10.5d), then there exist (V_j, I_{jk}) that satisfies (10.10) (or (10.6)). In particular, $V_j V_j^{\mathsf{H}} = v_j$, $I_{jk} I_{jk}^{\mathsf{H}} = \ell_{jk}$, and $V_j I_{jk}^{\mathsf{H}} = S_{jk}$. They are unique only up to an arbitrary phase, but we do not need $\tilde{x}$ to be unique for the mapping to be surjective. Let $\tilde{x} := (s, V, I, \ell, S)$. We now show that $\tilde{x}$ satisfies (10.2) and hence the mapping $\tilde{x} \mapsto x$ through (10.9) is surjective.

Clearly $\tilde{x}$ satisfies (10.2a), (10.2c) and (10.2d). To prove (10.2b), consider

$$V_k V_k^{\mathsf{H}} \,-\, (V_j - z_{jk} I_{jk})(V_j - z_{jk} I_{jk})^{\mathsf{H}} \;\;=\;\; v_k - \left(v_j - \left(z_{jk} S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}}\right) + z_{jk}\ell_{jk} z_{jk}^{\mathsf{H}}\right) \;\;=\;\; 0$$

where the last equality follows from (10.5b). Hence we have shown

$$V_k V_k^{\mathsf{H}} \;\;=\;\; (V_j - z_{jk} I_{jk})(V_j - z_{jk} I_{jk})^{\mathsf{H}}$$

Since rank-1 decomposition is unique up to an arbitrary phase, we have

$$V_j - V_k e^{\mathbf{i}\varphi_{jk}} \;\;=\;\; z_{jk} I_{jk}, \qquad j \to k \in E$$

for any φ_{jk}; in particular, choosing $\varphi_{jk} = 0$ shows that x satisfies (10.2b). Hence the mapping $\tilde{x} \mapsto x$ is surjective, and hence bijective. $\square$

10.3 Overall model and examples

10.3.1 Overall model

Suppose assumption C10.1 holds. The overall model of a network of three-phase devices connected by three-phase lines, its specification and analysis are similar to that in the bus injection model discuss in Chapter 9.2. The only difference is that the power flow equations are those for BFM rather than BIM. Specifically the overall model consists of:

1. A network model that relates terminal voltage, current, and power (V, I, s). Any equivalent model can be used, whichever is convenient for the problem under study, including:
   - the BFM (10.5) for radial networks; or
   - the BFM (10.2) for general networks.
2. A device model for each three-phase device j. For ideal devices, this can either be:
   - Its internal model (7.29) and the conversion rules (7.8) and (7.9)(7.10); or
   - Its external model summarized in Tables 7.3 and 7.4 when only terminal quantities are needed.

   For non-ideal devices, this can either be:
   - Its internal model summarized in Table 7.2 and the conversion rules (7.8) and (7.9)(7.10); or
   - Its external model summarized in Table 7.2 when only terminal quantities are needed.

Unlike the models of Chapter 9.1.5 where, if only voltage sources, current sources and impedances are involved, then the overall model is linear, consisting of the nodal current balance equation (9.5)(9.6) and linear device models. Here the BFM equations (10.5) and (10.2) are quadratic, leading to a nonlinear overall model even if power sources are absent.

A typical three-phase analysis problem can be specified and analyzed the same way as described in Chapter 9.2 for BIM. A solution typically takes the following steps:

1. Write down the models of the given collection of three-phase devices, either their internal models and conversion rules or their external models (if internal variables are not required).
2. Write down a network equation that relates the terminal variables, either the current balance equation or a power flow equation.
3. Steps 1 and 2 specify a system of nonlinear equations that relate relevant external and internal variables as well as given parameters. It generally needs to be solved numerically. We will describe in Chapter 10.4 such an algorithm for radial networks, the three-phase backward-forward sweep (BFS).
4. Usually we first compute the terminal variables $\left(V_j, I_j, s_j\right)$ using network equations, together with some of $\left(\gamma_j, \beta_j\right)$, and then determine the internal variables $\left(V_j^{Y/\Delta}, I_j^{Y/\Delta}, s_j^{Y/\Delta}\right)$ using the conversion rules.

10.3.2 Examples

We now illustrate with examples three-phase BFMs and the analysis procedure. Suppose assumption C10.1 holds.

Example 10.1 (Power source in Y configuration)**.** Consider the system in Figure 10.1 where a constant-power source $\sigma_j^Y \in \mathbb{C}^3$ is connected through a three-phase line to an impedance load z_k^Y, both in Y configuration. For simplicity we assume that both neutrals are directly grounded and all voltages are defined with respect to the ground, so that the neutral voltages $\gamma_j := V_j^n = \gamma_k := V_k^n = 0$. Suppose the following are given:

- The constant-power source $\sigma_j^Y := \left(\sigma_j^{an}, \sigma_j^{bn}, \sigma_j^{cn}\right)$ with $\angle V_0^a := 0°$.
- The impedance load $z_k^Y := \text{diag}\left(z_k^{an}, z_k^{bn}, z_k^{cn}\right)$.
- The series impedance matrix $z_{jk} \in \mathbb{C}^{3\times 3}$ of the line. Its shunt admittance matrices are assumed zero.

Derive the $\left(s_k^Y, v_k, \ell_{jk}, S_{jk}\right)$ in terms of the given parameters.

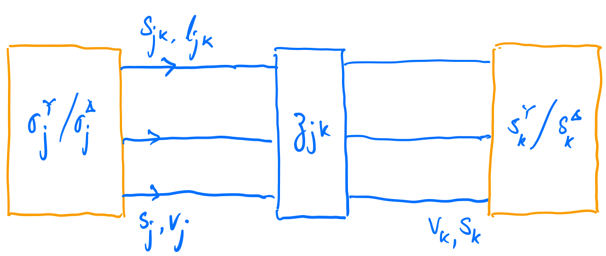


Figure 10.1: Example 10.1.

Solution. The system is specified by:

1. *Netowrk model*: The power flow equation (10.5) that relates terminal variables, specialized to the two-bus system in Figure 10.1, is:

$$\text{diag}(S_{jk}) = s_j, \qquad \text{diag}\left(S_{jk} - z_{jk}\ell_{jk}\right) = -s_k \tag{10.12a}$$

$$v_j - v_k = \left(z_{jk} S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}}\right) - z_{jk}\,\ell_{jk}\,z_{jk}^{\mathsf{H}} \tag{10.12b}$$

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \succeq 0, \qquad \text{rank}\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} = 1 \tag{10.12c}$$

2. *Device model*: The internal model of Y-configured impedance is (since $V_k = V_k^Y = z_k^Y I_k^Y$ and $I_k^Y = I_{jk}$):

$$v_k = z_k^Y \ell_{jk} z_k^{Y\mathsf{H}}, \qquad s_k^Y = \text{diag}\left(z_k^Y \ell_{jk}\right) \tag{10.13a}$$

and the conversion rule (7.8) between internal and terminal variables is:

$$s_j \;=\; -\left(\sigma_j^Y + V_j^n \bar{I}_{jk}\right) \;=\; -\sigma_j^Y, \qquad s_k \;=\; -\left(s_k^Y + V_k^n(-\bar{I}_{jk})\right) \;=\; -s_k^Y \tag{10.13b}$$

The system of quadratic equations (10.12)(10.13) cannot generally be solved in closed form, but can be solved numerically for $\left(s_k^Y, v_k, \ell_{jk}, S_{jk}\right)$ (see Chapter 10.4).

To better appreciate the structure of the three-phase model we now reduce (10.12)(10.13) to three quadratic equations in three unknowns $I_{jk} \in \mathbb{C}^3$. Relate ℓ_{jk} to σ_j^Y by eliminating the terminal powers (s_j, s_k), line power S_{jk} and internal power s_k^Y from (10.12a) (10.13):

$$-\sigma_j^Y \;=\; \operatorname{diag}\left(\left(z_k^Y + z_{jk}\right)\ell_{jk}\right) \tag{10.14}$$

This is a system of three complex quadratic equations in three unknown line currents $I_{jk} := \left(I_{jk}^a, I_{jk}^b, I_{jk}^c\right)$ because (10.12c) means that ℓ_{jk} has a rank-1 decomposition $\ell_{jk} = I_{jk} I_{jk}^{\mathsf{H}}$ (from (10.6)). Let $Z_k^Y := z_k^Y + z_{jk}$. Then (10.14) is explicitly:

$$-\sigma_j^Y \;=\; \operatorname{diag}\left(\begin{bmatrix} Z_k^{aa} & Z_k^{ab} & Z_k^{ac} \\ Z_k^{ba} & Z_k^{bb} & Z_k^{bc} \\ Z_k^{ca} & Z_k^{cb} & Z_k^{cc} \end{bmatrix} \begin{bmatrix} I_{jk}^a \\ I_{jk}^b \\ I_{jk}^c \end{bmatrix} \begin{bmatrix} I_{jk}^{a\mathsf{H}} & I_{jk}^{b\mathsf{H}} & I_{jk}^{c\mathsf{H}} \end{bmatrix}\right)$$

or

$$\begin{aligned} -\sigma_j^{an} &= Z_k^{aa} I_{jk}^a I_{jk}^{a\mathsf{H}} + Z_k^{ab} I_{jk}^b I_{jk}^{a\mathsf{H}} + Z_k^{ac} I_{jk}^c I_{jk}^{a\mathsf{H}} \\ -\sigma_j^{bn} &= Z_k^{ba} I_{jk}^a I_{jk}^{b\mathsf{H}} + Z_k^{bb} I_{jk}^b I_{jk}^{b\mathsf{H}} + Z_k^{bc} I_{jk}^c I_{jk}^{b\mathsf{H}} \\ -\sigma_j^{cn} &= Z_k^{ca} I_{jk}^a I_{jk}^{c\mathsf{H}} + Z_k^{cb} I_{jk}^b I_{jk}^{c\mathsf{H}} + Z_k^{cc} I_{jk}^c I_{jk}^{c\mathsf{H}} \end{aligned}$$

There is a power flow solution for (10.12)(10.13) if and only if (10.14) has a solution for I_{jk}, up to an angle to be determined (from the given $\angle V_0^a = 0°$).

Once I_{jk} and hence ℓ_{jk} are determined from (10.14), all other variables can be obtained. Specifically since $V_k = V_k^Y + V_k^n = V_k^Y$ by assumption, the load voltage and power are given by (10.13a):

$$v_k \;=\; v_k^Y \;=\; z_k^Y \ell_{jk} z_k^{Y\mathsf{H}} \;=\; \left(z_k^Y I_{jk}\right)\left(z_k^Y I_{jk}\right)^{\mathsf{H}}, \qquad\qquad s_k^Y \;=\; \operatorname{diag}\left(z_k^Y \ell_{jk}\right)$$

Since v_k has a rank-1 decomposition due to (10.12c), $V_k := \left(V_k^a, V_k^b, V_k^c\right)$ can be obtained from the first equation as $V_k = z_k^Y I_{jk}$, up to an angle to be determined. Finally we obtain V_j from $-\sigma_j^Y = s_j = \operatorname{diag}\left(V_j I_{jk}^{\mathsf{H}}\right)$ due to (10.13b) and then $S_{jk} = V_j I_{jk}^{\mathsf{H}}$. The given $\angle V_j^a = 0°$ then fixes the angles of $\left(V_j, V_k, I_{jk}\right)$. $\square$

The next example illustrates two solution approaches for constant-power source in Δ configuration. Both relate the terminal variables of each device to its parameters and then relates these terminal variables by the power flow equation. The first approach boils down to computing the internal current I_j^Δ from a system of quadratic equations, which then yields $\left(I_j, \beta_j\right)$ and all other variables. The second approach boils down to computing the terminal current and its zero-sequence component $\left(I_j, \beta_j\right)$ and then other variables.

As for Example 9.8, only γ_j of the source needs to be given. All other variables including $\left(\beta_j, \gamma_k, \beta_k\right)$ can then be determined. The solution method of these two examples is similar because the overall models in these examples differ only in their power flow equations, BIM (9.12) versus BFM (10.2). The positive definite and rank-1 condition in (10.12c) leads to the equivalence of BFM (10.2) to (10.12) and BIM (9.12) (Theorems 10.2 and 10.1).

Example 10.2 (Power source in Δ configuration)**.** Consider a three-phase power source and an impedance, both in Δ configuration, connected by a three-phase line (as in Example 9.8) with the following given parameters:

- The constant-power source $\left(\sigma_j^\Delta, \gamma_j\right)$ with $\angle V_j^{ab} := 0°$.
- The impedance load z_k^Δ. (Note that β_k need not be specified for an impedance and can be derived.)
- The series impedance matrix z_{jk} of the line. Its shunt admittance matrices are assumed zero.

Solve for the remaining variables.

Solution 1: compute I_j^Δ. The system is specified by:

1. *Netowrk model*: The power flow equation that relates terminal variables remains (10.12).
2. *Device model for power source* σ_j^Δ: At bus j we use the model (7.25b) and the conversion rule that relates the terminal variables $\left(V_j, I_j, s_j\right)$ to internal power σ_j^Δ and internal current I_j^Δ:

$$s_j \;:=\; \text{diag}\left(V_j I_j^{\mathsf{H}}\right) \tag{10.15a}$$

$$\sigma_j^\Delta \;:=\; \text{diag}\left(V_j^\Delta I_j^{\Delta\mathsf{H}}\right) \;=\; \text{diag}\left(\Gamma V_j I_j^{\Delta\mathsf{H}}\right), \qquad I_j \;=\; -\Gamma^{\mathsf{T}} I_j^\Delta \tag{10.15b}$$

3. *Device model for impedance* z_k^Δ: At bus k the external model in Table 7.4 relates the terminal variables (V_k, I_k, s_k) to impedance z_k^Δ through the admittance matrix Z_k^Δ defined in (7.27b):[3]

$$s_k \;:=\; \text{diag}\left(V_k I_k^{\mathsf{H}}\right), \qquad V_k \;=\; -Z^\Delta I_k + \gamma_k \mathbf{1}, \qquad \mathbf{1}^{\mathsf{T}} I_k \;=\; 0 \tag{10.15c}$$

The device models (10.15) relate terminal variables $\left(V_j, I_j, s_j\right)$ and (V_k, I_k, s_k) to the internal parameters $\left(\sigma_j^\Delta, z_k^\Delta\right)$ of the devices through γ_k (which is to be determined). The power flow equation (10.12) relates these terminal variables.

[3]Using the equivalent impedance model in terms of the impedance matrix Y_k^Δ defined in (7.27a) here does not . in which case (10.15c) is replaced by:

$$s_k \;:=\; \text{diag}\left(V_k I_k^{\mathsf{H}}\right), \qquad I_k \;=\; -Y^\Delta V_k$$

The rank-1 condition (10.12c) (as well as KCL) connects these terminal variables and the variables $(v_j, v_k, \ell_{jk}, S_{jk})$ of (10.12):

$$\begin{aligned} I_j &= I_{jk} = -I_k, & S_{jk} &= V_j I_{jk}^{\mathsf{H}} & & \quad (10.16a) \\ \ell_{jk} &= I_{jk} I_{jk}^{\mathsf{H}}, & v_j &= V_j V_j^{\mathsf{H}}, & v_k &= V_k V_k^{\mathsf{H}} \quad (10.16b) \end{aligned}$$

The equations (10.12)(10.15)(10.16) are a system of quadratic equations in variables $\left(V_j, I_j, s_j, I_j^{\Delta}\right)$, $(V_k, I_k, s_k, \gamma_k)$, and $(I_{jk}, v_j, v_k, \ell_{jk}, S_{jk})$. They can be solved numerically. Once these terminal variables are determined, the internal variables $(\beta_j, V_k^{\Delta}, I_k^{\Delta}, s_k^{\Delta}, \beta_k)$ can be determined. In particular once V_k is determined from the network equations we can obtain $V_k^{\Delta} = \Gamma V_k$ and then $I_k^{\Delta} = z_{jk}^{-1} V_k^{\Delta}$ and hence β_k.

To better appreciate the structure of this model we now reduce (10.12)(10.15)(10.16) to 3 quadratic equations in 3 variables I_{jk}^{Δ} for each link $j \to k \in E$. Theorem 10.2 implies the equivalence of BFMs (10.12) and (10.2). In particular (from (10.2b))

$$V_j - V_k = z_{jk} I_{jk}$$

which can also be derived by substituting (10.16) into (10.12b). Substitute V_k from (10.15c) and $I_k = -I_{jk}$ into this equation to eliminate V_k:

$$V_j = \hat{Z}_k^{\Delta} I_{jk} + \gamma_k \mathbf{1}, \qquad \mathbf{1}^{\mathsf{T}} I_{jk} = 0 \tag{10.17}$$

where $\hat{Z}_k^{\Delta} := Z_k^{\Delta} + z_{jk}$ is the equivalent of the line impedance in series with the load impedance. Substituting $I_{jk} = I_j = -\Gamma^{\mathsf{T}} I_j^{\Delta}$ into (10.17) and substituting the resulting V_j into (10.15b), we obtain a quadratic equation in I_j^{Δ} (using $\Gamma \mathbf{1} = 0$):

$$\sigma_j^{\Delta} := -\mathrm{diag}\left(\left(\Gamma \hat{Z}_k^{\Delta} \Gamma^{\mathsf{T}}\right) I_j^{\Delta} I_j^{\Delta \mathsf{H}}\right), \qquad j \in \overline{N} \tag{10.18}$$

There is a power flow solution to (10.12)(10.15)(10.16) if and only if (10.18) has a solution for I_j^{Δ}. Once I_j^{Δ} is determined it yields $I_{jk} = I_j = -\Gamma^{\mathsf{T}} I_j^{\Delta}$ and $\beta_j := \frac{1}{3} \mathbf{1}^{\mathsf{T}} I_j^{\Delta}$. Since (10.18) is the same equation as (9.24) in Example 9.8, we can follow the same procedure there to derive all variables (V_j, I_j, s_j, β_j) and $(V_k, I_k, s_k, \gamma_k)$. Then we can obtain internal variables $(V_k^{\Delta}, I_k^{\Delta}, s_k^{\Delta}, \beta_k)$ and the BFM variables $(I_{jk}, v_j, v_k, \ell_{jk}, S_{jk})$ from (10.16). In particular, V_k yields V_k^{Δ} and hence I_k^{Δ} and β_k. (To get more insight on its solution, see the solution of the balanced case in Exercise 9.10.)

Solution 2: compute I_j. Instead of the power source model (10.15b), we can also use the external model in Table 7.4 to relate the terminal current I_j direclty to the internal power σ_j^{Δ}:

$$\sigma_j^{\Delta} := \mathrm{diag}\left(V_j^{\Delta} I_j^{\Delta \mathsf{H}}\right) = -\mathrm{diag}\left(\Gamma \left(V_j I_j^{\mathsf{H}}\right) \Gamma^{\dagger}\right) + \overline{\beta}_j \Gamma V_j, \qquad \mathbf{1}^{\mathsf{T}} I_j = 0 \tag{10.19}$$

where the internal variable β_j is to be determined. Substituting (10.17) into (10.19) and noting $I_j = I_{jk}$ we have

$$\sigma_j^{\Delta} = -\frac{1}{3} \mathrm{diag}\left(\Gamma \hat{Z}_k^{\Delta} I_{jk} I_{jk}^{\mathsf{H}} \Gamma^{\mathsf{T}}\right) + \overline{\beta}_j \Gamma \hat{Z}_k^{\Delta} I_{jk}, \qquad \mathbf{1}^{\mathsf{T}} I_{jk} = 0 \tag{10.20}$$

There is a power flow solution to (10.12)(10.15)(10.16) if and only if there is a solution $I_{jk} := I_{jk}(\sigma_j^\Delta)$ and $\beta_j := \beta_j(\sigma_j^\Delta)$ to (10.20). Given a solution $\left(I_{jk}, \beta_j\right)$ and hence I_{jk}^Δ, all other variables can be derived as in Solution 1. $\square$

Remark 10.1. Even though the analysis in Example 10.2 makes heavy use of BFM (10.12) with phasor variables such as $\left(V_j, I_{jk}\right)$ instead of variables of BFM (10.2) such as $\left(v_j, \ell_{jk}, S_{jk}\right)$, the model (10.2) is useful for solving optimal power flow problems through semidefinite relaxation; see Chapter . $\square$

10.4 Backward forward sweep

In this section we extend the backward forward sweep (BFS) of Chapter 5.4 for the computation of power flow solutions from single-phase radial networks to three-phase radial networks. As explained in Chapter 5.4.1 BFS can be interpreted as a Gauss-Siedel algorithm that computes a fixed point of BFM equations. It has two special structures that exploit the tree topology of the network. First it partitions the power flow variable into two vectors x and y and updates them iteratively in an outer loop. Typically x consists of branch variables, e.g., branch currents or powers, and y consists of nodal variables, e.g., nodal voltages. Second, for each outer iteration, it computes iteratively each component of (x,y) in an inner loop that makes use of a spatially recursive structure enabled by the tree topology. Specifically it computes the components of x iteratively from leaf nodes towards the root of the tree (backward sweep) and then computes the components of y iteratively from the root towards the leaf nodes (forward sweep). The design of BFS involves the choice of power flow equations and variables (x,y) based on what information is given in a power flow problem. These choices are not unique and may have different convergence properties. The general algorithmic structure described in Chapter 5.4.1 applies to three-phase as well as single-phase radial networks. We have presented two BFS algorithms in Chapters 5.4.2 and 5.4.3 that use different branch flow models. In this section we describe an algorithm that extends both single-phase algorithms to the three-phase setting. As we will see, the main addition is the computation of internal variables associated with each three-phase device.

Recall that we assume C10.1 holds throughout this chapter.

10.4.1 Complex form BFM

Consider a radial network modeled as a directed graph G, rooted at bus 0 and with each line pointing *away* from the root bus 0. Each line is characterized by 3×3 admittance matrices $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$. Suppose there is exactly one three-phase power source at each bus j either in Y or Δ configuration. At every non-root bus $j \in N$, the internal power $\sigma_j^{Y/\Delta} \in \mathbb{C}^3$ of the power source is given and its terminal voltage and current (V_j, I_j) are to be determined.[4] At bus 0, $V_0 \in \mathbb{C}^3$ is given and the current injection I_0 and the internal power injection $s_0^{Y/\Delta}$ are to be determined. We assume for simplicity that C7.1 with $z_j^n = 0$ holds at every bus $j \in \overline{N}$ that has a Y-configured power source so that $V_j^n = 0$ (see Remark 10.3 on the case when $z_j^n \neq 0$ so that $V_j^n = -z_j^n \left(\mathbf{1}^\mathsf{T} I_j\right)$).

4

As for the single-phase BFS, let $\left(I^s_{jk},\ j \to k \in E\right)$ be the branch current through the series admittance matrix $y^s_{jk} \in \mathbb{C}^{3\times 3}$ (see Exercise 10.3 for a BFS algorithm that computes the sending-end current I_{jk} instead). The receiving current at bus j from its parent i is $\left(I^s_{ij} - y^m_{ji} V_j\right) \in \mathbb{C}^3$ (see Figure 10.2). The current

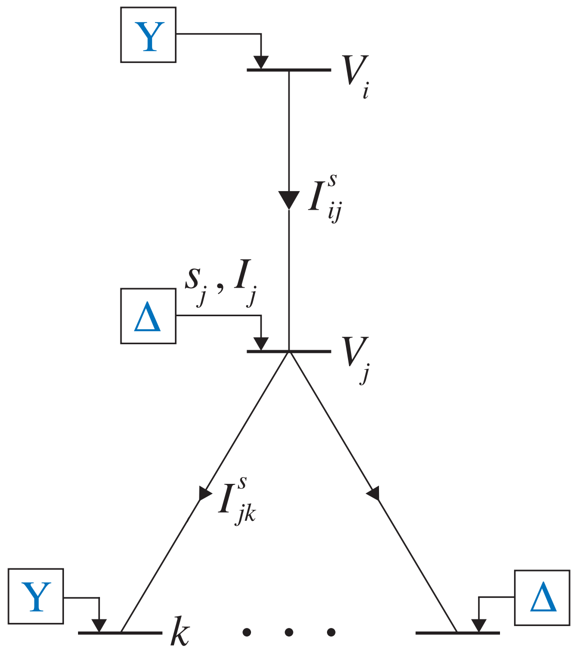


Figure 10.2: Notation for BFS on unbalanced three-phase radial networks.

balance equation is then

$$I_j \;+\; \left(I^s_{ij} - y^m_{ji} V_j\right) \;\;=\;\; \sum_{k:j\to k} \left(I^s_{jk} + y^m_{jk} V_j\right)$$

Rewriting this in a form suitable for backward sweep, we obtain the following three-phase branch flow model in terms of branch variables $\left(I^s_{jk}, j \to k \in E\right)$ and nodal variables $\left(V_j, I_j,\ j \in \overline{N}\right)$:

$$I^s_{ij} \;=\; \sum_{k:j\to k} I^s_{jk} \;-\; \left(I_j - y^m_{jj} V_j\right), \qquad j \in N \tag{10.21a}$$

$$V_j \;=\; V_i \;-\; z^s_{ij} I^s_{ij}, \qquad j \in N \tag{10.21b}$$

where $y^m_{jj} := y^m_{ji} + \sum_{k:j\to k} y^m_{jk}$ are the total shunt admittances incident on j and $z^s_{ij} := \left(y^s_{ij}\right)^{-1}$ are the series impedances. These network equations relate the branch currents I^s_{jk} as well as the terminal voltages and currents $\left(V_j, I_j\right)$ at buses across the network.

Each terminal variable $\left(V_j, I_j\right)$ is related to the internal power $\sigma^{Y/\Delta}_j$ through a three-phase device model. We adopt the following device models for reasons discussed in Remark 10.2 (from (7.17b) and

(7.25b) and recall that $V_j^n = 0$ by assumption):

$$\begin{aligned} &Y \text{ configuration:} && \sigma_j^Y \;=\; \operatorname{diag}\left(V_j I_j^{Y\mathsf{H}}\right), \quad I_j \;=\; -I_j^Y && (10.22a)\\ &\Delta \text{ configuration:} && \sigma_j^\Delta \;=\; \operatorname{diag}\left(\Gamma V_j I_j^{\Delta\mathsf{H}}\right), \quad I_j \;=\; -\Gamma^\mathsf{T} I_j^\Delta && (10.22b) \end{aligned}$$

Hence, for a non-root bus j, the given internal power $\sigma_j^{Y/\Delta}$ determines, through its internal current $I_j^{Y/\Delta}$, its terminal voltage and current (V_j, I_j) according to (10.22). These terminal variables interact across the network according to the network equations (10.21). Given V_j, the forward sweep function g_j in (5.20a) to update $\left(I_j^{Y/\Delta}, I_j\right)$ is:

$$\begin{aligned} &Y: && I_j^Y \;=\; \left(\operatorname{diag}\bar{V}_j\right)^{-1}\bar{\sigma}_j^Y, && I_j \;=\; -I_j^Y, && j \in N && (10.23a)\\ &\Delta: && I_j^\Delta \;=\; \left(\operatorname{diag}\left(\Gamma\bar{V}_j\right)\right)^{-1}\bar{\sigma}_j^\Delta, && I_j \;=\; -\Gamma^\mathsf{T} I_j^\Delta, && j \in N && (10.23b) \end{aligned}$$

where $\bar{v}$ denotes the componentwise complex conjugate of a vector v. Here, we have used, for vectors $v, w \in \mathbb{C}^n$, $\operatorname{diag}(vw^\mathsf{H}) = \operatorname{diag}(v)\bar{w} = \operatorname{diag}(\bar{w})v \in \mathbb{C}^n$ where $\operatorname{diag}(v)$ is the diagonal matrix whose diagonal is the vector v.

To construct the backward forward sweep, identify lines $j \to k \in E$ by the non-root buses $k \in N$. Given V_0 and $\sigma := \left(\sigma_j^{Y/\Delta},\ j \in N\right)$, the BFS will compute the following branch and nodal variables respectively:

$$x \;:=\; \left(I_{ij}^s,\ j \in N\right), \qquad\qquad y \;:=\; \left(V_j, I_j, I_j^{Y/\Delta},\ j \in N\right)$$

All other variables, such as injections $I_0, s_0, s_0^{Y/\Delta} \in \mathbb{C}^3$, branch flow matrices $S_{jk} \in \mathbb{C}^{3\times 3}$, and $\left(\gamma_j, \beta_j\right) \in \mathbb{C}^2$ of power sources σ_j^Δ, can be computed once (x, y) are determined. The update function f in the backward sweep to update x is defined by (10.21a) and the update function g in the forward sweep to update y is defined by (10.21b) and (10.23). The function f is jointly linear in (x, y). The function g is linear in x but nonlinear in y because of the power source model (10.23).

The boundary conditions are

$$V_0 \in \mathbb{C}^3 \;\text{ is given },\qquad I_{jk}^s \;:=\; 0 \;\text{ for all leaf nodes } j, \qquad V_j(0) \;:=\; V_0,\ j \in N \tag{10.24a}$$

In addition, given the initial voltages $\left(V_j(0), j \in N\right)$, the terminal and internal currents $\left(I_j(0), I_j^{Y/\Delta(0)}, j \in N\right)$ are determined using (10.23):

$$\begin{aligned} &Y: && I_j^Y(0) \;=\; \left(\operatorname{diag}\bar{V}_j(0)\right)^{-1}\bar{\sigma}_j^Y, && I_j(0) \;=\; -I_j^Y(0), && j \in N && (10.24b)\\ &\Delta: && I_j^\Delta(0) \;=\; \left(\operatorname{diag}\left(\Gamma\bar{V}_j(0)\right)\right)^{-1}\bar{\sigma}_j^\Delta, && I_j(0) \;=\; -\Gamma^\mathsf{T} I_j^\Delta(0), && j \in N && (10.24c) \end{aligned}$$

Specifically the BFS algorithm defined by (10.21) (10.23) (10.24) proceeds as follows.

0. *Input*: voltage V_0 pu and internal power $\left(\sigma_j^{Y/\Delta}, j \in N\right)$.

1. *Initialization.*
   - $I^s_{jk}(t) := 0$ for all leaf nodes j for all iterations $t = 1, 2, \ldots$.
   - $V_0(t) := V_0$ for all $t = 0, 1, \ldots$.
   - $V_j(0) := V_0$ at all buses $j \in N$. Compute $\left(I_j(0), I_j^{Y/\Delta}(0)\right)$ using (10.24b)(10.24c).

2. *Backward forward sweep.* Iterate for $t = 1, 2, \ldots$ until a stopping criterion (see below) is satisfied:

   (a) *Backward sweep.* Starting from the leaf nodes and iterating towards bus 0, compute

$$I^s_{ij}(t) \;\leftarrow\; \sum_{k: j\to k} I^s_{jk}(t) \;-\; \left(I_j(t-1) - y^m_{jj} V_j(t-1)\right), \quad i \to j \in E \tag{10.25a}$$

   where $y^m_{jj} := y^m_{ji} + \sum_{k: j\sim k} y^m_{jk}$.

   (b) *Forward sweep.* Starting from bus 0 and iterating towards the leaf nodes, compute for $j \in N$

$$V_j(t) \;\leftarrow\; V_i(t) \;-\; z^s_{ij} I^s_{ij}(t) \tag{10.25b}$$

$$Y: \qquad I^Y_j(t) \;\leftarrow\; \left(\text{diag}\, \bar{V}_j(t)\right)^{-1} \bar{\sigma}^Y_j, \qquad I_j(t) \;\leftarrow\; -I^Y_j(t) \tag{10.25c}$$

$$\Delta: \qquad I^\Delta_j(t) \;\leftarrow\; \left(\text{diag}\left(\Gamma \bar{V}_j(t)\right)\right)^{-1} \bar{\sigma}^\Delta_j, \qquad I_j(t) \;\leftarrow\; -\Gamma^{\mathsf{T}} I^\Delta_j(t) \tag{10.25d}$$

   where $z^s_{ij} := \left(y^s_{ij}\right)^{-1}$.

3. *Output*: branch variable $x := \left(I^s_{ij}(t),\ j \in N\right)$ and nodal variable $y := \left(V_j(t), I_j(t), I_j^{Y/\Delta(t)},\ j \in N\right)$.

A stopping criterion can be based on the discrepancy between the given internal powers $\sigma_j^{Y/\Delta}$ and those implied by the nodal variable $\left(V_j(t), I_j(t), I_j^{Y/\Delta(t)},\ j \in N\right)$ in each iteration t. From the device model (10.23), let

$$\hat{\sigma}_j(t) \;:=\; \begin{cases} \text{diag}\left(V_j(t) I_j^{Y\mathsf{H}}(t)\right) & \text{for} \quad Y \text{ configuration} \\ \text{diag}\left(\Gamma V_j(t) I_j^{\Delta\mathsf{H}}(t)\right) & \text{for} \quad \Delta \text{ configuration} \end{cases}$$

Then a stopping criterion can be

$$\|\hat{\sigma}(t) - \sigma^{Y/\Delta}\|_2^2 \;:=\; \sum_{j\in N} \left(\hat{\sigma}_j(t) - \sigma_j^{Y/\Delta}\right)^2 \;<\; \varepsilon$$

for a given tolerance $\varepsilon > 0$.

Remark 10.2 (Choice of variables)**.** 1. We have used the current balance equation (10.21a) to relate terminal voltages and currents (V_j, I_j) across the network. This leads to a linear update function (10.21a) for x in backward sweep. Nonlinearity shows up in the device model (10.23) for the nodal variable $y := \left(V_j, I_j, I_j^{Y/\Delta},\ j \in N\right)$ in the forward sweep (together with (10.21b)).

2. A direct extension of the single-phase BFS in [24] to the three-phase setting is the approach in [41] which substitutes I_j in (10.21a) by $I_j = \left(\text{diag}\bar{V}_j\right)^{-1} \bar{s}_j$ to obtain a nonlinear update function for x:

$$I^s_{ij} \;=\; \sum_{k:j\to k} I^s_{jk} \;-\; \left(\left(\text{diag}\,\bar{V}_j\right)^{-1} \bar{s}_j - y^m_{jj} V_j\right), \qquad\qquad j \in N \tag{10.26a}$$

In this case the nodal variable becomes $y := \left(V_j, s_j, I^{Y/\Delta}_j,\ j \in N\right)$ and the update functions (10.23) become

$$Y: \qquad I^Y_j \;=\; \left(\text{diag}\,\bar{V}_j\right)^{-1} \bar{\sigma}^Y_j, \qquad s_j \;=\; -\sigma^Y_j, \qquad j \in N \tag{10.26b}$$

$$\Delta: \qquad I^\Delta_j \;=\; \left(\text{diag}\left(\Gamma\bar{V}_j\right)\right)^{-1} \bar{\sigma}^\Delta_j, \qquad s_j \;=\; -\text{diag}\left(V_j I^{\Delta\mathsf{H}}_j \Gamma\right), \qquad j \in N \tag{10.26c}$$

The three-phase BFS of [41] includes only Y-configured power sources and therefore its update functions simplifies to only (10.26a) (10.21b), with $s_j = -\sigma^Y_j$ that is fixed and given. The addition of Δ-configured power sources requires the nodal variable I^Δ_j and update function (10.26c).

3. For Δ configuration, the device model (10.23) relates σ^Δ to (V_j, I_j) through I^Δ_j. Since $\left(V_j, I^\Delta_j\right)$ are determined directly from the overall model, the quantities $\left(\gamma_j, \beta_j\right)$ can be computed and need not be specified. Note however that V_0 is given.

□

Remark 10.3 (Nonzero z^n_j)**.** If we had assumed C7.1 with $z^n_j \neq 0$ so that $V^n_j = -z^n_j\left(\mathbf{1}^\mathsf{T} I\right)$, then the device model (10.23a) for a Y-configured power source becomes nonlinear in I_j (from (7.17b)):

$$Y: \qquad V_j \;=\; -\left(\text{diag}\left(\bar{I}_j\right)\right)^{-1} \sigma^Y_j \;-\; z^n_j\left(\mathbf{1}\mathbf{1}^\mathsf{T}\right) I_j, \qquad j \in N$$

Given voltage V_j this is a system of three quadratic equations in three unknowns $I_j \in \mathbb{C}^3$:

$$z^n_j\left(\mathbf{1}^\mathsf{T} I_j\right)\bar{I}_j \;+\; \text{diag}\left(V_j\right)\bar{I}_j \;+\; \sigma^Y_j \;\;=\;\; 0$$

The linear update functions (10.23a) (10.24b) then become nonlinear. Moreover the update of I_j is defined only implicitly by a solution of this system of quadratic equations. □

Remark 10.4 (Specification)**.** Unlike in Examples 10.1 and 10.2, the BFS method here does not required γ_j be specified, but it requires that V_0 be specified. □

10.4.2 DistFlow model

Consider a three-phase radial network modeled by a directed graph with every link $k \to j \in E$ points *away from* the root bus 0. Assume for simplicity zero shunt admittances, $y^m_{jk} = y^m_{kj} = 0$. The three-phase DistFlow equations for the down orientation are (10.5). Given V_0, hence $v_0 := V_0 V_0^\mathsf{H}$, and internal power $\sigma := \left(\sigma^{Y/\Delta}_j, j \in N\right)$, we wish to compute the other variables from (10.5).

The nonlinear equation $v_j \ell_{jk} = |S_{jk}|^2$ in (10.4c) for the single-phase model is replaced by (10.5c)(10.5d) in the three-phase model, reproduced here

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \geq 0, \qquad\qquad \text{rank} \begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} = 1$$

These equations are an implicit description and do not directly yield an update equation for a BFS algorithm, as $v_j \ell_{jk} = |S_{jk}|^2$ does in the single-phase model. Instead, they imply that there exist voltage and current phasors $(V, \tilde{I})$ that satisfy the rank-1 decomposition in (10.6). In order to compute DistFlow variables (v, ℓ, S) we have to compute iteratively the voltages V_j and (sending-end) line currents $\tilde{I}_{jk}$ in the process. Here we use $\tilde{I}_{jk}$ to denote a line current to differentiate it from the terminal current I_j in a device model (see below). Therefore, instead of designing an BFS algorithm based on (10.5), we will use the following network equations derived from (10.5) to compute $\left(V, I, \tilde{I}\right)$:

$$\tilde{I}_{ij} = -I_j + \sum_{k: j \to k} \tilde{I}_{jk} \tag{10.27a}$$

$$V_k = V_j - z_{jk} \tilde{I}_{jk} \tag{10.27b}$$

All other terminal variables such as $v_j = V_j V_j^{\mathsf{H}}$, $\ell_{jk} = \tilde{I}_{jk} \tilde{I}_{jk}^{\mathsf{H}}$, and $S_{ij} = V_i \tilde{I}_{ij}^{\mathsf{H}}$, can then be derived. Note that we have replaced the power balance equation (10.5a) by the current balance equation in (10.27a). The network equation (10.27) is the same as (10.21) with $\tilde{I}_{jk} = I_{jk}^s$ when $y_{jk}^m = y_{kj}^m = 0$. Hence the three-phase DistFlow model can be solved using the BFS algorithm of Chapter 10.4.1.

10.5 Linear model

We generalize the linear DistFlow model from single-phase to unbalanced multiphase radial networks. The key assumptions in our linear approximation are:

1. The real and reactive line losses $z_{jk} \ell_{jk}$ are much smaller than line flows S_{jk} on each line $j \to k$, so that we can assume $\ell_{jk} = 0$ in (10.5).

2. The voltages are approximately balanced, so that we can assume

$$\frac{V_j^a}{V_j^b} = \frac{V_j^b}{V_j^c} = \frac{V_j^c}{V_j^a} = e^{\mathbf{i} 2\pi/3}$$

Recall that we adopt, without loss of generality, the graph orientation in which all lines point away from bus 0. Then, as for the single-phase model, we set $\ell_{jk} = 0$ in (10.5a)(10.5b) to obtain

$$\sum_{k: j \to k} \text{diag}(S_{jk}) = \text{diag}\left(S_{ij}\right) + s_j, \qquad j \in \overline{N}$$

$$v_j - v_k = z_{jk} S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}} \qquad j \to k \in E$$

where bus $i := i(j)$ is the unique parent of bus j. Given injections s_j for all non-slack buses $j \in N$, the first set of equations determines uniquely s_0 and the diagonal entries of S_{jk}, but not the off-diagonal entries of

S_{jk}. The second assumption of balanced voltage is needed to determine the off-diagonal entries of S_{jk}. Specifically the assumption means that the vector V_j is determined by a scalar (say) V_j^a. Let

$$\alpha \;:=\; e^{-\mathbf{i}2\pi/3}, \qquad \alpha_+ \;:=\; \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix} \tag{10.28a}$$

Then, assuming positive sequence,

$$V_j \;=\; V_j^a \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix} \;=\; V_j^a\,\alpha_+ \tag{10.28b}$$

This makes it possible to determine the off-diagonal entries of S_{jk} from its diagonal entries, as follows. Let $\lambda_{jk} := \mathrm{diag}(S_{jk})$ denote the vector consisting of the diagonal entries of S_{jk}:

$$\lambda_{jk} \;:=\; \begin{bmatrix} V_j^a \left(I_{jk}^a\right)^{\mathsf{H}} \\ V_j^b \left(I_{jk}^b\right)^{\mathsf{H}} \\ V_j^c \left(I_{jk}^c\right)^{\mathsf{H}} \end{bmatrix}$$

Using (10.28), the 3×3 line flow matrix S_{jk} is given by:

$$S_{jk} \;:=\; V_j I_{jk}^{\mathsf{H}} \;=\; V_j^a\,\alpha_+ \begin{bmatrix} I_{jk}^a & I_{jk}^b & I_{jk}^c \end{bmatrix}^{\mathsf{H}}$$

This expression says that the columns of S_{jk} are in $\mathrm{span}(\alpha_+)$. The first column of the right-hand side is

$$\alpha_+ \underbrace{V_j^a \left(I_{jk}^a\right)^{\mathsf{H}}}_{[S_{jk}]_{11}} \;=\; \alpha_+ \left[\lambda_{jk}\right]_1$$

The second column is

$$\alpha_+ V_j^a \left(I_{jk}^b\right)^{\mathsf{H}} \;=\; \frac{1}{\alpha}\,\alpha_+ \left(\alpha V_j^a\right) \left(I_{jk}^b\right)^{\mathsf{H}} \;=\; \frac{1}{\alpha}\,\alpha_+ \underbrace{V_j^b \left(I_{jk}^b\right)^{\mathsf{H}}}_{[S_{jk}]_{22}} \;=\; \frac{1}{\alpha}\,\alpha_+ \left[\lambda_{jk}\right]_2$$

The third column is

$$\alpha_+ V_j^a \left(I_{jk}^c\right)^{\mathsf{H}} \;=\; \frac{1}{\alpha^2}\,\alpha_+ \left(\alpha^2 V_j^a\right) \left(I_{jk}^c\right)^{\mathsf{H}} \;=\; \frac{1}{\alpha^2}\,\alpha_+ \underbrace{V_j^c \left(I_{jk}^c\right)^{\mathsf{H}}}_{[S_{jk}]_{33}} \;=\; \frac{1}{\alpha^2}\,\alpha_+ \left[\lambda_{jk}\right]_3$$

Putting all this together define

$$\gamma \;:=\; \begin{bmatrix} \alpha_+ & \tfrac{1}{\alpha}\,\alpha_+ & \tfrac{1}{\alpha^2}\,\alpha_+ \end{bmatrix} \;=\; \begin{bmatrix} 1 & \alpha^2 & \alpha \\ \alpha & 1 & \alpha^2 \\ \alpha^2 & \alpha & 1 \end{bmatrix}$$

and we can determine the line flow matrix S_{jk} in terms of its diagonal entries:

$$S_{jk} \;=\; \gamma \, \mathrm{diag}\left(\lambda_{jk}\right), \qquad j \to k \in E$$

where $\mathrm{diag}(x)$ is a diagonal matrix whose diagonal consists of entries of vector x. Then the linear model that generalizes the single phase linear DistFlow model to three-phase radial networks is (graph is oriented so that all lines point away from bus 0):

$$\sum_{k: j\to k} \lambda_{jk} \;=\; \lambda_{ij} + s_j, \qquad j \in \overline{N} \tag{10.29a}$$

$$S_{jk} \;=\; \gamma \, \mathrm{diag}\left(\lambda_{jk}\right), \qquad j \to k \in E \tag{10.29b}$$

$$v_j - v_k \;=\; z_{jk} S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}}, \qquad j \to k \in E \tag{10.29c}$$

where $i := i(j)$ is the unique parent node of j.

Linear solution. Given $\left(v_0, s_j, j \in N\right)$, (10.29) can be used to determine explicitly $\left(s_0, v_j, j \in N\right)$ and $\left(\lambda_{jk}, S_{jk}, j \to k \in E\right)$, as follows (Exercise 10.5):

$$\begin{aligned}
s_0 &= -\sum_{j \in N} s_j \\
\lambda_{ij} &= -\sum_{k \in \mathsf{T}_j} s_k, \qquad S_{ij} \;=\; \gamma \, \mathrm{diag}\left(\lambda_{ij}\right), && i \to j \in E \\
v_j &= v_0 - \sum_{(i,k) \in \mathsf{P}_j} \left(z_{ik} S_{ik}^{\mathsf{H}} + S_{jk} z_{ik}^{\mathsf{H}}\right), && j \in N
\end{aligned}$$

where T_j is the subtree rooted at bus j, including j, and P_k is the set of lines on the unique path from bus 0 to bus k; see Figure 10.3.

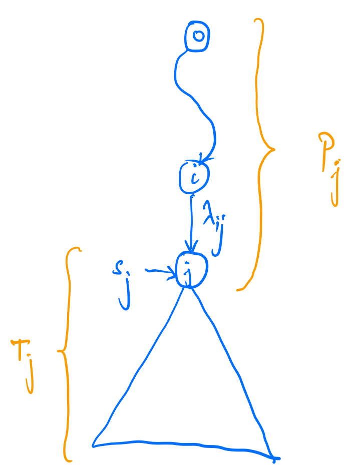


Figure 10.3: Linear solution of branch flow model for unbalanced three-phase radial networks.

10.6 Bibliographical notes

Algorithms for solving power flows in three-phase radial networks are developed in [37, 38, 39, 41, 43, 46]. For backward forward sweep methods for radial networks, both single-phase and three-phase networks, see bibliographical notes in Chapter 5.6.

Chapter 10.1.

Exercise 10.1. Derive the three-phase BFM (10.1) for general networks without line charging from the BFM (5.1) with line charging.

Chapter 10.2.

Exercise 10.2. Show that $(Ab) \otimes \mathbb{I} = (A \otimes \mathbb{I})\,(b \otimes \mathbb{I})$ where $A \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$, and $\mathbb{I}$ is the identity matrix of size 3. (Hint: Use Lemma 9.6.)

Chapter 10.3.

Chapter 10.4.

Exercise 10.3 (Backward forward sweep)**.** This exercise solves the same overall model as the BFS described in Chapter 10.4.1, but here, instead of $I^s_{jk} \in \mathbb{C}^3$ over the series impedance, we are to derive a BFS algorithm to compute the sending-end current $I_{jk} \in \mathbb{C}^3$ for every line $j \to k$, as well as the nodal variable $y := \left(V_j, I_j, I^\Delta_j\ j \in N\right)$. It extends Exercise 5.7 from single-phase radial networks to three-phase radial networks.

Exercise 10.4 (Backward forward sweep)**.** Extend the BFS described in Chapter 5.4.3 from single-phase to three-phase radial networks. This allows the inclusion of PV buses where real power and voltage magnitudes are given instead of internal powers.

Chapter 10.5.

Exercise 10.5 (Three-phase BFM linear solution)**.** Given $\left(v_0, s_j, j \in N\right)$, show that an explicit solution $\left(s_0, v_j, j \in N, S_{jk}, j \to k \in E\right)$ of (10.29) is

$$
\begin{aligned}
s_0 &= -\sum_{j \in N} s_j \\
\lambda_{ij} &= -\sum_{k \in \mathsf{T}_j} s_k, \qquad S_{ij} \;=\; \gamma\,\mathrm{diag}\left(\lambda_{ij}\right), && i \to j \in E \\
v_j &= v_0 - \sum_{(i,k) \in \mathsf{P}_j} \left(z_{ik} S_{ik}^{\mathsf{H}} + S_{jk} z_{ik}^{\mathsf{H}}\right), && j \in N
\end{aligned}
$$

where T_j is the subtree rooted at bus j, including j, and P_k is the set of lines on the unique path from bus 0 to bus k.

Part III

Power flow optimization

Chapter 11

Smooth convex optimization

In this chapter we introduce basic concepts in convex optimization, describe iterative algorithms that are widely used for solving such problems, and explain basic techniques for analyzing their convergence properties. Convexity is a simplifying structure that enables a rich theory on algorithm design and analysis for convex optimization. Even though optimal power flow problems are nonconvex, convex optimization theory is useful for two reasons. First, iterative algorithms that have been designed and analyzed for convex problems are often used also for solving nonconvex problems. Unlike for convex problems, there is typically no guarantee on optimality or convergence for nonconvex problems, but they often perform well nonetheless. Second, an important method to deal with a nonconvex problem is solving its convex relaxation where an approximate convex problem is solved instead. We will study optimal power flow problems in Chapter 13 and their convex relaxations in Chapters 14 and 15.

11.1 Convex programs and duality

This section mostly follows [72, Section 2.1]. We start by introducing some basic concepts in convex optimization.

A convex program is defined by a convex set and a convex function, as we now define.

11.1.1 Convex set

A set is called convex if, given any two points in the set, every point in between lies in the set.

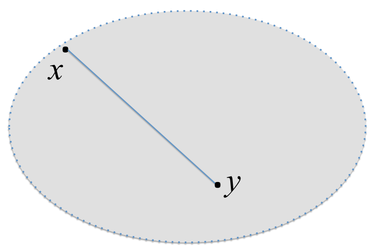
Definition 11.1. A set $D \subseteq \mathbb{R}^n$ is *convex* if, given any $x, y \in D$,

$$\alpha x + (1-\alpha) y \ \in \ D, \qquad \forall \alpha \in [0,1]$$

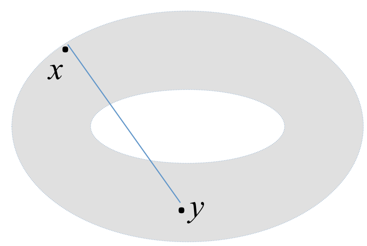
For instance for any $x_0 \in D$ there exists $r > 0$ such that the r-ball around x_0,

$$B_r(x_0) \quad := \quad \{\ x \in D \mid \|x - x_0\|_2 \leq r\ \}$$

is contained in D, where $\|x\|_2 := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ is the Euclidean norm. Moreover $B_r(x_0)$ is convex for any $r > 0$, $x_0 \in D$. The definition is illustrated in Figure 11.1.



(a) Convex set.



(b) Nonconvex set.

Figure 11.1: Definition of a convex set: every point in between two points in the set lies in the set.

Three types of convex sets are the most useful in engineering applications. The first is a set specified by linear inequalities:

$$\text{Affine set:} \qquad C \;:=\; \{x \in \mathbb{R}^n \mid Ax \le b\}, \;\; n \ge 1$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, $m \ge 1$. The second is a second-order cone (SOC) defined as:

$$\text{SOC:} \qquad C \;:=\; \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \le t\}, \;\; n \ge 1$$

A ball $B_t(0)$ is a cross section of the second-order cone defined by $\|x\|_2 \le t$ for a fixed t. The third is a set of semidefinite matrices defined as follows. A real matrix $X \in \mathbb{R}^{n\times n}$ is *symmetric* if $X = X^{\mathsf{T}}$, i.e., $X_{ij} = X_{ji}$ for all $i, j = 1, \ldots, n$. A real matrix X is *positive semidefinite* (psd) if X is symmetric and $x^{\mathsf{T}}Xx = \sum_{i,j} X_{ij}x_ix_j \ge 0$ for all $x \in \mathbb{R}^n$. We write $X \succeq 0$ to denote that X is positive semidefinite. Given a symmetric matrix $X \in \mathbb{R}^{n\times n}$ the following are equivalent:

1. X is positive semidefinite.
2. All eigenvalues of X are nonnegative.
3. $X = BB^{\mathsf{T}}$ for some matrix $B \in \mathbb{R}^{n\times m}$ and some natural number m.

The set of all positive semidefinite matrices is:

$$\text{psd matrices:} \qquad \mathbb{S}^n_+ \;:=\; \{X \in \mathbb{R}^{n\times n} \mid X \succeq 0\}, \;\; n \ge 1$$

The proof that these three types of sets are convex is left as an exercise. Efficient algorithms exist to solve constrained optimization problems that minimize a certain cost function over an affine set, second-order cones, or semidefinite matrices.

Given these three basic convex sets we can create other convex sets through simple *convexity-preserving* operations. Let $\mathbb{X}$ and $\mathbb{Y}$ be linear subspaces. For example $\mathbb{X} := \mathbb{R}^n$ and $\mathbb{Y} := \mathbb{R}^m$.

1. *Linear transformation:* Let $f : \mathbb{X} \to \mathbb{Y}$ be linear.
   (a) If $A \subseteq \mathbb{X}$ is convex then $f(A) := \{f(x) \mid x \in A\} \subseteq \mathbb{Y}$ is convex.
   (b) If $B \subseteq \mathbb{Y}$ is convex then $f^{-1}(B) = \{x \mid f(x) \in B\} \subseteq \mathbb{X}$ is convex.
2. *Direct product:* Let $A \subseteq \mathbb{X}$, $B \subseteq \mathbb{Y}$ be convex. Then $A \times B := \{(x,y) \mid x \in A, y \in B\}$ is convex. In fact the direct product of an arbitrary (e.g., uncountably many) number of convex sets is convex.
3. *Finite sum:* Let $A, B \subseteq \mathbb{X}$ be convex. Then $A + B := \{a + b \mid a \in A, b \in B\}$ is convex. Therefore the sum of any finite number of convex sets is convex.
4. *Arbitrary intersection:* Let $A, B \subseteq \mathbb{X}$ be convex. Then the intersection $A \cap B$ is convex. In fact the intersection of an arbitrary collection of (e.g., uncountably many) convex sets is convex.

The proof that these set operations preserve convexity is left as an exercise. In contrast to intersection the union of two convex sets can be nonconvex. Note that if A, B are convex, then $A \cap B$ is convex. The converse may not be true; e.g., $A := \{x : x \geq 0\} \cup \{x : x \leq 0\} \subseteq \mathbb{R}^n$ and $B := \{x : x_1 \geq 0\} \subseteq \mathbb{R}^n$.

Example 11.1. Consider the ellipsoid

$$E \quad := \quad \{x \in \mathbb{R}^n \mid x^\mathsf{T} A x \leq c\}$$

where $A \in \mathbb{R}^{n \times n}$ is a psd matrix and $c > 0$. E is convex because it can be derived from an application of convexity-preserving operation on a convex set as follows. Since A is psd it can be expressed as $A := BB^\mathsf{T}$ for some $B \in \mathbb{R}^{n \times m}$. Hence $x^\mathsf{T} A x = x^\mathsf{T} B B^\mathsf{T} x = \|B^\mathsf{T} x\|_2^2$.

Let $y = B^\mathsf{T} x$. Then the set $C := \{(y,t) \in \mathbb{R}^{m+1} \mid \|y\|_2 \leq t\}$ is a (convex) SOC. Hence the set $D := \{y \in \mathbb{R}^m \mid \|y\|_2 \leq c\}$ is convex since it is the intersection of two convex sets:

$$D \quad = \quad C \cap (\mathbb{R}^m \times \{t = c\})$$

Then $E = f^{-1}(D)$ where $f(x) := B^\mathsf{T} x$ is a linear function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Hence E is convex as desired. $\square$

For several important properties of convex sets see Chapter **??**.

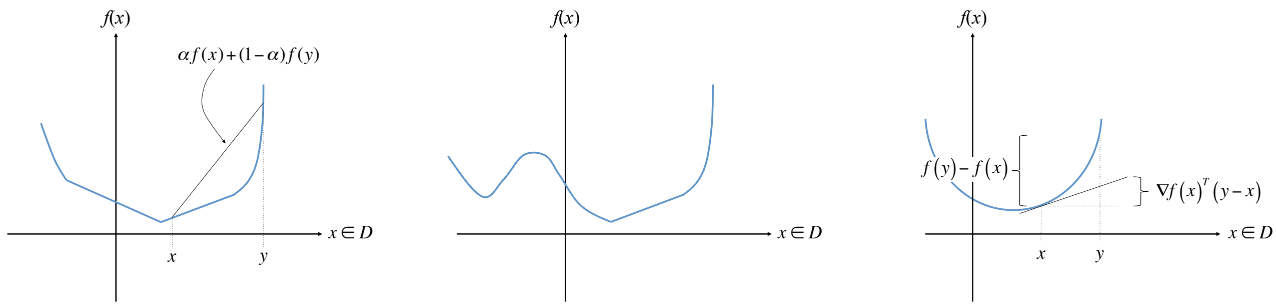
11.1.2 Convex function

Definition 11.2. A function $f : D \to \mathbb{R}$ defined over a convex domain $D \subseteq \mathbb{R}^n$ is *convex* if, for all $x, y \in D$ and all $\alpha \in [0,1]$,

$$f(\alpha x + (1-\alpha)y) \quad \leq \quad \alpha f(x) + (1-\alpha) f(y)$$

It is *strictly convex* if the inequality is strict for $x \neq y$ and $\alpha \in (0,1)$. A function f is *concave* (*strictly concave*) if $-f$ is convex (strictly convex). $\square$

The definition says that the straight line connecting $f(x)$ and $f(y)$ lies above the function f between x and y, as illustrated in Figure 11.2(a).



(a) Convex function. (b) Nonconvex function. (c) Differentiable convex function.

Figure 11.2: Definition of a convex function: The straight line connection $f(x)$ and $f(y)$ lies above f between x and y. The linear approximation of a differentiable convex function f lies below f.

Example 11.2. If $f(x) = x^2$ then for any x, y and $\alpha \in [0,1]$

$$\alpha f(x) + (1-\alpha) f(y) - f(\alpha x + (1-\alpha) y) \quad = \quad \alpha(1-\alpha)(x-y)^2 \; > \; 0$$

for $x \neq y$ and $\alpha \in (0,1)$. Hence f is strictly convex. □

Checking if a function is convex by verifying the convexity definition is often difficult. The following theorem provides three different ways to check the convexity of a function. Consider $f : D \to \mathbb{R}$ over a convex domain $D \subseteq \mathbb{R}^n$. Let $\nabla f(x)$ denote the *column* vector of partial derivatives of f (whereas $\frac{\partial f}{\partial x}$ denotes the row vector of partial derivatives). Let

$$\nabla^2 f(x) \; := \; \frac{\partial^2 f}{\partial x^2} \quad := \quad \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]$$

denote the $n \times n$ Hessian matrix.

Theorem 11.1 (Convex function)**.** Consider a function f defined on a convex open domain $D \subseteq \mathbb{R}^n$. The function f is convex if and only if any one of the following holds:

1. For $x \in D$ and all $v \in \mathbb{R}^n$ the function

$$g(t) \quad := \quad f(x + tv) \tag{11.1}$$

   is convex on $\{t \in \mathbb{R} \mid x + tv \in D\}$.

2. For a differentiable function f,

$$f(y) - f(x) \quad \geq \quad \nabla f(x)^{\mathsf{T}} (y - x), \qquad \forall x, y \in D \tag{11.2}$$

3. For a twice differentiable function f,

$$\nabla^2 f(x) \;\; \succeq \;\; 0, \qquad \forall x \in D$$

   i.e., the Hessian matrix is positive semidefinite (all eigenvalues are nonnegative).

□

The condition in Theorem 11.1.1 does not require differentiability of f and says that, if we take any cross section of the surface f defined by (x, v), i.e., from x in the direction of v or $-v$, the corresponding *scalar* function $g(t)$ is convex. The first-order condition in Theorem 11.1.2 says that the function f always lies above its linear approximation, i.e., $f(y)$ is always greater than or equal to the tangent plane to f at any point x. This is illustrated in Figure 11.2(c). The second-order condition in Theorem 11.1.3 roughly says that the gradient at any point x is increasing around x.

Proof of Theorem 11.1.

1. Suppose f is convex. Fix any $x \in D$ and any $v \in \mathbb{R}^n$. We will show that $g(t) := f(x+tv)$ is convex, i.e., for $s < u$ such that $x+sv$ and $x+uv$ are both in D, we have, for any $t := \alpha s + (1-\alpha)u$ with $\alpha \in [0,1]$,

$$g(t) \;\leq\; \alpha g(s) + (1-\alpha) g(u)$$

From Figure 11.3 we have

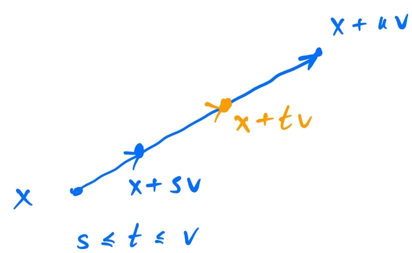


Figure 11.3: Proof of Theorem 11.1.1.

$$x + tv \;=\; \alpha(x+sv) + (1-\alpha)(x+uv)$$

Hence, since f is convex,

$$g(t) \;=\; f(x+tv) \;=\; f\left(\alpha(x+sv) + (1-\alpha)(x+uv)\right) \;\leq\; \alpha g(s) + (1-\alpha) g(u)$$

i.e., g is convex. Conversely suppose g is convex but f is not, i.e., there exists two points $x, y \in D$ and a point $z := (1-\alpha)x + \alpha y$, $\alpha \in [0,1]$, in between such that

$$f(z) \;>\; (1-\alpha) f(x) + \alpha f(y)$$

Define $g(t) := x + tv$ where $v := y - x$. Then $z = x + \alpha v$ and, since g is convex,

$$f(z) \;=\; g(\alpha) \;\leq\; (1-\alpha) g(0) + \alpha g(1) = (1-\alpha) f(x) + \alpha f(y)$$

contradicting that f is not convex.

2. We first prove the result for a scalar differentiable function $g : \mathbb{R} \to \mathbb{R}$. Then we use the result to prove the theorem for a differentiable function $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$ with $n \geq 1$.

   Consider first $g : \mathbb{R} \to \mathbb{R}$. We prove that the following are equivalent:

(a) g is convex.

(b) $g(t) - g(s) \geq g'(s)(t-s)$ for any $s \neq t \in \mathbb{R}$.

(c) $g'(t) \geq g'(s)$ for any $t \geq s$ in $\mathbb{R}$, i.e. g has nondecreasing slope.

Suppose (a): g is convex. Fix any $s, t \in D$. For any $\alpha \in [0,1]$ we have $g(s + \alpha(t-s)) \leq (1-\alpha)g(s) + \alpha g(t)$ and hence

$$g(t) - g(s) \quad \geq \quad \frac{g(s+\alpha(t-s)) - g(s)}{\alpha}$$

Taking limit

$$\lim_{\alpha \downarrow 0} \frac{g(s+\alpha(t-s)) - g(s)}{\alpha(t-s)} \, (t-s) \quad = \quad g'(s)\,(t-s)$$

we have (b). Conversely suppose (b) and we want to prove (a), i.e.

$$\alpha g(t) + (1-\alpha) g(s) \; - \; g(z) \quad \geq \quad 0 \tag{11.3}$$

for any $z := s + \alpha(t-s)$, $\alpha \in [0,1]$. Compare the difference $g(t) - g(z)$ and $g(s) - g(z)$ in terms of gradient at the common point z:

$$g(t) - g(z) \; \geq \; g'(z)(t-z) \quad \text{and} \quad g(s) - g(z) \; \geq \; g'(z)(s-z)$$

To obtain (11.3), multiply the first inequality by α and the second inequality by $1-\alpha$ and sum, noting that $t - z = (1-\alpha)(t-s)$ and $s - z = -\alpha(t-s)$ so that the right-hand sides of these two inequalities sum to zero. This proves (a) $\Leftrightarrow$ (b).

Now suppose (b). Fix any $t \geq s$ and compare $g(t) - g(s)$ in terms of slope at s and at t:

$$g'(s)(t-s) \quad \leq \; g(t) - g(s) \; \leq \quad g'(t)(t-s)$$

yielding (c). Conversely suppose (c) and fix any $t \geq s$. By the mean value theorem we have, for some $z \in [s,t]$, $g(t) - g(s) \; = \; g'(z)(t-s) \; \geq \; g'(s)(t-s)$, which is (b). This proves (b) $\Leftrightarrow$ (c).

Now consider $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$ with $n \geq 1$. We use the result above on scalar functions to prove the theorem. Suppose f is convex and fix any $x, y \in D$. Define the scalar function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(s) \quad := \quad f(x + sy), \qquad \text{for } s \in \mathbb{R} \text{ such that } x + sy \in D \tag{11.4}$$

It is easy to show that $g(s)$ is convex. By the mean value theorem there exists an $s \in [0,1]$ such that

$$f(x+y) - f(x) \quad = \quad g(1) - g(0) \quad = \quad g'(s)$$

By (c) above we have $g'(s) \geq g'(0) = (\nabla f(x))^{\mathsf{T}} y$ and hence

$$f(x+y) - f(x) \quad \geq \quad (\nabla f(x))^{\mathsf{T}} y$$

establishing (11.2). Moreover if f is strictly convex then the inequalities above are strict.

Conversely suppose (11.2) holds. To prove the convexity of f, use the same proof above for (b) $\Rightarrow$ (a). Take $z := x+\alpha(y-x)$ for any $\alpha \in [0,1]$. We have

$$f(y)-f(z) \;\geq\; (\nabla f(z))^\mathsf{T}(y-z) \quad \text{and} \quad f(x)-f(z) \;\geq\; (\nabla f(z))^\mathsf{T}(x-z)$$

Multiply the first inequality by α and the second inequality by $1-\alpha$ and sum to obtain:

$$\alpha f(y)+(1-\alpha)f(x) \;-\; f(z) \;\;\geq\;\; (\nabla f(z))^\mathsf{T}(\alpha(y-z)-(1-\alpha)(z-x)) \;\;=\;\; 0$$

proving the convexity of f. Moreover if the inequalities above are strict then f is strictly convex.

3. To prove the second-order condition, fix any $x,y \in D$, and define the scalar function $g(s) := f(x+s(y-x))$. Applying the second-order Taylor expansion to g:

$$\begin{aligned} f(y)-f(x) \;&=\; g(1)-g(0) \;=\; g'(0)+\frac{1}{2}g''(s) \\ &=\; (\nabla f(x))^\mathsf{T}(y-x)+\frac{1}{2}(y-x)^\mathsf{T}\nabla^2 f(x+s(y-x))(y-x) \end{aligned}$$

for some $s \in [0,1]$. If $\nabla^2 f(z) \succeq 0$ for all $z \in D$, then $f(y)-f(x) \geq (\nabla f(x))^\mathsf{T}(y-x)$ which is equivalent to the convexity of f from part 2.

Conversely, suppose f is convex but $\nabla^2 f(x) \prec 0$ for some $x \in D$. Then there exists a vector $v \in \mathbb{R}^n$ such that $v^\mathsf{T}\nabla^2 f(x)v < 0$. Since f is convex, part 1 shows that the scalar function $g(t) := f(x+tv)$ is convex in t. Then the proof of part 2(c) shows that, when g is twice differentiable, $g''(t) \geq 0$ for all $t \in \mathbb{R}$ such that $x+tv \in D$. But $g''(t) = v^\mathsf{T}\nabla^2 f(x+tv)v$ and hence $v^\mathsf{T}\nabla^2 f(x)v < 0$ means $g''(0) < 0$, contradicting that g is convex.

□

Theorem 11.1 provides an exact characterization for convexity. For *strict* convexity, the second-order characterization is sufficient but not necessary: e.g., $f(x) = x^4$ is strictly convex but $f''(x) = 0$ at $x = 0$.

Corollary 11.2 (Strictly convex function)**.** Consider a function f defined on a convex open domain $D \subseteq \mathbb{R}^n$.

1. The function f is strictly convex if and only if the function $g(t)$ in (11.1) is strictly convex on $\{t \in \mathbb{R} \mid x+tv \in D\}$.
2. For a differentiable function f, f is strictly convex if and only if strictly inequality holds in (11.2) for $x \neq y$.
3. For a twice differentiable function f, f is strictly convex if $\nabla^2 f(x) \succ 0$ for all $x \in D$.

A common mistake is to confuse the second-order condition in Theorem 11.1.3 that $\nabla^2 f(x)$ is positive semidefinite with the condition that

$$x^\mathsf{T}\nabla^2 f(x)x \;\;\geq\;\; 0 \qquad \text{for all } x \in D$$

For any $x \in D$, $\nabla^2 f(x) \succeq 0$ if and only if

$$y^\mathsf{T}\nabla^2 f(x)y \;\;\geq\;\; 0 \qquad \text{for all } y \in \mathbb{R}^n$$

i.e., regardless of what D is, the test on $\nabla^2 f(x)$ is for *all* $y \in \mathbb{R}^n$. This is illustrated in the next example.

Example 11.3. Consider the function

$$f(x_1, x_2) \;=\; x_1 x_2$$

over the domain

$$D \;:=\; \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0, x_2 > 0\right\}$$

with

$$\nabla^2 f(x) \;=\; \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We have

$$x^{\mathsf{T}} \nabla^2 f(x) x \;=\; 2 x_1 x_2 \;>\; 0 \qquad \text{for all } x \in D$$

This however does *not* imply that f is strictly convex over D. The eigenvalues of $\nabla^2 f(x)$ are 1 and -1, and hence f is neither convex nor concave. Indeed the function value along the direction $x_1 = x_2$ corresponding to the eigenvalue-eigenvector pair $\left(1, [1\ 1]^{\mathsf{T}}\right)$ is given by

$$g(t) \;:=\; f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \;=\; (x_1 + t)(x_2 + t), \qquad t > -\min\{x_1, x_2\}$$

Hence $g(t)$ is convex in t, i.e. f is convex along $x_1 = x_2$. Along the direction $x_1 = -x_2$ corresponding to the eigenvalue-eigenvector pair $\left(-1, [1\ -1]^{\mathsf{T}}\right)$ the function value is

$$g(t) \;:=\; f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + t \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \;=\; (x_1 + t)(x_2 - t), \qquad -x_1 \le t \le x_2$$

Therefore $g(t)$ is concave in t, i.e., f is concave along $x_1 = -x_2$. This is illustrated in Figure 11.4. □

Example 11.4. We illustrate Theorem 11.1 using $f(x) = \log x$ for $x > 0$.

1. We have $f'(x) = x^{-1}$ and for $x \neq y > 0$ (such that $\frac{y}{x} \neq 1$)

$$f(y) - f(x) \;=\; \log\frac{y}{x} \;<\; \frac{y}{x} - 1 \;=\; \frac{1}{x}(y - x) \;=\; f'(x)(y - x)$$

   where the inequality follows from $\log z < z - 1$ for $z > 0$ and $z \neq 1$. Hence f is strictly concave by Theorem 11.1.2.

2. To use Theorem 11.1.3 we have

$$f''(x) \;=\; -\frac{1}{x^2} \;<\; 0$$

   implying strict concavity of f.

□

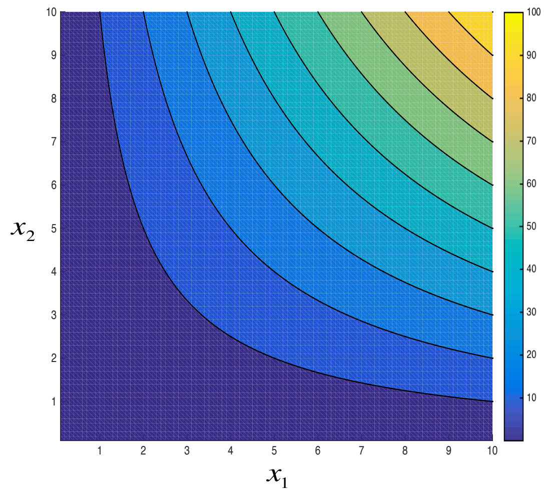


Figure 11.4: Contour plot of $f(x) = x_1x_2$ which is neither convex nor concave over $D := \{(x_1,x_2) \,|\, x_1 > 0, x_2 > 0\}$.

Example 11.5. We illustrate the three sufficient conditions of Theorem 11.1 using the convex $f : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$f(x) \ := \ f(x_1,x_2) \ := \ x_1^2 - 4x_1x_2 + 4x_2^2 \ = \ (x_1 - 2x_2)^2$$

For the first-order condition we have

$$\nabla f(x) \ := \ \nabla f(x_1,x_2) \ = \ 2(x_1 - 2x_2)\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

and hence

$$\begin{aligned} & f(y) - f(x) - \nabla f(x)^{\mathsf{T}}(y - x) \\ = \ & (y_1 - 2y_2)^2 - (x_1 - 2x_2)^2 \ - \ 2(x_1 - 2x_2)((y_1 - x_1) - 2(y_2 - x_2)) \\ = \ & (y_1 - 2y_2)^2 - 2(x_1 - 2x_2)(y_1 - 2y_2) + (x_1 - 2x_2)^2 \\ = \ & ((y_1 - 2y_2) - (x_1 - 2x_2))^2 \ \geq \ 0 \end{aligned}$$

satisfying the condition of Theorem 11.1.2.

For Theorem 11.1.3 we have

$$\nabla^2 f(x) \ = \ 2\begin{bmatrix} 1 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \end{bmatrix}$$

Therefore $\nabla^2 f(x)$ is positive semidefinite as

$$y^{\mathsf{T}} \nabla^2 f(x) y \ = \ 2\left(\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)^2 \ \geq \ 0$$

for any $y \in \mathbb{R}^2$.

For Theorem 11.1.1 we have

$$g(t) \;:=\; f(x+tv) \;=\; ((x_1+tv_1)-2(x_2+tv_2))^2 \;=\; ((v_1-2v_2)t+(x_1-2x_2))^2$$

which is clearly a convex function in t for any fixed x and v. □

The addition, multiplication by a positive constant, and supremum operations preserve convexity. Specifically suppose f_1 and f_2 are two convex functions on the same domain. Then

1. $f := \alpha f_1 + \beta f_2$, $\alpha, \beta \geq 0$, is convex.
2. $f := \max\{f_1, f_2\}$ is convex. In fact $f(x) := \sup_{y\in Y} f(x;y)$ is convex in x for arbitrary set Y, provided that, for every $y \in Y$ fixed, $f(x;y)$ is convex in x.
3. $f(x,y) := |x| + |y|$ defined on $\mathbb{R}^2$ is convex as it can be expressed in terms of the supremum and addition operations $(f(x,y) = \max\{x,-x\} + \max\{y,-y\})$.

Convex functions define another important class of convex sets. Let $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$. If D is a convex set and f a convex function then for each $\alpha \in \mathbb{R}$ the *level set* $\{x \in D \mid f(x) \leq \alpha\}$ is convex. Let $f : D \to \mathbb{R}^m$ where $D \subseteq \mathbb{R}^n$ be a vector-valued function where $f := (f_1, \ldots, f_m)$ with $f_i : D \to \mathbb{R}$. Then the set specified by:

$$X \;:=\; \{x \in D \mid f(x) \leq b\} \quad \text{for some } b \in \mathbb{R}^m$$

is convex if each f_i is convex. This is because the level sets

$$X_i \;:=\; \{x \in D \mid f_i(x) \leq b_i\}, \quad i = 1, \ldots, m$$

are all convex and $X = \cap_{i=1}^m X_i$ and hence is convex since intersection preserves convexity.

11.1.3 Convex program

Consider an optimization problem of the form:

$$\min_x \; f(x) \quad \text{subject to} \quad x \in X \tag{11.5}$$

$X \subseteq \mathbb{R}^n$ is called the *feasible set* and $f : \mathbb{R}^n \to \mathbb{R}$ the *objective function*. The problem (11.5) is called a *convex program/problem* if f is a convex function and X is a convex set. For instance

$$X \;:=\; \{x \in \mathbb{R}^n \mid g(x) \leq b\} \qquad \text{for some } b \in \mathbb{R}^m$$

for a vector-valued convex function $g : \mathbb{R}^n \to \mathbb{R}^m$. An $x \in X$ is called a *feasible solution* of (11.5). A feasible solution x^* that attains the minimum of f over X (i.e., $f(x^*) \leq f(x)$ for all $x \in X$) is called a (global) *optimal solution/optimum* or a (global) *minimizer*. A feasible solution x^* that attains the minimum of f over a neighborhood of x^* (i.e., $f(x^*) \leq f(x)$ for all $x \in B_r(x^*) \cap X$ for some $r > 0$) is called a *local optimal solution/optimum* or a *local minimizer*. By setting $U(x) = -f(x)$, the following maximization problem is called a convex program if $U(x)$ is a concave function and X is a convex set:

$$\max_x \; U(x) \quad \text{subject to} \quad x \in X$$

An optimal solution may not exist and when it does, it may not be unique.

Theorem 11.3 (Existence and uniqueness of primal solution)**.** Consider the problem (11.5).

1. An optimal solution x^* exists if X is nonempty and compact (closed and bounded) and f is continuous.

2. The optimal solution x^* is unique if f is strictly convex.

Theorem 11.3 provides a sufficient condition for the existence of an optimal solution x^* and is a consequence of the Weierstrass theorem. For an exact condition see Theorem 17.21 in Chapter 17.3.3. See Examples 11.7 and 11.8 for cases where the feasible sets are not compact and optimal solutions do not exist, even if the optimal objective value is finite.

Note that the existence of an optimal solution x^* requires only that f be continuous, not necessarily convex. Convexity is important for the efficient computation of an optimal solution. This is because for a convex objective function, local optimality implies global optimality. Moreover only the first-order condition is required to guarantee local optimality. Specifically, for an unconstrained minimization problem

$$\min_{x\in\mathbb{R}^n} \; f(x)$$

a necessary condition for a point x^* to be a *local* minimizer is (assuming f is differentiable)

$$\nabla f(x^*) \;=\; 0$$

If f is convex then this is also sufficient for x^* to be *globally* optimal, as illustrated in Figure 11.2. For constrained minimization problem (11.5) where X is nonempty, closed and convex, the first-order necessary condition for $x^* \in X$ to be a local minimizer becomes: there is a neighborhood $B_r(x^*)$ for some $r > 0$ such that

$$(\nabla f(x^*))^{\mathsf{T}} (x - x^*) \;\geq\; 0 \qquad \forall x \in B_r(x^*) \cap X \tag{11.6}$$

i.e., moving away from x^* to any other feasible point x in $B_r(x^*)$ can only locally increase the function value f. If f is convex then this is both necessary and sufficient for x^* to be *globally* optimal. To see this, suppose (11.6) holds but there is another $\hat{x} \in X$ such that $f(\hat{x}) < f(x^*)$. Consider $z(\alpha) := \alpha\hat{x} + (1-\alpha)x^*$. Since X is convex $z(\alpha)$ is feasible for $\alpha \in [0,1]$. Since f is convex we have, for *any* $\alpha \in (0,1]$,

$$f(z(\alpha)) \;\leq\; \alpha f(\hat{x}) + (1-\alpha) f(x^*) \;<\; f(x^*)$$

But, for small enough $\alpha > 0$ so that $z(\alpha) \in B_r(x^*)$, this contradicts

$$f(z(\alpha)) \;\geq\; f(x^*) + \nabla^{\mathsf{T}} f(x^*)(z(\alpha) - x^*) \;\geq\; f(x^*)$$

where the first inequality follows from Theorem 11.1.2 and the second inequality from (11.6). Hence x^* is globally optimal in X.

Example 11.6 (KKT condition for constrained optimization)**.** Consider

$$\min_{x\in\mathbb{R}} \; f(x) := x^2 \qquad \text{subject to} \quad x \geq a$$

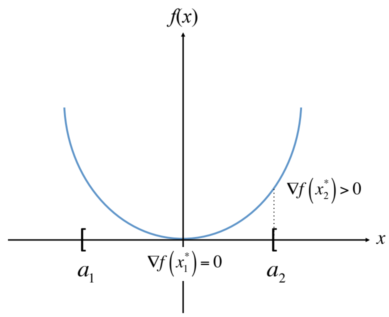


Figure 11.5: Example 11.6: $\min_{x \geq a} x^2$. If $a \leq 0$ then the unique minimizer is $x_1^* = 0$ where $f'(x^*) = 0$. If $a > 0$ then the unique minimizer is $x_2^* = a$ where $f'(x^*) > 0$.

See Figure 11.5. It is clear from the figure that the unique minimizer is 0 where $f'(0) = 0$ if $a \leq 0$ and a where $f'(a) > 0$ if $a > 0$. We will derive this conclusion from the optimality condition (11.6) which is

$$f'(x^*)(x - x^*) \;\; \geq \;\; 0, \qquad \forall x \geq a \tag{11.7}$$

First suppose $a \leq 0$. If $a \leq x^* < 0$ then $f'(x^*) < 0$ and there exists a feasible $x > x^*$ where (11.7) cannot be satisfied. Similarly if $x^* > 0 \geq a$ then $f'(x^*) > 0$ and there exists a feasible $a \leq x < x^*$ where (11.7) cannot be satisfied. Hence the unique optimal is $x^* = 0$ where $f'(x^*) = 0$. Suppose next $a > 0$. Then $f'(x) > 0$ for any feasible $x \geq a$. Then the only way (11.7) can be satisfied is if $x^* = a$.

Therefore the optimality condition reduces for this example (for any $a \in \mathbb{R}$) to: x^* is optimal if and only if there exists a p^* such that

$$x^* \geq a, \;\; p^* \geq 0, \;\; f'(x^*) = p^*, \;\; p^*(x^* - a) = 0$$

This is called the Karush-Kuhn-Tucker (KKT) condition for optimality. We next generalize this optimality condition. □

11.1.4 Dual problem and saddle point theorem

We now study the case where the feasible set $X \subseteq \mathbb{R}^n$ is specified by a set of equality and inequality constraints. Consider

$$f^* \;\; := \;\; \min_{x \in \mathbb{R}^n} \; f(x) \qquad \text{s.t.} \qquad g(x) = 0, \; h(x) \leq 0 \tag{11.8}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ are continuously differentiable, not necessarily convex. We will call this problem the *primal problem*. Associated with every constrained optimization problem (11.8) specified by equality and inequality constraints is a dual problem, defined as follows.

Dual problem. Associated with the equality constraint is the *dual variable* $\lambda \in \mathbb{R}^m$ and associated with the inequality cosntraint is the dual variable $\mu \in \mathbb{R}^l_+$. Define the *Lagrangian function* or the *Lagrangian* associated with the (11.8) as the function $L : \mathbb{R}^{n+m+l} \to \mathbb{R}$:

$$L(x, \lambda, \mu) \;\; := \;\; f(x) \, + \, \lambda^{\mathsf{T}} g(x) \, + \, \mu^{\mathsf{T}} h(x) \tag{11.9a}$$

For any (λ, μ) define the *dual function* by the unconstrained minimization of the Lagrangian over the primal variable x:

$$d(\lambda, \mu) \;:=\; \min_{x\in\mathbb{R}^n} L(x, \lambda, \mu) \tag{11.9b}$$

The *dual problem* of (11.8) is defined to be:

$$d^* \;:=\; \max_{\lambda\in\mathbb{R}^m,\, \mu\in\mathbb{R}^l} d(\lambda, \mu) \quad \text{s.t} \quad \mu \geq 0 \tag{11.9c}$$

Let $X := \{x \in \mathbb{R}^n : g(x) = 0, h(x) \leq 0\}$ denote the primal feasible set and $Y := \{(\lambda, \mu) \in \mathbb{R}^{m+l} : \mu \geq 0\}$ the dual feasible set. A primal feasible point $x^* \in X$ is called *primal optimal* if x^* solves (11.8) and a dual feasible point $(\lambda^*, \mu^*) \in Y$ is called *dual optimal* if (λ^*, μ^*) solves (11.9). We also called such an (x^*, λ^*, μ^*) *primal-dual* optimal.

The dual problem (11.9) always provides a lower bound on the primal problem (11.8).

Theorem 11.4 (Weak duality)**.** If $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in X \times Y$ is a primal-dual feasible point then $d(\bar{\lambda}, \bar{\mu}) \;\leq\; f(\bar{x})$.

Proof. Since $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is primal-dual feasible we have $\bar{\lambda}^\mathsf{T} g(\bar{x}) = 0$ and $\bar{\mu}^\mathsf{T} h(\bar{x}) \leq 0$ and hence $L(\bar{x}, \bar{\lambda}, \bar{\mu}) \leq f(\bar{x})$ from (11.9a). Therefore

$$d(\bar{\lambda}, \bar{\mu}) \;:=\; \min_{x\in\mathbb{R}^n} L(x, \bar{\lambda}, \bar{\mu}) \;\leq\; L(\bar{x}, \bar{\lambda}, \bar{\mu}) \;\leq\; f(\bar{x})$$

as desired. $\square$

The weak duality Theorem 11.4 implies in particular that the dual objective value d^* lower bounds the primal objective value f^*:

$$d^* \;:=\; \max_{\lambda, \mu\geq 0} d(\lambda, \mu) \;\leq\; \min_{x\in X} f(x) \;=:\; f^* \tag{11.10}$$

This holds whether or not the primal problem is convex and whether or not these values are bounded: if the primal optimal value is $f^* = -\infty$ then the dual problem is infeasible; if the dual optimal value is $d^* = \infty$ then the primal problem is infeasible. The gap $f^* - d^*$ is called the *duality gap*. For general nonlinear optimization the duality gap can be strictly positive, and even unbounded. If the primal problem (11.8) is convex then the duality gap is zero provided a constraint qualification is satisfied, as we explain in Chapter 11.1.5.

Saddle point. For the duality gap to be zero and for the primal and dual problems to both attain their optimal values, it is necessary and sufficient that a saddle point exists. To define a saddle point we first claim that the primal problem can be written in terms of L:

$$f^* \;=\; \min_{x} \max_{\lambda, \mu\geq 0} L(x, \lambda, \mu) \tag{11.11}$$

It is important that the minimization over $x \in \mathbb{R}^n$ is unconstrained. We can therefore interpret the dual objective function (11.9b) as converting the constrained primal problem (11.8) into an unconstrained minimization where the primal constraints are replaced by the penalty terms $\lambda^\mathsf{T} g(x) + \mu^\mathsf{T} h(x)$ in the Lagrangian $L(x, \lambda, \mu)$.

To prove (11.11), note that given any infeasible $x \notin X := \{x : g(x) = 0,\ h(x) \le 0\}$, it is clear that $\max_{\lambda,\mu\ge 0} L(x,\lambda,\mu)$ is unbounded. Therefore

$$\min_{x} \max_{\lambda,\mu\ge 0} L(x,\lambda,\mu) \;=\; \min_{x\in X} \max_{\lambda,\mu\ge 0} L(x,\lambda,\mu) \tag{11.12a}$$

Fix any $x \in X$. On the one hand, $L(x,\lambda,\mu) \le f(x)$ for any $\mu \ge 0$, and hence

$$\min_{x\in X} \max_{\lambda,\mu\ge 0} L(x,\lambda,\mu) \;\le\; \min_{x\in X} f(x) \;=:\; f^* \tag{11.12b}$$

On the other hand, $\max_{\lambda,\mu\ge 0} L(x,\lambda,\mu) \ge L(x,\lambda,0) = f(x)$, and hence

$$\min_{x\in X} \max_{\lambda,\mu\ge 0} L(x,\lambda,\mu) \;\ge\; \min_{x\in X} f(x) \;=:\; f^* \tag{11.12c}$$

Then (11.12) proves (11.11).

In view of (11.11), the weak duality (11.10) admits a game interpretation:

$$\max_{(\lambda,\mu)\in Y} \min_{x\in X} L(x,\lambda,\mu) \;\;\le\;\; \min_{x\in X} \max_{(\lambda,\mu)\in Y} L(x,\lambda,\mu) \tag{11.13}$$

This can be interpreted as a two-person game where a player tries to maximize $L(x,\lambda,\mu)$ over $(\lambda,\mu) \in Y$ and the other player tries to minimize $L(x,\lambda,\mu)$ over $x \in X$. The inequality (11.13) expresses the second-mover advantage: the player that makes the first move is generally disadvantaged.

Definition 11.3 (Saddle point)**.** A primal-dual feasible point $(x^*,\lambda^*,\mu^*) \in X \times Y$ is called a *saddle point* of the Lagrangian L if it satisfies

$$\max_{(\lambda,\mu)\in Y} L(x^*,\lambda,\mu) \;\;=\;\; L(x^*,\lambda^*,\mu^*) \;\;=\;\; \min_{x\in X} L(x,\lambda^*,\mu^*)$$

i.e., (x^*,λ^*,μ^*) is a Nash equilibrium of the two-person game (11.13). □

The existence of a saddle point (x^*,λ^*,μ^*) of L is necessary and sufficient for the duality gap to be zero and both the primal and dual problems to attain their optimal values, as stated in Theorem 11.5. This is the case even if none of the functions f,g,h in the primal problem (11.8) are convex.

Theorem 11.5 (Saddle point theorem)**.** A primal-dual feasible point $(x^*,\lambda^*,\mu^*) \in X \times Y$ is a saddle point if and only if

$$d(\lambda^*,\mu^*) \;=\; d^* \;=\; f^* \;=\; f(x^*) \tag{11.14}$$

i.e., if and only if

1. (x^*,λ^*,μ^*) is primal-dual optimal, and hence both the primal and dual objective values (f^*,d^*) are attained.

2. The duality gap is zero.

Proof. Suppose the primal-dual feasible point $(x^*, \lambda^*, \mu^*) \in X \times Y$ is a saddle point. Since $x^* \in X$ we have from the definition (11.9a) of L:

$$f(x^*) \; = \; L(x^*, \lambda, 0) \; \le \; \max_{(\lambda,\mu)\in Y} L(x^*, \lambda, \mu) \; = \; \min_{x\in X} L(x, \lambda^*, \mu^*) \; = \; \min_{x} L(x, \lambda^*, \mu^*) \; = \; d(\lambda^*, \mu^*)$$

where the second equality follows from Definition 11.3 for saddle point and the third equality follows from (11.12a). The weak duality Theorem 11.4 then implies that

$$f(x^*) \; = \; d(\lambda^*, \mu^*)$$

The definition of f^* and d^* and weak duality (11.10) then imply

$$d(\lambda^*, \mu^*) \; \le \; d^* \; \le \; f^* \; \le \; f(x^*) \; = \; d(\lambda^*, \mu^*)$$

which is (11.14).

Conversely suppose the primal-dual feasible point $(x^*, \lambda^*, \mu^*) \in X \times Y$ satisfies (11.14). Since $g(x^*) = 0$ and $\mu^{*\mathsf{T}} h(x^*) \le 0$ we have

$$L(x^*, \lambda^*, \mu^*) \; \le \; f(x^*) \; = \; d(\lambda^*, \mu^*) \; = \; \min_{x} L(x, \lambda^*, \mu^*) \; \le \; \min_{x\in X} L(x, \lambda^*, \mu^*)$$

where the first equality follows from (11.14) and the second equality follows from the definition of the dual objective function d. But $\min_{x\in X} L(x, \lambda^*, \mu^*) \le L(x^*, \lambda^*, \mu^*)$ since $x^* \in X$ and hence $L(x^*, \lambda^*, \mu^*) = \min_{x\in X} L(x, \lambda^*, \mu^*)$, proving the second equality in Definition 11.3 of a saddle point. For the first equality we have

$$L(x^*, \lambda^*, \mu^*) \; \ge \; \min_{x} L(x, \lambda^*, \mu^*) \; = \; d(\lambda^*, \mu^*) \; = \; f(x^*) \; \ge \; \max_{(\lambda,\mu)\in Y} L(x^*, \lambda, \mu)$$

where the first equality follows from the definition of d, second equality from (11.14), and the last inequality because $\lambda^{\mathsf{T}} g(x^*) = 0$ and $\mu^{\mathsf{T}} h(x^*) \le 0$ for any $(\lambda, \mu) \in Y$. This means that (λ^*, μ^*) attains equality in the inequalities above, proving the first equality in Definition 11.3. □

Theorem 11.5 only characterizes a saddle point when it exists. In general a saddle point may not exist. It does if the primal problem (11.8) is convex and a certain constraint qualification is satisfied, as we explain next.

11.1.5 KKT theorem, constraint qualifications, strong duality

KKT condition. The *KKT condition* on (x, λ, μ) associated with the primal and dual problems (11.8)(11.9) is defined by the following system of equations:

$$\begin{aligned}
&\text{Stationarity :} && \nabla_x L(x, \lambda, \mu) \; = \; 0 && (11.15a)\\
&\text{Primal feasibility :} && g(x) \; = \; 0, \quad h(x) \; \le \; 0 && (11.15b)\\
&\text{Dual feasibility :} && \mu \; \ge \; 0 && (11.15c)\\
&\text{Complementary slackness :} && \mu^{\mathsf{T}} h(x) \; = \; 0 && (11.15d)
\end{aligned}$$

where $\nabla_x L$ is the column vector whose ith entry is $\frac{\partial L}{\partial x_i}$. The stationarity (11.15a) is explicitly:

$$\text{Stationarity :} \qquad \nabla f(x) + \nabla g(x)\lambda + \nabla h(x)\mu \ = \ 0$$

A primal variable x^* is called a *stationary point* and a dual variable (λ^*, μ^*) a *Lagrange multiplier* (vector) of (11.8) if (x^*, λ^*, μ^*) satisfies (11.15), i.e.,

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0, \quad g(x^*) = 0, \quad h(x^*) \leq 0, \quad \mu^* \geq 0, \quad \mu^{*\mathsf{T}} h(x^*) = 0 \tag{11.16}$$

We also call such a point $(x^*, \lambda^*, \mu^*) \in X \times Y$ a *KKT point*. If f, g, h are convex functions then a stationary point x^* is an unconstrained minimizer of $L(x, \lambda^*, \mu^*)$ over $x \in \mathbb{R}^n$. Otherwise a stationary point x^* can be a local minimizer, a local maximizer or an inflection point of $L(x, \lambda^*, \mu^*)$. If f, g, h are convex then a primal-dual feasible $(x^*, \lambda^*, \mu^*) \in X \times Y$ satisfies the KKT condition if and only if it is a saddle point, as proved in the next result. Note that the primal problem (11.8) is still nonconvex if $g(x)$ is convex but not affine.

Theorem 11.6 (KKT theorem)**.** Suppose f, g, h are convex and differentiable functions . Consider a primal-dual feasible point $(x^*, \lambda^*, \mu^*) \in X \times Y$. The following are equivalent:

1. (x^*, λ^*, μ^*) is a saddle point.
2. (x^*, λ^*, μ^*) satisfies the KKT condition (11.16).
3. (x^*, λ^*, μ^*) is primal-dual optimal and closes the duality gap, i.e., $d(\lambda^*, \mu^*) = d^* = f^* = f(x^*)$.

Proof. The equivalence of the first and the third assertions is proved in Theorem 11.5. To show the equivalence of the first two assertions, since (x^*, λ^*, μ^*) is primal-dual feasible, we only need to show the complementary slackness condition (11.15d) and the stationarity condition (11.15a).

Suppose (x^*, λ^*, μ^*) is a saddle point, i.e.,

$$\max_{(\lambda,\mu)\in Y} L(x^*, \lambda, \mu) \ = \ L(x^*, \lambda^*, \mu^*) \ = \ \min_{x\in X} L(x, \lambda^*, \mu^*) \tag{11.17}$$

We now show that the first equality implies (11.15d) and the second equality implies (11.15a). The first equality in (11.17) reads, substituting $g(x^*) = 0$,

$$f(x^*) \ + \ \max_{(\lambda,\mu)\in Y} \mu^{\mathsf{T}} h(x^*) \ = \ f(x^*) \ + \ \mu^{*\mathsf{T}} h(x^*)$$

But $\max_{(\lambda,\mu)\in Y} \mu^{\mathsf{T}} h(x^*) = 0$ since $h(x^*) \leq 0$, and hence $f(x^*) = f(x^*) + \mu^{*\mathsf{T}} h(x^*)$, implying the complementary slackness condition $\mu^{*\mathsf{T}} h(x^*) = 0$. This means that $L(x^*, \lambda^*, \mu^*) = f(x^*)$. Hence the second equality in (11.17) reads

$$f(x^*) \ = \ \min_{x\in X} L(x, \lambda^*, \mu^*) \ \geq \ \min_{x} L(x, \lambda^*, \mu^*)$$

Since f, g, h are convex and differentiable functions, the inequality above holds with equality if and only if $\nabla_x L(x, \lambda^*, \mu^*) = 0$. This proves the stationarity condition (11.15a).

Conversely suppose $(x^*, \lambda^*, \mu^*) \in X \times Y$ satisfies the KKT condition (11.16). We now show that the saddle point condition (11.17) is satisfied. First $g(x^*) = 0$ and $\mu^{*\mathsf{T}} h(x^*) = 0$ implies that $f(x^*) = L(x^*, \lambda^*, \mu^*)$. Second, since f, g, h are convex and differentiable functions, the stationarity condition $\nabla_x L(x, \lambda^*, \mu^*) = 0$ implies that $L(x^*, \lambda^*, \mu^*) = \min_x L(x, \lambda^*, \mu^*)$. These two properties combine to yield

$$f(x^*) \;=\; L(x^*, \lambda^*, \mu^*) \;=\; \min_x L(x, \lambda^*, \mu^*) \;\le\; \min_x \max_{(\lambda,\mu)\in Y} L(x, \lambda, \mu) \;=\; \min_{x\in X} \max_{(\lambda,\mu)\in Y} L(x, \lambda, \mu) \;=\; f^*$$

where the third equality follows from (11.12a) and the last equality from (11.11). Since $f(x^*) \ge f^*$ by the definition of f^*, the inequality above must hold with equality. This implies that (λ^*, μ^*) attains the maxima above and hence

$$L(x^*, \lambda^*, \mu^*) \;=\; \min_{x\in X} \max_{(\lambda,\mu)\in Y} L(x, \lambda, \mu) \;=\; \min_{x\in X} L(x, \lambda^*, \mu^*)$$

which is the second equality in (11.17). For the first equality, since $g(x^*) = 0$ and $h(x^*) \le 0$, $f(x^*) \ge f(x^*) + \tilde{\lambda}^{\mathsf{T}} g(x^*) + \tilde{\mu}^{\mathsf{T}} h(x^*)$ for any $(\tilde{\lambda}, \tilde{\mu}) \in Y$. Hence, using again $L(x^*, \lambda^*, \mu^*) = f(x^*)$, we have

$$L(x^*, \lambda^*, \mu^*) \;=\; f(x^*) \;\ge\; \max_{(\tilde{\lambda},\tilde{\mu})\in Y} \; f(x^*) + \tilde{\lambda}^{\mathsf{T}} g(x^*) + \tilde{\mu}^{\mathsf{T}} h(x^*) \;=\; \max_{(\lambda,\mu)\in Y} L(x^*, \lambda, \mu)$$

proving $L(x^*, \lambda^*, \mu^*) = \max_{(\lambda,\mu)\in Y} L(x^*, \lambda, \mu)$. This completes the proof of the theorem. □

Remark 11.1 (Comparison: Saddle point and KKT theorems**).**

1. The saddle point Theorem 11.5 holds without requiring that f, g, h in the primal problem (11.8) be convex or differentiable. It says that a saddle point (x^*, λ^*, μ^*) is primal-dual optimal and closes the duality gap.

2. The KKT Theorem 11.6 requires that f, g, h are convex and differentiable. It implies that, for a primal-dual feasible point (x^*, λ^*, μ^*), the saddle point condition (11.17) is equivalent to stationarity and complementary slackness conditions:

$$\nabla_x L(x^*, \lambda^*, \mu^*) \;=\; 0, \qquad \mu^{*\mathsf{T}} h(x^*) = 0$$

The consequence of $\nabla_x L(x^*, \lambda^*, \mu^*) = 0$ is that x^* is an unconstrained minimizer of L, i.e., $L(x^*, \lambda^*, \mu^*) = \min_x L(x, \lambda^*, \mu^*)$. As mentioned above, (11.8) remains nonconvex if g is convex but not affine.

3. Like Theorem 11.5, Theorem 11.6 only shows that a KKT point (x^*, λ^*, μ^*) is primal-dual optimal and closes the duality gap, but does not guarantee its existence. We now study the existence and uniqueness of a KKT point.

□

Existence and uniqueness of primal optimal x^*. A sufficient condition for the existence of a primal optimal solution x^* is that the cost function $f(x)$ is continuous (not necessarily convex) and the primal feasible set X is compact, according to Theorem 11.3 (see Theorem 17.21 in Chapter 17.3.3 for an exact condition). The optimal solution x^* is unique if the cost function $f(x)$ is strictly convex. See Examples 11.7 and 11.8 for cases where optimal primal solutions do not exist even though f^* is finite and dual optimal solutions exist.

Existence and uniqueness of Lagrange multiplier (λ^*, μ^*)**.** Conditions that guarantee the existence and uniqueness of Lagrange multipliers (λ^*, μ^*) are called constraint qualification conditions. We describe three of them.

Suppose x^* is a local optimal of (11.8). Let $\bar{Y}(x^*)$ be the set of Lagrange multipliers associated with x^*:

$$\bar{Y}(x^*) \ := \ \left\{ (\lambda, \mu) \in \mathbb{R}^{m+l} \ : \ (x^*, \lambda^*, \mu^*) \text{ satisfies (11.16)} \right\}$$

If $\bar{Y}(x^*)$ is nonempty then it is a convex polyhedral set whether or not (11.8) is a convex program. (A set $B \subseteq \mathbb{R}^n$ is a polyhedral set if $B = \{x \in \mathbb{R}^n : Ax \leq b\}$ for some matrix A and vector b of appropriate sizes).

The set $\bar{Y}(x^*)$ of Lagrange multipliers associated with a local optimal x^* is nonempty if and only if the following condition holds at x^*:

$$\text{rank } \frac{\partial g}{\partial x}(x^*) \ = \ m, \qquad \exists \bar{\xi} \in N\left(\frac{\partial g}{\partial x}(x^*)\right) \ \text{ s.t. } \ \frac{\partial h_{\bar{I}}}{\partial x}(x^*)\, \bar{\xi} < 0 \tag{11.18}$$

where $N(A)$ is the null space of matrix A and $\bar{I}$ is the set of indices of inequality constraints that are active at x^*:

$$\bar{I} \ := \ \bar{I}(x^*) \ := \ \{i \ : \ h_i(x^*) = 0\}$$

The condition (11.18) is called the *Mangasarian-Fromovitz constraint qualification* (MFCQ). In particular $\bar{Y}(x^*)$ can be empty if MFCQ is not satisfied. The second condition of MFCQ says that the local optimal x^* can move infinitesimally in the direction of $\bar{\xi}$ and become strictly feasible.

Another condition that guarantees not only existence, but also uniqueness, of Lagrangian multiplier associated with a local optimal x^* is the following:

$$\text{the rows of } \frac{\partial g}{\partial x}(x^*), \ \frac{\partial h_{\bar{I}}}{\partial x}(x^*) \text{ are linearly independent} \tag{11.19}$$

This is called the *linear independence constraint qualification* (LICQ) and it guarantees that $\bar{Y}(x^*)$ is a singleton. Using the Farkas Lemma 17.14 it can be shown that LICQ implies MFCQ (Exercise 11.8).

Both LICQ and MFCQ presumes the existence of an optimal solution x^* for the primal problem (11.8). When x^* exists and if one of the condition is satisfied then a Lagrange multiplier $(\lambda^*, \mu^*) \in \bar{Y}(x^*)$ exists. Theorem 11.6 then implies that (x^*, λ^*, μ^*) is a saddle point that closes the duality gap, and is a KKT point if f, g, h are convex functions, even if g is not affine and (11.8) remains nonconvex.

We next discuss the third constraint qualification, called the Slater condition, that does not require the existence of a primal optimal solution x^*. For the case where g is affine and f, h are convex, the Slater condition ensures that strong duality holds and a Lagrange multiplier (λ^*, μ^*) exists that attains the dual optimal value $d(\lambda^*, \mu^*) = d^*$.

Convex duality. Consider the following convex program:

$$f^* \ := \ \min_{x \in \mathbb{R}^n} \ f(x) \qquad \text{s.t.} \qquad Ax = b, \ \ h(x) \leq 0 \tag{11.20}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ are convex and continuously differentiable, and $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Suppose $h_1(x), \ldots, h_{\bar{l}}(x)$ are affine functions and $h_{\bar{l}+1}(x), \ldots, h_l(x)$ are nonlinear convex functions.

Theorem 11.7 (Strong duality and dual optimality)**.** Consider the convex program (11.20). Suppose

1. *Finite primal value*: f^* is finite, i.e., $-\infty < f^* < \infty$.
2. *Slater condition*: There exists $\bar{x}$ such that

$$A\bar{x} \;=\; b, \qquad h_i(\bar{x}) \;\leq\; 0, \quad i = 1, \ldots, \bar{l}, \qquad h_i(\bar{x}) \;<\; 0, \quad i = \bar{l}+1, \ldots, l$$

Then $f^* = d^*$ and there exists $(\lambda^*, \mu^*) \in Y$ such that $d(\lambda^*, \mu^*) = d^*$. □

The theorem is the smooth version of (the first part of) Theorem Theorem 17.23 in Chapter **??**. A proof is provided there for the similar Theorem 17.22. As will be seen in Chapter 17.4 smoothness is not essential and hence the proof there is directly applicable to Theorem 11.7 here. Since f^* is finite, weak duality implies that the dual problem can only be finite feasible or infeasible. The Slater condition in Theorem 11.7 guarantees that it is feasible and attained. It is often stated as having a strictly feasible point $\bar{x}$ because $\bar{x}$ satisfies the nonlinear inequality constraints strictly. If all $h_i(x)$ are affine then the Slater condition reduces to primal feasibility.

Theorem 11.7 does not guarantee that the primal optimal is attained, i.e., there may not be an $x^* \in X$ such that $f(x^*) = f^*$, as the following two examples show. In both examples the feasible set X is not compact.

Example 11.7 (Nonexistence of primal optimal)**.** Consider

$$f^* \quad := \quad \min_{x \in \mathbb{R}} \; f(x) := x^2 \quad \text{ s.t. } \quad x > 1$$

Clearly both conditions in Theorem 11.7 are satisfied with $f^* = 1$, but no primal optimal x^* exists such that $f(x^*) = f^*$.

The Lagrangian is $L(x, \mu) := x^2 + \mu(1 - x) = x^2 - \mu x + \mu$, the dual function is

$$d(\mu) \;:=\; \min_x L(x, \mu) \;=\; -\frac{\mu^2}{4} + \mu$$

and hence $d^* := \max_{\mu \geq 0} d(\mu) = d(2) = 1 = f^*$, i.e., strong duality holds and $\mu^* = 2$ attains the dual optimal. □

The reason the primal optimal is not attained in Example 11.7 is that the primal feasible set is not closed. The next example possesses a closed (but unbounded) feasible set and has no primal optimal solution either.

Example 11.8 (Nonexistence of primal optimal)**.** Consider

$$f^* \quad := \quad \min_{x \in \mathbb{R}} \; f(x) := e^{-x} \quad \text{ s.t. } \quad x \geq 0$$

Clearly both conditions in Theorem 11.7 are satisfied with $f^* = 0$, but no finite $x^* \in \mathbb{R}$ exists such that $f(x^*) = f^*$.

The Lagrangian is $L(x,\mu) := e^{-x} - \mu x$, the dual function is

$$d(\mu) := \min_x e^{-x} - \mu x = \begin{cases} 0, & \mu = 0 \\ -\infty, & \mu > 0 \end{cases}$$

and hence $d^* := \max_{\mu \geq 0} d(\mu) = d(0) = 0 = f^*$, i.e., strong duality holds and $\mu^* = 0$ attains the dual optimal. □

11.2 Optimization algorithms

Even though OPF can be formulated as an optimization problem in the complex domain using the complex form of power flow equations (e.g., in (13.9) or (13.15) for single-phase OPF in BIM), in computing a solution, it is first converted into a problem in the real domain; see Remark 13.2. OPF can also be formulated directly in the real domain using the polar form (4.22) or the Cartesian form (4.23) of the power flow equations. We therefore present and analyze algorithms for solving OPF in the real domain.

Consider the problem

$$\min_x \ f(x) \quad \text{subject to} \quad x \in X \tag{11.21}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and $X \subseteq \mathbb{R}^n$ is nonempty, closed and convex. Let the column vector $\nabla f(x)$ denote the gradient of f evaluated at x, i.e., $[\nabla f(x)]_i := \partial f / \partial x_i$, $i = 1, \ldots, n$. Recall that a point x^* is a *local minimizer* if $f(x^*)$ is minimum on a neighborhood of x^*, i.e., there exists $r > 0$ such that $f(x^*) \leq f(x)$ for all $x \in B_r(x^*) \cap X$. A necessary optimality condition for general f is: if $x^* \in X$ is a local minimizer for (11.21) then there is a neighborhood $B_r(x^*)$ for some $r > 0$ such that

$$(\nabla f(x^*))^{\mathsf{T}} (x - x^*) \ \geq \ 0 \qquad \forall x \in B_r(x^*) \cap X \tag{11.22}$$

i.e., moving away from x^* to any other feasible point x in $B_r(x^*)$ can only increase the function value f. If f is a convex function (X is assumed convex) then this is both necessary and sufficient for x^* to be a *global* minimum of (11.21). This is illustrated in Figure 11.6.

11.2.1 Steepest descent algorithm

Steepest descent is the most widely used class of iterative algorithms for solving optimization problems. For (11.21), it is given by the following iteration: starting from an initial point $x(0) = x_0$,

$$x(t+1) \ = \ [\, x(t) - \gamma \nabla f(x(t)) \,]_X \tag{11.23}$$

where $\gamma > 0$ is a stepsize. Here $[x]_X$ denotes the projection of x onto the nonempty, closed and convex set X, i.e., for any $x \in \mathbb{R}^n$,

$$[x]_X \ := \ \arg\min_{y \in X} \ \|x - y\|_2 \tag{11.24}$$

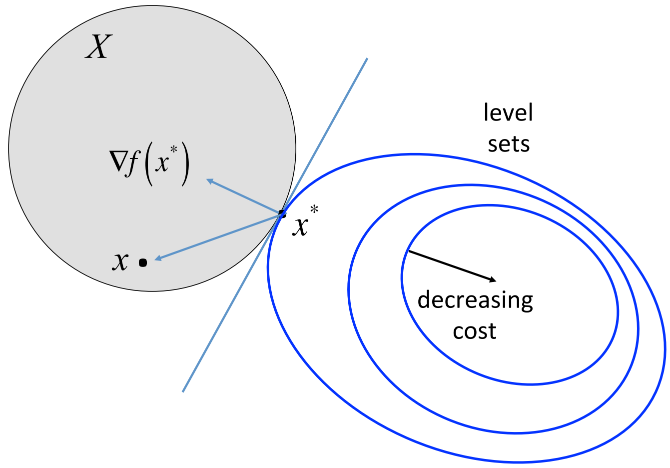


Figure 11.6: Moving away from an optimal point x^* to any other feasible point x can only locally increase the cost.

where $\|\cdot\|_2$ is the *Euclidean* norm. Hence $[x]_X$ is the unique point in X that is closest to $x \in \mathbb{R}^n$ in the Euclidean norm. The steepest descent algorithm is called a first-order algorithm because it uses only the first derivative of the objective function f. A second-order algorithm, such as the Newton-Raphson algorithm widely used for solving optimal power flow problems, also uses the second derivative, as we now explain.

11.2.2 Newton-Raphson algorithm

As explained in Chapter 4.5.2, Newton-Raphson is an iterative algorithm for solving nonlinear equations $F(y) = 0$ where $F : \mathbb{R}^n \to \mathbb{R}^n$. It computes iteratively

$$y(t+1) \;=\; y(t) \;+\; \Delta y(t) \qquad \text{where} \qquad J(y(t))\Delta y(t) \;=\; -F(y(t)) \tag{11.25}$$

where $J(y) := \frac{\partial F}{\partial y}(y)$ is the Jacobian of F at y. In this section we apply it to optimization problems where the equation $F(y) = 0$ represents the KKT condition. A solution y^{opt} of $F(y) = 0$ then produces an optimal solution if the underlying optimization problem is convex. For simplicity we assume solutions exist for all the optimization problems considered unless otherwise specified.

In this subsection we focus on solving problems with equality constraints. Specifically

1. *Linear equality constrained problems*. The idea is to approximate the cost function by a quadratic function around the next iterate (to be determined). This results in a quadratic program in each iteration whose KKT condition is a system of linear equations that can be solved analytically for the next iterate. We will also describe another algorithm that generalizes to nonlinear constraints.

2. *Nonlinear equality constrained problems*. In contrast to the KKT condition of an approximating quadratic program, the KKT condition of these problems is generally nonlinear and cannot be solved analytically. The idea is to solve the KKT condition iteratively using the Newton-Raphson method.

3. *Inequality constrained problems*. The KKT condition of these problems involves inequalities and Newton-Raphson is not directly applicable. The idea is to replace the inequality constraint by a penalty term in the cost function to obtain an approximate problem that has no inequality constraints.

Nonlinear program with linear equality constraint. Consider the following problem with an equality constraint:

$$\min_{x\in\mathbb{R}^n} \quad f(x) \qquad \text{s.t.} \qquad Ax \;=\; b \tag{11.26}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable and $A \in \mathbb{R}^{m\times n}$. We will derive two equivalent algorithms. The first algorithm relies on the linearity of the constraint and is generally not applicable to problems with nonlinear constraints. It approximates the cost function $f(x)$ by a quadratic function in each iteration and solves the resulting quadratic program directly. The second algorithm solves the KKT condition for (11.26) and extends directly to problems with nonlinear equality constraints.

For the first algorithm, given the current iterate $x(t)$, approximate the cost $f(x(t)+\Delta x(t))$ at the *next* iterate by

$$\hat{f}(x(t)+\Delta x(t)) \;:=\; f(x(t)) + \frac{\partial f}{\partial x}(x(t))\,\Delta x(t) + \frac{1}{2}\Delta x(t)^{\mathsf{T}}\frac{\partial^2 f}{\partial x^2}(x(t))\,\Delta x(t) \tag{11.27a}$$

and consider the optimization over $\Delta x(t)$

$$\min_{\Delta x\in\mathbb{R}^n} \quad \hat{f}(x(t)+\Delta x(t)) \qquad \text{s.t.} \qquad A(x(t)+\Delta x(t)) \;=\; b \tag{11.27b}$$

This is a quadratic program in $\Delta x(t)$ with a fixed $x(t)$ and can be solved analytically. Let $\lambda(t) \in \mathbb{R}^m$ be the Lagrange multiplier of (11.27). If f is convex then (11.27) is a convex program and the KKT condition is both necessary and sufficient for optimality. The KKT condition is (Exercise 11.10)

$$\underbrace{\begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x(t)) & A^{\mathsf{T}} \\ A & 0 \end{bmatrix}}_{K(t)} \begin{bmatrix} \Delta x(t) \\ \lambda(t) \end{bmatrix} \;=\; -\underbrace{\begin{bmatrix} \nabla f(x(t)) \\ Ax(t) - b \end{bmatrix}}_{d(x(t))} \tag{11.28a}$$

This is system of $n+m$ linear equations in $n+m$ unknowns $(\Delta x(t), \lambda(t))$. The matrix $K(t)$ on the left-hand side of (11.28a) is called a KKT matrix. If $K(t)$ is nonsingular[1] then $\Delta x(t)$ can be computed directly. If $K(t)$ is singular but the given vector $d(x(t))$ on the right-hand side is orthogonal to the null space of $K(t)$, then there is a subspace of solutions $(\Delta x(t), \lambda(t))$ to (11.28a) and $-K^{\dagger}(t)d(x(t))$ is the minimum-norm solution where $K^{\dagger}(t)$ is the pseudo inverse of $K(t)$. Neither $K(t)$ nor $d(x(t))$ depends on $\lambda(t)$. Hence in both cases $\Delta x(t)$ can be computed from just the current iterate $x(t)$ and (11.28a) always allows pure primal iterations,

$$x(t+1) \;=\; x(t) + \Delta x(t) \tag{11.28b}$$

for solving (11.26).

[1] See [73, Chapter 10.1, p.523] for equivalent conditions of the nonsingularity of the KKT matrix $K(t)$.

The second algorithm does not use the second-order approximation of $f(x)$ and considers (11.26) directly. Specifically let $\lambda \in \mathbb{R}^m$ denote the Lagrange multiplier associated with the m constraints in (11.26). The Lagrangian is

$$L(x;\lambda) \;:=\; f(x) \,+\, \lambda^{\mathsf{T}}(Ax-b)$$

Let $y := (x,\lambda) \in \mathbb{R}^{n+m}$ and define $F : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by

$$F(y) \;:=\; \begin{bmatrix} \nabla_x L(x,\lambda) \\ \nabla_\lambda L(x,\lambda) \end{bmatrix} \;=\; \begin{bmatrix} \nabla f(x) + A^{\mathsf{T}}\lambda \\ Ax-b \end{bmatrix}$$

The KKT condition is $F(y) = 0$. This specifies a system of $n+m$ nonlinear equations in $n+m$ unknowns (x,λ), in contrast to the linear KKT condition (11.28a) for the second-order approximation (11.27). It generally needs to be solved iteratively. The Jacobian $J(y) := \frac{\partial F}{\partial y}$ of F is:

$$J(y) \;=\; \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x) & A^{\mathsf{T}} \\ A & 0 \end{bmatrix}$$

(which is the KKT matrix $K(t)$ in (11.28a).) Hence the Newton-Raphson iteration is

$$\begin{bmatrix} x(t+1) \\ \lambda(t+1) \end{bmatrix} \;=\; \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} \Delta x(t) \\ \Delta\lambda(t) \end{bmatrix} \tag{11.29a}$$

where the increment $\Delta y(t)$ is given by $J(y(t))\,\Delta y(t) = -F(y(t))$, i.e.,

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x(t)) & A^{\mathsf{T}} \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x(t) \\ \Delta\lambda(t) \end{bmatrix} \;=\; -\begin{bmatrix} \nabla f(x(t)) + A^{\mathsf{T}}\lambda(t) \\ Ax(t)-b \end{bmatrix} \tag{11.29b}$$

We compare the two algorithms (11.28) and (11.29). Both algorithms solve a linear equation with the KKT matrix $K(t)$ in each iteration. As mentioned above, the approach of (11.28) solves the KKT condition for the second-order approximation (11.27) directly. This is possible because the linearity of the constraint allows a second-order approximation of only the cost function but not of the constraint, resulting in a quadratic program that can be solved analytically. It leads to a primal algorithm that iterates only on $x(t)$. This is generally inapplicable if the constraint is nonlinear. The approach of (11.29), on the other hand, solves the KKT condition $F(x,\lambda) = 0$ for the original problem (11.26) iteratively using the Newton-Raphson algorithm. It leads to a primal-dual algorithm that updates both the primal and the dual variables. It will be extended to problems with a nonlinear constraint in (11.31).

These two algorithms are equivalent in that both produce the same sequence of $(x(t),\lambda(t))$ starting from the same initial point. Indeed, given the current iterate $(x(t),\lambda(t))$ of the primal and dual variables, $(\Delta x(t),\Delta\lambda(t))$ satisfies (11.29) if and only if $(\Delta x(t),\lambda := \lambda(t)+\Delta\lambda(t))$ satisfies (11.28). To see this, suppose $(\Delta x(t),\lambda(t)+\Delta\lambda(t))$ satisfies (11.28), i.e.,

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x(t)) & A^{\mathsf{T}} \\ A & 0 \end{bmatrix} \left(\begin{bmatrix} \Delta x(t) \\ \Delta\lambda(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda(t) \end{bmatrix} \right) \;=\; -\begin{bmatrix} \nabla_x f(x(t)) \\ Ax(t)-b \end{bmatrix}$$

which yields (11.29). Suppose the converse holds. Write the right-hand side of (11.29) as

$$\begin{bmatrix} \nabla_x f(x(t)) + A^{\mathsf T}\lambda(t) \\ Ax(t) - b \end{bmatrix} \;=\; \begin{bmatrix} \nabla_x f(x(t)) \\ Ax(t) - b \end{bmatrix} + \begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x(t)) & A^{\mathsf T} \\ A & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \lambda(t) \end{bmatrix}$$

which, together with (11.29), yields (11.28). The only difference between these algorithms is that (11.29) computes $\Delta\lambda(t)$ from $(x(t), \lambda(t))$ and forms $\lambda(t+1)$ whereas (11.28) computes $\lambda(t+1)$ directly from $x(t)$.

Nonlinear program with equality constraint. Consider the following problem with a possibly nonlinear equality constraint

$$\min_{x\in\mathbb{R}^n} \quad f(x) \qquad \text{s.t.} \qquad g(x) \;=\; 0 \tag{11.30}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable. The approach of (11.29) generalizes directly to this problem. Let $\lambda \in \mathbb{R}^m$ denote the Lagrange multiplier associated with the m constraints. The Lagrangian is

$$L(x;\lambda) \;:=\; f(x) \,+\, \lambda^{\mathsf T} g(x)$$

Let $y := (x, \lambda) \in \mathbb{R}^{n+m}$ and define $F : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by

$$F(y) \;:=\; \begin{bmatrix} \nabla_x L(x,\lambda) \\ \nabla_\lambda L(x,\lambda) \end{bmatrix} \;=\; \begin{bmatrix} \nabla f(x) \,+\, \frac{\partial g}{\partial x}(x)^{\mathsf T}\lambda \\ g(x) \end{bmatrix} \tag{11.31a}$$

The KKT condition is $F(y) = 0$ which specifies a system of $n+m$ nonlinear equations in $n+m$ unknowns (x, λ). Hence the Jacobian $J(y) := \frac{\partial F}{\partial y}$ of F is:

$$J(y) \;:=\; \begin{bmatrix} \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial \lambda \partial x} \\ \frac{\partial^2 L}{\partial x \partial \lambda} & \frac{\partial^2 L}{\partial \lambda^2} \end{bmatrix} \;=\; \left[\begin{array}{c|c} \frac{\partial^2 f}{\partial x^2}(x) \,+\, \sum_k \frac{\partial^2 g_k}{\partial x^2}\lambda_k & \frac{\partial g}{\partial x}(x)^{\mathsf T} \\ \hline \frac{\partial g}{\partial x}(x) & 0 \end{array}\right] \tag{11.31b}$$

which reduces to the Jacobian in (11.29b) when $g(x) = Ax - b$ is linear. Here $\frac{\partial^2 L}{\partial \lambda \partial x} = \left(\frac{\partial^2 L}{\partial \lambda \partial x}\right)^{\mathsf T}$ is $n \times m$. The Newton-Raphson algorithm for solving (11.30) is the iteration (11.25) where $F(y)$ and its Jacobian $J(y)$ are given by (11.31). It is a primal-dual algorithm that iterates on both $x(t)$ and $\lambda(t)$.

When the cost function $f(x)$ or the feasible set $\{x \in \mathbb{R}^n : g(x) = 0\}$ is nonconvex, there is generally no guarantee that the Newton-Raphson algorithm will converge and if it does, it will produce a local or global optimum. In practice, for OPF problems, the algorithm often converges to a local, and even global, optimum despite their nonconvexity.

When f and g are homogeneous quadratic functions the nonlinear program reduces to the following QCQP with equality constraints:

$$\min_{x\in\mathbb{R}^n} \quad \frac{1}{2}x^{\mathsf T} C_0 x \qquad\qquad \text{s.t.} \qquad\qquad \frac{1}{2}x^{\mathsf T} C_l x \;=\; b_l, \qquad l = 1, \ldots, m$$

where $C_l \in \mathbb{R}^{n\times n}$, $l \geq 0$, are real symmetric matrices and $b_l \in \mathbb{R}$, $l \geq 1$. Then (11.31) reduces to:

$$F(y) \;:=\; \begin{bmatrix} \nabla_x L(y) \\ \nabla_\lambda L(y) \end{bmatrix} \;=\; \left[\begin{array}{c} A(\lambda)^\mathsf{T} x \\ \hline \frac{1}{2} x^\mathsf{T} C_1 x - b_1 \\ \vdots \\ \frac{1}{2} x^\mathsf{T} C_m x - b_m \end{array}\right]$$

where $A(\lambda) := C_0 + \sum_l \lambda_l C_l$ and

$$J(y) \;:=\; \begin{bmatrix} \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial \lambda \partial x} \\ \frac{\partial^2 L}{\partial x \partial \lambda} & \frac{\partial^2 L}{\partial \lambda^2} \end{bmatrix} \;=\; \left[\begin{array}{c|c} A(\lambda)^\mathsf{T} & C_1^\mathsf{T} x \;\; \cdots \;\; C_m^\mathsf{T} x \\ \hline x^\mathsf{T} C_1 & \\ \vdots & 0 \\ x^\mathsf{T} C_m & \end{array}\right]$$

Nonlinear program with inequality constraint. Consider the following problem with an inequality constraint

$$\min_{x\in\mathbb{R}^n} \quad f(x) \qquad \text{s.t.} \qquad g(x) \;\leq\; 0 \tag{11.32}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are twice continuously differentiable. Let $\lambda \in \mathbb{R}^m$ denote the Lagrange multiplier associated with the m constraints. The KKT condition involves inequalities, of the form

$$\begin{aligned} \nabla_x L(x,\lambda) \;&=\; \nabla_x f(x) + \frac{\partial g}{\partial x}(x)^\mathsf{T}\lambda \;=\; 0, \qquad & \nabla_\lambda L(x,\lambda) \;&=\; g(x) \;\leq\; 0 \\ \lambda \;&\geq\; 0, & \lambda^\mathsf{T} g(x) \;&=\; 0 \end{aligned}$$

The standard Newton-Raphson method cannot be applied directly to solve this system of equalities and inequalities. There are however many Newton-like methods that have been developed for inequality constrained problems.

One approach is to introduce a slack variable $z \in \mathbb{R}^m$ and convert (11.32) into a problem with a 'simple' inequality constraint:

$$\min_{(x,z)\in\mathbb{R}^{n+m}} \quad f(x) \qquad \text{s.t.} \qquad g(x) + z \;=\; 0, \quad z \;\geq\; 0$$

Algorithms for solving equality constrained problems can be modified by projecting $z(t)$ to the nonnegative quadrant in each iteration; see e.g. [74]. Another approach is to replace the constraint $g(x) \leq 0$ in (11.32) by a penalty term $(1/t)\phi(x)$ in the cost function and solve the resulting unconstrained approximate problem

$$\min_{x\in\mathbb{R}^n} \quad f(x) \;+\; \frac{1}{t}\phi(x)$$

where $t > 0$ is a parameter that controls the accuracy of the approximation. Newton-Raphson can be applied to solve the optimality condition $\nabla f(x) + (1/t)\nabla\phi(x) = 0$. This is the approach of the interior point methods which we describe next.

11.2.3 Interior-point algorithm

Consider the following problem with an equality and an inequality constraints:

$$\min_{x\in\mathbb{R}^n} \quad f_0(x) \qquad \text{s.t.} \qquad f(x) \le 0, \quad g(x) = 0 \tag{11.33}$$

where $f_0 : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^m$, and $g : \mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable. The idea is to approximate (11.33) by an equality constrained problem by replacing the inequality constraint $f(x) \le 0$ by a penalty term in the cost function, and then solving the equality constrained problem using Newton methods.

Log barrier function. A popular barrier function is $\varphi : \mathbb{R}_- \to \mathbb{R}$ defined by:

$$\varphi_t(u) \; := \; -\frac{1}{t}\log(-u), \qquad u \; < \; 0$$

where $t > 0$ is a parameter. For each $t > 0$, the function $\varphi_t(u)$ is convex increasing over its domain $u < 0$ and approaches ∞ as $u \to 0$. It is an approximation of the indicator function which takes the value 0 if $u \le 0$ and ∞ if $u > 0$. The larger the parameter t is, the more accurate the approximation will be. While the indicator function is discontinuous, the log barrier function $\varphi_t(u)$ is continuously differentiable over its domain $u < 0$ for each $t > 0$.

The *logarithmic barrier* $\phi : \mathbb{R}^n \to \mathbb{R}$ is

$$\phi(x) \; := \; -\sum_{i=1}^{m} \log(-f_i(x)) \tag{11.34a}$$

over the domain

$$\mathrm{dom}\phi \; := \; \{x \in \mathbb{R}^n : f_i(x) < 0, i = 1, \ldots, m\}$$

The log barrier $\phi(x)$ grows without bound as $f_i(x) \to 0$ for any i. Its gradient and Hessian are (Exercise **??**):

$$\nabla\phi(x) \; = \; \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) \tag{11.34b}$$

$$\frac{\partial^2 \phi}{\partial x^2}(x) \; = \; \sum_i \frac{1}{f_i^2(x)} \nabla f_i(x) \nabla f_i^{\mathsf{T}}(x) \; + \; \sum_i \frac{1}{-f_i(x)} \frac{\partial^2 f_i}{\partial x^2}(x) \tag{11.34c}$$

The approximate problem. Fix any $t > 0$. An approximate problem to (11.33) with an equality constraint is

$$\min_{x\in\mathbb{R}^n} \quad f_0(x) \; + \; \frac{1}{t}\phi(x) \qquad \text{s.t.} \qquad g(x) \; = \; 0$$

It is more convenient to consider the following equivalent approximate problem (they have the same minimizers):

$$\text{Problem}(t): \qquad \min_{x\in\mathbb{R}^n} \; t f_0(x) + \phi(x) \qquad \text{s.t.} \qquad g(x) = 0 \tag{11.35}$$

Unlike (11.33) the problem (11.35) has only equality constraints and therefore can be solved using the Newton-Raphson algorithm defined by (11.25)(11.31). If f_0 is convex and g is linear then (11.35) is a convex problem. In that case, if the Newton-Raphson algorithm converges to a solution $(x(t), \lambda(t))$, then the solution satisfies the KKT condition and is therefore primal and dual optimal, i.e., $x(t)$ solves (11.35) and $\lambda(t)$ solves its dual. Otherwise, (11.35) is nonconvex and there is generally no guarantee that the Newton-Raphson algorithm will converge. If it does converge, it will produce a feasible solution but there is no guarantee that it is a local or global optimum. In practice, for OPF problems, the algorithm often converges to a local, and even global, optimum despite nonconvexity.

A popular interior point method, called the barrier method, is based on solving a sequence of the approximate problems (11.35) with increasing t until the approximation is sufficiently accurate. To describe it we first explain how to estimate the gap between the optimal value of the original problem (11.33) and the objective value of a solution of its approximation (11.35).

Suboptimality gap. The theory of the barrier method is most complete for convex problems. For simplicity, we make the following assumptions:

C13.1: The original problem (11.33) is convex, i.e., $f_0, f_1, \ldots, f_m$ are convex functions and $g(x) = Ax - b$ for some $A \in \mathbb{R}^{p\times n}$ and $b \in \mathbb{R}^p$.

C13.2: For every $t > 0$ the approximate problem (11.35) has a unique primal solution $x(t)$ and the Newton-Raphson algorithm converges to $x(t)$.

We call the optimal solution $x(t)$ of (11.35) a *central point* and the set $\{x(t) : t > 0\}$ of central points the *central path*. The assumption of unique $x(t)$ for each $t > 0$ means that there is a unique central path. In this case the barrier method will use the Newton-Raphson algorithm to follow this unique path, as we will see.

Let f_0^* denote the optimal value of the original problem (11.33). The next result shows that a central point $x(t)$ is a feasible solution of (11.33) with a suboptimality gap that is strictly decreasing in $t > 0$. A certificate for the suboptimality gap is provided by a dual feasible solution for (11.33) associated with a central point $x(t)$.

Theorem 11.8 (Central point $x(t)$)**.** Under assumptions C13.1 and C13.2, for each $t > 0$:

1. The central point $x(t)$ is feasible for the original problem (11.33).
2. Its objective value is at most m/t away from the optimal value f_0^*, i.e.,

$$f_0(x(t)) - f_0^* \leq \frac{m}{t}$$

In particular $f_0(x(t)) \to f_0^*$ as $t \to \infty$.

Proof. Since (11.35) is convex by assumption, the optimality of $x(t)$ means there exists an optimal dual variable $\hat{\lambda}(t) \in \mathbb{R}^p$ such that $\left(x(t), \hat{\lambda}(t)\right)$ satisfies the KKT condition for (11.35):

$$t\nabla f_0(x(t)) + \nabla\phi(x(t)) + \frac{\partial g^{\mathsf{T}}}{\partial x}(x(t))\,\hat{\lambda}(t) \;=\; 0, \qquad g(x(t)) \;=\; Ax(t) - b \;=\; 0 \tag{11.36a}$$

This follows from Theorem 17.23 in Appendix 17.4. Because of the log barrier ϕ we must have $f_i(x(t)) < 0$ for all $i = 1, \ldots, m$. This means that $x(t)$ is also (strictly) feasible for the original problem (11.33), i.e., $x(t)$ satisfies

$$f(x(t)) \;<\; 0, \qquad g(x(t)) \;=\; Ax(t) - b \;=\; 0 \tag{11.36b}$$

We now use (11.36) to estimate the suboptimality gap of $x(t)$. Define the Lagrangian of the original problem (11.33)

$$L(x, \mu, \lambda) \;:=\; f_0(x) + \mu^{\mathsf{T}} f(x) + \lambda^{\mathsf{T}} g(x)$$

where the dual variables are $\mu \in \mathbb{R}^m_+$, $\lambda \in \mathbb{R}^p$. Let the dual function be

$$d(\mu, \lambda) \;:=\; \min_{x \in \mathbb{R}^n} L(x, \mu, \lambda)$$

Define

$$\mu_i(t) \;:=\; \frac{1}{-t f_i(x(t))}, \qquad \lambda_i(t) \;:=\; \frac{\hat{\lambda}_i(t)}{t}$$

and let $\mu(t) := (\mu_i(t), i = 1, \ldots, m)$, $\lambda(t) := (\lambda_i(t), i = 1, \ldots, p)$. Since $f_i(x(t)) < 0$, we have $\mu_i(t) > 0$ and hence $(\mu(t), \lambda(t))$ is dual feasible for (11.33). Dividing by t the first condition in (11.36a) and substituting (11.34b) we have

$$\nabla_x L(x, \mu(t), \lambda(t)) \;=\; \nabla f_0(x(t)) + \sum_{i=1}^m \mu_i(t)\nabla f_i(x(t)) + \frac{\partial g^{\mathsf{T}}}{\partial x}(x(t))\,\lambda(t) \;=\; 0$$

which implies that $x(t)$ minimizes $L(x, \mu(t), \lambda(t))$ over x. Hence the dual function evaluated at $(\mu(t), \lambda(t))$ is

$$d(\mu(t), \lambda(t)) \;=\; L(x(t), \mu(t), \lambda(t)) \;=\; f_0(x(t)) + \mu^{\mathsf{T}}(t) f(x(t)) + \lambda^{\mathsf{T}}(t) g(x(t))$$

But $g(x(t)) = 0$ from (11.36a) and $d(\mu(t), \lambda(t)) \le f_0^*$ from weak duality for (11.33). Hence

$$f_0(x(t)) \;-\; f_0^* \;\le\; -\sum_{i=1}^m \mu_i(t) f_i(x(t)) \;=\; \frac{m}{t}$$

as desired. $\square$

The barrier method. Theorem 11.8 says that, when (11.33) is convex, the central point $x(t)$ computed by the Newton-Raphson algorithm is feasible for the original problem (11.33) and its objective value $f_0(x(t))$ is at most m/t away from the optimal value f_0^*. This motivates the *barrier method*, also known as the *path-following method*, that solves Problem(t) in (11.35) to compute a central point $x(t)$, sequentially for increasing $t > 0$.

Specifically the barrier method solves a sequence of the approximate problems (11.35) with increasing $t > 0$, using the solution of the previous problem as the initial point for the current problem, as follows. Fix a parameter $\gamma > 1$ and solve Problem(t) in (11.35) with parameter t using the Newton-Raphson algorithm. Geometrically increase the parameter t by multiplying it by $\gamma > 1$ and solve (11.35) again starting from the solution of the previous problem. Repeat until t is sufficiently large so that the solution produced by Newton-Raphson is an accurate enough solution to the original problem (11.33). This method is described more precisely as Algorithm 3. Even though optimality of the barrier method is guaranteed only when

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Algorithm 3: Barrier method

Input: *strictly* feasible x, initial $t := t_0$, scaling factor $\gamma > 1$, tolerance ε.
Output: an approximate solution x for (11.33).

1. **while** $t \leq \frac{m}{\varepsilon}$ **do**
   (a) Solve Problem(t) in (11.35) to compute $x(t)$ using the Newton-Raphson algorithm starting from x.
   (b) $x \leftarrow x(t)$.
   (c) $t \leftarrow \gamma t$.
2. **Return**: x.

---

the problem is convex and the Newton-Raphson converges for each $t > 0$ (assumptions C13.1 and C13.2), the method is also widely applied to problems that do not satisfy these conditions.

In principle one can solve Problem(t) in (11.35) with parameter $t := m/\varepsilon$ instead of solving a sequence of (11.35) with increasing t as in Algorithm 3. In practice this method does not work well unless the problem is small, the required accuracy ε is moderate and a good starting point is available. Therefore the barrier method is usually preferred.

Strictly feasible initial point. Algorithm 3 requires an initial point x that is *strictly* feasible for the original problem (11.33), i.e. x satisfies

$$f(x) \; < \; 0, \qquad g(x) \; = \; 0$$

There are various methods to produce a strictly feasible point and we explain a simplest one (see [73, Chapter 11.4] for others). When necessary, such a method can be used to compute a strictly feasible x before the barrier method is executed. Starting from such an initial point, all subsequent iterates, across Problem(t) for different t, will remain strictly feasible because of the log barrier ϕ.

Consider the feasibility problem

$$\inf_{(x,s)\in\mathbb{R}^{n+1}} \quad s \qquad \text{s.t.} \qquad f_i(x) \ \leq \ s, \ \ i=1,\ldots,m, \quad g(x) \ = \ 0 \tag{11.37}$$

where $s \in \mathbb{R}^n$, and as before, $f : \mathbb{R}^n \to \mathbb{R}^m$, and $g : \mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable. Suppose we are given an initial x_0 such that $g(x_0) = 0$ and $x_0 \in \text{dom}(f_1) \cap \cdots \cap \text{dom}(f_m)$, i.e., $f_i(x_0) < \infty$, $i = 1,\ldots,m$. Then (11.37) is feasible because (x_0, s_0) is a feasible point with $s_0 := \max_{i=1}^m f_i(x_0)$. Note that the feasible set is closed but not necessarily bounded and hence an optimal point of (11.37) may not exist or the infimum may not be attained by any x.

A strictly feasible point x for (11.33) exists if and only if the optimal value s^{opt} of (11.37) is strictly negative (can be $-\infty$). Indeed solving (11.37) either produces such an x or proves that none exists, according to the sign of s^{opt} (Exercise 11.11):

1. $s^{\text{opt}} < 0$: An x exists that is strictly feasible for (11.33) (hence the minimum s^{opt} of (11.37) is attained).
2. $s^{\text{opt}} > 0$: The problem (11.33) is infeasible, whether or not the minimum s^{opt} of (11.37) is attained.
3. $s^{\text{opt}} = 0$: If the minimum $s^{\text{opt}} = 0$ is attained at an $(x^{\text{opt}}, s^{\text{opt}})$ then (11.33) is feasible but not strictly feasible. Otherwise (i.e., there is no finite x^{opt} that attains $s^{\text{opt}} = 0$) then (11.33) is infeasible. In both cases, Algorithm 3 is not applicable as a strictly feasible x does not exist.

Application to OPF.

11.2.4 Mixed integer linear program / branch and bound methods

LP relaxation.

Dual relaxation.

Branch and bound.

11.2.5 Benders decomposition

11.3 Convergence analysis

Consider the problem (11.21), reproduced here:

$$\min_x \ \ f(x) \quad \text{subject to} \quad x \in X \tag{11.38}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and $X \subseteq \mathbb{R}^n$ is nonempty, closed and convex. We assume:

C11.3: The objective function f is lower bounded on X, continuously differentiable and convex. The feasible set X is nonempty, closed and convex.

C11.3 guarantees that (11.38) is feasible and the gradient based algorithms, such as (11.23), are well defined.

11.3.1 Convergence theorems

In this subsection we prove some basic results that are widely used for convergence analysis of constrained optimization.

We start with the following properties of the projection operation onto a convex set X, defined by (11.24) and reproduced here:

$$[x]_X \quad := \quad \arg\min_{y \in X} \; \|x - y\|_2 \tag{11.39}$$

They are illustrated in Figure 11.7. The Projection Theorem is extremely useful when X is convex. For instance it is used to prove the Separating Hyperplane Theorems 17.11 and 17.12 in Chapter **??**, which are then used to prove the Farkas's Lemma 17.14 which can be used to prove strong duality of linear programs.

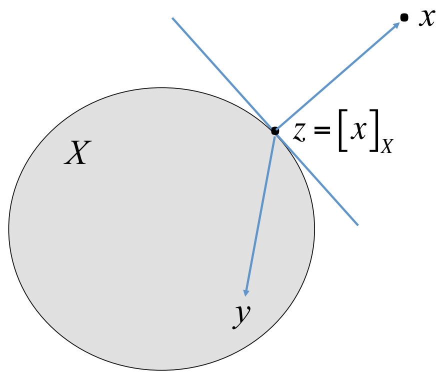

Figure 11.7: The point $z := [x]_X$ is the unique closest point to x in the convex set X under the *Euclidean* norm. For all other points $y \in X$, the inner product of $y - z$ and $x - z$ is nonpositive.

Theorem 11.9 (Projection Theorem)**.** Suppose $X \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set.

1. For every $x \in \mathbb{R}^n$ there exists a unique $[x]_X$ defined by (11.39).
2. For every $x \in \mathbb{R}^n$, $z = [x]_X$ if and only if $z \in X$ and $(y - z)^{\mathsf{T}} (x - z) \;\leq\; 0$ for all $y \in X$.
3. The projection mapping $T : \mathbb{R}^n \to \mathbb{X}$ defined by $T(x) := [x]_X$ is continuous and *nonexpansive* under the Euclidean norm, i.e.,

$$\|[y]_X - [x]_X\|_2 \quad \leq \quad \|y - x\|_2 \qquad \forall x, y \in \mathbb{R}^n$$

Note that Theorem 11.9 does not require X to be bounded (compact), only closed. This is because since X is nonempty there is an $w \in X$. Hence the minimization in the projection (11.39) can be equivalently restricted to the compact set $\{y \in X \mid \|x-y\|_2 \leq \|x-w\|_2\}$.

Since the feasible set X in (11.38) is not necessarily compact (bounded), the optimal may not be attained (e.g., $X = \mathbb{R}$ and $f(x) = e^{-x}$). Moreover the sequence $(x(t), t = 0, 1, \dots)$ generated by the gradient projection algorithm (11.23) may not stay bounded and hence may not have any convergent subsequence (the Bolzano-Weierstrass theorem states that a sequence $(x(t), t = 0, 1, \dots)$ has a convergent subsequence if it is bounded). To guarantee that the gradient projection algorithm makes progress towards minimizing f, we need:

C11.4: The gradient of f is Lipschitz continuous with a Lipschitz constant K, i.e.,

$$\|\nabla f(y) - \nabla f(x)\|_2 \quad \leq \quad K\,\|y-x\|_2 \qquad \forall x, y \in \mathbb{R}^n$$

Note that the norm is Euclidean.[2] C11.4 implies the following useful result which will be used in Theorem 11.13 to prove the optimality of gradient projection algorithm (11.23).

Lemma 11.10 (Descent Lemma.)**.** If $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and satisfies C11.4 then

$$f(x+y) \quad \leq \quad f(x) + y^{\mathsf{T}} \nabla f(x) + \frac{K}{2}\|y\|_2^2 \qquad \forall x, y \in \mathbb{R}^n$$

Proof. We estimate the difference $f(x+y) - f(x)$ by considering the *scalar* function $g(s)$ defined by the intersection of the $f(x)$ surface with the vertical plane at x in the direction y. Fix any $x, y \in \mathbb{R}^n$ and define

$$g(s) \quad := \quad f(x+sy) \qquad \text{for } s \in [0,1]$$

Then

$$f(x+y) - f(x) \quad = \quad g(1) - g(0) \quad = \quad \int_0^1 g'(s)ds$$

Using

$$g'(s) \quad = \quad y^{\mathsf{T}} \nabla f(x+sy)$$

we have

$$\begin{aligned}
f(x+y) - f(x) \quad &= \quad \int_0^1 y^{\mathsf{T}} \nabla f(x+sy) ds \\
&= \quad \int_0^1 \Big(y^{\mathsf{T}} \nabla f(x) \; + \; y^{\mathsf{T}} \left(\nabla f(x+sy) - \nabla f(x) \right) \Big) ds \\
&\leq \quad y^{\mathsf{T}} \nabla f(x) \; + \; \int_0^1 \|y\|_2 \, \|\nabla f(x+sy) - \nabla f(x)\|_2 \, ds \\
&\leq \quad y^{\mathsf{T}} \nabla f(x) \; + \; \|y\|_2 \int_0^1 K \, \|sy\|_2 \, ds \\
&= \quad y^{\mathsf{T}} \nabla f(x) \; + \; \frac{K}{2} \, \|y\|_2^2
\end{aligned}$$

[2] In contrast, the norm that defines a *contraction mapping* can be arbitrary (see Definition 11.4 below).

where the first inequality follows from the Cauchy-Schwarz inequality and the second inequality follows from condition C11.4. $\square$

When f satisfies a stronger form of convexity then the gradient projection algorithm indeed converges and does so geometrically. This is because the stronger form of convexity (condition C11.5 below) implies that the gradient projection algorithm is a contraction mapping, as we now explain.

Definition 11.4 (Contraction)**.** Consider a function $T : X \to X$ from a subset X of $\mathbb{R}^n$ into itself. T is called a *contraction mapping* or simply a *contraction* if there exists an $\alpha \in [0,1)$ such that

$$\|T(y) - T(x)\| \;\; \leq \;\; \alpha \|y - x\| \qquad \forall x, y \in X$$

for an arbitrary norm $\|\cdot\|$.

A function T can be a contraction under a certain norm, but not under a different norm, so the proper choice of norm is critical.

Theorem 11.11 (Contraction mapping theorem)**.** Suppose $T : X \to X$ is a contraction mapping on a closed subset X of $\mathbb{R}^n$. Then

1. There exists a unique fixed point x^* such that $x^* = T(x^*)$.

2. Starting from any initial point $x(0) \in X$, the contraction iteration $x(t+1) := T(x(t))$ converges geometrically to x^*; in particular

$$\|x(t) - x^*\| \;\; \leq \;\; \alpha^{\mathsf{T}} \|x(0) - x^*\| \qquad \forall t \geq 0$$

Proof. Consider the contraction iteration $x(t+1) := T(x(t))$. Definition 11.4 implies

$$\|x(t+1) - x(t)\| \; \leq \; \alpha \|x(t) - x(t-1)\| \;\; \leq \;\; \cdots \;\; \leq \;\; \alpha^{\mathsf{T}} \|x(1) - x(0)\|$$

Hence, for all $t \geq 0$ and $s \geq 1$, we have

$$\begin{aligned} \|x(t+s) - x(t)\| &= \left\| \sum_{m=0}^{s-1} (x(t+m+1) - x(t+m)) \right\| \\ &\leq \sum_{m=0}^{s-1} \|x(t+m+1) - x(t+m)\| \;\; \leq \;\; \|x(1) - x(0)\| \, \alpha^{\mathsf{T}} \sum_{m=0}^{s-1} \alpha^m \\ &\leq \frac{\alpha^{\mathsf{T}}}{1-\alpha} \|x(1) - x(0)\| \end{aligned}$$

Since $\alpha \in [0,1)$, $x(t)$ is a Cauchy sequence and hence must converge to a point x^* in $\mathbb{R}^n$. Since X is closed, $x^* \in X$. Since T is continuous,

$$x^* \;\; = \;\; \lim_t x(t+1) \;\; = \;\; \lim_t T(x(t)) \;\; = \;\; T(\lim_t x(t)) \;\; = \;\; T(x^*)$$

and hence x^* is a fixed point of T. Moreover, the fixed point is unique for, otherwise, if x^* and y^* are both fixed points then

$$\|y^* - x^*\| \;=\; \|T(y^*) - T(x^*)\| \;\leq\; \alpha \|y^* - x^*\|$$

implying $y^* = x^*$ since $\alpha \in [0,1)$. This completes the proof of part 1.

For part 2, we have for all $t \geq 1$,

$$\|x(t) - x^*\| \;=\; \|T(x(t-1)) - T(x^*)\| \;\leq\; \alpha \|x(t-1) - x^*\|$$

Hence $\|x(t) - x^*\| \;\leq\; \alpha^{\mathsf{T}} \|x(0) - x^*\|$. □

Suppose the cost function f is twice continuously differentiable (not just continuously differentiable as guaranteed by condition C11.3). Then f is strictly convex if $\nabla^2 f(x) \succ 0$ for all x according to Corollary 11.2. The curvature of a strictly convex function may be arbitrarily flat, i.e., $y^{\mathsf{T}} \nabla^2 f(x) y > 0$ can be arbitrarily close to zero. A stronger form of convexity bounds this away from zero uniformly in x, i.e., for some $\alpha > 0$, $\nabla^2 f(x) \succeq \alpha I$ for all $x \in \mathbb{R}^n$. Consider:

C11.5: For some $\alpha > 0$, f satisfies

$$(\nabla f(y) - \nabla f(x))^{\mathsf{T}} (y - x) \;\geq\; \alpha \|y - x\|_2^2 \qquad \forall x, y \in \mathbb{R}^n \tag{11.40}$$

We say f is *strongly convex* if it satisfies condition C11.5. The next result shows that it is stronger than strict convexity.

Lemma 11.12 (Strong convexity)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. If f satisfies C11.5 then f is strictly convex. Indeed (11.40) is equivalent to $\nabla^2 f(x) \succeq \alpha I$ for all $x \in \mathbb{R}^n$ when f is twice continuously differentiable.

Proof. We first use Corollary 11.2.2 to prove that if f satisfies C11.5 then f is strictly convex. As in the proof of Lemma 11.10, fix any $x, y \in \mathbb{R}^n$ and consider the (scalar) function along the path from x to y:

$$g(s) \;:=\; f(x + sy) \qquad \text{for } s \in [0,1]$$

Then

$$\begin{aligned}
f(x+y) - f(x) \;&=\; \int_0^1 g'(s) ds \;=\; \int_0^1 y^{\mathsf{T}} \nabla f(x+sy) ds \\
&=\; \int_0^1 \Big(y^{\mathsf{T}} \nabla f(x) \;+\; y^{\mathsf{T}} (\nabla f(x+sy) - \nabla f(x)) \Big) ds \\
&\geq\; y^{\mathsf{T}} \nabla f(x) \;+\; \int_0^1 \frac{1}{s} \alpha \|sy\|_2^2 \, ds \\
&=\; y^{\mathsf{T}} \nabla f(x) \;+\; \frac{\alpha}{2} \|y\|_2^2
\end{aligned}$$

where the inequality follows from C11.5 . Since $\alpha > 0$, Corollary 11.2.2 implies the strict convexity of f.

We now show that if $\nabla^2 f(x) \succeq \alpha I$ for all $x \in \mathbb{R}^n$ then f is strongly convex, i.e., f satisfies C11.5. Fix any x, y and let

$$h(s) \;:=\; \nabla f(x+s(y-x))^{\mathsf{T}}(y-x)$$

Then

$$h'(s) \;=\; (y-x)^{\mathsf{T}}\, \nabla^2 f(x+s(y-x))\,(y-x)$$

and

$$\begin{aligned}(\nabla f(y)-\nabla f(x))^{\mathsf{T}}(y-x) \;&=\; h(1)-h(0) \;=\; \int_0^1 h'(s)ds \\ &=\; \int_0^1 (y-x)^{\mathsf{T}}\, \nabla^2 f(x+s(y-x))\,(y-x)\; ds \\ &\geq\; \alpha \, \|y-x\|_2^2\end{aligned}$$

where the inequality follows from $\nabla^2 f(x) \succeq \alpha I$. Hence $f(x)$ is strongly convex.

Conversely suppose f is strongly convex. To estimate $\nabla^2 f(x)$ we have for any $x, y \in \mathbb{R}^n$

$$\begin{aligned}y^{\mathsf{T}}\, \nabla^2 f(x)\, y \;&=\; \lim_{\lambda \to 0} \frac{1}{\lambda} \left(\frac{\partial f}{\partial x}(x+\lambda y) - \frac{\partial f}{\partial x}(x) \right) y \\ &\geq\; \lim_{\lambda \to 0} \frac{1}{\lambda^2} \left(\alpha \, \|\lambda y\|_2^2 \right) \;=\; \alpha \, \|y\|_2^2\end{aligned}$$

where the inequality follows from the strong convexity of f. Hence $\nabla^2 f(x) \succeq \alpha I$ as desired. This completes the proof of Lemma 11.12. □

If a function f satisfies both C11.4 (Lispschit continuity of ∇f with parameter K) and C11.5 (strong convexity of f with parameter α) then the proof of Lemma 11.12 and that of Lemma 11.10 show that

$$y^{\mathsf{T}}\, \nabla f(x) + \frac{\alpha}{2} \|y\|_2^2 \;\leq\; f(x+y) - f(x) \;\leq\; y^{\mathsf{T}}\, \nabla f(x) + \frac{K}{2} \|y\|_2^2$$

A consequence is that the gradient projection algorithm (11.23) is a contraction mapping and therefore converges geometrically to the unique optimal point, as we explain next.

11.3.2 Steepest descent algorithm

Conditions C11.3 and C11.4 do not guarantee that the sequence $(x(t), t = 0, 1, \dots)$ generated by the gradient projection algorithm has any convergent subsequence, but if it does then it converges to an optimal point x^* of (11.38) provided the stepsize γ is sufficiently small. This implies that, when f is *strictly* convex so that the optimal point x^* is unique, then $(x(t), t = 0, 1, \dots)$ itself converges to x^*.

Theorem 11.13 (Optimality of gradient projection algorithm)**.** Suppose conditions C11.3 and C11.4 hold, and suppose $0 < \gamma < 2/K$. If the sequence $(x(t), t = 0, 1, \dots)$ produced by the gradient projection algorithm (11.23) has a convergent subsequence $(x(t_k), k = 1, 2, \dots)$ then its limit x^* is an optimal solution of (11.38).

Proof. We prove the theorem in three steps. First we show the sequence $(f(x(t)), t = 0, 1, \dots)$ of objective values produced by the gradient projection algorithm (11.23) converges monotonically. Moreover the difference sequence $(x(t+1) - x(t), t = 0, 1, \dots)$ converges to zero. Specifically, by the Descent Lemma 11.10, we have

$$f(x(t+1)) \;\le\; f(x(t)) \;+\; (\,x(t+1) - x(t)\,)^{\mathsf{T}} \nabla f(x(t)) \;+\; \frac{K}{2} \|x(t+1) - x(t)\|_2^2 \tag{11.41}$$

Theorem 11.9.2 implies that for all t

$$(\,y - x(t+1)\,)^{\mathsf{T}} \,(\, x(t) - \gamma \nabla f(x(t)) - x(t+1)\,) \;\le\; 0 \qquad \forall y \in X \tag{11.42}$$

In particular let $y = x(t)$ and we have, after rearranging,

$$(\,x(t+1) - x(t)\,)^{\mathsf{T}} \nabla f(x(t)) \;\le\; -\frac{1}{\gamma} \|x(t+1) - x(t)\|_2^2$$

Substituting into (11.41) we have

$$f(x(t+1)) \;\le\; f(x(t)) \;-\; \left(\frac{1}{\gamma} - \frac{K}{2}\right) \|x(t+1) - x(t)\|_2^2 \tag{11.43}$$

Hence the sequence $(f(x(t)), t = 0, 1, \dots)$ is strictly decreasing as long as $x(t+1) \neq x(t)$ provided $\gamma < 2/K$. Since f is lower bounded on X (condition C11.3), the sequence $(f(x(t)), t = 0, 1, \dots)$ is bounded and monotone and thus converges. Rearranging (11.43), we also have

$$\|x(t+1) - x(t)\|_2^2 \;\le\; \left(\frac{1}{\gamma} - \frac{K}{2}\right)^{-1} (f(x(t)) - f(x(t+1)))$$

Since $f(x(t))$ converges this means that the *differences* $x(t+1) - x(t)$ converge to zero (though this does not guarantee that $x(t)$ itself converges).

Second suppose there is a subsequence $(x(t_k), k = 1, 2, \dots)$ that converges to x^*. Consider the sequence $(x(t_k+1), k = 1, 2, \dots)$. By Theorem 11.9.3, the iteration $x(t+1) = [x(t) - \gamma \nabla f(x(t))]_X$ defined by (11.23) is a projection and hence a continuous function of $x(t)$. Hence the sequence $(x(t_k+1), k = 1, 2, \dots)$, being the image of a continuous function on $x(t_k)$, also converges. We now show that it converges to x^* as $k \to \infty$. Fix any $\varepsilon > 0$. We have to show that there exists an K such that

$$\|x(t_k+1) - x^*\|_2 \;<\; \varepsilon \qquad \forall k > K$$

Since $x(t_k) \to x^*$ there exists an K' such that

$$\|x(t_k) - x^*\|_2 \;<\; \frac{\varepsilon}{2} \qquad \forall k > K' \tag{11.44a}$$

Step 1 above shows that $x(t_k+1) - x(t_k)$ converges to zero and hence there exists K'' such that

$$\|x(t_k+1) - x(t_k)\|_2 \;<\; \frac{\varepsilon}{2} \qquad \forall k > K'' \tag{11.44b}$$

Combining (11.44) we have for $k > K := \max\{K', K''\}$

$$\|x(t_k+1) - x^*\|_2 \;\leq\; \|x(t_k+1) - x(t_k)\|_2 + \|x(t_k) - x^*\|_2 \;<\; \varepsilon$$

as desired.

Finally note that (11.42) holds for all t. In particular consider $t = t_k, k = 1, 2, \ldots$. Taking $k \to \infty$, (11.42) yields

$$\left(y - \lim_k x(t_k+1)\right)^{\mathsf{T}} \left(\lim_k x(t_k) - \gamma \lim_k \nabla f(x(t_k)) - \lim_k x(t_k+1)\right) \leq 0, \quad \forall y \in X$$

Since f is continuously differentiable and $\lim_k x(t_k) = \lim_k x(t_k+1) = x^*$, we have

$$\gamma\, (y - x^*)^{\mathsf{T}}\, \nabla f(x^*) \;\geq\; 0 \qquad \forall y \in X$$

Hence x^* satisfies the optimality condition (11.22) and is globally optimal since f is a convex function over a convex set X. □

As mentioned above, if the objective function f satisfies both C11.4 (Lispschit continuity of ∇f with parameter K) and C11.5 (strong convexity of f with parameter α) then the mapping defined by the gradient projection algorithm (11.23) is a contraction. Theorem 11.11 then implies that the gradient projection algorithm converges geometrically to the unique optimal solution of (11.38). In particular condition C11.4 guarantees strict descent for sufficiently small stepsize $\gamma > 0$ and condition C11.5 guarantees geometric convergence. The bound $2/K$ on the stepsize γ in Theorem 11.13 depends only on the first-order information (the Lipschitz constant K of the gradient ∇f). The bound $2\alpha/K^2$ on the stepsize γ in Theorem 11.14 depends also on the second-order information α, the strength of the convexity of f.

Theorem 11.14 (Convergence rate of gradient projection algorithm)**.** Suppose conditions C11.3–C11.5 hold. Then there is a unique optimal solution x^* for (11.38) and the gradient projection algorithm (11.23) converges geometrically to x^*, provided the stepsize γ satisfies:

$$\begin{aligned} &\text{if } \alpha < K: \quad && 0 < \gamma < \frac{2\alpha}{K^2} \\ &\text{if } \alpha \geq K: \quad && 0 < \gamma < \frac{\alpha}{K^2} - d \quad \text{or} \quad \frac{\alpha}{K^2} + d \;<\; \gamma \;<\; \frac{2\alpha}{K^2} \end{aligned}$$

where $d := \sqrt{\alpha^2 - K^2}/K^2$. Then

$$\|x(t) - x^*\| \;\leq\; \beta^{\mathsf{T}}\, \|x(0) - x^*\| \qquad \forall t \geq 0$$

where $\beta := \sqrt{K^2\gamma^2 - 2\alpha\gamma + 1} \in (0,1)$.

Proof. The gradient project algorithm (11.23) is the iteration $x(t+1) = T(x(t))$ where $T : X \to X$ is defined by $T(x) := [x - \gamma \nabla f(x)]_X$. We will show that T is a contraction under conditions C11.4 and C11.5. Then the assertions follow from Theorem 11.11.

We have under the Euclidean norm

$$\begin{aligned}\|T(y)-T(x)\|_2^2 &= \|\,[y-\gamma\nabla f(y)]_X \;-\; [x-\gamma\nabla f(x)]_X\,\|_2^2 \\ &\le \|\,(y-x)-\gamma(\nabla f(y)-\nabla f(x))\|_2^2 \\ &= \|y-x\|_2^2 \;-\; 2\gamma(\nabla f(y)-\nabla f(x))^{\mathsf{T}}(y-x) \;+\; \gamma^2\|\nabla f(y)-\nabla f(x))\|_2^2\end{aligned}$$

where the inequality above follows from the fact that the projection operation is nonexpansive (Theorem 11.9.3). Conditions C11.5 and C11.4 guarantee that $(\nabla f(y)-\nabla f(x))^{\mathsf{T}}(y-x)\ge\alpha\|y-x\|_2^2$ and $\|\nabla f(y)-\nabla f(x))\|_2^2\le K^2\|y-x\|_2^2$ respectively. Hence

$$\|T(y)-T(x)\|_2^2 \;\le\; \left(1-2\alpha\gamma+\gamma^2K^2\right)\|y-x\|_2^2$$

Hence T is a contraction if and only if $\beta^2(\gamma):=1-2\alpha\gamma+\gamma^2K^2\in[0,1)$. The function $\beta^2(\gamma)$ is shown in Figure 11.8. Hence the condition on the stepsize γ in the theorem guarantees T is a contraction with

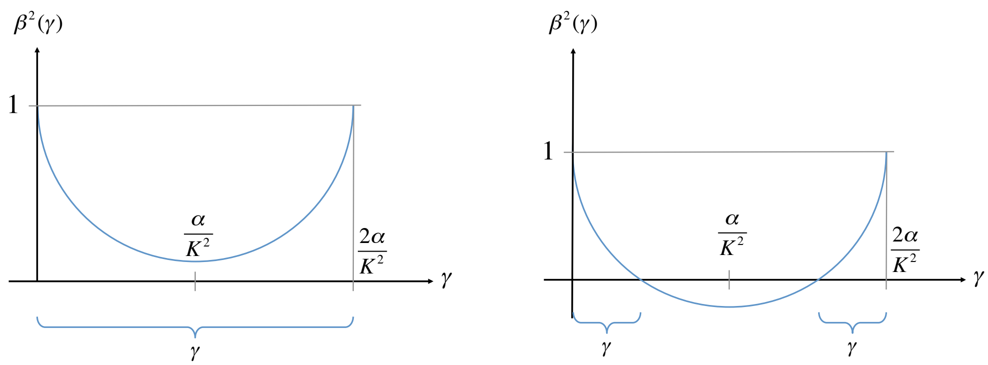

Figure 11.8: Proof of Theorem 11.14. The function $\beta^2(\gamma)$. (a) If $\alpha<K$ then T is a contraction for any stepsize $\gamma\in(0,2\alpha/K^2)$. (b) If $\alpha\ge K$ then T is a contraction if $\gamma\in(0,\alpha/K^2-d)$ or if $\gamma\in(\alpha/K^2+d,2\alpha/K^2)$ where $d:=\sqrt{\alpha^2-K^2}/K^2$.

parameter $\beta(\gamma)\in(0,1)$. Theorem 11.11 then implies that $x(t)$ converges geometrically to x^*. Theorem 11.13 then guarantees that the unique limit point x^* is the optimal solution of (11.38). □

11.3.3 Newton-Raphson algorithm

11.3.4 Interior-point algorithm

11.4 Bibliographical notes

There are many excellent texts on convex analysis and optimization, including nonsmooth convex analysis (e.g. Rockafellar, Clarke, ...). We have used materials from [75, 73, 76]. Many texts in Section 11.1 and the associated exercises are taken from [72, Section 2.1].

Materials in Chapter **??** on convergence theorems are mainly taken from [26]. The Projection Lemma **??** is proved in [26, Proposition 3.2], the Descent Lemma 11.10 in [26, Proposition A.32], and Lemma 11.12 on strong convexity in [26, Proposition A.41].

The materials in Chapters **??** and 17.6 on nonsmooth convex optimization and conic programming respectively mostly follow [76, Chapter 5], [77]. Books on nonsmooth analysis include [78, 77, 79] with [78] focuses more on control theory for applications of nonsmooth analysis and [77, 79] more on nonsmooth convex optimization. The emphasis of [77] is on $\mathbb{R}^n$ whereas that of [79] is on infinite dimensional vector spaces.

11.5 Problems

Chapter 11.1.

Exercise 11.1 (Convex sets)**.** Prove that the following sets are convex:

1. *Affine set:* $C = \{x \in \mathbb{R}^n \mid Ax = b\}$ where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, $m,n \geq 1$.
2. *Second-order cone:* $C = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \leq t\}$, $n \geq 1$. Here $\|x\|_2 := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ is the Euclidean norm.
3. *Semidefinite matrices:* $C = \{A \in \mathbb{S}^{n\times n} \mid A \succeq 0\}$, $n \geq 1$. where $\mathbb{S}^{n\times n}$ is the set of symmetric $n \times n$ real matrices and $A \succeq 0$ means $x^\mathsf{T} Ax \geq 0$ for any $x \in \mathbb{R}^n$. Such a matrix is called *positive semidefinite*.

Exercise 11.2 (Operations preserving set convexity)**.** Operations that preserve convexity are of fundamental importance to the convex optimization theory. Let $\mathbb{X}$ and $\mathbb{Y}$ be linear subspaces. For example $\mathbb{X} := \mathbb{R}^n$ and $\mathbb{Y} := \mathbb{R}^m$.

1. *Linear transformation:* Let $f : \mathbb{X} \to \mathbb{Y}$ be linear. Prove:
   (a) If $A \subseteq \mathbb{X}$ is convex then $f(A) := \{f(x) \mid x \in A\}$ is convex.
   (b) If $B \subseteq \mathbb{Y}$ is convex then $f^{-1}(B) = \{x \in \mathbb{R}^n \mid f(x) \in B\}$ is convex.
2. *Arbitrary direct product:* Let $A \subseteq \mathbb{X}$, $B \subseteq \mathbb{Y}$ be convex.
   (a) Prove that the product space
   $$\mathbb{X} \times \mathbb{Y} := \{(x,y) \mid x \in \mathbb{X},\ y \in \mathbb{Y}\}$$
   with $+$ and $\cdot$ defined by
   $$\begin{aligned} (x_1,y_1) + (x_2,y_2) &:= (x_1 + x_2, y_1 + y_2) \qquad \forall (x_1,y_1),\ (x_2,y_2) \in \mathbb{X} \times \mathbb{Y}; \\ \lambda(x,y) &:= (\lambda x, \lambda y) \qquad \forall \lambda \in \mathbb{R},\ \forall (x,y) \in \mathbb{X} \times \mathbb{Y} \end{aligned}$$
   is also a linear space. For example, if $\mathbb{X} = \mathbb{R}^m$ and $\mathbb{Y} = \mathbb{R}^n$ for some $m,n \geq 1$, then $\mathbb{X} \times \mathbb{Y} = \mathbb{R}^{m+n}$.

(b) Prove that the direct product

$$A \times B := \{(x,y) \mid x \in A, y \in B\}$$

is convex. In fact the direct product of an arbitrary number of convex sets is convex.

3. *Finite sum:* Let $A, B \subseteq \mathbb{X}$ be convex. Prove that the set

$$A + B := \{a + b \mid a \in A, b \in B\}$$

is convex. Therefore the sum of any finite number of convex sets is convex.

4. *Arbitrary intersection:* Let $A, B \subseteq \mathbb{X}$ be convex. Prove that the intersection $A \cap B$ is convex. In fact the intersection of an arbitrary collection of convex sets is convex.

5. *Union can be nonconvex.* Let $A, B \subseteq \mathbb{X}$ be convex. Give an example where the union $A \cup B$ is nonconvex. [Hint: Consider $\mathbb{X} = \mathbb{R}$].

Exercise 11.3 (Convex functions)**.** Prove that the following functions are convex:

1. *Exponential:* $f(x) := e^{ax}$ where $a, x \in \mathbb{R}$.
2. *Entropy:* $f(x) := x \ln x$ defined on $\mathbb{R}_{++} := (0, \infty)$.
3. *Log-exponential:* $f(x_1, x_2) := \ln(e^{x_1} + e^{x_2})$, $x_i \in \mathbb{R}$.

Exercise 11.4 (Convex functions)**.** [73, Exercise 3.6]

For each of the following functions determine if it is convex, concave, or neither.

- $f(x) = e^x - 1$ on $\mathbb{R}$.
- $f(x) = x_1 x_2$ on $\left\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 > 0, x_2 > 0\right\}$.
- $f(x) = \frac{1}{x_1 x_2}$ on $\left\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 > 0, x_2 > 0\right\}$.
- $f(x) = x_1 / x_2$ on $\left\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_1 > 0, x_2 > 0\right\}$.

Exercise 11.5 (Strict convexity)**.** Prove Corollary 11.2.

Exercise 11.6 (Operations preserving function convexity)**.** Prove that addition, multiplication by nonnegative constants, and supremum operations preserve convexity. Specifically suppose f_1 and f_2 are two convex functions on the same domain. Prove that:

1. $f := \alpha f_1 + \beta f_2$, $\alpha, \beta \geq 0$, is convex.
2. $f := \max\{f_1, f_2\}$ is convex.
3. $f(x,y) := |x| + |y|$ defined on $\mathbb{R}^2$ is convex. [Hint: use result in 2.]

Exercise 11.7 (Level set and convex problem)**.** 1. *Level set.* Let $f : C \to \mathbb{R}$ where $C \subseteq \mathbb{R}^n$. Prove that the level set $\{x \in C \mid f(x) \leq \alpha\}$ is convex for any $\alpha \in \mathbb{R}$ provided that C is a convex set and f is a convex function.

2. *Convex problem.* Consider

$$\min_x \ f(x) \quad \text{s.t.} \quad Ax = b, \ g_i(x) \leq 0, \quad i = 1, \ldots, k$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $k \geq 1$, and f, $g_1, \ldots, g_k$ are scalar functions defined on $\mathbb{R}^n$. Prove that if $f, g_1, g_2, \ldots, g_k$ are convex then the feasible set

$$X \ := \ \{x \in \mathbb{R}^n \mid Ax = b, \ g_i(x) \leq 0, i = 1, \ldots, k\}$$

is convex.

Exercise 11.8 (LICQ implies MFCQ)**.** Suppose x^* is a local optimal of the constrained optimization problem (11.8). Let $\bar{Y}(x^*)$ be the set of Lagrange multipliers associated with x^*:

$$\bar{Y}(x^*) \ := \ \left\{ (\lambda, \mu) \in \mathbb{R}^{m+l} \ : \ \frac{\partial L}{\partial x}(x^*, \lambda, \mu) = 0, \ g(x^*) = 0, \ h(x^*) \leq 0, \ \mu \geq 0, \ \mu^\mathsf{T} h(x^*) = 0 \right\}$$

Prove that the linear independence constraint qualification (11.19) implies the Mangasarian-Fromovitz constraint qualification (11.18). (Hint: Use the Farkas Lemma 17.14.)

Exercise 11.9 (KKT condition)**.** This problem derives the KKT condition for the constrained optimization problem:

$$\textbf{(P)} : \qquad \min_{x\in\mathbb{R}^n} \ f(x) \quad \text{s.t.} \quad Ax = b, \ h_i(x) \leq 0, \quad i = 1, \ldots, l$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $k \geq 1$, and f, $h_1, \ldots, h_l$ are scalar functions defined on $\mathbb{R}^n$. Let $\mu \in \mathbb{R}^m, \lambda \in \mathbb{R}^l_+ = [0, \infty)^l$, and define

$$L(x, \lambda, \mu) := f(x) + \lambda^\mathsf{T}(Ax - b) + \mu^\mathsf{T} h(x)$$

where $h(x) = (h_1(x), h_2(x), \ldots, h_l(x))^\mathsf{T}$.

1. *Unconstrained optimization.* Let $d(\lambda, \mu) := \min_{x\in\mathbb{R}^n} L(x, \lambda, \mu)$ denote the unconstrained optimization over x for fixed (λ, μ). Assume that Problem (P) has an optimal solution and denote it by x^*. Show that $d(\lambda, \mu) \leq f(x^*)$ for any $\lambda \in \mathbb{R}^m$ and $\mu \in \mathbb{R}^l_+$.

2. *Dual problem.* Consider the dual problem

$$\textbf{(D)} : \qquad \max_{(\lambda,\mu)\in\mathbb{R}^{m+l}} \; d(\lambda,\mu) \quad \text{s.t.} \quad \mu \geq 0$$

Assume (D) has an optimal solution (λ^*,μ^*).

(a) Show that $d(\lambda^*,\mu^*) - f(x^*) \leq \sum_{i=1}^{l} \mu_i^* h_i(x^*) \leq 0$. It implies that Problem (D) provides a lower bound for Problem (P). Note that this holds whether or not $f, h_1, \ldots, h_l$ are convex.

(b) Assume now $f, h_1, \ldots, h_l$ are convex and differentiable. Show that the equality is attained, i.e., $d(\lambda^*,\mu^*) = f(x^*) + \sum_{i=1}^{l} \mu_i^* h_i(x^*)$, if and only if

$$\nabla_x L(x^*,\lambda^*,\mu^*) = 0$$

(c) Show that if there exists (x,λ,μ) such that x is feasible for (P), (λ,μ) is feasible for (D), $\nabla_x L(x,\lambda,\mu) = 0$, and $\mu_i h_i(x) = 0$ for $i = 1,\ldots,l$, then x solves (P) and (λ,μ) solves (D). These are the KKT conditions.

Chapter 11.2.

Exercise 11.10 (Linear equality constraint)**.** Consider the quadratic program (11.27) over $\Delta x(t)$, with a given $x(t)$, reproduced here:

$$\begin{aligned} \min_{\Delta x\in\mathbb{R}^n} \quad & \hat{f}(x(t)+\Delta x(t)) \qquad \text{s.t.} \qquad A(x(t)+\Delta x(t)) \;=\; b \\ \text{where} \quad & \hat{f}(x(t)+\Delta x(t)) \;:=\; f(x(t)) + \frac{\partial f}{\partial x}(x(t))\,\Delta x(t) + \frac{1}{2}\Delta x(t)^{\mathsf{T}}\frac{\partial^2 f}{\partial x^2}(x(t))\,\Delta x(t) \end{aligned}$$

Let $\lambda \in \mathbb{R}^m$ be the Lagrange multiplier associated with the linear constraint. Show that its KKT condition is given by (11.28):

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(x(t)) & A^{\mathsf{T}} \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x(t) \\ \lambda \end{bmatrix} \;=\; -\begin{bmatrix} \nabla f(x(t)) \\ Ax(t)-b \end{bmatrix}$$

Exercise 11.11 (Interior-point method - strictly feasible point)**.** Consider the following problem to compute a strictly feasible point for (11.33):

$$\min_{(x,s)\in\mathbb{R}^{n+1}} \quad s \qquad \text{s.t.} \qquad f_i(x) \;\leq\; s, \;\; i=1,\ldots,m, \quad g(x) \;=\; 0 \tag{11.45}$$

Assume (11.45) is feasible. Show that such a strictly feasible point exists if and only if the optimal value s^{opt} of (11.45) is strictly negative (possibly $-\infty$), whether or not the minimum of (11.45) is attained.

Chapter 11.3.

Chapter 12

Power system operations

The primary function of a power system is to deliver electricity reliably, and, subject to reliable operation, economically. In Part III we study the mathematical problem of optimal power flow (OPF) that underlies various power system operations. This chapter overviews main operational components and provides context for OPF which will be studied in detail in subsequent chapters. In Part IV we study electricity markets which is an integral part of network operation.

After a brief overview in Chapter 12.1 we describe three control mechanisms at different timescales to balance power supply and demand. In Chapter 12.2 we explain the problem of unit commitment that decides a day in advance which bulk generating units will be turn on the next day. In Chapter 12.3 we explain the problem of optimal dispatch that decides every 5-15 minutes the generation levels of units that are online. Both unit commitment and optimal dispatch are formulated as OPF problems. In Chapter 12.4 we explain frequency control that balances power on a second by second basis and regulates system frequency tightly around its nominal value. Finally in Chapter 12.5 we explain security constrained OPF that schedules responses to contingency events such as the outage of a generator, transmission line or transformer.

12.1 Overview

12.1.1 Operation

Electricity has two important differences from most commodities such as rice and minerals. First there is not yet large-scale energy storage in our power system so that inventory control as a means to match supply and demand for most commodities is not applicable. Instead generation and load must be balanced on a second-by-second basis at all points on the network. Second electricity cannot yet be routed from generators to loads at will but must follow paths determined power flow equations. The nonlinearity of power flow equations introduces computational challenges. These differences have strong implications on how the network is operated and how markets are organized.

The central control problem is to balance supply and demand, continuously and everywhere, without violating operational constraints such as capacity limits of generators and loads, bounds on voltage magnitudes, and thermal and stability limits of transmission lines and transformers. Thermal generators such as gas, coal and nuclear generators still generate the majority of electricity today. For example, in 2020, fossil fuels generated 60.6% and nuclear generated 19.7% of all electricity in the US [80, Table 1.1]. They are fully controllable and can produce a specified amount of electricity at a specified time and location. Traditionally a power system operator forecasts demand, which is assumed inelastic, and schedules bulk generators to meet the forecast demand. As we decarbonize our energy system by replacing fossil fuel generators by wind and solar farms, our ability to control generation decreases and we must also exploit flexibility in demand to match volatile supply. Difficulties arise from the variability and uncertainty of undispatchable demand and supply, the need to match the speed of our control and the speed of disturbances, as well as random unscheduled outages of generators, loads, lines and transformers. Engineering operation and market operation are tightly integrated in a power system. In this chapter we explain network operation. Electricity markets will be studied in Part IV.

A transmission network is a high-voltage long-distance network that connects bulk power producers to power consumers. These consumers are called load centers and represent aggregate loads such as substations of a local utility company that feeds a city. The operation of a transmission network is typically coordinated by an independent system operator that commits and dispatches generation units to meet demand at timescales ranging from hours to minutes to seconds. Bulk generators such as gas, coal, and nuclear generators need nontrivial amounts of time and cost to start up and shut down, e.g., the startup time for a nuclear plant can be hours. This motivates a day-ahead market which usually closes 12–36 hours in advance of energy delivery and determines which generators will be online and their output levels for each hour or half hour over a 24-hour horizon . This is the problem of *unit commitment* and is discussed in Chapter 12.2. The commitment decisions are determined based on forecast of loads and variable generations such as wind and solar power 12–36 hours in advance. A real-time market computes, every 5–15 minutes in advance of energy delivery, adjustments to generation and consumption levels relative to the schedules produced by the day-ahead market as uncertainty in consumption, generation, and network state is resolved. This is the problem of *economic dispatch* and is discussed in Chapters 12.3. Balancing on a second-by-second basis within a real-time dispatch interval takes the form of *frequency control* and is discussed in Chapter 12.4.

12.1.2 Optimal power flow

The problems of day-ahead unit commitment and real-time economic dispatch can be formulated as a constrained optimization of the form

$$\min_{u,x} \ c(u,x) \quad \text{s.t.} \quad f(u,x) = 0, \ \ g(u,x) \le 0$$

This is called an optimal power flow (OPF) problem and it is a basic building block that underlies numerous power system applications. The optimization variable (u,x) consists of control u and network state x and can span multiple time periods, e.g., in unit commitment problems. The cost function c and the constraint functions f, g depend on the application under study. There are usually two types of constraint. The first is power flow equations in various forms studied in Chapters 4 and 5 for single-phase networks and Chapters 9 and 10 for unbalanced multiphase networks. The second type of constraint consists of operational limits such as voltage limits, capacity limits on generators and loads, and thermal and stability limits on transmission lines and transformers.

Part III of this book focuses on OPF especially its computational properties. In Chapters 12.2 and 12.3 we formulate unit commitment and economic dispatch as OPF problems. In Chapter 12.5 we extend the economic dispatch problem to include system security. In Chapter **??** we present other applications of OPF. Finally we describe in Chapter **??** popular algorithms to solve OPF and in Chapter **??** techniques for scalable solution of large practical OPF problems.

12.2 Unit commitment

In this section we formulate the unit commitment problem as OPF. As mentioned above the problem is typically solved by the system operator in the day-ahead market 12–36 hours in advance of energy delivery to decide which units will be turned on for each hour or half hour over a 24-hour period. Integral to the commitment decision is also a dispatch decision that determines the output levels of those units that will be online. The commitment decision is made assuming that the dispatch decision will be optimized at delivery time. This can be formulated as a two-stage optimization problem. For most day-ahead markets, the commitment decision is binding but the dispatch decision can be binding or advisory, to be adjusted by economic dispatch in the real-time market. We will discuss in detail the problem of optimal dispatch in Chapter 12.3, so we will focus on formulating the commitment decision in this section.

Consider a time horizon $T := \{1, 2, \ldots, T\}$ and a power network represented as a graph $G := (\overline{N}, E)$ as before. For example, each time t represents an hour and $T = 24$. For each period $t \in T$ let $u(t) := (u_j(t), j \in \overline{N})$ denote controllable real and reactive power injections at time t, $V(t) := (V_j(t), j \in \overline{N})$ the voltage phasor, $S(t) := \big(S_{jk}(t), S_{kj}(t), (j,k) \in E\big)$ the complex line flows. We call $u(t)$ a *dispatch* and $x(t) := (V(t), S(t))$ a *network state* at time t. Let $u := (u(t), t \in T)$ and $x := (x(t), t \in T)$. They are complex vectors of appropriate sizes. Let $\kappa_j(t) \in \{0, 1\}$ be the binary variable indicating that unit j will be on at time t if $\kappa_j(t) = 1$ and off otherwise. Let $\kappa(t) := \big(\kappa_j(t), j \in \overline{N}\big)$ and $\kappa := (\kappa(t), t \in T)$.

Our OPF formulation includes only two features of the unit commitment problem. The first is injection bounds on a unit when it is turned on. This can be expressed as the constraint:

$$\underline{u}_j(t)\kappa_j(t) \;\leq\; u_j(t) \;\leq\; \overline{u}_j(t)\kappa_j(t), \qquad j \in \overline{N} \tag{12.1a}$$

where $\underline{u}_j(t)$ and $\overline{u}_(t)j$ are given bounds on the active and reactive injections respectively at bus j at time t.[1] The second feature is the startup and shut down costs incurred by a bulk unit when it is turned on or off. This can be expressed as a cost function d_t that is positive when the on/off status of the unit changes:

$$d_{jt}(\kappa_j(t-1), \kappa_j(t)) \;:=\; \begin{cases} \text{startup cost} & \text{if} \quad \kappa_j(t) - \kappa_j(t-1) = 1 \\ \text{shutdown cost} & \text{if} \quad \kappa_j(t) - \kappa_j(t-1) = -1 \\ 0 & \text{if} \quad \kappa_j(t) - \kappa_j(t-1) = 0 \end{cases} \tag{12.1b}$$

Unit commitment problems in practice include many other features. For instance, once turned on or off, a generator must stay in the same on/off state for a minimum amount time. This can be expressed with an additional state variable that keeps track of the time since the last on/off state change.

[1] All variables are complex and, by $a \leq \overline{a}$ where $a, \overline{a} \in \mathbb{C}$, we mean separate bounds on the real and imaginary parts, $\text{Re}\, a \leq \text{Re}\, \overline{a}$ and $\text{Im}\, a \leq \text{Im}\, \overline{a}$.

We illustrate how unit commitment can be posed as an OPF using the simplest formulation that includes only the two features in (12.1). Unit commitment is then the following two-stage optimization problem:

$$\min_{\kappa \in \{0,1\}^{(N+1)T}} \quad \sum_t \sum_j d_{jt}\left(\kappa_j(t-1), \kappa_j(t)\right) \;+\; c^*(\kappa) \tag{12.2a}$$

where the startup/shut down costs d_{jt} are given by (12.1b). Given a commitment decision κ, $c^*(\kappa)$ is the optimal dispatch cost over the entire optimization horizon:

$$\begin{aligned} c^*(\kappa) \;:=\; \min_{(u,x)} \quad & \sum_t c_t(u(t), x(t); \kappa(t)) && (12.2b) \\ \text{s.t.} \quad & f_t(u(t), x(t); \kappa(t)) = 0, \;\; g_t(u(t), x(t); \kappa(t)) \le 0, \quad t \in T && (12.2c) \\ & \tilde{f}(u, x) = 0, \;\; \tilde{g}(u, x) \le 0 && (12.2d) \end{aligned}$$

Here c_t is the dispatch cost, e.g., fuel cost, at time t. The constraints (12.2c) include power flow equations and capacity limits such as (12.1a) at each time t, and the constraints (12.2d) are inter-temporal constraints such as ramp rate limits of the form $|u_j(t) - u_j(t-1)| \le \rho_j$. Hence the commitment decision κ is chosen in (12.2a) in anticipation that the dispatch decisions $(u(t), x(t))$ will be optimized in the second-stage problem (12.2b)(12.2c)(12.2d). The constraint functions $f_t, g_t, \tilde{f}$ may include uncontrollable injections, e.g., forecast loads, as parameters as we will see in Chapter 12.3.

Remark 12.1 (Unit commitment in practice)**.** The unit commitment problem (12.2) is nonconvex and computationally challenging for large networks. Nonconvexity is due both to the binary variable κ and the nonlinear power flow equations. In practice these nonlinear power flow equations are usually replaced by their linear approximations such as the DC power flow model. This reduces the problem to a mixed integer linear program (MILP) and can often be solved within the available time. The solution (κ^*, u^*, x^*) of the MILP however may not satisfy the original nonlinear constraints. Typically the nonlinear power flow model is then used to check if the commitment and dispatch decisions (κ^*, u^*) will produce a state x that satisfies operational constraints such as voltage and line limits. This involves solving nonlinear power flow equations. If operational constraints are violated, the MILP is modified and the procedure is repeated.

Active effort is underway in the R&D community and industry to scale computation methods for mixed integer nonlinear programs to large networks, so that the OPF problem (12.2) can be applied in day-ahead markets. See Chapter 13.5 for an example. □

12.3 Optimal dispatch

After the on-off status of generating units and large controllable loads have been determined by a day-ahead market, a real-time market computes every 5-15 minutes optimal injection levels of those units that are online. This is the problem of optimal, or economic, dispatch. While the control, or dispatch, interval t for unit commitment is typically an hour or half hour, the control, or dispatch, interval t for unit commitment is 5-15 minutes. The most common, and simplest, form of the problem computes an optimal dispatch in each control interval without taking into account decisions in future control intervals. We hence fix a control interval and drop the time index t in our notation.

In this section we formulate the optimal dispatch problem and discuss causes for intra-interval imbalance. In the next section we describe frequency control mechanisms that balance power within a dispatch interval.

12.3.1 OPF formulation

Consider a set of buses $\overline{N}$ and assume there is a generator or controllable load at each bus $j \in \overline{N}$. Let $u := (u_j, j \in \overline{N})$ denote the complex controllable injections, $V := (V_j, j \in \overline{N})$ the voltage phasors, and $S := \left(S_{jk}, S_{kj}, (j,k) \in E\right)$ the complex line flows. We call u a *dispatch* and $x := (V, S)$ a *network state*. They are complex vectors of appropriate sizes. Let $\sigma := \left(\sigma_j, j \in \overline{N}\right)$ be given complex uncontrollable injections. For optimal dispatch the objective function $c(u,x)$ may represent fuel cost which may be convex quadratic in real power generation:

$$c(u,x) \quad = \sum_{\text{generators } j} \left(a_j \left(\text{Re}(u_j)\right)^2 \,+\, b_j \text{Re}(u_j) \right)$$

for some $a_j \geq 0$, $b_j \geq 0$.

The relation between the line flows $S := \left(S_{jk}, (j,k) \in E\right)$ and voltages $V := \left(V_j, j \in \overline{N}\right)$ is specified by the power flow equation

$$S \quad = \quad S(V) \tag{12.3a}$$

where we have abused notation to use S_{jk} to denote both a line flow and a function of voltages. For example we can write the line flow S_{jk} in terms of V in the complex form (4.2) reproduced here:

$$\begin{aligned} S_{jk}(V) &= \left(y^s_{jk}\right)^H \left(|V_j|^2 - V_j V_k^H\right) \,+\, \left(y^m_{jk}\right)^H |V_j|^2, \quad (j,k) \in E \\ S_{kj}(V) &= \left(y^s_{jk}\right)^H \left(|V_k|^2 - V_k V_j^H\right) \,+\, \left(y^m_{kj}\right)^H |V_k|^2, \quad (j,k) \in E \end{aligned}$$

where $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right)$ are series and charging admittances of line (j,k), or in polar form (see (4.22)):

$$\begin{aligned} P_{jk}(V) &= \left(g^s_{jk} + g^m_{jk}\right) |V_i|^2 \,-\, |V_i||V_j| \left(g^s_{jk} \cos(\theta_j - \theta_k) \,-\, b^s_{jk} \sin(\theta_j - \theta_k)\right), \quad (j,k) \in E \\ Q_{jk}(V) &= \left(b^s_{jk} + b^m_{jk}\right) |V_i|^2 \,-\, |V_i||V_j| \left(b^s_{jk} \cos(\theta_j - \theta_k) \,+\, g^s_{jk} \sin(\theta_j - \theta_k)\right), \quad (j,k) \in E \end{aligned}$$

where (g^s_{jk}, b^s_{jk}) and (g^m_{jk}, b^m_{jk}) are series and charging admittances of line (j,k) and $\theta_j := \angle V_j$. Similarly for $\left(P_{kj}(V), Q_{kj}(V)\right)$ in the opposite direction on line (j,k). Different power flow equations lead to different OPF formulations with different computational properties. Then power balance is expressed as[2]

$$u_j \,+\, \sigma_j \quad = \sum_{k:j\sim k} S_{jk}, \qquad j \in \overline{N} \tag{12.3b}$$

The most common operational constraints are:

[2] If $y^s_{jk} = y^s_{kj}$ and $y^m_{jk} = y^m_{kj} = 0$ then we can model the network by a directed graph described by a node-by-line incidence matrix C. In this case (12.3b) takes the form $u + \sigma = CS$.

- *Injection limits* (e.g., generator or load capacity limits):

$$\underline{u}_j \;\le\; u_j \;\le\; \overline{u}_j, \qquad j \in \overline{N} \tag{12.3c}$$

where $\underline{u}_j$ and $\overline{u}_j$ are given bounds on the active and reactive injections respectively at buses j.[3]

- *Voltage limits*:

$$\underline{v}_j \;\le\; |V_j|^2 \;\le\; \overline{v}_j, \qquad j \in \overline{N} \tag{12.3d}$$

where $\underline{v}_j$ and $\overline{v}_j$ are given lower and upper bounds on the squared voltage magnitudes. We assume $\underline{v}_j > 0$ to avoid triviality (in practice $v_j \approx 1$ pu).

- *Line limits*: Thermal limits can be expressed as upper bounds on the magnitudes of line currents, on the magnitudes of real and reactive line power, or on the apparent line power, as:

$$|S_{jk}| \;\le\; \overline{S}_{jk}, \qquad |S_{kj}| \;\le\; \overline{S}_{kj}, \qquad\qquad (j,k) \in E \tag{12.3e}$$

The real-time optimal dispatch problem is then the following constrained optimization

$$\min_{u,x} \;\; c(u,x) \quad \text{ s.t. } \quad (12.3) \tag{12.4}$$

where $(u,x) := (u,V,S) \in \mathbb{C}^{2(N+1+M)}$ and $N+1, M$ are the numbers of buses and lines respectively. It is solved by the system operator for every control interval (e.g., every 5 minutes). We call u a *feasible dispatch* if $(u,x) := (u,V,S)$ satisfies (12.3) for some network state x. We call u^{opt} an *optimal dispatch* if $(u^{\text{opt}}, x^{\text{opt}}) := (u^{\text{opt}}, V^{\text{opt}}, S^{\text{opt}})$ is an optimal solution of (12.4) for some network state x^{opt}. The key parameter of (12.4) is the uncontrollable injection σ in (12.3b). We often abuse notation and write $u^{\text{opt}}(\sigma)$ for an optimal dispatch as a function of σ. We also say that the optimal dispatch $u^{\text{opt}}(\sigma)$ is *driven by* σ.

The interpretation of an optimal $(u^{\text{opt}}, x^{\text{opt}})$ is that the controllable generators and loads will produce and consume according to the dispatch command u^{opt} from the system operator. The injection u^{opt} will drive the voltage V^{opt} and line flow S^{opt} on the network to a solution of the power flow equations (12.3a) (12.3b) that satisfies the operational constraints (12.3c) (12.3d) (12.3e). In particular this should guarantee power balance at all points of the network given an uncontrollable injection σ. The reality is more complicated as we will see in Chapter 12.3.2.

Remark 12.2. We have assumed without loss of generality that there is at most one controllable generator or load at each bus with injection u_j. It is straightforward to extend to the case where there are multiple generators and loads at buses j. If there is no controllable injection at bus j then we can set $\underline{u}_j = \overline{u}_j = 0$ or remove u_j as an optimization variable. □

Remark 12.3 (Economic dispatch in practice)**.** The nonlinearity of power flow equations (12.3a) makes the optimal dispatch problem (12.4) nonconvex and the standard economic theory inapplicable. Most markets today adopt a linear approximation of (12.3a), e.g., the DC power flow model together with methods to determine reactive injections, to compute electricity prices together with a reasonable dispatch

[3] All variables are complex and, by $a \le \overline{a}$ where $a, \overline{a} \in \mathbb{C}$, we mean separate bounds on the real and imaginary parts, $\text{Re}\, a \le \text{Re}\, \overline{a}$ and $\text{Im}\, a \le \text{Im}\, \overline{a}$.

u. This problem is usually called *DC OPF* or *economic dispatch*. Given an optimal dispatch $\hat{u}$ from an economic dispatch problem a system operator may check using AC power flow equations (12.3a) (12.3b) whether a resulting network state $\hat{x} := (\hat{V}, \hat{S})$ satisfies the operational constraints (12.3c) (12.3d) (12.3e), i.e., whether $(\hat{u}, \hat{V}, \hat{S})$ is feasible for (12.4). If it is, then the system operator may price electricity according to a dual optimal solution of the economic dispatch problem (see Part IV) and dispatch the injection $\hat{u}$. Otherwise the system operator may adjust the parameters of the DC OPF problem and repeat the cycle. Even though this procedure may not produce an optimal solution of (12.4) it avoids the complication of nonconvex pricing. We study economic dispatch in detail in Chapter 20. $\square$

12.3.2 Imbalance and error model

Recall that the optimal dispatch problem (12.4) is solved for every control interval. We now describe a simple error model in order to understand how imbalance arises within a control interval even when controllable generators and loads follow the system operator's dispatch. In the next section we explain frequency control mechanisms that correct the imbalance.

Suppose the uncontrollable injection (vector) $\sigma := (\sigma(t), t \in \mathbb{R}_+)$ is a continuous-time stochastic process with the mean process $m(t) := E\sigma(t)$. This can model wind or solar generation or inelastic demand. A realization $\sigma(\xi) := (\sigma(\xi, t), t \in \mathbb{R}_+)$ of the process is indexed by ξ associated with a probability space, though we may omit ξ and use σ or $\sigma(t)$ to refer to a realization when there is no risk for confusion. For each realization ξ and time $t \geq 0$ let $u(\sigma(\xi, t))$ denote an actual injection that can maintain power balance at all points of the network at time t. For instance $u(\sigma(\xi, t))$ is an optimal dispatch driven by the realization $\sigma(\xi, t)$, i.e., there exists a network state $x(\sigma(\xi, t))$ such that $(u(\sigma(\xi, t)), x(\sigma(\xi, t)))$ is an optimal solution of the (deterministic) problem

$$\min_{(u,x):=(s,V,S)} \; c(x) \quad \text{s.t.} \quad \text{(12.3a)(12.3c)(12.3d)(12.3e)} \tag{12.5a}$$

$$u_j \, + \, \sigma_j(\xi, t) \; = \; \sum_{k: j \sim k} S_{jk}, \quad \forall j \tag{12.5b}$$

It is of course impractical to compute such an optimal dispatch for each realization ξ at each time $t \geq 0$.[4]

Instead, a dispatch is computed by the real-time market in each discrete time period $n\delta$, $n = 0, 1, \ldots$, where δ is the duration of each control interval, e.g., $\delta = 5$ minutes. Suppose the system operator's dispatch for the nth control interval is an optimal solution $u^{\text{opt}}(\hat{m}(n))$ of (12.4), or its linear approximation, driven by a certain estimate $\hat{m}(n)$ of the uncontrollable injection $\sigma(\xi, t)$ over the interval. The imbalance at time t is then the difference between the actual injection and the operator's dispatch:

$$\Delta u(\xi, t) \;\; := \;\; u(\sigma(\xi, t)) \; - \; u^{\text{opt}}(\hat{m}(n)), \qquad t \in [n\delta, (n+1)\delta), \; n = 0, 1, \ldots \tag{12.6}$$

The imbalance $\Delta u(\xi, t)$ can be interpreted as consisting of three errors:

$$\Delta u(\xi, t) \;\; = \;\; \Delta_1(\xi, t) \, + \, \Delta_2(t) \, + \, \Delta_3(\xi, t)$$

These errors are:

[4] This will correspond to choosing an equilibrium injection of the dynamic model of Chapter 12.4 at each time t.

1. *Random error* $\Delta_1(\xi,t)$. The optimal dispatch (12.4) is a deterministic problem driven by an estimate of the random injection $\sigma(t)$. If the estimate is the mean process $m(t)$, it will lead to a random imbalance between the actual injection and the dispatch driven by $m(t)$, resulting in a dispatch error $\Delta_1(\xi,t) := u(\sigma(\xi,t)) - u^{\text{opt}}(m(t))$. This assumes however that it were possible to solve for an optimal dispatch $u^{\text{opt}}(m(t))$ at each time $t \geq 0$.

2. *Discretization error* $\Delta_2(t)$. An optimal dispatch however is computed only in each discrete time period $n\delta$, driven by a vector that approximates the behavior of the function $m(t)$ over that interval. Assume this is the time average of $m(t)$ over $[n\delta,(n+1)\delta)$:

$$\bar{m}(n) \quad := \quad \frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} m(t)\,dt, \qquad n = 0,1,\ldots \tag{12.7}$$

   For instance $m_j(t)$ may model the mean uncontrollable load at bus j at time t and $\bar{m}_j(n)$ is then its time average over the nth interval. Approximating the continuous-time mean process $m(t)$ by the discrete-time process $\bar{m}(n)$ leads to a dispatch error $\Delta_2(t) := u^{\text{opt}}(m(t)) - u^{\text{opt}}(\bar{m}(n))$ at each time $t \in [n\delta,(n+1)\delta)$.

3. *Prediction error* $\Delta_3(\xi,t)$. The computation of $\bar{m}(n)$ needs the ensemble average $m(t)$ over $[n\delta,(n+1)\delta)$. This is difficult because the statistics of the stochastic process $\sigma(t)$ may not be known accurately and because the optimal dispatch for the nth control interval must be computed during the $n-1$st interval. The system operator therefore must use an estimate $\hat{m}(n)$ of $\bar{m}(n)$ in (12.3b) when solving the optimal dispatch problem (12.4) for the nth interval. This leads to a dispatch error $\Delta_3(\xi,t) := u^{\text{opt}}(\bar{m}(n)) - u^{\text{opt}}(\hat{m}(n))$ at each time $t \in [n\delta,(n+1)\delta)$.

   In general the estimate $\hat{m}(n) = \hat{m}(\xi,n)$ depends on the realization ξ and is a random variable. For instance a common strategy is to set the estimate to be the uncontrollable injection realized in the previous time interval

$$\hat{m}(\xi,n) \quad := \quad \frac{1}{\delta}\int_{(n-1)\delta}^{n\delta} \sigma(\xi,t)\,dt, \qquad n = 0,1,\ldots \tag{12.8}$$

   e.g., the forecast wind energy in the next period is the actual wind energy in the current period. In this case its mean $E\hat{m}(n) = \bar{m}(n-1)$. The estimate $\hat{m}(n)$ may also be independent of the realization ξ. This is a special case where $\hat{m}(\xi,n)$ is a deterministic quantity for all ξ. For instance the forecast of uncontrollable energy injection over the interval 7:00–7:05pm on Wednesday is the mean energy estimated from historical data for 7:00–7:05pm on Wednesdays.

Typically the random error $\Delta_1(t)$ tends to have zero mean. The time average of the discretization error $\Delta_2(t)$ over each control interval tends to be zero. This means that the energy discrepancy over each control interval due to discretization tends to be small. If the statistics of the uncontrollable injection $\sigma(t)$ is slowly time-varying then the prediction error $\Delta_3(t)$ tends to be small. The next example describes a simple model where these observations can be made precise.

Example 12.1. Consider a 2-bus network described by the DC power flow model. Bus 1 has an uncontrollable load $\sigma := (\sigma(t), t \in \mathbb{R}_+)$ with mean $(m(t), t \in \mathbb{R}_+)$ and bus 2 has a controllable generator with output level $u(t)$. Suppose the generator and line capacities are high so that the injection and line limits are never active. The actual generation $u(\sigma(t)) = -\sigma(t)$ balances the actual load at time t. Since the DC power flow model is lossless the optimal dispatch is simply $u^{\text{opt}}(\hat{m}(n)) = -\hat{m}(n)$ for the nth control interval. It balances the predicted mean load over that interval. Suppose we use the prediction $\hat{m}(n) := \hat{m}(\xi, n)$ given by (12.8). Then the random imbalance at time $t \in [n\delta, (n+1)\delta)$ is

$$\Delta u(\xi,t) \ := \ u(\sigma(t)) \ - \ u^{\text{opt}}(\hat{m}(n)) \ = \ -\sigma(\xi,t) \ + \ \frac{1}{\delta}\int_{(n-1)\delta}^{n\delta} \sigma(\xi,\tau)\,d\tau$$

i.e., the imbalance at time t is the difference between the actual load at time t and the time average load over the pervious interval.

To gain further insight into the imbalance $\Delta u(t)$ and the constituent errors, suppose σ is a (possibly non-stationary) white Gaussian process with mean $E\sigma(t) = m(t)$ and correlation function $K(t,t') = \nu^2$ if $t = t'$ and $K(t,t') = 0$ if $t \neq t'$ for $t,t' \geq 0$. Then, under appropriate integrability assumptions, $w(\tau) := \int_0^\tau \sigma(t)dt$ is a Wiener process with the property that non-overlapping increments are independent Gaussian random variables, i.e., for any $t' < t \leq \tau' < \tau$, the random variables

$$w(t) - w(t') \ := \ \int_{t'}^{t} \sigma(s)ds \qquad \text{and} \qquad w(\tau) - w(\tau') \ := \ \int_{\tau'}^{\tau} \sigma(s)ds$$

are independent and Gaussian with means $\int_{t'}^{t} m(s)ds$ and $\int_{\tau'}^{\tau} m(s)ds$ respectively and variance $\nu^2(t-t')$ and $\nu^2(\tau-\tau')$ respectively. Then the prediction given by (12.8) is

$$\hat{m}(n) \ = \ \frac{1}{\delta}\left(w(n\delta) - w((n-1)\delta)\right)$$

Therefore $(\hat{m}(n), n = 0,1,\dots)$ are independent Gaussian random variables whose means are the time averages of the mean $m(t)$ over the previous control intervals:

$$E\left(\hat{m}(n)\right) \ = \ \frac{1}{\delta}\int_{(n-1)\delta}^{n\delta} m(t)dt \ =: \ \bar{m}(n-1), \qquad n = 1,2,\dots$$

and whose variances are time invariant:

$$\text{var}\left(\hat{m}(n)\right) \ = \ \frac{\nu^2}{\delta}, \qquad n = 1,2,\dots$$

Here $\bar{m}(n)$ is the time average of the mean process $m(t)$ over the nth interval defined in (12.7). The system operator's dispatch $u^{\text{opt}}(\hat{m}(n)) = -\hat{m}(n)$ for the nth interval is a Gaussian random variable with mean $-\bar{m}(n-1)$ and variance ν^2/δ. The actual load $\sigma(t)$ at time t is a Gaussian random variable with mean $m(t)$ and variance ν^2. Note that $\sigma(t)$ and $\hat{m}(n)$ are independent because of the independent increment property of Wierner process. Hence the imbalance $\Delta u(t) = -\sigma(t) + \hat{m}(n)$ at time $t \in [n\delta, (n+1)\delta)$ is a Gaussian random variable with mean and variance

$$E\left(\Delta u(t)\right) \ = \ -m(t) \ + \ \bar{m}(n-1), \qquad \text{var}\left(\Delta u(t)\right) \ = \ \nu^2\left(1 + \frac{1}{\delta}\right)$$

In particular if the Gaussian process σ is stationary then the imbalance $\Delta u(t)$ has zero mean.

We now calculate the various errors underlying the imbalance $\Delta u(t)$. The random error at time t is

$$\Delta_1(\xi,t) \;:=\; u(\sigma(\xi,t)) - u^{\text{opt}}(m(t)) \;=\; -\sigma(\xi,t) + m(t)$$

which is a Gaussian random variable with zero mean and variance ν^2. The discretization error at time $t \in [n\delta, (n+1)\delta)$ is

$$\Delta_2(t) \;:=\; u^{\text{opt}}(m(t)) - u^{\text{opt}}(\bar{m}(n)) \;=\; -m(t) + \bar{m}(n)$$

i.e., $\Delta_2(t)$ is the deviation of the mean process $m(t)$ from its time average $\bar{m}(n)$. The prediction error is

$$\Delta_3(\xi,t) \;:=\; u^{\text{opt}}(\bar{m}(n)) - u^{\text{opt}}(\hat{m}(\xi,n)) \;=\; -\bar{m}(n) + \hat{m}(\xi,n)$$

which is a Gaussian random variable with mean $E(\Delta_3(t)) = -\bar{m}(n) + \bar{m}(n-1)$ and variance ν^2/δ. The imbalance $\Delta u(\xi,t) = \Delta_1(\xi,t) + \Delta_2(t) + \Delta_3(\xi,t)$. These observations are summarized in Table 12.1. We

	Expression	Random Var	Mean	Variance
Random error $\Delta_1(\xi,t)$	$-\sigma(\xi,t)+m(t)$	Gaussian	zero	ν^2
Discretiz. error $\Delta_2(t)$	$-m(t)+\bar{m}(n)$	constant	$-m(t)+\bar{m}(n)$	0
Prediction error $\Delta_3(\xi,t)$	$-\bar{m}(n)+\hat{m}(\xi,n)$	Gaussian	$-\bar{m}(n)+\bar{m}(n-1)$	ν^2/δ
Imbalance $\Delta u(\xi,t)$	$\Delta_1(t)+\Delta_2(t)+\Delta_3(t)$	Gaussian	$-m(t)+\bar{m}(n-1)$	$\nu^2(1+1/\delta)$

Table 12.1: Example 12.1: Imbalance and underlying errors.

note the following properties:

1. As noted above, the ensemble average of the random error $\Delta_1(t)$ is zero.
2. The time average of the discretization error $\Delta_2(t)$ is zero over each control interval:

$$\frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} \Delta_2(t) \;=\; -\frac{1}{\delta}\int_{n\delta}^{(n+1)\delta} m(t)dt \,+\, \bar{m}(n) \;=\; 0$$

3. The mean prediction error $E\Delta_3(t) = -\bar{m}(n) + \bar{m}(n-1)$ is small if the mean process $m(t)$ is slowly time-varying. In particular if σ is stationary then the prediction error has zero mean.

$\square$

Imbalance due to random error $\Delta_1(t)$, discretization error $\Delta_2(t)$ and prediction error $\Delta_3(t)$ is handled by frequency control. The operator dispatch $u^{\text{opt}}(\hat{m}(n))$ is not the actual power injection but provides setpoints for controllable generators and loads for the nth control interval. While these setpoints $\hat{u}(n)$ are updated every δ amount of time, frequency control operates continuously to determine the actual power injection. The definition (12.6) and (12.3b) suggest the following power flow model at a fast timescale:

$$u^{\text{opt}}(\hat{m}(n)) + \Delta u_i(t) + \sigma_i(t) \;=\; \sum_{j:i\sim j} S_{ij}(t), \qquad t \in [n\delta,(n+1)\delta), \;\; i \in \overline{N}$$

where we have fixed a realization ξ and suppressed the index ξ. This model however is incorrect. It ignores two important features of frequency control, the fast timescale generator and frequency dynamics and the feedback control to maintain frequency around its nominal value (the correct equation is (12.11b)). Before we describe in Chapter 12.4 how an optimal dispatch $u^{\text{opt}}(\hat{m}(n))$ is realized at a fast timescale, we remark on two other sources of imbalance.

Remark 12.4. Imbalance can also result from two other types of errors:

1. *Contingency error.* Unanticipated outages of generators, transmission lines or transformers or the switching on or off of large loads may occur within a control interval, creating imbalance. Unlike other errors, such contingency events occur rarely and when they do occur, the model in the original problem (12.4) must be updated in order to compute a new dispatch. Imbalance due to contingency error is discussed in Chapter 12.5.

2. *Modeling error.* The power flow model as expressed in (12.3a) is only an approximation of the reality. A transmission network model is highly aggregated. A bus j may represent a substation where the uncontrollable injection $\sigma_j(t)$ is a coarse model of the aggregate demand on the underlying feeder. It may also represent a balancing area where the single control $u_j(t)$ is an approximation of aggregate output of multiple generators controlled by multiple organizations. A line (i, j) may be produced by Kron reduction and represents the connectivity between two sets of buses. Finally network parameters such as line admittances may not be known accurately, e.g., parameters for an aggregate model may have to be estimated experimentally, the parameters of a device may depend on the operating condition or change due to aging. Modeling error is of a different nature than the other errors and we assume its effect can be incorporated as randomness in σ.

$\square$

12.4 Frequency control

The power delivered by a thermal generator is determined by the mechanical power output of a prime mover such as a steam turbine or water turbine. The output level is controlled by opening or closing valves that regulate steam or water flow. For example if the load increases the valve of a generator must open wider to increase the generated power. When there is excess supply the rotating machines in bulk generators will speed up and the system frequency will rise. When there is a shortage the rotating machines will slow down and the system frequency will drop. If power is not re-balanced by adjusting generators or flexible loads, frequency excursion will continue which can disconnect generators to protect them from damage, potentially leading to involuntary load shedding and even system collapse. Frequency deviation from its nominal value is used as a control signal for generators, and controllable loads, that participate in frequency control to adjust their power.

Frequency control, also referred to as *automatic generation control*, consists of three mechanisms operating at timescales from seconds to minutes. A generating unit that participates in the *primary control*, also called *droop control*, uses a governor to automatically adjusts the mechanical power output of a turbine in proportion to its local frequency deviation. Primary frequency control is decentralized. It rebalances power and stabilizes the frequency to a new equilibrium value in 30 seconds or so. The *secondary* *control* adjusts generator setpoints around their dispatch values in order to restore system frequency to its nominal value within a few minutes, e.g., up to 10 minutes after a contingency event. In an interconnected power system consisting of multiple balancing areas, each managed by a single operator, the secondary control additionally restores interchanges of tie-line power between areas to their scheduled values. The adjustments are determined centrally within each area based on real-time measurements of tie-line flow deviations. The dispatched setpoint and scheduled tie-line flows are determined by the *tertiary control* that operates on a timescale of 5–15 minutes. They are chosen to attain economic efficiency as well as restoring the reserve capacities deployed in primary and secondary control so that they are available for contingency response. This is typically determined by solving an optimal dispatch problem as discussed in Chapter 12.3.

We now present a linear dynamic model of the primary and secondary control that clarifies the relation between system operator's dispatch $u^{\text{opt}}(\hat{m}(n))$ for each interval and the actual (active) power generation. A description of the physical system, including a generator, a turbine-governor system, a frequency control system, and a voltage control system, as well as their detailed models, are beyond the scope of this book. Our goal in this section is to use a simple model to connect optimal dispatch studied in Chapter 12.3 with its realization at a fast timescale.

12.4.1 Assumptions and notations

Consider a control interval $[n\delta, (n+1)\delta)$ for which the tertiary control has determined an optimal dispatch $u^{\text{opt}}(\hat{m}(n))$ with the associated network state $x(n)$ including scheduled tie-line flows. We assume that the primary and secondary control converges on a much faster timescale than δ so that the dispatch remains unchanged and serves as the operating point for our incremental model below. We fix a random realization ξ of the uncontrollable injection $\sigma(\xi, t)$. The dynamic model is deterministic with this fixed realization. We hence omit the indices n and ξ in the rest of this section.

We make several simplifying assumptions:

- There is a synchronous generator at each bus that determines the frequency dynamics at the bus. This assumption is only to simplify exposition and can be removed.
- Voltage regulation operates at a faster timescale so that voltage magnitudes $|V_j|$ are fixed for the analysis of frequency control. The effect of voltage regulation can be incorporated into the inertia constant M and damping constant D of (the rotor angle transfer function of) the generator; see below.
- The rotor angles, the internal and terminal (bus) voltage phase angles of generators swing together, i.e., the deviations of these angles from their operating points are equal at all times.
- The lines are lossless, i.e., their shunt admittances $\left(y_{jk}^m, y_{kj}^m\right)$ are zero and series admittances are inductive $y_{jk}^s = \mathbf{i}b_{jk}$ with $b_{jk} < 0$.

With these assumptions our dynamic model focuses on how active power in generating units change the voltage angles and their derivatives, i.e., frequencies. It makes similar assumptions to those in the DC power flow model. In fact the DC power flow describes the steady state of the dynamic model.

The tertiary control determines active power dispatch u_j^0 for the generators and the associated voltage angles θ_j^0 and active line flows P_{jk}^0 driven by estimates σ_j^0 of uncontrollable real power injections. They define the operating point around which we linearize our dynamic model. In particular they satisfy power balance:

$$u_j^0 \,+\, \sigma_j^0 \;\;=\;\; \sum_{k:j\sim k} P_{jk}^0, \qquad j \in \overline{N}$$

Define the following variables and their perturbations around the operating point:

- $u_j(t)$ denotes the setpoint of generator j at time t. Let $\Delta u_j(t) := u_j(t) - u_j^0$ denote the adjustment to the optimal dispatch u_j^0. The adjustment will be computed by the secondary frequency control.
- $\theta_j(t)$ denotes the (terminal) voltage angle at bus j at time t, relative to a rotating frame of the operating-point frequency ω^0 (which is expected to be close but not necessarily equal to the nominal frequency), i.e., the instantaneous voltage is $v_j(t) = \sqrt{2}|V_j|\cos\left(\omega^0 t + \theta_j(t)\right)$. Define the incremental angle $\Delta\theta_j(t) := \theta_j(t) - \theta_j^0$.
- $\omega_j(t)$ denotes the voltage frequency at bus j defined to be the derivative of the phase angle $\omega^0 t + \theta_j(t)$, i.e., $\omega_j(t) = \omega^0 + \dot{\theta}_j(t)$. Hence the frequency deviation $\Delta\omega_j(t) := \omega_j(t) - \omega_j^0$ satisfies $\Delta\omega_j(t) = \Delta\dot{\theta}_j(t)$.
- $P_{jk}(t)$ denotes the line flow from bus j to bus k on line (j,k). Let $P_{kj}(t) := -P_{jk}(t)$. Let $\Delta P_{jk}(t) := P_{jk}(t) - P_{jk}^0$ and similarly for $\Delta P_{kj}(t)$.
- $p_j^M(t)$ denotes the mechanical power output of the prime mover (e.g., gas or water turbine). Let P_j^{M0} denote its value associated with the operating point $\left(u_j^0, \theta_j^0, \omega^0, P_{jk}^0, \sigma_j^0,\, j \in \overline{N},\, (j,k) \in E\right)$ and $\Delta p_j^M(t) := p_j^M(t) - P_j^{M0}$.
- $a_j(t)$ denotes the valve position of the turbine-governor at bus j. Let a_j^0 denote its value associated with the operating point $\left(u_j^0, \theta_j^0, \omega^0, P_{jk}^0, \sigma_j^0,\, j \in \overline{N},\, (j,k) \in E\right)$ and $\Delta a_j(t) := a_j(t) - a_j^0$.

We will remark on $\left(a_j^0, p_j^{M0}\right)$ below when we describe the turbine-governor model. A common model of the instantaneous line flow $P_{jk}(t)$ as a function of voltage angles $\theta(t) := \left(\theta_j(t), j \in \overline{N}\right)$ is (cf. the polar form power flow equation (4.22a)):

$$P_{jk}(t) \;\;=\;\; |V_j||V_k|\left(-b_{jk}\right)\sin\left(\theta_j(t) - \theta_k(t)\right), \qquad (j,k) \in E$$

where $\left(-b_{jk}\right) > 0$. We will adopt its linearization around the operating point as our model:

$$P_{jk}(t) \;\;=\;\; \underbrace{|V_j||V_k|\left(-b_{jk}\right)\sin\left(\theta_j^0 - \theta_k^0\right)}_{P_{jk}^0} \;+\; T_{jk}\left(\Delta\theta_j(t) - \Delta\theta_k(t)\right), \qquad (j,k) \in E$$

where $T_{jk} := |V_j||V_k|\left(-b_{jk}\right)\cos\left(\theta_j^0-\theta_k^0\right)$ are called stiffness coefficients. Hence

$$\Delta P_{jk}(t) \;=\; T_{jk}\left(\Delta\theta_j(t)-\Delta\theta_k(t)\right), \qquad (j,k)\in E \tag{12.9}$$

The coefficient T_{jk} measures power exchange over line (j,k) with respect to changes in phase angles.

The model has three components (see Figure 12.1): (i) a turbine-governor that produces the mechanical power $p_j^M(t)$ based on the setpoint $u_j(t)$; (ii) a power generator that converts the mechanical power output $p_i^M(t)$ of the turbine-governor into electric power that serves the local load $-\sigma_j(t)$ and injects power $\sum_k P_{jk}(t)$ into the transmission system; and (iii) two feedback control mechanisms for primary and secondary frequency control. It describes the dynamics of the incremental variables $\Delta\theta_j$, $\Delta\omega_j$, etc.

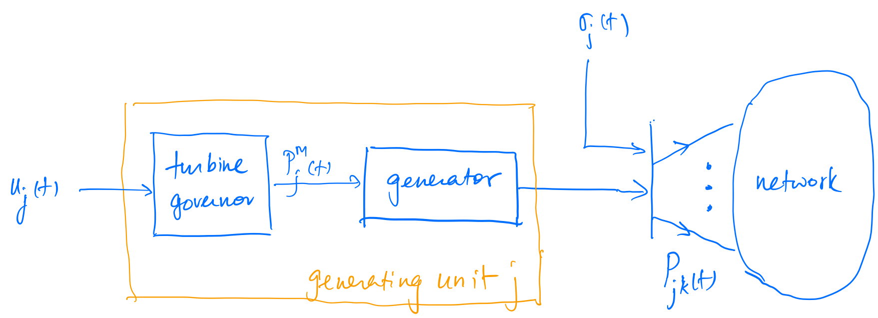

Figure 12.1: A schematic diagram of generating unit j, its setpoint $u_j(t)$, local injection $\sigma_j(t)$, and line power $P_{jk}(t)$ to the transmission system.

12.4.2 Primary control

Turbine-governor model. A second-order model of the turbine-governor with droop control is:

$$\begin{aligned} T_{gj}\,\dot a_j &= -a_j(t) + u_j(t) - \frac{\Delta\omega_j(t)}{R_j}, && j\in\overline{N} \\ T_{tj}\,\dot p_j^M &= -p_j^M(t) + a_j(t), && j\in\overline{N} \end{aligned}$$

where the states $a_j(t)$ and $P_j^M(t)$ are the valve position and mechanical power output of the turbine respectively. The constant R_j is called a regulation constant or a droop constant. The term $-\omega_j(t)/R_j$ increases the valve position when the frequency drops below ω^0 and decreases it otherwise. This is referred to as the droop control or the primary frequency control. This model makes several simplifying assumptions, e.g., it ignores the saturation of the valve position $a_j(t)$, but is reasonable when the frequency deviation $\Delta\omega_j(t)$ is small.

We define $\left(a_j^0, P_j^{M0}\right)$ to be the equilibrium point, defined by $\dot a_j = \dot p_j^M = 0$, when frequency deviations $\Delta\omega_j(t) = 0$ and setpoint $u_j(t) = u_j^0$ is the optimal dispatch, i.e.,

$$p^{M0} \;=\; a_j^0 \;=\; u_j^0, \qquad j\in\overline{N}$$

Then the incremental variable $\left(\Delta a_j, \Delta P_j^M\right) := \left(a_j - a_j^0,\, P_j^M - P_j^{M0}\right)$ satisfies the same equations:

$$T_{gj}\,\Delta\dot{a}_j \;=\; -\Delta a_j(t) \;+\; \Delta u_j(t) \;-\; \frac{\Delta\omega_j(t)}{R_j}, \qquad j \in \overline{N} \tag{12.10a}$$

$$T_{tj}\,\Delta\dot{p}_j^M \;=\; -\Delta p_j^M(t) \;+\; \Delta a_j(t), \qquad j \in \overline{N} \tag{12.10b}$$

This incremental model is what we will use. The block diagram representation of (12.10) is in Figure 12.2.

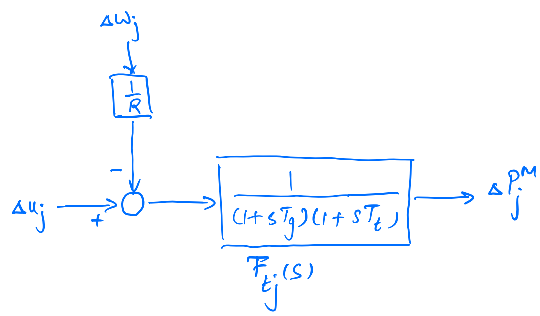

Figure 12.2: Block diagram in Laplace domain of the turbine-governor dynamic (12.10).

As we will see in Chapter 12.4.3 the setpoint adjustment $\Delta u_j(t)$ is changed by the secondary control at a much slower timescale (several minutes) than that of the primary control (approximately 30 secs). Hence a quasi steady-state of (12.10) is defined by a constant value of the setpoint adjustment $\Delta u_j(t) = \Delta u_j$. In this steady state, the frequency deviation $\Delta\omega_j^*$ is generally nonzero and the incremental mechanical power output Δp_j^{M*} is related to the frequency deviation by

$$\Delta p_j^{M*} \;=\; \Delta a_j^* \;=\; \Delta u_j \;-\; \frac{1}{R_j}\Delta\omega_j^*, \qquad j \in \overline{N}$$

Remark 12.5. The time constants T_{gi}, T_{ti} characterize the responsiveness of the governor and turbine respectively to a change in their input. Typical value of T_{gi} and T_{ti} are approximately 0.1 second and 0.5 second respectively. Since the governor responds much faster than the turbine the model is sometimes simplified to a first-order model

$$T_{tj}\,\Delta\dot{p}_j^M \;=\; -\Delta p_j^M(t) \;+\; \Delta u_j(t) - \frac{\Delta\omega_j(t)}{R_j}, \qquad j \in \overline{N}$$

□

Generator model. The frequency deviation $\Delta\omega_j(t)$ is determined by the rotating speed of a generator driven by the mechanical power output $p_j^M(t)$ of the turbine. A dynamic model of the generator in terms of the incremental variables is:

$$\Delta\dot{\theta}_j \;=\; \Delta\omega_j(t), \qquad j \in \overline{N} \tag{12.11a}$$

$$M_j\Delta\dot{\omega}_j \;+\; D_j\Delta\omega_j(t) \;=\; \Delta p_j^M(t) \;+\; \Delta\sigma_j(t) \;-\; \sum_{k:j\sim k}\Delta P_{jk}(t), \qquad j \in \overline{N} \tag{12.11b}$$

where $\Delta\sigma_j(t)$ is the deviation of the uncontrollable injection from its prediction σ_j^0 and $\Delta P_{jk}(t)$ are the incremental line flows given by (12.9). The block diagram representation of (12.11) is in Figure 12.3. Here M_j is the inertia constant of generator j, and D_j is the sum of damping constant of generator j and

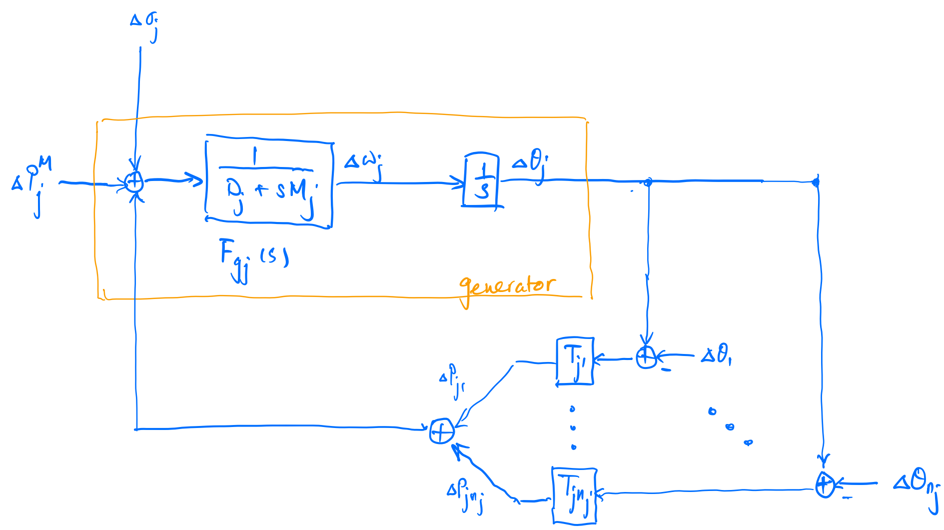

Figure 12.3: Block diagram in Laplace domain of the generator dynamic (12.11). Buses adjacent to bus j are labeled $1, \ldots, n_j$.

the frequency sensitivity of motor-type injection at bus j, as we now explain.

If $\sigma_j(t) < 0$ represents a load, a common model consists of both frequency sensitive load $\sigma_{1j}\left(\omega_j^0 + \omega_j(t)\right)$ such as a motor and frequency insensitive load $\sigma_{2j}(t)$ due to the switching on or off of an electrical device that draws a specified amount of power. Approximate the frequency sensitive load by its linear approximation $\sigma_{1j}\left(\omega^0\right) + \frac{\partial \sigma_{1j}}{\partial \omega_j}\left(\omega^0\right)\Delta\omega_j(t)$ and write the frequency insensitive load as $\sigma_{2j}(t) = \sigma_{2j}^0 + \Delta\sigma_{2j}(t)$. Then the deviation $\frac{\partial \sigma_{1j}}{\partial \omega_j}\left(\omega^0\right)\Delta\omega_j(t)$ of the frequency sensitive load is absorbed into $D_j\Delta\omega_j(t)$ in (12.11b). The uncontrollable load $\sigma_j(t)$ is then the sum of the remaining terms:

$$\sigma_j(t) \quad = \quad \underbrace{\left(\sigma_{1j}\left(\omega^0\right) + \sigma_{2j}^0\right)}_{\sigma_j^0} \; + \; \underbrace{\Delta\sigma_{2j}(t)}_{\Delta\sigma_j(t)}$$

In summary the primary frequency control is modeled by (12.9) (12.10) (12.11) reproduced here:

$$T_{gj}\,\Delta\dot{a}_j \;=\; -\Delta a_j(t) \;+\; \Delta u_j(t) \;-\; \frac{\Delta\omega_j(t)}{R_j}, \qquad j \in \overline{N} \tag{12.12a}$$

$$T_{tj}\,\Delta\dot{p}_j^M \;=\; -\Delta p_j^M(t) \;+\; \Delta a_j(t), \qquad j \in \overline{N} \tag{12.12b}$$

$$M_j\Delta\dot{\omega}_j \;+\; D_j\Delta\omega_j(t) \;=\; \Delta p_j^M(t) \;+\; \Delta\sigma_j(t) \;-\; \sum_{k:j\sim k}\Delta P_{jk}(t), \qquad j \in \overline{N} \tag{12.12c}$$

$$\Delta P_{jk}(t) \;=\; T_{jk}\left(\Delta\theta_j(t) - \Delta\theta_k(t)\right), \qquad (j,k) \in E \tag{12.12d}$$

$$\Delta\dot{\theta}_j \;=\; \Delta\omega_j(t), \qquad j \in \overline{N} \tag{12.12e}$$

This closes the droop control loop. The block diagram representation combines those in Figures 12.2 and 12.3. It is shown in Figure 12.4. The input to the system are external disturbance $\Delta\sigma_j(t)$ at each each

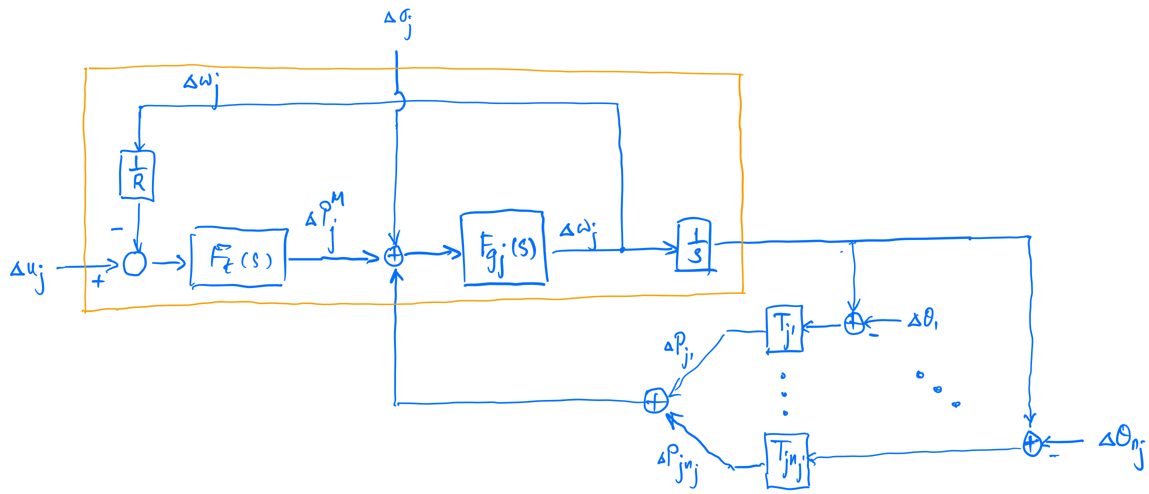

Figure 12.4: Block diagram of primary frequency control (12.12). Buses adjacent to bus j are labeled $1,\ldots,n_j$.

generator j and the adjustment $\Delta u_j(t)$ to the dispatch setpoint. Since the secondary control that updates the setpoint operates at a much slower timescale than the primary frequency control timescale, we can understand the behavior of the (quasi) steady state of the primary control by assuming a constant setpoint adjustment $\Delta u_j(t) = \Delta u_j$.

Consider then a step disturbance in the uncontrollable injection where $\Delta\sigma_j(t)$ changes at time $t = 0$ from 0 to a constant value $\Delta\sigma_j$. We say that $x^* := \left(\Delta\omega^*, \Delta P^*, \Delta\theta^*, \Delta a^*, \Delta p^{M*}\right)$ is an *equilibrium point* of (12.12) driven by the step change $\Delta\sigma$ and constant setpoint Δu_j if, at x^*,

$$\Delta\dot{\omega}_j \;=\; \Delta\dot{a}_j \;=\; \Delta\dot{p}_j^M \;=\; 0, \qquad j \in \overline{N}$$

We do not require $\Delta\dot{\theta} = 0$ in the definition of equilibrium point. Indeed $\Delta\dot{\theta}$ is generally nonzero when primary control converges. Recall the bus-by-line incidence matrix C defined by:

$$C_{jl} \;:=\; \begin{cases} 1 & \text{if } l = j \to k \text{ for some bus } k \\ -1 & \text{if } l = i \to j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}, \qquad\qquad j \in \overline{N},\ l \in E$$

The next result calculates the equilibrium frequency and line flows (its proof is left as Exercise 12.1). It motivates secondary control discussed in Chapter 12.4.3.

Theorem 12.1 (Steady state of primary control)**.** Suppose the network is connected. If x^* is an equilibrium point of (12.12) driven by a step changes $\Delta\sigma$ and constant setpoints Δu then:

1. Local frequency deviations converge to a new value equal to the total disturbance divided by the

system damping:[5]

$$\Delta\omega_j^* \;=\; \Delta\omega^* \;:=\; \frac{\sum_k (\Delta u_k + \Delta\sigma_k)}{\sum_k (D_k + 1/R_k)}, \qquad j \in \overline{N}$$

2. Line flow deviations converge to

$$\Delta P^* \;=\; TC^T L^\dagger \left(\Delta u \,+\, \Delta\sigma \,-\, \Delta\omega^* d\right)$$

where $T := \text{diag}(T_{jk}, (j,k) \in E)$, $L^\dagger$ is the pseudo inverse of the Laplacian matrix $L := CTC^T$, and $d := (D_j + 1/R_j, j \in \overline{N})$.

Remark 12.6. 1. Intuitively the larger the disturbance or the smaller the system damping, the larger will frequency deviation $\Delta\omega^*$ be. Theorem 12.1 clarifies precisely the simple relationship among them. Droop control R_j adds to the system damping and reduces frequency deviation.

2. The theorem says that frequency can be restored to the operating-point value, i.e., $\Delta\omega^* = 0$, only if we change the setpoints so that the total setpoint changes cancel out the total disturbances

$$\sum_k (\Delta u_k + \Delta\sigma_k) \;=\; 0$$

3. To restore all line flows, i.e., $\Delta P^* = 0$, requires canceling disturbances locally at each bus,

$$\Delta u_k + \Delta\sigma_k \;=\; 0, \qquad k \in \overline{N}$$

The next example illustrates a benefit of interconnecting multiple areas.

Example 12.2 (Interconnected system)**.** Consider $N+1$ balancing areas each modeled as a single bus. Suppose $\Delta u_j = 0$ for all areas j and that there is a step change of the uncontrollable injection where $\Delta\sigma_j(t)$ changes at time 0 from 0 to a value $\Delta\sigma_j$. Suppose $\Delta\sigma_j$ are independent random variables with mean $\Delta\bar{\sigma}_j$ and variance ν_j^2. We will evaluate the equilibrium frequency deviation $\Delta\omega^*$ using Theorem 12.1 when the primary frequency control converges.

Case 1: Independent operation. Suppose these buses are not connected. Then the equilibrium frequency deviation in each area j is

$$\Delta\omega_j^* \;=\; \frac{\Delta\sigma_j}{d_j}, \qquad j \in \overline{N}$$

where $d_j := D_j + 1/R_j$ with mean $\Delta\bar{\sigma}_j/d_j$ and variance ν_j^2/d_j^2.

Case 1: Interconnected system. Suppose these buses are connected. Then the equilibrium frequency deviation for the entire interconnected system is

$$\Delta\omega^* \;=\; \frac{\sum_j \Delta\sigma_j}{\sum_j d_j} \;=\; \frac{1}{N+1}\sum_j \frac{\Delta\sigma_j}{\hat{d}}$$

[5]We abuse notation to use $\Delta\omega^*$ to both denote a scalar and the vector whose entries are all $\Delta\omega^*$. The meaning should be clear from the context.

where $\hat{d} := \sum_j d_j/(N+1)$ is the average system damping. Define the average mean and variance of $\Delta\sigma_j$ respectively:

$$\Delta\hat{\sigma} \;:=\; \frac{1}{N+1}\sum_j \Delta\bar{\sigma}_j, \qquad\qquad \hat{v}^2 \;:=\; \frac{1}{N+1}\sum_j v_j^2$$

Then the mean and variance of $\Delta\omega^*$ are respectively

$$\text{mean}\,(\Delta\omega^*) \;=\; \frac{\Delta\hat{\sigma}}{\hat{d}}, \qquad\qquad \text{var}\,(\Delta\omega^*) \;=\; \frac{1}{N+1}\frac{\hat{v}^2}{\hat{d}}$$

The simple case when the random variables $\Delta\sigma_j$ are i.i.d. (independently and identically distributed) with mean $\Delta\bar{\sigma}_1$ and variance v_1^2. Suppose also $d_j = d_1$ for all j. Then $\Delta\hat{\sigma} = \Delta\bar{\sigma}_1$, $\hat{v}^2 = v_1^2$, and $\hat{d} = d_1$. Hence the mean of the interconnected system is the same as that of each area in independent operation, but the variance is reduced by a factor of $N+1$. The bigger the interconnection, i.e., larger N, the smaller the variance in equilibrium frequency deviation $\Delta\omega^*$. $\square$

12.4.3 Secondary control

The first objective of the secondary control is to restore system frequency, i.e., to drive $\Delta\omega(t)$ to zero. The second objective is to restore line flows to their scheduled values, i.e., to drive $\Delta P(t)$ to zero. This is less important and sometimes not pursued for an island system managed by a single operator. In an interconnected system consisting of multiple areas managed by separate operators the interchanges of tie-line power between areas have financial implications. Such a system usually operates under the principle that (i) each area absorbs its own load changes, and (ii) scheduled tie-line flows are maintained. If each bus in (12.12) models an entire area this requires driving $\Delta P(t)$ to zero.

Theorem 12.1 suggests that the objectives of the secondary control can only be achieved by adjusting the setpoints $u(t)$ of the generators to cancel the disturbances (see Remark 12.6). Suppose each bus j in (12.12) represents an area and the setpoint adjustment $\Delta u_j(t)$ represents an aggregate adjustment that will then be shared by all generators in area j that participate in the secondary control. The adjustment is based on the *area control error* (ACE) which is a weighted sum of frequency and line flow deviations:

$$\text{ACE}_j(t) \;:=\; \sum_{k:j\sim k}\Delta P_{jk}(t) \;+\; \beta_j\Delta\omega_j(t), \qquad j\in\overline{N}$$

where $\beta_j > 0$ is called a frequency bias setting. The setpoint adjustment $\Delta u_j(t)$ integrates ACE_j in order to drive it to zero:

$$\Delta\dot{u}_j \;=\; -\gamma_j\left(\sum_{k:j\sim k}\Delta P_{jk}(t) \;+\; \beta_j\Delta\omega_j(t)\right), \qquad j\in\overline{N} \tag{12.13}$$

The computation (12.13) requires real-time measurement of tie-line flow deviations $\Delta P_{jk}(t)$ with all neighboring areas k. This information is sent to area j's system operator which centrally computes the aggregate adjustment $\Delta u_j(t)$ for the entire area using (12.13). It then dispatches in real time setpoint adjustments

$\alpha_{ji}\Delta u_j(t)$ with $\alpha_{ji} \geq 0$ and $\sum_i \alpha_{ji} = 1$ to participating generators i in area j. The weights α_{ji} are called participation factors.

In summary the primary and secondary frequency control in area j is modeled by the system (12.12) (12.13). It is driven by the uncontrollable injection $\Delta\sigma_j(t)$ and consists of two feedback control mechanisms, the droop control with regulation parameter R_j and setpoint adjustment based on $\mathrm{ACE}_j(t)$. Its block diagram is shown in Figure 12.5.

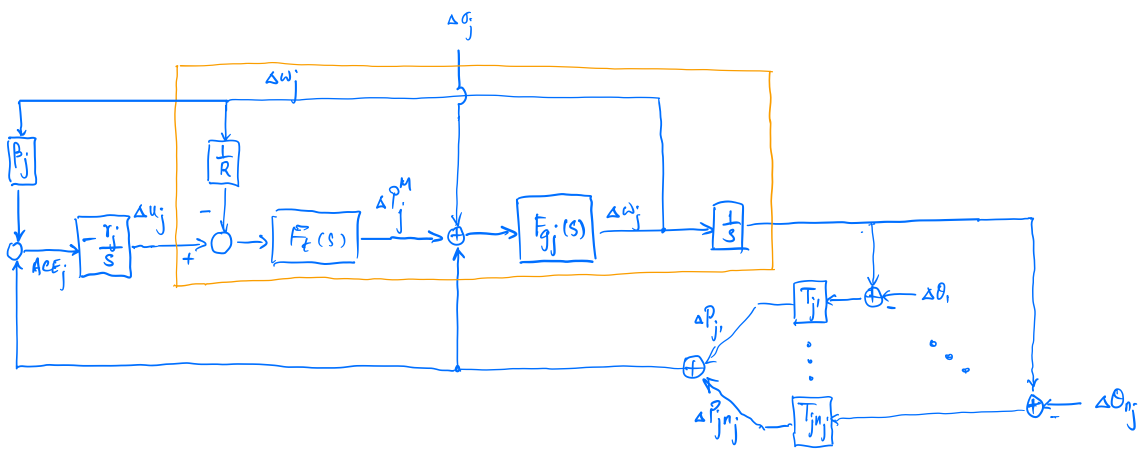

Figure 12.5: Block diagram of primary and secondary frequency control (12.12) (12.13) in area j.

To understand the behavior of the entire interconnected system it is convenient to write (12.12) (12.13) in vector form:

$$
\begin{aligned}
T_g\,\Delta\dot{a} &= -\Delta a(t) + \Delta u(t) - R^{-1}\Delta\omega_j(t) && (12.14a)\\
T_t\,\Delta\dot{p}^M &= -\Delta p^M(t) + \Delta a(t) && (12.14b)\\
M\Delta\dot{\omega} + D\Delta\omega(t) &= \Delta p^M(t) + \Delta\sigma(t) - C\Delta P(t) && (12.14c)\\
\Delta P(t) &= TC^T\Delta\theta(t) && (12.14d)\\
\Delta\dot{\theta} &= \Delta\omega(t) && (12.14e)\\
\Delta\dot{u} &= -\Gamma\left(C\Delta P(t) + B\Delta\omega(t)\right) && (12.14f)
\end{aligned}
$$

where T_g, T_t, T, Γ, B are diagonal gain matrices, R is the diagonal matrix of droop parameters, M, D are diagonal matrices of generator parameters, and C is the $(N+1)\times M$ incidence matrix.

Consider a step change in uncontrollable injection where $\Delta\sigma(t)$ changes at time 0 from the 0 vector to a constant vector $\Delta\sigma$. We say that $x^* := \left(\Delta u^*, \Delta\omega^*, \Delta P^*, \Delta\theta^*, \Delta a^*, \Delta p^{M*}\right)$ is an *equilibrium point* of (12.14) driven by the step change $\Delta\sigma$ if, at x^*,

$$\Delta\dot{u} = \Delta\dot{\omega} = \Delta\dot{a} = \Delta\dot{p}^M = 0$$

Note that we do not require $\Delta\dot{\theta} = 0$ in the definition of equilibrium point. The next result proves that indeed the objectives of the secondary control are achieved (its proof is left as Exercise 12.2).[6] Furthermore $\Delta\dot{\theta} = \Delta\omega^* = 0$ in equilibrium when frequency deviation is driven to zero.

[6] Even though Theorem 12.2 asserts that $\Delta\omega^* = 0$, in practice, it is possible that the frequency is not restored to ω^0 even when ACE's are driven to zero. This is because models are only approximations of reality.

Theorem 12.2 (Steady state of secondary control)**.** Suppose the network is connected. If x^* is an equilibrium point of (12.14) driven by a step change $\Delta\sigma$ then:

1. Frequencies are restored to ω^0 and $\Delta\omega^* = 0$.
2. Line flows are restored to their scheduled values P^0 and $\Delta P^* = 0$.
3. Disturbances are compensated for locally at each bus $\Delta u^* + \Delta\sigma = 0$.

12.5 System security

Power system security refers to the ability to withstand large disturbances. The small random imbalances are handled by real-time optimal dispatch and frequency control mechanisms discussed in Chapters 12.3 and 12.4 respectively. In this section we explain techniques to handle large disturbances due to contingency events such as the loss of a bulk generator or wind or solar farm, the switching on or off of a large industrial load, or the outage of a transmission line or transformer in the transmission network.

12.5.1 Secure operation

Contingency events are rare but their potential impacts are large. North American Electric Reliability Corporation's (NERC) $N-1$ rule states that the outage of a single piece of equipment (e.g., generator, line, transformer) should not result in flow or voltage limit violations. As volatile generation from wind and solar farms continues to displace thermal generators, a large deviation of such nondispatchable generation from its predicted value may also count as a contingency event in the future. For instance the random generation can be modeled as taking one of a finite number of values, each triggering a contingency response if it differs significantly from its predicted value.

Secure operation is achieved through three main mechanisms: (i) analyze credible contingencies that may lead to voltage or line limit violations, (ii) account for these contingencies in optimal commitment and dispatch schedules, and (iii) monitor system state in real time and take corrective actions when a contingency occurs. We summarize each of these functions.

Contingency analysis. When a generator or load contingency occurs the resulting power flows might violate line limits and lead to transmission outages where transmission lines or transformers are disconnected. If reserve capacity is insufficient to re-balance generation and demand, frequency excursion will continue which can disconnect other generators to protect them from damage, potentially leading to involuntary load shedding and even system collapse. When a transmission line or transformer is disconnected power flows in the network will redistribute and line limits can be violated, potentially leading to cascading line outages. Furthermore a transmission outage results in reactive losses in the network which can suppress voltage magnitudes, leading to voltage violations.

The impacts of these contingency events can be assessed by solving AC power flow equations that describe the network state after each contingency. Currently this set of post-contingency equations are solved in the industry mostly using Newton-Raphson or the decoupled power flow methods because they have good speed and convergence properties. Due to the large number of contingencies that must be assessed in order to satisfy $N-k$ security for $k \geq 1$, it is a common practice to first use the DC power flow model to quickly screen contingencies and select a much smaller subset that result in voltage or line limit violations for more detailed analysis using the AC power flow model, especially for contingency scenarios where voltage magnitudes and reactive flows are important. Contingency scenarios in which line or voltage limits are violated are called *credible contingencies*. Contingency screening uses the DC power flow model often makes use of power transfer distribution factor and line outage distribution factor analyzed in Chapter 24. These distribution factors are used to quickly estimate incremental line flow changes due to a contingency from the pre-contingency operating point determined by the AC power flow model. An advantage of this approach is that the impact of generator and transmission outages on the post-contingency networks can be analyzed using the common pre-contingency topology across contingency scenarios.

Security constrained dispatch and commitment. The credible contingencies that have been identified in contingency analysis are taken into account in day-ahead (e.g., 12–36 hours) unit commitment and real-time (e.g., 5–15 minutes) dispatch as well as automatic generation control (seconds to minutes). Capacities are reserved for normal operation (regulation and load-following reserves) and for contingencies (contingency reserves).

There are two approaches to account for credible contingencies in scheduling optimal dispatch. The *preventive approach* augments the optimal dispatch problem studied in Chapter 12.3 with additional constraints so that the network state under the optimal dispatch will satisfy operational constraints even after contingency events. This allows the dispatch to remain unchanged until the next real-time dispatch period even if a contingency occurs in the middle of the current period. The intra-period imbalance due to contingency will be handled by the frequency control mechanisms studied in Chapter 12.4. The *corrective approach*, on the other hand, will compute optimal dispatches both for normal operation and after each contingency event. This allows the system operator to dispatch a response immediately after a contingency is detected without having to wait till the next dispatch period. Both approaches can be formulated as security constrained OPF problems; see Chapter 12.5.2.

System monitoring. A system operator's energy management system collects and processes measurements of voltages, currents, line flows, and the status of circuit breakers and switches at all transmission substations. Other measurements such as frequencies, generator outputs, and transformer tap positions are also measured at various locations of a transmission network, e.g., using phasor measurement units. These measurements are used for state estimation, real-time commitment and dispatch, and automatic generation control, among other applications. Based on these measurements the system can be classified as in a normal state, an emergency state, or after a contingency, in a restoration state, with default actions in each of these states.

12.5.2 Security constrained OPF

We will refer to the problem of optimal dispatch without security constraints studied in Chapter 12.3:

$$\begin{aligned} \min_{(u_0,x_0)} \quad & c_0(u_0,x_0) \\ \text{s.t.} \quad & f_0(u_0,x_0) \;=\; 0, \quad g_0(u_0,x_0) \;\leq\; 0 \end{aligned} \tag{12.15}$$

as the base or pre-contingency case. Here u_0 is a vector representing controls such as real power injections of controllable generators and loads, generator voltage magnitudes, transformer tap positions, x_0 is a vector representing the network state such as bus voltage magnitudes and angles at load buses, $f_0(x_0,u_0)$ represents linear or nonlinear power flow equations, and $g_0(x_0,u_0)$ represents operational constraints such as voltage and line flow limits, all in the base case.

Let credible contingencies be indexed by $k = 1,\ldots,K$. After a contingency k, the dispatch u_0 remains unchanged in the short term (e.g., 1–5 mins). The network state however changes immediately from x_0 to a new system state $\tilde{x}_k$ determined by the post-contingency network and frequency control actions. The choice of pre-contingency dispatch u_0 can take the new network state into account, in three ways.

Some operational constraints such as thermal limits may be temporarily relaxed immediately after the contingency provided corrective actions will be implemented quickly. A preventive approach chooses u_0 so that emergency operational constraints in the short term are satisfied before corrective actions take effect. Let $\tilde{f}_k$ denote the power flow equations for the post-contingency network, and $\tilde{g}_k$ models the emergency operational constraints after contingency k. The pre-contingency control u_0 and the post-contingency network state $\tilde{x}_k$ in the short term must satisfy:

$$\tilde{f}_k(u_0,\tilde{x}_k) \;=\; 0, \quad \tilde{g}_k(u_0,\tilde{x}_k) \;\leq\; 0, \qquad k = 1,\ldots,K \tag{12.16}$$

A *preventive security-constrained OPF* (SCOPF) problem chooses an optimal control decision u_0 that will remain secure after each contingency $k = 1,\ldots,K$, before corrective actions are implemented, i.e., it is of the form

$$\min_{(u_0,x_0,\ \tilde{x}_k,\ k\geq 1)} \quad c_0(u_0,x_0) \qquad \text{s.t.} \qquad (12.15)(12.16)$$

In the corrective approach a new dispatch u_k is applied after contingency k. In addition to changes in injections, the corrective control u_k may also include changes to network topology such as line switching or circuit breaker actions. These changes are captured in new power flow equations f_k. While $\tilde{f}_k$ in (12.16) is determined only by the contingency, e.g., a line or generator outage, f_k may include topology changes as part of the corrective control. The operational constraints, modeled by g_k, are generally different from the pre-contingency constraints g and the emergency constraints $\tilde{g}_k$ immediately after contingency k. Besides constraints such as voltage and line limits under control u_k, g_k may also include constraints due to capacity reserves (see Chapters 21.1–21.4). The corrective control u_k and the resulting network state x_k therefore must satisfy

$$f_k(u_k,g_k) \;=\; 0, \quad g_k(u_k,x_k) \;\leq\; 0, \qquad k = 1,\ldots,K \tag{12.17a}$$

Often the corrective control u_k is constrained to be close to the base control u_0, e.g., because of limited ramp rates ρ_k of large generators or loads:

$$\|u_k - u_0\| \;\leq\; \rho_k, \qquad k = 1,\ldots,K \tag{12.17b}$$

Then a *corrective SCOPF* takes the form

$$\min_{(u_k, x_k,\ k \geq 0)} \sum_{k \geq 0} w_k\, c_k\,(u_k, x_k) \qquad \text{s.t.} \qquad (12.15)(12.17)$$

where c_k are costs that can depend on the contingency and $w_k \geq 0$ are nonnegative weights.

This corrective approach ignores the emergency constraints (12.16) and assumes the system will ride through the small delay between the time a contingency occurs and when the corrective control u_k takes effect. This allows more flexibility in the base control u_0 and lowers the cost of normal operation. A more secure and potentially more costly approach will impose both the emergency constraints as well as constraints on the corrective control:

$$\min_{(u_k, x_k,\ \tilde{x}_{k+1},\ k \geq 0)} \sum_{k \geq 0} w_k\, c_k\,(u_k, x_k) \qquad \text{subject to} \qquad (12.15)(12.16)(12.17)$$

Security constrained OPF are used in both control and market applications. We focus in the next few chapters on the control and optimization aspect of OPF. Market operation is described in Part IV of this book.

12.6 Bibliography

There are many excellent texts on various aspects of power system operations in much more detail than this book, e.g., [1, 3, 2]. Automatic generation control that encompasses voltage control and load frequency control is discussed in detail in e.g. [1, Chapter 11], [81].

12.7 Problems

Exercise 12.1 (Primary frequency control)**.** Proof Theorem 12.1.

Exercise 12.2 (Secondary frequency control)**.** Proof Theorem 12.2.

Exercise 12.3 (Optimality of primary frequency control)**.** Formulate underlying optimization problem solved by primary frequency control (c.f. Changhong2014TAC).

Exercise 12.4 (Optimality of secondary frequency control)**.** Formulate underlying optimization problem solved by secondary frequency control (c.f. LinaCZ paper).

Chapter 13

Optimal power flow

An optimal power flow (OPF) problem is a constrained optimization that takes the form

$$\min_{u,x} \ c(u,x) \qquad \text{subject to} \qquad f(u,x) = 0, \ \ g(u,x) \leq 0$$

The cost function c may represent generation cost, voltage deviation, power loss, or user disutility. The variable u collects control decisions such as generator commitment, generation setpoints, transformer taps, capacitor switch status, electric vehicle charging levels, thermostatic settings, or inverter reactive power. The variable x collects network state such as voltage levels, line currents, or power flows. The constraint functions f, g describe current or power balance, generation or consumption limits, voltage or line limits, and stability and security constraints, as well as other operational requirements. OPF is a fundamental problem because it underlies numerous power system operation and planning applications.

In this chapter we formulate OPF generically using the device and network models studied in Parts I and II of the book. In Chapter 13.1 we describe different device models and the resulting optimization variables and formulate OPF in the bus injection model. In Chapter 13.2 we formulate OPF in the branch flow model and show that it is equivalent to OPF in the bus injection model. We have seen in Chapter 12 the application of OPF to transmission system applications such as unit commitment, economic dispatch and state estimation. In Chapter 13.3 we present example applications in distribution systems. We show in Chapter 13.4 that OPF is NP-hard. We describe in Chapter 11.2 optimization algorithms that are often used for solving OPF problems and in Chapter 13.5 techniques for scaling OPF solutions.

13.1 Bus injection model

In Chapter 13.1.1 we describe how to represent different devices in terms of their terminal voltage and power injection (V_j, s_j). The interaction of these terminal variables is described by power flow equations. Different power flow equations lead to different OPF formulations with different computational properties. We first formulate in Chapter 13.1.2 OPF in the bus injection model for single-phase networks and then express it in Chapter 13.1.3 as a standard quadratically constrained quadratic program. Finally we extend in Chapters 13.1.4–13.1.6 these device models and OPF formulations from single-phase to three-phase networks.

13.1.1 Single-phase devices

For simplicity we will assume voltages are defined with respect to the ground and every single-phase device is connected between its terminal and the ground. We will model the devices we encounter by one of the following:

1. *Voltage source j:* An ideal voltage source j fixes its voltage $V_j \in \mathbb{C}$ if it is uncontrollable and it adjusts V_j if it is controllable. Its current and power injections $\left(I_j, s_j\right)$ are then determined by the interaction with other devices through the network equation $I = YV$ or power flow balance.

2. *Current source j:* An ideal current source fixes its current $I_j \in \mathbb{C}$ if it is uncontrollable and it adjusts I_j if it is controllable. Its terminal voltage and power injection $\left(V_j, s_j\right)$ are then determined through network equations. An example of current source (load) is an electric vehicle charger whose charging current is controllable.

3. *Power source j:* An ideal power source fixes its power injection $s_j \in \mathbb{C}$ if it is uncontrollable and adjusts s_j if it is controllable. Its voltage and current $\left(V_j, I_j\right)$ are then determined through network equations.

4. *Impedance j:* An impedance z_j fixes the relationship between its voltage and current $V_j = -z_j I_j$ where the negative sign indicates that I_j is defined in the direction of ground-to-terminal.

The bus injection model focuses on the terminal voltages and power injections $\left(V_j, s_j\right)$ of these devices and describe their interaction over a network through power flow equations. We now formulate OPF for single-phase systems.

13.1.2 Single-phase OPF

Consider a single-phase network modeled as an undirected graph $G := (\overline{N}, E)$ where there are $N+1$ buses $j \in \overline{N}$ and M lines in E. Each line $(j,k) \in E$ is characterized by admittances $\left(y^s_{jk}, y^m_{jk}, y^m_{kj}\right) \in \mathbb{C}^3$. We now explain the variables, power flow equations, cost function, and constraints that define an OPF problem. As we will see the OPF formulation (13.5) below does not require assumption C4.1 that $y^s_{jk} = y^s_{kj}$. It can therefore accommodate single-phase transformers that have complex turns ratios.

Simple OPF. Without loss of generality we first make the following assumptions and present a simplest OPF formulation:

1. The OPF involves only voltage sources and power sources.

2. There is exactly one single-phase device (voltage or power source) at each bus j. We will then interchangeably refer to j as a bus or a device.

We will explain below how to relax these assumptions. We now describe the optimization variable, cost function, and constraints that define a simple OPF.

Under the assumptions above, associated with each bus j is its bus (nodal) voltage V_j and power injection s_j. The vectors $V := (V_j, j \in \overline{N})$ and $s := (s_j, j \in \overline{N})$ are the optimization variables. The cost function $C_0(V,s)$ may represent the cost of generation (e.g. in economic dispatch), estimation error (e.g. in state estimation), line loss (e.g. in volt/var control in distribution systems), and user disutility (e.g., in demand response). For instance to minimize a weighted sum of real power generations we can use

$$C_0(V,s) \ := \ \sum_{j:\text{gens}} c_j \operatorname{Re}(s_j)$$

To minimize the total real power loss over the network we can use

$$C_0(V,s) \ := \ \sum_{j} \operatorname{Re}\left(s_j\right)$$

There are two type constraints on (V,s). The first is power flow equations, the complex form of which is derived in Chapter 4.2 as follows. The sending-end line currents from buses j to k in terms of V and that from buses k to j are given in (4.1a) and reproduced here:

$$I_{jk}(V) \ = \ y^s_{jk}(V_j - V_k) \ + \ y^m_{jk} V_j, \qquad I_{kj}(V) \ = \ y^s_{kj}(V_k - V_j) \ + \ y^m_{kj} V_k, \qquad (j,k) \in E \quad (13.1)$$

The sending-end complex power flow from buses j to k and that from buses k to j are respectively (from (4.2)):

$$S_{jk}(V) \ := \ V_j I^{\mathsf{H}}_{jk}(V) \ = \ \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \ + \ \left(y^m_{jk}\right)^{\mathsf{H}} |V_j|^2, \qquad (j,k) \in E \qquad (13.2a)$$

$$S_{kj}(V) \ := \ V_k I^{\mathsf{H}}_{kj}(V) \ = \ \left(y^s_{kj}\right)^{\mathsf{H}} \left(|V_k|^2 - V_k V_j^{\mathsf{H}}\right) \ + \ \left(y^m_{kj}\right)^{\mathsf{H}} |V_k|^2 \qquad (j,k) \in E \qquad (13.2b)$$

The bus injection model in complex form is therefore (from (4.20a)):

$$s_j \ = \ \sum_{k: j\sim k} S_{jk}(V) \ := \ \sum_{k: j\sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \ \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \ + \ \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2, \qquad j \in \overline{N} \qquad (13.3)$$

where $y^m_{jj} := \sum_{k: j\sim k} y^m_{jk}$ are the total shunt admittances incident on buses j. Instead of the complex form (13.3), we can also use the polar form or the Cartesian form of power flow equations.

The second type of constraints on (V,s) is operational constraints. We will consider only three constraints:

1. *Injection limits*: These can represent generation or load capacity limits and take the form:

$$s_j^{\min} \ \leq \ s_j \ \leq \ s_j^{\max}, \qquad j \in \overline{N} \qquad (13.4a)$$

   where $s_j^{\min}, s_j^{\max} \in \mathbb{C}$ are given bounds on the injections at buses j.

2. *Voltage limits*: These are limits on voltage magnitudes:

$$v_j^{\min} \ \leq \ |V_j|^2 \ \leq \ v_j^{\max}, \qquad j \in \overline{N} \qquad (13.4b)$$

   where $v_j^{\min}, v_j^{\max} \in \mathbb{R}$ are given lower and upper bounds on the squared voltage magnitudes. We assume $\underline{v}_j > 0$ to avoid triviality.

3. *Line limits*: Thermal limits can be expressed in terms of line currents $\left(I_{jk}(V), I_{kj}(V)\right)$ in (13.1):

$$\left|y^s_{jk}(V_j - V_k) \; + \; y^m_{jk} V_j\right|^2 \; \leq \; \ell^{\max}_{jk}, \qquad \left|y^s_{kj}(V_k - V_j) \; + \; y^m_{kj} V_k\right|^2 \; \leq \; \ell^{\max}_{kj}, \qquad (j,k) \in E \tag{13.4c}$$

   which are quadratic inequalities in V.

   Alternatively line limits can be expressed in terms of complex line power:

$$S^{\min}_{jk} \; \leq \; S_{jk}(V) \; \leq \; S^{\max}_{jk}, \qquad S^{\min}_{kj} \; \leq \; S_{kj}(V) \; \leq \; S^{\max}_{kj}, \qquad (j,k) \in E$$

   or in terms of apparent power:

$$|S_{jk}(V)| \; \leq \; S^{\max}_{jk}, \qquad |S_{kj}(V)| \; \leq \; S^{\max}_{kj}, \qquad (j,k) \in E$$

   where $\left(S_{jk}(V), S_{kj}(V)\right)$ are given by (13.2). The limits on apparent power can be expressed in terms of a degree four polynomial in V which can be converted into quadratic constraints with additional variables (see Exercise 13.7).

Depending on the application there can be many more constraints, e.g., stability and security constraints, ramp limits, limits on battery state of charge and charging rates. For illustration purpose we will mostly restrict ourselves to these three types of constraints.

A simple OPF problem in the bus injection model is then

$$\min_{(V,s)} \;\; C_0(V,s) \quad \text{s.t.} \quad (13.3)(13.4) \tag{13.5}$$

Since the constraints (13.3)(13.4c) do not require assumption C4.1 that $y^s_{jk} = y^s_{kj}$, the OPF formulation (13.5) can accommodate single-phase transformers that have complex turns ratios.

Remark 13.1 (Uncontrollable parameters and reference voltage)**.** This is a general formulation that allows the power injection s_j and voltages V_j at every bus j to be optimization variables. If there is practically no bound on the injection at bus j then $s^{\min}_j := -\infty - \mathbf{i}\infty$ or $s^{\max}_j := \infty + \mathbf{i}\infty$ which removes the lower or upper bound on the function $s_j(V)$ of V. On the other hand the inequality constraints also allow the case where a quantity is not an optimization variable but a parameter, by setting $s^{\min}_j = s^{\max}_j$ to the specified value. For instance $s_j(V) = s^{\min}_j = s^{\max}_j$ may represent a given uncontrollable constant-power load or a given renewable generation. For the slack bus 0, unless otherwise specified, we always assume $V_0 := 1\angle 0^\circ$ pu so that $v^{\min}_0 = v^{\max}_0 = 1$ and $s^{\min}_0 = -\infty - \mathbf{i}\infty$, $s^{\max}_0 = \infty + \mathbf{i}\infty$. Therefore we sometimes replace $j \in \overline{N}$ in (13.3)(13.4) by $j \in N$. □

Other devices. Single-phase devices other than voltage and power sources can also be included in the OPF formulation. For instance an electric vehicle charger can be modeled by a current source. If it is controllable then its current I_j is an additional optimization variable and it imposes a quadratic equality constraint on (V_j, s_j, I_j):

$$s_j \;\; = \;\; V_j I^{\mathsf{H}}_j$$

If the current source is uncontrollable with a fixed I_j, then the constraint above is a linear constraint on (V_j, s_j). An impedance z_j introduces a quadratic equality constraint on (V_j, s_j):

$$s_j \;=\; \frac{|V_j|^2}{z_j^{\mathsf{H}}}$$

A nodal admittance y_j, such as a capacitor tap, can be incorporated by including the the variable y_j and quadratic equality constraint on (V_j, s_j, y_j):

$$s_j \;=\; y_j^{\mathsf{H}} |V_j|^2$$

We assume in the OPF formulation (13.5) that each bus j has a single device with nodal variable (V_j, s_j). If multiple devices are connected to bus j in parallel with power injections s_{jk}, $k = 1, \ldots, K_j$, they introduce additional variables $(s_{jk}, k = 1, \ldots, K_j)$ and impose the linear constraint

$$s_j \;=\; \sum_k s_{jk}$$

Hence other devices can be incorporated and they impose a local equality constraint at each bus j. If the devices at bus j are controllable, an additional optimization variable u_j (e.g., I_j of a controllable current source) will be introduced and the local constraint is of the form

$$f_j(u_j, V_j, s_j) \;=\; 0, \qquad\qquad j \in \overline{N} \tag{13.6a}$$

Otherwise, they do not introduce any additional variable at bus j (e.g., impedance z_j) but only a local constraint of the form $f_j(V_j, s_j) = 0$ which can be considered a special case of (13.6a). When an additional optimization variable u_j is introduced, there may also be an operational constraint on u_j of the form

$$g_j(u_j) \;\le\; 0, \qquad\qquad j \in \overline{N} \tag{13.6b}$$

This constraint is vacuous if no u_j is introduced.

Most applications indeed involve other variables in addition to (V_j, s_j). For example, the unit commitment problem in Chapter 12.2 includes binary variables to indicate if a unit will be on or off. In distributed energy resource optimization, battery charging rates and their states of charge as well as the temperature setpoint of a thermostat may be additional variables. In volt/var control that optimizes over the reactive power output of an inverter given its real power input, the reactive power needs to satisfy a sector constraint. For single-phase networks, however, we will focus on the simple OPF (13.5) and study its computational properties. In particular we will omit variables u_j and the associated local constraints (13.6).

OPF in terms of V only. We can treat the power flow equation (13.3) as defining $s_j(V)$ as a function of V:

$$s_j(V) \;=\; \sum_{k:j\sim k} S_{jk}(V) \;:=\; \sum_{k:j\sim k} \left(y_{jk}^s\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y_{jj}^m\right)^{\mathsf{H}} |V_j|^2, \qquad j \in \overline{N} \tag{13.7}$$

where $y^m_{jj} := \sum_{k:j\sim k} y^m_{jk}$ are the total shunt admittances incident on buses j. Using (13.1)(13.2)(13.7) for single-phase networks, we can express powers and currents $\left(s_j, S_{jk}, I_{jk}\right)$ in terms of voltages V and formulate OPF as an optimization over V only.

For instance the cost function to minimize a weighted sum of real power generations is:

$$C_0(V) \;:=\; \sum_{j:\text{gens}} c_j \operatorname{Re}(s_j(V)) \;=\; \sum_{j:\text{gens}} c_j \operatorname{Re}\left(\sum_{k:j\sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2\right)$$

The cost function to minimize the total real power loss over the network is:

$$C_0(V) \;:=\; \sum_j \operatorname{Re}\left(s_j(V)\right) \;=\; \sum_j \operatorname{Re}\left(\sum_{k:j\sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2\right)$$

The total real power loss equals the total thermal $(r|I|^2)$ loss in the network lines if line shunt admittances are reactive, i.e., if y^m_{jk} and y^m_{kj} are pure imaginary:

$$C_0(V) \;:=\; \sum_{(j,k)\in E} r_{jk}\, |I^s_{jk}(V)|^2$$

where $r_{jk} := \operatorname{Re}\left(z^s_{jk}\right) = \operatorname{Re}\left(\left(y^s_{jk}\right)^{-1}\right)$ is the series resistance of the line and $I^s_{jk}(V) := y^s_{jk}(V_j - V_k)$ is the current through the series impedance of the line. All these costs are quadratic functions of V (Exercise 13.6).

For operational constraints, the voltage limits (13.4b) and the line limits (13.4c) are already quadratic inequalities in V. We can use (13.7) to express the injection limits $s^{\min}_j \le s_j(V) \le s^{\max}_j$ also as quadratic inequalities in V:

$$s^{\min}_j \;\le\; \sum_{k:j\sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2 \;\le\; s^{\max}_j, \qquad j \in \overline{N} \tag{13.8}$$

If we use the polar form (4.22) BIM then the injection limits become:

$$\begin{aligned} p^{\min}_j \;&\le\; \left(\sum_{k=0}^{N} g_{jk}\right)|V_j|^2 \;-\; \sum_{k\neq j} |V_j||V_k| \left(g_{jk}\cos\theta_{jk} - b_{jk}\sin\theta_{jk}\right) \;\le\; p^{\max}_j, \qquad j \in \overline{N} \\ p^{\min}_j \;&\le\; \left(\sum_{k=0}^{N} b_{jk}\right)|V_j|^2 \;-\; \sum_{k\neq j} |V_j||V_k| \left(b_{jk}\cos\theta_{jk} + g_{jk}\sin\theta_{jk}\right) \;\le\; q^{\max}_j, \qquad j \in \overline{N} \end{aligned}$$

For notational simplicity only, we will mostly use the complex form (13.8) as injection limits.

The simple OPF (13.5) can be equivalently formulated in terms of V only:

$$\min_V \;\; C_0(V) \quad \text{s.t.} \quad (13.4b)(13.4c)(13.8) \tag{13.9}$$

As mentioned before, this formulation does not require assumption C4.1 that $y^s_{jk} = y^s_{kj}$ and hence can accommodate single-phase transformers that have complex turns ratios.

13.1.3 OPF as QCQP

As we have seen above the constraints in OPF (13.9) are quadratic in V. We now explain how to express (13.9) as a quadratically constrained quadratic program (QCQP).

QCQP. A QCQP is the following problem:

$$\min_{x\in\mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x \tag{13.10a}$$

$$\text{s.t.} \quad x^{\mathsf{H}} C_l x \ \le\ b_l, \qquad l = 1,\dots,L \tag{13.10b}$$

where $x \in \mathbb{C}^n$ is a vector, $C_l \in \mathbb{S}^n$ for $l = 0,\dots,L$, are Hermitian matrices so that $x^{\mathsf{H}} C_l x$ are real values, and $b_l \in \mathbb{R}$ are given scalars. If C_l, $l = 0,\dots,L$, are positive semidefinite (psd) then (13.10) is a convex QCQP. Otherwise it is generally nonconvex. If x^{opt} is optimal for (13.10), so is $-x^{\text{opt}}$.

The inequality constraints (13.10b) can include equality constraints $(a = b \Leftrightarrow a \le b,\ b \le a)$. Sometimes equality constraints are specified explicitly as in

$$\begin{aligned} \min_{x\in\mathbb{C}^n} \quad & x^{\mathsf{H}} C_0 x \\ \text{s.t.} \quad & x^{\mathsf{H}} C_l x \ \le\ b_l, && l = 1,\dots,L \\ & x^{\mathsf{H}} \tilde{C}_l x \ =\ \tilde{b}_l, && l = 1,\dots,\tilde{L} \end{aligned}$$

Remark 13.2 (Real QCQP)**.** In computing a solution of (13.10), the QCQP is first converted into a problem in the real domain (we study common algorithms for solving OPF in Chapter 11.2). Indeed the complex QCQP (13.10) is equivalent to the following QCQP in the real domain of twice the dimension (Exercise 13.4):

$$\min_{y\in\mathbb{R}^{2n}} \quad y^{\mathsf{T}} D_0 y \qquad \text{s.t.} \qquad y^{\mathsf{T}} D_l y \ \le\ b_l, \quad l = 1,\dots,L$$

where

$$y \ :=\ \begin{bmatrix} \operatorname{Re}(x) \\ \operatorname{Im}(x) \end{bmatrix}, \qquad\qquad D_l \ :=\ \begin{bmatrix} \operatorname{Re}(C_l) & -\operatorname{Im}(C_l) \\ \operatorname{Im}(C_l) & \operatorname{Re}(C_l) \end{bmatrix}, \qquad l = 0,1,\dots,L$$

□

The problem (13.10) is called a homogeneous QCQP because each term, called a monomial, in the polynomial $x^{\mathsf{H}} C_l x$ is of degree 2. An inhomogeneous QCQP contains monomials with degree 1 and takes the form

$$\min_{x\in\mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x \ +\ \left(c_0^{\mathsf{H}} x + x^{\mathsf{H}} c_0\right) \tag{13.11a}$$

$$\text{s.t.} \quad x^{\mathsf{H}} C_l x \ +\ \left(c_l^{\mathsf{H}} x + x^{\mathsf{H}} c_l\right) \ \le\ b_l, \qquad l = 1,\dots,L \tag{13.11b}$$

Note that $\left(c_l^{\mathsf{H}} x + x^{\mathsf{H}} c_l\right)$ are real numbers. This problem can be homogenized by introducing a scalar complex variable $t \in \mathbb{C}$. The idea can be illustrated using a scalar complex variable $x \in \mathbb{C}$. The following characterizations are equivalent:

$$|x|^2 \ +\ \left(c^{\mathsf{H}} x + x^{\mathsf{H}} c\right) \ \le\ b \qquad \Longleftrightarrow \qquad |x + ct|^2 \ -\ |c|^2 |t|^2 \ \le\ b, \quad |t|^2 \ =\ 1$$

The second characterization with t consists of a homogeneous quadratic equality and inequality in (x,t). A solution for t is $t = e^{\mathbf{i}\theta}$ for any $\theta \in \mathbb{R}$ and hence

$$|x+ct|^2 \;-\; |c|^2|t|^2 \;=\; |x|^2 \;+\; \left(x^{\mathsf{H}} c e^{\mathbf{i}\theta} + c^{\mathsf{H}} e^{-\mathbf{i}\theta} x\right)$$

Therefore these two characterizations are equivalent in that, if $(x,t) \in \mathbb{C}^2$ satisfies the second characterization, then their product $xe^{\mathbf{i}\theta}$ satisfies the first if $t = e^{\mathbf{i}\theta}$. The extension of this idea to vector variable $x \in \mathbb{C}^n$ can be used to homogenize (13.11). Consider the following homogeneous QCQP with equality and inequality constraints:

$$\min_{x\in\mathbb{C}^n, t\in\mathbb{C}} \quad \begin{bmatrix} x^{\mathsf{H}} & t^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} C_0 & c_0 \\ c_0^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \tag{13.12a}$$

$$\text{s.t.} \quad \begin{bmatrix} x^{\mathsf{H}} & t^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} C_l & c_l \\ c_l^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \;\le\; b_l, \qquad l = 1,\ldots,L \tag{13.12b}$$

$$\begin{bmatrix} x^{\mathsf{H}} & t^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \;=\; 1 \tag{13.12c}$$

Problem (13.12) is equivalent to (13.11) in the sense that, if $(x^{\text{opt}}, t^{\text{opt}}) \in \mathbb{C}^{n+1}$ is optimal for (13.12), then their product $x^{\text{opt}}t^{\text{opt}} = x^{\text{opt}}e^{\mathbf{i}\theta^{\text{opt}}}$ is optimal for (13.11) when $t^{\text{opt}} = e^{\mathbf{i}\theta^{\text{opt}}}$.

We will hence study, without loss of generality, homogeneous QCQP (13.10) with inequality constraints.

Remark 13.3 (Real QCQP)**.** If the variable x is in $\mathbb{R}^n$ instead of $\mathbb{C}^n$ and C_l are $n \times n$ real symmetric matrices, $l = 0,\ldots,L$, then (13.10) is a real homogeneous QCQP:

$$\min_{x\in\mathbb{R}^n} \; x^{\mathsf{T}} C_0 x \qquad \text{s.t.} \qquad x^{\mathsf{T}} C_l x \;\le\; b_l, \qquad l \;=\; 1,\ldots,L$$

A real inhomogeneous QCQP

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & x^{\mathsf{T}} C_0 x \;+\; \left(c_0^{\mathsf{T}} x + x^{\mathsf{T}} c_0\right) \\ \text{s.t.} \quad & x^{\mathsf{T}} C_l x \;+\; \left(c_l^{\mathsf{T}} x + x^{\mathsf{T}} c_l\right) \;\le\; b_l, \qquad l = 1,\ldots,L \end{aligned}$$

is equivalent to the following real homogeneous QCQP

$$\begin{aligned} \min_{x\in\mathbb{R}^n, t\in\mathbb{R}} \quad & \begin{bmatrix} x^{\mathsf{T}} & t \end{bmatrix} \begin{bmatrix} C_0 & c_0 \\ c_0^{\mathsf{T}} & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \\ \text{s.t.} \quad & \begin{bmatrix} x^{\mathsf{T}} & t \end{bmatrix} \begin{bmatrix} C_l & c_l \\ c_l^{\mathsf{T}} & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \;\le\; b_l, \qquad l = 1,\ldots,L \\ & \begin{bmatrix} x^{\mathsf{T}} & t \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \;=\; 1 \end{aligned}$$

in that, if $(x^{\text{opt}}, t^{\text{opt}}) \in \mathbb{R}^{n+1}$ is optimal for the homogeneous QCQP, then $x^{\text{opt}}t^{\text{opt}}$ is optimal for the original nonhomogeneous QCPQ where $x^{\text{opt}}t^{\text{opt}} = x^{\text{opt}}$ if $t^{\text{opt}} = 1$ and $x^{\text{opt}}t^{\text{opt}} = -x^{\text{opt}}$ if $t^{\text{opt}} = -1$. $\square$

Remark 13.4 (Linear and bilinear cost or constraints)**.** For any $l \geq 0$, $C_l = 0$ corresponds to a linear cost or constraint. It can be homogenized in exactly the same way above, i.e., (13.12) allows any of the matrices C_l to be zero. For example, in the scalar case $n = 1$, a linear constraint is homogenized through

$$c^{\mathsf{H}} x \,+\, x^{\mathsf{H}} c \;\leq\; b \qquad \Longleftrightarrow \qquad c^{\mathsf{H}} x t^{\mathsf{H}} \,+\, c x^{\mathsf{H}} t \;\leq\; b, \quad |t|^2 \;=\; 1 \tag{13.13a}$$

As before, if (x,t) is a solution to the second inequality then xt is a solution to the first inequality. Note that the two linear terms must be complex conjugates of each other so that they sum to a real number. For a linear inequality $d^{\mathsf{H}} x \leq b$ where $b := b_r + \mathbf{i} b_i$ is complex, we can rewrite it as two real inequalities:

$$\frac{1}{2}\left(d^{\mathsf{H}} x + x^{\mathsf{H}} d\right) \;\leq\; b_r, \qquad\qquad \frac{1}{2\mathbf{i}}\left(d^{\mathsf{H}} x - x^{\mathsf{H}} d\right) \;\leq\; b_i \tag{13.13b}$$

The first inequality takes the form of (13.13a) with $c := d/2$. The second inequality takes the form of (13.13a) with $c := -d/2\mathbf{i}$.

A block bilinear term of the form $x^{\mathsf{H}} C y$ can be homogenized as follows. For any variables $(x,y) \in \mathbb{C}^{2n}$ and any matrices $C, D \in \mathbb{C}^{n \times n}$

$$x^{\mathsf{H}} C y \,+\, y^{\mathsf{H}} D x \;=\; \begin{bmatrix} x^{\mathsf{H}} & y^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{13.14}$$

Note that C and D may not be Hermitian of each other so that the product $x^{\mathsf{H}} C y + y^{\mathsf{H}} D x$ may be a complex number. Its real and imaginary parts can be written as quadratic forms of (x,y) in terms of the following Hermtian matrices respectively:

$$\Phi \;:=\; \frac{1}{2}\begin{bmatrix} 0 & C + D^{\mathsf{H}} \\ C^{\mathsf{H}} + D & 0 \end{bmatrix}, \qquad\qquad \Psi \;:=\; \frac{1}{2\mathbf{i}}\begin{bmatrix} 0 & C - D^{\mathsf{H}} \\ -C^{\mathsf{H}} + D & 0 \end{bmatrix}$$

We emphasize that we convert QCQPs to their homogenized form mainly so that we can focus only on homogeneous QCQP in our study of structural properties. In computation, one may not convert an inhomogeneous constraint, especially a linear constraint, into a homogeneous quadratic constraint. $\square$

Example 13.1 (Polynomial cost or constraints)**.** A polynomial can be expressed as a quadratic with auxiliary variables. Write the following as quadratic constraints:

1. $\left(|V_j|^2 - 1\right)^2 \leq \varepsilon$.
2. $a_0 x^3 + a_1 x^2 + a_2 x \leq \alpha$ with $a_i, x \in \mathbb{C}$.

Solution.

1. We have $\left(|V_j|^2 - 1\right)^2 \leq \varepsilon$ if and only if there exist $t_j \in \mathbb{C}$ such that $\left(V_j, t_j\right)$ satisfies

$$\left|t_j - 1\right|^2 \;\leq\; \varepsilon, \qquad\qquad t_j \;=\; |V_j|^2$$

which are quadratic equality and inequality constraints that can be homogenized as discussed above. Note that $t_j = V_j^2$ is not a quadratic form when $\left(V_j, t_j\right)$ are complex.

2. Let $x =: y + \mathbf{i}z$ with $y, z \in \mathbb{R}$. First convert the constraint into two real polynomial constraints in y and z, each of the form

$$\sum_{(i,j):i+j=3} b_{ij} y^i z^j \; + \sum_{(i,j):i+j=2} c_{ij} y^i z^j \; + \sum_{(i,j):i+j=1} d_{ij} y^i z^j \;\; \leq \;\; \beta$$

for some real coefficients b_{ij}, c_{ij}, d_{ij} and real β. To write this as a quadratic constraint in $(y,z) \in \mathbb{R}^2$, introduce auxiliary variables $t = y^2$, $u = z^2$. Then write $y^3 = ty$, $y^2 z = tz$, $yz^2 = yu$, $z^3 = uz$. These quadratic expressions can then be homogenized as discussed above.

□

OPF as QCQP. We now assume the cost function $C_0(V) := V^{\mathsf{H}} C_0 V$ is a quadratic form in V for some positive semidefinite matrix C_0. We can then express OPF (13.9) as a QCQP, by deriving the cost matrices C_l underlying the quadratic constraints (13.4b)(13.4c)(13.8).

1. *Injection limits*: To express the injection s_j in (13.8) as a quadratic form, use $I = YV$ to write

$$s_j \;\; = \;\; V_j I_j^{\mathsf{H}} \;\; = \;\; \left(e_j^{\mathsf{H}} V\right) \left(e_j^{\mathsf{H}} I\right)^{\mathsf{H}} \;\; = \;\; e_j^{\mathsf{H}} \, V V^{\mathsf{H}} Y^{\mathsf{H}} e_j$$

where e_j is the $(N+1)$-dimensional vector with 1 in the jth entry and 0 elsewhere. Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, we have[1]

$$s_j \; = \; \mathrm{tr}\left(e_j^{\mathsf{H}} V V^{\mathsf{H}} Y^{\mathsf{H}} e_j\right) \;\; = \;\; \mathrm{tr}\left(\left(Y^{\mathsf{H}} e_j e_j^{\mathsf{H}}\right) V V^{\mathsf{H}}\right) \;\; =: \; V^{\mathsf{H}} Y_j^{\mathsf{H}} V$$

where $Y_j := e_j e_j^{\mathsf{H}} Y$ is an $(N+1) \times (N+1)$ matrix with its jth row equal to the jth row of the admittance matrix Y and all other rows equal to the zero vector. Y_j is not Hermitian so that $V^{\mathsf{H}} Y_j^{\mathsf{H}} V$ is in general a complex number. Its real and imaginary parts can be expressed in terms of the Hermitian and skew Hermitian components of Y_j^{H} defined as:

$$\Phi_j \; := \; \frac{1}{2}\left(Y_j^{\mathsf{H}} + Y_j\right) \qquad \text{and} \qquad \Psi_j \; := \; \frac{1}{2\mathbf{i}}\left(Y_j^{\mathsf{H}} - Y_j\right)$$

Then Φ_j and Ψ_j are Hermitian matrices and (Exercise 13.2)

$$\mathrm{Re}(s_j) \; = \; V^{\mathsf{H}} \Phi_j V \qquad \text{and} \qquad \mathrm{Im}(s_j) \; = \; V^{\mathsf{H}} \Psi_j V$$

They will be upper and lower bounded by

$$\begin{aligned} p_j^{\min} \; &:= \; \mathrm{Re}\, s_j^{\min} \qquad \text{and} \qquad & p_j^{\max} \; &:= \; \mathrm{Re}\, s_j^{\max} \\ q_j^{\min} \; &:= \; \mathrm{Im}\, s_j^{\min} \qquad \text{and} \qquad & q_j^{\max} \; &:= \; \mathrm{Im}\, s_j^{\max} \end{aligned}$$

These quantities will be used to rewrite below OPF as a standard QCQP of the form (13.10).

[1] The inner product of two complex matrices is defined to be $A \cdot B := \mathrm{tr}(A^{\mathsf{H}} B) = \sum_{i,j} \overline{A}_{ij} B_{ij}$ and is not equal to $\mathrm{tr}(AB) = \sum_{i,j} A_{ij} B_{ji}$ unless A is Hermitian; see Exercise 13.1.

2. *Voltage limits*: Let $J_j := e_j e_j^{\mathsf{H}}$ denote the Hermitian matrix with a single 1 in the (j,j)th entry and 0 everywhere else. Then squared voltage magnitude $|V_j|^2 = V^{\mathsf{H}} J_j V$ is a quadratic form. It will be lower and upper bounded by $v_j^{\min}$ and $v_j^{\max}$ in (13.4b) respectively.

3. *Line limits*: For the first set of constraints in (13.4c), use (13.1) to write

$$I_{jk} \;=\; y_{jk}^s(V_j - V_k) \,+\, y_{jk}^m V_j \;=\; \left(y_{jk}^s(e_j - e_k)^{\mathsf{T}} \,+\, y_{jk}^m e_j^{\mathsf{T}}\right) V$$

Hence $|I_{jk}|^2 = V^{\mathsf{H}} \hat{Y}_{jk} V$, which will be upper bounded by $\ell_{jk}^{\max}$, where

$$\hat{Y}_{jk} \;:=\; \left(y_{jk}^s(e_j - e_k)^{\mathsf{T}} \,+\, y_{jk}^m e_j^{\mathsf{T}}\right)^{\mathsf{H}} \left(y_{jk}^s(e_j - e_k)^{\mathsf{T}} \,+\, y_{jk}^m e_j^{\mathsf{T}}\right)$$

The matrix $\hat{Y}_{jk}$ is Hermitian and hence $V^{\mathsf{H}} \hat{Y}_{jk} V$ is indeed a real number. Similarly for bounds on $|I_{kj}|^2$.

Putting all this together, OPF (13.9) can be written as a standard QCQP

$$\textbf{OPF}: \qquad \min_{V \in \mathbb{C}^{N+1}} \quad V^{\mathsf{H}} C_0 V \tag{13.15a}$$
$$\text{s.t.} \quad p_j^{\min} \;\le\; V^{\mathsf{H}} \Phi_j V \;\le\; p_j^{\max}, \qquad j \in \overline{N} \tag{13.15b}$$
$$q_j^{\min} \;\le\; V^{\mathsf{H}} \Psi_j V \;\le\; q_j^{\max}, \qquad j \in \overline{N} \tag{13.15c}$$
$$v_j^{\min} \;\le\; V^{\mathsf{H}} J_j V \;\le\; v_j^{\max}, \qquad j \in \overline{N} \tag{13.15d}$$
$$V^{\mathsf{H}} \hat{Y}_{jk} V \;\le\; \ell_{jk}^{\max}, \qquad (j,k) \in E \tag{13.15e}$$
$$V^{\mathsf{H}} \hat{Y}_{kj} V \;\le\; \ell_{kj}^{\max}, \qquad (j,k) \in E \tag{13.15f}$$

This form will be used to derive a convex relaxation in Chapter 14.2. As mentioned above the OPF formulation here does not require assumption C4.1 that $y_{jk}^s = y_{kj}^s$, and hence can accommodate single-phase transformers that have complex turns ratios.

Instead of (13.15f), line limits are sometimes expressed in terms of line power flows. The next example shows how to express such limits on real and reactive line flows as quadratic constraints. See Exercise 13.7 on how to express limits on apparent powers $|S_{jk}(V)|$, $|S_{kj}(V)|$ as inhomogeneous quadratic constraints.

Example 13.2 (Quadratic line power limit)**.** Use (13.2) to write the line limit

$$S_{jk}^{\min} \le S_{jk}(V) \le S_{jk}^{\max}, \qquad S_{kj}^{\min} \le S_{kj}(V) \le S_{kj}^{\max}, \qquad (j,k) \in E \tag{13.16}$$

in terms of quadratic forms in V.

Solution. We will rewrite the first constraint in (13.16) on $S_{jk}(V)$ as a quadratic constraint; the constraint on $S_{kj}(V)$ can be similarly converted. Using the expression of I_{jk}, $S_{jk}(V)$ in quadratic form is:

$$\begin{aligned} S_{jk}(V) \;&=\; V_j I_{jk}^{\mathsf{H}} \;=\; \left(e_j^{\mathsf{H}} V\right)\left(y_{jk}^s(e_j - e_k)^{\mathsf{T}} V \,+\, y_{jk}^m e_j^{\mathsf{T}} V\right)^{\mathsf{H}} \\ &=\; e_j^{\mathsf{H}}\left(VV^{\mathsf{H}}\right)\left(\left(y_{jk}^s + y_{jk}^m\right)^{\mathsf{H}} e_j \,-\, \left(y_{jk}^s\right)^{\mathsf{H}} e_k\right) \\ &=\; \operatorname{tr}\left(\tilde{Y}_{jk}\left(VV^{\mathsf{H}}\right)\right) \;=:\; V^{\mathsf{H}} \tilde{Y}_{jk} V \end{aligned}$$

where

$$\tilde{Y}_{jk} \;:=\; \left(\left(y^s_{jk} + y^m_{jk}\right)^{\mathsf{H}} e_j \;-\; \left(y^s_{jk}\right)^{\mathsf{H}} e_k \right) e_j^{\mathsf{H}} \tag{13.17a}$$

$\tilde{Y}_{jk}$ is not Hermitian and hence $V^{\mathsf{H}}\tilde{Y}_{jk}V$ is a complex number. Define the Hermitian and skewed Hermitian components of $\tilde{Y}_{jk}$:

$$\tilde{\Phi}_{jk} \;:=\; \frac{1}{2}\left(\tilde{Y}^{\mathsf{H}}_{jk} + \tilde{Y}_{jk}\right) \qquad \text{and} \qquad \tilde{\Psi}_{jk} \;:=\; \frac{1}{2\mathbf{i}}\left(\tilde{Y}^{\mathsf{H}}_{jk} - \tilde{Y}_{jk}\right) \tag{13.17b}$$

so that

$$\mathrm{Re}\left(S_{jk}\right) \;=\; V^{\mathsf{H}}\tilde{\Phi}_{jk}V \qquad \text{and} \qquad \mathrm{Im}\left(S_{jk}\right) \;=\; V^{\mathsf{H}}\tilde{\Psi}_{jk}V \tag{13.17c}$$

Hence the constraint $S^{\min}_{jk} \leq S_{jk}(V) \leq S^{\max}_{jk}$ becomes a pair of quadratic constraints:

$$\begin{aligned}
\mathrm{Re}\left(S^{\min}_{jk}\right) \;&\leq\; V^{\mathsf{H}}\tilde{\Phi}_{jk}V \;\leq\; \mathrm{Re}\left(S^{\max}_{jk}\right) \\
\mathrm{Im}\left(S^{\min}_{jk}\right) \;&\leq\; V^{\mathsf{H}}\tilde{\Psi}_{jk}V \;\leq\; \mathrm{Im}\left(S^{\max}_{jk}\right)
\end{aligned}$$

We can also write down the entries of matrix $\tilde{Y}_{jk}$ explicitly using (4.2):

$$\begin{aligned}
S_{jk}(V) \;&:=\; V_j I^{\mathsf{H}}_{jk} \;=\; \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V^{\mathsf{H}}_k\right) \;+\; \left(y^m_{jk}\right)^{\mathsf{H}} |V_j|^2 \\
&=\; \left(y^s_{jk} + y^m_{jk}\right)^{\mathsf{H}} |V_j|^2 \;+\; \left(-y^s_{jk}\right)^{\mathsf{H}} V_j V^{\mathsf{H}}_k \;=:\; V^{\mathsf{H}}\tilde{Y}s_{jk}V
\end{aligned}$$

where, since $V^{\mathsf{H}}\tilde{Y}_{jk}V = \sum_{m,n} [\tilde{Y}_{jk}]_{mn} V^{\mathsf{H}}_m V_n$,

$$\left[\tilde{Y}_{jk}\right]_{mn} \;:=\; \begin{cases} \left(y^s_{jk} + y^m_{jk}\right)^{\mathsf{H}} & m = n = j \\ \left(-y^s_{jk}\right)^{\mathsf{H}} & m = k,\; n = j \\ 0 & \text{otherwise} \end{cases}$$

□

13.1.4 Three-phase devices

A key assumption underlying our OPF formulation is that all controllable devices are the single-phase devices that make up three-phase devices. Therefore internal variables u_j are optimization variables (i.e., $V_j^{Y/\Delta}$ for voltage sources, $I_j^{Y/\Delta}$ for current sources, $\left(s_j^{Y/\Delta}, I_j^{\Delta}\right)$ for power sources). Their values determine the terminal variables (V_j, I_j, s_j) through conversion rules. These terminal variables interact over the network through either the current balance equation $I = YV$ or the power balance equation, but they are typically not directly controllable. In this chapter we mostly use the power balance equation to relate the terminal voltages and power injections (V_j, s_j). We therefore use the conversion rules (and external models of impedances) of Chapter 7.3 to relate an internal variable u_j of a three-phase device j to its terminal voltage and power (V_j, s_j).

1. *Voltage source $u_j := V_j^{Y/\Delta}$*: An ideal voltage source introduces its internal voltage $V_j^{Y/\Delta} \in \mathbb{C}^3$ as an additional optimization variable and a linear constraint to relate it to the terminal voltage V_j (from the conversion rules (7.8) and (7.9a)):

$$\begin{aligned} Y \text{ configuration:} \qquad V_j &= V_j^Y + \gamma_j^Y \mathbf{1} && (13.18a)\\ \Delta \text{ configuration:} \qquad \Gamma V_j &= V_j^\Delta && (13.18b)\end{aligned}$$

We assume here that the neutral voltage $\gamma_j^Y := V_j^n$ of a Y-configured device is a given parameter. For example, $\gamma_j^Y = 0$ if the neutral of the Y-configured device directly grounded and all voltages are defined with respect to the ground.

2. *Current source $u_j := I_j^{Y/\Delta}$*: An ideal current source introduces its internal current $I_j^{Y/\Delta} \in \mathbb{C}^3$ as an additional optimization variable and a quadratic constraint to relate $I_j^{Y/\Delta}$ to the terminal variables (V_j, s_j) (from the conversion rules (7.8) and (7.10c)):

$$\begin{aligned} Y \text{ configuration:} \qquad s_j &= -\text{diag}\left(V_j I_j^{Y\mathsf{H}}\right) && (13.18c)\\ \Delta \text{ configuration:} \qquad s_j &= -\text{diag}\left(V_j I_j^{\Delta\mathsf{H}}\Gamma\right) && (13.18d)\end{aligned}$$

3. *Power source $u_j := \left(s_j^{Y/\Delta}, I_j^{Y/\Delta}\right)$*: For an ideal power source, we assume that the internal power and current $\left(s_j^{Y/\Delta}, I_j^{Y/\Delta}\right)$ are additional optimization variables. We assume the neutral voltage $\gamma_j^Y := V_j^n$ of a Y-configured power source is a given parameter. They are related to terminal voltage and power (V_j, s_j) according to the conversion rules (7.8) and (7.10c):

$$\begin{aligned} Y \text{ configuration:} \qquad s_j &= -\text{diag}\left(V_j I_j^{Y\mathsf{H}}\right), \qquad & s_j &= -s_j^Y - \gamma_j^Y \bar{I}_j^Y && (13.18e)\\ \Delta \text{ configuration:} \qquad s_j &= -\text{diag}\left(V_j I_j^{\Delta\mathsf{H}}\Gamma\right), \qquad & s_j^\Delta &= \text{diag}\left(\Gamma V_j I_j^{\Delta\mathsf{H}}\right) && (13.18f)\end{aligned}$$

For a Y-configured power source, if $\gamma_j^Y = 0$, then additional optimization variable is s_j^Y and the conversion rule reduces to

$$Y \text{ configuration:} \qquad s_j = -s_j^Y$$

4. *Impedance $\left(z_j^Y, \gamma_j^Y\right)$ or z_j^Δ*: An impedance, if not controllable, does not introduce addition optimization variable but imposes an additional constraint on the terminal variables (V_j, s_j) (from (7.19a) and Theorem 7.4):

$$\begin{aligned} Y \text{ configuration:} \qquad s_j &= -\text{diag}\left(V_j \left(V_j - \gamma_j^Y \mathbf{1}\right)^{\mathsf{H}} y_j^{Y\mathsf{H}}\right) && (13.18g)\\ \Delta \text{ configuration:} \qquad s_j &= -\text{diag}\left(V_j V_j^{\mathsf{H}} Y_j^{\Delta\mathsf{H}}\right) && (13.18h)\end{aligned}$$

where $y_j^{Y/\Delta} := \left(z_j^{Y/\Delta}\right)^{-1}$, $Y_j^\Delta := \Gamma^{\mathsf{T}} y^\Delta \Gamma$. The neutral voltage $\gamma_j^Y := V_j^n$ is usually a fixed parameter.

The conversion rule (13.18) takes the form $f_j^{Y/\Delta}\left(V_j, s_j, u_j\right) = 0$. Note the structural similarity between Y and Δ configurations when $\gamma_j^Y := V_j^n = 0$. Once an optimal solution $\left(V_j^{\text{opt}}, s_j^{\text{opt}}, u_j^{\text{opt}}\right)$ of an OPF problem is chosen, other internal variables for each device j can be derived (possibly requiring additional information e.g. β_j of an ideal voltage source).

Remark 13.5 (Implicit optimization over (γ_j, β_j))**.** The constraint (13.18b) for a Δ-configured device does not determined the terminal voltage V_j uniquely and therefore an optimal V_j also determines an optimal zero-sequence voltage $\gamma_j^\Delta := \frac{1}{3}\mathbf{1}^\mathsf{T} V_j$. If γ_j^Δ is given instead, then (13.18b) should be replaced by $V_j = \Gamma^\dagger V_j^\Delta + \gamma_j \mathbf{1}$. Similarly for other devices, e.g., Δ-configured impedance.

Optimization over I_j^Δ in current source and power source implicitly chooses an optimal zero-sequence current $\beta_j := \frac{1}{3}\mathbf{1}^\mathsf{T}\beta_j^\Delta$. If β_j is given then it imposes an additional constraint through the conversion rule $I_j^\Delta = -\frac{1}{3}\Gamma I_j + \beta_j \mathbf{1}$ (and express I_j in terms of $\left(V_j, s_j\right)$). $\square$

13.1.5 Three-phase OPF

We still assume without loss of generality that there is a single three-phase device connected to each bus j. We now generalize the single-phase OPF formulation (13.5) to the three-phase setting. A key assumption underlying our formulation is that all controllable devices are the single-phase devices that make up three-phase devices. A three-phase OPF problem is defined by its optimization variables, its device model with operational constraints, its network equations and constraints, as well as its cost function. We describe them in turn.

Optimization variables. There are two types of optimization variables (u, x). The internal variable $u := (u_j, j \in \overline{N})$ represents controllable quantities of the three-phase devices such as the internal voltage $V_j^{Y/\Delta}$ of a voltage source or the internal power $s_j^{Y/\Delta}$ of a power source. The terminal variable $x := \left(V_j, s_j, j \in \overline{N}\right)$ represents the terminal voltages and power injections. These variables interact over the network through either the current balance equation $I = YV$ or the power balance equation, but they are typically not directly controllable.

Device models. The device models are described in Chapter 13.1.4 with internal variables u_j that depend on the types of devices and their configurations:

1. *Voltage source* $u_j := V_j^{Y/\Delta}$.
2. *Current source* $u_j := I_j^{Y/\Delta}$.
3. *Power source* $u_j := \left(s_j^{Y/\Delta}, I_j^{Y/\Delta}\right)$.
4. *Impedance* $\left(z_j^Y, \gamma_j^Y\right)$ *or* z_j^Δ.

These internal variables u_j are related to terminal voltages and powers $x_j := \left(V_j, s_j\right)$ according to (13.18). The conversion rule (13.18) is local at each bus j and takes the form $f_j^{Y/\Delta}\left(u_j, V_j, s_j\right) = 0$.

The operational constraints on the internal variables u_j are:

1. *Voltage source* $u_j := V_j^{Y/\Delta}$*:*

$$v_j^{Y/\Delta \min} \;\leq\; \text{diag}\left(u_j u_j^{\mathsf{H}}\right) \;\leq\; v_j^{Y/\Delta \max} \tag{13.19a}$$

2. *Current source* $u_j := I_j^{Y/\Delta}$:

$$\text{diag}\left(u_j u_j^{\mathsf{H}}\right) \;\leq\; \ell_j^{Y/\Delta \max} \tag{13.19b}$$

3. *Power source* $u_j := (u_{j1}, u_{j2}) := \left(s_j^{Y/\Delta}, I_j^{Y/\Delta}\right)$:

$$s_j^{Y/\Delta \min} \;\leq\; u_{j1} \;\leq\; s_j^{Y/\Delta \max}, \qquad\qquad \text{diag}\left(u_{j2}\, u_{j2}^{\mathsf{H}}\right) \;\leq\; \ell_j^{Y/\Delta \max} \tag{13.19c}$$

These constraints are local at each bus j and takes the form $g_j^{Y/\Delta}\left(u_j\right) \leq 0$.

Power flow equations and constraints. The power flow equations relate the terminal variables $x := (V, s)$ (from (9.12)):

$$s_j \;=\; \sum_{k: j \sim k} \text{diag}\left(V_j (V_j - V_k)^{\mathsf{H}} \left(y_{jk}^s\right)^{\mathsf{H}} \;+\; V_j V_j^{\mathsf{H}} \left(y_{jk}^m\right)^{\mathsf{H}}\right), \qquad j \in \overline{N} \tag{13.20}$$

which directly extend the single-phase equations (13.3). This constraint is global as it couples voltages and powers (V_j, s_j) at all buses j.

The operational constraints on $x := (V, s)$ are the same as (13.4) for single-phase OPF, except that the variables and their bounds are 3-dimensional vectors, rather than scalars, for three-phase networks:

$$
\begin{aligned}
&\text{injection limits:} && s_j^{\min} \;\leq\; s_j \;\leq\; s_j^{\max}, && j \in \overline{N} && (13.21a)\\
&\text{voltage limits:} && v_j^{\min} \;\leq\; \text{diag}\left(V_j V_j^{\mathsf{H}}\right) \;\leq\; v_j^{\max}, && j \in \overline{N} && (13.21b)\\
&\text{line limits:} && \text{diag}\left(I_{jk}(V)\, I_{jk}^{\mathsf{H}}(V)\right) \;\leq\; \ell_{jk}^{\max}, \;\; \text{diag}\left(I_{kj}(V)\, I_{kj}^{\mathsf{H}}(V)\right) \;\leq\; \ell_{kj}^{\max}, && (j,k) \in E && (13.21c)
\end{aligned}
$$

where $\left(I_{jk}(V), I_{kj}(V)\right)$ in (13.21c) are given by (8.4a) reproduced here:

$$I_{jk}(V) \;=\; y_{jk}^s \left(V_j - V_k\right) \;+\; y_{jk}^m V_j, \qquad\qquad I_{kj}(V) \;=\; y_{kj}^s \left(V_k - V_j\right) \;+\; y_{kj}^m V_k$$

The constraint (13.21a) can be due to limits on the busbar to which the three-phase device is connected. The constraints (13.21a)(13.21b) are local at each bus j but (13.21c) is global.

Cost function. As for single-phase OPF, the cost function $C_0(u,x)$ may represent generation cost, real power loss, estimation error, voltage deviations, or user disutility, depending on applications. For instance to minimize the cost of real power generations we can use

$$C_0(u,x) \; := \; C_0(u,V,s) \; := \sum_{\text{gens. } j} c_j \mathbf{1}^{\mathsf{T}} \mathrm{Re}\left(s_j^{Y/\Delta}\right)$$

Other example costs include estimation error in state estimation, and user disutility in demand response.

Define the feasible set

$$\mathbb{V}_{3p} \; := \; \{(u,x) := (u,V,s) \mid (u,x) \text{ satisfies (13.18)(13.19)(13.20)(13.21)}\} \tag{13.22a}$$

Then the simple OPF formulation in the three-phase setting is

$$\min_{(u,x)} \; C_0(u,x) \quad \text{s.t.} \quad (u,x) \in \mathbb{V}_{3p} \tag{13.22b}$$

Since the constraints (13.20)(13.21c) do not require assumption C9.1 that $y_{jk}^s = y_{kj}^s$, the OPF formulation (13.22) can accommodate three-phase transformers whose admittance matrices Y are not block symmetric, e.g., transformers in ΔY and $Y\Delta$ configurations.

Remark 13.6 (Uncontrollable parameters)**.** As for single-phase OPF, the formulation (13.22) allows the case where a quantity is not an optimization variable but a given parameter. For instance a given uncontrollable constant-power load or a given renewable generation at bus j can be represented by setting $s_j^{Y/\Delta} = s_j^{Y/\Delta\,\min} = s_j^{Y/\Delta\,\max}$ to the specified value. □

Structurally the three-phase OPF (13.22) takes the form with $x := (V,s)$:

$$\min_{(u,x)} \; C_0(u,x) \tag{13.23a}$$

$$\text{s.t.} \quad f_j^{Y/\Delta}\left(u_j, V_j, s_j\right) \; = \; 0, \quad g_j^{Y/\Delta}\left(u_j\right) \; \leq \; 0, \qquad j \in \overline{N} \tag{13.23b}$$

$$f(x) \; = \; 0, \quad g(x) \; \leq \; 0 \tag{13.23c}$$

where (13.23b) represents the conversion rule (13.18) and operational constraint (13.19) on the internal variable u, and (13.23c) represents the power flow equation (13.20) and operational constraint (13.21) on the terminal variable $x := (V,s)$. The local constraints (13.23b) generalize (13.6) from single-phase systems to three-phase systems.

13.1.6 Three-phase OPF as QCQP

The three-phase OPF (13.22) can be written as a QCQP in (V,u), following the same process for the single-phase OPF. Specifically we treat the power flow equation (13.20) as defining the terminal powers $s_j(V)$ as functions of V. This allows us to eliminate s from (13.23), as well as the power flow equation $f(V,s) = 0$ in (13.23c).

Operational constraints as quadratic forms. First we reduce the two constraints in (13.23c) into a single constraint in V of the form

$$g(V, s(V)) \leq 0$$

where g consists of quadratic forms in V. The operational constraints (13.21) on the terminal variables (V, s) can be converted into quadratic forms in V following the same derivation in Chapter 13.1.3, but applied to the single-phase equivalent circuit.

1. *Injection limits*: Let $Y \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ denote the single-phase equivalent admittance matrix. Define the matrix $Y_j^\phi := e_j^\phi e_j^{\phi \mathsf{H}} Y$ where $e_j^\phi \in \{0,1\}^{3(N+1)}$ is the unit vector with a single 1 at the (j,ϕ)th entry and 0 elsewhere. Define the Hermitian and skew Hermitian components of $Y_j^{\phi \mathsf{H}}$:
$$\Phi_j^\phi := \frac{1}{2}\left(Y_j^{\phi \mathsf{H}} + Y_j^\phi\right) \qquad \text{and} \qquad \Psi_j^\phi := \frac{1}{2\mathbf{i}}\left(Y_j^{\phi \mathsf{H}} - Y_j^\phi\right) \tag{13.24a}$$
Then
$$p_j^\phi := \mathrm{Re}\left(s_j^\phi\right) = V^{\mathsf{H}}\Phi_j^\phi V \qquad \text{and} \qquad q_j^\phi := \mathrm{Im}(s_j^\phi) = V^{\mathsf{H}}\Psi_j^\phi V$$
Then the injection limits become
$$p_j^{\phi\,\min} \leq V^{\mathsf{H}}\Phi_j^\phi V \leq p_j^{\phi\,\max}, \qquad q_j^{\phi\,\min} \leq V^{\mathsf{H}}\Psi_j^\phi V \leq q_j^{\phi\,\max}, \qquad j \in \overline{N} \tag{13.24b}$$

2. *Voltage limits*: Let $J_j^\phi := e_j^\phi \left(e_j^\phi\right)^{\mathsf{H}}$ denote the $3(N+1)\times 3(N+1)$ diagonal Hermitian matrix with a single 1 in the $(j\phi, j\phi)$th entry and 0 everywhere else. Then terminal voltage limits are
$$v_j^{\phi\,\min} \leq V^{\mathsf{H}} J_j^\phi V \leq v_j^{\phi\,\max}, \qquad j \in \overline{N} \tag{13.24c}$$

3. *Line limits*: The same derivation as that for single-phase OPF shows that the limit on the sending-end current I_{jk}^ϕ in the phase-a line is (Exercise 13.8)
$$\left|I_{jk}^\phi\right|^2 := V^{\mathsf{H}}\hat{Y}_{jk}^\phi V \leq \ell_{jk}^{\phi\,\max}, \qquad (j,k) \in E \tag{13.24d}$$
where $\hat{Y}_{jk}^\phi := \tilde{Y}_{jk}^{\mathsf{H}} E^\phi \tilde{Y}_{jk}$ is a $3(N+1)\times 3(N+1)$ matrix and $\tilde{Y}_{jk}$ is a $3 \times 3(N+1)$ matrix given by
$$\tilde{Y}_{jk} := \left((e_j - e_k)^{\mathsf{T}} \otimes y_{jk}^s + e_j^{\mathsf{T}} \otimes y_{jk}^m\right)$$
Here $e_j \in \{0,1\}^{N+1}$ and $e^\phi \in \{0,1\}^3$ are unit vectors of different sizes with a single 1 at the jth and ϕth position respectively, $E^\phi := e^\phi e^{\phi \mathsf{T}}$, and $\mathbb{I}$ is the 3×3 identity matrix. The matrix $\hat{Y}_{jk}$ is Hermitian and hence $V^{\mathsf{H}}\hat{Y}_{jk}^\phi V$ is indeed a real number. Similarly for $\left|I_{kj}^\phi\right|^2$.

Conversion rules as quadratic forms. Next we reduce the equality constraints $f_j^{Y/\Delta}(u_j, V_j, s_j) = 0$ in (13.23b) into constraints in (u, V) of the form

$$f_j^{Y/\Delta}(u_j, V, s_j(V)) \;=\; 0, \qquad j \in \overline{N}$$

where $f_j^{Y/\Delta}$ consist of quadratic forms in (u_j, V). Here $s_j(V) := \left(s_j^a(V), s_j^b(V), s_j^c(V)\right)$ and

$$s_j^{\phi}(V) \;=\; V^{\mathsf{H}}\left(Y_j^{\phi\mathsf{H}}\right)V \;=\; V^{\mathsf{H}}\left(\Phi_j^{\phi} + \mathbf{i}\Psi_j^{\phi}\right)V, \qquad \phi \in \{a,b,c\},\; j \in \overline{N} \tag{13.25}$$

where $Y_j^{\phi} := e_j^{\phi} e_j^{\phi\mathsf{H}} Y$ and Φ_j^{ϕ} and Ψ_j^{ϕ} are defined in (13.24a). This transforms the original local constraints into global constraints since the function $s_j(V)$ depends on V_k at all neighbors k of j.

Let

$$e^a \;:=\; (1,0,0), \qquad e^b \;:=\; (0,1,0), \qquad e^c \;:=\; (0,0,1), \qquad E^{\phi} := e^{\phi} e^{\phi\mathsf{H}} \in \mathbb{C}^{3\times 3} \tag{13.26a}$$

$$e_j \;\in\; \{0,1\}^{N+1}, \qquad e_j^{\phi} \;\in\; \{0,1\}^{3(N+1)}, \qquad \phi \;\in\; \{a,b,c\} \tag{13.26b}$$

where e_j has a single 1 in the jth position and e_j^{ϕ} has a single 1 in the $j\phi$th position. Then $V_j \in \mathbb{C}^3$ can be written in terms of $V \in \mathbb{C}^{3(N+1)}$ as follows:

$$V_j \;=\; \left(e_j \otimes \mathbb{I}\right)^{\mathsf{H}} V \;=\; \left(e_j^{\mathsf{H}} \otimes \mathbb{I}\right) V, \qquad V_j^{\phi} \;=\; e_j^{\phi\mathsf{H}} V, \quad \phi \in \{a,b,c\} \tag{13.27}$$

where $\mathbb{I}$ is the identity matrix of size 3.

We now use (13.25)(13.26)(13.27) to convert the conversion rules $f_j^{Y/\Delta}$ in (13.18) into inhomogeneous quadratic forms in (u_j, V). They can then be homogenized using the identity (13.14) in Remark 13.4.

1. *Voltage source* $u_j := V_j^{Y/\Delta}$: Application of (13.27) to the conversion rules (13.18a) (13.18b) leads to the following linear constraints in (u_j, V):

$$Y \text{ configuration:} \qquad \left(e_j^{\mathsf{H}} \otimes \mathbb{I}\right) V \;=\; u_j + \gamma_j^Y \mathbf{1} \tag{13.28a}$$

$$\Delta \text{ configuration:} \qquad \Gamma\left(e_j^{\mathsf{H}} \otimes \mathbb{I}\right) V \;=\; u_j \tag{13.28b}$$

   where $\gamma_j^Y := V_j^n$ is assumed given (e.g., $\gamma_j^Y = 0$).

2. *Current source* $u_j := I_j^{Y/\Delta}$: The conversion rules (13.18c)(13.18d) for a current source are equivalent to the following inhomogeneous quadratic equations in (u_j, V) (Exercise 13.9):

$$Y \text{ configuration:} \qquad s_j^{\phi}(V) \;=\; -u_j^{\mathsf{H}}\left(e_j^{\mathsf{H}} \otimes E^{\phi}\right) V \tag{13.28c}$$

$$\Delta \text{ configuration:} \qquad s_j^{\phi}(V) \;=\; -u_j^{\mathsf{H}}\left(e_j^{\mathsf{H}} \otimes (\Gamma E^{\phi})\right) V \tag{13.28d}$$

   where $s_j(V)$ is given in (13.25) and e_j is defined in (13.26b).

3. *Power source* $u_j := \left(s_j^{Y/\Delta}, I_j^{Y/\Delta}\right)$: For a Y-configured power source let $u_j =: (u_{j1}, u_{j2})$ where $u_{j1} := s_j^Y$ and $u_{j2} := I_j^Y$. Then the conversion rule (13.18e) is equivalent to the following inhomogeneous quadratic equations in (u_j, V) (Exercise 13.10):

$$Y: \quad s_j^{\phi}(V) \;=\; -u_{j2}^{\mathsf{H}}\left(e_j^{\mathsf{H}} \otimes E^{\phi}\right)V, \qquad s_j(V) \;=\; -u_{j1} - \gamma_j^Y \bar{u}_{j2}, \qquad \phi \in \{a,b,c\} \tag{13.28e}$$

where $s_j(V)$ is given in (13.25) and $\gamma_j^Y := V_j^n$ is assumed given (e.g., $\gamma_j^Y = 0$).

For a Δ-configured power source let $u_j =: (u_{j1}, u_{j2})$ where $u_{j1} := s_j^{\Delta}$ and $u_{j2} := I_j^{\Delta}$. Then the conversion rule (13.18f) is equivalent to the following inhomogeneous quadratic equations in (u_j, V):

$$\Delta: \quad s_j^{\phi}(V) \;=\; -u_{j2}^{\mathsf{H}}\left(e_j^{\mathsf{H}} \otimes (\Gamma E^{\phi})\right)V, \quad u_{j1}^{\phi\varphi} \;=\; u_{j2}^{\mathsf{H}}\left(e_j^{\mathsf{H}} \otimes \left(E^{\phi}\Gamma\right)\right)V, \quad \phi\varphi \in \{ab, bc, ca\} \tag{13.28f}$$

where $s_j(V)$ is given in (13.25).

4. *Impedance* $\left(z_j^Y, \gamma_j^Y\right)$ *or* z_j^{Δ}: The equality constraints imposed by an impedance $\left(z_j^Y, \gamma_j^Y\right)$ or z_j^{Δ} are equivalent to the following inhomogeneous quadratic equations in V (Exercise 13.11):

$$Y \text{ configuration:} \qquad s_j^{\phi}(V) \;=\; V^{\mathsf{H}}\left(\left(e_j e_j^{\mathsf{H}}\right) \otimes \left(y_j^{Y\mathsf{H}} E^{\phi}\right)\right)V \;-\; \gamma_j^{Y\mathsf{H}}\left(e_j^{\mathsf{H}} \otimes \left(\mathbf{1}^{\mathsf{H}} y_j^{Y\mathsf{H}} E^{\phi}\right)\right)V \tag{13.28g}$$

$$\Delta \text{ configuration:} \qquad s_j^{\phi}(V) \;=\; -V^{\mathsf{H}}\left(\left(e_j e_j^{\mathsf{H}}\right) \otimes \left(Y_j^{\Delta\mathsf{H}} E^{\phi}\right)\right)V \tag{13.28h}$$

where $\gamma_j^Y := V_j^n$ is assumed given (e.g., $\gamma_j^Y = 0$), $Y_j^{\Delta} := \Gamma^{\mathsf{T}} y^{\Delta} \Gamma$ and $y_j^{\Delta} := \left(z_j^{\Delta}\right)^{-1}$.

Note the structural similarity between Y and Δ configurations when $\gamma_j^Y := V_j^n = 0$

Three-phase OPF as QCQP. We have thus eliminated the power flow equation $f(V,s) = 0$ and expressed the operational constraint $g(V, s(V)) \le 0$ and conversion rules $f_j^{Y/\Delta}\left(u_j, V_j, s_j(V)\right) = 0$ as quadratic forms in (u, V) in (13.23). This shows that (13.23) is equivalent to:

$$\min_{(u,V)} \quad C_0(u, V, s(V)) \tag{13.29a}$$

$$\text{s.t.} \quad (13.24)\ (13.28), \ \ g_j^{Y/\Delta}\left(u_j\right) \;\le\; 0, \ \ j \in \overline{N} \tag{13.29b}$$

where $s(V)$ is given by (13.25) and $g_j^{Y/\Delta}\left(u_j\right) \le 0$ are given in (13.19). All constraints in (13.29b) are inhomogeneous quadratic constraints which can be homogenized. This will express the problem (13.29) as a standard QCQP, assuming the cost function C_0 can also be expressed as a quadratic form in (u, V).

13.2 Branch flow model

The device models are the same as those for the bus injection model and described in Chapter 13.1.1 for single-phase networks and Chapter 13.1.4 for three-phase networks. OPF in the branch flow model differs only in the variables and power flow equations.

The branch flow model is most useful for radial networks, both single-phased and three-phased, on which we will focus. For simplicity we assume through this section:

- $z^s_{jk} = z^s_{kj}$, or equivalently $y^s_{jk} = y^s_{kj}$, for every line (j,k) (assumption C5.1 for single-phase BFM and assumption C10.1 for three-phase BFM).
- $y^m_{jk} = y^m_{kj} = 0$ for every line (j,k) (see Remark 13.8 for the case of nonzero y^m_{jk}, y^m_{kj}). This is a reasonable assumption on distribution lines where y^m_{jk} and y^m_{kj} are typically much smaller in magnitude than the series admittance y^s_{jk}.

These two assumptions allow us to assume $G = (\overline{N}, E)$ is directed and includes branch variables in only one direction. We denote a line in E from bus j to bus k either by $(j,k) \in E$ or $j \to k$. It is characterized by its series impedance $z_{jk} := z^s_{jk}$. Without loss of generality we take bus 0 as the root of the tree.

13.2.1 Single-phase OPF

Associated with each line (j,k) are branch variables (ℓ_{jk}, S_{jk}). Let $(s,v) := (s_j, v_j, j \in \overline{N})$ and $(\ell, S) := (\ell_{jk}, S_{jk}, j \to k \in E)$. Let $x := (s, v, \ell, S)$ in $\mathbb{R}^{3(N+1+M)}$ with $M = N$ since G is a tree. In particular we use the model (5.6) with the down orientation (all lines point away from bus 0), reproduced here:

$$\sum_{k: j \to k} S_{jk} = S_{ij} - z_{ij}\ell_{ij} + s_j, \qquad j \in \overline{N} \tag{13.30a}$$

$$v_j - v_k = 2\,\mathrm{Re}\left(z^{\mathsf{H}}_{jk} S_{jk}\right) - |z_{jk}|^2 \ell_{jk}, \qquad j \to k \in E \tag{13.30b}$$

$$v_j \ell_{jk} = |S_{jk}|^2, \qquad j \to k \in E \tag{13.30c}$$

where, in (13.30a), bus $i := i(j)$ denotes the unique adjacent node of j on the path from node 0 to node j, with the understanding that when $j = 0$ then $S_{i0} := 0$ and $\ell_{i0} := 0$. The vector v includes v_0 and s includes s_0. The injection, voltage and line limits can be expressed in terms of the BFM variable x:

$$s_j^{\min} \le s_j \le s_j^{\max}, \qquad v_j^{\min} \le v_j \le v_j^{\max}, \qquad \ell_{jk} \le \ell_{jk}^{\max}, \qquad j \in \overline{N},\ (j,k) \in E \tag{13.30d}$$

Let the cost function in the branch flow model be $C(x)$. Let the feasible set be

$$\mathbb{T} := \{x : (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies } (13.30)\} \tag{13.31a}$$

Then the optimal power flow problem in the branch flow model is:
OPF:

$$\min_x \ C(x) \quad \text{subject to} \quad x \in \mathbb{T} \tag{13.31b}$$

We assume the cost functions $C(x)$ here and $C_0(V)$ in the single-phase OPF problem (13.9) or (13.15) in the bus injection model represent the same function but in terms of different variables. Since $\mathbb{T} \equiv \mathbb{V}$ by Theorem 5.2, the single-phase OPF problem (13.31) in the branch flow model is equivalent to (13.9) or (13.15) in the bus injection model.

Remark 13.7 (Current sources and impedances)**.** The model (13.30) includes only voltage and power sources whose controllable variables are v_j and s_j respectively. A current source will introduce its current $I_j \in \mathbb{C}$ as an additional variable and an equality constraint $|s_j|^2 = v_j|I_j|^2$ that relate I_j to (s_j, v_j). An impedance z_j will introduce an equality constraint $s_j = v_j/z_j^{\mathsf{H}}$ on (s_j, v_j). If z_j is controllable, e.g., representing a switched capacitor, then z_j is an additional variable. For simplicity we restrict ourselves to voltage and power sources only. □

Remark 13.8 (With shunt admittances)**.** The feasible set $\mathbb{T}$ is based on the DistFlow equations (13.30a)–(13.30c) that assume zero shunt admittances on the lines. Radial networks mostly model distribution systems where the shunt admittances y_{jk}^m and y_{kj}^m are typically much smaller in magnitude than the series admittance y_{jk}^s.

An OPF problem that includes line shunts can be based on the power flow model (5.4) that includes branch variables $\ell := \big(\ell_{jk}, \ell_{kj}, (j,k) \in E\big)$, $S := \big(S_{jk}, S_{kj}, (j,k) \in E\big)$ in both directions. The feasible set is

$$\mathbb{X}_{\text{tree}} \;:=\; \{x : (s, v, \ell, S) \in \mathbb{R}^{9N+3} \mid x \text{ satisfies } (5.4), (13.30\text{d})\}$$

and the OPF problem is:

$$\min_x \; C(x) \quad \text{subject to} \quad x \in \mathbb{X}_{\text{tree}}$$

This is studied in Exercise 13.13. □

13.2.2 Three-phase OPF

We now extend the single-phase OPF (13.31) to the three-phase setting. We make the same assumptions $y_{jk}^s = y_{kj}^s$ and $y_{jk}^m = y_{kj}^m = 0$ as for single-phase OPF. We describe the three-phase optimization variables, device models, power flow equations, operational constraints, and the cost function that define an OPF problem. As in BIM, a key assumption underlying our formulation is that all controllable devices are the single-phase devices that make up three-phase devices. Both BIM and BFM use the same device models and their operational constraints. Their difference lies in the power flow equations that, for BFM, include line variables as well.

Optimization variables. There are two types of optimization variables (u, x). The internal variable $u := (u_j, j \in \overline{N})$ represents controllable quantities of the three-phase devices, as in BIM. The variable x represents both the terminal variables (e.g., a nodal voltage V_j) as well as the line variables (e.g., a line power S_{jk}). The variables x interact over the network through the power balance equation.

Device models. The device models for three-phase BFM are the same as those for three-phase BIM. They are described in Chapter 13.1.4 with the operational constraints (13.19) on the internal variables u_j, $j \in \overline{N}$.

Power flow equations and constraints. Power flow equations relate the following terminal variables and line variables:

$$\begin{array}{llll} s_j \in \mathbb{C}^3, & v_j \in \mathbb{S}_+^3, & V_j \in \mathbb{C}^3, & j \in \overline{N} \\ \ell_{jk} \in \mathbb{S}_+^3, & S_{jk} \in \mathbb{C}^{3\times 3}, & \tilde{I}_{jk} \in \mathbb{C}^3, & j \to k \in E \end{array}$$

where $\mathbb{S}_+^n \subseteq \mathbb{C}^{n\times n}$ is the set of of $n \times n$ complex (Hermitian and) positive semidefinte matrices. Let $s := (s_j, j \in \overline{N}), v := (v_j, j \in \overline{N}), \ell := (\ell_{jk}, (j,k) \in E), S := (S_{jk}, (j,k) \in E)$. Here (s, v, ℓ, S) directly generalize the corresponding variables in the single-phase model. The voltage phasor $V := \left(V_j, j \in \overline{N}\right)$ is needed to express the conversion rule (13.18) for three-phase devices and the line current phasor $\tilde{I} := \left(\tilde{I}_{jk}, j \to k \in E\right)$ is introduced for convenience. Let $x := (s, v, \ell, S, V, \tilde{I})$.

The power flow equations we use are (10.5) in Chapter 10.2, reproduced here, augmented with (13.32e):

$$\sum_{k: j\to k} \operatorname{diag}(S_{jk}) = \operatorname{diag}\left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j, \qquad j \in \overline{N} \tag{13.32a}$$

$$v_j - v_k = \left(z_{jk} S_{jk}^{\mathsf{H}} + S_{jk} z_{jk}^{\mathsf{H}}\right) - z_{jk}\ell_{jk} z_{jk}^{\mathsf{H}}, \qquad j \to k \in E \tag{13.32b}$$

$$\begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \succeq 0, \qquad j \to k \in E \tag{13.32c}$$

$$\operatorname{rank} \begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} = 1, \qquad j \to k \in E \tag{13.32d}$$

$$v_j = V_j V_j^{\mathsf{H}}, \qquad \ell_{jk} = \tilde{I}_{jk}\tilde{I}_{jk}^{\mathsf{H}}, \quad S_{jk} = V_j \tilde{I}_{jk}^{\mathsf{H}}, \qquad j \to k \in E \tag{13.32e}$$

where bus $i := i(j)$ is the unique parent of bus j in (13.32a). Given matrices $\left(v_j, \ell_{jk}, S_{jk}\right)$, the vectors $\left(V_j, \tilde{I}_{jk}\right)$, $j \in \overline{N}$, $j \to k \in E$, are determined uniquely up to a reference angle. These constraints are global.

Remark 13.9. The equation (13.32e) and (13.32c)(13.32d) are equivalent, and therefore redundant, up to an arbitrary reference angle. We keep (13.32c)(13.32d) in the three-phase OPF formulation for the purpose of semidefinite relaxation studied in Chapter 15. □

The operational constraints on x are the three-phase version of the (13.30d):

$$\begin{array}{llll} \text{injection limits:} & s_j^{\min} \le s_j \le s_j^{\max}, & j \in \overline{N} & (13.33a) \\ \text{voltage limits:} & v_j^{\min} \le \operatorname{diag}\left(v_j\right) \le v_j^{\max}, & j \in \overline{N} & (13.33b) \\ \text{line limits:} & \operatorname{diag}\left(\ell_{jk}\right) \le \ell_{jk}^{\max}, & (j,k) \in E & (13.33c) \end{array}$$

The constraint (13.33a) can be due to limits on the busbar to which the three-phase device is connected. All constraints in (13.33) are local at each bus j or on each line (j,k).

Cost function. Let $C(u,x)$ denote the cost function. For instance to minimize the thermal loss in the network we can use

$$C(u,x) := \sum_{(j,k)\in E} \operatorname{diag}^{\mathsf{T}}\left(z_{jk}\right) \operatorname{diag}\left(\ell_{jk}\right)$$

OPF. We assume $V_0 \in \mathbb{C}^3$ is given and impose $v_0 = V_0 V_0^{\mathsf{H}}$. Let the feasible set be

$$\mathbb{T}_{3p} \;:=\; \left\{ (u,x) := (u,s,v,\ell,S,V,\tilde{I}) \mid (u,x) \text{ satisfies (13.18)(13.19)(13.32)(13.33)}, v_0 = V_0 V_0^{\mathsf{H}} \right\} \tag{13.34a}$$

Then the three-phase OPF problem is:

$$\min_{u,x} \; C(u,x) \quad \text{subject to} \quad (u,x) \in \mathbb{T}_{3p} \tag{13.34b}$$

By Theorems 10.1 and 10.2, The feasible set $\mathbb{T}_{3p}$ of the three-phase OPF (13.34) in BFM is equivalent to the feasible set $\mathbb{V}_{3p}$ of the three-phase OPF (13.22) in BIM. Hence these problems are equivalent, provided their cost functions $C(u,x)$ and $C_0(u,x)$ are the same.

13.3 Applications of OPF

13.4 NP hardness

Since the feasible set $\mathbb{V}$ of OPF is generally a nonconvex set, OPF formulated in (**??**) or (13.15) is a nonconvex problem and has been shown to be NP-hard.

13.5 Techniques for scalability

Practical OPF problems can be difficult to solve. This can be due to the sheer number of variables and constraints relative to available solution time. It can also arise from the nonsmoothness or the nonconvexity of the objective or constraint functions that often lead to numerical issues. The nonsmoothness or nonconvexity can take different forms, e.g., nonlinear power flow equations, discrete variables, nondifferentiability of the objective or constraint functions, complementary or disjunctive constraints. All of these features are embodied in security constrained OPF (SCOPF). Practical solutions for a large-scale optimization problem require not only the understanding of basic optimization theory, but also the development of many heuristics tailored to the structure of the specific problem.

In this section we illustrate these computational challenges and some solution techniques through an SCOPF problem proposed by the US Advanced Research Projects Agency - Energy (ARPA-E) in a multi-year Grid Optimization (GO) Competition. The GO Competition aims to accelerate the development of algorithms and software for solving large OPF problems. It was staged as a series of challenges. Challenge 1, which was conducted over the course of 2019, focused on real-time SCOPF [82]. We will present some of the techniques used by the top three winners of the GO Challenge 1 in addressing the scalability, nonconvexity, and nonsmoothness of SCOPF [83, 84, 85]. As we will see effective treatment of complementarity constraints, efficient contingency screening, and robust parallelization of computation have proved to be essential in devising a practical solution.

Even though these techniques are introduced for concreteness in the context of large-scale SCOPF, they are much more widely applicable.

13.5.1 SCOPF formulation

The detailed SCOPF formulation is described in the official specification [82]. We present a highly simplified version to illustrate the main algorithmic ideas in [83, 84, 85] to overcome some of the computational challenges.

Constraints. We start by formulating the constraints of the GO Challenge 1 problem. It can sometimes be difficult to exactly satisfy equality and inequality constraints in a realistic problem. This can be due to modeling or numerical errors, not just the lack of computational resources. Energy management systems in practice however must recommend a decision even when it is impossible to satisfy all constraints of the model. One way to deal with this is to allow some constraint violations in order to practically eliminate infeasibility, but penalize them in the objective.

Let $k = 0$ denote the base case and $k = 1, \ldots, K$ denote contingencies, though we will often refer to the base case also as contingency $k = 0$. Let $\left(p^u_{ki}, q^u_{ki}\right)$ denote uncontrollable loads (or generations) and (p_{ki}, q_{ki}) denote controllable generation levels at buses $i \in \overline{N}$ in contingencies $k \geq 0$. For notational simplicity we assume without loss of generality that there is exactly one uncontrollable injection and one controllable generator at each bus i. We impose the standard voltage and generation limits:

$$\underline{v}_{ki} \leq |V_{ki}| \leq \overline{v}_{ki}, \quad \underline{p}_i \leq p_{ki} \leq \overline{p}_i, \quad \underline{q}_i \leq q_{ki} \leq \overline{q}_i, \qquad k \geq 0,\ i \in \overline{N} \tag{13.35}$$

where $\underline{v}_{ki} \leq \overline{v}_{ki}$, $\underline{p}_i \leq \overline{p}_i$, and $\underline{q}_i \leq \overline{q}_i$ are given constants.

For each line $(i, j) \in E$, let $\left(P_{k,ij}, Q_{k,ij}\right)$ denote the sending-end real and reactive power from buses i to j and $\left(P_{k,ji}, Q_{k,ji}\right)$ denote the sending-end line power in the opposite direction in contingencies $k \geq 0$. Instead of exact real and reactive power balance at bus i, we impose

$$p_{ki} - p^u_{ki} = \sum_{j:j\sim i} P_{k,ij} + \sigma^{p+}_{ki} - \sigma^{p-}_{ki}, \quad \left(\sigma^{p+}_{ki}, \sigma^{p-}_{ki}\right) \geq 0, \quad k \geq 0,\ i \in \overline{N} \tag{13.36a}$$

$$q_{ki} - q^u_{ki} = \sum_{j:j\sim i} Q_{ki} + \sigma^{q+}_{ki} - \sigma^{q-}_{ki}, \quad \left(\sigma^{q+}_{ki}, \sigma^{q-}_{ki}\right) \geq 0, \quad k \geq 0,\ i \in \overline{N} \tag{13.36b}$$

where the nonnegative variables $\left(\sigma^{p+}_{ki}, \sigma^{p-}_{ki}\right)$ are slack variables for real power violations and $\left(\sigma^{q+}_{ki}, \sigma^{q-}_{ki}\right)$ are slack variables for reactive power violations. These slack variables will be penalized in the objective as we will see below.

With a slight abuse of notation we use $(P_{k,ij}(\theta_k, |V_k|), Q_{k,ij}(\theta_k, |V_k|))$ to denote the line power as functions of voltage magnitudes and angles in contingencies $k \geq 0$ defined by:

$$P_{k,ij}(\theta_k, |V_k|) = \left(g^s_{ij} + g^m_{ij}\right)|V_{ki}|^2 - |V_{ki}||V_{kj}|\left(g^s_{ij}\cos(\theta_{ki} - \theta_{kj}) - b^s_{ij}\sin(\theta_{ki} - \theta_{kj})\right) \tag{13.37a}$$

$$Q_{k,ij}(\theta_k, |V_k|) = \left(b^s_{ij} + b^m_{ij}\right)|V_{ki}|^2 - |V_{ki}||V_{kj}|\left(b^s_{ij}\cos(\theta_{ki} - \theta_{kj}) + g^s_{ij}\sin(\theta_{ki} - \theta_{kj})\right) \tag{13.37b}$$

where $\left(g^s_{ij}, b^s_{ij}\right)$ and $\left(g^m_{ij}, b^m_{ij}\right)$ are series and shunt admittances of line (i, j). Similarly for $\big(P_{k,ji}(\theta_k, |V_k|),$ $Q_{k,ji}(\theta_k, |V_k|)\big)$ in the opposite direction on line (i, j). Then we impose the constraints

$$(P_{k,ij}, Q_{k,ij}) = (P_{k,ij}(\theta_k, |V_k|), Q_{k,ij}(\theta_k, |V_k|)), \qquad k \geq 0,\ (i, j) \in E \tag{13.37c}$$

$$(P_{k,ji}, Q_{k,ji}) = (P_{k,ji}(\theta_k, |V_k|), Q_{k,ji}(\theta_k, |V_k|)), \qquad k \geq 0,\ (i, j) \in E \tag{13.37d}$$

Line limits are expressed in terms of apparent power and the sending-end voltage magnitudes, on both ends of the lines $(i, j) \in E$:

$$\begin{aligned}
\sqrt{P_{k,ij}^2 + Q_{k,ij}^2} &\le P_{k,ij}^{\max} |V_{ki}| + \sigma_{k,ij}^e, & k \ge 0,\ (i,j) \in E \qquad (13.38a)\\
\sqrt{P_{k,ji}^2 + Q_{k,ji}^2} &\le P_{k,ij}^{\max} |V_{kj}| + \sigma_{k,ij}^e, & k \ge 0,\ (i,j) \in E \qquad (13.38b)\\
\sigma_{k,ij}^e &\ge 0, & k \ge 0,\ (i,j) \in E \qquad (13.38c)
\end{aligned}$$

where $P_{k,ij}^{\max}$ are given parameters and $\sigma_{k,ij}^e$ are slack variables that measure line limit violations.

When contingency $k \ge 1$ occurs the generators will adjust their real and reactive power to rebalance. This may be necessary even if the contingency is a transmission outage, i.e, the disconnection of a line or a transformer, instead of a generator outage, because the redistribution of line flows may result in different amounts of losses that need to be compensated for by these generators. Moreover the outage may also lead to deviation of tie-line flows from their scheduled values and hence nonzero area control error that must be corrected. The rebalancing is carried out at a fast timescale by frequency control mechanisms (see Chapter 12.4). The effect of the frequency control actions is modeled as follows. The real power at the generators is adjusted proportionally within their generation capacities:

$$p_{ki} = [p_{0i} + \alpha_i \Delta_k]_{\underline{p}_i}^{\overline{p}_i}, \qquad k \ge 1,\ i \in \overline{N} \qquad (13.39a)$$

where p_{0i} are the output levels of generators i in the base case $k = 0$, $\left(\underline{p}_i, \overline{p}_i\right)$ are their lower and upper capacity limits, Δ_k are the total real power contingency response, and $\alpha_i \ge 0$ are called the participation factors of generators i with $\sum_i \alpha_i = 1$. (If generator i does not participate in contingency response then $\alpha_i = 0$.) Here, for real scalars x, $a \le b$, we define $[x]_a^b := \max(a, \min(x, b))$. The reactive power of generators i is adjusted within their capacity limits in an attempt to restore the voltage magnitudes $|V_{ki}|$ to their pre-contingency values, as expressed in:

$$\left\{\underline{q}_i \le q_{ki} \le \overline{q}_i,\ |V_{ki}| = |V_{0i}|\right\} \cup \left\{q_{ki} = \underline{q}_i,\ |V_{ki}| \ge |V_{0i}|\right\} \cup \left\{q_{ki} = \overline{q}_i,\ |V_{ki}| \le |V_{0i}|\right\}, \quad k \ge 1,\ i \in \overline{N} \qquad (13.39b)$$

Variables. To simplify notation define the following nodal vector variables for each contingency:

$$(p_k, q_k, |V_k|, \theta_k) := \left(p_{ki}, q_{ki}, |V_{ki}|, \theta_{ki},\ i \in \overline{N}\right), \quad \sigma_k^{p+} := \left(\sigma_{ki}^{p+}, i \in \overline{N}\right), \quad k \ge 0 \qquad (13.40a)$$

and similarly for $\left(\sigma_k^{p-}, \sigma_k^{q+}, \sigma_k^{q-}\right)$. Define the following branch variables for each contingency:

$$(P_k, Q_k) := \left(P_{k,ij}, Q_{k,ij}, P_{k,ji}, Q_{k,ji},\ (i,j) \in E\right), \quad \sigma_k^e := \left(\sigma_{k,ij}^e,\ (i,j) \in E\right), \quad k \ge 0 \qquad (13.40b)$$

Let

$$\sigma_k \ := \ \left(\sigma_k^{p+}, \sigma_k^{p-}, \sigma_k^{q+}, \sigma_k^{q-}, \sigma_k^{e}\right), \qquad k \geq 0 \tag{13.40c}$$

$$x_k \ := \ (p_k, q_k, |V_k|, \theta_k, \ P_k, Q_k, \ \sigma_k), \qquad k \geq 0 \tag{13.40d}$$

$$y_k \ := \ (x_k, \Delta_k), \qquad k \geq 1 \tag{13.40e}$$

The vector x_0 collects base-case decisions and y_k collect responses to contingencies $k \geq 1$.

SCOPF formulation. The SCOPF problem in the GO Challenge 1 takes the form:

$$\min \quad \sum_i c_i^g(p_{0i}) \ + \ \delta c_0(\sigma_0) \ + \ (1-\delta)\frac{1}{|K|}\sum_{k\geq 1} c_k(\sigma_k) \tag{13.41a}$$

$$\text{over} \quad x_0, (y_k, \ k \geq 1) \tag{13.41b}$$

$$\text{s.t.} \quad (13.35)(13.36)(13.37)(13.38)(13.39) \tag{13.41c}$$

where $c_i^g(p_{0i})$ are the generation costs at buses i in the base case, $c_0(\sigma_0)$ and $c_k(\sigma_k)$ are the penalty functions for constraint violations in the base case $k = 0$ and contingencies $k \geq 1$ respectively, defined as:

$$c_k\left(\sigma_k\right) \ := \ \sum_{i\in\overline{N}} \left(c_{ki}^p\left(\sigma_{ki}^{p+} + \sigma_{ki}^{p-}\right) + c_{ki}^q\left(\sigma_{ki}^{q+} + \sigma_{ki}^{q-}\right)\right) \ + \sum_{(i,j)\in E} c_{k,ij}^e\left(\sigma_{k,ij}^e\right), \quad k \geq 0 \tag{13.41d}$$

and $\delta \in [0,1]$ is the weight to trade off the penalty in the base case against the average contingency penalty. The functions c_{ki}^p, c_{ki}^q, $c_{k,ij}^e$, $k \geq 0$, are convex piecewise linear, each with three segments of increasing slopes.

Two-stage formulation. The problem (13.41) can also be treated as a two-stage optimization where the first-stage optimization is over the base-case decision x_0 and the second-stage optimization is over the contingency response y_k in each contingency $k \geq 1$. It can be rewritten as

$$\min_{x_0} \quad \sum_i c_i^g(p_{0i}) \ + \ \delta c_0(\sigma_0) \ + \ (1-\delta)\frac{1}{|K|}\sum_{k\geq 1} r_k\left(x_0\right) \tag{13.42a}$$

$$\text{s.t.} \quad (13.35)(13.36)(13.37)(13.38) \ \text{ with } \ k := 0 \tag{13.42b}$$

where the recourse functions from the second-stage optimization are: for $j \geq 1$,

$$r_j\left(x_0\right) \ := \ \min_{y_j} \quad c_j\left(\sigma_j\right) \tag{13.43a}$$

$$\text{s.t.} \quad (13.35)(13.36)(13.37)(13.38)(13.39) \ \text{ with } \ k := j \tag{13.43b}$$

where the penalty functions $c_k(\sigma_k)$ are defined in (13.41d). The second-stage problem is used for contingency evaluation.

Remark 13.10 (Key structures of SCOPF). 1. Even though the capacity limits on (p_{ki}, q_{ki}) are already enforced in (13.35), they are explicitly included in (13.39) because we may approximate (13.39) to enhance scalability.

2. The constraints (13.35) and (13.36) are linear. The constraint (13.37) is smooth but nonconvex. The constraints (13.38) (13.39) are nonsmooth and computationally difficult especially for interior-point methods (e.g., Ipopt [86]) used by all three teams [83, 84, 85]. All three teams devise methods to effectively handle these nonsmooth constraints.

3. The constraints (13.35) (13.36) (13.37) (13.38) apply to both the base case $k = 0$ and contingencies $k \geq 1$, but (13.39) where complementarity constraints must be dealt with applies only to contingencies $k \geq 1$ and hence only appears in the second-stage problem (13.43). As noted above (13.39) models the steady-state effect of frequency control actions after a contengency.

4. All constraints except (13.39) are separable in k. The constraint (13.39) couples the base case variables x_0 and contingency response y_k for each k. The SCOPF problem is therefore highly parallelizable and this is exploited by all three teams.

□

13.5.2 Computational challenges

The GO Challenge 1 includes a real-time SCOPF test where a base case decision x_0 must be computed within 10 or 45 minutes depending on the category of competition. It includes another test that computes contingency responses given the base-case decision x_0 with a time limit corresponding to 2 seconds per contingency.

The problem (13.41) does not include unit commitment decisions or switched devices such as transformer taps, capacitor banks and switchable transmission lines. They are included in Challenge 2 of the GO Competition that was conducted in 2021 and introduce discrete variables that add to the computational difficulty. We now discuss three main types of computational challenges of (13.41): large problem size, nonconvexity, and nonsmoothness.

Large problem size. For a network with G generators and M transmission lines or transformers[2], if we are to evaluate security against the outage of every single generator or line/transformer, it can increase the number of constraints by a factor of $G+M$ under $N-1$ security. If the dispatch has to be secure against $N-k$ security then the number of constraints will be increased by a factor of $(G+M)!/(k!(G+M-k)!)$. For example the largest network used in the GO Challenge 1 has 30,000 buses, 3,526 generators, 32,020 transmission lines, 3,373 transformers [85, Table EC.1], yielding $G+M = 3,526+32,020+3,373 = 38,919$. This would have increased the number of constraints by 4 orders of magnitude under $N-1$ security, or almost 9 orders of magnitude under $N-2$ security ($(G+M)!/(k!(G+M-k)!) = 757,324,821$). The GO Competition adopts $N-1$ security and specifies about 16,000 contingency scenarios which is still an increase of 4 orders of magnitude. For real-time SCOPF any practical solution must include methods to efficiently rank contingencies and solve an approximate problem with only a few highly ranked contingencies.

[2]The official GO Challenge 1 formulation models transformers with slightly different capacity limits than (13.38).

Nonconvexity. The constraints (13.37) are nonconvex. Nonconvexity implies that, in general, OPF problems are NP-hard and the best one can hope for is computing a local optimal. All three teams let the underlying interior-point solver (Ipopt [86]) handle nonconvexity. Methods to deal with nonconvexity through convex relaxations are explained in Chapters **??**–**??**. A challenge with semidefinite relaxations is the difficulty in scalaing these methods to large problems.

Nonsmoothness. Interior-point solvers, which all three winning teams use, by default require the problem to be smooth. The constraints (13.38) and (13.39) are computationally challenging. The difficulty can be illustrated from various angles by considering several equivalent representations. In particular they can be represented as nonsmooth constraints (as in (13.38) and (13.39a)), as logical constraints (as in (13.39b)), as complementarity constraints, or as mixed integer constraints. As we will see below effective treatment of (13.39) is an important component of all three teams' solutions.

1. *Nonsmooth constraints.* The function on the left-hand side of the constraint (13.38) is nondifferentiable at $(P_{k,ij}, Q_{k,ij}) = (0,0)$. Indeed along (say) $P_{k,ij} = 0$ the constraint is equivalent to following piecewise affine constraint:

$$\left|Q_{k,ij}\right| \;\;\leq\;\; P_{k,ij}^{\max} \left|V_{ki}\right| \;+\; \sigma_{k,ij},$$

   The constraint (13.39a) is nondifferentiable at $p_{ki} = \underline{p}_i$ or $p_{ki} = \overline{p}_i$.

2. *Logical constraints.* The logical constraint (13.39b) is disjunctive and computationally inconvenient. The other two constraints can also be expressed as a disjunctive constraints. For example (13.39a) is equivalent to:

$$\{\underline{p}_i \leq p_{ki} \leq \overline{p}_i,\ p_{ki} = p_{0i} + \alpha_i \Delta_k\} \,\cup\, \{p_{0i} + \alpha_i \Delta_k \leq \underline{p}_i = p_{ki}\} \,\cup\, \{p_{ki} = \overline{p}_i \leq p_{0i} + \alpha_i \Delta_k\}$$

3. *Complementarity constraints.* The constraints (13.38) and (13.39) can be reformulated as complementarity constraints. Introduce two slack variables $\rho_{ki}^- \geq 0$, $\rho_{ki}^+ \geq 0$. Then (13.39a) is equivalent to (Exercise 13.18):

$$p_{ki} + \rho_{ki}^+ - \rho_{ki}^- \;=\; p_{0i} + \alpha_i \Delta_k, \quad 0 \;\leq\; \rho_{ki}^- \;\perp\; p_{ki} - \underline{p}_{ki} \;\geq\; 0, \quad 0 \;\leq\; \rho_{ki}^+ \;\perp\; \overline{p}_{ki} - p_{ki} \;\geq\; 0 \tag{13.44a}$$

   where, for scalars $a, b \in \mathbb{R}$, "$0 \leq a \perp b \geq 0$" means "$a \geq 0, b \geq 0, ab = 0$". Note that the set $\{(a,b) \in \mathbb{R}^2 : 0 \leq a \perp b \geq 0\}$ is a nonconvex set. Similarly the constraint (13.39b) is equivalent to (Exercise 13.19):

$$|V_{ki}| + \mu_{ki}^+ - \mu_{ki}^- \;=\; |V_{0i}|, \quad 0 \;\leq\; \mu_{ki}^- \;\perp\; q_{ki} - \underline{q}_{ki} \;\geq\; 0, \quad 0 \;\leq\; \mu_{ki}^+ \;\perp\; \overline{q}_{ki} - q_{ki} \;\geq\; 0 \tag{13.44b}$$

   with slack variables $\left(\mu_{ki}^-, \mu_{ki}^+\right)$.

4. *Mixed integer constraints.* The constraint (13.39a) is equivalent to the following big-M mixed integer constraints:

$$\begin{aligned} \underline{p}_i \;\le\; p_{ki} \;\le\; \overline{p}_i, & \qquad\qquad \underline{z}_{ki}, \overline{z}_{ki} \;\in\; \{0,1\} \\ p_{ki} - \underline{p}_i \;\le\; M\underline{z}_{ki}, & \qquad p_{ki} - (p_{0i} + \alpha_i \Delta_k) \;\le\; M(1 - \underline{z}_{ki}) \\ \overline{p}_i - p_{ki} \;\le\; M\overline{z}_{ki}, & \qquad (p_{0i} + \alpha_i \Delta_k) - p_{ki} \;\le\; M(1 - \overline{z}_{ki}) \end{aligned}$$

where M is a sufficiently large constant and $(\underline{z}, \overline{z})$ are auxiliary binary variables (Exercise 13.18). The constraint (13.39b) is equivalent to (Exercise 13.19):

$$\begin{aligned} \underline{q}_i \;\le\; q_{ki} \;\le\; \overline{q}_i, & \qquad \underline{z}_{ki}, \overline{z}_{ki} \;\in\; \{0,1\} & \\ q_{ki} - \underline{q}_i \;\le\; M\underline{z}_{ki}, & \qquad |V_{0i}| - |V_{ki}| \;\le\; M\underline{z}_{ki}, & \qquad |V_{ki}| - |V_{0i}| \;\le\; M(1 - \underline{z}_{ki}) \\ \overline{q}_i - q_{ki} \;\le\; M\overline{z}_{ki}, & \qquad |V_{ki}| - |V_{0i}| \;\le\; M\overline{z}_{ki}, & \qquad |V_{0i}| - |V_{ki}| \;\le\; M(1 - \overline{z}_{ki}) \end{aligned}$$

Some properties of the complementarity and big-M mixed integer constraints are studied in Exercise 13.14–13.19.

Remark 13.11 (Linear complementarity problem)**.** Given a matrix $M \in \mathbb{R}^{m \times n}$ and vector $q \in \mathbb{R}^m$ the standard linear complementarity problem LCP(M, q) is to find vectors $z, w \in \mathbb{R}^m$ such that

$$z \ge 0, \quad w \ge 0, \quad z^{\mathsf{T}} w = 0, \quad w = Mz + q$$

A sufficient condition for the existence and uniqueness of such a solution (z, w) is that M satisfies $x^{\mathsf{T}} M x \ge 0$ for all $x \in \mathbb{R}^m$ whether or not M is symmetric.[3] M being positive definite or symmetric is however not necessary (Exercise 13.17). We can eliminate the variable w to write LCP(M, q) as finding z such that

$$z \ge 0, \quad Mz + q \ge 0, \quad z^{\mathsf{T}} (Mz + q) = 0$$

LCP arises, e.g., from the KKT condition of a quadratic program (Exercise 13.14). The equations (13.44a) can be transformed into a standard linear complementarity problem (Exercise 13.16). □

13.5.3 Computational techniques

We now describe some of the techniques used in each of the three winning teams of the GO Challenge 1 [83, 84, 85] in dealing with the computational challenges discussed in Chapter 13.5.2.

Besides algorithmic techniques, efficient software implementation is also critical, especially how to effectively use multi-core platforms for parallel computation, how to detect and reduce numerical instability, and how to handle software failures such as solver divergence or convergence to an infeasible point even when the problem is provably feasible. For example, the number of nonlinear subproblems that needs to be solved in [83] can be as high as 100,000, each with 2,000,000 variables and constraints. Software implementation issues in such a large-scale computational regime are highly nontrivial.

[3] For a matrix M over the field $\mathbb{R}$, we define M to be positive (semi)definite only for symmetric M; see Definition 25.2 in Chapter 25.1.2 and the discussion there.

We will however focus only on algorithmic techniques. The approach of [83] approximates the second-stage recourse function $r_k(x_0)$ by an explicit quadratic function that is iteratively computed. The approach of [84] continuously and iteratively evaluates contingencies quickly and includes only the top three contingencies in the SCOPF problem in each iteration. The approach of [85] uses smooth approximation of constraint (13.39) and develops an ADMM-based algorithm to exploit the problem's distributed structure. All three teams use an off-the-shelf interior-point optimization solver. Our description is a highly simplified version of [83, 84, 85] that illustrates some of the ideas.

13.5.3.1 Approximating recourse function $r_k(x_0)$ [83]

The approach of [83] uses the two-stage formulation (13.42) (13.43) of the SCOPF problem. A two-stage problem is computationally difficult because an explicit form of the second-stage recourse function $r_k(x_0)$ is generally not available. The key idea of [83] is to approximate $r_k(x_0)$ as an explicit polynomial function $\hat{r}_k(x_0;\pi_k)$ where π_k is a scaling factor in the approximation to be computed in each iteration, so that the (approximate) first-stage problem and the second-stage problem can be solved iteratively according to a

Meta algorithm: for $t = 0, 1, \ldots$, repeat until a stopping criterion is satisfied ($\pi_k(0) = 0$, i.e., start with the base case):

1. Given $\hat{r}_k(x_0;\pi_k(t))$, approximate $r_k(x_0)$ by $\hat{r}_k(x_0;\pi_k(t))$ in (13.42) and solve the approximate first-stage problem to obtain an optimal solution $x_0(t+1)$;
2. Given $x_0(t+1)$, solve the second-stage problem (13.43) to obtain $\pi_k(t+1)$ and construct $\hat{r}_k(x_0;\pi_k(t+1))$.

Two main algorithmic techniques are used to simplify the two subproblems in the meta algorithm: (i) Contingency screening to quickly identify and include only contingencies that are likely to have large recourse costs $r_k(x_0)$; and (ii) Dealing with the nonsmoothness of (13.38) (13.39) through relaxation. We now explain each of these ideas: approximating $r_k(x_0)$, contingency screening, and handling nonsmoothness.

Approximating $r_k(x_0)$. Recall that the GO Challenge 1 considers only $N-1$ security, so there is exactly a single device (a generator or line) that is disconnected in each contingency. If the device in contingency $k \geq 1$ is a generator then index it by $i(k)$ and let $\left(p_{0i(k)}, q_{0i(k)}\right)$ denote the real and reactive power it generates in the base case. Its recourse function $r_k(x_0)$ is approximated by

$$\hat{r}_k(x_0;\pi_k) \quad := \quad \pi_k \cdot \left(p_{0i(k)}^2 \; + \; q_{0i(k)}^2\right)^2 \tag{13.45a}$$

where π_k is a constant to be determined. If the disconnected device in contingency $k \geq 1$ is a line then index it by $(i(k), j(k))$. Let $(P_{0,i(k)j(k)}, Q_{0,i(k)j(k)})$ denote the sending-end line power from bus $i(k)$ to bus $j(k)$ and let $(P_{0,j(k)i(k)}, Q_{0,j(k)i(k)})$ denote the sending-end line power in the opposite direction. Then its recourse function $r_k(x_0)$ is approximated by

$$\hat{r}_k(x_0;\pi_k) \quad := \quad \pi_k \cdot \left(\max\left\{ P_{0,i(k)j(k)}^2 \; + \; Q_{0,i(k)j(k)}^2, \; P_{0,j(k)i(k)}^2 \; + \; Q_{0,j(k)i(k)}^2 \right\}\right)^2 \tag{13.45b}$$

i.e., $\hat{r}_k(x_0;\pi_k)$ is a polynomial in the greater of the apparent sending-end line power. The approximation (13.45a) is motivated by three observations; the motivation for (13.45b) is similar. First the generation cost in SCOPF tends to select generators with lowest marginal costs and schedule them to produce at their upper capacity limits in the base case unless this is prevented by network congestion where some line limits are active. This suggests that pre-contingency injection of the disconnected generator $i(k)$ is a good approximation for the penalty σ_j in the second-stage problem. This observation is also used in [85] for fast contingency evaluation. Second the recourse cost $r_k(x_0)$ tends to be an increasing function of the pre-contingency generation levels of the disconnected generator. Moreover the rate of increase tends to be steep and local. Replacing the second-stage problem (13.43) with the convex polynomial $\hat{r}_k(x_0; \pi_k)$ greatly simplifies the SCOPF problem (13.42) (13.43).

Contingency screening. Recall that each contingency is defined by the outage of a generator or line. To reduce the number of contingencies in the first-stage problem (13.42), [83] first considers using machine learning to rank contingencies according to their impact on the objective value, but found it difficult to generalize machine learning models to networks that have not been seen in the training dataset. It does not adopt other ranking approaches that may be sophisticated but too computationally intensive for real-time applications. Instead it chooses a subset g of generators with the largest generation capacities and a subset e of lines with the largest line limits to define the set of credible contingencies such that $|g| + |e|$ is no more than the number of parallel processors available for contingency screening. This simple heuristic has been found to produce some false positives (contingencies that seem impactful but are not) but no false negatives (contingencies that seem harmless but are not). Only the set $\hat{K}$ of contingencies defined by this set of devices are included in the objective and constraints.

Approximating nonsmooth constraint (13.38) **and costs** $c_k(\sigma_k)$**.** Instead of (13.38a), consider the smooth constraint:

$$P_{k,ij}^2 + Q_{k,ij}^2 \;\; \leq \;\; \left(P_{k,ij}^{\max} |V_{ki}| \; + \; \sigma_{k,ij}^e\right)^2, \qquad k \geq 0, \; (i,j) \in E$$

This constraint is nonconvex but the log-barrier function

$$\log\left(\left(P_{k,ij}^{\max} |V_{ki}| \; + \; \sigma_{k,ij}^e\right)^2 \; - \; P_{k,ij}^2 \; - \; Q_{k,ij}^2\right)$$

is convex. In fact, it is a self-concordant barrier function for the second-order cone and is used to eliminate the nonsmooth constraint (13.38a). The constraint (13.38b) is treated similarly. Specifically the problem (13.42) (13.43) is modified by removing (13.38) and adding log-barrier penalty functions $\tilde{c}_k(x_k)$ to the objective for each credible contingency:

$$\begin{aligned} \tilde{c}_k(x_k) \;\; := \;\; & \sum_{(i,j)\in E} \log\left(\left(P_{k,ij}^{\max} |V_{ki}| + \sigma_{k,ij}^e\right)^2 - P_{k,ij}^2 - Q_{k,ij}^2\right) \; + \\ & \sum_{(i,j)\in E} \log\left(\left(P_{k,ji}^{\max} |V_{kj}| + \sigma_{k,ij}^e\right)^2 - P_{k,ji}^2 - Q_{k,ji}^2\right), \qquad k \; \in \; \{0\} \cup \hat{K} \end{aligned} \tag{13.46}$$

Recall that the constraint violation functions $c_{ki}^p(s)$ in (13.41d) are piecewise linear and hence nonsmooth. They are approximated by quadratic functions $\hat{c}_{ki}^p(s) := a_{ki}^p s^2 + b_{ki}^p s$ for appropriate constants

$\left(a_{ki}^p, b_{ki}^p\right)$. Similarly for the functions $c_{ki}^q(s)$, $c_{k,ij}^e(s)$. The nonsmooth penalty function $c_k(\sigma_k)$ in (13.41d) is replaced by the quadratic function

$$\hat{c}_k(\sigma_k) \quad := \quad \sum_{i\in\overline{N}} \left(\hat{c}_{ki}^p\left(\sigma_{ki}^{p+} + \sigma_{ki}^{p-}\right) + \hat{c}_{ki}^q\left(\sigma_{ki}^{q+} + \sigma_{ki}^{q-}\right)\right) \; + \sum_{(i,j)\in E} \hat{c}_{k,ij}^e\left(\sigma_{k,ij}^e\right), \quad k \geq 0 \tag{13.47}$$

These three types of approximations (13.45) (13.46) (13.47) reduce the first-stage problem (13.42) to a much simpler approximate problem with an explicit recourse function (given π_k):

$$\min_{x_0} \quad \sum_i c_i^g(p_{0i}) \; + \; \delta\hat{c}_0(\sigma_0) \; + \; (1-\delta)\frac{1}{|\hat{K}|}\sum_{k\in\hat{K}} \hat{r}_k(x_0;\pi_k) \; + \; \tilde{\delta}_0\,\tilde{c}_0(x_0) \tag{13.48a}$$

$$\text{s.t.} \quad (13.35)(13.36)(13.37) \quad \text{with} \quad k := 0 \tag{13.48b}$$

where the approximate base-case penalty function $\hat{c}_0(\sigma_0)$ is defined in (13.47), the approximate recourse functions $\hat{r}_k(x_0;\pi_k)$ are defined in (13.45), the line limit penalty functions $\tilde{c}_0(x_0)$ are defined in (13.46) for $k = 0$, and $\tilde{\delta}_0$ is an appropriate weight. In particular the nonsmooth constraint (13.38) is replaced by the penalty $\tilde{c}_k(x_k)$ for $k = 0$.

Relaxing nonsmooth constraint (13.39). The nonsmooth constraint (13.39) that describes contingency response appears only in the second-stage problem (13.43), not the approximate first-stage problem (13.48). Hence (13.39) affects only step 2 of the meta algorithm that determines π_k and the approximation $\hat{r}_k(x_0;\pi_k)$. Indeed π_k is determined iteratively as follows. After obtaining an optimal solution x_0 in step 1 of the meta algorithm by solving the approximate first-stage problem (13.48), solve an approximate version of the second-stage problem (13.43) to obtain an (approximate) optimal $r_k(x_0)$ and set

$$\pi_k(x_0) \quad := \quad \frac{r_k(x_0)}{\left(p_{0i(k)}^2 \; + \; q_{0i(k)}^2\right)^2}$$

for π_k in (13.45a); similarly for π_k in (13.45b). Repeat the cycle until some stopping criterion is satisfied.

We now explain how to solve an approximate version of the second-stage problem (13.43) given a first-stage solution x_0. As in (13.48), the approximation of (13.43) replaces $c_k(\sigma_k)$ by its quadratic approximation $\hat{c}_k(\sigma_k)$ defined in (13.47) and replaces the nonsmooth constraint (13.38) by the log-barrier penalty function $\tilde{c}_k(x_k)$ defined in (13.46) in the objective. The additional feature is how to handle the nonsmooth constraint (13.39).

For this purpose, [83] uses the complementarity representation (13.44). For scalars $a, b \in \mathbb{R}$, a complementarity constraint $0 \leq a \perp b \geq 0$ can be approximated by the Fischer-Burmeister function $\phi(a,b) := a + b - \sqrt{a^2 + b^2}$ so that $0 \leq a \perp b \geq 0$ if and only if $\phi(a,b) = 0$. A standard way to handle the complementarity constraint is to replace it with the Fischer-Burmeister function as a penalty term in the objective. Even though the function ϕ is convex and Lipschitz continuous, a difficulty is that it is not differentiable at $(0,0)$. Finding a solution $\left(p_{ki},\ \rho_{ki}^+, \rho_{ki}^-\right)$ that satisfies (13.44a) is called a linear complementarity problem (see Remark 13.11). There are algorithms to solve such problems exactly, but [83] finds this approach numerically unstable for the SCOPF problem.

At the end [83] relaxes the complementarity condition $0 \leq a \perp b \geq 0$ to the constraints

$$a \;\geq\; 0, \;\; b \;\geq\; 0, \;\; ab \;\leq\; \varepsilon(\overline{a}-\underline{a})(\overline{b}-\underline{b})$$

where $\varepsilon \geq 0$ is a small constant, $\underline{a}, \overline{a}$ are lower and upper bounds respectively on a, and $\underline{b}, \overline{b}$ are corresponding bounds on b. The nonlinear relaxation is exact if $\varepsilon = 0$ but numerical issues may arise if ε is too small. To apply this relaxation to the linear complementarity constraint (13.44a), let $\underline{\Delta}$ and $\overline{\Delta}$ with $\underline{\Delta} < 0 < \overline{\Delta}$ be apriori lower and upper bounds on Δ_k for all $k \in \hat{K}$ (e.g., $\overline{\Delta}$ can be the upper limit of the largest generator). Then $0 \leq \rho_{ki}^- \leq -\alpha_i \underline{\Delta}$ and $0 \leq \rho_{ki}^+ \leq \alpha_i \overline{\Delta}$. Hence the nonlinear relaxation of (13.44a) is

$$\begin{aligned}
p_{ki} + \rho_{ki}^+ - \rho_{ki}^- \;&=\; p_{0i} + \alpha_i \Delta_k, \qquad k \geq 1, \; i \in \overline{N} && (13.49a)\\
0 \;\leq\; \rho_{ki}^-, \qquad \rho_{ki}^-\left(p_{ki} - \underline{p}_{ki}\right) \;&\leq\; -\varepsilon\, \alpha_i \underline{\Delta}\left(\overline{p}_i - \underline{p}_i\right), \qquad k \geq 1, \; i \in \overline{N} && (13.49b)\\
0 \;\leq\; \rho_{ki}^+, \qquad \rho_{ki}^+\left(\overline{p}_{ki} - p_{ki}\right) \;&\leq\; \varepsilon\, \alpha_i \overline{\Delta}\left(\overline{p}_i - \underline{p}_i\right), \qquad k \geq 1, \; i \in \overline{N} && (13.49c)
\end{aligned}$$

Similarly the relaxation of (13.44b) is

$$\begin{aligned}
|V_{ki}| + \mu_{ki}^+ - \mu_{ki}^- \;&=\; |V_{0i}|, \qquad k \geq 1, \; i \in \overline{N} && (13.49d)\\
0 \;\leq\; \mu_{ki}^-, \qquad \mu_{ki}^-\left(q_{ki} - \underline{q}_{ki}\right) \;&\leq\; \varepsilon\,(\overline{v}_{ki} - \underline{v}_{0i})\left(\overline{q}_i - \underline{q}_i\right), \qquad k \geq 1, \; i \in \overline{N} && (13.49e)\\
0 \;\leq\; \mu_{ki}^+, \qquad \mu_{ki}^+\left(\overline{q}_{ki} - q_{ki}\right) \;&\leq\; \varepsilon\,(\overline{v}_{0i} - \underline{v}_{ki})\left(\overline{q}_i - \underline{q}_i\right), \qquad k \geq 1, \; i \in \overline{N} && (13.49f)
\end{aligned}$$

These relaxations are used in the approximate version of the second-stage problem.

Overall algorithm. Putting all these together, the overall algorithm in [83] for solving an approximation of the SCOPF is as follows.

1. *Initialization*: $\pi_k := 0$ for all $k \in \hat{K}$ (no contingencies).
2. *Iterate* until stopping criterion:
   (a) Given $\left(\pi_k, k \in \hat{K}\right)$, solve (13.48) to obtain an optimal x_0.
   (b) Given x_0, for each contingency $j \in \hat{K}$:
      i. Solve (recall that y_j denotes contingency response and is defined in (13.40e))[4]

$$\begin{aligned}
\tilde{r}_j(x_0) \;\;:=\;\; \min_{y_j} \quad & \hat{c}_j\left(\sigma_j\right) \;+\; \tilde{\delta}_j \tilde{c}_j(x_j)\\
\text{s.t.} \quad & (13.35)(13.36)(13.37)(13.49) \quad \text{with} \quad k := j
\end{aligned}$$

      where $\hat{c}_j(\sigma_j)$ and $\tilde{c}_j(x_j)$ are defined in (13.47) and (13.46) respectively with $k = j$.

[4]The method in [83] includes an algorithm that recovers from a solution y_j of the relaxation here a contingency response that is feasible for the original second-stage problem (13.43). This algorithm can be repeatedly used to generate a feasible solution to the original SCOPF problem (13.41).

ii. If contingency j is due to the disconnection of generator $i := i(j)$ then

$$\pi_j \quad := \quad \frac{\tilde{r}_j(x_0)}{\left(p_{0i}^2 \,+\, q_{0i}^2\right)^2}$$

iii. If contingency j is due to the disconnection of line $(i,l) := (i(j), l(j))$ then

$$\pi_j \quad := \quad \frac{\tilde{r}_j(x_0)}{\left(\max\left\{P_{0,il}^2 \,+\, Q_{0,il}^2,\, P_{0,li}^2 \,+\, Q_{0,li}^2\right\}\right)^2}$$

3. *Return*: a first-stage decision x_0, second-stage decisions y_k and their associated costs $\tilde{r}_k(x_0)$, $k \in \hat{K}$.

13.5.3.2 Fast contingency selection [84]

The approach of [84] focuses on continuously and iteratively evaluate contingencies and include only the top three contingencies in the solution of SCOPF (13.41) in each iteration. This requires techniques to handle complementarity constraints, evaluate contingencies quickly, remove dominated contingencies, and effectively parallelize the solution of a master problem and contingency evaluation.

One can eliminate a large number of variables and equality constraints by substituting these equalities into the cost and constraint functions. For instance eliminating line flow variables $\left(P_{k,ij}, Q_{k,ij}\right)$ using (13.37) can reduce the problem size by the number of lines times the number of contingencies. This however will greatly increase the density of the constraint Jacobian. The team has decided to keep these variables and let the (interior-point) optimization solver to exploit sparsity. Like [83], [84] also uses the squared version of the line limit (13.38) because of its smoothness. It uses complementarity constraints to represent contingency responses (13.39).

Handling complementarity constraints. The second-stage problem is solved during contingency evaluation. The complementarity constraints representing the contingency responses are handled using an active set method. Note that each complementary constraint in (13.44) involves an injection variable $\chi \in [\underline{\chi}, \overline{\chi}]$ (e.g., p_{ki}, q_{ki}), a linear condition η (e.g., $p_{ki} - (p_{0i} + \alpha_i \Delta_k)$), and a slack variable ρ. It takes the form: $\eta + \rho = 0$ and one of the following linear constraint:

- $\chi = \underline{\chi}$ and $\rho \leq 0$.
- $\chi = \overline{\chi}$ and $\rho \geq 0$.
- $\chi \in [\underline{\chi}, \overline{\chi}]$ and $\rho = 0$.

The strategy for solving the second-stage problem (13.43) is based on active set prediction, as follows: (i) For each complementarity constraint, predict which of these three conditions holds at optimality; (ii) Replace each complementarity constraint by the predicted linear constraints and solve the resulting second-stage problem; (iii) update the prediction based on the multiplier values obtained from the solution in step (ii). The process is repeated until a certain stopping criterion is attained.

For instance, suppose step (i) predicts that $\chi = \underline{\chi}$ and $\rho \leq 0$ and the second-stage problem is solved in step (ii) with these constraints in place of complementarity constraints. If the Lagrange multiplier $\lambda > 0$ corresponding to the constraint $\rho \leq 0$ is positive and $-\rho/\lambda$ is close to zero (say, $0 < -\rho/\lambda < 10^{-6}$), then the constraint $\rho \leq 0$ is considered active and the prediction is changed in the next iteration.

Another method that [84] tried for handling a complementarity constraint $0 \leq a \perp b \geq 0$ is to replace it by the constraint $(a,b) \geq 0$ and a penalty term $\beta(ab)$ in the cost function. The team has found however that this approach does not work well for their approach.

Fast contingency selection. The approach of [84] uses three main contingency selection techniques:

1. *Initial ranking using ML.* Initial contingency ranking uses supervised learning to predict the importance of a contingency on overall cost based on various features, such as different expressions of generation levels and line power, generator ratings, degrees of buses, etc. It finds that the apparent line power

$$\max\left\{\sqrt{P_{0,i(k)j(k)}^2 + Q_{0,i(k)j(k)}^2}, \sqrt{P_{0,j(k)i(k)}^2 + Q_{0,j(k)i(k)}^2}\right\}$$

   has the best predictive power. This is consistent with the intuition used to approximate the recourse function $r_k(x_0)$ in [83] (cf. (13.45b)).

2. *Contingency evaluation.* Each contingency k identified by the initial ranking as credible is then evaluated more carefully by solving the second-stage problem (13.43), in two steps.

   First, given a first-stage decision x_0, an upper bound on the second-stage cost $r_k(x_0)$ is computed by solving a reduced problem with only the power flow equations and linear constraints associated with complementarity constraints predicted by the active set method described above. In particular this reduced problem does not include any operational constraints. Only if this upper bound exceeds a certain threshold will a full evaluation of the contingency be carried out by solving the second-stage problem using the active set method.

3. *Dominated contingency.* Inclusion of the constraints due to contingency j may cause the constraints due to other contingencies k to be automatically (possibly approximately) satisfied. To identify these constraints, let $\sigma_k^{\max}$ be the largest entry of the vector σ_k defined in (13.40c), i.e., $\sigma_k^{\max}$ is the largest slack variable measuring the violation of power balance or a line limit in contingency k. We say that *contingency k is dominated by contingency j* if $\sigma_j^{\max} > \sigma_k^{\max}$. Only contingencies that are not dominated by another contingency are included in the solution of the master problem (13.41).

Overall algorithm. Putting all these techniques together, the overall algorithm of [84] is as follows (see Figure 13.1).

1. *Initialization*:

   - Solve the SCOPF problem (13.41) without contingency with flat start: $p_0 := \overline{p}_0$, $q_0 := 0$, $|V_0| := (\underline{v}_0 + \overline{v}_0)/2$, $|\theta_0| := 0$, $P_0 = Q_0 = 0$.

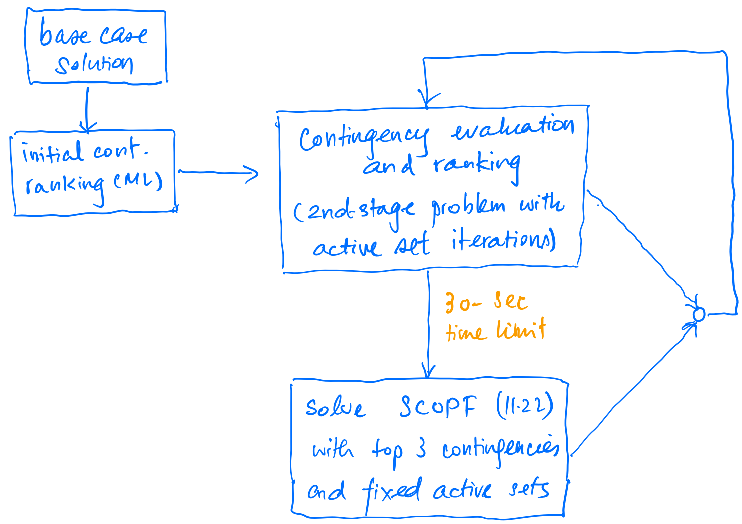

Figure 13.1: Overall algorithm of [84] to compute the first-stage optimal solution x_0^*.

- Given the first-stage solution x_0, use machine learning model to produce an initial contingency priority list.
- Perform fast evaluation of contingencies in the order of the current priority list for up to 1 minute by solving a reduced second-stage problem with active set iterations. Update the priority list.

2. *Iterate* until time limit (10 minutes):

   - Perform full evaluation of contingencies in the order of the current priority list for 30 seconds and update the priority list.
   - Include only the top three contingencies in the SCOPF problem (13.41), avoiding dominated contingencies, and solve it with the complementarity constraints replaced by the same active sets identified for these contingencies during contingency evaluation.
   - Meanwhile continue to evaluate contingencies *in parallel* until a solution of the approximate SCOPF problem is obtained. Specifically each contingency is first evaluated by solving a reduced second-stage problem with active set iterations. If its cost upper bound exceeds a threshold then it is further evaluated by solving the full second-stage problem with active set iterations.
   - Update the priority list.

3. *Return* a first-stage decision x_0^*.

4. Given x_0^*, evaluate every contingency in parallel using fast evaluation with active set iterations followed by a full evaluation if its cost upper bound exceeds a threshold. *Return* second-stage decisions y_k and their associated costs $r_k(x_0)$, $k \geq 1$.

13.5.3.3 Smoothed ADMM algorithm [85]

The approach of [85] employs four main ideas: a smooth approximation of the nonsmooth constraint (13.39), a two-level distributed algorithm based on ADMM, an algorithm to solve the second-stage problem for contingency responses, and an efficient parallel computation structure. In the following we explain the first three ideas.

Smooth approximation of (13.39)**.** The function $f(x) := \max(0,x)$, $x \in \mathbb{R}$, can be approximated by the smooth function

$$f^{\varepsilon}(x) \quad := \quad \varepsilon \ln\left(1 + e^{x/\varepsilon}\right), \qquad \varepsilon > 0$$

See Figure 13.2. Indeed $f^{\varepsilon}(x)$ upper bounds $f(x)$ with $f^{\varepsilon}(x) - \varepsilon \ln 2 \leq f(x) < f^{\varepsilon}(x)$ for all $x \in \mathbb{R}$. This

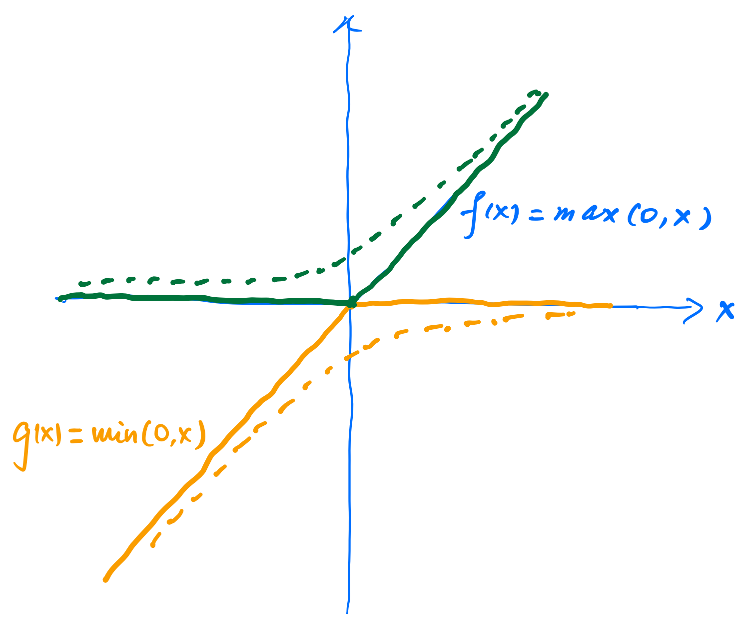

Figure 13.2: The nonsmooth functions $f(x) := \max(0,x)$, $g(x) := \min(0,x)$ and their smooth approximations. See Exercise 13.20 for more properties.

method leads to a smooth approximation of $h(x) := \max(a, \min(x,b))$ given by

$$h^{\varepsilon}(x) \quad := \quad a \,+\, \varepsilon \ln\left(1 \,+\, \frac{e^{(b-a)/\varepsilon}}{1 \,+\, e^{(b-x)/\varepsilon}}\right)$$

See Exercise 13.20 for approximations of $\max(a,x)$, $\min(x,b)$ and $\max(a, \min(x,b))$.

Applying this to the contingency response (13.39a) reproduced here:

$$p_{ki} \quad = \quad [p_{0i} \,+\, \alpha_i \Delta_k]_{\underline{p}_i}^{\overline{p}_i}, \qquad k \geq 1,\ i \in \overline{N}$$

gives the smooth approximation

$$p_{ki}^{\varepsilon}(p_i) \quad := \quad \underline{p}_i \, + \, \varepsilon \ln\left(1 \, + \, \frac{e^{(\overline{p}_i - \underline{p}_i)/\varepsilon}}{1 \, + \, e^{(\overline{p}_i - p_i)/\varepsilon}}\right), \qquad k \geq 1, \; i \in \overline{N} \tag{13.50}$$

To apply this method to (13.39b) reproduced here:

$$\left\{\underline{q}_i \leq q_{ki} \leq \overline{q}_i, \; |V_{ki}| = |V_{0i}|\right\} \cup \left\{q_{ki} = \underline{q}_i, \; |V_{ki}| \geq |V_{0i}|\right\} \cup \left\{q_{ki} = \overline{q}_i, \; |V_{ki}| \leq |V_{0i}|\right\}, \quad k \geq 1, \; i \in \overline{N}$$

let R_{ki} be the set of $(q_{ki}, |V_{ki}|)$ that satisfy this constraint. Then $(q_{ki}, |V_{ki}|) \in R_{ki}$ if and only if there exist slack variable $\left(\mu_{ki}^-, \mu_{ki}^+\right)$ such that

$$\begin{aligned}
|V_{ki}| \, + \, \mu_{ki}^+ \, - \, \mu_{ki}^- \quad &= \quad |V_{0i}| && (13.51a)\\
\min\left(\mu_{ki}^-, \; q_{ki} - \underline{q}_{ki}\right) \quad &\leq \quad 0 && (13.51b)\\
\min\left(\mu_{ki}^+, \; \overline{q}_{ki} - q_{ki}\right) \quad &\leq \quad 0 && (13.51c)\\
\underline{q}_{ki} \; \leq \; q_{ki} \; \leq \; \overline{q}_{ki}, \qquad &\left(\mu_{ki}^-, \mu_{ki}^+\right) \; \geq \; 0 && (13.51d)
\end{aligned}$$

Here (13.51a) (13.51d) are linear constraints in $\left(q_{ki}, |V_{ki}|, \mu_{ki}^-, \mu_{ki}^+\right)$ but (13.51b) (13.51c) are nonconvex constraints. We can visualize this in Figure 13.3 that shows the set

$$\{(x,y) : \min(x,y) \leq a\} \quad = \quad \{(x,y) : x \leq a \; \text{ or } \; y \leq a\}$$

Using Exercise 13.20, we can approximate $\min(x,y)$ by $y - \varepsilon \ln\left(1 + e^{(y-x)/\varepsilon}\right)$ and relax (13.51) to:[5]

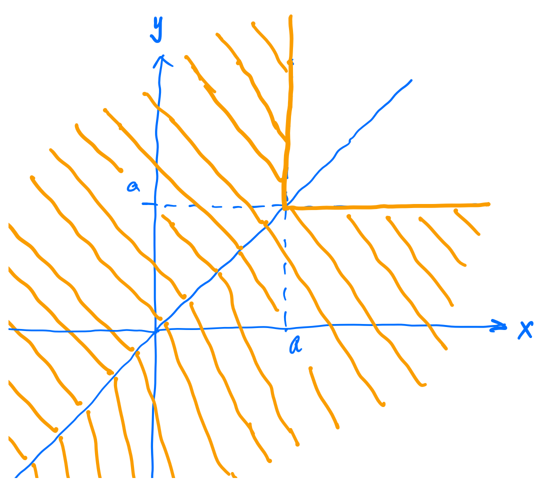

Figure 13.3: The set $\{(x,y) : \min(x,y) \leq a\}$ is nonconvex. The set $\{(x,y) : \min(x,y) = a\}$ consists of two lines $x = a$ and $y = a$.

$$|V_{ki}| + \mu_{ki}^{+} - \mu_{ki}^{-} = |V_{0i}| \tag{13.52a}$$

$$\mu_{ki}^{-} - \varepsilon \ln\left(1+\exp\frac{1}{\varepsilon}\left(\mu_{ki}^{-} - q_{ki} + \underline{q}_{ki}\right)\right) \leq 0 \tag{13.52b}$$

$$\mu_{ki}^{+} - \varepsilon \ln\left(1+\exp\frac{1}{\varepsilon}\left(\mu_{ki}^{+} - \overline{q}_{ki} + q_{ki}\right)\right) \leq 0 \tag{13.52c}$$

$$\underline{q}_{ki} \leq q_{ki} \leq \overline{q}_{ki}, \qquad \left(\mu_{ki}^{-}, \mu_{ki}^{+}\right) \geq 0 \tag{13.52d}$$

Let R_{ki}^{ε} be the set of $(q_{ki}, |V_{ki}|)$ such that there exist slack variables $\left(\mu_{ki}^{-}, \mu_{ki}^{+}\right)$ so that $(q_{ki}, |V_{ki}|, \mu_{ki}^{-}, \mu_{ki}^{+})$ satisfies (13.52). Then it is shown in [85] that R_{ki}^{ε} is a relaxation of R_{ki} and approaches R_{ki} as $\varepsilon \to 0$. Specifically, for each $k \geq 1$, $i \in \overline{N}$, we have (Exercise 13.21):

$$R_{ki} \subseteq R_{ki}^{\varepsilon} \tag{13.53a}$$

$$\sup_{(q_{ki},|V_{ki}|)\in R_{ki}^{\varepsilon}} \ \inf_{(q'_{ki},|V'_{ki}|)\in R_{ki}} \left\|(q_{ki}, |V_{ki}|) - (q'_{ki}, |V'_{ki}|)\right\| \to 0 \quad \text{as} \quad \varepsilon \to 0 \tag{13.53b}$$

While choosing a small ε gives a better approximation of the feasible set, a small ε may also cause numerical issues since the second derivative $\frac{d^2}{dx^2} f^{\varepsilon}(0)$ of the approximation $f^{\varepsilon}(x)$ of $f(x) := \max(0,x)$ evaluated at $x = 0$ diverges as $\varepsilon \to 0$.

ADMM-based distributed algorithm. As mentioned above, the base case $k = 0$ and the contingencies $k \geq 1$ are decoupled only through the first-stage decision x_0 in the constraint (13.39) that appears in the set of second-stage problems (13.43), one for each contingency $k \geq 1$. By introducing a local copy x_k^0 of x_0 for each contingency subproblem these second-stage problems are decoupled and can therefore be computed in parallel, with a consensus constraint that all local copies equal to x_0 at optimality. Hence the SOCP problem (13.41) can be equivalently reformulated into the form

$$\min \quad f_0(x_0) + \sum_{k\geq 1} f_k\left(x_k^0, y_k\right) \tag{13.54a}$$

$$\text{over} \quad x_0, \left(x_k^0, y_k,\ k \geq 1\right) \tag{13.54b}$$

$$\text{s.t.} \quad x_0 \in X_0, \ \left(x_k^0, y_k\right) \in X_k,\ k \geq 1 \tag{13.54c}$$

$$x_k^0 = x_0, \ k \geq 1 \tag{13.54d}$$

where the constraint $x_0 \in X_0$ means that x_0 satisfies (13.35)–(13.38), and the constraint $\left(x_k^0, y_k\right) \in X_k$ means that y_k satisfies (13.35)–(13.38) and $\left(x_k^0, y_k\right)$ satisfies the smooth approximations (13.50) (13.52) of (13.39). This is a form that is suitable for distributed solution using the standard ADMM (alternating direction method of multipliers).

To explain ADMM consider the general optimization problem

$$\min_{x\in\mathbb{R}^n,\ y\in\mathbb{R}^p} \quad f(x) + g(y) \tag{13.55a}$$

$$\text{s.t.} \quad x \in X, \ y \in Y \tag{13.55b}$$

$$Ax + By = c \tag{13.55c}$$

[5]The right-hand sides of (13.52b) (13.52c) are further relaxed from 0 to $\varepsilon \ln 2$ in [85].

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times p}$ and $c \in \mathbb{R}^m$ are given. The key features of (13.55) that make it amenable to distributed solution are that the objective function in (13.55a) is separable in the variables x and y, and the possibly nonlinear constraints (13.55b) are separable in x and y. The only coupling between x and y is through the linear constraints (13.55c). For this type of separable problems, ADMM combines the distributed structure of dual decomposition with better convergence properties of augmented Lagrangian methods.

Specifically define the augmented Lagrangian function that relaxes the coupling constraint:

$$L_\rho(x,y,\lambda) \;:=\; f(x) \,+\, g(y) \,+\, \lambda^{\mathsf{T}}(Ax+By-c) \,+\, \frac{\rho}{2}\,\|Ax+By-c\|_2^2$$

The ADMM algorithm is

$$x_{t+1} \;:=\; \arg\min_{x\in X} L_\rho\left(x,\, y_t,\, \lambda_t\right) \tag{13.56a}$$

$$y_{t+1} \;:=\; \arg\min_{y\in Y} L_\rho\left(x_{t+1},\, y,\, \lambda_t\right) \tag{13.56b}$$

$$\lambda_{t+1} \;:=\; \lambda_t \,+\, \rho\left(Ax_{t+1}+By_{t+1}-c\right) \tag{13.56c}$$

The subproblems (13.56a) (13.56b) can be solved in parallel, given the dual variable λ_t. These two subproblems are coordinated by the dual update (13.56c). It is important that the subproblem (13.56b) uses x_{t+1} instead of x_t (this is called one pass of a Gauss-Seidel method) and that the stepsize for the dual update is ρ; see Exercise 13.22.

While the example (13.55) involves two variables x and y that are almost decoupled except for the linear coupling constraint, the SCOPF (13.54) has $K+1$ variables x_0 and $\left(x_k^0, y_k, k\geq 1\right)$ with K coupling constraints (13.54d). Given the Lagrange multipliers λ_k associated with these K coupling constraints, the $K+1$ subproblems that optimize over x_0 and $\left(x_k^0, y_k, k\geq 1\right)$ can be computed in parallel as in (13.56a) (13.56b). The algorithm of [85] applies this idea to SCOPF (13.54) with two main refinements. First it relaxes the coupling constraint (13.54d) with a slack variable z_k for each contingency $k\geq 1$ which is penalized in the objective function with a term $\beta\|z_k\|_2^2$. As a result the solution returned by the ADMM algorithm may violate by a large amount the coupling constraint and is therefore infeasible for the original SCOPF. The second refinement is an outer loop where the weight β on the penalty is increased if the worst violation $\max_{k\geq 1}\|z_k\|$ across contingencies is too large and the approximate SCOPF is solved again using ADMM. The outer loop terminates when $\max_{k\geq 1}\|z_k\|$ is small enough (and the stationarity condition is sufficiently satisfied). Even though the problem is nonconvex it is proved in [85] that the two-level ADMM algorithm with both the inner and outer loops converges under the condition that each inner-loop iteration (13.56a) (13.56b) produces sufficient descent.

Solving second-stage problem (13.43)**.** After a base-case solution x_0^* is obtained using the two-level ADMM algorithm, [85] solves for second-stage solutions y_k^* for each contingency $k\geq 1$, in two steps. For each $k\geq 1$, it first solves an approximate version of (13.43) in which the nonsmooth constraint (13.39) has been replaced by its smooth approximations (13.50) (13.52). If the resulting solution $(\hat{y}_k, \hat{\Delta}_k)$ violates (13.39) by a significant amount (because ε is not small enough), then it uses the solution $(\hat{y}_k, \hat{\Delta}_k)$ to define a set of linear constraints to replace the smooth approximations (13.50) (13.52) and solve the approximate second-stage problem again to obtain the final second-stage solution y_k^* and its cost $r_k(x_0^*)$.

Specifically, for the nonsmooth constraint (13.39a), use the solutions x_0^* and $(\hat{y}_k, \hat{\Delta}_k)$ to partition the set of generators that remain connected in contingency k into three groups,

$$\hat{N}^{p-} := \{i : p_{0i}^* + \alpha_i \hat{\Delta}_k \leq \underline{p}_i + \nu\}, \qquad \hat{N}^{p+} := \{i : p_{0i}^* + \alpha_i \hat{\Delta}_k \geq \overline{p}_i - \nu\}$$

and their complements. Here $\nu > 0$ is a violation threshold. Then, instead of (13.50), the following linear constraints are used to approximate the nonsmooth constraint (13.39a):

$$\begin{array}{lll} p_{ki} = \underline{p}_i, & \alpha_i \Delta_k \leq \underline{p}_i - p_{0i}^*, & i \in \hat{N}^{p-} \\ p_{ki} = \overline{p}_i, & \alpha_i \Delta_k \geq \overline{p}_i - p_{0i}^*, & i \in \hat{N}^{p+} \\ \underline{p}_i \leq p_{ki} \leq \overline{p}_i, & \alpha_i \Delta_k = p_{ki} - p_{0i}^*, & i \notin \hat{N}^{p-} \cup \hat{N}^{p+} \end{array}$$

Similarly, instead of (13.52), a set of linear constraints defined by x_0^* and $(\hat{y}_k, \hat{\Delta}_k)$ can be used to approximate the nonsmooth constraint (13.39b). Then (13.43) is solved approximately with the nonsmooth constraint (13.39) replaced by these linear constraints to obtain y_k^* and $r_k(x_0^*)$. These linear constraints are an inner approximation of (13.39) and hence the cost $r_k(x_0^*)$ is an upper bound on the optimal second-stage cost.

Overall algorithm. In summary the overall algorithm of [85] is as follows.

1. Solve SCOPF for a base-case solution x_0^* using the two-level ADMM algorithm with smooth approximations (13.50) (13.52) and fast contingency screening.
2. Given x_0^*, use fast contingency screening to sort contingencies into a priority list.
3. For each contingency k in the order of the priority list, solve the second-stage problem (13.43) approximately for y_k^* and its cost $r_k(x_0^*)$, using the smooth approximations (13.50) (13.52) followed by another solve using linear approximations if necessary.

13.6 Bibliographical notes

As for most chapters, this section is now a placeholder with references collected in a somewhat random fashion during the writing of the text. Major rewrite later.

There has been a great deal of research on OPF since Carpentier's first formulation in 1962 [87]. An early solution appears in [88] and extensive surveys can be found in e.g. [89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 36, 102]. It is nonconvex and has been shown to be NP-hard in general [103, 104, 105].

Many references for 3-phase OPF: e.g. [69, 70, 106]

There are many excellent texts on optimization theory especially for convex problems, e.g., [75, 73, 76]. Optimization texts with power system applications include [107, 108]. In particular Chapter 11.2.3 mostly follows the presentation in [73, Chapter 11]. A popular interior-point solver for OPF problems is [109].

OPF has been shown to be NP-hard in general [103, 104, 105, 110, 111]. [112] surveys combinatorial OPF and proves approximation results and conditions for exactness (when there are no discrete variables). It shows that OPF with discrete injections cannot be efficiently approximated. The hardness results complement those in [113, 114, 104, 105]; see [112, Chapter 5] and its Section 5.6 for comparison.

[115] shows that, by dualizing clique tree conversion, a class of nonconvex problems, including OPF problems, the per-iteration cost of an interior-point method is linear $O(n)$ in time and in memory, so an ε-accurate and ε-feasible iterate is obtained after $O(\sqrt{n}\log(1/\varepsilon))$ iterations in $O(n^{1.5}\log(1/\varepsilon))$ time.

13.7 Problems

Chapter 13.1

Exercise 13.1 (Inner product and trace)**.** Let $A, B \in \mathbb{C}^{n\times n}$ be square complex matrices. The inner product of A, B is defined to be $A \cdot B := \mathrm{tr}(A^{\mathsf{H}} B)$. Show that:

1. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
2. $A \cdot B \ := \ \mathrm{tr}(A^{\mathsf{H}} B) \ = \ \mathrm{tr}(AB)$ if A is Hermitian. The converse does not necessarily hold.
3. If A and B are both Hermitian then $A \cdot B = B \cdot A$.

Exercise 13.2 (Hermitian components)**.** Let $A \in \mathbb{C}^{n\times n}$ and $x \in \mathbb{C}^n$. Define the Hermitian and skewed Hermitian components of A:

$$B_r \ := \ \frac{1}{2}\left(A + A^{\mathsf{H}}\right), \qquad\qquad B_i \ := \ \frac{1}{2\mathbf{i}}\left(A - A^{\mathsf{H}}\right)$$

Show that

1. B_r and B_i are both Hermitian for arbitrary A, so that $x^{\mathsf{H}} B_r x$ and $x^{\mathsf{H}} B_i x$ are both real numbers.
2. Moreover $x^{\mathsf{H}} A x \ = \ x^{\mathsf{H}} B_r x + \mathbf{i} x^{\mathsf{H}} B_i x$.

Exercise 13.3 (Skew-symmetric and Hermitian matrices)**.** Show that:

1. If $C \in \mathbb{R}^{n\times n}$ is a skew symmetric matrix (i.e., $C^{\mathsf{T}} = -C$) then $x^{\mathsf{T}} C x = 0$ for any $x \in \mathbb{R}^n$.
2. If $C \in \mathbb{C}^{n\times n}$ is a Hermitian matrix (i.e., $C^{\mathsf{H}} = C$) then $x^{\mathsf{H}} C x \in \mathbb{R}$ for any $x \in \mathbb{C}^n$.
3. If $C \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, then $\mathrm{tr}(CX) \in \mathbb{R}$ for any rank-1 matrix $X \in \mathbb{C}^{n\times n}$ (psd or nsd).
4. Let $C := C_r + \mathbf{i} C_i$ where $C_r, C_i \in \mathbb{R}^{n\times n}$. If C is Hermitian then $C_r^{\mathsf{T}} = C_r$ and $C_i^{\mathsf{T}} = -C_i$.

Exercise 13.4 (Real QCQP)**.** Consider the complex QCQP

$$\min_{x\in\mathbb{C}^n} \quad x^{\mathsf{H}} C_0 x \qquad \text{s.t.} \qquad x^{\mathsf{H}} C_l x \ \le\ b_l, \quad l = 1,\dots,L$$

where $x \in \mathbb{C}^n$ is a vector, $C_l \in \mathbb{S}^n$ for $l = 0,\dots,L$, are Hermitian matrices. Show that it is equivalent to the real QCQP of twice the dimension:

$$\min_{y\in\mathbb{R}^{2n}} \quad y^{\mathsf{T}} D_0 y \qquad \text{s.t.} \qquad y^{\mathsf{T}} D_l y \ \le\ b_l, \quad l = 1,\dots,L$$

where

$$y \ :=\ \begin{bmatrix}\mathrm{Re}(x)\\ \mathrm{Im}(x)\end{bmatrix}, \qquad\qquad D_l \ :=\ \begin{bmatrix}\mathrm{Re}(C_l) & -\mathrm{Im}(C_l)\\ \mathrm{Im}(C_l) & \mathrm{Re}(C_l)\end{bmatrix}, \qquad l = 0,1,\dots,L$$

Note that D_l are symmetric matrices.

Exercise 13.5 (Homogenization)**.** Let $x,a,b \in \mathbb{C}^n$.

1. Let $e_j \in \{0,1\}^n$ be the unit vector with a single 1 at the jth position. Show that the set of inequalities $a_j \le x_j \le b_j$, $j = 1,\dots,n$, is equivalent to the following set of homogeneous quadratic inequalities in $(\hat{x},t)$ with $x := \hat{x}t$: for $j = 1,\dots,n$,

$$\mathrm{Re}(a_j) \ \le\ \begin{bmatrix}\hat{x}\\ t\end{bmatrix}^{\mathsf{H}} \begin{bmatrix}0 & \zeta_j\\ \zeta_j^{\mathsf{H}} & 0\end{bmatrix} \begin{bmatrix}\hat{x}\\ t\end{bmatrix} \ \le\ \mathrm{Re}(b_j), \quad \mathrm{Im}(a_j) \ \le\ \begin{bmatrix}\hat{x}\\ t\end{bmatrix}^{\mathsf{H}} \begin{bmatrix}0 & \mathbf{i}\zeta_j\\ -\mathbf{i}\zeta_j^{\mathsf{H}} & 0\end{bmatrix} \begin{bmatrix}\hat{x}\\ t\end{bmatrix} \ \le\ \mathrm{Im}(b_j) \tag{13.57a}$$

$$1 \ \le\ \begin{bmatrix}\hat{x}\\ t\end{bmatrix}^{\mathsf{H}} \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \begin{bmatrix}\hat{x}\\ t\end{bmatrix} \ \le\ 1 \tag{13.57b}$$

   where $\zeta_j = e_j/2$.

2. Let $c_j \in \mathbb{C}^n$ for $j = 1,\dots,n$. Show that the set of inequalities $a_j \le c_j^{\mathsf{H}} x \le b_j$, $j = 1,\dots,n$, is equivalent to (13.57) with $\zeta_j = e_j/2$ replaced by $\zeta_j = c_j/2$.

Exercise 13.6 (Single-phase OPF: power losses as quadratic form)**.** For each line $(j,k) \in E$, let its admittances be $y^s_{jk} = g^s_{jk} + \mathbf{i}b^s_{jk}$ and $y^m_{jk} = g^m_{jk} + \mathbf{i}b^m_{jk}$. Suppose $y^s_{jk} = y^s_{kj}$ and $g^s_{jk} \ge 0$, $g^m_{jk} \ge 0$ (these conditions are satisfied if (j,k) models a transmission line).

1. Define the total real power loss as:

$$C_0(V) \ :=\ \sum_j \mathrm{Re}\left(s_j(V)\right) \ =\ \sum_j \mathrm{Re}\left(\sum_{k:j\sim k} \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \ +\ \left(y^m_{jj}\right)^{\mathsf{H}} |V_j|^2\right)$$

   Show that $C_0(V)$ is a quadratic form $C_0(V) = V^{\mathsf{H}} C_0 V$ where the cost matrix $C_0 := \frac{1}{2}\left(Y^{\mathsf{H}} + Y\right)$ is the Hermitian component of the admittance matrix Y. Conclude that C_0 is a positive definite matrix when $g^m_{jk} + g^m_{kj} > 0$ for at least one line $(j,k) \in E$.

2. Suppose $y^m_{jk} = y^m_{kj} = 0$. Define the total thermal loss as:

$$C_0(V) \;:=\; \sum_{(j,k)\in E} r^s_{jk} \, |I_{jk}(V)|^2 \;=\; \sum_{(j,k)\in E} r^s_{jk} \left| y^s_{jk}(V_j - V_k) \right|^2$$

where $z^s_{jk} = r^s_{jk} + \mathbf{i}x^s_{jk} := 1/y^s_{jk}$. Show that $C_0(V)$ is a quadratic form $C_0(V) = V^{\mathsf{H}} C_0 V$ where the cost matrix $C_0 = \mathrm{Re}(Y)$ when $y^m_{jk} = y^m_{kj} = 0$. Conclude that C_0 is a positive semidefinite matrix.

3. Suppose $y^m_{jk} = y^m_{kj} = 0$. Show that the total real power loss in part 1 reduces to the total thermal loss in part 2.

Exercise 13.7 (Single-phase OPF: quadratic line limit)**.** Consider the line limit

$$|S_{jk}(V)|^2 \;\le\; \overline{S}^2_{jk}, \qquad |S_{kj}(V)|^2 \;\le\; \overline{S}^2_{kj}, \qquad (j,k)\in E$$

where

$$\begin{aligned}
S_{jk}(V) \;&:=\; V_j I^{\mathsf{H}}_{jk}(V) \;=\; \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_j|^2 - V_j V_k^{\mathsf{H}}\right) \;+\; \left(y^m_{jk}\right)^{\mathsf{H}} |V_j|^2, \qquad (j,k)\in E \\
S_{kj}(V) \;&:=\; V_k I^{\mathsf{H}}_{kj}(V) \;=\; \left(y^s_{jk}\right)^{\mathsf{H}} \left(|V_k|^2 - V_k V_j^{\mathsf{H}}\right) \;+\; \left(y^m_{kj}\right)^{\mathsf{H}} |V_k|^2 \qquad (j,k)\in E
\end{aligned}$$

Show that the line limit can be written as an inhomogeneous quadratic form.

Exercise 13.8 (3-phase OPF: line limit)**.** Show that the line limit in three-phase OPF is

$$\left| I^{\phi}_{jk} \right|^2 \;:=\; V^{\mathsf{H}} \hat{Y}^{\phi}_{jk} V \;\le\; \ell^{\phi\,\max}_{jk}$$

where $\hat{Y}^{\phi}_{jk} := \tilde{Y}^{\mathsf{H}}_{jk} E^{\phi} \tilde{Y}_{jk}$ is a $3(N+1)\times 3(N+1)$ matrix and $\tilde{Y}_{jk}$ is a $3\times 3(N+1)$ matrix given by

$$\tilde{Y}_{jk} \;:=\; \left((e_j - e_k)^{\mathsf{T}} \otimes y^s_{jk} \;+\; e_j^{\mathsf{T}} \otimes y^m_{jk} \right)$$

Here $e_j \in \{0,1\}^{N+1}$ and $e^{\phi} \in \{0,1\}^3$ are unit vectors of different sizes with a single 1 at the jth and ϕth position respectively, $E^{\phi} := e^{\phi} e^{\phi\mathsf{T}}$, and $\mathbb{I}$ is the 3×3 identity matrix. Similarly for $\left| I^{\phi}_{kj} \right|^2$.

Exercise 13.9 (3-phase OPF as QCQP: current source)**.** Show that the conversion rules (13.18c)(13.18d) for a current source are equivalent to the following inhomogeneous quadratic forms:

$$Y \text{ configuration:} \qquad s_j(V) \;=\; -u_j^{\mathsf{H}} \left(e_j^{\mathsf{H}} \otimes E^{\phi} \right) V \tag{13.58}$$

$$\Delta \text{ configuration:} \qquad s_j(V) \;=\; -u_j^{\mathsf{H}} \left(e_j^{\mathsf{H}} \otimes (\Gamma E^{\phi}) \right) V \tag{13.59}$$

where $s_j(V)$ is given by (13.25).

Exercise 13.10 (3-phase OPF as QCQP: power source)**.** Recall the quadratic form $s_j(V)$ given by (13.25).

1. *Y-configured power source*: Let $u_j =: (u_{j1}, u_{j2})$ where $u_{j1} := s_j^Y$ and $u_{j2} := I_j^Y$ be the optimization variable. Show that the conversion rule (13.18e) is equivalent to the following inhomogeneous quadratic equations in (V, u_j):

$$Y: \qquad s_j^\phi(V) \;=\; -u_{j2}^{\mathsf{H}}\left(e_j^{\mathsf{H}} \otimes E^\phi\right)V, \qquad s_j(V) \;=\; -u_{j1} - \gamma_j^Y \overline{u}_{j2}, \qquad \phi \in \{a,b,c\}$$

2. *Δ-configured power source*: Let $u_j =: (u_{j1}, u_{j2})$ where $u_{j1} := s_j^\Delta$ and $u_{j2} := I_j^\Delta$ be the optimization variable. Show that the conversion rule (13.18f) is equivalent to the following inhomogeneous quadratic equations in (V, u_j):

$$\Delta: \quad s_j^\phi(V) \;=\; -u_{j2}^{\mathsf{H}}\left(e_j^{\mathsf{H}} \otimes (\Gamma E^\phi)\right)V, \quad u_{j1}^{\phi\varphi} \;=\; u_{j2}^{\mathsf{H}}\left(e_j^{\mathsf{H}} \otimes \left(E^\phi \Gamma\right)\right)V, \quad \phi\varphi \in \{ab, bc, ca\}$$

Exercise 13.11 (3-phase OPF as QCQP: impedance)**.** Show that the equality constraints imposed by an impedance $\left(z_j^Y, \gamma_j^Y\right)$ or z_j^Δ are equivalent to the following inhomogeneous quadratic equations in V:

$$\begin{aligned}
&Y \text{ configuration:} \qquad & s_j^\phi(V) \;&=\; V^{\mathsf{H}}\left(\left(e_j e_j^{\mathsf{H}}\right) \otimes \left(y_j^{Y\mathsf{H}} E^\phi\right)\right)V \;-\; \gamma_j^{Y\mathsf{H}}\left(e_j^{\mathsf{H}} \otimes \left(\mathbf{1}^{\mathsf{H}} y_j^{Y\mathsf{H}} E^\phi\right)\right)V \\
&\Delta \text{ configuration:} \qquad & s_j^\phi(V) \;&=\; -V^{\mathsf{H}}\left(\left(e_j e_j^{\mathsf{H}}\right) \otimes \left(Y_j^{\Delta\mathsf{H}} E^\phi\right)\right)V
\end{aligned}$$

where the neutral voltage $\gamma_j^Y := V_j^n$ is given (e.g., $\gamma_j^Y = 0$) and $Y_j^\Delta := \Gamma^{\mathsf{T}} y^\Delta \Gamma$.

The next exercise studies a Y-configured power source when the optimization variable is taken to be $u_j := s_j^Y$ instead of $u_j := (s_j^Y, I_j^Y)$. It suggests that the formulation in the text that uses $u_j := (s_j^Y, I_j^Y)$ as the optimization variable seems simpler.

Exercise 13.12 (3-phase OPF as QCQP: power source)**.** For a Y-configured ideal power source, suppose the optimization variable is the internal power (only) $u_j := s_j^Y$ and its neutral voltage $\gamma_j^Y := V_j^n$ is given. If $\gamma_j^Y = 0$ then $s_j = -s_j^Y$. Suppose $\gamma_j^Y \neq 0$.

1. Show that u_j is related to the terminal voltage and current $\left(V_j, s_j\right)$ as:

$$s_j \;=\; -\text{diag}\left(\frac{V_j^\phi}{V_j^\phi - \gamma_j^Y}, \phi = a,b,c\right) u_j$$

2. *Y configuration*: Show that the conversion rule in part 1 is equivalent to the following set of inhomogeneous equality constraints on $\left(V, u_j, w_j^\phi, \phi \in \{a,b,c\}\right) \in \mathbb{C}^{12(N+1)+3}$: for each $j \in \overline{N}$,

$$\begin{aligned}
V^{\mathsf{H}}\left(\gamma_j^Y Y_j^{\phi\mathsf{H}}\right)V \;&=\; \overline{u}_j^{\mathsf{H}}\left(e^\phi e_j^{\phi\mathsf{H}}\right)V \;+\; w_j^{\phi\mathsf{H}}\left(Y_j^{\phi\mathsf{H}}\right)V, \qquad && \phi \in \{a,b,c\} \\
e_k^{\varphi\mathsf{H}} w_j^\phi \;&=\; V^{\mathsf{H}}\left(e_j^\phi e_j^{\varphi\mathsf{H}}\right)V, && k \in \overline{N},\ \phi, \varphi \in \{a,b,c\}
\end{aligned}$$

where $w_j^\phi \in \mathbb{C}^{3(N+1)}$ is an auxiliary variable, one for each $\phi \in \{a,b,c\}$. For each $j \in \overline{N}$, this is a set of $9(N+1)+3$ quadratic equations in $\left(V, u_j, w_j^\phi, \phi \in \{a,b,c\}\right)$.

Chapter 13.2.

Exercise 13.13 (DistFlow with nonzero shunt admittances)**.** Formulate single-phase OPF with generalized DistFlow model with nonzero shunt admittances $\left(y_{jk}^m, y_{kj}^m\right)$.

Chapter 13.5 Given a matrix $M \in \mathbb{R}^{m\times n}$ and vector $q \in \mathbb{R}^m$ the standard linear complementarity problem LCP(M,q) is to find vectors $z, w \in \mathbb{R}^m$ such that

$$z \geq 0, \quad w \geq 0, \quad z^{\mathsf{T}} w = 0, \quad w = Mz + q \tag{13.60a}$$

or, equivalently, to find z such that

$$z \geq 0, \quad Mz + q \geq 0, \quad z^{\mathsf{T}}(Mz+q) = 0 \tag{13.60b}$$

Complementarity constraints arise frequently in OPF problems (see Chapter 13.5.2). The next few exercises are on linear complementarity problems.

Exercise 13.14 (Linear complementarity problem)**.**

1. Consider the quadratic optimization:

$$\min_{x\in\mathbb{R}^n} \ \frac{1}{2}x^{\mathsf{T}}Qx + c^{\mathsf{T}}x \quad \text{s.t.} \quad Ax \leq b, \ x \geq 0 \tag{13.61a}$$

   Show that solving the associated KKT condition is a standard LCP.

2. Consider the quadratic optimization without the nonnegativity constraint on x:

$$\min_{x\in\mathbb{R}^n} \ \frac{1}{2}x^{\mathsf{T}}Qx + c^{\mathsf{T}}x \quad \text{s.t.} \quad Ax \leq b \tag{13.61b}$$

   If Q is positive definite show that solving the associated KKT condition is a standard LCP.

Exercise 13.15 (Linear complementarity problem)**.** Suppose $A, B \in \mathbb{R}^{n\times n}$ are square matrices and $a, b \in \mathbb{R}^n$. Consider

$$0 \leq Az + a \perp Bz + b \geq 0 \tag{13.62}$$

Show that if A is nonsingular then (13.62) is a standard LCP.

Exercise 13.16 (Linear complementarity problem)**.** Show that the system of equations in (y, ρ^-, ρ^+):

$$y + \rho^+ - \rho^- \ = \ x, \qquad 0 \ \leq \ \rho^- \ \perp \ y - \underline{a} \ \geq \ 0, \qquad 0 \ \leq \ \rho^+ \ \perp \ \overline{a} - y \ \geq \ 0 \tag{13.63}$$

where $\underline{a} < \overline{a}$ is equivalent to a standard LCP.

Exercise 13.17 (Linear complementarity problem)**.** Let

$$M \ := \ \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \qquad q \ := \ \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solve the LCP(M, q): find $x := [x_1 \ x_2]^\mathsf{T}$ such that

$$x \ \geq \ 0, \quad Mx + q \ \geq \ 0, \quad x^\mathsf{T}(Mx + q) \ = \ 0$$

Note that there exists a unique solution even though M is neither positive definite nor symmetric.

Exercise 13.18 (SCOPF: nonsmooth constraints)**.** Consider the nonsmooth constraint $y = [x]_{\underline{a}}^{\overline{a}} := \max\{\underline{a}, \min\{x, \overline{a}\}\}$ on variables $(x, y) \in \mathbb{R}^2$ where $\underline{a} < \overline{a}$ are given constants.

1. Show that it is equivalent to the following complementarity constraints:

$$y + \rho^+ - \rho^- \ = \ x, \qquad 0 \ \leq \ \rho^- \ \perp \ y - \underline{a} \ \geq \ 0, \qquad 0 \ \leq \ \rho^+ \ \perp \ \overline{a} - y \ \geq \ 0$$

   where (ρ^-, ρ^+) are slack variables. Finding a solution (y, ρ^-, ρ^+) to this system of equations is a linear complementarity problem; see Exercise 13.16.

2. Show that it is equivalent to the following set of big-M mixed integer constraints:

$$\underline{a} \ \leq \ y \ \leq \ \overline{a}, \qquad \underline{z}, \overline{z} \ \in \ \{0, 1\} \tag{13.64a}$$
$$y - \underline{a} \ \leq \ M\underline{z}, \quad y - x \ \leq \ M(1 - \underline{z}) \tag{13.64b}$$
$$\overline{a} - y \ \leq \ M\overline{z}, \quad x - y \ \leq \ M(1 - \overline{z}) \tag{13.64c}$$

   where M is a sufficiently large constant and $(\underline{z}, \overline{z})$ are auxiliary binary variables. What value of $(\underline{z}, \overline{z})$ will result in infeasibility?

3. Show that it is also equivalent to:

$$x - \underline{a} \ \leq \ M\underline{z}, \qquad \underline{a} - x \ \leq \ M(1 - \underline{z}) \tag{13.65a}$$
$$\overline{a} - x \ \leq \ M\overline{z}, \qquad x - \overline{a} \ \leq \ M(1 - \overline{z}) \tag{13.65b}$$

   together with (the nonlinear equality)

$$(y - \underline{a})(1 - \underline{z}) \ + \ (y - \overline{a})(1 - \overline{z}) \ + \ (y - x)\underline{z}\overline{z} \ = \ 0, \qquad \underline{z}, \overline{z} \ \in \ \{0, 1\} \tag{13.65c}$$

   What value of $(\underline{z}, \overline{z})$ will result in infeasibility?

Exercise 13.19 (SCOPF: nonsmooth constraints)**.** Consider the following disjunctive constraint on $(q,V) \in \mathbb{R}^2$:

$$\left\{\underline{q} \le q \le \overline{q},\ V = V_0\right\} \cup \left\{q = \underline{q},\ V \ge V_0\right\} \cup \left\{q = \overline{q},\ V \le V_0\right\}$$

where $\underline{q} < \overline{q}$.

1. Show that it is equivalent to:

   $$V + \mu^+ - \mu^- \ = \ V_0, \qquad 0 \ \le \ \mu^- \ \perp \ q - \underline{q} \ \ge \ 0, \quad 0 \ \le \ \mu^+ \ \perp \ \overline{q} - q \ \ge \ 0$$

   where (μ^-, μ^+) are slack variables. Unlike the complementarity problem in Exercise 13.18.1, the equality constraint here involves another variable V, not q.

2. Show that it is equivalent to:

   $$\begin{array}{lll} \underline{q} \ \le \ q \ \le \ \overline{q}, & \underline{z}, \overline{z} \ \in \ \{0,1\} & \\ q - \underline{q} \ \le \ M\underline{z}, & V_0 - V \ \le \ M\underline{z} & V - V_0 \ \le \ M(1-\underline{z}) \\ \overline{q} - q \ \le \ M\overline{z}, & V - V_0 \ \le \ M\overline{z} & V_0 - V \ \le \ M(1-\overline{z}) \end{array}$$

   where M is a sufficiently large constant and $(\underline{z}, \overline{z})$ are auxiliary binary variables.

Exercise 13.20 (Smooth approximation)**.** This problem considers smooth approximations of $\max(a,x)$ and $\min(a,x)$.

1. Let $f(x) := \max(0,x)$ and its approximation $f^\varepsilon(x) := \varepsilon \ln\left(1 + e^{x/\varepsilon}\right)$ for $x \in \mathbb{R}$ and $\varepsilon > 0$. For any $\varepsilon > 0$ show that $f^\varepsilon(x) - \varepsilon \ln 2 \le f(x) < f^\varepsilon(x)$ for all $x \in \mathbb{R}$.
2. What is the corresponding approximation for $\tilde{f}(x) := \max(a,x)$ for any $a \in \mathbb{R}$?
3. Let $g(x) := \min(0,x)$. Justify its approximation $g^\varepsilon(x) := -\varepsilon \ln\left(1 + e^{-x/\varepsilon}\right)$ for $x \in \mathbb{R}$ and $\varepsilon > 0$. For any $\varepsilon > 0$ show that $g^\varepsilon(x) < g(x) \le g^\varepsilon(x) + \varepsilon \ln 2$ for all $x \in \mathbb{R}$.
4. What is the corresponding approximation for $\tilde{g}(x) := \min(x,b)$ for any $b \in \mathbb{R}$?
5. What is the approximation for $h(x) := \max(a, \min(x,b))$ for $a < b$ if we apply the approximations for $\tilde{f}^\varepsilon$ and $\tilde{g}^\varepsilon$?

Exercise 13.21 (Smooth approximation)**.** Prove the properties (13.53).

Exercise 13.22 (ADMM)**.** Consider the problem

$$\min_{x\in\mathbb{R}^n,\ y\in\mathbb{R}^p} \quad f(x) + g(y) \tag{13.67a}$$
$$\text{s.t.} \quad x \in X, \ \ y \in Y \tag{13.67b}$$
$$Ax + By = c \tag{13.67c}$$

with the augmented Lagrangian

$$L_\rho(x,y,\lambda) \;:=\; f(x) + g(y) + \lambda^\mathsf{T}(Ax+By-c) + \frac{\rho}{2}\|Ax+By-c\|_2^2$$

and the ADMM algorithm:

$$x_{t+1} \;:=\; \arg\min_{x\in X} L_\rho(x,\ y_t,\ \lambda_t) \tag{13.68a}$$
$$y_{t+1} \;:=\; \arg\min_{y\in Y} L_\rho(x_{t+1},\ y,\ \lambda_t) \tag{13.68b}$$
$$\lambda_{t+1} \;:=\; \lambda_t + \rho\,(Ax_{t+1}+By_{t+1}-c) \tag{13.68c}$$

Suppose the objective functions f and g are closed proper convex and differentiable.

1. Write down the first-order optimality condition.
2. Show that the Gauss-Siedel step (13.68b) and the choice of the step size ρ for the dual update ensure that (x_t, y_t, λ_t) satisfies one of the two stationarity conditions in part 1 in every iteration t.
3. Show that (under appropriate assumptions) ADMM converges, i.e.,
    - The other stationarity condition is satisfied as $t \to \infty$.
    - $Ax_t + By_t - c \to 0$ as $t \to \infty$.
    - $f(x_t) + g(y_t)$ converges to the optimal value as $t \to \infty$.

Chapter 14

Semidefinite relaxations: BIM

Chapter 12 motivates optimal power flow (OPF) problems through various control decisions in power system operations. Chapter 13 studies generic OPF as a nonconvex constrained optimization problem. In particular we have formulated OPF as a nonconvex quadratically constrained quadratic program (QCQP) and shown that it is NP-hard in general. Numerous methods have been proposed for solving OPF in different applications. Algorithms such as those discussed in Chapter 11.2 have been used for computing a local solution of the nonconvex OPF (often called AC OPF because of the nonlinear power flow equations). Instead of the nonconvex problem, its linear or convex approximations have also been solved. For instance DC OPF is a linear program approximation of the nonconvex problem that is widely used for dispatching generators in electricity markets. In this and the next chapters we study a convex approximation, called semidefinite relaxation, of OPF.

There is a rich theory and extensive empirical experiences in applying semidefinite relaxation to many engineering problems. Due to nonconvexity of OPF, algorithms typically computes a local optimal without assurance on the quality of the solution. A semidefinite relaxation provides the ability to check if a feasible solution is globally optimal. If it is not, the solution of a relaxation provides a lower bound on the minimum cost and hence a bound on how far any feasible solution is from optimality. Unlike approximations, if a relaxed problem is infeasible, it is a certificate that the original OPF is infeasible. In Chapter 14.1 we define semidefinite relaxation of QCQP in general and explain how to use the concept of partial matrices and their psd rank-1 completion to reduce the computational complexity of the semidefinite relaxation for large sparse networks. In Chapter 14.2 we apply these results to formulate single-phase OPF as QCQP in the bus injection model. In Chapters 14.3 and 14.4 we describe two sufficient conditions for the semidefinite relaxation of OPF to be exact for single-phase radial network. In Chapter 14.5 we extend semidefinite relaxations of OPF to unbalanced three-phase networks.

14.1 Semidefinite relaxations of QCQP

OPF is formulated in (13.15) as a standard homogeneous QCQP. The computational difficulty arises from the nonconvex feasible set of OPF. Informally one can regard a relaxation of OPF as minimizing the same cost function over a convex superset (though in a lifted space). Different choices of convex supersets lead to different relaxations, but they all provide lower bounds to OPF. If an optimal solution of a relaxation happens to lie in the feasible set of the original OPF problem, then it is optimal for the original OPF. In this case we say the relaxation is exact. In this section we describe three types of semidefinite relaxation of OPF and explain equivalence relations among them. In the next section we present sufficient conditions that guarantee exact relaxations

14.1.1 SDP relaxation

Since these methods are not restricted to OPF, we will discuss them using the general QCQP formulation (13.10), reproduced here:

$$\min_{x\in\mathbb{C}^n} \quad x^{\mathsf{H}}C_0x \tag{14.1a}$$

$$\text{s.t.} \quad x^{\mathsf{H}}C_lx \ \le\ b_l, \quad l=1,\ldots,L \tag{14.1b}$$

Using $x^{\mathsf{H}}C_lx = \operatorname{tr}\left(C_lxx^{\mathsf{H}}\right)$ we can rewrite (14.1) as

$$\begin{aligned} \min_{X\in\mathbb{S}^n, x\in\mathbb{C}^n} \quad & \operatorname{tr}(C_0X) \\ \text{s.t.} \quad & \operatorname{tr}(C_lX) \ \le\ b_l, \quad l=1,\ldots,L \\ & X \ =\ xx^{\mathsf{H}} \end{aligned}$$

Any positive semidefinite (psd) rank-1 matrix $X\in\mathbb{S}_+^{n\times n}$ has a spectral decomposition $X = xx^{\mathsf{H}}$ for some $x\in\mathbb{C}^n$; see Chapter 25.1.5. The factor x is unique *up to a rotation*, i.e., x satisfies $X = xx^{\mathsf{H}}$ if and only if $xe^{j\theta}$ does for any $\theta\in\mathbb{R}$. Hence (14.1) is equivalent to the following problem where the optimization is over the set $\mathbb{S}^n$ of Hermitian matrices X:

$$\min_{X\in\mathbb{S}^n} \quad \operatorname{tr}(C_0X) \tag{14.2a}$$

$$\text{s.t.} \quad \operatorname{tr}(C_lX) \ \le\ b_l, \quad l=1,\ldots,L \tag{14.2b}$$

$$X \ \succeq\ 0, \quad \operatorname{rank}(X) \ =\ 1 \tag{14.2c}$$

Recall that $\operatorname{tr}(C_lX) = \sum_{j,k}[C_l]_{jk}X_{kj} = \sum_{j,k}[C_l]_{jk}X_{jk}^{\mathsf{H}}$ where the second equality follows when X is Hermitian. While the objective function and the constraints in (14.1) are quadratic in x, they are linear in X in (14.2a)(14.2b). The constraint $X\succeq 0$ in (14.2c) is convex ($\mathbb{S}_+^n$ is a convex cone). The rank constraint in (14.2c) is the only nonconvex constraint. These two problems are equivalent in the sense that, given a feasible (or optimal) solution x to QCQP (14.1), there is an $X := xx^{\mathsf{H}}$ that is feasible (or optimal) to the semidefinite program (14.2). Conversely, given an X that is feasible (or optimal) to (14.2), a solution x to (14.1) can be recovered through rank-1 factorization $X = xx^{\mathsf{H}}$. It is in this sense that we also say that the feasible sets of (14.1) and (14.2) are equivalent. This is referred to as *lifting* the original QCQP problem from n dimensional space $\mathbb{C}^n$ to the higher-dimensional space of $n\times n$ Hermitian matrices.

Removing the rank constraint (14.2c) results in a semidefinite program (SDP):

$$\min_{X\in\mathbb{S}^n} \quad \operatorname{tr}(C_0X) \tag{14.3a}$$

$$\text{s.t.} \quad \operatorname{tr}(C_lX) \ \le\ b_l, \quad l=1,\ldots,L \tag{14.3b}$$

$$X \ \succeq\ 0 \tag{14.3c}$$

See Chapter 25.3.2 for more details of the convex problem SDP. We call (14.3) a *semidefinite relaxation* or an *SDP relaxation* of QCQP (14.1) because the feasible set of the equivalent problem (14.2) is a subset of the feasible set of SDP (14.3). A strategy for solving QCQP (14.1) is to solve SDP (14.3) for an optimal matrix X^{opt} and check its rank. If $\text{rank}(X^{\text{opt}}) = 1$ then X^{opt} is feasible and hence optimal for (14.2) as well and an optimal solution x^{opt} of QCQP (14.1) can be recovered from X^{opt} through spectral decomposition $X^{\text{opt}} = x^{\text{opt}}(x^{\text{opt}})^{\mathsf{H}}$. If $\text{rank}\, X^{\text{opt}} > 1$ then, in general, no feasible solution of QCQP can be directly obtained from X^{opt} but the optimal objective value of SDP provides a lower bound on that of QCQP.

14.1.2 Partial matrices and completions

Even though the relaxation (14.3) is a convex problem computing its solution can still be challenging if the problem size n is large. If the underlying network is sparse, much more efficient relaxations can be used. To develop these ideas precisely, the key is to study the feasible sets of QCQP and its relaxations.

We start with the concept of partial matrices and their completions. An instance of QCQP (14.1) is specified by a set of matrices and scalars $(C_0, C_l, b_l, l = 1, \ldots, L)$. We assume the matrices $C_l, l = 0, 1, \ldots, L$, are Hermitian so that $x^{\mathsf{H}} C_l x$ are real. They define an underlying undirected graph $F := (N, E)$ with n nodes and m edges where distinct nodes j and k are *adjacent* (i.e., $(j,k) \in E$) if and only if there exists an $l \in \{0, 1, \ldots, L\}$ such that $[C_l]_{jk} = [C_l]_{kj}^{\mathsf{H}} \neq 0$. Assume without loss of generality that the graph F is connected. For any $x \in \mathbb{C}^n$ note that the quadratic forms $x^{\mathsf{H}} C_l x$ depends on $|x_j|^2$ and on $x_j^{\mathsf{H}} x_k$ if and only if $(j,k) \in E$ is a link in F, i.e., if and only if there exists an l such that the coefficient of $x_j^{\mathsf{H}} x_k$ is nonzero. Indeed

$$x^{\mathsf{H}} C_l x \;=\; \sum_{j,k} [C_l]_{jk} x_j^{\mathsf{H}} x_k \;=\; \sum_j [C_l]_{jj} |x_j|^2 \,+\, 2 \sum_{(j,k) \in E} \text{Re}\left([C_l]_{jk} x_j^{\mathsf{H}} x_k \right)$$

where the last equality follows from $[C_l]_{kj} x_k^{\mathsf{H}} x_j = [C_l]_{jk}^{\mathsf{H}} x_k^{\mathsf{H}} x_j = \left([C_l]_{jk} x_j^{\mathsf{H}} x_k \right)^{\mathsf{H}}$ since C_l is Hermitian. Hence the constraints $x^{\mathsf{H}} C_l x \leq b_l$ do not depend on $x_j^{\mathsf{H}} x_k$ if $(j,k) \notin E$ for any l, and X_{jk} of the lifted variable X are not directly constrained by $\text{tr}(C_l X) \leq b_l$ if $(j,k) \notin E$. This can be used to relax the psd and rank-1 constraints on the entire matrix X using the concept of partial matrices, greatly simplifying computation when the underlying graph F of the QCQP is sparse.

A *partial matrix* X_F is a set of $2m + n$ complex numbers *defined on* $F := (N, E)$:

$$X_F \;:=\; \left\{ \, [X_F]_{jj}, [X_F]_{jk}, [X_F]_{kj} \; : \; \text{nodes } j \in N \text{ and links } (j,k) \in E \, \right\}$$

X_F can be interpreted as a matrix with entries partially specified by these complex numbers. The (j,k)th entry of X_F that does not correspond to an edge in F is not specified. If F is a complete graph (in which there is an edge between every pair of vertices) then X_F is a fully specified $n \times n$ matrix. A *completion* X of X_F is any fully specified $n \times n$ matrix that agrees with X_F on graph F, i.e.,

$$[X]_{jj} \;=\; [X_F]_{jj}, \qquad [X]_{jk} \;=\; [X_F]_{jk}, \qquad [X]_{kj} \;=\; [X_F]_{kj}, \qquad j \in N, \; (j,k) \in E$$

Given an $n \times n$ matrix X we use X_F to denote the *submatrix of X on F,* i.e., the partial matrix consisting of the entries of X defined on graph F. If q is a clique (a fully connected subgraph) of F then let $X_F(q)$

denote the fully-specified principal submatrix of X_F defined on q, i.e., if the clique q has k nodes then $X(q)$ is a $k \times k$ matrix and, for every node j and link (j,k) in the clique q,

$$[X(q)]_{jj} \; := \; [X_F]_{jj}, \qquad [X(q)]_{jk} \; := \; [X_F]_{jk}, \qquad [X(q)]_{kj} \; := \; [X_F]_{kj}$$

We extend the definitions of Hermitian, psd, and rank-1 for matrices to partial matrices, as follows. A partial matrix X_F is *Hermitian*, denoted by $X_F = X_F^{\mathsf{H}}$, if $[X_F]_{kj} = [X_F]_{jk}^{\mathsf{H}}$ for all $(j,k) \in F$; it is *psd*, denoted by $X_F \succeq 0$, if X_F is Hermitian and the principal submatrices $X_F(q)$ are psd for all cliques q of F; it is *rank-1*, denoted by $\mathrm{rank}(X_F) = 1$, if the principal submatrices $X_F(q)$ are rank-1 for all cliques q of F. We say X_F is 2×2 *psd (rank-1)* if, for all edges $(j,k) \in F$, the 2×2 principal submatrices

$$X_F(j,k) \quad := \quad \begin{bmatrix} [X_F]_{jj} & [X_F]_{jk} \\ [X_F]_{kj} & [X_F]_{kk} \end{bmatrix}$$

are psd (rank-1). The condition $X_F(j,k) \succeq 0$ is equivalent to: the matrix $X_F(j,k)$ is Hermitian, i.e., $X_F(j,k) = X_F(j,k)^{\mathsf{H}}$, its diagonal entries $[X_F]_{jj}, [X_F]_{kk}$ are real, and

$$[X_F]_{jj} \geq 0, \; [X_F]_{kk} \geq 0, \; [X_X]_{jj}[X_X]_{kk} \geq \left|[X_F]_{jk}\right|^2$$

This is a second-order cone (see (17.48)). The condition $\mathrm{rank}(X_F(j,k)) = 1$ is equivalent to $X_F(j,k)$ is not a zero matrix and

$$[X_F]_{jj}\,[X_F]_{kk} \quad = \quad \left|[X_F]_{jk}\right|^2$$

We extend the trace operation to partial matrices X_F:

$$\mathrm{tr}\,(C_l X_F) \; := \; \sum_{j \in N} [C_l]_{jj}\,[X_F]_{jj} \; + \; \sum_{(j,k) \in E} \left([C_l]_{jk}\,[X_F]_{kj} + [C_l]_{kj}\,[X_F]_{jk}\right)$$

If both C_l and X_F are Hermitian then $[C_l]_{kj}[X_F]_{jk} = \left([C_l]_{jk} X_{kj}\right)^{\mathsf{H}}$ and hence

$$\mathrm{tr}\,(C_l X_F) \quad = \quad \sum_{j \in N} [C_l]_{jj}\,[X_F]_{jj} \; + 2 \sum_{(j,k) \in E} \mathrm{Re}\left([C_l]_{jk}\,[X_F]_{kj}\right)$$

is a real scalar.

We call F a *chordal graph* if either F has no cycle or all its minimal cycles (ones without chords) are of length three. A *chordal extension* $c(F)$ of F is a chordal graph that contains F, i.e., $c(F)$ has the same vertex set as F but an edge set that is a superset of F's edge set. In that case we call the partial matrix $X_{c(F)}$ a *chordal extension* of the partial matrix X_F. Every graph F has a chordal extension, generally nonunique. In particular a complete supergraph of F is a trivial chordal extension of F. Chordal graphs are important for us because of the result [116, Theorem 7] that every psd partial matrix has a psd completion if and only if the underlying graph is chordal. When a positive definite completion exists, there is a *unique* positive definite completion, in the class of all positive definite completions, whose determinant is maximal. Before extending this to rank-1 partial matrices, we present an example.

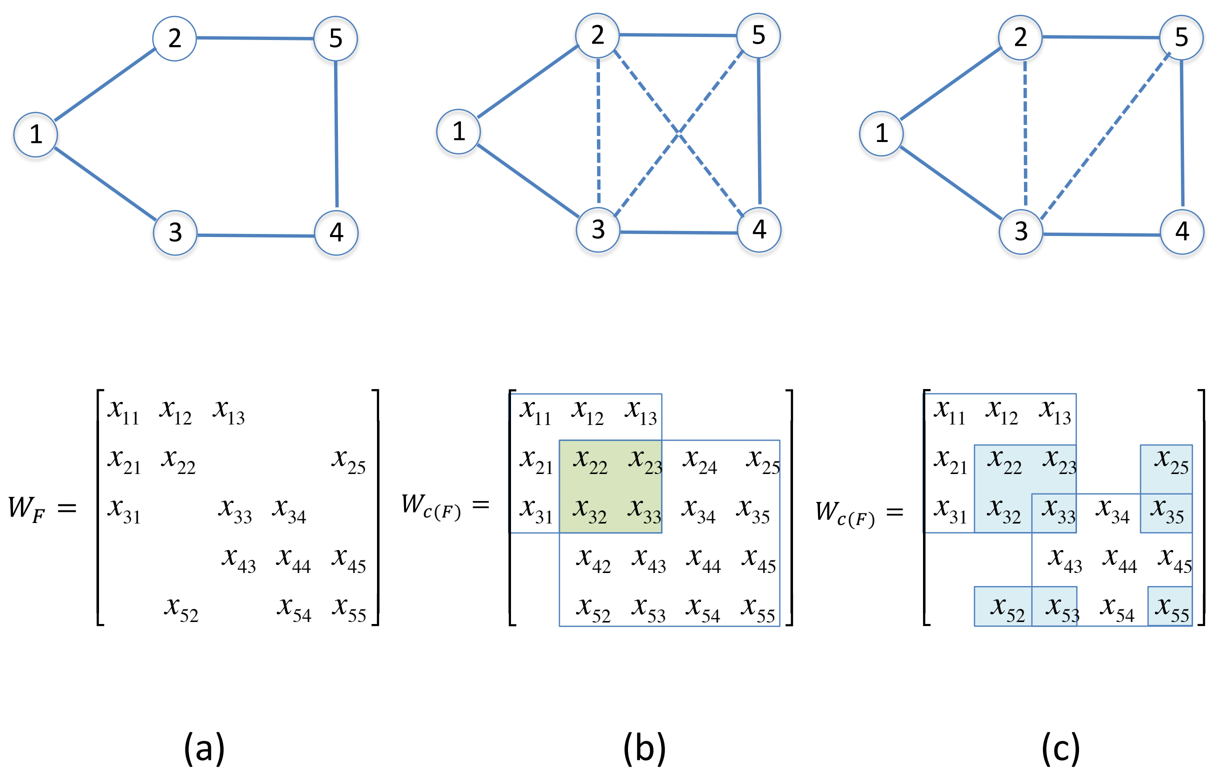

Figure 14.1: Example 14.1: (a) Partial matrix X_F. (b)(c) Two chordal extensions $X_{c(F)}$ and their overlapping maximal cliques.

Example 14.1 (Partial matrices and definitions)**.** Consider the graph F and the partial matrix X_F in Figure 14.1(a). X_F is Hermitian if $x_{jk} = x_{kj}^{\mathsf{H}}$. The only cliques in F consist of two nodes that are adjacent, and hence X_F is psd if it is 2×2 psd and X_F is rank-1 if it is 2×2 rank-1. X_F is not chordal as it contains a cycle of length greater than 3.

Figure 14.1(b) and (c) depict two chordal extensions $c(F)$ of F and their corresponding partial matrices $X_{c(F)}$. The chordal extension in Figure 14.1(b) has 2 maximal cliques, $q_1 := (1,2,3)$ and $q_2 := (2,3,4,5)$. These cliques share two nodes, 2 and 3. The corresponding cliques are outlined in $X_{c(F)}(q_1)$ in the figure with the overlapping entries shaded in green. The chordal extension in Figure 14.1(c) has 3 maximal cliques, outlined and shaded in blue in $X_{c(F)}(q_2)$.

□

Consider the following conditions on $n\times n$ matrices X and partial matrices $X_{c(F)}$ and X_F:

$$X \succeq 0, \quad \text{rank}(X) = 1 \tag{14.4a}$$

$$X_{c(F)} \succeq 0, \quad \text{rank}(X_{c(F)}) = 1 \tag{14.4b}$$

$$X_F(j,k) \succeq 0, \quad \text{rank}(X_F(j,k)) = 1, \quad (j,k)\in E \tag{14.4c}$$

We say that a partial matrix X_F satisfies the *cycle condition* if for every cycle c in F

$$\sum_{(j,k)\in c} \angle[X_F]_{jk} \;=\; 0 \quad \mod 2\pi \tag{14.5}$$

where $x = \phi \mod 2\pi$ means $x = \phi + 2k\pi$ for some integer k. For instance if $\angle[X_F]_{jk}$ represent the voltage phase differences across lines (j,k) then the cycle condition imposes that they sum to zero (mod 2π) around any cycle c. The next theorem implies that X_F has a psd rank-1 completion X if and only if X_F has a chordal extension $X_{c(F)}$ that is psd rank-1, if and only if X_F is 2×2 psd rank-1 on F and satisfies the cycle condition (14.5). [1]

Theorem 14.1 (Rank-1 characterization)**.** Fix a connected graph $F := (N,E)$ with $n := |N|$ nodes. Consider any chordal extension $c(F)$ of F. Suppose $X_{jj} > 0$, $\left[X_{c(F)}\right]_{jj} > 0$ and $[X_F]_{jj} > 0$, $j\in N$, for the matrix X and submatrices X_F and $X_{c(F)}$ below. Then

(1) Given a $n\times n$ matrix X that satisfies (14.4a), its submatrix $X_{c(F)}$ satisfies (14.4b).

(2) Given a partial matrix $X_{c(F)}$ that satisfies (14.4b), its submatrix X_F satisfies (14.4c) and the cycle condition (14.5).

(3) Given a partial matrix X_F that satisfies (14.4c) and the cycle condition (14.5), there is a completion X of X_F that satisfies (14.4a).

Informally Theorem 14.1 says that (14.4a) is equivalent to (14.4b) which is equivalent to (14.4c)(14.5). It implies in particular that, for a chordal graph, X is psd rank-1 if and only if the principal submatrix $X(q)$ of X is psd rank-1 for every maximal clique q of the graph. It characterizes a property of the full matrix X (that X is psd and rank-1) in terms of its submatrices $X_{c(F)}$ and X_F. This is important because the submatrices are typically much smaller than X for large sparse networks and much easier to compute. The theorem thus allows us to solve smaller problems in terms of partial matrices as we now explain.

[1]The theorem also holds with psd replaced by negative semidefinite.

14.1.3 Feasible sets

To develop semidefinite relaxations of QCQP we start by studying their feasible sets. Fix C_l, $l = 0, 1, \ldots L$, and its underlying graph F. Define the feasible set of the QCQP (14.1) as:

$$\mathbb{V} \;:=\; \{x \in \mathbb{C}^n \,|\, x^{\mathsf{H}} C_l x \;\leq\; b_l,\; l = 1, \ldots, L\} \tag{14.6}$$

Given an $x \in \mathbb{V}$, it defines a unique (up to a rotation) psd rank-1 matrix $X := xx^{\mathsf{H}}$ and therefore a unique psd rank-1 partial matrix X_F that satisfies $\mathrm{tr}(C_l X_F) \leq b_l$. The converse is not always true: given a partial matrix X_F that is psd rank-1 and satisfies $\mathrm{tr}(C_l X_F) \leq b_l$, it is not always possible to recover an x in $\mathbb{V}$. This is possible if and only if X_F has a psd rank-1 completion X that satisfies $\mathrm{tr}(C_l X) \leq b_l$. We now characterize the set of partial matrices from which $x \in \mathbb{V}$ can be recovered.

Define the set of Hermitian matrices:

$$\mathbb{X} \;:=\; \{\, X \in \mathbb{S}^n \mid X \text{ satisfies } \mathrm{tr}(C_l X) \leq b_l,\; l = 1, \ldots, L,\;\; (14.4a) \,\} \tag{14.7a}$$

Fix a connected graph F. Fix any chordal extension $c(F)$ of F and define the set of Hermitian partial matrices $X_{c(F)}$:

$$\mathbb{X}_{c(F)} \;:=\; \{\, X_{c(F)} \mid X_{c(F)} \text{ satisfies } \mathrm{tr}\left(C_l X_{c(F)}\right) \leq b_l,\; l = 1, \ldots, L,\;\; (14.4b) \,\} \tag{14.7b}$$

Finally define the set of Hermitian partial matrices X_F:

$$\mathbb{X}_F \;:=\; \{\, X_F \,|X_F \text{ satisfies } \mathrm{tr}(C_l X_F) \leq b_l,\; l = 1, \ldots, L,\;\; (14.4c)(14.5) \,\} \tag{14.7c}$$

Note that the definition of psd for partial matrices implies that $X_{c(F)}$ and X_F are Hermitian partial matrices.

Theorem 14.1 implies that given a partial matrix $X_{c(F)} \in \mathbb{X}_{c(F)}$ or a partial matrix $X_F \in \mathbb{X}_F$ there is a psd rank-1 completion $X \in \mathbb{X}$. Moreover the completion X is unique.

Corollary 14.2 (Uniqueness of rank-1 completion)**.** Fix a connected graph F. Given a partial matrix $X_{c(F)} \in \mathbb{X}_{c(F)}$ or $X_F \in \mathbb{X}_F$ there is a unique psd rank-1 completion $X \in \mathbb{X}$.

The corollary implies that, given any Hermitian partial matrix $X_F \in \mathbb{X}_F$, the set of *all* completions of X_F consists of a single psd rank-1 matrix and infinitely many indefinite or non-rank-1 matrices.

We say two sets A and B are *equivalent*, denoted $A \equiv B$, if there is a bijection between them. Even though $\mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F$ are different kinds of spaces, Theorem 14.1 and Corollary 14.2 imply that they are all equivalent to the feasible set of QCQP (14.1).

Theorem 14.3 (Equivalence)**.** $\mathbb{V} \equiv \mathbb{X} \equiv \mathbb{X}_{c(F)} \equiv \mathbb{X}_F$.

Since the cost function $x^{\mathsf{H}} C_0 x$ of (14.1) depends on X only through the partial matrix X_F, Theorem 14.3 suggests three problems that are equivalent to QCQP (14.1): for $\hat{\mathbb{X}} \in \left\{\mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F\right\}$,

$$\min_X \, C(X_F) \quad \text{subject to} \quad X \in \hat{\mathbb{X}} \tag{14.8}$$

Specifically, given an optimal solution X^{opt} in $\mathbb{X}$, it can be decomposed into $X^{\text{opt}} = x^{\text{opt}}(x^{\text{opt}})^{\mathsf{H}}$ where x^{opt} is unique up to an arbitrary reference angle. Then x^{opt} is in $\mathbb{V}$ and an optimal solution of QCQP (14.1). Alternatively given an optimal solution $X_F^{\text{opt}} \in \mathbb{X}_F$ or $X_{c(F)}^{\text{opt}} \in \mathbb{X}_{c(F)}$, Corollary 14.2 guarantees that it has a unique psd rank-1 completion X^{opt} in $\mathbb{X}$ from which an optimal $x^{\text{opt}} \in \mathbb{V}$ can be recovered. This suggests solving the QCQP (14.1) by computing X_F^{opt} or $X_{c(F)}^{\text{opt}}$ instead of X^{opt} because both of them are typically much smaller in size than X^{opt} for a large sparse network.

Given a partial matrix $X_F \in \mathbb{X}_F$ (or $X_{c(F)} \in \mathbb{X}_{c(F)}$), however, there is a more direct construction of a feasible solution $x \in \mathbb{V}$ of QCQP than through its completion, as we will see in Chapter 14.1.4.

Remark 14.1 (Graph $\hat{F}$ underlying QCQP)**.** Note that the feasible sets $\mathbb{V}, \mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F$ defined in (14.6) (14.7) depend only on the constraint matrices C_l, $l = 1, \ldots, L$, but not on the cost matrix C_0. Equivalence among these sets will therefore hold if we replace F in Theorem 14.1, Corollary 14.2 and Theorem 14.3 with a subgraph $\hat{F}$ that is induced by C_l only for $l \geq 1$, i.e., two nodes j and k in $\hat{F}$ are adjacent if and only if $[C_l]_{jk} \neq 0$ for some $l \in \{1, \ldots, L\}$.

The matrix F is needed for the proper definition of cost function. For the optimization problems in (14.8) to be equivalent, we need to compute the partial matrices X_F and $X_{c(F)}$. The partial matrices $X_{\hat{F}}$ will have missing terms $[X_{\hat{F}}]_{jk}$ in the cost function if (j,k) is in F but not in $\hat{F}$, i.e., if $[C_0]_{jk} \neq 0$ but $[C_l]_{jk} = 0$ for all $l \geq 1$. Similarly for $X_{c(\hat{F})}$. □

14.1.4 Semidefinite relaxations and solution recovery

Hence solving QCQP (14.1) is equivalent to solving (14.8) over any of $\mathbb{X}, \mathbb{X}_{c(F)}, \mathbb{X}_F$ for an appropriate matrix variable. The difficulty with solving (14.8) is that the feasible sets $\mathbb{X}$, $\mathbb{X}_{c(F)}$, and $\mathbb{X}_F$ are still nonconvex due to the rank-1 constraint and the cycle condition (14.5). Their removal leads to SDP, chordal, and SOCP relaxations of QCQP (14.1) respectively.

Semidefinite relaxations. Relax $\mathbb{X}$, $\mathbb{X}_{c(F)}$ and $\mathbb{X}_F$ to the following convex supersets:

$$\begin{aligned}
\mathbb{X}^+ &:= \{X \in \mathbb{S}^n \mid X_F \text{ satisfies } \operatorname{tr}(C_l X) \leq b_l,\ l = 1, \ldots, L,\ \ X \succeq 0\} \\
\mathbb{X}^+_{c(F)} &:= \{X_{c(F)} \mid X_F \text{ satisfies } \operatorname{tr}\left(C_l X_{c(F)}\right) \leq b_l,\ l = 1, \ldots, L,\ \ X_{c(F)} \succeq 0\} \\
\mathbb{X}^+_F &:= \{X_F \mid X_F \text{ satisfies } \operatorname{tr}(C_l X_F) \leq b_l,\ l = 1, \ldots, L,\ \ X_F(j,k) \succeq 0,\ (j,k) \in E\}
\end{aligned}$$

These feasible sets are defined for different (partial) matrices and differ in the definition of psd. Remark 14.1 applies to these relaxed feasible sets regarding the underlying graph and the corresponding partial matrices.

The following problems are semidefinite relaxations of QCQP (14.1) with different sizes and tightness:
QCQP-sdp:

$$\min_{X} C(X_F) \quad \text{subject to} \quad X \in \mathbb{X}^+ \tag{14.9a}$$

QCQP-ch:

$$\min_{X_{c(F)}} C(X_F) \quad \text{subject to} \quad X_{c(F)} \in \mathbb{X}^+_{c(F)} \tag{14.9b}$$

QCQP-socp:

$$\min_{X_F} C(X_F) \quad \text{subject to} \quad X_F \in \mathbb{X}_F^+ \tag{14.9c}$$

Solution recovery. When the semidefinite relaxations OPF-sdp, OPF-ch, OPF-socp are exact, i.e., if their optimal solutions X^{sdp}, $X_{c(F)}^{\text{ch}}$, X_F^{socp} happen to lie in $\mathbb{X}$, $\mathbb{X}_{c(F)}$, $\mathbb{X}_F$ respectively, then an optimal solution $x^{\text{opt}} \in \mathbb{V}$ of the original QCQP can be recovered from these solutions. Indeed the recovery method works not just for an optimal solution, but any feasible solution that lies in $\mathbb{X}$, $\mathbb{X}_{c(F)}$ or $\mathbb{X}_F$. Moreover, given an $X \in \mathbb{X}$ or an $X_{c(F)} \in \mathbb{X}_{c(F)}$, the construction of x depends on X or $X_{c(F)}$ only through their submatrix X_F. We hence describe a method for recovering an $x \in \mathbb{V}$ from an X_F, which may be a partial matrix in $\mathbb{X}_F$ or the submatrix of a (partial) matrix in $\mathbb{X}$ or $\mathbb{X}_{c(F)}$. The solution x is unique if F is connected and, say, $\angle x_1$ is fixed.

Let T be an arbitrary spanning tree of F rooted at bus 1. Let $\mathbb{P}_j$ denote the unique path from bus 1 to bus j in T with orientation pointing away from bus 1. Set $|x_1| := \sqrt{[X_F]_{11}}$ and $\angle x_1$ to an arbitrary value. For $j = 2, \ldots, n$,

$$|x_j| \ := \ \sqrt{[X_F]_{jj}}, \qquad\qquad \angle x_j \ := \ \angle V_1 \ - \sum_{(i,k) \in \mathbb{P}_j} \angle [X_F]_{ik}$$

Then, on link (j,k), $\angle x_j - \angle x_k = \angle [X_F]_{jk}$ and $[X_F]_{jk} = x_j x_k^{\mathsf{H}}$ since X_F is 2×2 psd rank-1. It can be checked that x is in the feasible set $\mathbb{V}$ of QCQP, i.e., $x^{\mathsf{H}} C_l x \leq b_l$, $l = 1, \ldots, L$ (Exercise 14.1). This method for recovering x from X_F is generally more efficient than computing the psd rank-1 completion X of X_F and factorizing X, as suggested in Theorem 14.3, and is used in the proof of Theorem 14.1 (see Chapter 14.1.6).

14.1.5 Tightness of relaxations

Recall that $\mathbb{V} \equiv \mathbb{X} \equiv X_{c(F)} \equiv X_F$ (Theorem 14.3). Since $\mathbb{X} \subseteq \mathbb{X}^+$, $\mathbb{X}_{c(F)} \subseteq \mathbb{X}_{c(F)}^+$, $\mathbb{X}_F \subseteq \mathbb{X}_F^+$, the relaxations OPF-sdp, OPF-ch, OPF-socp all provide lower bounds on OPF (13.9) OPF-socp is the simplest computationally. OPF-ch usually requires heavier computation than OPF-socp but much lighter than OPF-sdp for large sparse networks (even though OPF-ch can be as complex as OPF-sdp in the worse case [117, 118]). The relative tightness of the relaxations depends on the network topology. For a general network that may contain cycles, OPF-ch is as tight a relaxation as OPF-sdp and they are strictly tighter than OPF-socp. For a tree (radial) network the hierarchy collapses and all three are equally tight. We now make this precise.

Consider the relaxed feasible sets $\mathbb{X}^+$, $\mathbb{X}_{c(F)}^+$ and $\mathbb{X}_F^+$. Consider two sets A and B and the corresponding cost functions $C_A : A \to \mathbb{R}$ and $C_B : B \to \mathbb{R}$. For instance $A := \mathbb{C}^n$, $B := \mathbb{S}^n$, $C_A(x) := \text{tr}\left(Cxx^{\mathsf{H}}\right)$ and $C_B(X) := \text{tr}\,(CX)$ for a given Hermitian matrix C. We say that A is an *effective subset* of B with respect to the cost functions C_A, C_B, denoted by $A \sqsubseteq B$, if, given any $a \in A$, there is a $b \in B$ that has the same cost $C_A(a) = C_B(b)$. We say A is *similar to* B with respect to the cost functions C_A, C_B, denoted by $A \simeq B$, if $A \sqsubseteq B$ and $B \sqsubseteq A$. Note that $A \equiv B$ implies $A \simeq B$ but the converse may not hold. Even though effective subset and similarity are defined with respect to some cost functions C_A, C_B, we often omit the cost functions when their existence is understood and unimportant for the discussion, and simply say A is an effective subset of B or A is similar to B.

The following result says that feasible set of QCQP (14.1) is an effective subset of the feasible sets of its relaxations; moreover these relaxations have similar feasible sets when the network is radial.

Theorem 14.4 (Tightness of relaxations)**.**

1. $\mathbb{V} \sqsubseteq \mathbb{X}^+ \simeq \mathbb{X}^+_{c(F)} \sqsubseteq \mathbb{X}^+_F$.
2. If F is a tree then $\mathbb{V} \sqsubseteq \mathbb{X}^+ \simeq \mathbb{X}^+_{c(F)} \simeq \mathbb{X}^+_F$.

The reason $\mathbb{X}^+_{c(F)}$ is similar, but not equivalent, to $\mathbb{X}^+$ is that psd completions of a psd submatrix $X \in \mathbb{X}^+_{c(F)}$ are generally nonunique. In contrast, the psd rank-1 completion of a psd rank-1 submatrix $X \in \mathbb{X}_{c(F)}$ is unique according to Corollary 14.2.

Let $C^{\text{qcqp}}, C^{\text{sdp}}, C^{\text{ch}}, C^{\text{socp}}$ be the optimal values of QCQP (14.1), QCQP-sdp (14.9a), QCQP-ch (14.9b), QCQP-socp (14.9c) respectively. Theorem 14.3 and Theorem 14.4 directly imply

Corollary 14.5.

1. $C^{\text{qcqp}} \geq C^{\text{sdp}} = C^{\text{ch}} \geq C^{\text{socp}}$.
2. If F is a tree then $C^{\text{qcqp}} \geq C^{\text{sdp}} = C^{\text{ch}} = C^{\text{socp}}$.

Remark 14.2 (Tightness)**.** Theorem 14.4 and Corollary 14.5 imply that for radial networks one should always solve QCQP-socp, not QCQP-sdp or QCQP-ch, since it is the tightest and the simplest relaxation of the three. For networks that contain cycles there is a tradeoff between QCQP-socp and QCQP-ch/QCQP-sdp: the latter is tighter but requires heavier computation. Between QCQP-ch and QCQP-sdp, QCQP-ch is preferable as they are equally tight but QCQP-ch is usually much faster to solve for large sparse networks. □

Theorem 14.1 through Corollary 14.5 apply to *any* chordal extension $c(F)$ of F. The choice of $c(F)$ does not affect the optimal value of the chordal relaxation but determines its complexity. Unfortunately the optimal choice that minimizes the complexity of QCQP-ch is NP-hard to compute. This difficulty is due to two conflicting factors in choosing a $c(F)$. Recall that the constraint $X_{c(F)} \succeq 0$ in the definition of $\mathbb{X}^+_{c(F)}$ consists of multiple constraints that the principal submatrices $X_{c(F)}(q) \succeq 0$, one for each maximal clique q of $c(F)$. When two cliques q and q' share a node their submatrices $X_{c(F)}(q)$ and $X_{c(F)}(q')$ share entries that must be decoupled by introducing auxiliary variables and equality constraints on these variables. The choice of $c(F)$ determines the number of these submatrices $X_{c(F)}(q)$ and their sizes as well as the number of auxiliary variables and decoupling constraints. On the one hand if $c(F)$ contains few cliques q then the submatrices $X_{c(F)}(q)$ tend to be large and expensive to compute (e.g. if $c(F)$ is the complete graph then there is a single clique, but $X_{c(F)} = X$ and QCQP-ch is identical to QCQP-sdp). On the other hand if $c(F)$ contains many small cliques q then there tends to be more overlap and chordal relaxation tends to require more decoupling constraints. Hence choosing a good chordal extension $c(F)$ of F is important but nontrivial.

Example 14.2. Example of chordal extension, chordal relaxations, decoupling overlap variables, etc.

14.1.6 Proofs

Proof of Theorem 14.1: Rank-1 characterization. We will prove (1) ⇒ (2) ⇒ (3) ⇒ (1). If X is psd rank-1 then all its principle submatrices are psd and of rank 1 (the submatrix cannot be of rank 0 because, by assumption, $X_{jj} > 0$ for all $j \in \overline{N}$). This implies that its submatrix $X_{c(F)}$ is psd and rank-1. Hence (1) ⇒ (2).

Fix a partial matrix $X_{c(F)}$ that is psd and rank-1 and consider its submatrix X_F. Since each link $(j,k) \in E$ is a clique of $c(F)$ the 2×2 principle submatrix $X_F(j,k)$ is psd and rank-1. Therefore to prove that (2) ⇒ (3), it suffices to show that X_F satisfies the cycle condition (14.5). We now prove the following statement by induction on k: for all cycles $c := (j_1,\ldots,j_k)$ of length $3 \le k \le n$ in $c(F)$, such that the lines $(j_i, j_{i+1}) \in c$ with $j_{k+1} := j_1$, we have

$$\sum_{i=1}^{k} \angle [X_F]_{j_i j_{i+1}} \quad = \quad 0 \quad \mod 2\pi \tag{14.10}$$

For $k = 3$, a cycle $c := (n_1,n_2,n_3)$ is a clique of $c(F)$ and therefore the following principle submatrix of $X_{c(F)}$:

$$X_{c(F)}(n_1,n_2,n_3) \quad := \quad \begin{bmatrix} [X_{c(F)}]_{n_1n_1} & [X_{c(F)}]_{n_1n_2} & [X_{c(F)}]_{n_1n_3} \\ [X_{c(F)}]_{n_2n_1} & [X_{c(F)}]_{n_2n_2} & [X_{c(F)}]_{n_2n_3} \\ [X_{c(F)}]_{n_3n_1} & [X_{c(F)}]_{n_3n_2} & [X_{c(F)}]_{n_3n_3} \end{bmatrix}$$

defined on the cycle is psd rank-1. Hence $X_{c(F)}(n_1,n_2,n_3) = xx^{\mathsf{H}}$ for some $x := (x_1,x_2,x_3) \in \mathbb{C}^3$. Then

$$\sum_{i=1}^{3} \angle [X_F]_{j_i j_{i+1}} \quad = \quad \angle\left[\left(x_1x_2^{\mathsf{H}}\right)\left(x_2x_3^{\mathsf{H}}\right)\left(x_3x_1^{\mathsf{H}}\right)\right] \;=\; 0 \quad \mod 2\pi$$

Suppose (14.10) holds for all cycles in $c(F)$ of length up to $k > 3$. Consider now a cycle $(j_1,\ldots,j_{k+1})$ of length $k+1$ in $c(F)$. Since $c(F)$ is chordal there is a chord, say, $(j_1, j_l) \in E$ for some $1 < l < k+1$. Since both cycles $(j_1,\ldots,j_l)$ and $(j_1, j_l,\ldots,j_{k+1})$ satisfy (14.10) we have

$$\begin{aligned} \sum_{i=1}^{l-1} \angle [X_F]_{j_i j_{i+1}} + \angle [X_F]_{j_l j_1} \quad &= \quad 0 \quad \mod 2\pi \\ \angle [X_F]_{j_1 j_l} + \sum_{i=l}^{k+1} \angle [X_F]_{j_i j_{i+1}} \quad &= \quad 0 \quad \mod 2\pi \end{aligned}$$

where $j_{k+2} := j_1$. Since X_F is Hermitian, $\angle [X_F]_{j_l j_1} = -\angle [X_F]_{j_1 j_l}$ and hence adding the above equations yields

$$\sum_{i=1}^{k+1} \angle [X_F]_{j_i j_{i+1}} \quad = \quad 0 \quad \mod 2\pi$$

proving (14.10) for $k+1$. This completes the proof of (2) ⇒ (3).

For (3) $\Rightarrow$ (1), fix any partial matrix X_F that is 2×2 psd rank-1 and satisfies the cycle condition (14.5). We now construct a psd rank-1 completion X of X_F, by constructing a vector $x \in \mathbb{C}^n$ such that $X = xx^{\mathsf{H}}$. Let

$$|x_j| \;\; := \;\; \sqrt{[X_F]_{jj}}, \qquad j \in \{1,\dots,n\}$$

For the angles $\angle x_j$, note that the graph F need not be connected. For each connected component, pick an arbitrary spanning tree for that connected component, and assume its nodes are indexed by $\{1,\dots,k\}$. Without loss of generality let $\angle x_1 = 0°$. Going down the spanning tree from node 1, we have $\angle x_j - \angle x_k = [X_F]_{jk}$ and hence set

$$\angle x_j \;\; := \;\; -\sum_{(i,i')\in\mathbb{P}_j} \angle [X_F]_{ii'}, \qquad j \in \{1,\dots,k\}$$

where $\mathbb{P}_j$ is *any* path from node 1 to node j in that spanning tree. This is well defined because X_F satisfies the cycle condition (14.5). This defines x_j for all $j \in \{1,\dots,n\}$. Clearly $X = xx^{\mathsf{H}}$ is a psd rank-1 completion of X_F. This completes the proof. □

Proof of Corollary 14.2: Uniqueness of rank-1 completion. The proof of Theorem 14.1 shows that given a partial matrix $X_{c(F)} \in \mathbb{X}_{c(F)}$, the (unique) submatrix X_F of $X_{c(F)}$ has a psd rank-1 completion $X \in \mathbb{X}$. Therefore to prove the corollary it suffices to prove that any partial matrix $X_F \in \mathbb{X}_F$ has a unique psd rank-1 completion $X \in \mathbb{X}$. To this end fix an $X_F \in \mathbb{X}_F$ and suppose there are two psd rank-1 completions $X := xx^{\mathsf{H}}$ and $\hat{X} := \hat{x}\hat{x}^{\mathsf{H}}$ in $\mathbb{X}$. Since $X_F = \hat{X}_F$ we have

$$\left|x_j\right| \;=\; \left|\hat{x}_j\right| \;=\; \sqrt{[X_F]_{jj}}, \quad j \in \overline{N}, \qquad \theta_j - \theta_k \;=\; \hat{\theta}_j - \hat{\theta}_k \;=\; \angle [X_F]_{jk}, \quad (j,k) \in E$$

In particular $C^{\mathsf{T}}\theta = C^{\mathsf{T}}\hat{\theta}$ where C is the $|N| \times |E|$ incidence matrix of the graph $G := (N,E)$:

$$C_{jl} \;\; := \;\; \begin{cases} \;\;1 & \text{if } l = j \to k \text{ for some bus } k \\ -1 & \text{if } l = i \to j \text{ for some bus } i \\ \;\;0 & \text{otherwise} \end{cases}, \qquad\qquad j \in N,\ l \in E$$

This means that $C^{\mathsf{T}}\left(\hat{\theta} - \theta\right) = 0$ and hence, since the graph F is connected and hence the null space of C is span$(\mathbf{1})$, $\hat{\theta} = \theta + \gamma\mathbf{1}$ for any $\gamma \in \mathbb{R}$. Therefore $\hat{x} = xe^{\mathbf{i}\gamma}$. This implies that

$$\hat{X} \;=\; \hat{x}\hat{x}^{\mathsf{H}} \;=\; \left(xe^{\mathbf{i}\gamma}\right)\left(xe^{\mathbf{i}\gamma}\right)^{\mathsf{H}} \;=\; X$$

i.e., the psd rank-1 completion is unique. □

Proof of Theorem 14.4: Tightness of relaxations. First $\mathbb{V} \sqsubseteq \mathbb{X}^+ \sqsubseteq \mathbb{X}^+_{c(F)} \sqsubseteq \mathbb{X}^+_F$ follows from Theorem 14.3 and the definitions of $\mathbb{X}^+$, $\mathbb{X}^+_{c(F)}$, $\mathbb{X}^+_F$ (recall that by assumption the cost function C depends on $V, X, X_{c(F)}$ only through the submatrix X_F). Since $c(F)$ is chordal, [116, Theorem 7] implies that every $X_{c(F)}$ in $\mathbb{X}^+_{c(F)}$ has a psd completion X in $\mathbb{X}^+$, i.e., $\mathbb{X}^+_{c(F)} \sqsubseteq \mathbb{X}^+$. Hence $\mathbb{X}^+ \simeq \mathbb{X}^+_{c(F)}$.

Suppose F is a tree and consider any chordal extension $c(F)$. We need to show that $\mathbb{X}_F^+ \sqsubseteq \mathbb{X}_{c(F)}^+$, i.e., given any $X_F \in \mathbb{X}_F^+$ there is a $X_{c(F)} \in \mathbb{X}_{c(F)}^+$ with the same cost. Since F is itself chordal, [116, Theorem 7] implies that X_F has a psd completion X in $\mathbb{X}^+$. The submatrix $X_{c(F)}$ of X defined on $c(F)$ is the desired partial matrix in $\mathbb{X}_{c(F)}^+$ with the same cost. This proves $\mathbb{X}_F^+ \sqsubseteq \mathbb{X}_{c(F)}^+$ and hence $\mathbb{X}_F^+ \simeq \mathbb{X}_{c(F)}^+$ for radial networks. □

14.1.7 Strong SOCP relaxations: mesh network

1. Strong SOCP relaxations are proposed and their relation with SOCP and SDP relaxations are studied in [119].
2. SDP, SOCP and strong SOCP relaxations are applied to a two-stage robust AC OPF problem, and column-and-constraint generation method of [120, 121] are used to solve these relaxations.
3. Check out LIngling Fan's recent paper: A sparse Convex AC OPF Solver and Convex Iteration Implementation Based on 3-Node Cycles Minyue Ma, Lingling Fan, Zhixin Miao, Bo Zeng, Hossein Ghassempour.

14.2 Single-phase OPF

In this section we apply the results of Chapter 14.1 to single-phase OPF problems in the bus injection model. In Chapter 14.2.1 we write OPF (13.15) as a standard QCQP but expressed in terms of the partial matrix defined on the network graph G. Its semidefinite relaxations then follow from (14.9). In Chapter 14.2.2 we define exact relaxation of OPF. Sufficient conditions for exact relaxations of OPF for radial networks will be studied in Chapters 14.3 and 14.4. In Chapter 14.5 we extend semidefinite relaxations of OPF to unbalanced three-phase networks.

14.2.1 Semidefinite relaxations

Constraints. Recall the undirected connected graph $G = (\overline{N}, E)$ that models a power network with $N+1$ buses and M lines. Given a voltage vector $V \in \mathbb{V}$ define the partial matrix $W_G := W_G(V)$:

$$[W_G]_{jj} := |V_j|^2, \quad j \in \overline{N}; \qquad [W_G]_{jk} := V_j V_k^{\mathsf{H}} =: [W_G]_{kj}^{\mathsf{H}}, \quad (j,k) \in E$$

It can then be shown that the constraints in OPF (13.15) as a QCQP can be written in terms of the partial matrix W_G as:

$$p_j^{\min} \le \operatorname{tr}\left(\Phi_j W_G\right) \le p_j^{\max} \tag{14.12a}$$
$$q_j^{\min} \le \operatorname{tr}\left(\Psi_j W_G\right) \le q_j^{\max} \tag{14.12b}$$
$$v_j^{\min} \le \operatorname{tr}\left(J_j W_G\right) \le v_j^{\max} \tag{14.12c}$$
$$\operatorname{tr}\left(\hat{Y}_{jk} W_G\right) \le \ell_{jk}^{\max} \tag{14.12d}$$
$$\operatorname{tr}\left(\hat{Y}_{kj} W_G\right) \le \ell_{kj}^{\max} \tag{14.12e}$$

Cost function. Common cost functions can also be expressed in terms of the partial matrix W_G. For example if the cost is a weighted sum of real generation power then

$$C(W_G) \;=\; \sum_{j:\text{gens}} c_j \, \text{Re}(s_j) \;=\; \sum_{j:\text{gens}} c_j \, \text{tr}\left(\Phi_j W_G\right)$$

In particular the real line loss in the network is:

$$C(W_G) \;=\; \sum_j \text{Re}(s_j) \;=\; \sum_j \text{tr}\left(\Phi_j W_G\right)$$

We present a less obvious example.

Example 14.3 (Cost function)**.** Consider the problem of minimizing the total deviation of squared voltage magnitudes from their squared nominal values $a_j \in \mathbb{R}$

$$\min_{V \in \mathbb{C}^{N+1}} \; \sum_j \left(|V_j|^2 - a_j \right)^2 \qquad \text{s.t.} \qquad V \in \mathbb{V} \tag{14.13}$$

where the feasible set $\mathbb{V}$ is defined by quadratic constraints in terms of the partial matrix W_G: $V \in \mathbb{V}$ if and only if

$$V^{\mathsf{H}} C_l V \;=\; \text{tr}\left(C_l W_G\right) \;\le\; b_l, \qquad l = 1, \ldots, L$$

with some matrices C_l and real numbers b_l such that $[C_l]_{jk} = 0$ if $(j,k) \notin E$. Even though the cost function is not a quadratic form in terms of W_G, show that the problem can be equivalently expressed as a QCQP in terms of W_G with additional variables and constraints.

Solution. The cost function is $\sum_j \left(|V_j|^4 \,-\, 2a_j |V_j|^2 \,+\, a_j^2 \right)$. We can omit the constants a_j^2 in the cost and hence (14.13) is equivalent to the following problem:

$$\min_{V \in \mathbb{C}^{N+1}} \; \sum_j \left(|U_j|^2 \,-\, 2a_j U_j \right) \qquad \text{s. t.} \qquad V \;\in\; \mathbb{V}, \qquad U_j \;=\; |V_j|^2, \quad j \;\in\; \overline{N} \tag{14.14a}$$

Let $V := \left(V_j, j \in \overline{N}\right) \in \mathbb{C}^{N+1}$, $U := \left(U_j, j \in \overline{N}\right) \in \mathbb{C}^{N+1}$, $a := \left(a_j, j \in \overline{N}\right)$, and $e_j \in \{0,1\}^{N+1}$ with a single 1 at the jth entry and 0 elsewhere. In terms of the variable $x := (V, U) \in \mathbb{C}^{2(N+1)}$, we will rewrite (14.14a) as an inhomogeneous QCQP of the form:

$$\min_{x \in \mathbb{C}^{2(N+1)}} \; x^{\mathsf{H}} C_0 x \,+\, \left(c_0^{\mathsf{H}} x + x^{\mathsf{H}} c_0 \right) \quad \text{s. t.} \quad V \;\in\; \mathbb{V}, \quad x^{\mathsf{H}} C_j x \,+\, \left(c_j^{\mathsf{H}} x + x^{\mathsf{H}} c_j \right) \;=\; 0, \quad j \in \overline{N} \tag{14.14b}$$

Indeed

$$\sum_j \left(|U_j|^2 \,-\, 2a_j U_j \right) \;=\; U^H U \,-\, \left(a^H U + U^H a \right)$$

$$|V_j|^2 \,-\, U_j \;=\; V^{\mathsf{H}} \left(e_j e_j^{\mathsf{H}} \right) V \,-\, \frac{1}{2} \left(e_j^{\mathsf{H}} U_j + U_j^{\mathsf{H}} e_j \right), \qquad j \in \overline{N}$$

since a_j and $U_j = |V_j|^2$ are real numbers. Therefore (14.14a) is an inhomogeneous QCQP of the form (14.14b) with

$$\begin{aligned} C_0 &:= \begin{bmatrix} 0 & 0 \\ 0 & I_{N+1} \end{bmatrix}, & c_0 &:= \begin{bmatrix} 0 \\ -a \end{bmatrix} & & \\ C_j &:= \begin{bmatrix} e_j e_j^{\mathsf{H}} & 0 \\ 0 & 0 \end{bmatrix}, & c_j &:= \begin{bmatrix} 0 \\ -\frac{1}{2} e_j \end{bmatrix}, & j &\in \overline{N} \end{aligned}$$

where I_{N+1} is the identity matrix of size $N+1$. Since the cost function and the new constraints depends on V only through $|V_j|^2$, in particular, it does not depend on $V_j V_k^{\mathsf{H}}$, $j \neq k$, the problem (14.14b) depends only on W_G. Indeed W_G appears only in the term $V^{\mathsf{H}} \left(e_j e_j^{\mathsf{H}} \right) V = \mathrm{tr}\left(\left(e_j e_j^{\mathsf{H}} \right) V V^{\mathsf{H}} \right) = \mathrm{tr}\left(\left(e_j e_j^{\mathsf{H}} \right) W_G \right)$.

As explained in Chapter 13.1.3, the inhomogeneous QCQP (14.14b) is equivalent to the following homogeneous QCQP with an auxiliary scalar variable $t \in \mathbb{C}$:

$$\begin{aligned} \min_{x \in \mathbb{C}^{2(N+1)}, t \in \mathbb{C}} \quad & \begin{bmatrix} x^{\mathsf{H}} & t^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} C_0 & c_0 \\ c_0^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \\ \text{s. t.} \quad & V \in \mathbb{V} \\ & \begin{bmatrix} x^{\mathsf{H}} & t^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} C_j & c_j \\ c_j^{\mathsf{H}} & 0 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 0, \qquad j \in \overline{N} \\ & \begin{bmatrix} x^{\mathsf{H}} & t^{\mathsf{H}} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} = 1 \end{aligned}$$

in the sense that, if $(x^{\text{opt}}, t^{\text{opt}}) \in \mathbb{C}^{2N+3}$ is optimal for the homogeneous QCQP, then their product $x^{\text{opt}} e^{\mathbf{i}\theta^{\text{opt}}}$ is optimal for the inhomogeneous problem (14.14b) when $t^{\text{opt}} = e^{\mathbf{i}\theta^{\text{opt}}}$. □

Henceforth we will abuse notation and use C_0 to denote the cost function both as a function $C_0(V)$ of a voltage vector $V \in \mathbb{C}^{N+1}$ and as a function $C_0(W_G)$ of a partial matrix W_G. When the cost is quadratic then C_0 also denotes the cost matrix as in $C_0(V) := V^{\mathsf{H}} C_0 V$ or $C_0(W_G) := \mathrm{tr}(C_0 W_G)$.

OPF and relaxations. Recall the OPF problem (13.15) as a QCQP, reproduced here

$$\min_V \ C_0(V) \qquad \text{s.t.} \qquad V \in \mathbb{V} := \left\{ V \in \mathbb{C}^{N+1} \mid V^{\mathsf{H}} C_l V \leq b_l, \ l = 1, \ldots, L \right\} \tag{14.15}$$

where the constraint matrices C_l are given explicitly in (14.12). Define the set of Hermitian matrices:

$$\mathbb{W} \ := \ \{ \, W \in \mathbb{S}^{N+1} \mid W \text{ satisfies (14.12) with } W_G \text{ replaced by } W, \text{ (14.4a)} \, \}$$

Fix any chordal extension $c(G)$ of G and define the set of Hermitian partial matrices $W_{c(G)}$:

$$\mathbb{W}_{c(G)} \ := \ \{ \, W_{c(G)} \mid W_{c(G)} \text{ satisfies (14.12) with } W_G \text{ replaced by } W_{c(G)}, \text{ (14.4b)} \, \}$$

Finally define the set of Hermitian partial matrices W_G:

$$\mathbb{W}_G \ := \ \{ \, W_G \mid W_G \text{ satisfies (14.12)(14.4c)(14.5)} \, \}$$

Then Theorem 14.3 implies that OPF (14.15) is equivalent to

$$\min_{W} C_0(W_G) \quad \text{s.t.} \quad W \in \hat{\mathbb{W}}$$

where $\hat{\mathbb{W}}$ is any one of the equivalent feasible sets $\mathbb{W}, \mathbb{W}_{c(G)}, \mathbb{W}_G$. Its semidefinite relaxation relaxes $\hat{\mathbb{W}}$ to semidefinite cones:

$$\begin{aligned}
\mathbb{W}^+ &:= \{ W \in \mathbb{S}^{N+1} \mid W_G \text{ satisfies (14.12)}, \ W \succeq 0 \} \\
\mathbb{W}^+_{c(G)} &:= \{ W_{c(G)} \mid W_G \text{ satisfies (14.12)}, \ W_{c(G)} \succeq 0 \} \\
\mathbb{W}^+_G &:= \{ W_G \mid W_G \text{ satisfies (14.12)}, \ W_G(j,k) \succeq 0, \ (j,k) \in E \}
\end{aligned}$$

i.e., the semidefinite relaxations of OPF (14.15) is:

$$\min_{W} C_0(W_G) \quad \text{s.t.} \quad W \in \hat{\mathbb{W}}^+$$

where $\hat{\mathbb{W}}^+$ is any one of the feasible sets $\mathbb{W}^+, \mathbb{W}^+_{c(G)}, \mathbb{W}^+_G$. Explicitly, these semidefinite relaxations are (c.f. (14.9)):

OPF-sdp:

$$\min_{W \in \mathbb{S}^{N+1}} \ C_0(W_G) \qquad \text{s.t.} \qquad \operatorname{tr}(C_l W) \ \le \ b_l, \quad l \ = \ 1, \ldots, L, \qquad W \ \succeq \ 0 \tag{14.16a}$$

OPF-ch:

$$\min_{W_{c(G)}} \ C_0(W_G) \qquad \text{s.t.} \qquad \operatorname{tr}\left(C_l W_{c(G)}\right) \ \le \ b_l, \quad l \ = \ 1, \ldots, L, \qquad W_{c(G)} \ \succeq \ 0 \tag{14.16b}$$

OPF-socp:

$$\min_{W_G} \ C_0(W_G) \quad \text{s.t.} \quad \operatorname{tr}(C_l W_G) \ \le \ b_l, \quad l \ = \ 1, \ldots, L, \quad W_G(j,k) \ \succeq \ 0, \quad (j,k) \in E \tag{14.16c}$$

where C_l are given explicitly in (14.12).

As discussed in Remark 14.2, if the network graph G is a tree, then we should solve OPF-socp to compute the partial matrix W_G because it will be as tight as OPF-sdp that computes the entire matrix W, but much simpler computationally. Otherwise we can solve OPF-ch to compute $W_{c(G)}$ corresponding to a chordal extension $c(G)$ of G which is usually much simpler than OPF-sdp for large sparse network but as tight.

Example 14.4 (Two-bus network). □

14.2.2 Exact relaxation: definition

Consider the single-phase OPF (14.15) as a standard QCQP and its semidefinite relaxations (14.16).

Definition 14.1 (Strong exactness)**.** We say that

1. OPF-sdp (14.16a) is *exact* if every optimal solution W^{sdp} of OPF-sdp is psd rank-1;

2. OPF-ch (14.16b) is *exact* if every optimal solution $W^{\text{ch}}_{c(G)}$ of OPF-ch is psd rank-1, i.e., the principal submatrices $W^{\text{ch}}_{c(G)}(q)$ of $W^{\text{ch}}_{c(G)}$ are psd rank-1 for all maximal cliques q of the chordal extension $c(G)$ of graph G;

3. OPF-socp (14.16c) is *exact* if every optimal solution W^{socp}_G of OPF-socp

   - is 2×2 psd rank-1, i.e., the 2×2 principal submatrices $W^{\text{socp}}_G(j,k)$ are psd rank-1 for all $(j,k)\in E$; and
   - satisfies the cycle condition (14.5).

To recover an optimal solution V^{opt} of OPF (14.15) from an optimal solution W^{sdp} or $W^{\text{ch}}_{c(G)}$ or W^{socp}_G of its relaxations, see Chapter 14.1.4. The strong exactness notion in Definition 14.1 is convenient because it ensures that any algorithm that solves an exact relaxation always produces a globally optimal solution to the OPF problem. If exactness were defined to mean that an optimal solution of OPF can be recovered from some, but not necessarily all, optimal solutions of its relaxation, then an algorithm may not be guaranteed to produce an optimal solution of OPF by solving its relaxation. This strong notion of exactness is however more stringent than necessary under the sufficient exactness conditions of Chapters 14.3 and 14.4 for radial networks. See Remark 14.3 after Theorem 14.6 and Remark 14.4 after Theorem 14.9 (and Remarks 15.1 and 15.3 for BFM). These conditions guarantee that an optimal solution to OPF can always be recovered from *any* optimal solution of OPF-socp for radial networks, even when the OPF-socp is not exact under Definition 14.1.

In the rest of this chapter we present sufficient conditions for exact semidefinite relaxations when the network is radial, i.e., the network graph is a tree. We restrict our discussion to single-phase networks though exactness conditions exist in the literature for three-phase radial networks.

14.3 Exactness condition: linear separability

Theorem 14.4 implies that, for a single-phase radial network whose graph G is a tree, if SOCP relaxation is exact then SDP and chordal relaxations are also exact. We hence focus on the exactness of OPF-socp (14.16c). Since the cycle condition (14.5) is vacuous for radial networks, OPF-socp (14.16c) is exact if all of its optimal solutions are 2×2 rank-1. To avoid triviality we assume OPF (14.15) is feasible.

We will first present a general result on the exactness of the SOCP relaxation of general QCQP on a tree graph G and then apply it to OPF-socp (14.16c) for single-phase radial networks.

14.3.1 Sufficient condition for QCQP

Fix an undirected graph $G = (N, E)$ where $|N| = n$ and $E \subseteq N \times N$. Fix Hermitian matrices $C_l \in \mathbb{S}^n$, $l = 0, \ldots, L$, defined on G, i.e., $[C_l]_{jk} = 0$ if $(j,k) \notin E$. Consider QCQP:

$$\min_{x \in \mathbb{C}^n} \quad x^H C_0 x \tag{14.17a}$$

$$\text{s.t.} \quad x^H C_l x \leq b_l, \quad l = 1, \ldots, L \tag{14.17b}$$

where $b_l \in \mathbb{R}$, $l = 1, \ldots, L$, and its SOCP relaxation where the optimization variable ranges over Hermitian partial matrices X_G:

$$\min_{X_G} \quad \operatorname{tr}(C_0 X_G) \tag{14.18a}$$

$$\text{s.t.} \quad \operatorname{tr}(C_l X_G) \leq b_l, \quad l = 1, \ldots, L \tag{14.18b}$$

$$X_G(j,k) \succeq 0, \quad (j,k) \in E \tag{14.18c}$$

The following result can be regarded as an extension of [122] on the SOCP relaxation of QCQP from the real domain to the complex domain. Consider: [2]

C14.1: The cost matrix C_0 is positive definite.

C14.2: For each link $(j,k) \in E$ there exists an α_{jk} such that $\angle [C_l]_{jk} \in [\alpha_{ij}, \alpha_{ij} + \pi]$ for all $l = 0, \ldots, L$.

Condition C14.2 is illustrated in Figure 14.2. Let C^{opt} and C^{socp} denote the optimal values of QCQP (14.17) and SOCP (14.18) respectively.

Theorem 14.6. Suppose G is a tree and C14.2 holds. Then $C^{\text{opt}} = C^{\text{socp}}$ and an optimal solution of QCQP (14.17) can be recovered from every optimal solution of SOCP (14.18).

Remark 14.3 (Strong exactness)**.** The proof of Theorem 14.6 prescribes a simple procedure to recover an optimal solution of QCQP (14.17) from any optimal solution x^{socp} of its SOCP relaxation (14.18), whether or not x^{socp} is 2×2 rank-1. Hence the SOCP relaxation may not be exact according to our definition of exactness, i.e., some optimal solutions of (14.18) may be 2×2 psd but not 2×2 rank-1, but our exactness condition still guarantees that an optimal solution of QCQP can be recovered from x^{socp}. If the objective function is strictly convex however then the optimal solution sets of QCQP (14.17) and SOCP (14.18) are indeed equivalent and the SOCP is exact under our definition.

Corollary 14.7. Suppose G is a tree and C14.1–C14.2 hold. Then SOCP (14.18) is exact.

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[2] All angles should be interpreted as "mod 2π", i.e., projected onto $(-\pi, \pi]$.

14.3.2 Application to OPF

We now apply Theorem 14.6 to our OPF problem (13.15). To simplify illustration we ignore the branch constraints (13.15e)(13.15f) and consider:

$$\begin{aligned}
\min_{x\in\mathbb{C}^n} \quad & V^{\mathsf{H}} C_0 V \\
\text{s.t.} \quad & V^{\mathsf{H}}\Phi_j V \le p_j^{\max},\ V^{\mathsf{H}}(-\Phi_j)V \le -p_j^{\min} && \text{(14.19a)}\\
& V^{\mathsf{H}}\Psi_j V \le q_j^{\max},\ V^{\mathsf{H}}(-\Psi_j)V \le -q_j^{\min} && \text{(14.19b)}\\
& V^{\mathsf{H}} J_j V \le v_j^{\max},\ V^{\mathsf{H}}(-J_j)V \le -v_j^{\min}
\end{aligned}$$

for some Hermitian matrices C_0, Φ_j, Ψ_j, J_j where $j \in \overline{N}$. Condition C14.2 depends only on the off-diagonal entries of C_0, Φ_j, Ψ_j (J_j are diagonal matrices). It implies a simple pattern on the power injection constraints (14.19a)(14.19b). Write the series admittances in terms of its real and imaginary parts $y_{jk}^s =: g_{jk}^s + \mathbf{i} b_{jk}^s$ with $g_{jk}^s > 0, b_{jk}^s < 0$. (Note that C14.2 does not depend on the shunt admittances $\left(y_{jk}^m, y_{kj}^m\right)$.) Then we have

$$[\Phi_k]_{ij} \ = \ \begin{cases} \frac{1}{2}Y_{ij} \ = \ -\frac{1}{2}(g_{ij}^s + \mathbf{i} b_{ij}^s) & \text{if } k = i \\ \frac{1}{2}Y_{ij}^{\mathsf{H}} \ = \ -\frac{1}{2}(g_{ij}^s - \mathbf{i} b_{ij}^s) & \text{if } k = j \\ 0 & \text{if } k \notin \{i,j\} \end{cases}$$

$$[\Psi_k]_{ij} \ = \ \begin{cases} \frac{-1}{2\mathbf{i}}Y_{ij} \ = \ \frac{1}{2}(b_{ij}^s - \mathbf{i} g_{ij}^s) & \text{if } k = i \\ \frac{1}{2\mathbf{i}}Y_{ij}^{\mathsf{H}} \ = \ \frac{1}{2}(b_{ij}^s + \mathbf{i} g_{ij}^s) & \text{if } k = j \\ 0 & \text{if } k \notin \{i,j\} \end{cases}$$

Hence for each line $(j,k) \in E$ the relevant angles for C14.2 are those of $[C_0]_{jk}$ and

$$\begin{aligned}
\left[\Phi_j\right]_{jk} &= -\frac{1}{2}\left(g_{jk}^s + \mathbf{i} b_{jk}^s\right), & \left[\Phi_k\right]_{jk} &= -\frac{1}{2}\left(g_{jk}^s - \mathbf{i} b_{jk}^s\right)\\
\left[\Psi_j\right]_{jk} &= \frac{1}{2}\left(b_{jk}^s - \mathbf{i} g_{jk}^s\right), & \left[\Psi_k\right]_{jk} &= \frac{1}{2}\left(b_{jk}^s + \mathbf{i} g_{jk}^s\right)
\end{aligned}$$

as well as the angles of $-[\Phi_j]_{jk}, -[\Phi_k]_{jk}$ and $-[\Psi_j]_{jk}, -[\Psi_k]_{jk}$. These quantities are shown in Figure 14.2 with their magnitudes normalized to a common value and explained in the caption of the figure.

Condition C14.2 applied to OPF (14.19) takes the following form (see Figure 14.2):

> C14.2': For each link $(j,k) \in E$ there is a line in the complex plane through the origin such that $[C_0]_{jk}$ as well as those $\pm[\Phi_i]_{jk}$ and $\pm[\Psi_i]_{jk}$ corresponding to *finite* lower or upper bounds on (p_i, q_i), for $i = j, k$, are all on one side of the line, possibly on the line itself.

Let C^{opt} and C^{socp} denote the optimal values of OPF and OPF-socp respectively.

Corollary 14.8. Suppose G is a tree and C14.2' holds.

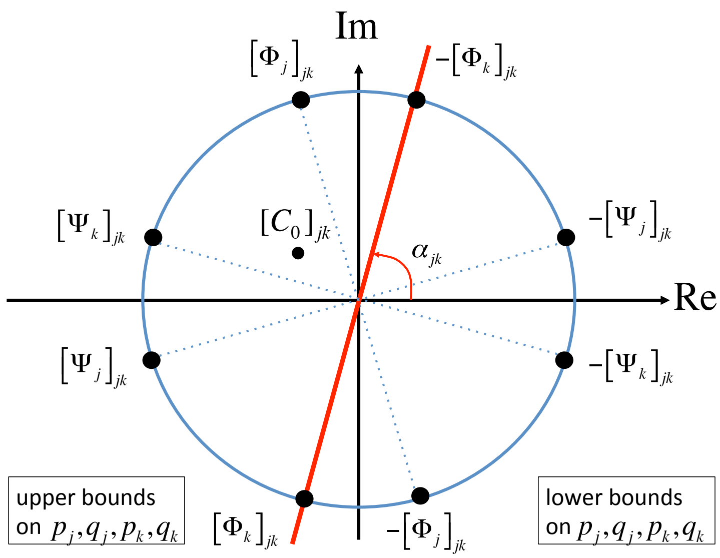


Figure 14.2: Condition C14.2' for OPF on a line $(j,k) \in E$. The quantities $([\Phi_j]_{jk}, [\Phi_k]_{jk}, [\Psi_j]_{jk}, [\Psi_k]_{jk})$ on the left-half plane correspond to finite upper bounds on (p_j, p_k, q_j, q_k) in (14.19a)(14.19b); $(-[\Phi_j]_{jk}, -[\Phi_k]_{jk}, -[\Psi_j]_{jk}, -[\Psi_k]_{jk})$ on the right-half plane correspond to finite lower bounds on (p_j, p_k, q_j, q_k).

1. $C^{\text{opt}} = C^{\text{socp}}$. Moreover an optimal solution V^{opt} of OPF can be recovered from every optimal solution X_G^{socp} of OPF-socp.

2. If, in addition, C14.1 holds then OPF-socp is exact.

It is clear from Figure 14.2 that condition C14.2' cannot be satisfied if there is a line where both the real and reactive power injections at both ends are both lower and upper bounded (8 combinations as shown in the figure). C14.2' requires that some of them be unconstrained. When the cost function is convex, this is the same as requiring that the constraints be inactive at optimality (see Exercise 14.3). The result proved in [123] also includes constraints on real branch power flows and line losses. Corollary 14.8 includes several sufficient conditions in the literature for exact relaxation as special cases. Referring to Figure 14.2, the load over-satisfaction condition in [124, 125] corresponds to the red line in the figure being the Im-axis that excludes all quantities on the right-half plane. The sufficient condition in [126, Theorem 2] corresponds to the red line in the figure that allows a finite lower bound on the real power at one end of the line, i.e., p_j or p_k but not both, and no finite lower bounds on reactive powers q_j and q_k.

14.3.3 Proofs

We now prove Theorem 14.6 and Corollary 14.7, following [127]. It is equivalent to the argument of [128] and simpler than the original duality proof in [123].

Proof of Theorem 14.6. Fix any partial matrix X_G that is feasible for SOCP (14.18). We will construct an $x \in \mathbb{C}^n$ that satisfies

$$x^{\mathsf{H}} C_l x \;\; \leq \;\; \text{tr}\, C_l X_G, \qquad\qquad l = 0, 1, \ldots, L$$

i.e., x is feasible for QCQP (14.17) and has an equal or lower cost than X_G. Since the minimum cost of QCQP is lower bounded by that of its SOCP relaxation this means that an optimal solution $x \in \mathbb{C}^n$ of QCQP (14.17) can be obtained from every optimal solution X_G of SOCP (14.18).

Now $X_G(j,k) \succeq 0$ for every $(j,k) \in E$ implies that $[X_G]_{jj} \geq 0$ for all $j \in N$ and

$$[X_G]_{jj}\,[X_G]_{kk} \;\; \geq \;\; \left|[X_G]_{jk}\right|^2, \;\; (j,k) \in E$$

Case 1: X_G is 2×2 psd rank-1. Suppose $[X_G]_{jj}[X_G]_{kk} = |[X_G]_{jk}|^2$ for all $(j,k) \in E$. We will construct an $x \in \mathbb{C}^n$ that is feasible for QCQP and has an equal cost. To construct such an x let $|x_j| := \sqrt{[X_G]_{jj}}$, $j \in N$. Recall that G is a (connected) tree with node 1 as its root. Let $\angle x_1 := 0$. Traversing the tree starting from the root the angles can be successively assigned: given $\angle x_j$ at one end of a link (j,k), let $\angle x_k := \angle x_j - \angle [X_G]_{jk}$ at the other end. Given any X_G which is 2×2 psd rank-1, angles $\angle x_j$ can always be consistently assigned if and only if G is a tree. (If G contains cycles then X_G must also satisfy the cycle condition according to Theorem 14.1).

With this x constructed from X_G we have, for $l = 0, 1, \ldots, L$,

$$x^{\mathsf{H}} C_l x \;\; = \;\; \sum_{j,k} [C_l]_{jk}\, x_j^{\mathsf{H}} x_k \;\; = \;\; \sum_{j,k} [C_l]_{jk}\, |x_j|\, |x_k|\, e^{\mathbf{i}\left(\angle x_k - \angle x_j\right)} \;\; = \;\; \sum_{j,k} [C_l]_{jk} \left|[X_G]_{jk}\right| e^{-\mathbf{i}\angle [X_G]_{jk}} \;\; = \;\; \text{tr}\left(C_l X_G\right)$$

where the last equality follows from $\text{tr}(C_l X_G)\, s = \sum_{j,k} [C_l]_{jk}\, [X_G]^{\mathsf{H}}_{jk}$. Hence x is feasible for QCQP (14.17) and has the same cost as X_G.

Case 2: X_G is 2×2 psd but not 2×2 rank-1. Suppose $[X_G]_{jj}[X_G]_{kk} > |[X_G]_{jk}|^2$ for some (j,k). We will

1. Construct an $\hat{X}_G$ that is 2×2 psd rank-1.

2. Show that C14.2 implies

$$\text{tr}\, C_l \hat{X}_G \;\; \leq \;\; \text{tr}\, C_l X_G, \qquad l = 0, 1, \ldots, L \tag{14.20}$$

Then an $x \in \mathbb{C}^n$ can be constructed from $\hat{X}_G$ as in the case above and step 2 ensures that for $l = 0, 1, \ldots, L$

$$x^{\mathsf{H}} C_l x \; = \; \text{tr}\, C_l \hat{X}_G \; \leq \; \text{tr}\, C_l X_G$$

i.e., x is feasible for QCQP (14.17) and has an equal or lower cost than X_G.

To construct such an $\hat{X}_G$ let $[\hat{X}_G]_{jj} = [X_G]_{jj}$, $j \in \overline{N}$. For each line $(j,k) \in E$ let

$$[\hat{X}_G]_{jk} - [X_G]_{jk} \;\; =: \;\; r_{jk} e^{-\mathbf{i}\left(\frac{\pi}{2} - \alpha_{jk}\right)}$$

for some $r_{jk} > 0$ to be determined and α_{jk} in assumption C14.2. For $\hat{X}_G$ to be 2×2 psd rank-1 we need to choose $r_{jk} > 0$ such that $[\hat{X}_G]_{jj}[\hat{X}_G]_{kk} = \left|[\hat{X}_G]_{jk}\right|^2$ for all $(j,k)\in E$, i.e.,

$$[X_G]_{jj}\,[X_G]_{kk} \;=\; \left|[X_G]_{jk} + r_{jk}e^{-\mathbf{i}\left(\frac{\pi}{2}-\alpha_{jk}\right)}\right|^2$$

or

$$r_{jk}^2 + 2b\,r_{jk} - c \;=\; 0$$

where

$$b \;:=\; \mathrm{Re}\left([X_G]_{jk}\,e^{\mathbf{i}\left(\frac{\pi}{2}-\alpha_{jk}\right)}\right), \qquad\qquad c \;:=\; [X_G]_{jj}\,[X_G]_{kk} - \left|[X_G]_{jk}\right|^2 \;>\; 0$$

Therefore setting $r_{jk} := \sqrt{b^2+c} - b > 0$ yields an $\hat{X}_G$ that is 2×2 psd rank-1.

To show that $\hat{X}_G$ is feasible for SOCP (14.18) and has an equal or lower cost than X_G, we have for $l = 0,1,\ldots,L$,

$$\begin{aligned}
\mathrm{tr}\, C_l\hat{X}_G - \mathrm{tr}\, C_l X_G &= \mathrm{tr}\left(C_l\left(\hat{X}_G - X_G\right)\right) \;=\; \sum_{(j,k)\in E} [C_l]_{jk}\left([\hat{X}_G]_{jk} - [X_G]_{jk}\right)^{\mathsf{H}} \\
&= 2\sum_{j<k} \mathrm{Re}\left([C_l]_{jk}\cdot r_{jk}\,e^{\mathbf{i}\left(\frac{\pi}{2}-\alpha_{jk}\right)}\right) \\
&= 2\sum_{j<k} \left|[C_l]_{jk}\right|\, r_{jk}\, \cos\left(\angle[C_l]_{jk} + \frac{\pi}{2} - \alpha_{jk}\right) \;\le\; 0
\end{aligned}$$

where the last inequality follows because assumption C14.2 implies

$$\frac{\pi}{2} \;\le\; \angle[C_l]_{jk} + \tfrac{\pi}{2} - \alpha_{jk} \;\le\; \frac{3\pi}{2}$$

and therefore $\cos\left(\angle[C_l]_{jk} + \frac{\pi}{2} - \alpha_{jk}\right) \le 0$. This completes the proof. □

Proof of Corollary 14.7. C14.1 implies that the objective function of SOCP (14.18) is strictly convex and hence has a unique optimal solution. Suppose X_G is an optimal solution of SOCP (14.18) but $[X_G]_{jj}[X_G]_{kk} > |[X_G]_{jk}|^2$ for some (j,k), i.e., X_G is 2×2 psd but not 2×2 psd rank-1. Then the above constructs another feasible solution $\hat{X}_G$ with equal cost. This contradicts the uniqueness of the optimal solution of SOCP (14.18), and hence X_G must be 2×2 psd rank-1. □

14.4 Exactness condition: small angle differences

The sufficient conditions in [126, 129, 130] require that the voltage angle difference across each line be small. We explain the intuition using a result in [129] for an OPF problem where $|V_j|$ are fixed for all $j\in\overline{N}$ and reactive powers are ignored. Under these assumptions, as long as the voltage angle difference is small, the power flow solutions form a locally convex surface that is the Pareto front of its relaxation. This implies that the relaxation is exact.

14.4.1 Sufficient condition

Consider the simple case where voltage magnitudes $|V_j|$ are fixed, reactive powers are ignored, and line charging admittances are assumed zero $y^m_{jk} = y^m_{kj} := 0$. Recall that $y_{jk} = g_{jk} + \mathbf{i}b_{jk}$ with $g_{jk} > 0, b_{jk} < 0$. Let $V_j = |V_j|e^{\mathbf{i}\theta_j}$ and suppose $|V_j|$ are given. Consider

$$\min_{p,P,\theta} \quad C(p) \tag{14.21a}$$

$$\text{s.t.} \quad p_j^{\min} \le p_j \le p_j^{\max}, \qquad j \in \overline{N} \tag{14.21b}$$

$$\theta_{jk}^{\min} \le \theta_{jk} \le \theta_{jk}^{\max}, \quad (j,k) \in E \tag{14.21c}$$

$$p_j = \sum_{k:k\sim j} P_{jk}, \qquad j \in \overline{N} \tag{14.21d}$$

$$P_{jk} = |V_j|^2 g_{jk} - |V_j||V_k| g_{jk}\cos\theta_{jk} - |V_j||V_k| b_{jk}\sin\theta_{jk}, \quad (j,k) \in E \tag{14.21e}$$

where $\theta_{jk} := \theta_j - \theta_k$ are the voltage angle differences across lines (j,k).

We comment on the constraints on angles θ_{jk} in (14.21). When the voltage magnitudes $|V_i|$ are fixed, constraints on real power flows, branch currents, line losses, as well as stability constraints can all be represented in terms of θ_{jk}. Indeed a line flow constraint of the form $|P_{jk}| \le \overline{P}_{jk}$ becomes a constraint on θ_{jk} using the expression for P_{jk} in (14.21e). A current constraint of the form $|I_{jk}| \le \bar{I}_{jk}$ is also a constraint on θ_{jk} since $|I_{jk}|^2 = |y_{jk}|(|V_j|^2 + |V_k|^2 - 2|V_jV_k|\cos\theta_{jk})$. The line loss over $(j,k) \in E$ is equal to $P_{jk} + P_{kj}$ which is again a function of θ_{jk}. Stability typically requires $|\theta_{jk}|$ to stay within a small threshold. Therefore given constraints on branch power or current flows, losses, and stability, appropriate bounds $\underline{\theta}_{jk}, \overline{\theta}_{jk}$ can be determined in terms of these constraints, assuming $|V_j|$ are fixed.

We can eliminate the branch flows P_{jk} and angles θ_{jk} from (14.21). Since $|V_j|, j \in \overline{N}$, are fixed we assume without loss of generality that $|V_j| = 1$ pu. Define the injection region

$$\mathbb{P}_\theta := \left\{ p \in \mathbb{R}^n \,\middle|\, p_j = \sum_{k:k\sim j} \left(g_{jk} - g_{jk}\cos\theta_{jk} - b_{jk}\sin\theta_{jk} \right),\, j \in \overline{N},\ \theta_{jk}^{\min} \le \theta_{jk} \le \theta_{jk}^{\max},\, (j,k) \in E \right\}$$

Let $\mathbb{P}_p := \{p \in \mathbb{R}^n \,|\, p_j^{\min} \le p_j \le p_j^{\max}, j \in N\}$. Then (14.21) is:
OPF:

$$\min_p C(p) \quad \text{subject to} \quad p \in \mathbb{P}_\theta \cap \mathbb{P}_p \tag{14.22}$$

This problem is hard because the set $\mathbb{P}_\theta$ is nonconvex. To avoid triviality we assume OPF (14.22) is feasible. For a set A let $\operatorname{conv} A$ denote the convex hull of A. Consider the following problem that relaxes the nonconvex feasible set $\mathbb{P}_\theta \cap \mathbb{P}_p$ of (14.22) to a convex superset:
OPF-socp:

$$\min_p C(p) \quad \text{s.t.} \quad p \in \operatorname{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p \tag{14.23}$$

We will show below that (14.23) is indeed an SOCP. It is said to be *exact* if every optimal solution of (14.23) lies in $\mathbb{P}_\theta \cap \mathbb{P}_p$ and is therefore also optimal for (14.22).

We say that a point $x \in A \subseteq \mathbb{R}^n$ is a *Pareto optimal point* in A if there does not exist another $x' \in A$ such that $x' \leq x$ with at least one strictly smaller component $x'_j < x_j$. The *Pareto front of* A, denoted by $\mathbb{O}(A)$, is the set of all Pareto optimal points in A. The significance of $\mathbb{O}(A)$ is that, for any increasing function, its minimizer, if exists, is necessarily in $\mathbb{O}(A)$ whether A is convex or not. If A is convex then x^{opt} is a Pareto optimal point in $\mathbb{O}(A)$ if and only if there is a nonzero vector $c := (c_1, \ldots, c_n) \geq 0$ such that x^{opt} is a minimizer of $c^{\mathsf{T}} x$ over A [73, pp.179–180].

Assume

C14.3: $C(p)$ is strictly increasing in each p_j.

C14.4: For all $(j,k) \in E$, $\tan^{-1} \frac{b_{jk}}{g_{jk}} < \theta_{jk}^{\min} \leq \theta_{jk}^{\max} < \tan^{-1} \frac{-b_{jk}}{g_{jk}}$.

says that (14.23) is exact provided θ_{jk} are suitably bounded.

Theorem 14.9. Suppose G is a tree and C14.3–C14.4 hold.

1. $\mathbb{P}_\theta \cap \mathbb{P}_p \;=\; \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$.
2. The problem (14.23) is an SOCP. Moreover it is exact.

Remark 14.4 (Strong exactness)**.** Condition C14.3 is needed to ensure that *every* optimal solution of OPF-socp (14.23) is optimal for OPF (14.22). If $C(p)$ is nondecreasing but not strictly increasing in all p_j, then $\mathbb{P}_\theta \cap \mathbb{P}_p \subseteq \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$ and OPF-socp may not be exact according to our definition. Even in that case it is possible to recover an optimal solution of OPF from any optimal solution of OPF-socp (see Exercise 14.9). □

14.4.2 Proof: 2-bus network

We now illustrate the geometric insight by proving the theorem for the case of a single line, following [129]. The case of a tree network is proved in Chapter 14.8.

Proof of Theorem 14.9: 2-bus network. Consider two buses j and k connected by a line with admittance $y_{jk} = g_{jk} + \mathbf{i} b_{jk}$ with $g_{jk} > 0, b_{jk} < 0$. Recall that we assume voltage magnitudes $|V_j| = 1$ pu are fixed for buses $j = 1, 2$, zero charging admittances, and we ignore reactive powers. Since $p_j = P_{jk}$ and $p_k = P_{kj}$ we will work with $P := (P_{jk}, P_{kj})$. Then

$$\begin{aligned} P_{jk} \;&:=\; P_{jk}(\theta_{jk}) \;:=\; g_{jk} - g_{jk} \cos\theta_{jk} - b_{jk} \sin\theta_{jk} \\ P_{kj} \;&:=\; P_{kj}(\theta_{jk}) \;:=\; g_{jk} - g_{jk} \cos\theta_{jk} + b_{jk} \sin\theta_{jk} \end{aligned}$$

where $\theta_{jk} := \theta_j - \theta_k$, or in vector form

$$P - g_{jk}\mathbf{1} \;=\; A \begin{bmatrix} \cos\theta_{jk} \\ \sin\theta_{jk} \end{bmatrix} \tag{14.24}$$

where $\mathbf{1} := [1\ 1]^\mathsf{T}$ and A is an invertible (indeed negative definite) matrix:

$$A \quad := \quad \begin{bmatrix} -g_{jk} & -b_{jk} \\ -g_{jk} & b_{jk} \end{bmatrix}$$

The proof will proceed in four steps:

1. We show that P traces out an ellipse in $\mathbb{R}^2$ as θ_{jk} ranges over $[-\pi, \pi]$. Since the feasible set is a subset of ellipse, it is nonconvex.
2. We show that condition C14.4 restricts the feasible set to the lower half of the ellipse.
3. We show that condition C14.3 implies that the Pareto front of the feasible set of the relaxed problem (14.23) coincides with the feasible set. This implies that the relaxation is exact.
4. Finally we show that the relaxation (14.23) is an SOCP.

Step 1: P that satisfies (14.24) *is an ellipse.* In general the set of points $x \in \mathbb{R}^k$ that satisfy

$$(x-c)^\mathsf{T} M (x-c) \quad = \quad \left\| M^{1/2}(x-c) \right\|_2^2 \quad = \quad 1$$

is an ellipse if $c \in \mathbb{R}^n$ and $M \succ 0$ is a real (symmetric) positive definite matrix. The center of the ellipsoid is c and the k principal axes are the k eigenvectors of M (see Exercise 14.4). To see that P describes an ellipse, write $v := [\cos\theta_{jk}\ \sin\theta_{jk}]^\mathsf{T} = A^{-1}\left(P - g_{jk}\mathbf{1}\right)$. Hence $\|v\|_2^2 = 1$, yielding

$$(P - g_{jk}\mathbf{1})^\mathsf{T} \left(AA^\mathsf{T}\right)^{-1} (P - g_{jk}\mathbf{1}) \quad = \quad 1 \tag{14.25}$$

Hence P is an ellipse centered at $g_{jk}\mathbf{1}$. From (14.24), the ellipse P passes through the origin when $\theta_{jk} = 0$, as shown in Figures 14.3. Since the feasible set is a subset of the ellipse P (without the interior), it is nonconvex.

Step 2: condition C14.4 restricts the feasible set to the lower half of the ellipse. Let $\pi_{jk}^{\min}$ denote the minimum $P_{jk}(\theta_{jk})$ and $\pi_{kj}^{\min}$ the minimum $P_{kj}(\theta_{jk})$ on the ellipse as shown in the figure. They are attained when θ_{jk} takes the values

$$\theta_{jk}^{\min} \ := \ \tan^{-1}\frac{b_{jk}}{g_{jk}} \quad \text{and} \quad \theta_{kj}^{\min} \ := \ \tan^{-1}\frac{-b_{jk}}{g_{jk}}$$

respectively (Exercise 14.7). The condition $\theta_{jk}^{\min} \le \theta_{jk} \le \theta_{kj}^{\min}$ restricts $P(\theta_{jk})$ to the darkened segment of the ellipse in Figures 14.3. Recall the sets

$$\mathbb{P}_\theta \ := \ \{\, p \mid p = P, P \text{ satisfies (14.24) for } \underline{\theta}_{jk} \le \theta_{jk} \le \overline{\theta}_{jk} \,\}, \qquad \mathbb{P}_p \ := \ \{p \,|\, \underline{p} \le p \le \overline{p}\}$$

and the feasible set $\mathbb{P}_\theta \cap \mathbb{P}_p$ of OPF (14.22). Condition C14.4 ensures $\theta_{jk}^{\min} \le \theta_{jk} \le \theta_{kj}^{\min}$ and hence restricts both $\mathbb{P}_\theta$ and the feasible set $\mathbb{P}_\theta \cap \mathbb{P}_p$ to the lower half of the ellipse.

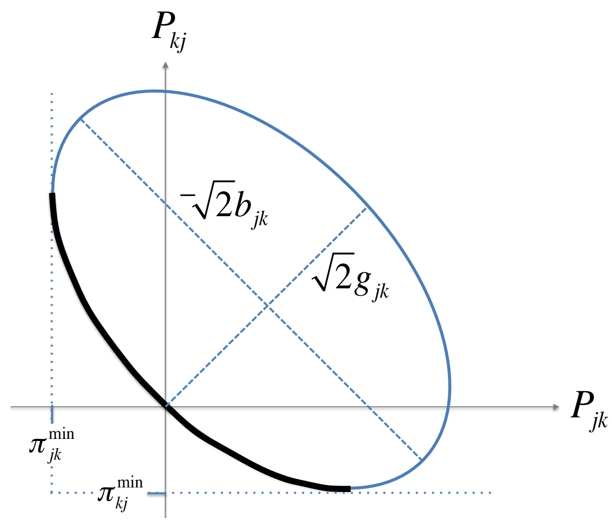


Figure 14.3: The feasible set of OPF (14.22) for the two-bus network is a subset of an ellipse without the interior, hence nonconvex. OPF-socp (14.23) includes the interior of the ellipse and is hence convex. If the cost function C is strictly increasing in (P_{jk}, P_{kj}) then the Pareto front of the SOCP feasible set will lie on the lower part of the ellipse, $\mathbb{O}(\mathbb{P}_\theta) = \mathbb{P}_\theta$, and hence OPF-socp is exact. The points $P :=(P_{jk}(\theta_{jk}), P_{kj}(\theta_{kj})) = 0$ when $\theta_{jk} = 0$, $P_{jk} = \pi_{jk}^{\min}$ when $\theta_{jk} = \theta_{jk}^{\min}$, and $P_{kj} = \pi_{kj}^{\min}$ when $\theta_{jk} = \theta_{kj}^{\min}$.

The implication is that, under condition C14.3 that the cost function C is strictly increasing in the injections $(p_j, p_k) = (P_{jk}, P_{kj})$, the nonconvex feasible sets $\mathbb{P}_\theta$ and $\mathbb{P}_\theta \cap \mathbb{P}_p$ coincide with the Parento fronts of their respectively convex hulls, i.e.,

$$\mathbb{P}_\theta \;=\; \mathbb{O}(\text{conv}\;\mathbb{P}_\theta), \qquad\qquad \mathbb{P}_\theta \cap \mathbb{P}_p \;=\; \mathbb{O}(\text{conv}(\mathbb{P}_\theta \cap \mathbb{P}_p)) \tag{14.26}$$

The first property is used in the proof of Theorem 14.9 for a tree network in Chapter 14.8.

Step 3: condition C14.3 implies that $\mathbb{P}_\theta \cap \mathbb{P}_p = \mathbb{O}(\textit{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$. Unfortunately the convex hull $\text{conv}(\mathbb{P}_\theta \cap \mathbb{P}_p)$ in (14.26) of the intersection of two sets generally does not have a simple algebraic representation. The feasible set $\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p$ of the relaxation OPF-socp (14.23) is the intersection of two convex hulls and is more amenable to computation. It is however a superset of $\text{conv}(\mathbb{P}_\theta \cap \mathbb{P}_p)$. To illustrate their relationship denote the points $P(\theta_{jk}) := \big(P_{jk}(\theta_{jk}), P_{kj}(\theta_{jk})\big)$ attained at $\underline{\theta}_{jk}$ and $\overline{\theta}_{jk}$ by

$$\big(\underline{\pi}_{jk}, \underline{\pi}_{kj}\big) \;:=\; P(\underline{\theta}_{jk}), \qquad\qquad \big(\overline{\pi}_{jk}, \overline{\pi}_{kj}\big) \;:=\; P(\overline{\theta}_{jk}) \tag{14.27}$$

The set $\mathbb{P}_\theta$ is the ellipse segment between these two points $\big(\underline{\pi}_{jk}, \underline{\pi}_{kj}\big)$ and $\big(\overline{\pi}_{jk}, \overline{\pi}_{kj}\big)$. As shown in Figure 14.4, the relationship between these two convex sets is:

$$\text{conv}\,(\mathbb{P}_\theta) \cap \mathbb{P}_p \;\supseteq\; \text{conv}\,(\mathbb{P}_\theta \cap \mathbb{P}_p)$$

Even though these two sets are generally different, it is clear from the figure that, if the cost function $C(p)$ is strictly increasing in each p_j (condition C14.3), then they share the same Pareto front, i.e.,

$$\mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p) \;=\; \mathbb{O}(\text{conv}(\mathbb{P}_\theta \cap \mathbb{P}_p)) \;=\; \mathbb{P}_\theta \cap \mathbb{P}_p$$

where the last equality follows from (14.26). This proves the first claim of Theorem 14.9.

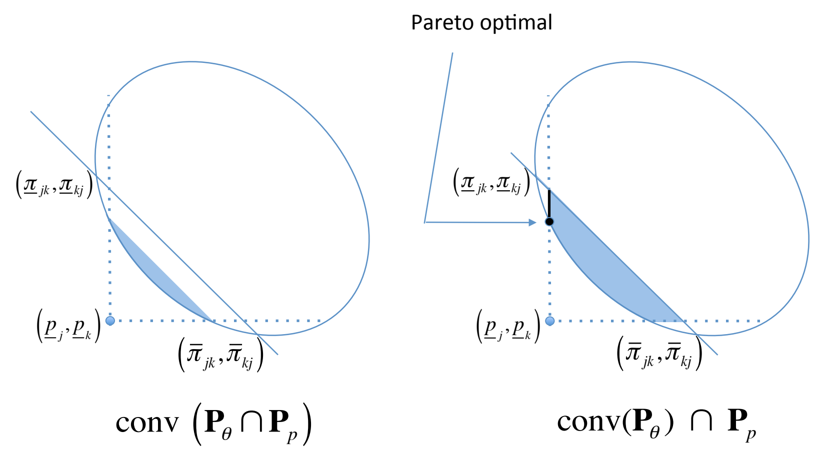


Figure 14.4: $\text{conv}(\mathbb{P}_\theta \cap \mathbb{P}_p) \subseteq \text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p$.

Step 4: (14.23) *is an SOCP and it is exact.* To show that the feasible set $\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p$ of OPF-socp is a second-order cone, we will show that it is the intersection of a second-order cone with several affine sets. First, from (14.25), the solid ellipse including the interior is the set of P satisfying

$$1 \quad \geq \quad \left(P - g_{jk}\mathbf{1}\right)^{\mathsf{T}} (AA^{\mathsf{T}})^{-1} \left(P - g_{jk}\mathbf{1}\right)$$

This is a second-order cone $t^2 \geq (P - g_{jk}\mathbf{1})^{\mathsf{T}}(AA^{\mathsf{T}})^{-1}(P - g_{jk}\mathbf{1})$ intersecting with the affine set $t = 1$. Second the set $\text{conv}(\mathbb{P}_\theta)$ is the intersection of this second-order cone with the following half space (see Figure 14.5)):

$$P_{kj} \quad \leq \quad \underline{\pi}_{kj} + \frac{\overline{\pi}_{kj} - \underline{\pi}_{kj}}{\overline{\pi}_{jk} - \underline{\pi}_{jk}} \left(P_{jk} - \underline{\pi}_{jk}\right)$$

where $(\underline{\pi}_{jk}, \underline{\pi}_{kj})$ and $(\overline{\pi}_{jk}, \overline{\pi}_{kj})$ are defined in (14.27). Finally intersecting this set with the affine set $\mathbb{P}_p$ produces the feasible set $\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p$ of OPF-socp. Hence the problem (14.23) is indeed an SOCP for the two-bus case.

In summary, the SOCP relaxation of OPF (14.22) enlarges the feasible set $\mathbb{P}_\theta \cap \mathbb{P}_p$ to the convex superset $\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p$. Under condition C14.3, *every* minimizer lies in its Pareto front and hence in the original nonconvex feasible set $\mathbb{P}_\theta \cap \mathbb{P}_p$, as proved in Step 3.

We have hence proved Theorem 14.9 for the two-bus case. $\square$

We illustrate the purpose of condition C14.4. If there are no constraints on the injections p, then SOCP relaxation (14.23) is exact under condition C14.3 due to $\mathbb{P}_\theta = \mathbb{O}(\text{conv } \mathbb{P}_\theta)$ in (14.26). As illustrated in Figure 14.6, upper bounds $\overline{p}$ on power injections p do not affect exactness whereas lower bounds $\underline{p}$ do.

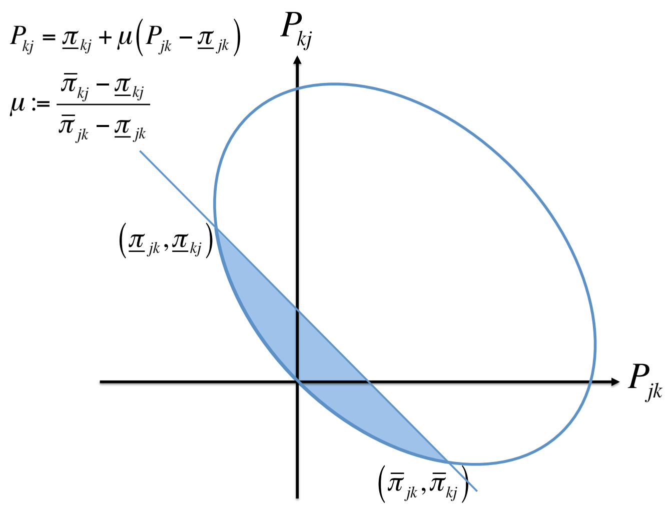
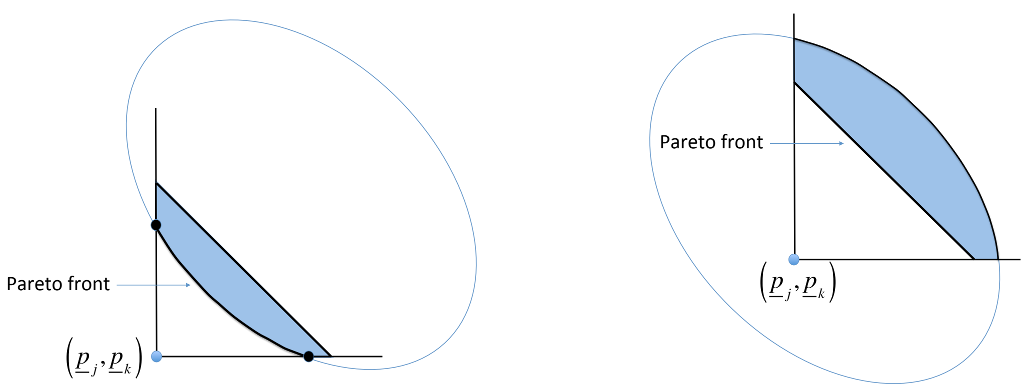


Figure 14.5: The set $\text{conv}(\mathbb{P}_\theta)$ is the intersection of the ellipse, including its interior, and a half-space.



(a) Exact relaxation with constraint

(b) Inexact relaxation with constraint

Figure 14.6: With lower bounds $\underline{p}$ on power injections, the feasible set of OPF-socp (14.23) is the shaded region. (a) When the feasible set of OPF (14.22) is restricted to the lower half of the ellipse (small $|\theta_{jk}|$), the Pareto front remains on the ellipse itself, $\mathbb{P}_\theta \cap \mathbb{P}_p = \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$, and hence the relaxation is exact. (b) When the feasible set of OPF includes upper half of the ellipse (large $|\theta_{jk}|$), the Pareto front may not lie on the ellipse if $\underline{p}$ is large, making the relaxation not exact.

The purpose of condition C14.4 is to restrict the angle θ_{jk} in order to eliminate the upper half of the ellipse from $\mathbb{P}_\theta$.

We close this subsection by illustrating the importance of tree topology.

Remark 14.5 (Tree topology)**.** The tree topology allows the extension of the argument for a single line to a radial network with multiple lines, in two ways. First let $\mathbb{F}_\theta^{jk}$ denotes the set of branch power flows on each line $(j,k) \in E$:

$$\mathbb{F}_\theta^{jk} \;\; := \;\; \{ \; (P_{jk}, P_{kj}) \mid (P_{jk}, P_{kj}) \text{ satisfies (14.24) for } \underline{\theta}_{jk} \leq \theta_{jk} \leq \overline{\theta}_{jk} \; \}$$

If the network is a tree, the set $\mathbb{F}_\theta$ of branch power flows on all lines is simply the product set, $\mathbb{F}_\theta = \prod_{(j,k)\in E} \mathbb{F}_\theta^{jk}$, because given any $(\theta_{jk}, (j,k) \in E)$ there is always a (unique) $(\theta_j, j \in \overline{N})$ that satisfies $\theta_{jk} = \theta_j - \theta_k$. (This is equivalent to the cycle condition (**??**).) If the network has cycles then this is not possible for some vectors $(\theta_{jk}, (j,k) \in E)$ and $\mathbb{F}_\theta$ is no longer a product set of $\mathbb{F}_\theta^{jk}$.

Second the power injections p are related to the branch flows P by a linear transformation $\mathbb{P}_\theta = A\mathbb{F}_\theta$ for some $(N+1) \times 2M$ dimensional matrix A. Matrix A has full row rank and it can be argued that there is a bijection between P_θ and F_θ using the fact that the graph is a tree. We can therefore freely work with either $p \in \mathbb{P}_\theta$ or the corresponding $P \in \mathbb{F}_\theta$. See the detailed proof in Chapter 14.8.

When the network is not radial or $|V_j|$ are not constants, then the feasible set can be much more complicated than ellipsoids [21, 22, 23, 130]. □

14.5 Three-phase OPF

Consider the three-phase OPF (13.22), reproduced here:

$$\min_{(u,V)} \;\; C_0(u, V, s(V)) \qquad \text{s.t.} \qquad (13.18)(13.19)(13.20)(13.21) \tag{14.28}$$

where the device models are given by (13.18)(13.19) and the power flow equations and constraints are given by (13.20)(13.21). In this section we reformulate these constraints to derive an SDP relaxation of three-phase OPF in the bus injection model.

14.5.1 Reformulation

We first reformulate the network equations and constraints (13.20) (13.21) as constraints on terminal variables. We then reformulate the device models (13.18) (13.19) as constraints on internal variables.

The power flow equations (13.20) are reproduced here:

$$s_j \; = \; \sum_{k: j \sim k} \text{diag}\left(V_j (V_j - V_k)^{\mathsf{H}} \left(y_{jk}^s \right)^{\mathsf{H}} \, + \, V_j V_j^{\mathsf{H}} \left(y_{jk}^m \right)^{\mathsf{H}} \right), \qquad j \in \overline{N} \tag{14.29}$$

Consider the $3(N+1) \times 3(N+1)$ matrix $W = VV^{\mathsf{H}}$ and its 3×3 submatrices W_{jj} and W_{jk} defined by:

$$W_{jj} \;=\; V_j V_j^{\mathsf{H}}, \qquad j \in \overline{N}, \qquad\qquad W_{jk} \;=\; V_j V_k^{\mathsf{H}}, \qquad (j,k) \in E \tag{14.30}$$

Then (14.29) is equivalent to the following equation that is linear in W:

$$s_j \;=\; \sum_{k:j\sim k} \mathrm{diag}\left(\left(W_{jj} - W_{jk}\right) \left(y_{jk}^s\right)^{\mathsf{H}} \;+\; W_{jj} \left(y_{jk}^m\right)^{\mathsf{H}} \right), \qquad j \in \overline{N} \tag{14.31a}$$

The operational constraints (13.21) can be expressed also as linear functions of (s, W):

$$\begin{aligned}
&\text{injection limits:} && s_j^{\min} \;\le\; s_j \;\le\; s_j^{\max}, && j \in \overline{N} && \text{(14.31b)}\\
&\text{voltage limits:} && v_j^{\min} \;\le\; \mathrm{diag}\left(W_{jj}\right) \;\le\; v_j^{\max}, && j \in \overline{N} && \text{(14.31c)}\\
&\text{line limits:} && \mathrm{diag}\left(\ell_{jk}\left(W_{jj}, W_{jk}, W_{kk}\right)\right) \;\le\; \ell_{jk}^{\max}, && (j,k) \in E && \text{(14.31d)}\\
& && \mathrm{diag}\left(\ell_{kj}\left(W_{jj}, W_{kj}, W_{kk}\right)\right) \;\le\; \ell_{kj}^{\max}, && (j,k) \in E && \text{(14.31e)}
\end{aligned}$$

where, using $I_{jk}(V) = \left(y_{jk}^s + y_{jk}^m\right) V_j - y_{jk}^s V_k$ and $I_{kj}(V) = \left(y_{kj}^s + y_{kj}^m\right) V_k - y_{kj}^s V_j$,

$$\begin{aligned}
\ell_{jk}\left(W_{jj}, W_{jk}, W_{kk}\right) &:= \left(y_{jk}^s + y_{jk}^m\right) W_{jj} \left(y_{jk}^s + y_{jk}^m\right)^{\mathsf{H}} - 2\mathrm{Re}\left(\left(y_{jk}^s + y_{jk}^m\right) W_{jk}\, y_{jk}^{s\mathsf{H}}\right) + y_{jk}^s W_{kk}\, y_{jk}^{s\mathsf{H}}\\
\ell_{kj}\left(W_{jj}, W_{kj}, W_{kk}\right) &:= \left(y_{kj}^s + y_{kj}^m\right) W_{kk} \left(y_{kj}^s + y_{kj}^m\right)^{\mathsf{H}} - 2\mathrm{Re}\left(\left(y_{kj}^s + y_{kj}^m\right) W_{kj}\, y_{kj}^{s\mathsf{H}}\right) + y_{kj}^s W_{jj}\, y_{kj}^{s\mathsf{H}}
\end{aligned}$$

Here the lower and upper bounds in (14.31b) – (14.31e) are 3-dimensional complex or real vectors. Instead of the quadratic equations (14.30) we use the following equivalent specification that is easy to convexify:

$$W \;\succeq\; 0, \qquad \mathrm{rank}(W) \;=\; 1 \tag{14.32}$$

Therefore the power flow equations and constraints (13.20)(13.21) are equivalent to the linear constraints (14.31) and the convex and nonconvex constraints in (14.32). These constraints are global. When deriving the semidefinite relaxation of the three-phase OPF (14.28), we will omit the nonconvex rank-1 constraint in (14.32).

We apply the same method to convexify the device models (13.18)(13.19). To simplify notation we assume:

- Only three-phase voltage and power sources are included, in Y or Δ configurations.
- The neutrals of all Y-configured devices are directly grounded and all voltages are defined with respect to the ground, so that all neutral voltages $\gamma_j^Y := V_j^n = 0$.

From Chapter 13.1.4 the conversion rules (13.18) are:

1. *Voltage source* $V_j^{Y/\Delta} \in \mathbb{C}^3$:

$$\begin{aligned}
&Y \text{ configuration:} && V_j \;=\; V_j^Y\\
&\Delta \text{ configuration:} && \Gamma V_j \;=\; V_j^\Delta
\end{aligned}$$

We reformulate this using a matrix variable $u_j := W_j^{Y/\Delta} \in \mathbb{C}^{3\times 3}$, as follows:

$$Y \text{ configuration:} \qquad W_{jj} \;=\; W_j^Y, \qquad W_j^Y \;\succeq\; 0, \qquad \text{rank}\left(W_j^Y\right) \;=\; 1 \tag{14.33a}$$

$$\Delta \text{ configuration:} \qquad \Gamma W_{jj} \Gamma^{\mathsf{T}} \;=\; W_j^{\Delta}, \qquad W_j^{\Delta} \;\succeq\; 0, \qquad \text{rank}\left(W_j^{\Delta}\right) \;=\; 1 \tag{14.33b}$$

Note that W_{jj} is the 3×3 principal submatrix of the $3(N+1)\times 3(N+1)$ matrix W associated with the vector V of terminal voltages while $W_j^{Y/\Delta}$ is a 3×3 matrix associated with the internal voltage $V_j^{Y/\Delta}$ of device j. The conditions (14.33a)(14.33b) ensure that there exists $V_j^{Y/\Delta}$, unique up to a rotation, so that $W_j^{Y/\Delta} = V_j^{Y/\Delta}\left(V_j^{Y/\Delta}\right)^{\mathsf{H}}$.

The operational constraints (13.19a) on the internal voltage magnitudes can be expressed as a linear function of the internal variable $u_j := W_j^{Y/\Delta}$:

$$v_j^{Y/\Delta \min} \;\leq\; \text{diag}\left(u_j\right) \;\leq\; v_j^{Y/\Delta \max} \tag{14.33c}$$

where the lower and upper bounds $\left(v_j^{Y/\Delta \min}, v_j^{Y/\Delta \max}\right) \in \mathbb{C}^6$ are given vectors.

2. *Power source* $\left(s_j^{Y/\Delta}, I_j^{Y/\Delta}\right) \in \mathbb{C}^6$:

$$Y \text{ configuration:} \qquad s_j \;=\; -\text{diag}\left(V_j I_j^{Y\mathsf{H}}\right), \qquad s_j \;=\; -s_j^Y$$

$$\Delta \text{ configuration:} \qquad s_j \;=\; -\text{diag}\left(V_j I_j^{\Delta\mathsf{H}}\Gamma\right), \qquad s_j^{\Delta} \;=\; \text{diag}\left(\Gamma V_j I_j^{\Delta\mathsf{H}}\right)$$

We reformulate this using three 3×3 matrix variables $u_j := \left(s_j^{Y/\Delta}, X_j^{\Delta}, \ell_j^{\Delta}\right)$, as follows:

$$Y \text{ configuration:} \qquad s_j \;=\; -s_j^Y \tag{14.33d}$$

$$\Delta \text{ configuration:} \qquad s_j \;=\; -\text{diag}\left(X_j^{\Delta}\Gamma\right), \qquad s_j^{\Delta} \;=\; \text{diag}\left(\Gamma X_j^{\Delta}\right) \tag{14.33e}$$

$$0 \;\preceq\; \begin{bmatrix} W_{jj} & X_j^{\Delta} \\ X_j^{\Delta\mathsf{H}} & \ell_j^{\Delta} \end{bmatrix}, \qquad 1 \;=\; \text{rank}\begin{bmatrix} W_{jj} & X_j^{\Delta} \\ X_j^{\Delta\mathsf{H}} & \ell_j^{\Delta} \end{bmatrix} \tag{14.33f}$$

For a Δ-configured power source, the conditions (14.33e)(14.33f) ensure that there exist V_j and I_j^{Δ} so that $W_{jj} = V_j V_j^{\mathsf{H}}$, $\ell_j^{\Delta} = I_j^{\Delta} I_j^{\Delta\mathsf{H}}$, and $X_j^{\Delta} = V_j I_j^{\Delta\mathsf{H}}$.

The operational constraints (13.19c) on the internal powers and currents can be expressed as linear functions of the internal variable $u_j := \left(s_j^{Y/\Delta}, X_j^{\Delta}, \ell_j^{\Delta}\right)$:

$$s_j^{Y/\Delta \min} \;\leq\; s_j^{Y/\Delta} \;\leq\; s_j^{Y/\Delta \max}, \qquad \text{diag}\left(\ell_j^{\Delta}\right) \;\leq\; \ell_j^{\Delta \max} \tag{14.33g}$$

where the lower and upper bounds are given vectors.

Therefore the conversion rules (13.18) and the internal operational constraints (13.19) of the device models are equivalent to the constraints (14.33). These constraints are local at each bus j. The rank-1 constraints in (14.33a)(14.33b)(14.33f) are nonconvex and the other constraints are convex (or linear). These rank-1 constraints will be omitted to derive a SDP relaxation of the three-phase OPF (13.22).

Let $s \in \mathbb{C}^{N+1}$ denote the terminal power injections and $W \in \mathbb{C}^{3(N+1)\times 3(N+1)}$ denote the terminal variable associated with terminal voltages. Let $u := (u_j, j \in \overline{N})$ denote the internal variables defined by

$$u_j \; := \; \begin{cases} W_j^{Y/\Delta} & \text{if} \quad \text{device } j \text{ is a voltage source} \\ \left(s_j^{Y/\Delta}, X_j^\Delta, \ell_j^\Delta\right) & \text{if} \quad \text{device } j \text{ is a power source} \end{cases}$$

Finally we assume the terminal voltage V_0 at bus 0 is given and imposes the constraint $W_{00} = V_0 V_0^{\mathsf{H}}$. Putting all this together the three-phase OPF (14.28) is equivalent to

$$\min_{(u,s,W)} \quad C_0(u,s,W) \qquad \text{s.t.} \qquad W_{00} \; = \; V_0 V_0^{\mathsf{H}}, \;\; (14.31)(14.32)(14.33) \tag{14.34}$$

where $V_0 \in \mathbb{C}^3$ is given.

14.5.2 SDP relaxation

Define the 6×6 matrix $M(A,B,C)$, as a function of 3×3 Hermitian matrices A, C, and a 3×3 arbitrary matrix B, by

$$M(A,B,C) \;\; := \;\; \begin{bmatrix} A & B \\ B^{\mathsf{H}} & C \end{bmatrix} \tag{14.35}$$

Then $M(A,B,C)$ is Hermitian and the matrix in (14.33f) is $M\left(W_{jj}, X_j^\Delta, \ell_j^\Delta\right)$.

Omitting the rank-1 constraints in (14.33a)(14.33b)(14.33f) yields an SDP relaxation of (14.34):

$$\min_{(u,s,W)} \quad C_0(u,s,W) \tag{14.36a}$$

$$\text{s.t.} \qquad W_{00} \; = \; V_0 V_0^{\mathsf{H}}, \;\; (14.31), \qquad W \succeq 0 \tag{14.36b}$$

$$W_{jj} \; = \; W_j^Y, \qquad W_j^Y \succeq 0, \qquad j \in N_v^Y \tag{14.36c}$$

$$\Gamma W_{jj} \Gamma^{\mathsf{T}} \; = \; W_j^\Delta, \qquad W_j^\Delta \succeq 0, \qquad j \in N_v^\Delta \tag{14.36d}$$

$$s_j \; = \; -s_j^Y, \qquad j \in N_p^Y \tag{14.36e}$$

$$s_j \; = \; -\text{diag}\left(X_j^\Delta \Gamma\right), \qquad j \in N_p^\Delta \tag{14.36f}$$

$$s_j^\Delta \; = \; \text{diag}\left(\Gamma X_j^\Delta\right), \qquad M\left(W_{jj}, X_j^\Delta, \ell_j^\Delta\right) \succeq 0, \qquad j \in N_p^\Delta \tag{14.36g}$$

where $V_0 \in \mathbb{C}^3$ is given and $M\left(W_{jj}, X_j^\Delta, \ell_j^\Delta\right)$ is defined in (14.35).

Let $(u^{\text{opt}}, s^{\text{opt}}, W^{\text{opt}})$ denote an optimal solution of the SDP relaxation (14.36). We say (14.36) is *exact* if the psd matrices of *every* optimal solution $(u^{\text{opt}}, s^{\text{opt}}, W^{\text{opt}})$ are of rank 1, i.e., $\text{rank}\left(W^{\text{opt}}\right) = 1$ and

$$\text{rank}\left(W_j^{Y\text{opt}}\right) = 1, \qquad \text{rank}\left(W_j^{\Delta\text{opt}}\right) = 1, \qquad \text{rank}\left(M\left(W_{jj}^{\text{opt}}, X_j^{\Delta\text{opt}}, \ell_j^{\Delta\text{opt}}\right)\right) = 1 \tag{14.37}$$

If $\text{rank}\left(W^{\text{opt}}\right) = 1$ then all its principal submatrices W_{jj}^{opt} are of rank 1 and therefore, by (14.36c)(14.36d), $W_j^{Y\text{opt}}$ and $W_j^{\Delta\text{opt}}$ are of rank 1 as well. The following result from [131, Lemma 1] implies that, in that case, $X_j^{\Delta\text{opt}}$ is of rank 1, but $\ell_j^{\Delta\text{opt}}$ and hence $M\left(W_{jj}^{\text{opt}}, X_j^{\Delta\text{opt}}, \ell_j^{\Delta\text{opt}}\right)$ may not be of rank 1.

Lemma 14.10 (Rank of $M(A,B,C)$)**.** Consider the matrix $M(A,B,C)$ defined in (14.35). Suppose $M(A,B,C) \succeq 0$ and $\text{rank}(A) = 1$. Then

1. A is psd rank-1.
2. B is rank-1 but may not be psd.
3. C is psd but may not be rank-1.

Hence $M(A,B,C)$ may not be of rank 1. □

Proof. Since $M(A,B,C) \succeq 0$ it can be decompose as

$$M(A,B,C) \; := \; \begin{bmatrix} A & B \\ B^{\mathsf{H}} & C \end{bmatrix} \; = \; \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \begin{bmatrix} M_1^{\mathsf{H}} & M_2^{\mathsf{H}} \end{bmatrix}$$

with $A = M_1 M_1^{\mathsf{H}}$, $B = M_1 M_2^{\mathsf{H}}$, and $C = M_2 M_2^{\mathsf{H}}$. Moreover $M(A,B,C) \succeq 0$ implies that its principle submatrices A and C are both psd. Hence A is psd rank-1.

Therefore there exists a vector x (unique up to a rotation) such that $A = xx^{\mathsf{H}}$. This means that the columns of M_1 are in $\text{span}(x)$, i.e., $M_1 = xy^{\mathsf{H}}$ for some vector y with $y^{\mathsf{H}} y = 1$. Hence $B = M_1 M_2^{\mathsf{H}} = x\left(M_2 y\right)^{\mathsf{H}}$, i.e., B is of rank 1, but not necessarily psd. The psd submatrix C can however take the form $C = zz^{\mathsf{H}} + KK^{\mathsf{H}}$ with $z := M_2 y$ for arbitrary matrix K. The resulting $M(A,B,C)$ remains psd, but not rank-1 unless $K = 0$ or the columns of K are in $\text{span}(z)$. □

Indeed the matrix $M(A,B,C) \succeq 0$ in Lemma 14.10 takes the form

$$M(A,B,C) \; = \; \begin{bmatrix} x \\ z \end{bmatrix} \begin{bmatrix} x^{\mathsf{H}} & z^{\mathsf{H}} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & KK^{\mathsf{H}} \end{bmatrix}$$

for arbitrary matrix K. The lemma has three implications on the exactness of SDP relaxation (14.36). First if there are no Δ-configured power sources, then (14.36) is exact if every optimal solution $(u^{\text{opt}}, s^{\text{opt}}, W^{\text{opt}})$ of (14.36) satisfies $\text{rank}\left(W^{\text{opt}}\right) = 1$. Second if there are Δ-configured power sources in N_p^{Δ}, then $\text{rank}\left(W^{\text{opt}}\right) = 1$ is not sufficient to guarantee exactness because the last condition in (14.37) may not be satisfied. Third, however, any optimal optimal solution $(u^{\text{opt}}, s^{\text{opt}}, W^{\text{opt}})$ with $\text{rank}\left(W^{\text{opt}}\right) = 1$ is sufficient for recovering an optimal solution of OPF (14.34), even if $\ell_j^{\Delta\text{opt}}$ in u_j^{opt} and hence $M\left(W_{jj}^{\text{opt}}, X_j^{\Delta\text{opt}}, \ell_j^{\Delta\text{opt}}\right)$ may not be of rank 1, provided the cost $C_0(u,s,W)$ does not depend on ℓ_j^{Δ} (e.g., C_0 depends only on $\left(s_j, s_j^{Y/\Delta}\right)$) [131, Theorem 1]. This is because Lemma 14.10 guarantees that there exists vectors $\left(V_j^{\text{opt}}, I_j^{\Delta\text{opt}}\right) \in \mathbb{C}^6$ such that, since W^{opt} is psd rank 1 and $M\left(W_{jj}^{\text{opt}}, X_j^{\Delta\text{opt}}, \ell_j^{\Delta\text{opt}}\right) \succeq 0$,

$$W_{jj}^{\text{opt}} = V_j^{\text{opt}}\left(V_j^{\text{opt}}\right)^{\mathsf{H}}, \qquad X_j^{\Delta\text{opt}} = V_j^{\text{opt}}\left(I_j^{\Delta\text{opt}}\right)^{\mathsf{H}}, \qquad j \in N_p^{\Delta} \tag{14.38a}$$

Then consider the point $(\tilde{u}, s^{\text{opt}}, W^{\text{opt}})$ obtained from $(u^{\text{opt}}, s^{\text{opt}}, W^{\text{opt}})$ by replacing $\ell_j^{\Delta\text{opt}}$ in u_j^{opt} by

$$\tilde{\ell}_j^{\Delta} := I_j^{\Delta\text{opt}}\left(I_j^{\Delta\text{opt}}\right)^{\mathsf{H}}, \qquad j \in N_p^{\Delta} \tag{14.38b}$$

It can then be checked that $(\tilde{u}, s^{\text{opt}}, W^{\text{opt}})$ is feasible for OPF (14.34). Since the cost C_0 is independent of $\tilde{\ell}_j^{\Delta}$, $(\tilde{u}, s^{\text{opt}}, W^{\text{opt}})$ is also optimal for OPF (14.34).

Remark 14.6 (Strong exactness)**.** As discussed in Remarks 14.3 and 14.4, even when a relaxation is not exact under our definition, an optimal solution of the original OPF problem may still be recoverable from an optimal solution of its relaxation under certain conditions. Theorems 14.6 and 14.9 provide two such conditions for single-phase radial network. The discussion above shows that $\text{rank}\left(W^{\text{opt}}\right) = 1$ is sufficient for recovering an optimal solution of the original three-phase OPF (14.34) from an optimal solution of its SDP relaxation (14.36) □

The method (14.38) to recover an optimal solution $(\tilde{u}, s^{\text{opt}}, W^{\text{opt}})$ of OPF (14.34) from an optimal solution of its relaxation may not work well in practice because of inevitable numerical errors. Even if W_{jj}^{opt} is close to being rank-1, i.e., its second largest eigenvalue is several orders of magnitude smaller than its largest eigenvalue, X_j^{Δ} can be far from being rank-1, e.g., its largest eigenvalues are multiple and of the same magnitude (see [131, Remark 1]). In this case $I_j^{\Delta\text{opt}}$ may not be obtained from $X_j^{\Delta\text{opt}}$ using (14.38a). Two methods are suggested in [131] to address this numerical issue. The first method substitutes V_j^{opt} obtained from $W_{jj}^{\text{opt}} = V_j^{\text{opt}}\left(V_j^{\text{opt}}\right)^{\mathsf{H}}$ into (14.36g):

$$s_j^{\Delta\text{opt}} = \text{diag}\left(\left(\Gamma V_j^{\Delta\text{opt}}\right)\left(I_j^{\Delta\text{opt}}\right)^{\mathsf{H}}\right) \qquad \Longrightarrow \qquad I_j^{\Delta\text{opt}} := \left(\text{diag}\left(\Gamma \overline{V}_j^{\Delta\text{opt}}\right)\right)^{-1} \overline{s}_j^{\Delta\text{opt}}$$

where for a vector x, $\overline{x}$ is its componentwise complex conjugate and $\text{diag}(x)$ is the diagonal matrix with x as its diagonal entries. The second method adds $\lambda \sum_j \text{tr}\left(\ell_j^{\Delta}\right)$ to the cost function of the SDP relaxation (14.36) for a positive but small weight $\lambda > 0$. This produces an optimal solution in which $\ell_j^{\Delta\text{opt}}$ tends to be of low rank.

14.5.3 Radial network

A special case that is particularly simple is a network where

- all three-phase devices are either voltage or power sources in Y configuration;
- all voltages are defined with respect to the ground and the neutral voltages $\gamma_j^Y := V_j^n$ of all these Y-configured devices are $\gamma_j^Y := 0$.

In this case the internal variables can be simply expressed in terms of terminal variables, $V_j^Y = V_j$, $I_j^Y = -I_j$, and $s_j^Y = -s_j$, and the operational constraints $g_j^{Y/\Delta}(u_j) \leq 0$ on u_j are included in the network constraints (13.24). Hence the internal variable u can be eliminated from the QCQP (13.29) which then consists of only network constraints (13.24) and no device models, as follows:

$$\min_V \quad C_0(V, s(V)) \qquad \text{s.t.} \qquad (13.24) \tag{14.39}$$

We now study the semidefinite relaxation of (14.39) when the network graph G is a tree.

Consider a network graph $G := (\overline{N}, E)$ with $N+1$ buses. Suppose each line $(j,k) \in E$ is characterized by three 3×3 admittance matrices $\left(y_{jk}^s, y_{jk}^m, y_{kj}^m\right)$. Recall its single-phase equivalent circuit described in Chapter 9.1.2 by a graph $G^{3\phi} := \left(\overline{N}^{3\phi}, E^{3\phi}\right)$ where $\overline{N}^{3\phi}$ contains $3(N+1)$ nodes identified by $j\phi$, $j \in \overline{N}$, $\phi \in \{a,b,c\}$. There is a link $(j\phi, k\phi')$ in $E^{3\phi}$ if and only if the $(j\phi, k\phi')$ entry $Y_{jk}^{\phi\phi'}$ of the three-phase admittance matrix Y is nonzero.

Even when G is a tree (i.e., the three-phase network is radial), its single-phase equivalent $G^{3\phi}$ contains cycles. The key observation is that $G^{3\phi}$ is a chordal graph. To see this, note that $G^{3\phi}$ has a maximal clique with 6 nodes consisting of the set $\left\{j\phi, k\phi' \in \overline{N}^{3\phi} : \phi, \phi' \in \{a,b,c\}\right\}$ of buses if and only if (j,k) is a line in G. See Figure 14.7 for an example. Two nodes $j\phi$ and $k\phi'$ in the equivalent circuit $G^{3\phi}$ are adjacent

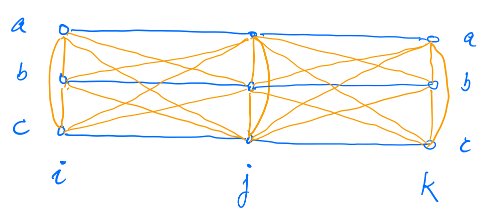


Figure 14.7: The graph $G^{3\phi}$ of the single-phase equivalent circuit of a radial network with three buses i, j, k connected by (three-wire) three-phase lines.

either because of a physical line between buses j and k in the graph G (in which case $\phi = \phi'$) or because of electromagnetic interactions across phases ϕ and ϕ' (in which case $\phi \neq \phi'$). Indeed $G^{3\phi}$ consists of a macro tree in which every link in the macro tree is such a clique and these are the only cliques in $G^{3\phi}$. This means that $G^{3\phi}$ is a chordal graph.

Theorem 14.4 suggests solving the chordal relaxation of (14.39). It computes a $(N+1) \times (N+1)$ Hermitian partial matrix $W_{G^{3\phi}}$:

$$W_{G^{3\phi}} \quad := \quad \left([W_{G^{3\phi}}]_{jj}^{\phi\phi}, j\phi \in \overline{N}^{3\phi}, \;\; [W_{G^{3\phi}}]_{jk}^{\phi\phi'}, (j\phi, k\phi') \in \overline{E}^{3\phi} \right)$$

The set of maximal cliques of $G^{3\phi}$ correspond to the following 6×6 principal submatrices of $W_{G^{3\phi}}$:

$$W_{G^{3\phi}}(j,k) \;=\; \begin{bmatrix} w_{jj} & w_{jk} \\ w_{kj} & w_{kk} \end{bmatrix} \in \mathbb{C}^{6\times 6}, \quad (j,k)\in E$$

where

$$w_{jj} \;:=\; \begin{bmatrix} [W_{G^{3\phi}}]_{jj}^{aa} & [W_{G^{3\phi}}]_{jj}^{ab} & [W_{G^{3\phi}}]_{jj}^{ac} \\ [W_{G^{3\phi}}]_{jj}^{ba} & [W_{G^{3\phi}}]_{jj}^{bb} & [W_{G^{3\phi}}]_{jj}^{bc} \\ [W_{G^{3\phi}}]_{jj}^{ca} & [W_{G^{3\phi}}]_{jj}^{cb} & [W_{G^{3\phi}}]_{jj}^{cc} \end{bmatrix}, \qquad w_{jk} \;:=\; \begin{bmatrix} [W_{G^{3\phi}}]_{jk}^{aa} & [W_{G^{3\phi}}]_{jk}^{ab} & [W_{G^{3\phi}}]_{jk}^{ac} \\ [W_{G^{3\phi}}]_{jk}^{ba} & [W_{G^{3\phi}}]_{jk}^{bb} & [W_{G^{3\phi}}]_{jk}^{bc} \\ [W_{G^{3\phi}}]_{jk}^{ca} & [W_{G^{3\phi}}]_{jk}^{cb} & [W_{G^{3\phi}}]_{jk}^{cc} \end{bmatrix},$$

The chordal relaxation of (14.39) is then (using (13.24)):

$$\min_{W_{G^{3\phi}}} \; \mathrm{tr}\left(C_0 W_{G^{3\phi}}\right) \tag{14.40a}$$

$$\text{s.t.} \quad p_j^{\phi\,\min} \;\le\; \mathrm{tr}\left(\Phi_j^{\phi} W_{G^{3\phi}}\right) \;\le\; p_j^{\phi\,\max}, \qquad j\in\overline{N},\ \phi\in\{a,b,c\} \tag{14.40b}$$

$$q_j^{\phi\,\min} \;\le\; \mathrm{tr}\left(\Psi_j^{\phi} W_{G^{3\phi}}\right) \;\le\; q_j^{\phi\,\max}, \qquad j\in\overline{N},\ \phi\in\{a,b,c\} \tag{14.40c}$$

$$v_j^{\phi\,\min} \;\le\; \mathrm{tr}\left(E_j^{\phi} W_{G^{3\phi}}\right) \;\le\; v_j^{\phi\,\max}, \qquad j\in\overline{N},\ \phi\in\{a,b,c\} \tag{14.40d}$$

$$\mathrm{tr}\left(\hat{Y}_{jk}^{\phi} W_{G^{3\phi}}\right) \;\le\; \ell_{jk}^{\phi\,\max}, \qquad (j,k)\in E,\ \phi\in\{a,b,c\} \tag{14.40e}$$

$$\mathrm{tr}\left(\hat{Y}_{kj}^{\phi} W_{G^{3\phi}}\right) \;\le\; \ell_{kj}^{\phi\,\max}, \qquad (j,k)\in E,\ \phi\in\{a,b,c\} \tag{14.40f}$$

$$W_{G^{3\phi}}(j,k) \;\succeq\; 0, \qquad (j,k)\in E \tag{14.40g}$$

$$w_{00} \;=\; V_0 V_0^{\mathsf{H}} \qquad (V_0 \text{ is given}) \tag{14.40h}$$

Let $W_{G^{3\phi}}^{\text{opt}}$ be an optimal solution of (14.40). If every 6×6 principal submatrix $W_{G^{3\phi}}^{\text{opt}}(j,k)$ of the partial matrix $W_{G^{3\phi}}^{\text{opt}}$ satisfies

$$\mathrm{rank}\left(W_{G^{3\phi}}^{\text{opt}}(j,k)\right) \;=\; 1, \qquad (j,k)\in E$$

then an optimal solution V^{opt} of (14.39) can be uniquely recovered from $W_{G^{3\phi}}^{\text{opt}}$ according to Theorem 14.3. This is because a chordal relaxation is exact if and only if the principle submatrix $W_{G^{3\phi}}^{\text{opt}}(q)$ of $W_{G^{3\phi}}^{\text{opt}}$ is psd rank-1 for every clique q of the chordal graph $G^{3\phi}$ (Theorem 14.1) and, as noted above, the only maximal cliques of $G^{3\phi}$ are those 6-node cliques corresponding to lines $(j,k)\in E$.

The method in Chapter 14.1.4 to recover an optimal V^{opt} from $W_{G^{3\phi}}^{\text{opt}}$ applies directly here. Since $\mathrm{rank}\left(W_{G^{3\phi}}^{\text{opt}}(j,k)\right)=1$ for all $(j,k)\in E$, they satisfy the cycle condition (Theorem 14.1). Take any spanning tree of $G^{3\phi}$ with root at, say, node $0a$. Let $\left|V_j^{\phi}\right| := \sqrt{[W_{G^{3\phi}}^{\text{opt}}]_{jj}^{\phi\phi}}$ for $j\in N, \phi\in\{a,b,c\}$. Let $\mathbb{P}_j^{\phi}$ be the unique path from the root $0a$ to the node $j\phi$. A link $(j'\phi', j''\phi'')$ in the path $\mathbb{P}_j^{\phi}$ is denoted by $(j'\phi', j''\phi'')\in\mathbb{P}_j^{\phi}$. Then for all nodes $j\phi$ in the equivalent single-phase network $G^{3\phi}$,

$$\angle V_j^{\phi} \;\;:=\;\; \angle V_0^{a} \;-\; \sum_{(j'\phi', j''\phi'')\in\mathbb{P}_j^{\phi}} \angle\left[W_{G^{3\phi}}^{\text{opt}}\right]_{j'j''}^{\phi'\phi''} \mod 2\pi$$

14.6 Conditions for global optimality

Even though OPF is NP hard in general, heuristics seem to work very well in practice in that its semidefinite relaxation tends to be exact (e.g. see simulation results in [132]) and local algorithms such as Newton-Raphson and interior-point methods tend to produce globally optimal solutions. Conditions are studied in [133] for such nonconvex problems to simultaneously have exact convex relaxation and no spurious local optima. These conditions help explain the empirical experience that local algorithms for OPF tend to work very well in practice.

14.7 Bibliographical notes

Solving OPF through semidefinite relaxation in the bus injection model is first proposed in [134] as a second-order cone program (SOCP) for radial (tree) networks and in [135] as a semidefinite program (SDP) for general networks. The exactness of semidefinite relaxations is first studied in [104]. By defining a new set of variables $v_j := |V_j|^2$, $R_{jk} := |V_j||V_k|\cos(\theta_j - \theta_k)$, and $I_{jk} := |V_j||V_k|\sin(\theta_j - \theta_k)$ where $\theta_j := \angle V_j$, [134] rewrites the bus injection model (4.22) in the polar form as a set of linear equations in these new variables and the following quadratic equations:

$$v_j v_k \quad = \quad R_{jk}^2 + I_{jk}^2$$

Relaxing these equalities to $v_j v_k \geq R_{jk}^2 + I_{jk}^2$ enlarges the solution set to a second-order cone that is equivalent to $\mathbb{W}_G^+$ in this chapter. Partial matrices and their completions are studied in [116, 117, 118]. Exploiting graph sparsity to simplify the SDP relaxation of OPF through chordal extension is first proposed in [136, 137, 138] and analyzed in [139, 140, 32]. Theorem 14.1 is from [32] and Corollary 14.2 is from [139]). The sufficient condition on angle differences for exact SOCP relaxation in Chapter 14.4 is from [126, 129] and our proof mostly follows that in [129]. The result in Chapter 14.4 assumes the voltage magnitudes are fixed and ignores reactive powers. These assumptions are relaxed in [130] although, without these assumptions, the feasible set may no longer be a convex surface that is the Pareto front of its relaxation.

The semidefinite relaxation of three-phase OPF in Chapter 14.5 follows the idea in [106, 131].

Simulations [132] show that the SDP relaxation of OPF is often exact and adding valid inequalities and bound tightening can further reduce the optimality gap to within 1%, though [119] also reports instances where the optimality gap of SDP relaxation is large. The results in Chapter 14.6 are from [133].

14.8 Appendix: Proof of Theorem 14.9: tree network

We prove the theorem when the network graph is a tree, following [129].

Let $\mathbb{F}_\theta^{jk}$ denote the set of branch power flows on each line $(j,k) \in E$:

$$\mathbb{F}_\theta^{jk} \quad := \quad \{\ (P_{jk}, P_{kj}) \mid (P_{jk}, P_{kj}) \text{ satisfies (14.24) for } \underline{\theta}_{jk} \leq \theta_{jk} \leq \overline{\theta}_{jk}\ \}$$

Since the network is a tree, the set $\mathbb{F}_\theta$ of branch power flows on all lines is simply the product set:

$$\begin{aligned}\mathbb{F}_\theta &:= \{P := (P_{jk}, P_{kj}, (j,k) \in E) \mid P \text{ satisfies (14.24) for } \underline{\theta}_{jk} \leq \theta_{jk} \leq \overline{\theta}_{jk}, (j,k) \in E \ \} \\ &= \prod_{(j,k) \in E} \mathbb{F}_\theta^{jk} \end{aligned} \tag{14.41}$$

because given any $(\theta_{jk}, (j,k) \in E)$ there is always a (unique) $(\theta_j, j \in \overline{N})$ that satisfies $\theta_{jk} = \theta_j - \theta_k$. (This is equivalent to the cycle condition (**??**).) If the network has cycles then this is not possible for some vectors $(\theta_{jk}, (j,k) \in E)$ and $\mathbb{F}_\theta$ is no longer a product set of $\mathbb{F}_\theta^{jk}$.

Since the power injections p are related to the branch flows P by $p_j = \sum_{k:j\sim k} P_{jk}$, the injection region $\mathbb{P}_\theta$ in (14.22) is a linear transformation of $\mathbb{F}_\theta$:

$$\mathbb{P}_\theta \;=\; A\mathbb{F}_\theta$$

for some $(N+1) \times 2M$ dimensional matrix A. Matrix A has full row rank and it can be argued that there is a bijection between P_θ and F_θ using the fact that the graph is a tree. We can therefore freely work with either $p \in \mathbb{P}_\theta$ or the corresponding $P \in \mathbb{F}_\theta$.

To prove the second assertion of Theorem 14.9, note that the argument for the two-bus case shows that $\text{conv}(\mathbb{F}_\theta^{jk})$, $(j,k) \in E$, is the intersection of a second-order cone with an affine set. This property, together with Lemma 14.11 below, the fact that $\mathbb{F}_\theta$ is a direct product of $\mathbb{F}_\theta^{jk}$, and the fact that A is of full rank, imply that $\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p$ is the intersection of a second-order cone with an affine set. Hence (14.23) is indeed an SOCP for a tree network. Therefore it suffices to prove the first assertion of Theorem 14.9:

$$\mathbb{P}_\theta \cap \mathbb{P}_p \;=\; \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p) \tag{14.42}$$

because it implies that, under C14.3, *every* minimizer of OPF-socp (14.23) lies in its Pareto front and hence is feasible and optimal for OPF (14.22). Hence SOCP relaxation is exact.

We are hence left to prove (14.42). Half of the equality follows from the following simple properties of Pareto front and convex hull (Their proof is left as Exercise 14.10).

Lemma 14.11. Let $\mathbb{B}, \mathbb{C} \subseteq \mathbb{R}^k$ be arbitrary sets, $\mathbb{D} := \{x \in \mathbb{R}^k | Mx \leq c\}$ be an affine set, and M a matrix and b a vector of appropriate dimensions.

(1) $\text{conv}(M\mathbb{B}) = M\,\text{conv}(\mathbb{B})$ and $\text{conv}(\mathbb{B} \times \mathbb{C}) = \text{conv}(\mathbb{B}) \times \text{conv}(\mathbb{C})$ where for any sets $A_1, A_2 \subseteq \mathbb{R}^k$, $(x_1, x_2) \in A_1 \times A_2$ if and only if $x_1 \in A_1$ and $x_2 \in A_2$.

(2) Suppose $\mathbb{B}$ and $\mathbb{C}$ are convex and a point is Pareto optimal over a set if and only if it minimizes $c^\mathsf{T} x$ over the set for some $c > 0$.[3] Then $\mathbb{O}(M\mathbb{B}) = M\,\mathbb{O}(\mathbb{B})$ and $\mathbb{O}(\mathbb{B} \times \mathbb{C}) = \mathbb{O}(\mathbb{B}) \times \mathbb{O}(\mathbb{C})$.

(3) If $\mathbb{B} = \mathbb{O}(\text{conv}\ \mathbb{B})$ then $\mathbb{B} \cap \mathbb{D} \subseteq \mathbb{O}(\text{conv}(\mathbb{B}) \cap \mathbb{D})$.

The next lemma says that the feasible set of OPF (14.22) is a subset of the feasible set of its SOCP relaxation (14.23).

[3] In general, a point is Pareto optimal over a convex set if and only if it minimizes $c^\mathsf{T} x$ over the set for some nonzero $c \geq 0$, as opposed to $c > 0$. In that case, $\mathbb{O}(\mathbb{B} \times \mathbb{C}) \supseteq \mathbb{O}(\mathbb{B}) \times \mathbb{O}(\mathbb{C})$; c.f. Exercise 14.9.

Lemma 14.12. $\mathbb{P}_\theta \cap \mathbb{P}_p \subseteq \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$.

Proof of Lemma 14.12. Recall that $F_\theta = \prod_{(j,k)\in E} \mathbb{F}_\theta^{jk}$ and $\mathbb{P}_\theta = A\mathbb{F}_\theta$. Lemma 14.11(1) implies

$$\text{conv}(\mathbb{P}_\theta)) \;=\; \text{conv}(A\,\mathbb{F}_\theta)) \;=\; A\,\text{conv}(\mathbb{F}_\theta) \;=\; A \prod_{(j,k)\in E} \text{conv}\left(\mathbb{F}_\theta^{jk}\right)$$

Hence their Pareto fronts satisfy

$$\mathbb{O}(\text{conv}(\mathbb{P}_\theta)) \;=\; \mathbb{O}\left(A \prod_{(j,k)\in E} \text{conv}\left(\mathbb{F}_\theta^{jk}\right)\right) \;=\; A \prod_{(j,k)\in E} \mathbb{O}\left(\text{conv}\left(\mathbb{F}_\theta^{jk}\right)\right) \;=\; A \prod_{(j,k)\in E} \mathbb{F}_\theta^{jk} \;=\; \mathbb{P}_\theta$$

where the second equality follows from Lemma 14.11(2), and the third equality follows from (14.26) where $\mathbb{F}_\theta^{jk}$ plays the role of $\mathbb{P}_\theta$. Lemma 14.11(3) then implies the lemma. □

Lemma 14.12 means that every optimal solution of OPF (14.22) is an optimal solution of its SOCP (14.23). For exactness of OPF-socp (14.23) we need the converse to hold as well. The remainder of the proof is to show this is indeed true, proving (14.42).

Lemma 14.13. $\mathbb{P}_\theta \cap \mathbb{P}_p \supseteq \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$.

The proof of Lemma 14.12 shows that $\mathbb{P}_\theta = \mathbb{O}(\text{conv}(\mathbb{P}_\theta))$, so the converse of Lemma 14.11(3) would imply Lemma 14.13. Figure 14.8 and the explanation in its caption, however, illustrate why the converse of Lemma 14.11(3) generally does not hold. To prove Lemma 14.13 we need to exploit the structure of $\mathbb{P}_\theta, \mathbb{F}_\theta, \mathbb{P}_p$.

Proof of Lemma 14.13. Take any point $p \in \mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p)$. We now show that $p \in \mathbb{P}_\theta \cap P_p$. By definition of Pareto optimality, p is a minimizer of

$$\min_{\hat{p}\in\text{conv}(\mathbb{P}_\theta)} \; c^{\mathsf{T}}\hat{p} \quad \text{subject to} \quad \underline{p} \;\le\; \hat{p} \;\le\; \overline{p}$$

for some $c > 0$. This minimization is equivalent to:

$$\begin{aligned}
\min_{\alpha_j, \hat{p}_j} \quad & c^{\mathsf{T}} \sum_j \alpha_j \hat{p}_j \\
\text{subject to} \quad & \alpha_j \;\ge\; 0,\; \sum_j \alpha_j \;=\; 1,\;\; \hat{p}_j \in \mathbb{P}_\theta \\
& \underline{p} \;\le\; \sum_j \alpha_j \hat{p}_j \;\le\; \overline{p}
\end{aligned}$$

We can uniquely express p and p_j in terms of branch flows in $\mathbb{F}_\theta$, $p = AP$ and $\hat{p}_j = A\hat{P}_j$. Then P is in $\text{conv}(\mathbb{F}_\theta)$ and a minimizer of

$$\min_{\hat{P}\in\text{conv}(\mathbb{F}_\theta)} \; c^{\mathsf{T}} A\hat{P} \quad \text{subject to} \quad \underline{p} \;\le\; A\hat{P} \;\le\; \overline{p}$$

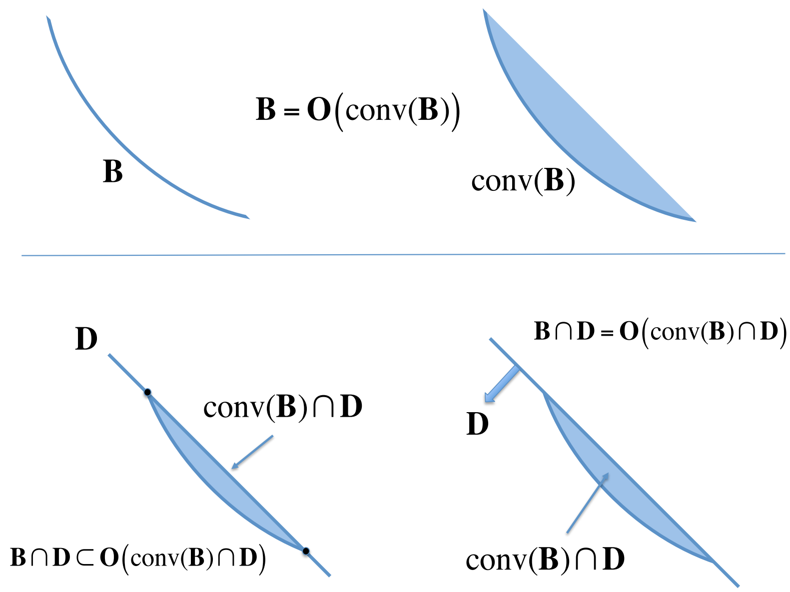


Figure 14.8: The upper panel shows a set $\mathbb{B}$ and its convex hull $\text{conv}(\mathbb{B})$ with the property that $\mathbb{B} = \mathbb{O}(\text{conv}(\mathbb{B}))$. The lower panel shows two affine sets $\mathbb{D}$. On the left $\mathbb{D}$ is a hyperplane; $\mathbb{B} \cap \mathbb{D}$ consists of two intersection points and is a strict subset of $\mathbb{O}(\text{conv}(\mathbb{B}) \cap \mathbb{D})$. On the right $\mathbb{D}$ is a halfspace and $\mathbb{B} \cap \mathbb{D} = \mathbb{O}(\text{conv}(\mathbb{B}) \cap \mathbb{D})$.

It suffices to prove that $P \in \mathbb{F}_\theta$, which then implies that $p = AP \in \mathbb{P}_\theta \cap P_p$.

The Slater's condition holds for OPF (14.22). By strong duality there exist Lagrange multipliers $\overline{\lambda} \geq 0$ and $\underline{\lambda} \geq 0$ such that P is a minimizer of the Lagrangian:

$$\min_{\hat{P} \in \text{conv}(\mathbb{F}_\theta)} \quad \left(c^\mathsf{T} + \overline{\lambda}^\mathsf{T} - \underline{\lambda}^\mathsf{T}\right) A\hat{P} \;-\; \overline{\lambda}^\mathsf{T}\overline{p} \;+\; \underline{\lambda}^\mathsf{T}\underline{p} \tag{14.43}$$

If $\overline{c} := c^\mathsf{T} + \overline{\lambda}^\mathsf{T} - \underline{\lambda}^\mathsf{T} \geq 0$ and is nonzero then $P \in \mathbb{F}_\theta$ since $\mathbb{O}(\text{conv}(\mathbb{F}_\theta)) = \mathbb{F}_\theta$. We are left to deal with the case where either $\overline{c} = 0$ (in which case every point in $\text{conv}(\mathbb{F}_\theta)$ is Pareto optimal) or there exists a j such that $\overline{c}_j < 0$.

Since $\mathbb{F}_\theta = \prod_{(j,k)\in E} \mathbb{F}_\theta^{jk}$, $P \in \mathbb{F}_\theta$ if and only if $(P_{jk}, P_{kj}) \in \mathbb{F}_\theta^{jk}$. Moreover (14.43) becomes separable by Lemma 14.11(1):

$$\min_{\hat{P} \in \text{conv}(\mathbb{F}_\theta)} \sum_{j\in \overline{N}} \overline{c}_j \sum_{k: j\sim k} \hat{P}_{jk} \;\equiv\; \sum_{(j,k)\in E} \min_{(\hat{P}_{jk},\hat{P}_{kj})\in \text{conv}(\mathbb{F}_\theta^{jk})} \left(\overline{c}_j \hat{P}_{jk} + \overline{c}_k \hat{P}_{kj}\right)$$

This reduces the problem to the two-bus case:

$$\min_{(\hat{P}_{jk},\hat{P}_{kj})\in \text{conv}(\mathbb{F}_\theta^{jk})} \left(\overline{c}_j \hat{P}_{jk} + \overline{c}_k \hat{P}_{kj}\right)$$

If either $\overline{c}_j > 0$ or $\overline{c}_k > 0$ then it can be seen from Figures 14.3 that the minimizer (P_{jk}, P_{kj}) is in $\mathbb{F}_\theta^{jk}$. We now show that $(P_{jk}, P_{kj}) \in \mathbb{F}_\theta^{jk}$ even when both $\overline{c}_j \leq 0$ and $\overline{c}_k \leq 0$.

Since $c > 0$, any node i with $\bar{c}_i \le 0$ has $\underline{\lambda}_i > 0$ and hence $p_i = \underline{p}_i$. Consider the biggest subtree T that contains link (j,k) in which every node i has $\bar{c}_i \le 0$ and $p_i = \underline{p}_i$. Call a node l in the subtree T a *boundary node* if it is a leaf or connected to another node l' outside T where $\bar{c}_{l'} > 0$. Without loss of generality, take one of the boundary nodes as the root of the network graph and assume this is node 0. For each line (l,i) in the graph, node i is called the *parent* of node l if i lies in the unique path from l to the root node 0.

Lemma 14.14. $(P_{li}, P_{il}) \in \mathbb{F}_\theta^{li}$ for every link (l,i) in the subtree T.

Proof of Lemma 14.14. Consider any $\tilde{P}$ that satisfies $(\tilde{P}_{li}, \tilde{P}_{il}) \in \mathbb{F}_\theta^{li}$ for every link $(l,i) \in T$ and $\underline{p} \le \tilde{p} = A\tilde{P} \le \bar{p}$. We will first prove that, for every link $(l,i) \in T$,

$$P_{li} \;\le\; \tilde{P}_{li} \quad \text{and} \quad P_{il} \;\ge\; \tilde{P}_{il} \tag{14.44}$$

We then use this to prove that $(P_{li}, P_{il}) = (\tilde{P}_{li}, \tilde{P}_{il}) \in \mathbb{F}_\theta^{li}$.

Consider first a boundary node l. If l is a leaf node then, since $\bar{c}_l \le 0$, $P_{li} = p_l = \underline{p}_l \le \tilde{p}_l = \tilde{P}_{li}$. Then, since $P_{li} \in \mathrm{conv}(\mathbb{F}_\theta^{li})$ and $\tilde{P}_{li} \in \mathbb{F}_\theta^{li}$, we have $P_{il} \ge \tilde{P}_{il}$; see Figure 14.9(a). Otherwise let l' outside T be a neighbor of l. Since $\bar{c}_l \le 0$ but $\bar{c}_{l'} > 0$, the minimization of $\bar{c}_{l'}\hat{P}_{l'l} + \bar{c}_l\hat{P}_{ll'}$ over $\mathrm{conv}(\mathbb{F}_\theta^{ll'})$ means $P_{ll'} = \underline{\pi}_{ll'}$; see Figure 14.9(b). Hence $P_{ll'} = \underline{\pi}_{ll'} \ge \tilde{P}_{ll'}$. This holds for all neighbors l' of l. Hence

$$P_{li} \;=\; p_l - \sum_{l'} P_{ll'} \;\;\le\;\; \tilde{p}_l - \sum_{l'} \tilde{P}_{ll'} \;=\; \tilde{P}_{li}$$

where l' ranges over all neighbors (outside T) of l except its parent i in T. From the region of possible values for (P_{li}, P_{il}) in Figure 14.9(c), we conclude that $P_{il} \ge \tilde{P}_{il}$. Hence the claim is true for all links (l,i) where l is a boundary node.

Consider node i one hop away from a boundary node towards root note 0 and let its parent be node h; see Figure 14.9(d). The above argument says that $P_{il} \ge \tilde{P}_{il}$ for all neighbors l of i except its parent h. This together with $p_i = \underline{p}_i$ (since $\bar{c}_i \le 0$) implies

$$P_{ih} \;=\; p_i - \sum_{l} P_{il} \;\;\le\;\; \tilde{p}_i - \sum_{l} \tilde{P}_{il} \;=\; \tilde{P}_{ih}$$

and hence as before $P_{hi} \ge \tilde{P}_{hi}$. Propagate towards the root node 0 and (14.44) follows by induction.

We now use (14.44) to show that $(P_{li}, P_{il}) \in \mathbb{F}_\theta^{li}$ for every link (l,i) in the subtree T. Now (14.44) implies that $P_{0l} \ge \tilde{P}_{0l}$ for all neighbors l of 0. Since node 0 has no parent, we have

$$\sum_{l} P_{0l} \;=\; p_0 \;=\; \underline{p}_0 \;\;\le\;\; \tilde{p}_0 \;=\; \sum_{l} \tilde{P}_{0l}$$

implying $P_{0l} = \tilde{P}_{0l}$ for all neighbors l of node 0. This implies $P_{l0} = \tilde{P}_{l0}$; see Figure 14.9(a) and (c). Repeat this argument propagating from node 0 towards the boundary nodes of the subtree T, and we conclude that $(P_{li}, P_{il}) = (\tilde{P}_{li}, \tilde{P}_{il}) \in \mathbb{F}_\theta^{li}$ for every link (l,i) in T. This completes the proof of Lemma 14.14. □

This completes the proof of Lemma 14.13. □

This completes the proof of Theorem 14.9.

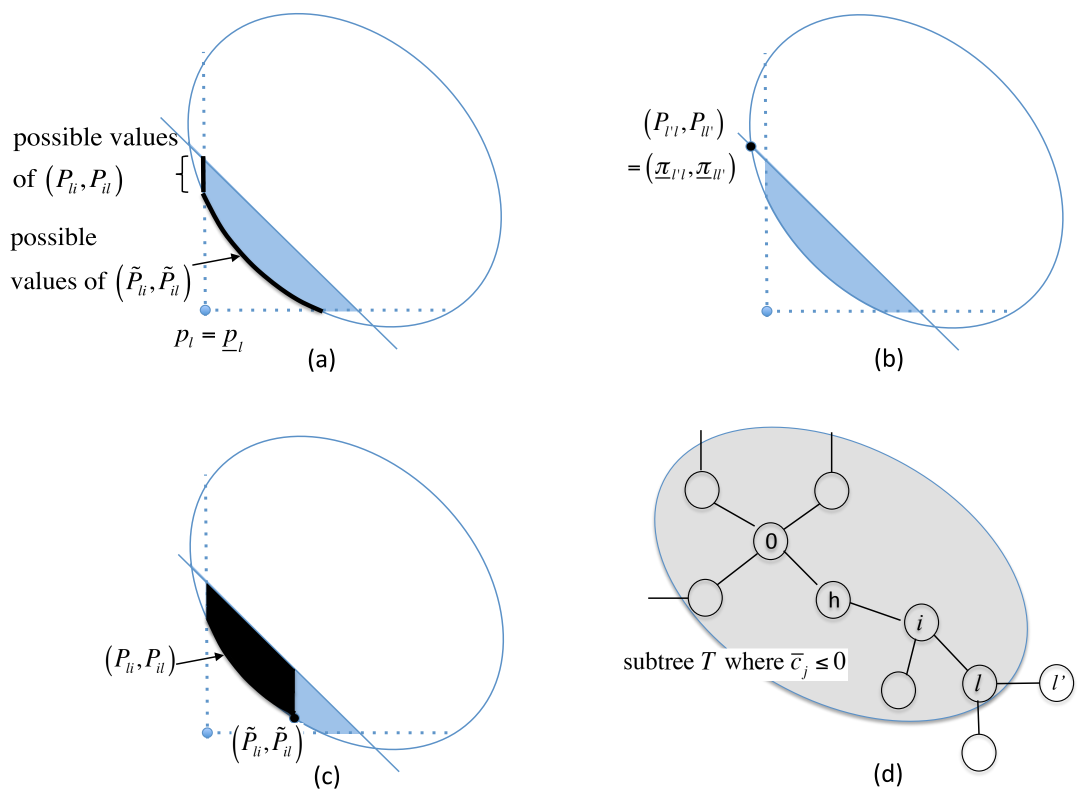


Figure 14.9: Illustration for the proof of Lemma 14.14.

14.9 Problems

Chapter 14.1

Exercise 14.1 (Solution recovery)**.** Given a partial matrix X_F in $\mathbb{X}_F$ defined in (14.7c), i.e., X_F satisfies

$$\begin{aligned} \mathrm{tr}(C_l X_F) \ &\leq \ b_l, \qquad l = 1, \ldots, L \\ X_F(j,k) \succeq 0, \ \mathrm{rank}(X_F(j,k)) &= 1, \qquad (j,k) \in E \\ \sum_{(j,k)\in c} \angle [X_F]_{jk} \ &= \ 0 \mod 2\pi, \qquad \text{all cycles } c \text{ in } F \end{aligned}$$

compute an $x \in \mathbb{C}^n$ as follows. Let T be an arbitrary spanning tree of F (assumed connected) rooted at node 1. Let $\mathbb{P}_j$ denote the unique path from node 1 to node j in T (with orientation pointing away from node 1). Set $|x_1| := \sqrt{[X_F]_{11}}$ and $\angle x_1$ to an arbitrary value. For $j = 2, \ldots, n$,

$$|x_j| \ := \ \sqrt{[X_F]_{jj}}, \qquad\qquad \angle x_j \ := \ \angle V_1 \ - \sum_{(i,k)\in\mathbb{P}_j} \angle [X_F]_{ik}$$

Prove that $x \in \mathbb{C}^n$ satisfies $[X_F]_{jk} = x_j \bar{x}_k$ and $x^{\mathsf{H}} C_l x \leq b_l$, $l = 1, \ldots, L$.

Chapter 14.2

Exercise 14.2 (Loss minimization and load power factor)**.** Consider a generator supplies a load through a transmission line. Let the complex voltage at the generator (reference) bus be fixed at $V_0 := 1\angle 0°$ p.u. and the transmission line be modeled as a series admittance $y := g - \mathbf{i}b$. Let the required load power be $s = p + \mathbf{i}q = |s|e^{\mathbf{i}\phi}$ with $p > 0$ specified, i.e., $-s$ is the power injection from the load bus. Let the load voltage be $V := ve^{\mathbf{i}\theta}$. The load current I is equal to the current through the transmission line Z since the line charging current is assumed zero. In this problem, we let v (and p) be specified and treat (θ, ϕ) as variables.

1. Show that the active line loss $r|I|^2 = g\left|1 - ve^{\mathbf{i}\theta}\right|^2$ where r is the line resistance.

2. Formulate the OPF problem that minimizes active line loss.

3. Solve the OPF and show that the active line loss is uniquely minimized when the reactive power is $p\tan\phi_{\min} = b(1 - v^2)$.

4. Now let v (in addition to θ, ϕ) be an optimization variable constrained to $v \in [v - \varepsilon, v + \varepsilon]$ and solve the OPF problem.

Since $v \approx 1$, the optimal reactive power $p\tan\phi_{\min} = b(1 - v^2) \approx 0$.

Solution 14.1. The current I is given by $I = y(V_0 - V)$ and hence

$$r|I|^2 \;=\; \mathrm{Re}\left(z\,|y|^2\,|V_0 - V|^2\right) \;=\; \mathrm{Re}\left(\frac{1}{z^*}\,|V_0 - V|^2\right) \;=\; g\,|1 - ve^{\mathbf{i}\theta}|^2$$

Therefore the OPF problem is

$$\min_{\theta,\phi}\; g\,|1 - ve^{\mathbf{i}\theta}|^2 \qquad \text{s. t.} \quad -s \;=\; y^*\left(v^2 - ve^{\mathbf{i}\theta}\right)$$

To eliminate θ, use the equality constraint to express $e^{\mathbf{i}\theta}$ in terms of ϕ:

$$y^*\left(v^2 - ve^{\mathbf{i}\theta}\right) \;=\; -s \;=\; -\frac{p}{\cos\phi}\,e^{\mathbf{i}\phi} \;=\; -p(1 + \mathbf{i}\tan\phi)$$

Hence $ve^{\mathbf{i}\theta} = v^2 \,+\, pz^*(1 + \mathbf{i}\tan\phi)$. Substituting into the cost function we have the active line loss as

$$\begin{aligned}
f(\phi) \;&:=\; g\left|1 - v^2 - p\,(r - \mathbf{i}x)(1 + \mathbf{i}\tan\phi)\right|^2 \;=\; g\left|\left(1 - v^2 - p(r + x\tan\phi)\right) \,+\, \mathbf{i}p(x - r\tan\phi)\right|^2 \\
&=\; g\left(\left(1 - v^2 - p(r + x\tan\phi)\right)^2 \,+\, p^2(x - r\tan\phi)^2\right)
\end{aligned}$$

Then (using $b = x/(r^2 + x^2)$ and $g(r^2 + x^2) = r$)

$$f'(\phi) \;=\; 2pg\sec^2\phi\,\left(p(r^2 + x^2)\tan\phi \,-\, x(1 - v^2)\right) \;=\; 2pr\sec^2\phi\,\left(p\tan\phi \,-\, b(1 - v^2)\right)$$

Therefore $f'(\phi_{\min}) = 0$ if and only if $\phi_{\min}$ satisfies

$$p\tan\phi_{\min} \;=\; b(1 - v^2)$$

Between $(-\pi/2, \pi/2)$ this $\phi_{\min}$ is unique. Moreover $f'(\phi) < 0$ for $\phi < \phi_{\min}$ and $f'(\phi) > 0$ for $\phi > \phi_{\min}$, implying that $\phi_{\min}$ is the unique minimizer of the line loss $f(\phi)$. Note that $p\tan\phi$ is the reactive load power.

When v is also a variable, the loss becomes $f(\phi, v)$. Simple analysis shows that there is a unique optimal $v_{\min}$ that minimizes f. The first order optimality condition provides two nonlinear equations in (θ, v).

Chapter 14.3 The linear separability condition C14.2' requires that some of power injections be unconstrained even though in practice they are always bounded. The next exercise discusses under what conditions can C14.2' be interpreted as requiring that the bounds on these power injections be inactive at optimality, as opposed to requiring that the optimal solutions obtained by ignoring these bounds turn out to satisfy these bounds.

Exercise 14.3 (Linear separability)**.** Consider the two problems:

$$\hat{x} \;\in\; \arg\min_{x\in X}\; f(x) \tag{14.45a}$$

$$x^* \;\in\; \arg\min_{x\in X}\; f(x) \quad \text{s. t.} \quad g(x) \;\le\; 0 \tag{14.45b}$$

where $X \subseteq \mathbb{R}^n$ is convex and $g : \mathbb{R}^n \to \mathbb{R}^m$ is a convex function.

1. Suppose f is strictly convex. Show that $g(\hat{x}) < 0$ if and only if $g(x^*) < 0$ in which case $f(\hat{x}) = f(x^*)$.

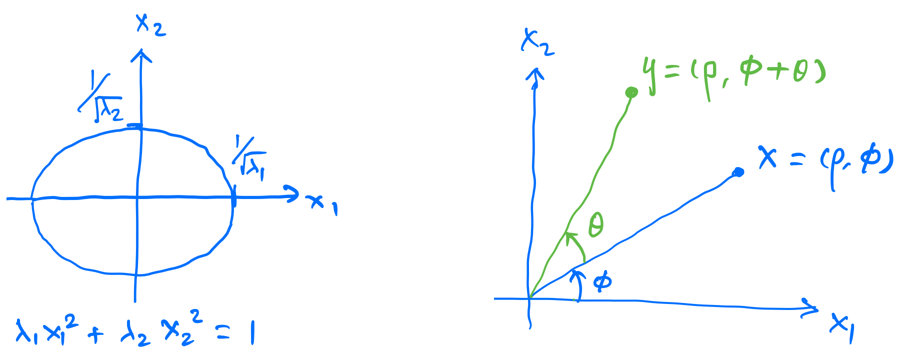
2. Show that if f is nonconvex, then it is possible that both $g(x^*) < 0$ and $g(\hat{x}) > 0$ hold in which case $f(\hat{x}) < f(x^*)$.

Chapter 14.4 The next few problems use a two-bus example to illustrate the geometry of solutions to the polar form power flow equations, convex relaxation and its exactness [126, 129].

Exercise 14.4 (Ellipsoid)**.** An ellipsoid in $\mathbb{R}^k$ (without the interior) in standard form are the points $x \in \mathbb{R}^k$ that satisfy

$$x^{\mathsf{T}} \Lambda x \;=\; 1 \tag{14.46a}$$

for a real positive definite diagonal matrix $\Lambda \succ 0$. The center of the ellipsoid is the origin 0 and the k principal axes are the coordinate axes. This is illustrated in Figure 14.10 for $k = 2$. In general the set of



(a) Ellipsoid in $\mathbb{R}^2$ (b) Rotation in $\mathbb{R}^2$

Figure 14.10: Exercise 14.4.

points $x \in \mathbb{R}^k$ that satisfy

$$(x-c)^{\mathsf{T}} M (x-c) \;=\; \left\| M^{1/2}(x-c) \right\|_2^2 \;=\; 1 \tag{14.46b}$$

is an ellipse if $c \in \mathbb{R}^n$ and $M \succ 0$ is a real (symmetric) positive definite matrix. The center of the ellipsoid is c and the k principal axes are the k eigenvectors of M. In this exercise, we show that a general ellipsoid (14.46b) can be obtained through simple transformations of the standard form ellipsoid (14.46a).

Given a standard form ellipsoid $x \in \mathbb{R}^k$ that satisfies (14.46a).

1. *Translation*: Let $y := x + x_0 \in \mathbb{R}^k$. Show that y is a standard form ellipsoid with its center translated to x_0. Illustrate y for $k = 2$.

2. *Scaling*: Let $y := ax$ where $a \in \mathbb{R}$ is nonzero. Show that y is a standard form ellipsoid with its size scaled by a in all the k dimensions. Illustrate y for $k = 2$.

3. *Scaling and rotation*: Let $y := Ax$. Show that y is an ellipsoid as long as A is real and invertible, i.e., y satisfies (14.46b) with a real (symmetric) positive definite matrix M.

4. *Inverse scaling and rotation*: Show that a general ellipsoid y that satisfies (14.46b) with the origin $c = 0$ as its center is a standard form ellipsoid x scaled and rotated by a matrix U, i.e., $y = Ux$. Derive U.

Exercise 14.5 (Rotation in $\mathbb{R}^2$)**.** Show that $y = R(\theta)x$ is a rotation of x by an angle θ in $\mathbb{R}^2$ where

$$R(\theta) \quad := \quad \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

as illustrated in Figure 14.11.

1. Show that $R^{-1}(\theta) = R(-\theta) = R^{\mathsf{T}}(\theta)$.

2. Show that $R(\theta)$ is normal and find its spectral decomposition for $\theta \neq 0$.

3. Suppose x is a standard form ellipse in $\mathbb{R}^2$ that satisfies (14.46a). Show that $y := R(\theta)x$ is an ellipse, i.e., y satisfies (14.46b) with a real (symmetric) positive definite matrix M.

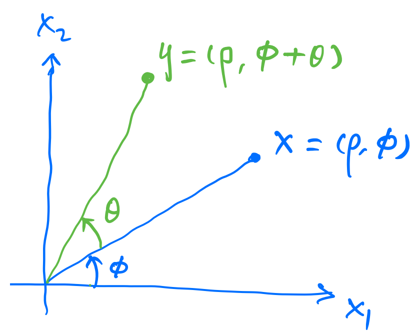


Figure 14.11: Exercise 14.5.

Exercise 14.6 (Geometric insight [126, 129])**.** Show that (14.24) can be rewritten as

$$1 = \left\| \begin{bmatrix} \cos\theta_{jk} \\ \sin\theta_{jk} \end{bmatrix} \right\|^2 \quad = \quad \hat{P}^{\mathsf{T}} \begin{bmatrix} \frac{1}{b_{jk}^2} & 0 \\ 0 & \frac{1}{g_{jk}^2} \end{bmatrix} \hat{P} \tag{14.47}$$

where $\hat{P} \in \mathbb{R}^2$ is related to $P = (P_{jk}, P_{kj})$ by

$$\begin{bmatrix} P_{jk} \\ P_{kj} \end{bmatrix} \quad = \quad \sqrt{2} \begin{bmatrix} \cos 45^\circ & \sin 45^\circ \\ -\sin 45^\circ & \cos 45^\circ \end{bmatrix} \cdot \hat{P} \; + \; \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This says that $\hat{P}$ defined by (14.47) is a standard form ellipse centered at the origin with its major axis of length $2b_{jk}$ on the x-axis and its minor axis of length $2g_{jk}$ on the y-axis. P is the ellipse obtained from $\hat{P}$ by scaling it by $\sqrt{2}$, rotating it by -45°, and shifting its center to (g_{jk}, g_{jk}), as shown in Figure **??**.

Exercise 14.7 (Geometric insight [126, 129])**.** Show that the two-bus network given by (14.24), reproduced here with subscript jk dropped:

$$p_1 \;=\; p_1(\theta) \;:=\; g - g\cos\theta - b\sin\theta \tag{14.48a}$$
$$p_2 \;=\; p_2(\theta) \;:=\; g - g\cos\theta + b\sin\theta \tag{14.48b}$$

We have shown that (p_1, p_2) forms an ellipse. Draw the ellipse and indicate on the ellipse values for θ where p_1 and p_2 attain minimum or maximum values. Conclude that the "lower half" of the ellipse corresponds to small $|\theta|$ and the "upper half" corresponds to large $|\theta|$.

Exercise 14.8 (Geometric insight [126, 129])**.** Consider the 2-bus network in Exercise 14.7. Let $x := (p_1, p_2, \theta)$. Let $c(p_1, p_2)$ be a cost function that is strictly increasing in (p_1, p_2), e.g., $c(p_1, p_2) := p_1 + p_2$.

1. Consider the OPF problem:

$$\min_x \; c(p_1, p_2) \quad \text{s.t.} \quad x \in X_1 \tag{14.49}$$

where the only constraint is the power flow equation:

$$X_1 \;:=\; \{x := (p_1, p_2, \theta) \;:\; x \text{ satisfies (14.48)}\}$$

The feasible set is nonconvex because it is an ellipse without its interior. Consider the convex relaxation:

$$\min_x \; c(p_1, p_2) \quad \text{s.t.} \quad x \in \operatorname{conv}(X_1) \tag{14.50}$$

Explain why the relaxation is exact, i.e., an optimal x^* for (14.50) is also optimal for (14.49).

2. Consider the constraints on injections (p_1, p_2) and constraints on θ:

$$X_2 \;:=\; \{x := (p_1, p_2, \theta) \in \mathbb{R}^3 \;:\; \underline{\theta} \le \theta \le \overline{\theta}\}$$
$$X_3 \;:=\; \{x := (p_1, p_2, \theta) \in \mathbb{R}^3 \;:\; \underline{p}_j \le p_j \le \overline{p}_j, \; j = 1, 2\}$$

Consider the OPF:

$$\min_x \; c(p_1, p_2) \quad \text{s.t.} \quad x \in \; X_1 \cap X_2 \cap X_3 \tag{14.51}$$

and its convex relaxation:

$$\min_x \; c(p_1, p_2) \quad \text{s.t.} \quad x \in \operatorname{conv}(X_1 \cap X_2) \,\cap\, X_3 \tag{14.52}$$

Indicate the feasible sets of (14.51) and (14.52) projected onto (p_1, p_2) plane, and explain why lower bounds $(\underline{p}_1, \underline{p}_2)$ on the injections (p_1, p_2) affect the exactness of SOCP relaxation, but not the upper bounds on $(\overline{p}_1, \overline{p}_2)$.

3. Explain why limiting $|\theta|$ to $[\underline{\theta}, \overline{\theta}]$ can ensure exact relaxation as long as (recall that $g > 0, b < 0$)

$$\tan^{-1}\left(\frac{b}{g}\right) \;\leq\; \underline{\theta} \;<\; \overline{\theta} \;\leq\; \tan^{-1}\left(\frac{-b}{g}\right)$$

Exercise 14.9 (Condition C14.3 and Pareto front)**.** In general, a point x^* is Pareto optimal over a convex set $A \subseteq \mathbb{R}^k$ if and only if it $x^* = \arg\min_{x\in A} c^{\mathsf{T}} x$ for some nonzero $c \geq 0$.

1. Show that, for the two-bus network in Exercise 14.7, $\mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p) \supseteq \mathbb{O}(\text{conv}(\mathbb{P}_\theta \cap \mathbb{P}_p))$ if condition C14.3 does not hold.
2. Show that if condition C14.3 holds, then we can define a Pareto optimal x^* as $x^* = \arg\min_{x\in A} c^{\mathsf{T}} x$ for some $c > 0$ and $\mathbb{O}(\text{conv}(\mathbb{P}_\theta) \cap \mathbb{P}_p) = \mathbb{O}(\text{conv}(\mathbb{P}_\theta \cap \mathbb{P}_p))$.

Exercise 14.10 (Convex hull and Pareto front)**.** Prove Lemma 14.11: Let $\mathbb{B}, \mathbb{C} \subseteq \mathbb{R}^k$ be arbitrary sets, $\mathbb{D} := \{x \in \mathbb{R}^k | Mx \leq c\}$ be an affine set, and M a matrix and b a vector of appropriate dimensions.

(1) $\text{conv}(M\mathbb{B}) = M\,\text{conv}(\mathbb{B})$ and $\text{conv}(\mathbb{B} \times \mathbb{C}) = \text{conv}(\mathbb{B}) \times \text{conv}(\mathbb{C})$ where for any sets $A_1, A_2 \subseteq \mathbb{R}^k$, $(x_1, x_2) \in A_1 \times A_2$ if and only if $x_1 \in A_1$ and $x_2 \in A_2$.

(2) Suppose $\mathbb{B}$ and $\mathbb{C}$ are convex and a point is Pareto optimal over a set if and only if it minimizes $c^{\mathsf{T}} x$ over the set for some $c > 0$. Then $\mathbb{O}(M\mathbb{B}) = M\,\mathbb{O}(\mathbb{B})$ and $\mathbb{O}(\mathbb{B} \times \mathbb{C}) = \mathbb{O}(\mathbb{B}) \times \mathbb{O}(\mathbb{C})$.

(3) If $\mathbb{B} = \mathbb{O}(\text{conv}\ \mathbb{B})$ then $\mathbb{B} \cap \mathbb{D} \subseteq \mathbb{O}(\text{conv}(\mathbb{B}) \cap \mathbb{D})$.

Chapter 14.5.

Exercise 14.11 (Lemma 14.10 [131])**.** aaa

Chapter 15

Semidefinite relaxations: BFM

15.1 Introduction

In Chapter 14 we study the semidefinite relaxation of OPF in the bus injection model. In this chapter we study the relaxation of OPF for radial networks in the branch flow model. In Chapter 15.2 we focus on single-phase radial networks. We formulate SOCP relaxation, prove its equivalence to the SOCP relaxation in BIM, and present two sufficient conditions for exact relaxation. In Chapter 15.3 we extend the relaxation to unbalanced three-phase networks.

15.2 Single-phase OPF

Branch flow model, in particular the original DistFlow model of [18, 19], is mostly used to model single-phase radial networks. In this section we describe the SOCP relaxation of DistFlow model. This model assumes that the series impedances $z^s_{jk} = z^s_{kj}$ of each line (j,k) are equal in each direction (assumption C5.1) and shunt admittances are zero $z^m_{jk} = z^m_{kj} = 0$. These two assumptions allow us to assume the network graph $G = (\overline{N}, E)$ is directed and includes branch variables in only one direction (see Chapter 5.2.2 for details). We denote a line in E from bus j to bus k either by $(j,k) \in E$ or $j \to k$. It is characterized by its series impedance $z_{jk} := z^s_{jk}$. Without loss of generality we take bus 0 as the root of the tree.

15.2.1 SOCP relaxation

Consider the OPF formulated in Chapter 13.2.1, reproduced here (but with a different graph orientation). Associated with each line (j,k) are branch variables (ℓ_{jk}, S_{jk}). Let $(s,v) := (s_j, v_j, j \in \overline{N})$ and $(\ell, S) := (\ell_{jk}, S_{jk}, j \to k \in E)$. Let $x := (s, v, \ell, S)$ in $\mathbb{R}^{3(N+1+M)}$ with $M = N$ since G is a tree. Bus 0 denotes the root of the tree. We adopt the graph orientation where every line points *towards* node 0. Then the DistFlow model (5.7) is:

$$
\begin{aligned}
S_{jk} &= \sum_{i:i\rightarrow j} \left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j, & j \in \overline{N} \qquad (15.1a)\\
v_j - v_k &= 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right) - |z_{jk}|^2 \ell_{jk}, & j \rightarrow k \in E \qquad (15.1b)\\
v_j \ell_{jk} &= |S_{jk}|^2, & j \rightarrow k \in E \qquad (15.1c)
\end{aligned}
$$

where $k := k(j)$ (15.1a) denotes the node adjacent to j on the unique path from bus j to bus 0. The boundary condition is: $S_{jk} := 0$ when $j = 0$ in (15.1a) and $S_{ij} = 0$, $\ell_{ij} = 0$ when j is a leaf node.[1] The injection, voltage and line limits can be expressed in terms of the BFM variable x:

$$
s_j^{\min} \leq s_j \leq s_j^{\max}, \qquad v_j^{\min} \leq v_j \leq v_j^{\max}, \qquad \ell_{jk} \leq \ell_{jk}^{\max}, \qquad j \in \overline{N},\ (j,k) \in E \qquad (15.2)
$$

Let the cost function in the branch flow model be $C(x)$. Let the feasible set be

$$
\mathbb{T} := \{x : (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies (15.1)(15.2)}\} \qquad (15.3a)
$$

Then an optimal power flow problem in the branch flow model is:

OPF:

$$
\min_x\ C(x) \quad \text{subject to} \quad x \in \mathbb{T} \qquad (15.3b)
$$

As remarked in Chapter 13.2.1, the model (15.1) includes only voltage and power sources whose controllable variables are v_j and s_j respectively. A current source will introduce its current $I_j \in \mathbb{C}$ as an additional variable and an equality constraint $|s_j|^2 = v_j |I_j|^2$ that relate I_j to (s_j, v_j). An impedance z_j will introduce an equality constraint $s_j = v_j / z_j^{\mathsf{H}}$ on (s_j, v_j). If z_j is controllable, e.g., representing a switched capacitor, then z_j is an additional variable. For simplicity we restrict ourselves to voltage and power sources only.

The constraints (15.1a)(15.1b) are linear in x. The constraint (15.1c) is however quadratic in x, making the feasible set of OPF (15.3) nonconvex. Relaxing the equality in (15.1c) into inequality

$$
v_j \ell_{jk} \geq |S_{jk}|^2, \qquad j \rightarrow k \in E \qquad (15.4)
$$

results in a (convex) second-order cone. Define

$$
\mathbb{T}^+ := \{x : (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies (15.1a)(15.1b)(15.4)(15.2)}\} \qquad (15.5a)
$$

Then the SOCP relaxation of OPF (15.3) is:

OPF-socp:

$$
\min_x\ C(x) \quad \text{subject to} \quad x \in \mathbb{T}^+ \qquad (15.5b)
$$

We say that OPF-socp (15.5) is *exact* if *every* optimal solution x^{socp} of (15.5) attains equalities in (15.4) and hence is an optimal solution of OPF (15.3). This is convenient because it ensures that any algorithm that solves an exact relaxation always produces a globally optimal solution to the OPF problem. This notion of strong exactness is however unnecessary under the sufficient exactness conditions of Chapters 15.2.3 and 15.2.4 for radial networks; see Remark 15.1 after Theorem 15.2 and Remark 15.3 after Theorem 15.3. These conditions guarantee that an optimal solution to OPF can be recovered from *any* optimal solution x^{socp} of OPF-socp whether or not x^{socp} attains equalities in (15.4).

[1] A node $j \in N$ is a *leaf node* if there is no i such that $i \rightarrow j \in \tilde{E}$.

15.2.2 Equivalence

The single-phase OPF (15.3) is equivalent to the single-phase OPF problem (13.9) or (13.15) in the bus injection model because their feasible sets $\mathbb{T}$ and $\mathbb{V}$ respectively are equivalent by Theorem 5.2. In this section we show that their SOCP relaxations are equivalent as well by establishing a bijection between the feasible sets of these relaxations.

The equivalence of the SOCP relaxations in these two models rests on the equivalence of their feasible sets. Recall that any sets A and B are *equivalent*, denoted by $A \equiv B$, if there is a bijection between them. When there is a one-one correspondence between their feasible sets, a feasible point is optimal for one problem if and only if its corresponding feasible point is optimal for the other problem. We now make this precise.

Recall from Chapter 14.2.1 that the SOCP relaxation (14.16c) of OPF in BIM is the minimization of $C_0(W_G)$ over Hermitian partial matrices $W_G \in \mathbb{C}^{2M+N+1}$ subject to operational and 2×2 psd constraints. The operational constraints are the injection limits, voltage limits, and line limits. In terms of the partial matrix W_G, they are respectively: (substituting $|V_j|^2 = [W_G]_{jj}$ and $V_j V_k^{\mathsf{H}} = [W_G]_{jk}$ into (13.8) (13.4b)(13.4c)):

$$s_j^{\min} \ \le \ \sum_{k:j\sim k} y_{jk}^{s\mathsf{H}} \left([W_G]_{jj} - [W_G]_{jk}\right) \ \le \ s_j^{\max}, \qquad j \in \overline{N} \tag{15.6a}$$

$$v_j^{\min} \ \le \ [W_G]_{jj} \ \le \ v_j^{\max}, \qquad j \in \overline{N} \tag{15.6b}$$

$$\left|y_{jk}^{s}\right|^2 \left([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj}\right) \ \le \ \ell_{jk}^{\max}, \qquad (j,k) \in E \tag{15.6c}$$

The 2×2 psd constraint $W_G(j,k) \succeq 0$, $(j,k) \in E$, is equivalent to

$$[W_G]_{jk} \ = \ [W_G]_{kj}^{\mathsf{H}}, \quad [W_G]_{jj} \ > \ 0, \quad [W_G]_{kk} \ > \ 0, \quad [W_G]_{jj}[W_G]_{kk} \ \ge \ \left|[W_G]_{jk}\right|^2, \quad (j,k) \in E \tag{15.6d}$$

Then the feasible set of the SOCP relaxation of OPF in BIM is

$$\mathbb{W}_G^+ \ := \ \left\{ \, W_G \in \mathbb{C}^{2M+N+1} \mid W_G \text{ satisfies (15.6)} \, \right\} \tag{15.7a}$$

and the SOCP relaxation is

$$\min_{W_G} \quad C_0(W_G) \qquad \text{s.t.} \qquad W_G \in \mathbb{W}_G^+ \tag{15.7b}$$

The feasible set of OPF-socp (15.5) in BFM is equivalent to that of (15.7) in BIM.

Theorem 15.1 (Equivalence of SOCPs). $\mathbb{T}^+ \equiv \mathbb{W}_G^+$.

The theorem implies that there is a bijection $g : \mathbb{W}_G^+ \to \mathbb{T}^+$. If the cost function in the SOCP relaxation (15.5) in BFM and that in (15.7) in BIM are equivalent, i.e., $C_0(W_G) = C\left(g(W_G)\right)$, then these SOCP relaxations are equivalent problems in the sense that W_G^{opt} is optimal for (15.7) if and only if $x^{\text{opt}} := g(W_G^{\text{opt}})$ is optimal for (15.5).

The proof of Theorem 15.1 below constructs a linear mapping $g : \mathbb{W}_G^+ \to \mathbb{T}^+$, motivated by the factorization $W = VV^{\mathsf{H}}$ of the psd rank-1 completion W of the partial matrix W_G when W_G is psd rank-1. Define the linear mapping $g : \mathbb{W}_G^+ \to \mathbb{T}^+$ with $x := (s, v, \ell, S) = g(W_G)$ where

$$\begin{aligned}
s_j &:= \sum_{k: j\sim k} y_{jk}^{s\mathsf{H}} \left([W_G]_{jj} - [W_G]_{jk}\right), && j \in \overline{N} && (15.8a)\\
v_j &:= [W_G]_{jj}, && j \in \overline{N} && (15.8b)\\
\ell_{jk} &:= |y_{jk}^s|^2 \left([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj}\right), && j \to k \in E && (15.8c)\\
S_{jk} &:= y_{jk}^{s\mathsf{H}} \left([W_G]_{jj} - [W_G]_{jk}\right), && j \to k \in E && (15.8d)
\end{aligned}$$

and the mapping $g^{-1} : \mathbb{T}^+ \to \mathbb{W}_G^+$ with $W_G = g^{-1}(x)$ where

$$\begin{aligned}
[W_G]_{jj} &:= v_j, && j \in \overline{N} && (15.9a)\\
[W_G]_{jk} &:= v_j - z_{jk}^{s\mathsf{H}} S_{jk} \;=\; [W_G]_{kj}^{\mathsf{H}}, && j \to k \in E && (15.9b)
\end{aligned}$$

The proof below establishes that g and g^{-1} are indeed inverses of each other. By restricting these mappings g and g^{-1} to subsets $\mathbb{W}_G \subseteq \mathbb{W}_G^+$ and $\mathbb{T} \subseteq \mathbb{T}^+$, the theorem immediately implies the equivalence of $\mathbb{T} \equiv \mathbb{W}_G$ and hence the equivalence of single-phase OPF (15.3) in BFM and the OPF (13.9) or (13.15) in BIM (since $\mathbb{W}_G \equiv \mathbb{V}$).

Since we assume $z_{jk}^m = z_{kj}^m = y_{jk}^m = y_{kj}^m = 0$, we often omit the superscript s in z_{jk}^s and y_{kj}^s.

Proof of Theorem 15.1. We will prove that g and g^{-1} are indeed inverses of each other in three steps: (1) g maps every point $W_G \in \mathbb{W}_G^+$ to a point in $\mathbb{T}^+$; (2) g^{-1} maps every point $x \in \mathbb{T}^+$ to a point in $\mathbb{W}_G^+$; and (3) $g(g^{-1}(x)) = x$ and $g^{-1}(g(W_G)) = W_G$. This defines a bijection between $\mathbb{W}_G^+$ and $\mathbb{T}^+$ and establishes $\mathbb{W}_G^+ \equiv \mathbb{T}^+$.

Step 1: $x := g(W_G) \in \mathbb{T}^+$. Given a $W_G \in \mathbb{W}_G^+$, we have to prove $x := g(W_G)$ satisfies (15.1a) (15.1b) (15.4) (15.2). We claim that (15.2) follows from (15.8) and (15.6). Specifically the injection limit follows from (15.8a) and (15.6a). The voltage limit follows from (15.8b) and and (15.6b). The line limit follows from (15.8c) and (15.6c). Hence x satisfies (15.2).

To prove (15.1a), we have for $j \in \overline{N}$

$$\begin{aligned}
& \sum_{i: i\to j} \left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j \\
=\; & \sum_{i: i\to j} \left(y_{ij}^{\mathsf{H}}\left([W_G]_{ii} - [W_G]_{ij}\right) - y_{ij}^{\mathsf{H}}\left([W_G]_{ii} + [W_G]_{jj} - [W_G]_{ij} - [W_G]_{ji}\right)\right) + s_j \\
=\; & \sum_{i: i\to j} \left(-y_{ij}^{\mathsf{H}}\left([W_G]_{jj} - [W_G]_{ji}\right)\right) + \sum_{i: i\to j} y_{ji}^{\mathsf{H}}\left([W_G]_{jj} - [W_G]_{ji}\right) + \sum_{k: j\to k} y_{jk}^{\mathsf{H}}\left([W_G]_{jj} - [W_G]_{jk}\right) \\
=\; & \sum_{k: j\to k} S_{jk}
\end{aligned}$$

where the last equality follows from $y_{ij} = y_{ji}$ by assumption C5.1. To prove (15.1b), we have for $j \to k \in E$

$$\begin{aligned}
2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right) - |z_{jk}|^2 \ell_{jk} &= 2\,\mathrm{Re}\left([W_G]_{jj} - [W_G]_{jk}\right) - \left([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj}\right) \\
&= \left([W_G]_{jj} - [W_G]_{kk}\right) - [W_G]_{jk}^{\mathsf{H}} + [W_G]_{kj} \\
&= v_j - v_k
\end{aligned}$$

where the last equality follows because the partial matrix W_G is Hermitian. Finally to prove (15.4), for each $j \to \in E$, we have from (15.6d) $[W_G]_{jj}[W_G]_{kk} \geq |[W_G]_{jk}|^2$. Hence

$$\begin{aligned} v_j \ell_{jk} &= \left|y_{jk}\right|^2 \; [W_G]_{jj} \left([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj}\right) \\ &\geq \left|y_{jk}\right|^2 \left([W_G]_{jj}^2 + \left|[W_G]_{jk}\right|^2 - [W_G]_{jj}[W_G]_{jk} - [W_G]_{jj}[W_G]_{jk}^{\mathsf{H}}\right) \\ &= \left|S_{jk}\right|^2 \end{aligned} \tag{15.10}$$

as desired. Hence g maps every $W_G \in \mathbb{W}_G^+$ to an $x \in \mathbb{T}^+$.

Step 2: $W_G := g^{-1}(x) \in \mathbb{W}_G^+$. Given an $x \in \mathbb{T}^+$, we have to prove that $W_G := g^{-1}(x)$ satisfies (15.6). Clearly (15.9a) and the voltage limit in (15.2) implies (15.6b).

To prove (15.6a), we have for each $j \in N^+$

$$\begin{aligned} \sum_{k:(j,k)\in E} y_{jk}^{\mathsf{H}} \left([W_G]_{jj} - [W_G]_{jk}\right) &= \sum_{i:i\to j} y_{ji}^{\mathsf{H}} \left([W_G]_{jj} - [W_G]_{ji}\right) + \sum_{k:j\to k} y_{jk}^{\mathsf{H}} \left([W_G]_{jj} - [W_G]_{jk}\right) \\ &= \sum_{i:i\to j} y_{ij}^{\mathsf{H}} \left(v_j - \left(v_i - z_{ij}^{\mathsf{H}} S_{ij}\right)^{\mathsf{H}}\right) + \sum_{k:j\to k} y_{jk}^{\mathsf{H}} \left(v_j - \left(v_j - z_{jk}^{\mathsf{H}} S_{jk}\right)\right) \\ &= \sum_{k:j\to k} S_{jk} - \sum_{i:i\to j} y_{ij}^{\mathsf{H}} \left(v_i - v_j - z_{ij} S_{ij}^{\mathsf{H}}\right) \\ &= \sum_{k:j\to k} S_{jk} - \sum_{i:i\to j} y_{ij}^{\mathsf{H}} \left(2\,\mathrm{Re}(z_{ij}^{\mathsf{H}} S_{ij}) - \left|z_{ij}\right|^2 \ell_{ij} - z_{ij} S_{ij}^{\mathsf{H}}\right) \end{aligned}$$

where the second equality follows from (15.9) and $y_{ji} = y_{ij}$ by assumption C5.1, and the last equality follows from (15.1b). But

$$\left(2\,\mathrm{Re}(z_{ij}^{\mathsf{H}} S_{ij}) - z_{ij} S_{ij}^{\mathsf{H}}\right) = \left(z_{ij}^{\mathsf{H}} S_{ij} + z_{ij} S_{ij}^{\mathsf{H}}\right) - z_{ij} S_{ij}^{\mathsf{H}} = z_{ij}^{\mathsf{H}} S_{ij}$$

and hence

$$\sum_{k:(j,k)\in E} y_{jk}^{\mathsf{H}} \left([W_G]_{jj} - [W_G]_{jk}\right) = \sum_{k:j\to k} S_{jk} - \sum_{i:i\to j} \left(S_{ij} - z_{ij}\ell_{ij}\right) = s_j$$

where the last equality follows from (15.1a). This and the injection limits in (15.2) imply (15.6a). To prove (15.6c), we have for each $(j,k) \in E$, from (15.9),

$$\begin{aligned} \left|y_{jk}\right|^2 \left([W_G]_{jj} + [W_G]_{kk} - [W_G]_{jk} - [W_G]_{kj}\right) &= \left|y_{jk}\right|^2 \left(v_j + v_k - \left(v_j - z_{jk}^{\mathsf{H}} S_{jk}\right) - \left(v_j - z_{jk}^{\mathsf{H}} S_{jk}\right)^{\mathsf{H}}\right) \\ &= \left|y_{jk}\right|^2 \left(-v_j + v_k + z_{jk}^{\mathsf{H}} S_{jk} + z_{jk} S_{jk}^{\mathsf{H}}\right) \\ &= \ell_{jk} \end{aligned}$$

where last equality follows from (15.1b). This and the line limit in (15.2) imply (15.6c). Finally to prove

(15.6d), note that $[W_G]_{jk} = [W_G]_{kj}^{\mathsf{H}}$, $[W_G]_{jj} > 0$, and $[W_G]_{kk} > 0$ follow directly from (15.9). Furthermore

$$\begin{aligned}
[W_G]_{jj}[W_G]_{kk} - \left|[W_G]_{jk}\right|^2 &= v_j v_k - \left|v_j - z_{jk}^{\mathsf{H}} S_{jk}\right|^2 \\
&= v_j v_k - \left(v_j^2 + \left|z_{jk}\right|^2 \left|S_{jk}\right|^2 - 2 v_j \operatorname{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right)\right) \\
&= v_j \left(v_k - v_j + 2 \operatorname{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right)\right) \; - \; \left|z_{jk}\right|^2 \left|S_{jk}\right|^2 \\
&= \left|z_{jk}\right|^2 \left(v_j \ell_{jk} - \left|S_{jk}\right|^2\right) \; \geq \; 0
\end{aligned}$$

where last equality follows from (15.1b) and the last inequality follows from (15.4). Therefore $W_G(j,k) \succeq 0$ for all $(j,k) \in E$, as desired. This shows that g^{-1} maps every $x \in \mathbb{T}^+$ to a $W_G \in \mathbb{W}_G^+$.

Step 3: $g(g^{-1}(x)) = x$ and $g^{-1}(g(W_G)) = W_G$. The proof uses (15.8)(15.9)(15.1a)(15.1b). It follows a similar argument used in Steps 1 and 2, and is omitted. This completes the proof that g and g^{-1} are indeed inverses of each other and establishes $\mathbb{W}_G^+ \equiv \mathbb{T}^+$.

This completes the proof of Theorem 15.1. □

15.2.3 Exactness condition: inactive injection lower bounds

Assume

C15.1: The cost function $C(x)$ is strictly increasing in ℓ, nondecreasing in $s = (p,q)$, and independent of branch flows $S = (P,Q)$.

C15.2: For $j \in \overline{N}$, $s_j^{\min} = -\infty - \mathbf{i}\infty$.

Popular cost functions in the literature include active power loss over the network or active power generations, both of which satisfy C15.1.

Theorem 15.2 (Inactive injection lower bounds)**.** Suppose the network graph G is a tree and C15.1, C15.2 hold. Then the SOCP relaxation (15.5) is exact, i.e., every optimal solution x^{socp} of (15.5) is optimal for OPF (15.3).

Remark 15.1 (Strong exactness)**.** If the cost function $C(x)$ in C15.1 is only nondecreasing, rather than strictly increasing, in ℓ, then C15.1, C15.2 still guarantee that all optimal solutions of OPF (15.3) are optimal solutions of its relaxation OPF-socp (15.5), but OPF-socp may have an optimal solution x^{socp} that maintains a strict inequality in (15.4) and hence is infeasible for OPF. Even though OPF-socp is not exact under the strong notion of exactness in Definition 14.1, an optimal solution of OPF (15.3) can still be constructed from such a solution x^{socp}; see explanation immediately after the proof of Theorem 15.2 below. □

Remark 15.2 (Convexity)**.** For exact relaxation, we do not require the cost function $c(x)$ to be convex in x; $c(x)$ needs to be convex for (15.5) to be a convex problem.

We can allow more general constraints on power injections s_j than $s_j \leq s_j^{\max}$ assumed in Theorem 15.2. The injection s_j can be in an *arbitrary* set $\mathbb{B}_j$ that satisfies C15.2. In particular $\mathbb{B}_j$ need not be convex nor even connected for OPF-socp to be exact. It (only) needs to be convex to be efficiently computable. Such a general constraint on s is useful in many applications. For instance it allows constraints of the form $|s_j|^2 \leq a$, $|\angle s_j| \leq \phi_j$ that is useful for volt/var control or $q_j \in \{0, a\}$ for capacitor configuration. □

Proof of Theorem 15.2. Fix any optimal solution $x := (s, v, \ell, S) \in \mathbb{R}^{3(M+N+1)}$ of OPF-socp (15.5). Since G is a tree, the cycle condition is vacuous and we only need to show that x attains equality in (15.4). For the sake of contradiction assume this is violated on line $j \to$k, i.e.,

$$v_j \ell_{jk} \;\; > \;\; |S_{jk}|^2 \tag{15.11}$$

We will construct an $\hat{x}$ that is feasible for OPF-socp and attains a strictly lower cost, contradicting the optimality of x.

For an $\varepsilon > 0$ to be determined below, consider the following $\hat{x}$ obtained by modifying only the current ℓ_{jk} and power flows S_{jk} on line $j \to k$ and the injections s_j, s_k at two ends of line $j \to k$:

$$\begin{aligned} \hat{\ell}_{jk} &:= \ell_{jk} - \varepsilon && (15.12a) \\ \hat{S}_{jk} &:= S_{jk} - z_{jk}\varepsilon/2 && (15.12b) \\ \hat{s}_j &:= s_j - z_{jk}\varepsilon/2 && (15.12c) \\ \hat{s}_k &:= s_k - z_{jk}\varepsilon/2 && (15.12d) \end{aligned}$$

and $\hat{v} := v$, $\hat{\ell}_{il} := \ell_{il}$ and $\hat{S}_{il} := S_{il}$ for $(i,l) \neq (j,k)$, $\hat{s}_i := s_i$ for $i \neq j,k$. By assumption C15.1 the objective function $C(x)$ is strictly increasing in ℓ and hence $\hat{x}$ has a strictly lower cost than x. It suffices to show that there exists an $\varepsilon > 0$ such that $\hat{x}$ is feasible for OPF-socp (15.5), i.e., $\hat{x}$ satisfies (15.1a)(15.1b)(15.4)(15.2). Moreover we can choose $\varepsilon > 0$ so that $\hat{x}$ attains equalities in (15.4) and is therefore feasible for OPF.

Assumption C15.2 ensures that $\hat{x}$ satisfies (15.2) since $z_{jk} > 0$ and $\varepsilon > 0$. Further $\hat{x}$ satisfies (15.1a) at buses $i \neq j,k$, and satisfies (15.1b)(15.4) over lines $(i,l) \neq (j,k)$. We now show that $\hat{x}$ also satisfies (15.1a)(15.1b)(15.4) at buses j,k and over the line (j,k).

For (15.1a) at bus j, we have from (15.12b)(15.12c)

$$\hat{S}_{jk} \;=\; S_{jk} - z_{jk}\frac{\varepsilon}{2} \;=\; \sum_{i:i \to j} \left(S_{ij} - z_{ij}\ell_{ij}\right) + s_j \;-\; z_{jk}\frac{\varepsilon}{2} \;=\; \sum_{i:i \to j} \left(\hat{S}_{ij} - z_{ij}\hat{\ell}_{ij}\right) + \hat{s}_j$$

as desired. For (15.1a) at bus k, one line $k \to l$ from k towards bus 0, we have from (15.12a)(15.12b)(15.12d)

$$\begin{aligned} \hat{S}_{kl} \;=\; S_{kl} &= \left(S_{jk} - z_{jk}\ell_{jk}\right) + \sum_{i \neq j: i \to k} \left(S_{ik} - z_{ik}\ell_{ik}\right) + s_k \\ &= \left(\hat{S}_{jk} - z_{jk}\hat{\ell}_{jk} - z_{jk}\frac{\varepsilon}{2}\right) + \sum_{i \neq j: i \to k} \left(\hat{S}_{ik} - z_{ik}\hat{\ell}_{ik}\right) + s_k \;=\; \sum_{i:i \to k} \left(\hat{S}_{ik} - z_{ik}\hat{\ell}_{ik}\right) + \hat{s}_k \end{aligned}$$

as desired. This shows that $\hat{x}$ satisfies (15.1a) at both buses j,k. For (15.1b) over line (j,k), we have from (15.12a)(15.12b)

$$\hat{v}_j - \hat{v}_k \;=\; v_j - v_k \;=\; 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right) - |z_{jk}|^2 \ell_{jk} \;=\; 2\,\mathrm{Re}\left(z_{jk}^{\mathsf{H}} \hat{S}_{jk}\right) - |z_{jk}|^2 \hat{\ell}_{jk}$$

as desired. For (15.4) over line (j,k), we have from (15.12a)(15.12b)

$$\hat{v}_j\hat{\ell}_{jk} - \left|\hat{S}_{jk}\right|^2 \;=\; -\frac{\left|z_{jk}\right|^2}{4}\varepsilon^2 \,-\, \left(v_j - \mathrm{Re}\left(z_{jk}^{\mathsf{H}} S_{jk}\right)\right)\varepsilon \,+\, \left(v_j\ell_{jk} - \left|S_{jk}\right|^2\right)$$

Hence (15.11) implies that we can always choose an $\varepsilon > 0$ such that $\hat{v}_j\hat{\ell}_{jk} = \left|\hat{S}_{jk}\right|^2$.

This completes the proof of Theorem 15.2. □

Note that the construction of $\hat{x}$ ensures that *equalities* are attained in (15.4) and therefore $\hat{x}$ is feasible for OPF (15.3), not just for its SOCP relaxation. If the cost function $C(x)$ in C15.1 is only nondecreasing, rather than strictly increasing, in ℓ, then it is possible that $C(\hat{x}) = C(x)$ and OPF-socp (15.5) has an optimal solution x that maintains a strict inequality in (15.4). Even in this case, the proof shows how to construct from such an x an optimal solution $\hat{x}$ for OPF (15.3) under C15.1 and C15.2.

15.2.4 Exactness condition: inactive voltage upper bounds

In this section we present a sufficient condition for the exactness of SOCP relaxation of single-phase OPF on a radial network, when the operational constraint (15.2) is replaced by the following set of slightly different constraints:

$$v_j^{\min} \;\leq\; v_j \;\leq\; v_j^{\max}, \qquad j \in N \tag{15.13a}$$

$$s_j \;\in\; \mathbb{B}_j \;\subseteq\; \{s_j \in \mathbb{C} \,|\, s_j \leq s_j^{\max}\}, \qquad j \in N \tag{15.13b}$$

for some given finite $s_j^{\max}$, $j \in N$. In particular we ignore line limits, but allow the injections $(s_j, j \in N)$ at non-root buses to be in an arbitrary set $\mathbb{B}_j$ that is bounded above (see Remark 15.2). We also assume v_0 is given and satisfies (15.13a) and s_0 is unconstrained.

Then OPF and its feasible set are:

$$\textbf{OPF:} \qquad \min_x \;\; C(x) \qquad \text{s.t.} \qquad x \in \mathbb{T} \tag{15.14a}$$

$$\text{where} \qquad \mathbb{T} \;:=\; \{x : (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies (15.1)(15.13)}\} \tag{15.14b}$$

Their SOCP relaxations are:

$$\textbf{OPF-socp:} \qquad \min_x \;\; C(x) \qquad \text{s.t.} \qquad x \in \mathbb{T}^+ \tag{15.15a}$$

$$\text{where} \qquad \mathbb{T}^+ \;:=\; \{x : (s, v, \ell, S) \in \mathbb{R}^{6N+3} \mid x \text{ satisfies (15.1a)(15.1b)(15.4)(15.13)}\} \tag{15.15b}$$

OPF-socp (15.15) is *exact* if every optimal solution x^{socp} of (15.15) attains equality in (15.4) and is hence optimal for OPF (15.14).

The main sufficient condition to be presented below for exact SOCP relaxation is that the voltage upper bounds are inactive at optimality. Before presenting it we first explain a simple intuition using a two-bus network that motivates this condition.

Example 15.1 (Geometric insight)**.** Consider bus 0 and bus 1 connected by a line with impedance $z := r + \mathbf{i}x$. Without loss of generality, let the direction of the line be from bus 1 to bus 0. Let ℓ be the sending-end squared current magnitude from buses 1 to 0 (recall that $S_{01} := 0$ in (15.1a)). Suppose also without loss of generality that $v_0 = 1$ pu. The model in (15.1) reduces to (Exercise 15.1):

$$p_0 - r\ell \;=\; -p_1, \quad q_0 - x\ell \;=\; -q_1, \quad p_0^2 + q_0^2 \;=\; \ell \tag{15.16a}$$

$$v_1 - v_0 \;=\; 2\,(rp_1 + xq_1) \;-\; (r^2 + x^2)\ell \tag{15.16b}$$

Suppose s_1 is given (e.g., a constant power load). Then the variables are $w := (p_0, q_0, v_1, \ell)$ and the feasible set consists of solutions of (15.16). subject to additional constraints on w. The case without any

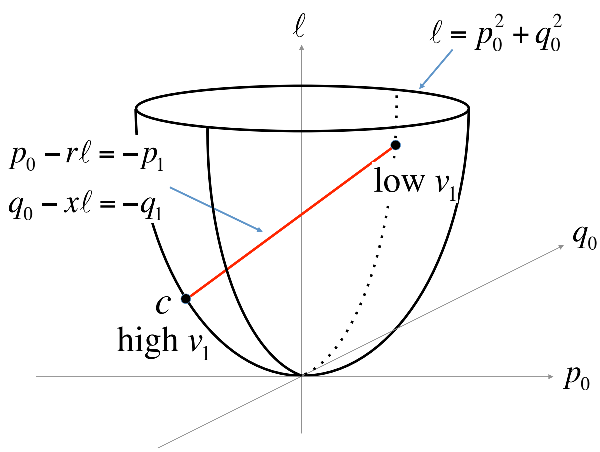


Figure 15.1: Feasible set of OPF for a two-bus network without any constraint. It consists of the (two) points of intersection of the line with the convex surface (without the interior), and hence is nonconvex. SOCP relaxation includes the interior of the convex surface and enlarges the feasible set to the line segment joining these two points. If the cost function C is increasing in ℓ or (p_0, q_0) then the optimal point over the SOCP feasible set (line segment) is the lower feasible point c, and hence the relaxation is exact.

constraint is instructive and shown in Figure 15.1 (see explanation in the caption). The point c in the figure corresponds to a power flow solution with a large v_1 (normal operation) whereas the other intersection corresponds to a solution with a small v_1 (fault condition). As explained in the caption, SOCP relaxation is exact if there is no voltage constraint and as long as constraints on (p_0, q_0, ℓ) do not remove the high-voltage solution c. Only when the system is stressed to a point where the high-voltage solution becomes infeasible will relaxation lose exactness. This agrees with conventional wisdom that power systems under normal operations are well behaved.

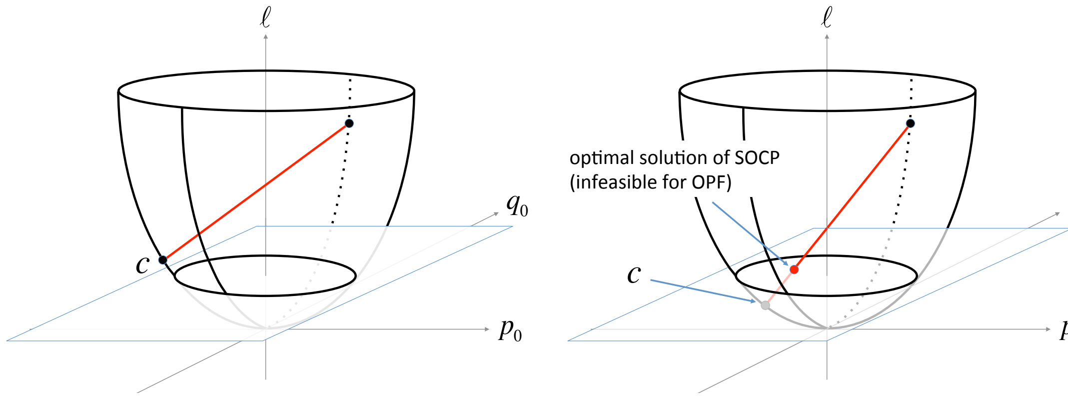
Consider now the voltage constraint $v_1^{\min} \le v_1 \le v_1^{\max}$. We have from (15.16b) and $v_0 = 1$

$$v_1 \;=\; (1 + 2rp_1 + 2xq_1) - |z|^2\ell$$

translating the constraint on v_1 into a box constraint on ℓ:

$$\frac{1}{|z|^2}\left(2rp_1 + 2xq_1 + 1 - v_1^{\max}\right) \;\le\; \ell \;\le\; \frac{1}{|z|^2}\left(2rp_1 + 2xq_1 + 1 - v_1^{\min}\right)$$

Figure 15.1 shows that the lower bound $v_1^{\min}$ (corresponding to an upper bound on ℓ) does not affect the exactness of SOCP relaxation. The effect of upper bound $v_1^{\max}$ (corresponding to a lower bound on ℓ) is illustrated in Figure 15.2. As explained in the caption of the figure SOCP relaxation is exact if the upper bound $v_1^{\max}$ does not exclude the high-voltage solution c and is not exact otherwise.



(a) Voltage constraint not binding (b) Voltage constraint binding

Figure 15.2: Impact of voltage upper bound $v_1^{\max}$ on exactness. (a) When $v_1^{\max}$ (corresponding to a lower bound on ℓ) is not binding, the power flow solution c is in the feasible set of SOCP and hence the relaxation is exact. (b) When $v_1^{\max}$ excludes c from the feasible set of SOCP, the optimal solution is infeasible for OPF and the relaxation is not exact.

See Exercises 15.2 and 15.3 for details of feasibility and exactness of OPF-socp. □

To state the exactness condition for a general radial network, recall the linear approximation of BFM studied in Chapter 5.5.1, obtained by setting $\ell_{jk} = 0$ in (15.1). Given v_0 and the injections $\hat{s} := (\hat{p}, \hat{q}) := (p_j, q_j, j \in N)$ at non-root buses, the line flow vector $S^{\text{lin}}(s) := \left(S_{jk}^{\text{lin}}, (j,k) \in E\right)$ and the voltage vector $\hat{v}^{\text{lin}}(s) := (v_j^{\text{lin}}, j \in N)$ at non-root buses in the linearized model are explicitly given by (from Theorem 5.3):

$$S^{\text{lin}}(s) \;=\; \hat{C}^{-1}\hat{s}, \qquad\qquad \hat{v}^{\text{lin}}(s) \;=\; v_0\mathbf{1} + 2\left(R\hat{p} + X\hat{q}\right) \tag{15.17}$$

for some given invertible matrices $\hat{C}$, R and X. The key property we will use is, from Corollary 5.4:

$$S_{jk} \;\leq\; S_{jk}^{\text{lin}}(s) \quad \text{and} \quad v_j \;\leq\; v_j^{\text{lin}}(s), \quad j \in N \tag{15.18}$$

Define the 2×2 matrix function

$$A_{jk}(S_{jk}, v_j) \;:=\; \mathbb{I}_2 - \frac{2}{v_j} z_{jk} \left(S_{jk}\right)^{\mathsf{T}} \tag{15.19}$$

where $\mathbb{I}_2$ is the identity matrix of size 2, $z_{jk} := [r_{jk}\ \ x_{jk}]^{\mathsf{T}}$ is the column vector of line impedance and $S_{jk} := [P_{jk}\ \ Q_{jk}]^{\mathsf{T}}$ is the column vector of branch power flows, so that $z_{jk}\left(S_{jk}\right)^{\mathsf{T}}$ is a 2×2 matrix with rank less or equal to 1. The matrices $A_{jk}(S_{jk}, v_j)$ describe how changes in branch power flows propagate towards the root node 0; see comments below. Evaluate the Jacobian matrix $A_{jk}(S_{jk}, v_j)$ at the boundary values:

$$\underline{A}_{jk} \;:=\; A_{jk}\left(\left[S_{jk}^{\text{lin}}(s^{\max})\right]^{+},\, v_j^{\min}\right) \;=\; \mathbb{I}_2 - \frac{2}{v_j^{\min}} z_{jk} \left(\left[S_{jk}^{\text{lin}}(s^{\max})\right]^{+}\right)^{\mathsf{T}} \tag{15.20}$$

Here $([a]^+)^\mathsf{T}$ is the row vector $[[a_1]^+ \ [a_2]^+]$ with $[a_j]^+ := \max\{0, a_j\}$.

For a radial network, for $j \neq 0$, every line $j \to k$ identifies a unique node k and therefore, to simplify notation, we refer to a line interchangeably by (j,k) or j and use A_j, $\underline{A}_j$, z_j etc. in place of A_{jk}, $\underline{A}_{jk}$, z_{jk} etc. respectively. Assume

C15.3: The cost function is $C(x) := \sum_{j=0}^{N} C_j\left(p_j\right)$ with $C_0(p_0)$ strictly increasing in p_0. There is no constraint on s_0.

C15.4: The set $\mathbb{B}_j$ of injections satisfies $\hat{v}_j^{\text{lin}}(s) \leq v_j^{\max}$, $j \in N$, where $\hat{v}_j^{\text{lin}}(s)$ is given by (15.17).

C15.5: For each leaf node $j \in N$ let the unique path from j to 0 have k lines and be denoted by $\mathbb{P}_j := ((i_k, i_{k-1}), \ldots, (i_1, i_0))$ with $i_k = j$ and $i_0 = 0$. Then $\underline{A}_{i_t} \cdots \underline{A}_{i_{t'}} z_{i_{t'+1}} > 0$ for all $1 \leq t \leq t' < k$, where $\underline{A}_j$ are defined in (15.20).

Theorem 15.3. Suppose the network graph G is a tree and C15.3–C15.5 hold. Then OPF-socp (15.15) is exact.

The proof of Theorem 15.3 is long and relegated to Appendix 15.4. It can be shown that Theorem 15.3 have the following simple and practical interpretation: OPF-socp is exact provided at least one of the following is satisfied:

- There are no reverse power flows in the network.
- The r/x ratios on all lines are equal.
- If the r/x ratios increase in the downstream direction from the substation (node 0) to the leaves then there are no reverse real power flows.
- If the r/x ratios decrease in the downstream direction then there are no reverse reactive power flows.

These properties are derived in [141, 142, 143] and are special cases of Theorem 15.3.

We now comment on the conditions C15.3–C15.5.

Remark 15.3 (Strong exactness)**.** Condition C15.3 requires that the cost functions C_j depend only on the injections p_j. For instance, if $C_j\left(p_j\right) = p_j$, then the cost is total active power loss over the network. It also requires that C_0 be strictly increasing but makes no assumption on $C_j, j > 0$. Common cost functions such as line loss or generation cost usually satisfy C15.3. If C_0 is only nondecreasing, rather than strictly increasing, in p_0 then C15.3–C15.5 still guarantee that all optimal solutions of OPF (15.14) are (effectively) optimal for OPF-socp (15.15), but OPF-socp may not be exact in our definition, i.e., it may also have an optimal solution that maintains a strict inequality in (15.4). In this case the proof of Theorem 15.3 can still construct from it another optimal solution that attains equalities in (15.4) and is hence optimal for OPF. □

C15.4 is affine in the injections $s := (p,q)$. It enforces the upper bounds on voltage magnitudes because of (15.18). C15.5 is a technical assumption and has a simple interpretation: the branch power flow S_{jk}

on all branches should move in the same direction. Specifically, given a marginal change in the complex power on line $j \to k$, the 2×2 matrix $\underline{A}_{jk}$ is (a lower bound on) the Jacobian and describes the effect of this marginal change on the complex power on the line immediately upstream from line $j \to k$. The product of $\underline{A}_i$ in C15.5 propagates this effect upstream towards the root. C15.5 requires that a small change, positive or negative, in the power flow on a line affects *all* upstream branch powers in the same direction. This seems to hold with a significant margin in practice.

The exactness of SOCP relaxation does not require convexity, i.e., the cost $C(x) = \sum_{j=0}^{n} C_j(\text{Re} s_j)$ need not be a convex function and the injection regions $\mathbb{B}_j$ need not be convex sets. Convexity allows polynomial-time computation. Moreover when it is convex the exactness of SOCP relaxation also implies the uniqueness of the optimal solution shows

Theorem 15.4 (Unique optimal of SOCP relaxation)**.** Suppose the network graph G is a tree. Suppose the costs C_j, $j \in N$, are convex functions and the injection regions $\mathbb{B}_j$, $j \in N$, are convex sets. If OPF-socp (15.15) is exact then its optimal solution is unique.

Proof. Suppose $\hat{x}$ and $\tilde{x}$ are distinct optimal solutions of the relaxation OPF-socp (15.15). Since the feasible set of OPF-socp is convex the point $x := (\hat{x} + \tilde{x})/2$ is also feasible for OPF-socp. Since the cost function C is convex and both $\hat{x}$ and $\tilde{x}$ are optimal for (15.15), x is also optimal for (15.15). The exactness of OPF-socp then implies that x attains equality in (15.4). This contradicts Theorem 5.1 that shows that if $\hat{x}$ and $\tilde{x}$ are feasible, then no convex combination of $\hat{x}$ and $\tilde{x}$ can be feasible. □

15.3 Three-phase OPF

15.3.1 Reformulation

Consider the three-phase OPF (13.34) in BFM for radial networks studied in Chapter 13.2.2, reproduced here:

$$\min_{(u,x)} \; C(u,x) \quad \text{s. t.} \quad (13.18)(13.19)(13.32)(13.33) \tag{15.21}$$

where $(u,x) := (u, s, v, \ell, S, V, \tilde{I})$, u denotes the internal variables of three-phase devices and x denotes the terminal variables that interact through power flow equations. The devices are modeled by the conversion rule (13.18) and the operational constraint (13.19) on u. The power flow equation is (13.32) and the operational constraint on x is (13.33).

To simplify notation we assume, as in Chapter 14.5.1, that:

- Only three-phase voltage and power sources are included, in Y or Δ configurations.
- The neutrals of all Y-configured devices are directly grounded and all voltages are defined with respect to the ground, so that all neutral voltages $\gamma_j^Y := V_j^n = 0$.

Then the internal variables for these devices are $u := (u_j, j \in \overline{N})$ where

$$u_j \;:=\; \begin{cases} v_j^{Y/\Delta} & \text{if} \quad \text{device } j \text{ is a voltage source} \\ \left(s_j^{Y/\Delta}, X_j^{\Delta}, \ell_j^{\Delta}\right) & \text{if} \quad \text{device } j \text{ is a power source} \end{cases} \tag{15.22}$$

The device models (13.18)(13.19) have been reformulated as (14.33) in Chapter 14.5.1, with the 3×3 matrix variables W_{jj} and $W_j^{Y/\Delta}$ in BIM replaced by v_j and $v_j^{Y/\Delta}$ respectively in BFM.

Since the voltage phasors V_j are no longer needed to relate with internal variables u_j, we can omit the quadratic constraints (13.32e), $v_j = V_j V_j^{\mathsf{H}}$, $\ell_{jk} = \tilde{I}_{jk}\tilde{I}_{jk}^{\mathsf{H}}$, and $S_{jk} = V_j \tilde{I}_{jk}^{\mathsf{H}}$. Let the BFM variables be $x := (s, v, \ell, S)$ where v_j, ℓ_{jk}, S_{jk} is each a 3×3 matrix. Finally we assume the terminal voltage V_0 at bus 0 is given and imposes the constraint $v_0 = V_0 V_0^{\mathsf{H}}$. Then the three-phase OPF (15.21) can be reformulated as follows. Let the feasible set be

$$\mathbb{T}_{3p} \;:=\; \left\{(u,x) := (u,s,v,\ell,S) \mid (u,x) \text{ satisfies } (13.32\text{a}) - (13.32\text{d})(13.33)(14.33), v_0 = V_0 V_0^{\mathsf{H}}\right\} \tag{15.23a}$$

where u is defined in (15.22). The three-phase OPF problem is equivalent to:

$$\min_{u,x} \; C(u,x) \quad \text{subject to} \quad (u,x) \in \mathbb{T}_{3p} \tag{15.23b}$$

15.3.2 Semidefinite relaxation

OPF (15.23) is nonconvex due to the rank-1 constraint (13.32d) in the power flow equations and the rank-1 constraints (14.33a)(14.33b)(14.33f) in the device models. Omitting these rank-1 constraints yields a semidefinite relaxation. Recall the function $M(A,B,C)$ that constructs a 6×6 matrix from 3×3 matrices A, B, C:

$$M(A,B,C) \;:=\; \begin{bmatrix} A & B \\ B^{\mathsf{H}} & C \end{bmatrix} \tag{15.24}$$

where A, C are 3×3 Hermitian matrices and B is a 3×3 arbitrary matrix. Then the psd constraints in (13.32c) and in (14.33f) can be written in terms of M as respectively.

$$\begin{aligned} M\left(v_j, S_{jk}, \ell_{jk}\right) &= \begin{bmatrix} v_j & S_{jk} \\ S_{jk}^{\mathsf{H}} & \ell_{jk} \end{bmatrix} \succeq 0, \qquad j \to k \in E \\ M\left(v_j, X_{jk}^{\Delta}, \ell_{jk}^{\Delta}\right) &= \begin{bmatrix} v_j & X_{jk}^{\Delta} \\ X_{jk}^{\Delta\mathsf{H}} & \ell_{jk}^{\Delta} \end{bmatrix} \succeq 0, \qquad j \to k \in E \end{aligned}$$

Then the feasible set of the semidefinite relaxation is defined by the following constraints:

$$\begin{aligned}
\text{network:} \quad && v_0 &= V_0 V_0^{\mathsf{H}}, \ \ (13.32\text{a})(13.32\text{b}), (13.33), && (15.25\text{a})\\
&& 0 &\preceq M\left(v_j, S_{jk}, \ell_{jk}\right), \quad (j,k) \in E && (15.25\text{b})\\
\text{devices:} \quad && v_j &= v_j^Y, \quad v_j^Y \succeq 0, \qquad j \in N_v^Y && (15.25\text{c})\\
&& \Gamma v_j \Gamma^{\mathsf{T}} &= v_j^\Delta, \quad v_j^\Delta \succeq 0, \qquad j \in N_v^\Delta && (15.25\text{d})\\
&& s_j &= -s_j^Y, \qquad j \in N_p^Y && (15.25\text{e})\\
&& s_j &= -\text{diag}\left(X_j^\Delta \Gamma\right), \ \ s_j^\Delta = \text{diag}\left(\Gamma X_j^\Delta\right), \ \ M\left(v_j, X_j^\Delta, \ell_j^\Delta\right) \succeq 0, \qquad j \in N_p^\Delta && (15.25\text{f})
\end{aligned}$$

where $V_0 \in \mathbb{C}^3$ is given. Define the feasible set as

$$\mathbb{T}_{3p}^+ \ := \ \{(u,x) := (u,s,v,\ell,S) \mid (u,x) \text{ satisfies } (15.25)\} \tag{15.26a}$$

where u is defined in (15.22). The set $\mathbb{T}_{3p}^+$ is a convex superset of $\mathbb{T}_{3p}$. The semidefinite relaxation of the three-phase OPF problem (15.23) is:

$$\min_{u,x} \ C(u,x) \quad \text{subject to} \quad (u,x) \in \mathbb{T}_{3p}^+ \tag{15.26b}$$

Let $(u^{\text{opt}}, x^{\text{opt}})$ denote an optimal solution of the SDP relaxation (15.26). We say (15.26) is *exact* if the psd matrices of *every* optimal solution $(u^{\text{opt}}, x^{\text{opt}})$ are of rank 1, i.e.,

$$\begin{aligned}
\text{rank}\left(M\left(v_j^{\text{opt}}, X_j^{\Delta\text{opt}}, \ell_j^{\Delta\text{opt}}\right)\right) &= 1, \qquad \text{rank}\left(v_j^{Y/\Delta\text{opt}}\right) = 1, \qquad j \in \overline{N} && (15.27\text{a})\\
\text{rank}\left(M\left(v_j^{\text{opt}}, S_{jk}^{\text{opt}}, \ell_{jk}^{\text{opt}}\right)\right) &= 1, \qquad (j,k) \in E && (15.27\text{b})
\end{aligned}$$

This means that $(u^{\text{opt}}, x^{\text{opt}})$ is feasible and therefore optimal for the original OPF (15.23).

Suppose $\text{rank}\left(v_j^{\text{opt}}\right) = 1$. Then $v_j^{Y/\Delta\text{opt}}$ is of rank 1 by (15.25c)(15.25d). Unfortunately $M\left(v_j^{\text{opt}}, X_j^{\Delta\text{opt}}, \ell_j^{\Delta\text{opt}}\right)$ and $M\left(v_j^{\text{opt}}, S_{jk}^{\text{opt}}, \ell_{jk}^{\text{opt}}\right)$ may not be of rank 1 because $\ell_j^{\Delta\text{opt}}$ and ℓ_{jk}^{opt} respectively may not be rank-1; see Lemma 14.10. As discussed after Lemma 14.10, even though the SDP relaxation (15.26) may not be exact, it is still possible to recover an optimal solution of OPF (15.23) from an optimal solution $(u^{\text{opt}}, x^{\text{opt}})$ of its relaxation (15.26) when $\text{rank}\left(v_j^{\text{opt}}\right) = 1$ for all $j \in \overline{N}$, if there were no numerical error.

Equivalence. When the network graph is a tree, then it can be shown that OPF OPF (15.23) and its relaxation (15.26) in BFM are equivalent to OPF (14.34) and its relaxation (14.36) respectively in BIM (see [131, Proposition 1]).

15.4 Appendix: Proof of Theorem 15.3: inactive voltage upper bounds

Given an optimal solution $x := (s,v,\ell,S)$ that maintains a strict inequality in (15.4), $v_j \ell_{jk} \geq |S_{jk}|^2$ for some line $j \to k \in E$, the proof of Theorem 15.2 in Section 15.2.3 constructs another feasible solution $\hat{x}$ that incurs a strictly smaller cost, contradicting the optimality of x. The modification is over a single line over which x maintains a strict inequality. The proof of Theorem 15.3 is also by contradiction but, unlike that of Theorem 15.2, the construction of $\hat{x}$ from x involves modifications on multiple lines, propagating from the line that is closest to bus 0 where strictly inequality holds all the way to bus 0. The proof relies crucially on the recursive structure of the branch flow model (15.1).

Proof of Theorem 15.3. To simplify notation we only prove the theorem for the case of a linear network representing a primary feeder without laterals. The proof for a general tree network follows the same idea but with more cumbersome notations; see [33] for details. We adopt the graph orientation where every line points *towards* the root node 0. The notation for the linear network is explained in Figure 15.3 (we refer to a line $j \to k$ by j and index the associated variables $z_{jk}, S_{jk}, \ell_{jk}$ with j). With this notation the branch

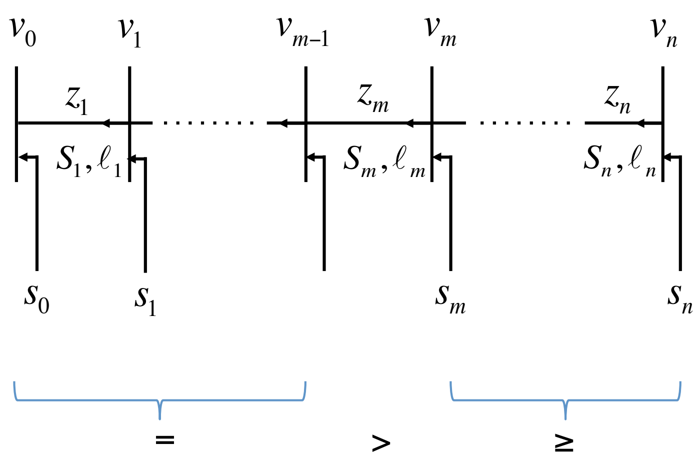


Figure 15.3: Linear network and notations. Line m in the proof is the line closest to bus 0 where the inequality in (15.29) is strict, i.e., (15.29) holds with equality at lines $j = 1, \ldots, m-1$, strict inequality at line m, and inequality at lines $j = m+1, \ldots, N$.

flow model (15.1) is the following recursion:

$$
\begin{aligned}
S_{j-1} &= S_j - z_j \ell_j + s_{j-1}, && j = 1, \ldots, N && (15.28a)\\
v_{j-1} &= v_j - 2\,\mathrm{Re}\left(z_j^{\mathsf{H}} S_j\right) + |z_j|^2 \ell_j, && j = 1, \ldots, N && (15.28b)\\
v_j \ell_j &= |S_j|^2, && j = 1, \ldots, N && (15.28c)\\
S_n &= s_n, \qquad S_0 := 0 && && (15.28d)
\end{aligned}
$$

where v_0 is given. The SOCP relaxation of (15.28c) is:

$$
v_j \ell_j \;\geq\; |S_j|^2, \qquad j = 1, \ldots, N \tag{15.29}
$$

OPF on the linear network in Figure 15.3 then becomes (s_0 is unconstrained by assumption C15.3):

OPF:

$$\min_x \quad C(x) := \sum_{j=0}^{N} C_j\left(p_j\right) \tag{15.30a}$$

$$\text{s.t.} \quad (15.13)(15.28) \tag{15.30b}$$

and its SOCP relaxation becomes:
OPF-socp:

$$\begin{aligned} \min_x \quad & C(x) := \sum_{j=0}^{N} C_j\left(p_j\right) \\ \text{s.t.} \quad & (15.13), (15.28a)(15.28b)(15.28d), (15.29) \end{aligned} \tag{15.31a}$$

For the linear network assumption C15.5 reduces:

C15.5': $\underline{A}_j \cdots \underline{A}_k\, z_{k+1} > 0$ for $1 \leq j \leq k < N$ where $\underline{A}_j$ are defined in (15.20).

Our goal is to prove OPF-socp (15.31) is exact, i.e., every optimal solution of (15.31) attains equality in (15.29) and hence is also optimal for OPF (15.30). Suppose on the contrary that there is an optimal solution $x := (S, \ell, v, s)$ of OPF-socp (15.31) that violates (15.28c). We will construct another feasible point $\hat{x} := (\hat{S}, \hat{\ell}, \hat{v}, \hat{s})$ of OPF-socp (15.31) that has a strictly lower cost than x, contradicting the optimality of x.

Let $m := \min\{j \in N \mid v_j \ell_j > |S_j|^2\}$ be the closest line from bus 0 where (15.28c) is violated; see Figure 15.3. Pick any $\varepsilon_m \in (0, \ell_m - |S_m|^2/v_m]$ and construct $\hat{x}$ as follows:

1. $\hat{s}_j := s_j$ for $j \neq 0$.

2. For $\hat{S}, \hat{\ell}, \hat{s}_0$:

   - For $j = N, \ldots, m+1$: $\hat{S}_j := S_j$ and $\hat{\ell}_j := \ell_j$.
   - For $j = m$: $\hat{S}_m := S_m$ and $\hat{\ell}_m := \ell_m - \varepsilon_m$.
   - For $j = m-1, \ldots, 1$:

$$\begin{aligned} \hat{S}_j &:= \hat{S}_{j+1} - z_{j+1}\hat{\ell}_{j+1} + \hat{s}_j \\ \hat{\ell}_j &:= \frac{|\hat{S}_j|^2}{v_j} \end{aligned}$$

   - $\hat{s}_0 := -\hat{S}_1 + z_1 \hat{\ell}_1$.

3. $\hat{v}_0 := v_0$. For $j = 1, \ldots, N$,

$$\hat{v}_j := \hat{v}_{j-1} + 2\,\mathrm{Re}\left(z_j^{\mathsf{H}} \hat{S}_j\right) - |z_j|^2 \hat{\ell}_j$$

Notice that the denomintor in $\hat{\ell}_j$ is defined to be v_j, not $\hat{v}_j$. This decouples the recursive construction of $(\hat{S}_j, \hat{\ell}_j)$ and $\hat{v}_j$ so that the former propagates from bus N towards bus 1 while the latter propagates in the opposite direction.

By construction $\hat{x}$ satisfies (15.28a), (15.28b), (15.28d), and (15.13b). We only have to prove that $\hat{x}$ satisfies (15.13a) and (15.29). Hence the proof of Theorem 15.3 is complete after Lemma 15.5 is established, which asserts that $\hat{x}$ is feasible and has a strictly lower cost under assumptions C15.3, C15.4, C15.5'.

Lemma 15.5. Under the conditions of Theorem 15.3 $\hat{x}$ satisfies

1. $C(\hat{x}) \ < \ C(x)$.
2. $\hat{v}_j \hat{\ell}_j \ \geq \ \left|\hat{S}_j\right|^2$, $j \in N$.
3. ${v_j}^{\min} \ \leq \ \hat{v}_j \ \leq \ v_j^{\max}$, $j \in N$.

To simplify notation redefine $S_0 := -s_0$ and $\hat{S}_0 := -\hat{s}_0$. Then for $j \in \overline{N}$ define $\Delta S_j := \hat{S}_j - S_j$ and $\Delta v_j := \hat{v}_j - v_j$. The key result that leads to Lemma 15.5 is:

$$\Delta S_j \ \geq \ 0 \quad \text{and} \quad \Delta v_j \ \geq \ 0, \qquad j \in \overline{N}$$

The first inequality is stated more precisely in Lemma 15.6 and proved after the proof of Lemma 15.5.

Lemma 15.6. Suppose $m > 1$ and C15.5' holds. Then $\Delta S_j \geq 0$ for $j \in \overline{N}$ with $\hat{S}_j > S_j$ for $j = 0, \ldots, m-1$. In particular $\hat{s}_0 < s_0$.

We now prove the second inequality together with Lemma 15.5 assuming Lemma 15.6 holds.

Proof of Lemma 15.5. 1) If $m = 1$ then, by construction, $\hat{s}_0 = s_0 - z_1 \varepsilon_1 < s_0$ since $z_1 > 0$. If $m > 1$ then $\hat{s}_0 < s_0$ by Lemma 15.6. Since $\hat{s} = s$ and $\hat{s}_0 < s_0$ we have

$$C(\hat{x}) - C(x) \ = \ \sum_{j=0}^{N} \left(C_j\left(\hat{p}_j\right) - C_j\left(p_j\right)\right) \ = \ C_0\left(\hat{p}_0\right) - C_0\left(p_0\right) \ < \ 0$$

as desired, since C_0 is strictly increasing.

2) To avoid circular argument we will first prove using Lemma 15.6

$$\hat{v}_j \ \geq \ v_j, \qquad j \in N \tag{15.32}$$

We will then use this and Lemma 15.6 to prove $\hat{v}_j \hat{\ell}_j \geq |\hat{S}_j|^2$ for all $j \in N$. We then use assumption C15.4 to prove $v_j^{\min} \leq \hat{v}_j \leq v_j^{\max}$, $j \in N$. This shows that $\hat{x}$ satisfies (15.29) and (15.13a) (in addition to (15.28a)(15.28b)(15.28d) and (15.13b)).

To prove (15.32), note that both $\hat{v}$ and v satisfy (15.28b) and hence we have, for $j = 1, \ldots, N$,

$$\Delta v_{j-1} \ = \ \Delta v_j - 2\,\mathrm{Re}\left(z_j^{\mathsf{H}} \Delta S_j\right) + |z_j|^2 \Delta \ell_j \tag{15.33}$$

where $\Delta\ell_j := \hat{\ell}_j - \ell_j$. From (15.28a) we have

$$z_j \Delta\ell_j \;=\; \Delta S_j - \Delta S_{j-1} + \Delta s_{j-1}$$

where $\Delta s_0 := \hat{s}_0 - s_0 < 0$ and $\Delta s_{j-1} = 0$ for $j > 1$. Multiplying both sides by z_j^{H} and noticing that both sides must be real, we conclude

$$|z_j|^2 \Delta\ell_j \;=\; \mathrm{Re}\,\left(z_j^{\mathsf{H}}\Delta S_j - z_j^{\mathsf{H}}\Delta S_{j-1} + z_j^{\mathsf{H}}\Delta s_{j-1}\right)$$

Substituting into (15.33) we have for $j = 1, \ldots, N$

$$\Delta v_j - \Delta v_{j-1} \;=\; \mathrm{Re}\, z_j^{\mathsf{H}}\Delta S_j \;+\; \mathrm{Re}\, z_j^{\mathsf{H}}\Delta S_{j-1} \;-\; \mathrm{Re}\, z_j^{\mathsf{H}}\Delta s_{j-1}$$

But Lemma 15.6 implies that $\mathrm{Re}\, z_j^{\mathsf{H}}\Delta S_j = r_j \Delta P_j + x_j \Delta Q_j \geq 0$. Similarly every term on the right-hand side is nonnegative and hence

$$\Delta v_j \;\geq\; \Delta v_{j-1} \qquad \text{for } j = 1, \ldots, N$$

implying that $\Delta v_j \geq \Delta v_0 = 0$, proving (15.32).

We now use (15.32) to prove the second assertion of the lemma. By construction, for $j = m+1, \ldots, N$,

$$\hat{\ell}_j \;=\; \ell_j \;\geq\; \frac{|S_j|^2}{v_j} \;\geq\; \frac{|\hat{S}_j|^2}{\hat{v}_j}$$

as desired, since $\hat{S}_j = S_j$ and $\hat{v}_j \geq v_j$. Similarly (15.29) holds for $\hat{x}$ for $j = m$ because of the choice of ε_m. For $j = 1, \ldots, m-1$, $\hat{v}_j \geq v_j$ again implies

$$\hat{\ell}_j \;=\; \frac{|\hat{S}_j|^2}{v_j} \;\geq\; \frac{|\hat{S}_j|^2}{\hat{v}_j}$$

3) The relation (15.32) means

$$\hat{v}_j \;\geq\; v_j \;\geq\; v_j^{\min}, \qquad j \in N$$

Assumption C15.4 and (15.18) imply that

$$\hat{v}_j \;\leq\; v_j^{\mathrm{lin}}(s) \;\leq\; v_j^{\max}, \qquad j \in N$$

This proves $\hat{x}$ satisfies (15.13a) and completes the proof of Lemma 15.5. □

The remainder of this subsection is devoted to proving the key result Lemma 15.6.

Proof of Lemma 15.6. By construction $\Delta S_j = 0$ for $j = m, \ldots, n$. To prove $\Delta S_j > 0$ for $j = 0, \ldots, m-1$, the key idea is to derive a recursion on ΔS_j in terms of the Jacobian matrix $A_j(S_j, v_j)$. The intuition is that, when the branch current ℓ_m is reduced by ε_m to $\hat{\ell}_m$, loss on line m is reduced and all upstream branch powers S_j will be increased to $\hat{S}_j$ as a consequence.

This is proved in three steps, of which we now give an informal overview. First we derive a recursion (15.35) on ΔS_j. This motivates a collection of linear dynamical systems w in (15.37) that contains the process $(\Delta S_j,\ j=0,\ldots,m-1)$ as a specific trajectory. Second we construct another collection of linear dynamical systems $\underline{w}$ in (15.38) such that assumption C15.5' implies $\underline{w} > 0$. Finally we prove an expression for the process $w - \underline{w}$ that shows $w \geq \underline{w}$ (in Lemmas 15.7, 15.8, 15.9). This then implies $\Delta S = w \geq \underline{w} > 0$ as desired. We now make these steps precise.

Since both x and $\hat{x}$ satisfy (15.28a) and $\hat{s}_j = s_j$ for all $j \in N$ we have (with the redefined $\Delta S_0 := -(\hat{s}_0 - s_0)$)

$$\Delta S_{j-1} \;=\; \Delta S_j - z_j \Delta\ell_j, \quad j = 1, 2, \ldots, N \tag{15.34}$$

where $\Delta\ell_j := \hat{\ell}_j - \ell_j$. For $j = 1,\ldots,m-1$ both x and $\hat{x}$ satisfy (15.28c). For these j, fix any $v_j \geq v_j^{\min}$ and consider $\ell_j := \ell_j(S_j)$ as functions of the real pair $S_j := (P_j, Q_j)$:

$$\ell_j(S_j) \;:=\; \frac{P_j^2 + Q_j^2}{v_j}, \quad j = 1,\ldots,m-1$$

whose Jacobian are the row vectors:

$$\frac{\partial \ell_j}{\partial S_j}(S_j) \;=\; \frac{2}{v_j}[P_j \;\; Q_j] \;=\; \frac{2}{v_j} S_j^{\mathsf{T}}$$

The mean value theorem implies for $j = 1,\ldots,m-1$

$$\Delta\ell_j \;=\; \ell_j(\hat{S}_j) - \ell_j(S_j) \;=\; \frac{\partial \ell_j}{\partial S_j}(\tilde{S}_j)\,\Delta S_j$$

where $\tilde{S}_j := \alpha_j S_j + (1-\alpha_j)\hat{S}_j$ for some $\alpha_j \in [0,1]$. Substituting it into (15.34) we obtain the recursion, for $j = 1,\ldots,m-1$,

$$\Delta S_{j-1} \;=\; \tilde{A}_j \Delta S_j \tag{15.35a}$$

$$\Delta S_{m-1} \;=\; \varepsilon_m z_m \;>\; 0 \tag{15.35b}$$

where the 2×2 matrix $\tilde{A}_j$ is the matrix function $A_j(S_j, v_j)$ defined in (15.19) evaluated at $(\tilde{S}_j, v_j)$:

$$\tilde{A}_j \;:=\; A_j(\tilde{S}_j, v_j) \;\;:=\;\; \mathbb{I}_2 - \frac{2}{v_j} z_j \tilde{S}_j^{\mathsf{T}} \tag{15.36}$$

which depends on $(S_j, \hat{S}_j)$ through $\tilde{S}_j$.

Note that $\tilde{A}_j$ and ΔS_j are not independent since both are defined in terms of $(S_j, \hat{S}_j)$, and therefore strictly speaking (15.35) does not specify a *linear* system. Given an optimal solution x of the relaxation OPF-socp (15.31) and our modified solution $\hat{x}$, however, the sequence of matrices $\tilde{A}_j,\ j = 1,\ldots,m-1$, are fixed. We can therefore consider the following collection of discrete-time linear time-varying systems (one for each τ), whose state at time t (going backward in time) is $w(t;\tau)$, when it starts at time $\tau \geq t$ in the initial state $z_{\tau+1}$: for each τ with $0 < \tau < m$,

$$w(t-1;\tau) \;=\; \tilde{A}_t\, w(t;\tau), \;\; t = \tau, \tau-1, \ldots, 1 \tag{15.37a}$$

$$w(\tau;\tau) \;=\; z_{\tau+1} \tag{15.37b}$$

Clearly $\Delta S_j = \varepsilon_m w(j;m-1)$. Hence, to prove $\Delta S_j > 0$, it suffices to prove $w(j;m-1) > 0$ for all j with $0 \leq j \leq m-1$.

To this end we compare the system $w(t;\tau)$ with the following collection of linear time-variant systems: for each τ with $0 < \tau < m$,

$$\underline{w}(t-1;\tau) \;=\; \underline{A}_t\, \underline{w}(t;\tau), \quad t = \tau, \tau-1, \ldots, 1 \tag{15.38a}$$

$$\underline{w}(\tau;\tau) \;=\; z_{\tau+1} \tag{15.38b}$$

where $\underline{A}_t$ is defined in (15.20) and reproduced here:

$$\underline{A}_t \;:=\; A_t\left(\left[S_t^{\text{lin}}(s^{\max})\right]^+,\, \underline{v}_t\right) \;=\; \mathbb{I}_2 - \frac{2}{v_t^{\min}}\, z_t \left(\left[S_t^{\text{lin}}(s^{\max})\right]^+\right)^{\mathsf{T}} \tag{15.39}$$

Note that $\underline{A}_t$ are *independent of* the OPF-socp solution x and our modified solution $\hat{x}$. Then assumption C15.5' is equivalent to

$$\underline{w}(t;\tau) \;>\; 0 \qquad \text{for all } 0 \leq t \leq \tau < m \tag{15.40}$$

We now prove, in Lemmas 15.7, 15.8, 15.9, that $w(t;\tau) \geq \underline{w}(t;\tau)$ and hence C15.5' implies $\Delta S_j = \varepsilon_m w(j;m-1) \geq \varepsilon_m \underline{w}(j;m-1) > 0$, establishing Lemma 15.6.

Lemma 15.7. For each $t = m-1, \ldots, 1$

$$\tilde{A}_t - \underline{A}_t \;=\; 2\, z_t\, \delta_t^{\mathsf{T}}$$

for some 2-dimensional vector $\delta_t \geq 0$.

Proof of Lemma 15.7. Fix any $t = m-1, \ldots, 1$. We have $S_t \leq S_t^{\text{lin}}(s)$ from (15.18). Even though we have not yet proved $\hat{S}_t$ is feasible for OPF-socp we know $\hat{S}_t$ satisfies (15.28a) by construction of $\hat{x}$. The same argument as in Corollary 5.4 then shows $\hat{S}_t \leq S_t^{\text{lin}}(s)$. Hence $\tilde{S}_t := \alpha_t S_t + (1-\alpha_t)\hat{S}_t$, $\alpha_t \in [0,1]$, satisfies $\tilde{S}_t \leq S_t^{\text{lin}}(s)$. Hence

$$\tilde{S}_t \;\leq\; S_t^{\text{lin}}(s) \;\leq\; S_t^{\text{lin}}(s^{\max}) \;\leq\; \left[S_t^{\text{lin}}(s^{\max})\right]^+ \tag{15.41}$$

Using the definitions of $\tilde{A}_t$ in (15.36) and $\underline{A}_t$ in (15.39) we have $\tilde{A}_t - \underline{A}_t = 2 z_t \delta_t^{\mathsf{T}}$ where

$$\delta_t^{\mathsf{T}} \;:=\; \begin{bmatrix} \dfrac{\left[P_t^{\text{lin}}(s^{\max})\right]^+}{v_t^{\min}} - \dfrac{\tilde{P}_t}{v_t} & \dfrac{\left[Q_t^{\text{lin}}(s^{\max})\right]^+}{v_t^{\min}} - \dfrac{\tilde{Q}_t}{v_t} \end{bmatrix}$$

Then (15.41) and $v_t \geq v_t^{\min}$ impy that $\delta_t \geq 0$. $\square$

For each τ with $0 < \tau < m$ define the scalars $a(t;\tau)$ in terms of the solution $\underline{w}(t;\tau)$ of (15.38) and δ_t in Lemma 15.7:

$$a(t;\tau) \;:=\; 2\,\delta_t^{\mathsf{T}}\, \underline{w}(t;\tau) \;>\; 0 \tag{15.42}$$

Lemma 15.8. Fix any τ with $0 < \tau < m$. For each $t = \tau, \tau - 1, \ldots, 0$ we have

$$w(t;\tau) - \underline{w}(t;\tau) \;=\; \sum_{t'=t+1}^{\tau} a(t';\tau)\, w(t;t'-1)$$

Proof of Lemma 15.8. Fix a τ with $0 < \tau < m$. We now prove the lemma by induction on $t = \tau, \tau-1, \ldots, 0$. The assertion holds for $t = \tau$ since $w(\tau;\tau) - \underline{w}(\tau;\tau) = 0$. Suppose it holds for t. Then for $t-1$ we have from (15.37) and (15.38)

$$\begin{aligned}
w(t-1;\tau) - \underline{w}(t-1;\tau) &= \tilde{A}_t\, w(t;\tau) - \underline{A}_t\, \underline{w}(t;\tau) \\
&= \left(\tilde{A}_t - \underline{A}_t\right) \underline{w}(t;\tau) \;+\; \tilde{A}_t \left(w(t;\tau) - \underline{w}(t;\tau)\right) \\
&= a(t;\tau)\, z_t \;+\; \sum_{t'=t+1}^{\tau} a(t';\tau)\, \tilde{A}_t\, w(t;t'-1) \\
&= a(t;\tau)\, z_t \;+\; \sum_{t'=t+1}^{\tau} a(t';\tau)\, w(t-1;t'-1) \\
&= \sum_{t'=t}^{\tau} a(t';\tau)\, w(t-1;t'-1)
\end{aligned}$$

where the first term on the right-hand side of the third equality follows from Lemma 15.7 and the definition of $a(t;\tau)$ in (15.42), and the second term from the induction hypothesis. The last two equalities follow from (15.37). □

Lemma 15.9. Suppose C15.5' holds. Then for each τ with $0 < \tau < m$ and each $t = \tau, \tau-1, \ldots, 0$,

$$w(t;\tau) \;\geq\; \underline{w}(t;\tau) \;>\; 0 \tag{15.43}$$

Proof of Lemma 15.9. We prove the lemma by induction on (t, τ).

1. *Base case:* For each τ with $0 < \tau < m$, (15.43) holds for $t = \tau$, i.e., for t such that $\tau - t = 0$.

2. *Induction hypothesis:* For each τ with $0 < \tau < m$, suppose (15.43) holds for $t \leq \tau$ such that $0 \leq \tau - t \leq k-1$.

3. *Induction:* We will prove that, for each τ with $0 < \tau < m$, (15.43) holds for $t \leq \tau$ such that $0 \leq \tau - t \leq k$. For $t = \tau - k$ we have from Lemma 15.8

   $$w(t;\tau) - \underline{w}(t;\tau) \;=\; \sum_{t'=t+1}^{\tau} a(t';\tau)\, w(t;t'-1)$$

   But each $w(t;t'-1)$ in the summands satisfies $w(t;t'-1) \geq \underline{w}(t;t'-1)$ by the induction hypothesis. Hence, since $a(t';\tau) > 0$,

   $$w(t;\tau) - \underline{w}(t;\tau) \;\geq\; \sum_{t'=t+1}^{\tau} a(t';\tau)\, \underline{w}(t;t'-1) \;>\; 0$$

   where the last inequality follows from (15.40) and (15.42).

This completes our induction proof. □

Lemma 15.9 implies, for $j = 0, \ldots, m-1$, $\Delta S_j = \varepsilon_m w(j; m-1) > 0$. This completes the proof of Lemma 15.6. □

This completes the proof of Theorem 15.3 for the linear network. For a general tree network the proof is almost identical, except with more cumbersome notations, by focusing on a path from the root to a first line m over which $v_j \ell_j > |S_j|^2$; see [33]. □

15.5 Bibliographical notes

SOCP relaxation of Chapter 15.2.1 for radial networks in the DistFlow model of [18, 19] is first proposed in [144, 30]. Theorem 15.1 is proved in [32] and the proof presented here follows that in [36, Theorem 11]. Theorem 15.2 is from [30, Part I] which generalizes an earlier result in [144] to allow convex objective functions, shunt elements, and line limits. Theorems 15.3 and 15.4 are from [33]. The semidefinite relaxation of three-phase OPF in Chapter 15.3 follows the idea in [106, 131].

15.6 Problems

Chapter 15.2. The next three problems use a two-bus example to illustrate solvability of BFM, SOCP relaxation of OPF, and the exactness of SOCP.

Exercise 15.1 (Geometric insight)**.** Consider the 2-bus example modeled by the following DistFlow equations (line direction from bus 1 to bus 0). The power balance at bus 0 (noting that $S_{0k} := 0$) and other power flow equations over line $1 \to 0$ are given by (15.1):

$$\begin{aligned} p_0 - r\ell &= -p_1, \qquad q_0 - x\ell \;=\; -q_1 && \text{(15.44a)} \\ v_1 - v_0 &= 2(rp_1 + xq_1) - (r^2 + x^2)\ell && \text{(15.44b)} \\ p_1^2 + q_1^2 &= v_1 \ell && \text{(15.44c)} \end{aligned}$$

where the voltage v_0 and the injections p_1, q_1 are given.

1. Show that (equations (15.16)):

$$p_0^2 + q_0^2 \;=\; v_0 \ell \tag{15.45}$$

2. Solutions (p_0, q_0, v_1, ℓ) to (15.44) exist if and only if

$$rp_1 + xq_1 + \frac{1}{4} \;\geq\; (rq_1 - xp_1)^2$$

Exercise 15.2 (Geometric insight)**.** For the 2-bus network in Exercise 15.1, suppose $q_1 = 0$ and $v_0 = r = x = 1$ pu.

1. Show that solutions (p_0, q_0, v_1, ℓ) to (15.44) exist if and only if

$$\frac{1}{2}\left(1-\sqrt{2}\right) \;\leq\; p_1 \;\leq\; \frac{1}{2}\left(1+\sqrt{2}\right) \tag{15.46}$$

2. Show that for each injection value p_1 that satisfies (15.46), there are two voltage solutions v_1 given by

$$v_1 \;=\; \frac{1}{2}\left(1+2p_1 \mp \sqrt{\Delta}\right) \;=\; \frac{1}{2}\left(1+2p_1 \mp \sqrt{4p_1(1-p_1)+1}\right)$$

   where

$$\Delta \;:=\; 4\left((rp_1 + xq_1) \;-\; (rq_1 - xp_1)^2\right) \;+\; 1$$

3. Show that the locus (v_1, p_1) that satisfies (15.44) is a (rotated) ellipse. Plot the two solutions for v_1 in Part 2 as functions of p_1. These two curves form the ellipse.

4. Show that the lowest voltage solution is $v_1 = 0$ pu attained at $p_1 = 0$ pu and the highest voltage solution is $v_1 = 2$ pu attained at $p_1 = 1$ pu.

Exercise 15.3 (Feasible set and relaxation)**.** For the 2-bus network in Exercise 15.2 with $q_1 = 0$ and $v_0 = r = x = 1$ pu, suppose the injection p_1 is controllable. With a slight overload of notation, let $w := (p_0, q_0, p_1, v_1, \ell)$. Consider the OPF problem:

$$\begin{aligned}
\min_{w} \quad & c(w) \\
\text{s.t.} \quad & p_0 - \ell \;=\; -p_1, \quad q_0 - \ell \;=\; 0 \\
& p_0^2 \;+\; q_0^2 \;\;=\;\; \ell \\
& v_1 \;-\; 1 \;\;=\;\; 2p_1 \;-\; 2\ell \\
& 0.9 \text{ pu} \;\leq\; v_1 \leq 1.1 \text{ pu} \\
& p_1^{\min} \;\leq\; p_1 \;\leq\; p_1^{\max}
\end{aligned}$$

where the cost function $c(w)$ is strictly increasing in ℓ. Its SOCP relaxation replaces the quadratic equality constraint with the convex constraint $p_0^2 + q_0^2 \;\leq\; \ell$.

1. Determine the largest range $R_1 := [p_1^{\min},\ p_1^{\max}]$ over which the SOCP relaxation is exact.

2. Determine the largest range $R_2 := [p_1^{\min},\ p_1^{\max}]$ over which the SOCP relaxation is inexact. Note that in this regime, bus 1 is generating power and causing a large amount of reverse power flow.

3. What happens if the range $[p_1^{\min},\ p_1^{\max}]$ for injection p_1 overlaps with neither R_1 nor R_2?

Part VI

Appendix: mathematical preliminaries

Chapter 25

Appendix: mathematical preliminaries

In this chapter we review some basic concepts in linear algebra, algebraic graph theory and optimization that we have used in this book. There are numerous excellent books on each of these topics and our goal is *not* to be comprehensive or systematic in coverage, but to collect concepts and properties used throughout this book in one place for convenience of the readers who have already had exposures to these topics.

More details (on semidefinite relaxations) and can be found in, e.g., [73, 255, 145, 117, 118, 256, 122, 116].

25.1 Linear algebra

25.1.1 Vector spaces, basis, rank, nullity

25.1.1.1 Vector spaces, subspaces, span

This subsection mostly follows [10, Chapter 0]. We restrict ourselves mostly to finite vector spaces. Underlying a vector space is its *field* F, which is a set of *scalars* that is closed under two binary operations, called "addition" $(a+b)$ and "multiplication" (ab). Most often, $F = \mathbb{R}$ or $\mathbb{C}$ for us, but in general F can be the set of rational numbers, or a set of integers modulo a specified prime number, etc. The two operations must be associative and commutative, and each must have an identity element in the set; inverses must exist in the set for all elements under addition and for all elements except the additive identity under multiplication; multiplication must distribute over addition.

Definition 25.1 (Vector space)**.** A *vector space* V, or *linear space*, over a field F is a set V of objects, called *vectors*, that is closed under two binary operations:

- *vector addition*: $+ : V \times V \to V$ denoted by $x + y$;
- *scalar multiplication*: $\cdot : F \times V \to V$ denoted by $a \cdot x =: ax$;

and satisfies the following properties: for all $x, y, z \in V$ and $a, b \in F$,

1. *Associativity of vector addition*: $x+(y+z)=(x+y)+z$.

2. *Commutativity of vector addition*: $x+y=y+x$.

3. *Identity element of vector addition*: There exists $0 \in V$, called the *zero vector*, such that $x+0=x$.

4. *Inverse elements of vector addition*: There exists $-x \in V$, called the *additive inverse* of x, such that $x+(-x)=0$.

5. *Associativity of scalar multiplication*: $a(bx)=(ab)x$.

6. *Identity element of scalar multiplication*: There exists $1 \in F$, called the *multiplicative identity* in F such that $1x=x$.

7. *Distributivity of scalar multiplication over vector addition*: $a(x+y)=ax+by$.

8. *Distributivity of scalar multiplication over field addition*: $(a+b)x=ax+bx$.

A *subspace* of a vector space V over a field F is a subset of V that is itself a vector space over F with the same binary operations as in V. $\square$

If $F=\mathbb{R}$ then V is called a *real vector space*. If $F=\mathbb{C}$ then V is called a *complex vector space*. Given F and an integer n the set $V := F^n$ of n-tuples with components from F forms a vector space over F where the vector addition "+" is defined by componentwise addition: $[x+y]_i = x_i + y_i$. The vector space F^n is important because any finite dimensional vector space can be identified with F^n for some integer n (see Example 25.1 and the next subsection for a formal definiteion). Note that $\mathbb{R}^n$ is a real vector space ($V=\mathbb{R}^n$ over $F=\mathbb{R}$) while $\mathbb{C}^n$ is both a real vector space ($V=\mathbb{C}^n$ over $F=\mathbb{R}$) and a complex vector space ($V=\mathbb{C}^n$ over $F=\mathbb{C}$).

A vector space V is however not restricted to $V=F^n$. An important finite dimensional vector space over F is the set $M_{m,n}(F)$ of $m\times n$ matrices whose entries $[M]_{ij}\in F$ for any finite m and n. We can vectorize $A\in M_{m,n}(F)$ and treat A as a vector in $V=F^{mn}$, but we will mostly treat A as an array of scalars in $V=F^{m\times n}$. Note that matrix multiplication is not involved in the definition of $V=F^{m\times n}$ as a vector space (it can be treated as a composition of linear transformations when a matrix is viewed as a linear transformation from F^n to F^m; see below). If $m=n$ we abbreviate $M_{m,n}(F)$ to $M_{m,n}$. If $F=C$ we abbreviate $M_{m,n}(C)$ to $M_{m,n}$.

The components x_i of vectors $x\in V$ may not be from F. Possibly infinite dimensional examples include: the set of polynomials with real or with complex coefficients (of up to a specified degree or of arbitrary degree) is a real or complex vector space respectively; the set of real-valued or complex-valued functions on subsets of $\mathbb{R}$ or $\mathbb{C}$ is a real or complex vector space respectively.

If $S\subseteq V$ is a nonempty subset of the vector space V over a field F then span(S) is the intersection of all subspaces of V that contain S. It consists of all linear combinations of finitely many vectors in S:

$$\text{span}(S) \;=\; \{a_1x_1+\cdots+a_kx_k \,:\, x_1,\ldots,x_k\in S,\ a_1,\ldots,a_k\in F,\ k=1,2,\ldots\}$$

It can be checked that span(S) is always a subspace whether or not S is a subspace. S is said to *span* V if span$(S) = V$. Let S_1 and S_2 be subspaces of a vector space over a field F. The *sum* of S_1 and S_2 is the subspace

$$S_1 + S_2 \ := \ \text{span}\{S_1 \cup S_2\} \ = \ \{x + y : x \in S_1, y \in S_2\}$$

If $S_1 \cap S_2 = \{0\}$ then $S_1 + S_2$ is called a *direct sum* and we write it as $S_1 \oplus S_2$. Every vector $z \in S_1 \oplus S_2$ can be uniquely written as $z = x + y$ with $x \in S_1$ and $y \in S_2$.

Example 25.1. Consider $S := \{1, t, t^2, \ldots, t^{n-1}\}$. Even though S is not a vector space its span

$$\text{span}(S) \ = \ \{a_0 + a_1 t + \cdots + a_{n-1} t^{n-1} : a_0, \ldots, a_{n-1} \in F\}$$

is an n-dimensional vector space V that can be identified with F^n where $x \in V$ is defined by $x_i = a_i$, $i = 0, \ldots, n-1$. □

25.1.1.2 Basis, dimension, linear transformation, rank and nullity

A finite set of vectors $x_1, \ldots, x_k$ in a vector space V over a field F is *linearly dependent* if and only if there are scalars $a_1, \ldots, a_k \in F$, not all zero, such that $a_1 x_1 + \cdots + a_k x_k = 0 \in V$. The vectors $x_1, \ldots, x_k$ are *linearly independent* if they are not linearly dependent. A linearly independent set $B := \{v_1, v_2, \ldots, \} \subseteq V$ of vectors that spans the vector space V is called a *basis*. Any vector $x \in V$ can be uniquely expressed as a linear combination of the basis, i.e., $x = \sum_k a_k v_k$ for a unique set of scalars $a_k \in F$, $k = 1, 2, \ldots$. If there is a positive integer n such that $B := \{v_1, \ldots, v_n\}$ is a basis of V, then all bases of V consists of exactly n vectors and n is the *dimension* of V, denoted by dim$V$. This is because adding any vector to a basis will render it linearly dependent and removing any vector from the basis will prevent it from spanning V. In this case V is *finite dimensional*. If no such integer n exists then V is *infinite dimensional*. For an infinite dimensional vector space, there is a one-to-one correspondence between the vectors in any two bases. A subspace of a (finite) n-dimensional vector space has dimension no more than n; it is a proper subspace if its dimension is strictly less than n.

The real vector space $\mathbb{R}^n$ has dimension n. The complex vector space C^n has dimension n over the field $F = \mathbb{C}$ but dimension $2n$ over the field $F = \mathbb{R}$. A *basis* of a vector space F^n is a set of vectors $\{v_1, \cdots, v_n\}$ such that any vector $x \in F^n$ can be expressed as a linear combination of vectors in the basis, i.e., $x = B\alpha$ for some $\alpha \in F^n$ where the columns of B are the vectors $\{v_1, \cdots, v_n\}$. If the basis vectors are orthogonal, i.e., $v_j^{\mathsf{H}} v_k = 0$ for $j \neq k$, then the basis is called an *orthogonal basis*. If the basis vectors are both orthogonal and of unit Euclidean norm ($\|v_j\|_2 = 1$ for all j), then the basis is called an *orthonormal basis*. The basis $\{e_1, \cdots, e_n\}$ of F^n in which the n-vector e_i has a 1 in its ith entry and 0s elsewhere is called the *standard basis* or the *unit basis*. It is an orthonormal basis. Two vector spaces U and V over the same filed F is called *isomorphic* if there is an invertible function $f : U \to V$ such that $f(ax + by) = af(x) + bf(y)$ for all $x, y \in U$ and $a, b \in F$. Then f is called an *isomorphism*. Any n-dimensional real vector space is isomorphic to $\mathbb{R}^n$ and any n-dimensional complex vector space is isomorphic to $\mathbb{C}^n$.

Let V be a finite-dimensional vector space and let S_1, S_2 be two given subspaces of V. Then

$$\dim(S_1 \cap S_2) \ + \ \dim(S_1 + S_2) \ \ = \ \ \dim S_1 \ + \ \dim S_2$$

Hence

$$\dim(S_1 \cap S_2) \;\geq\; \dim S_1 \,+\, \dim S_2 \,-\, \dim V$$

since $S_1 + S_2 := \text{span}\{S_1 \cup S_2\} \subseteq V$. By induction we have $\dim(S_1 \cap \cdots \cap S_k) \geq \dim S_1 + \cdots + \dim S_k - (k-1)\dim V$. If $\delta := \dim S_1 + \cdots + \dim S_k - (k-1)\dim V \geq 1$ then $S_1 \cap \cdots \cap S_k$ contains at least $\delta \geq 1$ linearly independent vectors. For example, for the vector space $V := \mathbb{R}^3$ and subspaces S_1, S_2 defined by two non-parallel planes, their intersection $S_1 \cap S_2$ is a line in V and has a dimension at least $2+2-3=1$. In fact its dimension is exactly 1 because $S_1 + S_2 = V$. If S_3 is a plane that is not parallel to S_1 or S_2, $\dim(S_1 \cap S_2 \cap S_3) \geq 2+2+2-(2)(3) = 0$. It is exactly 0 (their intersection is a point) because $S_1 + S_2 + S_3 = V$.

We can view a matrix $M_{m,n}(F)$ as a vector in the vector space F^{mn}, or an array of scalars F in the vector space $F^{m\times n}$. A third perspective is to view a matrix $A \in M_{m,n}(F)$ as a *linear transformation* $A : F^n \to F^m$ mapping x to Ax. Then

- The *domain* of A is F^n.
- The *range* of A is the subspace $\text{range}(A) := \{Ax \in F^m : x \in F^n\} \subseteq F^m$. The dimension of $\text{range}(A)$ is called the *rank* of A, denoted by $\text{rank}(A)$.
- The *null space* of A is the subspace $\text{null}(A) := \{x \in \mathbb{F}^n : Ax = 0\} \subseteq F^n$. The dimension of $\text{null}(A)$ is called the *nullity* of A, denoted by $\text{nullity}(A)$.

The span $\text{range}(A)$ is also called the *column space* of A. Similarly $\{y^{\mathsf{T}}A : y \in F^m\}$ is called the *row space* of A. The *rank-nullity theorem* states that

$$\text{rank}(A) \,+\, \text{nullity}(A) \;=\; n \;=\; \text{rank}\left(A^{\mathsf{H}}\right) \,+\, \text{nullity}(A) \tag{25.1}$$

where the last equality holds if $F = \mathbb{C}$ or $\mathbb{R}$ and follows since $\text{rank}(A) = \text{rank}\left(A^{\mathsf{H}}\right)$. Note that $\text{range}\left(A^{\mathsf{H}}\right) \subseteq F^n$ whereas $\text{range}(A) \subseteq F^m$.

Henceforth we use $M_{m,n} := M_{m,n}(\mathbb{C})$ to denote the set of $m \times n$ matrices whose elements are in $\mathbb{C}$. We abbreviate them to $M_n := M_n(\mathbb{C})$ if $m = n$ and use $M := M(\mathbb{C})$ when m and n are arbitrary. Similarly for $M_{m,n}(\mathbb{R})$, $M_n(\mathbb{R})$ and $M(\mathbb{R})$ for matrices whose elements are in $\mathbb{R}$. We often write $A \in \mathbb{C}^{m\times n}$ (or $A \in \mathbb{R}^{m\times n}$) and call A a complex (or real) matrix to mean a matrix A in M (or $M(\mathbb{R})$) of size $m \times n$.

25.1.2 Schur complement and matrix inversion formula

25.1.2.1 Schur complement

Let $M \in \mathbb{C}^{n\times n}$ and partition it into blocks:

$$M \;=\; \begin{bmatrix} A & B \\ D & C \end{bmatrix}$$

such that $C \in \mathbb{C}^{k\times k}$, $k < n$, is invertible and the other submatrices are of appropriate dimensions. The $(n-k)\times(n-k)$ matrix $M/C := A - BC^{-1}D$ is called the *Schur complement of block* C of matrix M. If A is invertible then the $k\times k$ matrix $M/A := C - DA^{-1}B$ is called the *Schur complement of block* A of matrix M.

Example 25.2 (Gaussian elimination)**.** Schur complement arises from applying Gaussian elimination to a system of linear equations such as:

$$\begin{bmatrix} A & B \\ D & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \;\Leftrightarrow\; \begin{bmatrix} Ax+By \\ Dx+Cy \end{bmatrix} \;=\; \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

When C is invertible, Gaussian elimination expresses y in terms of x by multiplying the second equation by BC^{-1} and subtracting the result from the first equation. This corresponds to multiplying the equations on the left by a block lower-triangular matrix:

$$\begin{bmatrix} I_{n-k} & -BC^{-1} \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ D & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; \begin{bmatrix} A-BC^{-1}D & 0 \\ C^{-1}D & I_k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} \tag{25.2a}$$

where

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} \;:=\; \begin{bmatrix} b_1 - BC^{-1}b_2 \\ C^{-1}b_2 \end{bmatrix}$$

If the Schur complement of C is invertible then the solutions for (x,y) can be read off equation (25.2a) as

$$\begin{aligned} x &= \left(A - BC^{-1}D\right)^{-1} \hat{b}_1 \;=\; (M/C)^{-1}\,\hat{b}_1 \\ y &= -C^{-1}Dx \,+\, \hat{b}_2 \;=\; -C^{-1}D\,(M/C)^{-1}\,\hat{b}_1 \,+\, \hat{b}_2 \end{aligned}$$

This means that

$$\begin{bmatrix} A-BC^{-1}D & 0 \\ C^{-1}D & I_k \end{bmatrix}^{-1} \;=\; \begin{bmatrix} (M/C)^{-1} & 0 \\ -C^{-1}D\,(M/C)^{-1} & I_k \end{bmatrix} \tag{25.2b}$$

□

Gaussian elimination can be represented as

$$\begin{bmatrix} I_{n-k} & -BC^{-1} \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A & B \\ D & C \end{bmatrix} \begin{bmatrix} I_{n-k} & 0 \\ -C^{-1}D & I_k \end{bmatrix} \;=\; \begin{bmatrix} A-BC^{-1}D & 0 \\ 0 & C \end{bmatrix} \tag{25.3}$$

This equation implies (since $\det(M_1M_2) \;=\; \det(M_1)\det(M_2)$)

$$\begin{aligned} \det(M) &= \det(C)\,\det(M/C) \\ \operatorname{rank}(M) &= \operatorname{rank}(C) \,+\, \operatorname{rank}(M/C) \end{aligned}$$

Theorem 25.1 (Schur complement)**.** Let $M \in \mathbb{C}^{n\times n}$ be partitioned as above with nonsingular C. Let $M/C := A - BC^{-1}D$ be the Schur complement of C of matrix M.

1. M is nonsingular if and only if M/C is nonsingular (given C is nonsingular).
2. $\det(M) = \det(C)\,\det(M/C)$.
3. $\text{rank}(M) = \text{rank}(C) + \text{rank}(M/C)$.
4. M is positive definite if and only if C and M/C are positive definite.
5. If M and C are invertible, then M/C is invertible and

$$M^{-1} \;=\; \begin{bmatrix} (M/C)^{-1} & -(M/C)^{-1}BC^{-1} \\ -C^{-1}D(M/C)^{-1} & C^{-1}+C^{-1}D(M/C)^{-1}BC^{-1} \end{bmatrix}$$

6. If M and A are invertible, then $M/A := C - DA^{-1}B$ is invertible and

$$M^{-1} \;=\; \begin{bmatrix} A^{-1}+A^{-1}B(M/A)^{-1}DA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}DA^{-1} & (M/A)^{-1} \end{bmatrix}$$

Proof. Assertions 1, 2, 3 follow from (25.3). Example 25.2 shows that (from (25.2a)):

$$\begin{bmatrix} I_{n-k} & -BC^{-1} \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ D & C \end{bmatrix} \;=\; \begin{bmatrix} A-BC^{-1}D & 0 \\ C^{-1}D & I_k \end{bmatrix} \tag{25.4}$$

M is singular if and only if there exists a nonzero vector (x, y) in null(M). Since the first matrix on the left-hand side of (25.4) is of full rank, this is equivalent to:

$$\begin{bmatrix} A-BC^{-1}D & 0 \\ C^{-1}D & I_k \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \;=\; 0 \quad\Leftrightarrow\quad (A-BC^{-1}D)x \;=\; 0, \quad y \;=\; C^{-1}Dx$$

Hence M is singular if and only if $A - BC^{-1}D$ is singular. Applying $\det(M_1M_2) \;=\; \det(M_1)\det(M_2)$ to (25.4) we have $\det(M) \;=\; \det(C)\,\det(A - BC^{-1}D) \;=\; \det(C)\,\det(M/C)$.

To prove 4, we have from (25.2)

$$\begin{bmatrix} A & B \\ D & C \end{bmatrix}^{-1} \begin{bmatrix} I_{n-k} & -BC^{-1} \\ 0 & C^{-1} \end{bmatrix}^{-1} \;=\; \begin{bmatrix} A-BC^{-1}D & 0 \\ C^{-1}D & I_k \end{bmatrix}^{-1} \;=\; \begin{bmatrix} (M/C)^{-1} & 0 \\ -C^{-1}D(M/C)^{-1} & I_k \end{bmatrix}$$

Hence

$$\begin{aligned} \begin{bmatrix} A & B \\ D & C \end{bmatrix}^{-1} \;&=\; \begin{bmatrix} (M/C)^{-1} & 0 \\ -C^{-1}D(M/C)^{-1} & I_k \end{bmatrix} \begin{bmatrix} I_{n-k} & -BC^{-1} \\ 0 & C^{-1} \end{bmatrix} \\ &=\; \begin{bmatrix} (M/C)^{-1} & -(M/C)^{-1}BC^{-1} \\ -C^{-1}D(M/C)^{-1} & C^{-1}D(M/C)^{-1}BC^{-1}+C^{-1} \end{bmatrix} \end{aligned}$$

The last assertion can be proved in the same way by eliminating x instead of y in Example 25.2; see Exercise 25.3. □

Let $I := \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, $J := \{j_1, \ldots, j_l\} \subseteq \{1, \ldots, n\}$, and A_{IJ} denote the submatrix obtained from deleting rows not in I and columns not in J.

- If $k = l$, i.e., A_{IJ} is square, then the *minor* M_{IJ} of A is the determinant of the submatrix A_{IJ}.
- If $I = J$, then A_{IJ} is called a *principal submatrix* and M_{IJ} a *principal minor* of A.
- If $I = J = \{1, \ldots, k\}$ with $k \le n$, then A_{IJ} is called a *leading principal submatrix* of order k and M_{IJ} a *leading principal minor* of order k.

Theorem 25.2 (Slyvester's criterion)**.** Suppose A is Hermitian. Then

1. A is positive definite if and only if all its leading principal minors are positive. This involves n determinants: those of the upper left 1×1 matrix, upper left 2×2 matrix, ..., $\det(A)$.
2. A is positive semidefinite if and only if all its principal minors are nonnegative. This involves $\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$ determinants.

□

25.1.2.2 Matrix inversion lemma

A useful identity is the matrix inversion lemma or Sherman-Morrison-Woodbury formula. Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times k}$, $C \in \mathbb{C}^{k\times k}$ and $D \in \mathbb{C}^{k\times n}$. Suppose A, C and the $k \times k$ matrix

$$\hat{C} \;:=\; C^{-1} + DA^{-1}B \tag{25.5a}$$

are invertible. Then

$$(A+BCD)^{-1} \;=\; A^{-1} - A^{-1}\left(B\hat{C}^{-1}D\right)A^{-1} \tag{25.5b}$$

An important case is when $k \ll n$. Then the $k \times k$ matrix C is much smaller than A and the multiplication of C by B and D on the left and right respectively produces an $n \times n$ matrix BCD of the right size for addition with A. Similarly reversing the order of multiplication produces a much smaller $k \times k$ matrix $DA^{-1}B$ for addition with C^{-1} to produce the matrix $\hat{C}$ in (25.5a). We can thus view the role of (B, D) as transforming between sizes n and k to simplify the inversion of large matrices. In many applications BCD represents a low-rank update of A in a dynamical system or an additive noise to a transmitted signal A so that $A+BCD$ is the received signal. Suppose A^{-1} has been precomputed. Then $\hat{C}$ is much smaller and easier to invert than $A+BCD$. The matrix inversion formula allows us to compute the inverse of the updated or noisy matrix $A+BCD$ in terms of A^{-1} and $\hat{C}^{-1}$ when they exist.

Many special cases are useful. For instance when $A = I_n$ and $C = I_k$ we have:

$$(I_n + BD)^{-1} \;=\; I_n - B(I_k + DB)^{-1}D$$

Note that BD is $n \times n$ while DB is $k \times k$ and hence the inverse on the right-hand side can be much easier to compute than that on the left-hand side. Using the push-through identity (see Exercise 25.4) this is equivalent to:

$$(I_n + BD)^{-1} \;=\; I_n - (I_n + BD)^{-1}BD \;=\; I_n - BD(I_n + BD)^{-1}$$

When $k = n$ and $B = D = I_n$ we have the inversion formula for sum of two matrices:

$$(A+C)^{-1} \;=\; A^{-1} - A^{-1}\left(C^{-1}+A^{-1}\right)^{-1}A^{-1}$$

Merging $A^{-1}\left(C^{-1}+A^{-1}\right)^{-1}A^{-1}$ we have Hua's identity:

$$(A+C)^{-1} \;=\; A^{-1} - \left(A+AC^{-1}A\right)^{-1}$$

25.1.3 Change of basis, diagonalizability, Jordan form

Recall that we can interpret any $m \times n$ complex matrix M as a linear transformation that maps a vector $x \in \mathbb{C}^n$ to a vector $y = Mx \in \mathbb{C}^m$, where the basis in the domain $\mathbb{C}^n$ is the standard basis consisting of the columns of the $n \times n$ identity matrix I_n and the basis in the range $\mathbb{C}^m$ is the standard basis consisting of the columns of I_m. Suppose we want to change the basis of the domain to (the columns of) an $n \times n$ nonsingular matrix V and the basis of the range to (the columns of) an $m \times m$ nonsingular matrix U. What is the new matrix $\tilde{M}$ that represents the same linear map with respect to the new bases?

25.1.3.1 Similarity transformation

Since V and U are bases of $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively we can express a ny $x \in \mathbb{C}^n$ in terms of V and any vector $y \in \mathbb{C}^m$ in terms of U as

$$I_n x \;=\; V\tilde{x} \qquad \text{and} \qquad I_m y \;=\; U\tilde{y}$$

Hence the linear transformation M that maps any vector $x \in \mathbb{C}^n$ to a vector $y = Mx \in \mathbb{C}^m$ with respect to the standard bases implies

$$U\tilde{y} \;=\; y \;=\; Mx \;=\; MV\tilde{x}$$

Hence

$$\tilde{y} \;=\; \underbrace{U^{-1}MV}_{\tilde{M}}\,\tilde{x}$$

This means that any vector $\tilde{x}$ in the domain $\mathbb{C}^n$ with respect to the new basis U is mapped to the (same) vector $\tilde{y}$ in the range $\mathbb{C}^m$ with respect to the new basis V by the matrix

$$\tilde{M} \;:=\; U^{-1}MV \qquad \text{or} \qquad M \;=\; U\tilde{M}V$$

For the special case where $n = m$ and the new bases for the domain and the range are the same, $U = V$,

$$\tilde{M} \;=\; U^{-1}MU \tag{25.6}$$

i.e., the new matrix $\tilde{M}$ represents the linear transformation under the new basis U. The mapping of M to $U^{-1}MU$ is called a *similarity transformation* of M by the nonsingular *similarity matrix* U. This is illustrated in Figure 25.1.

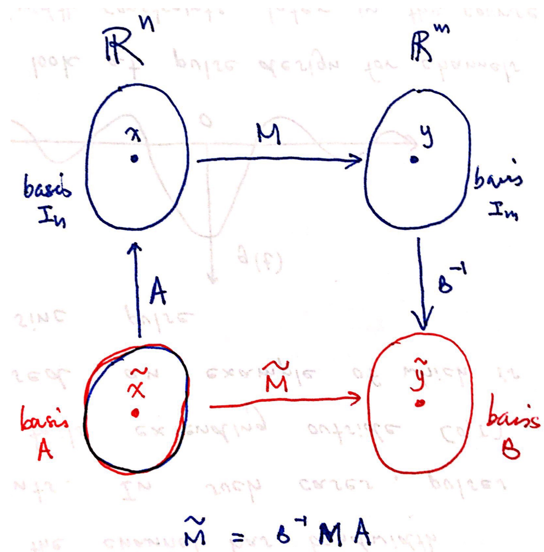


Figure 25.1: Change of bases. The new matrix $\tilde{M} = U^{-1}MU$ is similar to the original matrix M when $n = m$ and $U = V$.

25.1.3.2 Diagonalizabilty and Jordan form

For the case where $n = m$ and $U = V$, if the basis U in (25.6) is such that $\tilde{M} = \Lambda$ is diagonal then the diagonal entries λ_i of Λ are the eigenvalues of M with the ith columns u_i of U as their corresponding eigenvectors, since

$$MU \;=\; U\Lambda \qquad \text{or} \qquad Mu_i \;=\; \lambda_i u_i, \quad i = 1, \ldots, n$$

M is said to be *diagonalizable* in this case, i.e., by definition, M is diagonalizable if it is similar to a diagonal matrix Λ.

Not all $n \times n$ matrix M over the complex field is diagonalizable through a similarity transformation. We see above that M is diagonalizable if M has n linearly independent eigenvectors. Indeed having n linearly independent eigenvectors is also necessary for M's diagonalizability.[1] When M has fewer than n linearly independent eigenvectors, M is not similar to a diagonal matrix, but to a *Jordan form*, i.e., there exists an invertible matrix V such that

$$V^{-1}MV \;=\; J \;:=\; \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{bmatrix}$$

where J_i, $i = 1, \ldots, m$, are *Jordan blocks* of M:

$$J_i \;:=\; \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

[1]A square matrix $M \in \mathbb{C}^{n\times n}$ is said to be *unitarily* diagonalizable if $U^{-1} = U^H$ in (25.6). It can be shown that any $M \in \mathbb{C}^{n\times n}$ is unitarily diagonalizable if and only if it is normal ($MM^* = M^*M$); see Chapter 25.1.5.

To compute the columns of V, consider Jordan block J_i and suppose without loss of generality that it corresponds to columns $1, 2, \ldots, k_i$. Equate these k_i columns on both sides of $MV = VJ$ to get

$$M \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_{k_i} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_{k_i} \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

Therefore v_1 is the eigenvector corresponding to the eigenvalue λ_i and can be computed from

$$(M - \lambda_i I_n) v_1 = 0 \tag{25.7a}$$

The other columns $v_2, \ldots, v_{k_i}$ are not eigenvectors. They satisfy $Mv_j = v_{j-1} + \lambda_i v_j$, $j = 2, \ldots, k_i$, and can be computed from

$$(M - \lambda_i I_n) v_j = v_{j-1}, \qquad j = 2, \ldots, k_i \tag{25.7b}$$

Multiplying both sides by $M - \lambda_i I_n$ yields $(M - \lambda_i I_n)^2 v_j = v_{j-2}$. Repeated multiplications then imply that the columns $v_1, \ldots, v_{k_i}$ satisfy:

$$\begin{aligned} (M - \lambda_i I_n) v_1 &= 0 \qquad (v_1 \text{is eigenvector}) \\ (M - \lambda_i I_n)^2 v_2 &= 0 \qquad (v_j \text{ are } \textit{generalized eigenvectors}, \ j = 2, \ldots, k_i) \\ &\vdots \\ (M - \lambda_i I_n)^{k_i} v_{k_i} &= 0 \end{aligned}$$

The characteristic polynomial $p(x) := \det(xI_n - M)$ of M can be expressed in terms of the eigenvalues λ_i:

$$p(x) := \det(xI_n - VJV^{-1}) = \det\left(V(xI_n - J)V^{-1}\right) = \det(xI_n - J) = \prod_{i=1}^{m} \det(xI_{k_i} - J_i)$$

where J_i is the ith Jordan block of size $k_i \times k_i$, and I_{k_i} is the identity matrix of the same size. Since a Jordan block is upper triangular we have

$$\det(xI_{k_i} - J_i) = (x - \lambda_i)^{k_i}$$

and hence

$$p(x) = \prod_{i=1}^{m} (x - \lambda_i)^{k_i}$$

There can be more than one Jordan block whose diagonal entries are the repeated eigenvalue λ_i. Let q be the number of *distinct* eigenvalues λ_j, $j = 1, \ldots, q$, and let m_j be the number of Jordan blocks corresponding to the distinct eigenvalue λ_j, so that $m = \sum_{j=1}^{q} m_j$. Then the characteristic polynomial can also be expressed in terms of distinct eigenvalues as:

$$p(x) = \prod_{i=1}^{m} (x - \lambda_i)^{k_i} = \prod_{j=1}^{q} \prod_{i=1}^{m_j} (x - \lambda_j)^{k_i}$$

For each *distinct* eigenvalue λ_j, there are two quantities of interest:

1. *geometric multiplicity* of λ_j: This is the number m_j of Jordan blocks corresponding to λ_j. It is the dimension of the null space of $M - \lambda_j I_n$ since each such block yields a single eigenvector of M.

2. *algebraic multiplicity* of λ_j: This is the sum $\sum_{i=1}^{m_j} k_i$ of the sizes k_i of all these Jordan blocks. It is the maximum degree of the factor $x - \lambda_j$ in the characteristic polynomial $p(x)$ of M.

Hence for each distinct eigenvalue λ_j

$$\text{algebraic multiplicity } \sum_{i=1}^{m_j} k_i \quad \geq \quad \text{geometric multiplicity } m_j$$

We summarize implications of algebraic and geometric multiplicities on the diagonalizability of M in the following theorem.

Theorem 25.3. With the notations above,

1. For each distinct eigenvalue λ_j, algebraic multiplicity $=$ geometric multiplicity $= m_j$ if and only if all Jordan blocks corresponding to λ_j have sizes $k_i = 1$. In this case, there are m_j eigenvectors corresponding to λ_j, they are linearly independent, and the null space of $M - \lambda_j I_n$ has dimension m_j.

2. M is diagonalizable if and only if algebraic multiplicity $=$ geometric multiplicity for all eigenvalues, if and only if all Jordan blocks have sizes 1 and hence all superdiagonal entries are zero, if and only if M has n linearly independent eigenvectors.

3. As a special case, M is diagonalizable if M has n distinct eigenvalues (and hence all Jordan blocks are of size 1, $m_j = k_i = 1 =$ algebraic multiplicity $=$ geometric multiplicity).

25.1.4 Special matrices

Definition 25.2 (Square matrices)**.** 1. A real or complex matrix $A \in \mathbb{F}^{n\times n}$, with $F = \mathbb{R}$ or $\mathbb{C}$, is *symmetric* if $A^{\mathsf{T}} = A$, *skew-symmetric* if $A^{\mathsf{T}} = -A$, and *orthogonal* if $A^{\mathsf{T}} = A^{-1}$.

2. A complex matrix $A \in \mathbb{C}^{n\times n}$ is *Hermitian* if $A^{\mathsf{H}} = A$, *skew-Hermitian* if $A^{\mathsf{H}} = -A$, and *unitary* if $A^{\mathsf{H}} = A^{-1}$.

3. A complex matrix $A \in \mathbb{C}^{n\times n}$ is *normal* if $AA^{\mathsf{H}} = A^{\mathsf{H}}A$. If A is real, this reduces to $AA^{\mathsf{T}} = A^{\mathsf{T}}A$.

4. *Positive semidefiniteness.*

   - A complex matrix $A \in \mathbb{C}^{n\times n}$ is *positive semidefinite (psd)* (or *positive definite (pd)*) if $x^{\mathsf{H}}Ax$ is real and nonnegative (or real and positive) for all $x \in \mathbb{C}^n$.
   - A real symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *positive semidefinite (psd)* (or *positive definite (pd)*) if $x^{\mathsf{T}}Ax \geq 0$ (or $x^{\mathsf{T}}Ax > 0$) for all $x \in \mathbb{R}^n$.
   - A complex or real matrix A is *negative semidefinite (nsd)* (or *negative definite (nd)*) if $-A$ is psd (or pd). It is *indefinite* if there are vectors $y, z \in \mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ such that $y^*Ay < 0 < z^*Az$.

□

Remark 25.1.

1. A *real* orthogonal matrix or a unitary matrix has columns (or rows) that are an orthonormal list of vectors; indeed they are orthonormal basis of $\mathbb{R}^n$ or $\mathbb{C}^n$. A complex orthogonal matrix however is generally not unitary and their columns (or rows) are generally not orthonormal.

2. All Hermitian (symmetric), skew-Hermitian (skew-symmetric), or unitary complex matrices are normal, but the converse is not generally true. A *real* symmetric matrix is normal, but a complex symmetric matrix may or may not be normal (see Chapter 25.1.5.4). If A is both triangular and normal, then A is diagonal.

3. A complex Hermitian (skew-Hermitian) matrix behaves like a real symmetric (skew-symmetric) matrix, e.g., they have real eigenvalues and are normal matrices. It therefore has a spectral decomposition according to Theorem 25.10. A complex Hermitian matrix has real diagonal entries.

4. A complex symmetric matrix may or may not be normal. It therefore may or may not have a spectral decomposition (Theorem 25.10). It always have a singular value decomposition (Theorem 25.8) and a Takagi decomposition (Theorem 25.14), and these are generally different decompositions.

5. Our definition of psd (or pd) requires symmetry for real matrices, but does not require Hermitian for complex matrices. This is because, for a complex matrix $A \in \mathbb{C}^{n\times n}$, A is psd (or pd) if and only if A is Hermitian and its eigenvalues are nonnegative (or positive), so our Definition 25.2 for complex matrices implies Hermitian. For a real matrix $A \in \mathbb{R}^{n\times n}$, on the other hand, A can satisfy $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$ but not be symmetric (as long as its symmetric component $(A+A^T)/2$ is psd or pd). Following [10, Definition 4.1.11, p. 231], we therefore restrict our definition to real symmetric matrices. Then A is psd (or pd) if and only if all its eigenvalues are nonnegative (or positive) [10, Theorem 4.1.10, p.231].

□

Theorem 25.4 (Eigenvalues).

1. A matrix A, real or complex, is invertible if and only if all its eigenvalues are nonzero.

2. If a matrix A is real symmetric or complex Hermitian, then all its eigenvalues are real.

3. A matrix A, real or complex, is psd (pd) if and only if $A^{\mathsf{H}} = A$ and all its eigenvalues are real and nonnegative (positive).

□

Definition 25.3 (Diagonal dominance). A matrix $A \in \mathbb{C}^{n\times n}$ is *diagonally dominant* if

$$|A_{ii}| \quad \geq \quad \sum_{j:j\neq i} |A_{ij}| \qquad \text{for all rows } i$$

A is *strictly diagonally dominant* if the inequalities are strict for all rows i.

The Geršgorin disc theorem states that all eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$ lie in the union of n discs

$$\cup_{i=1}^{n} \left\{ z \in \mathbb{C}^n \ : \ |z - A_{ii}| \leq \sum_{j:j\neq i} |A_{ij}| \right\}$$

If A is strictly diagonally dominant then the origin is outside Geršgorin discs, i.e., all eigenvalues of A are nonzero. The geometry of the Geršgorin discs also implies the following property.

Theorem 25.5.

1. A strictly diagonally dominant matrix is invertible (but not necessarily positive definite).
2. Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitian with (real) nonnegative diagonal entries $A_{ii} \geq 0$.
   - If A is diagonally dominant then it is positive semidefinite.
   - If A is strictly diagonally dominant then it is positive definite and invertible.

Proof. Part 1 follows from the Geršgorin disc theorem. For part 2, for any $x \in \mathbb{C}^n$ we have

$$x^*Ax \;=\; \sum_{i,j} A_{ij}x_i^*x_j \;=\; \sum_i \left(A_{ii}|x_i|^2 \;+\; \sum_{j:j\neq i} A_{ij}x_i^*x_j \right)$$

Substitute $A_{ii} \geq \sum_{j:j\neq i} |A_{ij}|$ (diagonal dominance) to get

$$x^*Ax \;\geq\; \sum_i \sum_{j:j\neq i} \left(|A_{ij}||x_i|^2 \;+\; A_{ij}x_i^*x_j\right) \;=\; \sum_{(i,j):i\neq j} \left(|A_{ij}||x_i|^2 + |A_{ji}||x_j|^2 \;+\; A_{ij}x_i^*x_j + A_{ji}x_j^*x_i\right)$$

Since $A_{ji} = A_{ij}^*$ (A is Hermitian) we have

$$x^*Ax \;\geq\; \sum_{(i,j):i\neq j} |A_{ij}| \left(|x_i|^2 + |x_j|^2 \;-\; |x_i^*||x_j| - |x_j^*||x_i|\right) \;=\; \sum_{(i,j):i\neq j} |A_{ij}| \left(|x_i| - |x_j|\right)^2 \;\geq\; 0$$

If A is strictly diagonally dominant then the inequality is strict and therefore A is positive definite. □

Unitary matrices have the following properties (e.g. [10, Theorem 2.1.4, p.84]).

Lemma 25.6. Consider a complex matrix $U \in M_n := M_n(\mathbb{C})$. The following are equivalent:

- U is unitary.
- $U^{\mathsf{H}}U = I$.
- The columns of U are orthornormal.
- U^{H} is unitary.
- $UU^{\mathsf{H}} = I$.
- The rows of U are orthornormal.
- $\|Ux\|_2 = \|x\|_2$ for all $x \in \mathbb{C}^n$ where $\|\cdot\|_2$ is the Euclidean norm.

In fact, the Euclidean norm is the only vector norm that is unitarily invariant, i.e., $\|Ux\| = \|x\|$ for all $x \in \mathbb{C}^n$ and all unitary matrices U with $\|e_i\| = 1$; see Chapter 25.1.7.1.

Recall that a unitary matrix is normal because $UU^{\mathsf{H}} = U^{\mathsf{H}}U = I$, and hence unitarily diagonalizable (Theorem 25.10).

Lemma 25.7. Suppose $U \in M_n := M_n(\mathbb{C})$ is unitary and symmetric. Then

1. If $U = \text{diag}\,(a_1, \cdots, a_n)$ is diagonal then $a_j = e^{\mathbf{i}\theta_j}$ for some $\theta_j \in \mathbb{R}^n$.

2. *Spectral decomposition.* There exist real orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and real $\theta_1, \cdots, \theta_n$ in $[0, 2\pi)$ such that

$$U \;=\; Q\, \underbrace{\text{diag}\left(e^{\mathbf{i}\theta_1}, \cdots, e^{\mathbf{i}\theta_n}\right)}_{\Lambda} Q^{\mathsf{T}} \;=:\; Q\Lambda Q^{\mathsf{T}}$$

   where $\lambda_j := e^{\mathbf{i}\theta_j}$ are the eigenvalues of U and the columns of Q are an orthonormal set of corresponding (real) eigenvectors of U.

3. It has a square root, i.e., there is a unitary symmetric matrix $B := Q\Lambda^{1/2}Q^{\mathsf{T}}$ such that $Q = B^2$.

For proof that U is unitarily diagonalizable, i.e., $U = Q\Lambda Q$, see Corollary [10, 2.5.18, p.139]. The existence of the square root B relies on the fact that U is unitarily diagonalizable whose eigenvalues λ_j satisfy $|\lambda_j| = 1$, so that $BB^{\mathsf{H}} = I$. Lemma 25.7 justifies the interpretation of a unitary matrix as a rotation operator, i.e., the product Ux rotates the vector x without expanding its Euclidean norm, $\|Ux\| = \|Q\Lambda Qx\| = \|\Lambda Qx\| = \|Qx\| = \|x\|$.

25.1.5 SVD, spectral decompositions, complex symmetric matrices

In this subsection we review the various matrix decompositions and their relationship, as shown in Figure 25.2.

25.1.5.1 Singular value decomposition for any matrix

Consider a complex matrix $A \in \mathbb{C}^{m\times n}$. Suppose there exists a real value $\sigma \geq 0$ and nonzero vectors $v \in \mathbb{C}^m$, $w \in \mathbb{C}^n$ such that

$$Aw \;\; = \sigma v \tag{25.8}$$

In this case, (σ, v, w) are called respectively a *singular value*, associated *left singular vector* and *right singular vector* of A. The next result says that every matrix A has m orthonormal left singular vectors $v_1, \ldots, v_m \in \mathbb{C}^m$, n orthonormal right singular vectors $w_1, \ldots, w_n \in \mathbb{C}^n$, and at most $q := \min\{m, n\}$ strictly positive singular values $\sigma_1, \ldots, \sigma_q$. Like eigenvalues the singular values σ_i are unique. Like eigenvectors, left and right singular vectors (v_i, w_i) are generally not unique. As we will see below, they are eigenvectors of AA^{H} and $A^{\mathsf{H}}A$ respectively; but the converse may not hold, i.e., not every eigenvector of AA^{H} and that of $A^{\mathsf{H}}A$ may satisfy (25.8). For example, if (v_i, w_i) are singular vectors of unit Euclidean norm, so are $(e^{\mathbf{i}\theta}v_i, e^{\mathbf{i}\theta}w_i)$ for any $\theta \in \mathbb{R}$. Moreover the matrix A can be factorized as follows [10, Theorem 2.6.3, p.150].

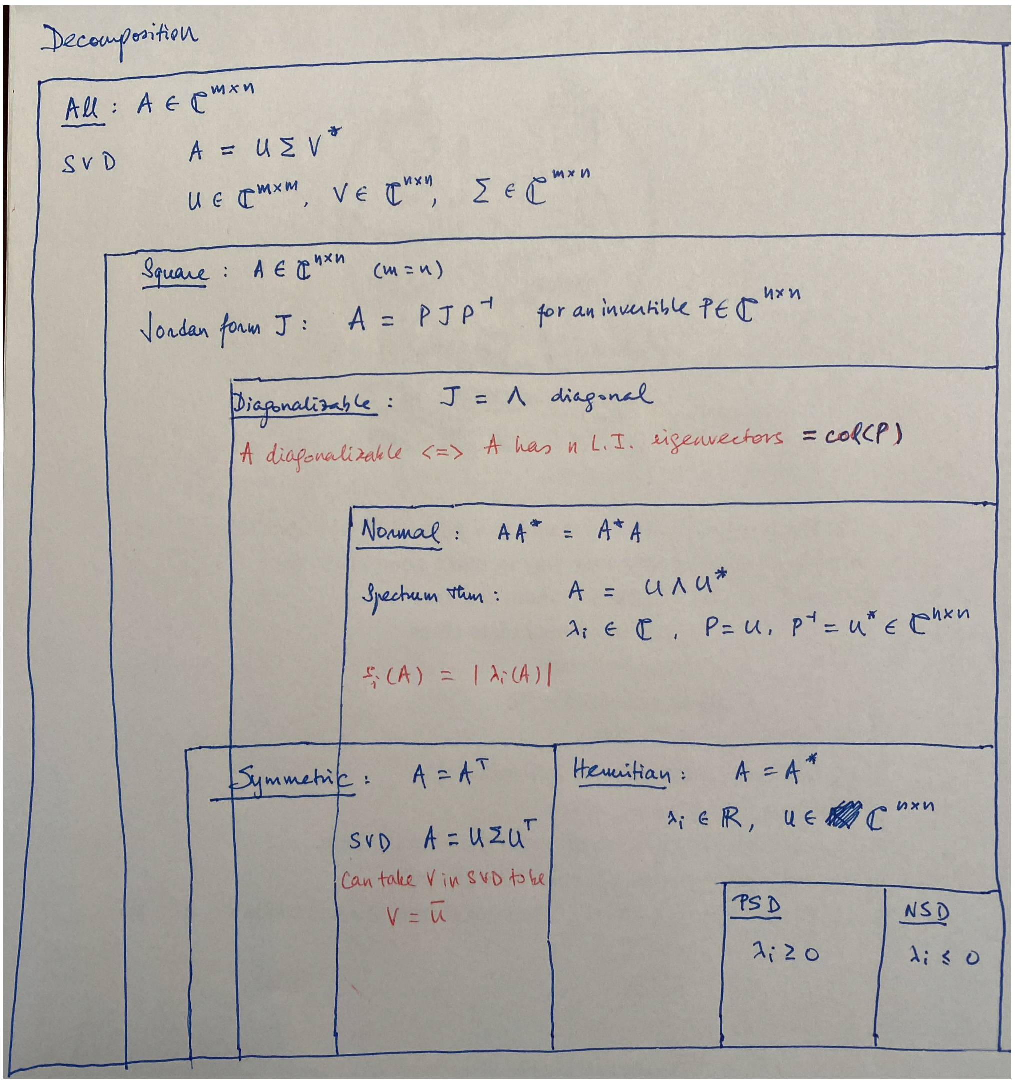


Figure 25.2: Matrix decompositions. Correction: For complex symmetric A, the unitary factor U in Takagi factorization is generally not eigenvectors of AA^{H}.

Consider an $m \times n$ matrix Σ and a diagonal matrix $\Sigma_q = \mathrm{diag}(\sigma_1, \cdots, \sigma_q)$ of size $q := \min\{m,n\}$. We will abuse notation and call Σ *diagonal*, even if $m \neq n$, if Σ is of the form:

$$\Sigma \;=\; \begin{cases} \Sigma_q & \text{if } m = n \\ \begin{bmatrix} \Sigma_q & 0 \end{bmatrix} & \text{if } n > m = q \\ \begin{bmatrix} \Sigma_q \\ 0 \end{bmatrix} & \text{if } m > n = q \end{cases} \tag{25.9}$$

Theorem 25.8 (Singular value decomposition)**.** For any matrix $A \in \mathbb{C}^{m\times n}$, there exists unitary matrices $V \in \mathbb{C}^{m\times m}$ and $W \in \mathbb{C}^{n\times n}$, and a real diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ of the form in (25.9) with

$$\sigma_1 \;\geq\; \sigma_2 \;\cdots\; \geq\; \sigma_q \;\geq 0$$

such that

$$AW \;=\; V\Sigma \qquad \text{or} \qquad A \;=\; V\Sigma W^{\mathsf{H}} \tag{25.10}$$

with $V^{-1} = V^{\mathsf{H}}$ and $W^{-1} = W^{\mathsf{H}}$. Moreover

1. The nonzero singular values of A are the positive square roots of the eigenvalues of AA^{H} (or equivalently of $A^{\mathsf{H}}A$):

$$\sigma_i \;=\; +\sqrt{\lambda_i(AA^{\mathsf{H}})} \;=\; +\sqrt{\lambda_i(A^{\mathsf{H}}A)}, \qquad i = 1, \ldots, q$$

2. If $r \leq q$ of the q singular values σ_i are positive, then A is of rank r and

$$A \;=\; \sum_{i=1}^{r} \sigma_i v_i w_i^{\mathsf{H}}$$

3. If V and W are unitary matrices such that $A = V\Sigma W^{\mathsf{H}}$ then
   - the columns of V are an orthonormal set of eigenvectors of AA^{H} because $AA^{\mathsf{H}} = V\Sigma^2 V^{\mathsf{H}}$, and
   - the columns of W are an orthonormal set of eigenvectors of $A^{\mathsf{H}}A$ because $A^{\mathsf{H}}A = W\Sigma^2 W^{\mathsf{H}}$;

   but the converse does not necessarily hold.

If A is real then V and W can be taken as real orthogonal matrices. □

The rank of A is the number its positive singular values, which is not less than (and can be greater than) the number of its nonzero eigenvalues of A. As we see will below (Theorem 25.10) $\mathrm{rank}(A)$ is equal to the number of nonzero (generally complex) eigenvalues if A is normal.

Theorem 25.8 does not provide a method to compute the unitary factors (V,W) in the singular value decomposition (25.10). This is because not every pair of orthonormal sets of eigenvectors of AA^{H} and $A^{\mathsf{H}}A$ respectively may be the unitary factors (V,W) in (25.10) when the eigenvalues associated with AA^{H} or with $A^{\mathsf{H}}A$ are not distinct. We describe how to compute unitary factors (V,W) in (25.10) when A is square

$(m = n)$ (see [10, Theorem 2.6.3, p.150] for details). When A is not normal, AA^{H} and $A^{\mathsf{H}}A$ are not equal, but they are unitarily similar since they have the same eigenvalues, i.e., there exists a unitary matrix Y such that $A^{\mathsf{H}}A = Y(AA^{\mathsf{H}})Y^{\mathsf{H}}$. Moreover YA is normal and hence it has a spectral decomposition according to Theorem 25.10, $YA = X\Lambda X^{\mathsf{H}}$ where $\Lambda := \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ consists of the eigenvalues of YA and the columns of X are an *arbitrary* orthonormal set of corresponding eigenvectors of YA. Let $\lambda_i = |\lambda_i|e^{\mathbf{i}\theta_i}$, $\Sigma_q := \mathrm{diag}(|\lambda_1|, \cdots, |\lambda_n|)$, $D := \mathrm{diag}\left(e^{\mathbf{i}\theta_1}, \cdots, e^{\mathbf{i}\theta_n}\right)$ so that $\Lambda = \Sigma_q D$. Then, since $YA = X\Sigma_q DX^{\mathsf{H}}$, we have

$$A = \underbrace{\left(Y^{\mathsf{H}}X\right)}_{V} \Sigma_q \underbrace{\left(DX^{\mathsf{H}}\right)}_{W^{\mathsf{H}}} \tag{25.11}$$

i.e., $V := Y^{\mathsf{H}}X$ and $W := XD^{\mathsf{H}}$. We illustrate this in the next example.

Example 25.3. Consider $A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Show that

1. Not arbitrary orthonormal sets of eigenvectors of AA^{H} and $A^{\mathsf{H}}A$ can be the unitary matrices (V, W) in the SVD (25.10).
2. Compute (V, W) according to the prescription (25.11). (Since A is real symmetric and hence normal, an alternative way to compute a (possibly different) pair (V, W) is given in Theorem 25.13; see Example 25.4.)

Solution. The matrices AA^{H} and $A^{\mathsf{H}}A$ are

$$AA^{\mathsf{H}} = A^{\mathsf{H}}A = A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore the eigenvalues of AA^{H} and those of $A^{\mathsf{H}}A$ are 1 and $\Sigma = I$. Moreover every vector x is an eigenvector of AA^{H} and of $A^{\mathsf{H}}A$, but not arbitrary orthonormal sets of eigenvectors can be (V, W) in SVD (25.10). For instance, if Q is any unitary matrix (and hence its columns are an orthonormal set of eigenvectors of AA^{H} and of $A^{\mathsf{H}}A$), $V = W = Q$ does not satisfy (25.10):

$$Q\Sigma Q^{\mathsf{H}} = QQ^{\mathsf{H}} = I \neq A$$

It is therefore necessary that V and W are different matrices in (25.10).

To compute (V, W) using (25.11), we choose $Y = I$ to be the identity matrix that relates AA^{H} and $A^{\mathsf{H}}A$ through unitary similarity, i.e., $A^{\mathsf{H}}A = I = Y(AA^{\mathsf{H}})Y^{\mathsf{H}}$. Next we compute the spectral decomposition of YA: the eigenvalues of $YA = A$ are $\lambda_1 := 1$, $\lambda_2 := -1$ with corresponding orthonormal set of eigenvectors (unique up to a rotation)

$$x_1 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad x_2 := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence

$$YA = A = X\Lambda X^{\mathsf{H}} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then $D := \text{diag}\left(e^{\mathbf{i}\theta_1}, e^{\mathbf{i}\theta_2}\right) = \text{diag}(1, -1)$ and hence

$$\Sigma_q := \text{diag}\left(|\lambda_1|, |\lambda_2|\right) = I, \quad V := Y^{\mathsf{H}} X = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad W := XD^{\mathsf{H}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

It can be verified that indeed $A = V\Sigma_q W^{\mathsf{H}}$. □

Suppose $m \leq n$ but $\text{rank}(A) =: r < m$. For a given V in the theorem, even though $A = V\Sigma W^{\mathsf{H}}$, W defined by $W^{\mathsf{H}} := \Sigma^{\dagger} V^{\mathsf{H}} A$ generally does not satisfy the singular value decomposition (25.10) because in that case $V\Sigma W^{\mathsf{H}} = V\Sigma\left(\Sigma^{\dagger} V^{\mathsf{H}} A\right) \neq A$ because $V\Sigma\Sigma^{\dagger} V^{\mathsf{H}} \neq I_m$; see Exercise 25.7. Here $\Sigma^{\dagger}$ is obtained from Σ by replacing its positive singular values σ_i by $1/\sigma_i$ and taking the transpose.

The set of singular values making up Σ is unique. The unitary factors (V, W) is non-unique, but given a pair, all possible pairs can be related, according to the following result from [10, Theorem 2.6.5, p.152].

Theorem 25.9 (Uniqueness of (V, W))**.** Let $A \in \mathbb{C}^{m\times n}$ have a singular value decomposition $A = V\Sigma W^{\mathsf{H}}$ as in Theorem 25.8. Then

1. $A = \hat{V}\Sigma\hat{W}$ for some unitary matrices $(\hat{V}, \hat{W})$ if and only if there are unitary block-diagonal matrices $\tilde{V}$ and $\tilde{W}$ such that

$$\hat{V} = V\tilde{V}, \qquad \hat{W} = W\tilde{W}$$

2. If A is square $(m = n)$ and nonsingular then $\tilde{V} = \tilde{W}$.

□

Properties of singular values.

1. Matrix transpose and conjugate: $\sigma_i(A) = \sigma_i(A^T) = \sigma_i(A^{\mathsf{H}}) = \sigma_i(\overline{A})$.
2. Unitary transformation: for any unitary matrices U and V, $\sigma_i(A) = \sigma_i(UAV)$. In particular $\sigma_i(A) = \sigma_i(UA) = \sigma_i(AV)$ (setting $V = I$ or $U = I$).
3. Interlacing properties:
   - If B denote A with one of its rows *or* columns deleted, then
   $$\sigma_{i+1}(A) \leq \sigma_i(B) \leq \sigma_i(A)$$
   - If B denote A with one of its rows *and* columns deleted, then
   $$\sigma_{i+2}(A) \leq \sigma_i(B) \leq \sigma_i(A)$$
   - If B denote any $(m-k)\times(n-l)$ submatrix of A, then
   $$\sigma_{i+k+l}(A) \leq \sigma_i(B) \leq \sigma_i(A)$$

4. Singular values of $A+B$: for any $A, B \in \mathbb{C}^{m \times n}$
   - $\sum_{i=1}^{k} \sigma_i(A+B) \ \leq \ \sum_{i=1}^{k} (\sigma_i(A) + \sigma_i(B))$, $k = \min\{m,n\}$.
   - $\sigma_{i+j-1}(A+B) \ \leq \ \sigma_i(A) + \sigma_j(B)$, $i+j-1 \leq \min\{m,n\}$.
5. Singular values of AB: for any $A, B \in \mathbb{C}^{m \times n}$
   - $\sigma_n(A)\sigma_i(B) \ \leq \ \sigma_i(AB) \ \leq \ \sigma_1(A)\sigma_i(B)$.
   - $\prod_{i=1}^{k} \sigma_i(AB) \ \leq \ \prod_{i=1}^{k} \sigma_i(A)\, \sigma_i(B)$.
6. Singular value and eigenvalues: For any matrix $A \in \mathbb{C}^{n \times n}$
   - If A is normal, then $\sigma_i(A) \ = \ |\lambda_i(A)|$, $i = 1, \ldots, n$. (Note that $\lambda_i(A) \in \mathbb{C}$.)
     Proof: Spectral theorem gives $A = U \Lambda U^{\mathsf{H}}$; hence $AA^{\mathsf{H}} = U \Lambda \overline{\Lambda} U^{\mathsf{H}} = U |\Lambda|^2 U^{\mathsf{H}}$. Hence $|\lambda_i(A)|^2$ are eigenvalues of AA^{H}, implying $\sigma_i(A) = \sqrt{\lambda_i(AA^{\mathsf{H}})} = |\lambda_i(A)|$.
   - Weyl's theorem: Assume eigenvalues satisfy $|\lambda_1(A)| \ \geq \ \cdots \ \geq \ |\lambda_n(A)|$. Then

$$\prod_{i=1}^{k} |\lambda_i(A)| \ \ \leq \ \ \prod_{i=1}^{k} \sigma_i(A), \qquad k = 1, \ldots, n$$

Consider the set of complex square matrices, i.e., $m = n$. Every square matrix $A \in \mathbb{C}^{n \times n}$ is similar to a Jordan form J, i.e., there exists an invertible matrix $P \in \mathbb{C}^{n \times n}$ such that

$$A \ = \ PJP^{-1}$$

A is said to be *diagonalizable* if its Jordan form $J =: \Lambda$ is diagonal. Therefore A is diagonalizable if and only if A has n linearly independent eigenvectors; see Theorem 25.3. In that case the columns of P are these eigenvectors, Λ has the corresponding eigenvalues on its diagonal, and $AP = P\Lambda$.

25.1.5.2 Spectral decomposition for normal matrices

Recall that A is *normal* if $AA^{\mathsf{H}} = A^{\mathsf{H}}A$ and that all unitary, Hermitian, or skew-Hermitian matrices are normal (the converse is not generally true). For any matrices $A, B \in \mathbb{C}^{n \times n}$, if $BA = I$ then B is unique and $B = A^{-1}$. This is because A being nonsingular means that $Ax = b$ and $x^T A = b^T$ has a unique solution x for any $b \in \mathbb{C}^n$; take b to be each column of I.

Normal matrices are exactly those that are *unitarily* diagonalizable to which the *spectral theorem* applies [10, Theorem 2.5.3, p.133].

Theorem 25.10 (Spectral theorem for normal matrices)**.** A complex square matrix $A \in \mathbb{C}^{n \times n}$ is normal if and only if it is unitarily diagonalizable, i.e., there exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ and a complex diagonal matrix $\Lambda \in \mathbb{C}^{n \times n}$ with

$$A \ = \ U \Lambda U^{\mathsf{H}} \ = \ \sum_{i=1}^{n} \lambda_i u_i u_i^{\mathsf{H}} \tag{25.12}$$

where

1. the diagonal entries of $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$ are eigenvalues of A (generally complex);
2. the columns of U are an *arbitrary* orthonormal set of corresponding eigenvectors of A.

Hence if A is normal, then rank A = number of nonzero eigenvalues and the sum in (25.12) becomes

$$A \;=\; U\Lambda U^{\mathsf{H}} \;=\; \sum_{i=1}^{\text{rank}\,A} \lambda_i u_i u_i^{\mathsf{H}}$$

□

Hence while A is diagonalizable if and only if it has n linearly independent eigenvectors, A is unitarily diagonalizable (or equivalently normal) if and only if it has an orthonormal set of n eigenvectors.

The eigenvalues Λ of A in Theorem 25.10 are unique, but the eigenspace of A always has more than one orthonromal basis. Since two basis U and V can always be related by a unitary matrix, we have the following uniqueness result from [10, Theorem 2.5.4, p.134].

Theorem 25.11 (Uniqueness of unitary U)**.** Let $A \in \mathbb{C}^{n\times n}$ be normal with spectral decomposition $A = U\Lambda U^{\mathsf{H}}$ where U is unitary and Λ is diagonal matrix consisting of the eigenvalues of A. Then

1. $A = V\Lambda V^{\mathsf{H}}$ for a unitary matrix V if and only if there is a block-diagonal unitary matrix W such that $U = VW$.
2. In particular, if A has n distinct eigenvalues then W is a diagonal unitary matrix of the form $W = \text{diag}\left(e^{\mathbf{i}\theta_1}, \cdots, e^{\mathbf{i}\theta_n}\right)$.
3. Two normal matrices A and B are unitarily similar, i.e., $A = WBW^{\mathsf{H}}$ for some unitary matrix W, if and only if they have the same eigenvalues.

□

For a normal matrix A the eigenvalues λ_i are complex in general. A normal matrix A is Hermitian if and only if all its eigenvalues are real. If A is Hermitian then the eigenvalues are real [257, Theorem 4.1.5, p.171].

Theorem 25.12 (Spectral theorem for Hermitian matrices)**.** A complex square matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if and only if it is unitarily diagonalizable with real eigenvalues, i.e., there exist a unitary matrices $U \in \mathbb{C}^{n\times n}$ and a real diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ with

$$A \;=\; U\Lambda U^{\mathsf{H}} \;=\; \sum_{i=1}^{n} \lambda_i u_i u_i^{\mathsf{H}} \tag{25.13}$$

where

1. $\Lambda =$diag$(\lambda_1, \ldots, \lambda_n)$ is real and consists of the eigenvalues of A;

2. the columns of U are an *arbitrary* orthonormal set of corresponding eigenvectors of A.

Hence if A is Hermitian, then rank A = number of nonzero eigenvalues and the sum in (25.13) becomes

$$A \;=\; U\Lambda U^{\mathsf{H}} \;=\; \sum_{i=1}^{\operatorname{rank} A} \lambda_i u_i u_i^{\mathsf{H}}$$

Moreover, if A is real and symmetric then U above can be taken as real and orthogonal. □

To explain the last statement let A be a real symmetric matrix. First a Hermitian matrix A has real eigvenvalues λ because if v are the corresponding eivenvectors, then $Av = \lambda v$ and hence $v^{\mathsf{H}}Av = \lambda\|v\|^2$. Taking Hermitian transpose shows $v^{\mathsf{H}}A^{\mathsf{H}}v = v^{\mathsf{H}}Av = \bar{\lambda}\|v\|^2$ where $\bar{\lambda}$ denotes the complex conjugate of λ. Therefore $\bar{\lambda} = \lambda$, i.e., λ is real. Next for eigenvector v, take the Hermitian transpose of $Av = \lambda v$ we have $v^{\mathsf{H}}A^{\mathsf{H}} = v^{\mathsf{H}}A = \lambda v^{\mathsf{H}}$ since λ is real. If A is real symmetric then taking the transpose we have $A\bar{v} = \lambda\bar{v}$ where $\bar{v}$ is the componentwise complex conjugate of v. Therefore if v is an eigenvector of a real symmetric matrix A corresponding to the real eigenvalue λ then so is its complex conjugate $\bar{v}$ as well as the real vector $v + \bar{v}$, i.e., the eigenvector of A can be taken to be real.

For general matrices, about the only characterization of its eigenvalues is that they are roots of the characteristic polynomial (see the discussion leading up to Theorem 25.3). For Hermitian matrices, however, the spectral theorem leads to a variational characterization of eigenvalues [257, Theorem 4.2.2, p.176]. If $A \in \mathbb{C}^{n\times n}$ is Hermitian then

$$\lambda_{\min} \;\le\; \tfrac{x^{\mathsf{H}}Ax}{x^{\mathsf{H}}x} \;\le\; \lambda_{\max}, \qquad \forall x \in \mathbb{C}^n$$

and

$$\lambda_{\min} \;=\; \min_{x\neq 0} \frac{x^{\mathsf{H}}Ax}{x^{\mathsf{H}}x} \quad \text{and} \quad \lambda_{\max} \;=\; \max_{x\neq 0} \frac{x^{\mathsf{H}}Ax}{x^{\mathsf{H}}x}$$

Theorem 25.12 implies that A is positive semidefinite if and only if A is Hermitian and all its eigenvalues are (real and) nonnegative, and that A is positive definite if and only if A is Hermitian and all its eigenvalues are (real and) positive.

25.1.5.3 SVD and unitary diagonalization

Consider a normal matrix $A \in \mathbb{C}^{n\times n}$. Since $AA^{\mathsf{H}} = A^{\mathsf{H}}A$, they have the same eigenvectors. This does *not* mean, in general, that $W = V$ in a singular value decomposition $A = V\Sigma W^{\mathsf{H}}$. Indeed, if $W = V$ then it is necessary that $A = V\Sigma V^{\mathsf{H}}$ is positive semidefinite, but a normal A may not be positive semidefinite. The eigenvalues of a normal matrix are complex, those of a Hermitian matrix are real, and those of a positive semedefinite matrix are real and nonnegative. The following relationship between singular value decomposition of a normal matrix A and its unitary diagonalization is proved in Exercise 25.9.

Theorem 25.13 (SVD and unitary diagonalization)**.** Consider a normal matrix $A \in \mathbb{C}^{n\times n}$ and let $A = U\Lambda U^{\mathsf{H}}$ be a unitary diagonalization of A described in Theorem 25.10 where $\Lambda := \operatorname{diag}(\lambda_i)$ has the eigenvalues $\lambda_i \in \mathbb{C}$ of A on its diagonal and the columns of U are an *arbitrary* orthonormal set of corresponding eigenvectors. Write $\lambda_i = |\lambda_i|\, e^{i\theta_i}$ for some $\theta_i \in \mathbb{R}$; set $\theta_i = 0$ if $\lambda_i = 0$. Let $D := \operatorname{diag}\left(e^{\mathbf{i}\theta_1}, \ldots, e^{\mathbf{i}\theta_m}\right)$. Then

1. $V := U, \Sigma := |\Lambda|, W := UD^{\mathsf{H}}$ form a singular value decomposition $A = V\Sigma W^{\mathsf{H}}$ of A.
2. The pseudo-inverse of A is $A^{\dagger} := U\Lambda^{\dagger}U^{\mathsf{H}}$ where the diagonal matrix $\Lambda^{\dagger}$ is obtained from Λ by replacing nonzero $\lambda_i \in \mathbb{C}$ by their reciprocals.
3. A is Hermitian if and only if D in W is a real matrix, i.e., $e^{\mathbf{i}\theta_i} = 1$ or -1.
4. A is positive semidefinite if and only if $V = W := U$ and $\Sigma := \Lambda$ forms a singular value decomposition $A = V\Sigma W^{\mathsf{H}} = U\Lambda U^{\mathsf{H}}$, i.e., SVD and unitary diagonalization of A coincide.

The theorem also prescribes a way to compute a singular value decomposition $A = V\Sigma W$ when A is normal. In this case we can take the columns of V to be an arbitrary orthonormal set of eigenvectors of A (which will also be eigenvectors of AA^*). This may not be the case if A is not normal and the more general method prescribed by (25.11) is needed to compute SVD (see Example 25.3). The theorem is illustrated in the following example.

Example 25.4. Use Theorem 25.13 to compute the SVD of the normal matrix A in Example 25.3.

Solution. Clearly $A = A^{\mathsf{H}} = A^T = \bar{A}$ and A is real symmetric and hence normal. Its eigenvalues are $\lambda_i = \pm 1$ with corresponding eigenvectors in the columns of U in the unitary diagonalization:

$$A \;=\; U\Lambda U^{\mathsf{H}} \;:=\; \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & \\ & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}$$

Note that A is not positive semidefinite and therefore $W \neq U$ in the singular value decomposition of A. According to Theorem 25.13, the angle matrix $D = \mathrm{diag}(1,-1)$ and the unitary factors (V,W) in the SVD $A = V\Sigma W^{\mathsf{H}}$ are given by

$$\Sigma \;:=\; |\Lambda| \;=\; I, \qquad V \;:=\; U \;=\; \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \qquad W \;:=\; UD^{\mathsf{H}} \;=\; \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

which agrees with those computed in Example 25.3. (The decomposition in these two examples agree because the matrix Y in Example 25.3 has been chosen to be $Y = I$ so that $YA = A$.) □

25.1.5.4 Complex symmetric matrices

Consider a complex symmetric matrix $A \in \mathbb{C}^{n\times n}$ with $A = A^{\mathsf{T}}$. Then $A^{\mathsf{H}} = \bar{A}$ where $\bar{A}$ is the matrix obtained from A by taking its complex conjugate componentwise. A is not Hermitian unless A is a real matrix. The following result, from [10, Corollary 2.6.6, p.153], is called the Takagi's factorization for complex symmetric matrices.

Theorem 25.14 (Takagi's decomposition)**.** A complex matrix $A \in \mathbb{C}^{n\times n}$ is symmetric $A = A^{\mathsf{T}}$ if and only if there is a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a real nonnegative diagonal matrix $\Sigma :=\mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ such that

$$A \;=\; U\Sigma U^{\mathsf{T}} \tag{25.14}$$

where Σ consists of the nonnegative square roots of the eigenvalues of $A\bar{A}$. □

The columns of the unitary matrix U in (25.14) are generally neither the singular vectors nor the eigenvectors of A; see the proof below. A Takagi decomposition of a complex symmetric matrix A is therefore generally different from its singular value decomposition. A Takagi decomposition of a real symmetric matrix may not have real factors. In contrast, its spectral decomposition in terms of its eigenvalues, rather than singular values, can always use real orthogonal factors according to Theorem 25.12.

We provide a sketch of the proof from [10, Corollary 2.6.6, p.153].

Proof sketch of Theorem 25.14. Let a singular value decomposition of A be $A = V\Sigma W^{\mathsf{H}}$ according to Theorem 25.8. Since $A = A^{\mathsf{T}}$ we have $A = V\Sigma W^{\mathsf{H}} = \bar{W}\Sigma\bar{V}^{\mathsf{H}}$ where $(\bar{V}, \bar{W})$ are componentwise complex conjugate of (V,W). The uniqueness Theorem 25.9 then implies the existence of unitary block-diagonal matrices $(\tilde{V}, \tilde{W})$ such that

$$\bar{V} \;=\; W\tilde{V}, \qquad\qquad \bar{W} \;=\; V\tilde{W} \tag{25.15a}$$

Indeed, according to Autonne's uniqueness theorem ([10, Theorem 2.6.5, p.152]), $\tilde{V}$ and $\tilde{W}$ can be taken to have identical blocks except the last block corresponding to the diagonal zero-block in (25.9). Specifically suppose A has rank r and d distinct positive singular values $s_1 > s_2 > \cdots > s_d > 0$ with (algebraic) multiplicities $n_1, \cdots, n_d$. Then $r := \sum_{i=1}^{d} n_i \leq n$. We can separate the diagonal of the $n \times n$ matrix Σ into $d+1$ diagonal blocks of diagonal submatrices $s_i I_{n_i}$ and 0_{n-r}:

$$\Sigma \;=\; \operatorname{diag}\left(s_1 I_{n_1}, \cdots, s_d I_{n_d}, 0_{n-r}\right) \tag{25.15b}$$

where I_k denotes the identity matrix of size k and 0_k denotes the $k \times k$ zero matrix. (If A is of full rank $r = n$ then the zero block 0_{n-r} is absent.) Then Autonne's uniqueness theorem ([10, Theorem 2.6.5, p.152]) implies that $A = V\Sigma W^{\mathsf{H}} = \bar{W}\Sigma\bar{V}^{\mathsf{H}}$ if and only if there are unitary matrices V_i of sizes n_i and V_{d+1}, W_{d+1} of size $n-r$ such that

$$\tilde{V} \;=\; \operatorname{diag}\left(V_1, \cdots, V_d, V_{d+1}\right), \qquad\qquad \tilde{W} \;=\; \operatorname{diag}\left(V_1, \cdots, V_d, W_{d+1}\right) \tag{25.15c}$$

and $\bar{V} = W\tilde{V}$, $\bar{W} = V\tilde{W}$. But $\tilde{V} \;=\; W^{\mathsf{H}}\bar{V} \;=\; \left(V^{\mathsf{H}}\bar{W}\right)^{\mathsf{T}} \;=\; \tilde{W}^{\mathsf{T}}$ and hence $V_i = V_i^{\mathsf{T}}$ are symmetric matrices for $i = 1, \ldots, d$.

Lemma 25.7 then implies that there exist unitary symmetric matrices $R_i \in \mathbb{C}^{n_i \times n_i}$ such that $V_i = R_i^2$ for $i = 1, \cdots, d$. Substitute this and (25.15) into $A = \bar{W}\Sigma\bar{V}^{\mathsf{H}}$, we have $A = \bar{W}\Sigma V^{\mathsf{T}} = V\tilde{W}\Sigma V^{\mathsf{T}}$. But (taking $W_{d+1} := I_{n-r}$)

$$\tilde{W}\Sigma \;=\; \operatorname{diag}\left(R_1^2, \cdots, R_d^2, I_{n-r}\right) \cdot \operatorname{diag}\left(s_1 I_{n_1}, \ldots, s_d I_{n_d}, 0_{n-r}\right) \;\;=: \;\; R\Sigma R$$

where $R := \operatorname{diag}\left(R_1, \cdots, R_d, I_{n-r}\right)$. Hence

$$A \;=\; V(\tilde{W}\Sigma)V^{\mathsf{T}} \;=\; V\left(R\Sigma R\right)V^{\mathsf{T}} \;=\; \underbrace{(VR)}_{U}\,\Sigma\,\underbrace{(VR)^{\mathsf{T}}}_{U^{\mathsf{T}}}$$

where the last equality uses the symmetry of R. This completes the proof. $\square$

A complex symmetric matrix $A \in \mathbb{C}^{n \times n}$ may or may not be normal. Complex symmetric matrices are useful for power systems because the admittance matrix Y (see Chapter 4.2) are complex symmetric, and generally not Hermitian. See Exercise 9.23 for a complex symmetric matrix that is not diagonalizable (and hence not normal). See Exercise 9.24 for a complex symmetric matrix that is normal and hence unitarily diagonalizable, and Exercise 4.2 for characterizations of symmetric and normal matrices.

25.1.6 Pseudo-inverse

Consider a matrix $A \in \mathbb{C}^{m\times n}$. Let null$(A)$ denote the *null space* (also called kernel) of A, i.e., null$(A) := \{x \in \mathbb{C}^n : Ax = 0\}$. Let range$(A)$ denote the *range space* (also called column space) of A, i.e., range$(A) := \{y \in \mathbb{C}^m : y = Ax$ for some $x \in \mathbb{C}^n\}$. In this subsection we treat A as a mapping from $\mathbb{C}^n$ to $\mathbb{C}^m$ and A^{H} a mapping from $\mathbb{C}^m$ to $\mathbb{C}^n$. Then null(A) and range(A^{H}) are linear spaces and they are orthogonal complements of each other because, if $x_1 \in$ null(A) and $x_2 \in$ range(A^{H}) so that $x_2 = A^{\mathsf{H}}y$ for some y, then

$$x_2^{\mathsf{H}} x_1 \;=\; y^{\mathsf{H}} A x_1 \;=\; 0$$

We denote this fact by the notation $\mathbb{C}^n = \text{range}(A^{\mathsf{H}}) \oplus \text{null}(A)$, as shown in the upper panel of Figure 25.3(a). This implies

$$\dim(\text{range}(A^{\mathsf{H}})) + \dim(\text{null}(A)) \;=\; n \tag{25.16}$$

The rank of a matrix $A \in \mathbb{C}^{m\times n}$, denoted rank A, is the largest number of linearly independent columns of A, or equivalently the largest number of linearly independent rows of A. By definition rank A = dim(range(A)). A square matrix $A \in \mathbb{C}^{n\times n}$ is called *nonsingular* if rank $A = n$; it is called *singular* if rank $A < n$. Some simple facts are collected in the following.

Theorem 25.15.

1. For any $A \in \mathbb{C}^{m\times n}$, rank A = rank A^{H} = rank A^T = rank $\bar{A}$.
2. For any $A \in \mathbb{C}^{m\times n}$, rank $A \le \min\{m,n\}$.
3. If $A \in \mathbb{C}^{m\times m}$ and $C \in \mathbb{C}^{n\times n}$ are nonsingular, then for any $B \in \mathbb{C}^{m\times n}$, rank B = rank ABC, i.e., left or/and right multiplication by a nonsingular matrix does not change rank.
4. For any $A \in \mathbb{C}^{m\times n}$, rank A + dim(null(A)) $= n$. This follows from substituting rank A^{H} = rank A into (25.16).

If we consider the matrix $A \in \mathbb{C}^{m\times n}$ as a mapping from $\mathbb{C}^n$ to $\mathbb{C}^m$ and restrict it to A : range$(A^{\mathsf{H}}) \to$ range(A), then A is surjective and injective (see Exercise 25.10). Hence an inverse always exists from range$(A) \to$ range(A^{H}). We will denote this inverse by $A^\dagger$; see Figure 25.3(b). Let $A = V\Sigma W^{\mathsf{H}}$ be its singular value decomposition and let rank $A = r \le \min\{m,n\}$. We will show that

$$A^\dagger \;=\; W\Sigma^\dagger V^{\mathsf{H}} \tag{25.17}$$

where $\Sigma^\dagger$ is a real diagonal $n\times m$ matrix of rank r obtained from the $m\times n$ diagonal matrix Σ by replacing the (positive) singular values σ_i by $1/\sigma_i$ and taking the transpose. When $r = m = n$, $\Sigma^\dagger = \text{diag}\left(\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_n}\right) = \Sigma^{-1}$ so that $A^\dagger = A^{-1}$ since

$$A^\dagger A \;=\; \left(W\Sigma^{-1}V^{\mathsf{H}}\right)\left(V\Sigma W^{\mathsf{H}}\right) \;=\; I_n$$

In general $A^\dagger A \ne I_n$ but the next result shows that $A^\dagger A$ equals I_n plus a matrix whose columns are in null(A). Specifically, let $A \in \mathbb{C}^{m\times n}$ with rank $A = r \le \min\{m,n\}$. Let $A = V\Sigma W^{\mathsf{H}}$ be its singular value

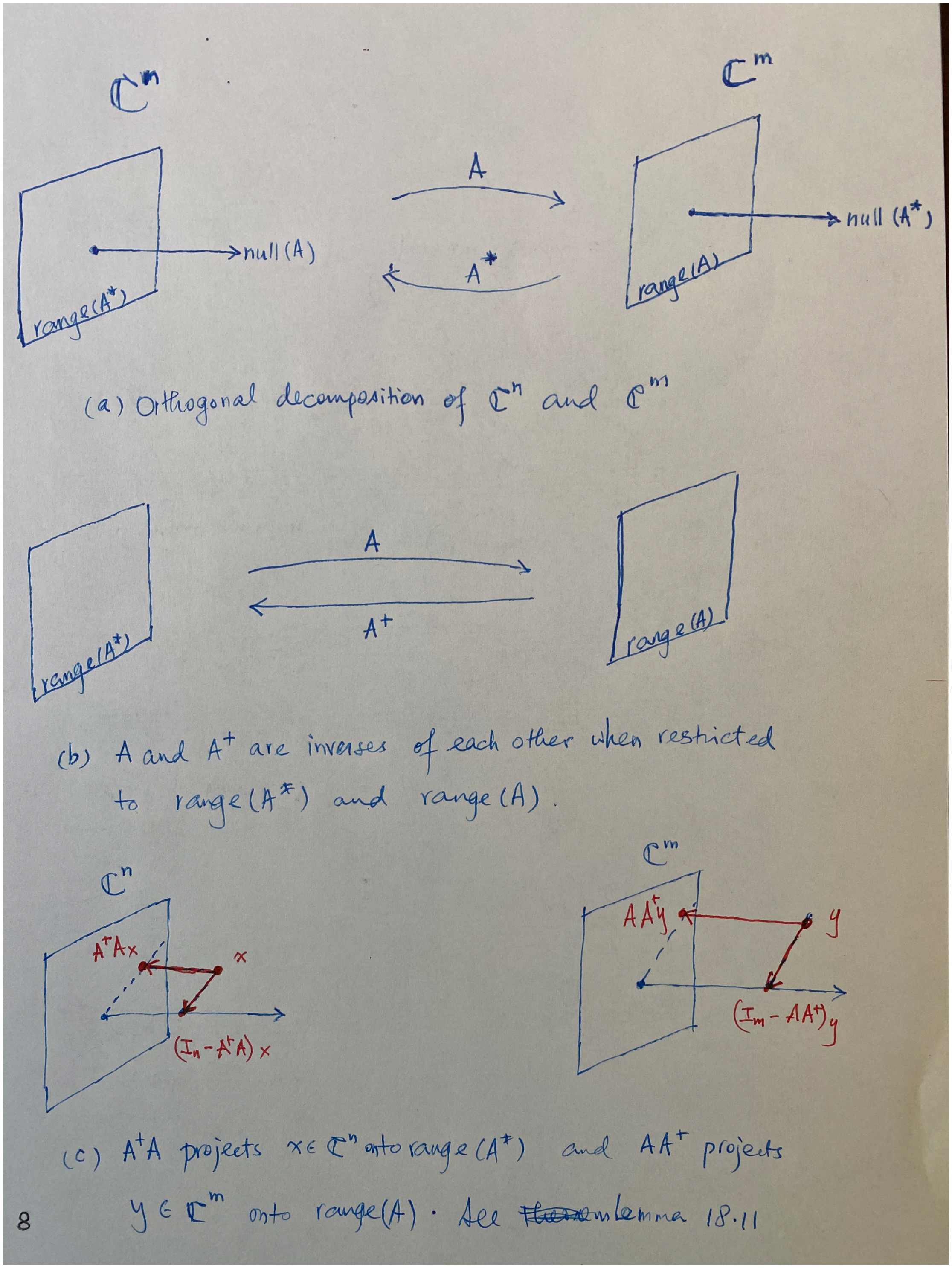


Figure 25.3: (a) Decomposition of $\mathbb{C}^n$ into orthogonal complements null(A) and range(A^{H}) and $\mathbb{C}^m$ into orthogonal complements null(A^{H}) and range(A). (b) A and $A^{\dagger}$ are bijective between range(A^{H}) and range(A) inverses of each other. (c) $A^{\dagger}A$ projects $x \in \mathbb{C}^n$ onto range(A^{H}) and $AA^{\dagger}$ projects $y \in \mathbb{C}^m$ onto range(A). See Theorem 25.16.

decomposition. Decompose the various matrices such that

$$\Sigma \;=\; \begin{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \;=:\; \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad V \;=:\; \begin{bmatrix} V_r & V_{m-r} \end{bmatrix}, \qquad W \;=:\; \begin{bmatrix} W_r & W_{n-r} \end{bmatrix}$$

where Σ_r is $r \times r$ diagonal matrix, the matrices $V_r \in \mathbb{C}^{m\times r}$ and $W_r \in \mathbb{C}^{n\times r}$ consist of the first r columns of V and W respectively, and the matrices $V_{m-r} \in \mathbb{C}^{m\times(m-r)}$ and $W_{n-r} \in \mathbb{C}^{n\times(n-r)}$ consist of the remaining columns of V and W respectively. Then

$$\begin{aligned} A &= \begin{bmatrix} V_r & V_{m-r} \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_r^{\mathsf{H}} \\ W_{n-r}^{\mathsf{H}} \end{bmatrix} &= V_r \Sigma_r W_r^{\mathsf{H}} \\ A^{\dagger} &= \begin{bmatrix} W_r & W_{n-r} \end{bmatrix} \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_r^{\mathsf{H}} \\ V_{m-r}^{\mathsf{H}} \end{bmatrix} &= W_r \Sigma_r^{-1} V_r^{\mathsf{H}} \end{aligned}$$

and $A^{\mathsf{H}} = W_r \Sigma_r V_r^{\mathsf{H}}$. Hence the range spaces of $A, A^{\dagger}, A^{\mathsf{H}}$ depend only on the nonzero singular values and the first r columns of V and W. The remaining columns V_{m-r}, W_{n-r} span their null spaces and can be interpreted as a measure of how different the pseudo-inverse $A^{\dagger}$ is from an inverse, as the following theorem shows. The theorem is illustrated in Figures 25.4 and 25.3(c).

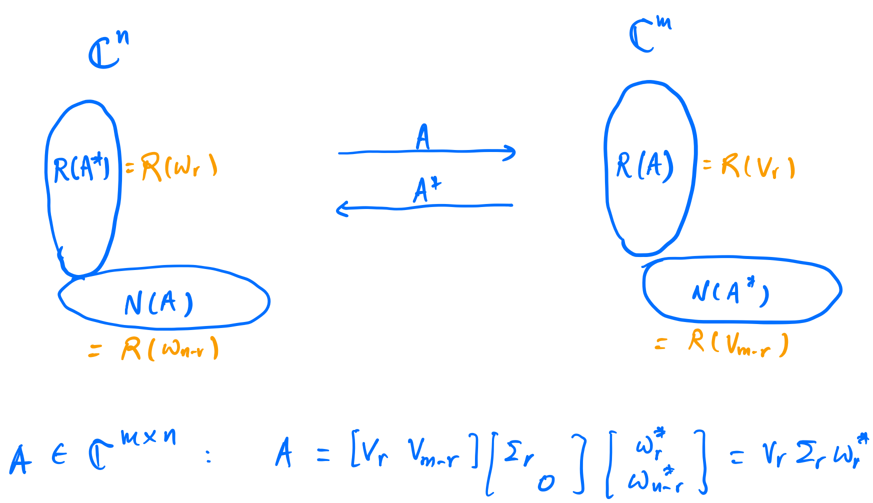


Figure 25.4: Orthogonal decomposition of $\mathbb{C}^n$ and $\mathbb{C}^m$ using singular value decomposition of matrix M.

Theorem 25.16. With the notations above,

1. $A^{\dagger} := W\Sigma^{\dagger}V^{\mathsf{H}}$ satisfies (I_n denotes the $n \times n$ identity matrix)

$$\begin{aligned} A^{\dagger}A &= I_n - W_{n-r}W_{n-r}^{\mathsf{H}} \\ AA^{\dagger} &= I_m - V_{m-r}V_{m-r}^{\mathsf{H}} \end{aligned}$$

2. $\text{null}(A) = \text{range}(W_{n-r})$ and $\text{range}(A^{\mathsf{H}}) = \text{range}(W_r)$.

3. null(A^{H}) = range(V_{m-r}) = null$(A^\dagger)$ and range(A) = range(V_r).

4. $A^\dagger A$ is the orthogonal projection of $x \in \mathbb{C}^n$ onto range(A^{H}). $I_n - A^\dagger A$ is the orthogonal projection of $x \in \mathbb{C}^n$ onto null(A), i.e., if $\hat{x} := (I_n - A^\dagger A)x$ then $A\hat{x} = 0$.

5. Similarly $AA^\dagger$ is the orthogonal projection of $y \in \mathbb{C}^m$ on to range(A) and $I_m - AA^\dagger$ is the orthogonal projection of $y \in \mathbb{C}^m$ onto null(A^{H}).

6. $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, and $A^{\mathsf{H}}AA^\dagger = A^{\mathsf{H}}$.

Proof. We have

$$\begin{aligned} A^\dagger A &= \begin{bmatrix} W_r & W_{n-r} \end{bmatrix} \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_r^{\mathsf{H}} \\ V_{n-r}^{\mathsf{H}} \end{bmatrix} \cdot \begin{bmatrix} V_r & V_{n-r} \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_r^{\mathsf{H}} \\ W_{n-r}^{\mathsf{H}} \end{bmatrix} \\ &= \begin{bmatrix} W_r & W_{n-r} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W_r^{\mathsf{H}} \\ W_{n-r}^{\mathsf{H}} \end{bmatrix} \;=\; W_r W_r^{\mathsf{H}} \end{aligned}$$

Even though $W^{\mathsf{H}} = W^{-1}$, W_r^{H} is not the inverse of W_r (unless $r = n \neq m$) since W_r is not even square. Since

$$WW^{\mathsf{H}} \;=\; \begin{bmatrix} W_r & W_{n-r} \end{bmatrix} \begin{bmatrix} W_r^{\mathsf{H}} \\ W_{n-r}^{\mathsf{H}} \end{bmatrix} \;=\; W_r W_r^{\mathsf{H}} + W_{n-r} W_{n-r}^{\mathsf{H}} \;=\; I_n$$

we have

$$A^\dagger A \;=\; I_n \;-\; W_{n-r} W_{n-r}^{\mathsf{H}}$$

Similarly $AA^\dagger = I_m \;-\; V_{m-r} V_{m-r}^{\mathsf{H}}$.

To show that null(A) = range(W_{n-r}) consider any $x \in \mathbb{C}^n$. Since columns of W are an orthonormal basis of $\mathbb{C}^n$ we can write $x = \sum_j b_j w_j$ for some $b_j \in \mathbb{C}$ where w_j are columns of W. Then

$$Ax \;=\; V\Sigma W^{\mathsf{H}} \sum_j b_j w_j \;=\; V\Sigma \sum_j b_j \begin{bmatrix} w_1^{\mathsf{H}} w_j \\ \vdots \\ w_n^{\mathsf{H}} w_j \end{bmatrix} \;=\; V\Sigma \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \;=\; V \begin{bmatrix} \sigma_1 b_1 \\ \vdots \\ \sigma_r b_r \\ 0_{n-r} \end{bmatrix}$$

where 0_{n-r} is the zero vector of size $n-r$. Since V is nonsingular and $\sigma_j > 0$, $Ax = 0$ if and only if $b_1 = \cdots = b_r = 0$. Hence null(A) = range(W_{n-r}) if and only if $x \in$ range(W_{n-r}). That range(A^{H}) = range(W_r) follows from $A^{\mathsf{H}} \;=\; W\Sigma^T V^{\mathsf{H}} \;=\; W_r \Sigma_r V_r^{\mathsf{H}}$.

The proof of null$(A^\dagger)$ = range(V_{m-r}) follows the same argument and is presented in the matrix notation as follows. Any $y \in \mathbb{C}^m$ can be written in terms of the columns of V, i.e., $y = V_r b_r + V_{m-r} b_{m-r}$ for some b_r, b_{m-r}. Then

$$A^\dagger y \;=\; W_r \Sigma_r^{-1} V_r^{\mathsf{H}} (V_r b_r + V_{m-r} b_{m-r}) \;=\; W_r \Sigma_r^{-1} V_r^{\mathsf{H}} V_r b_r$$

since $V_r^{\mathsf{H}} V_{m-r} = 0_{r \times (m-r)}$. Hence $A^\dagger y = 0$ if and only if $b_r = 0$ and $y = V_{m-r} b_{m-r}$. This means null$(A^\dagger)$ = range(V_{m-r}). Since $A^{\mathsf{H}} = W_r \Sigma_r V_r^{\mathsf{H}}$ the same argument shows that null(A^{H}) = range(V_{m-r}).

The remaining assertions follow from parts 1, 2, 3. For example

$$AA^\dagger A \;=\; A\left(I_n \,-\, W_{n-r}W_{n-r}^{\mathsf{H}}\right) \;=\; A \,-\, AW_{n-r}W_{n-r}^{\mathsf{H}} \;=\; A$$

Similarly $A^\dagger AA^\dagger = A^\dagger$, and $A^{\mathsf{H}}AA^\dagger = A^{\mathsf{H}}$. □

We remark on some implications of Theorem 25.16.

Remark 25.2 ($Ax = b$)**.** 1. The theorem implies that $A^\dagger$ in (25.17) and A are inverses of each other when restricted to range(A^{H}) and range(A) (see Exercise 25.11). Therefore, even though (V,W) in the singular value decomposition are generally not unique, $A^\dagger$ is uniquely defined. Treated as a mapping from $\mathbb{C}^m$ to $\mathbb{C}^n$, $A^\dagger$ is called a *pseudo-inverse* of A.

2. There is a solution x for $Ax = b$ if and only if b is in range(A) or equivalently b is orthogonal to null (A^{H}), in which case the set of solutions is given by

$$x \;=\; A^\dagger b \,+\, w, \qquad\qquad w \in \text{null}(A) = \text{range}(W_{n-r})$$

Moreover $A^\dagger b$ is the solution to $Ax = b$ with the smallest Euclidean norm $\|x\|_2 = \|A^\dagger b\|_2 + \|w\|_2$.

3. Consider $Ax = b$ when b is not in range(A) and therefore there is no x that satisfies this equation. The theorem says that $\hat{x} = A^\dagger b$ is a 'best estimate' of x from b in that $A\hat{x}$ equals the projection of b onto range(A) and the estimation error $b - A\hat{x} = (I_m - AA^\dagger)b$ is the projection of b onto null(A^{H}). This achieves the minimum estimation error under the Euclidan norm; see Exercise 25.14.

4. Theorem 25.16.6 is easy to understand given Lemmas 25.16.4 and 25.165. Consider any vector $y \in \mathbb{C}^m$. The operation $AA^\dagger$ removes y's component in the null space of A^{H}. We can first project y to range(A) to obtain $AA^\dagger y$ and then map it back into $\mathbb{C}^n$ to $A^\dagger(AA^\dagger y)$. Since $AA^\dagger y$ is already in range(A) over which $A^\dagger$ is an inverse of A, this operation should be the same as $A^\dagger$, i.e., $A^\dagger AA^\dagger y = A^\dagger y$ for all y. Similarly the projection operation $A^\dagger A$ to range(A^{H}) followed by the mapping A is the same operation as the mapping A.

For general matrix $A \in \mathbb{C}^{m\times n}$, its pseudo-inverse is given in terms of its singular value decomposition by (25.17). For special matrices the next result provide some explicit formulae.

Corollary 25.17. Consider a matrix $A \in \mathbb{C}^{m\times n}$ with rank $A = r \le \min\{m,n\}$. Let $A = V\Sigma W^{\mathsf{H}}$ be its singular value decomposition and $A^\dagger = W\Sigma^\dagger V^{\mathsf{H}}$ be its pseudo-inverse.

1. If $m = n$ and A is positive semidefinite then $A + V_{n-r}V_{n-r}^{\mathsf{H}}$ is invertible and

$$A^\dagger \;=\; \left(A + V_{n-r}V_{n-r}^{\mathsf{H}}\right)^{-1} - V_{n-r}V_{n-r}^{\mathsf{H}}$$

2. If $r = m \le n$ then $A^\dagger = A^{\mathsf{H}}\left(AA^{\mathsf{H}}\right)^{-1}$.

3. If $r = n \le m$ then $A^\dagger = \left(A^{\mathsf{H}}A\right)^{-1}A^{\mathsf{H}}$.

4. If $r = m = n$ then $A^\dagger = A^{-1}$.

Proof. Since A is positive semidefinite its singular value decomposition coincides with its spectral decomposition according to Theorem 25.13.3, so

$$A \;=\; V\Sigma W^{\mathsf{H}} \;=\; V\Lambda V^{\mathsf{H}} \;=\; V_r\Lambda_r V_r^{\mathsf{H}}$$

where V is a unitary matrix whose columns are orthonormal eigenvectors of A, $\Lambda := \mathrm{diag}(\lambda_i)$ is the diagonal matrix of eigenvalues

$$\lambda_1 \;\geq\; \cdots \;\geq\; \lambda_r \;>\; 0 \;=\; \lambda_{r+1} \;=\; \cdots \;=\; \lambda_n$$

and matrices are decomposed as before:

$$\Lambda \;=:\; \begin{bmatrix} \Lambda_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad V \;=:\; \begin{bmatrix} V_r & V_{n-r} \end{bmatrix}, \qquad x \;=:\; \begin{bmatrix} x_r \\ x_{n-r} \end{bmatrix} \in \mathbb{C}^n$$

To show that $A + V_{n-r}V_{n-r}^{\mathsf{H}}$ is invertible consider any $x \in \mathbb{C}^n$ in the null space of A expressed in terms of the basis V as $x = Va =: V_r a_r + V_{n-r}a_{n-r}$. We have

$$(A + V_{n-r}V_{n-r}^{\mathsf{H}})\,x \;=\; \left(V_r\Lambda_r V_r^{\mathsf{H}} + V_{n-r}V_{n-r}^{\mathsf{H}}\right)(V_r a_r + V_{n-r}a_{n-r}) \;=\; V_r\Lambda_r a_r \;+\; V_{n-r}a_{n-r}$$

where we have used $V_r^{\mathsf{H}}V_{n-r} = 0$. Hence

$$(A + V_{n-r}V_{n-r}^{\mathsf{H}})\,x \;=\; \begin{bmatrix} V_r & V_{n-r} \end{bmatrix} \begin{bmatrix} \Lambda_r a_r \\ a_{n-r} \end{bmatrix} \;=\; V \begin{bmatrix} \Lambda_r a_r \\ a_{n-r} \end{bmatrix}$$

Since V and Λ_r are nonsingular, $(A + V_{n-r}V_{n-r}^{\mathsf{H}})x = 0$ if and only if $a = 0$, proving the nonsingularity of $A + V_{n-r}V_{n-r}^{\mathsf{H}}$.

To show that $A^\dagger = \left(A + V_{n-r}V_{n-r}^{\mathsf{H}}\right)^{-1} - V_{n-r}V_{n-r}^{\mathsf{H}}$ we will prove that $A^\dagger + V_{n-r}V_{n-r}^{\mathsf{H}}$ is the inverse of $A + V_{n-r}V_{n-r}^{\mathsf{H}}$. We have (using again $V_r V_{n-r}^{\mathsf{H}} = 0$)

$$\begin{aligned}
\left(A^\dagger + V_{n-r}V_{n-r}^{\mathsf{H}}\right)\left(A + V_{n-r}V_{n-r}^{\mathsf{H}}\right) \;&=\; \left(V_r\Lambda_r^{-1}V_r^{\mathsf{H}} + V_{n-r}V_{n-r}^{\mathsf{H}}\right)\left(V_r\Lambda_r V_r^{\mathsf{H}} + V_{n-r}V_{n-r}^{\mathsf{H}}\right) \\
&=\; V_r V_r^{\mathsf{H}} \;+\; V_{n-r}V_{n-r}^{\mathsf{H}} \;=\; VV^{\mathsf{H}} \;=\; I_n
\end{aligned}$$

as desired.

If If $r = m \leq n$ then $V_r = V$ and

$$\Sigma \;=:\; \begin{bmatrix} \Sigma_r & 0 \end{bmatrix}, \qquad W \;=:\; \begin{bmatrix} W_r & W_{n-r} \end{bmatrix}$$

Then $A = V\Sigma W^{\mathsf{H}} = V\Sigma_r W_r^{\mathsf{H}}$ and hence $AA^{\mathsf{H}} = \left(V\Sigma_r W_r^{\mathsf{H}}\right)\left(W_r\Sigma_r V^{\mathsf{H}}\right) = V\Sigma_r^2 V^{\mathsf{H}}$ is invertible since $W_r^{\mathsf{H}}W_r = I_r$. Since V is unitary we have $\left(AA^{\mathsf{H}}\right)^{-1} = V\Sigma_r^{-2}V^{\mathsf{H}}$. Hence

$$A^{\mathsf{H}}\left(AA^{\mathsf{H}}\right)^{-1} \;=\; \left(W_r\Sigma_r V^{\mathsf{H}}\right)\left(V\Sigma_r^{-2}V^{\mathsf{H}}\right) \;=\; W_r\Sigma_r^{-1}V^{\mathsf{H}} \;=\; W\Sigma^\dagger V^{\mathsf{H}} \;=\; A^\dagger$$

The case of $r = n \leq m$ is similarly proved in Exercise 25.12. If $r = m = n$ then $\Sigma^\dagger = \Sigma^{-1}$ so that $A^\dagger = A^{-1}$ since $A^\dagger A = \left(W\Sigma^{-1}V^{\mathsf{H}}\right)\left(V\Sigma W^{\mathsf{H}}\right) = I_n$.

$\square$

Consider a partitioned matrix $A = [B\ C]$. In general $A^\dagger \neq \begin{bmatrix} B^\dagger \\ C^\dagger \end{bmatrix}$. [2] Several expressions for $A^\dagger$ in terms of $B^\dagger$ and $C^\dagger$ are derived in [258] under various necessary and sufficient conditions. The particularly simple case is the following result from [258, Corollary 1.4].

Lemma 25.18. Suppose $A = [B\ C]$. Then

$$A^\dagger = \begin{bmatrix} B^\dagger \\ C^\dagger \end{bmatrix}$$

if and only if $(I - BB^\dagger)C = C$ (i.e., if and only if C is in $\text{null}(B^{\mathsf{H}})$).

25.1.7 Norms and inequalities

25.1.7.1 Vector norms

This subsection mostly follows [10, Chapter 5].

Definition 25.4 (Normed linear space)**.** Let V be a vector space over the field F with $F = \mathbb{R}$ or $\mathbb{C}$. A function $\|\cdot\| : V \to \mathbb{R}$ is a *norm*, or *vector norm*, on V if, for all $x, y \in V$ and all $c \in F$,

1. *Positivity*: $\|x\| \geq 0$ and $\|x| = 0$ if and only if $x = 0$.
2. *Homogeneity*: $\|cx\| = |c|\,\|x\|$.
3. *Triangular inequality*: $\|x + y\| \leq \|x\| + \|y\|$.

The real or complex vector space together with a norm $(V, \|\cdot\|)$ is called a *normed linear space* or *normed vector space*. □

Examples of vector norms on $V = \mathbb{C}^n$ include: for any $x \in \mathbb{C}^n$,

- *Sum norm (l_1 norm)*: $\|x\|_1 := \sum_i |x_i|$.
- *Euclidean norm (l_2 norm)*: $\|x\|_2 := \sqrt{\sum_i |x_i|^2}$.
- *Max norm (l_∞ norm)*:$\|x\|_\infty := \max_i |x_i|$.
- *l_p norm*: $\|x\|_p := \left(\sum_i |x_i|^p\right)^{1/p}$, $p \geq 1$.

It can be shown that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$ for all $x \in \mathbb{C}^n$. We therefore often define l_p norms for $p \in [1, \infty]$. The Euclidean norm, and positive scalar multiples of the Euclidean norm, are the only norms on $\mathbb{C}^n$ that

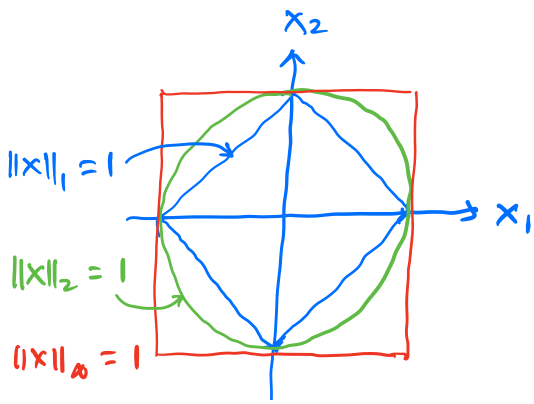


Figure 25.5: The boundaries of unit balls for l_1, l_2 and l_∞ norms.

are *unitarily invariant*: $\|Ux\|_2 = \|x\|_2$ for any $x \in \mathbb{C}^n$ and any unitary matrix $U \in \mathbb{C}^{n\times n}$ (Exercise 25.17). The unit balls $B := \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ for l_1, l_2 and l_∞ norms are shown in Figure 25.5.

An example of infinite dimensional normed vector spaces is the set $C[a,b]$ of all continuous real or complex-valued functions $f : [a,b] \to \mathbb{R}$ or $f : [a,b] \to \mathbb{C}$ on the real interval $[a,b]$. The L_p norms on $C[a,b]$ are

- L_1 *norm*: $\|f\|_1 := \int_a^b |f(t)|dt$.
- L_2 *norm*: $\|f\|_2 := \sqrt{\int_a^b |f(t)|^2 dt}$.
- L_p *norm*: $\|f\|_p := \left(\int_a^b |f(t)|^p dt\right)^{1/p}$, $p \geq 1$.
- L_∞ *norm*: $\|f\|_\infty := \max\{|f(x)| : x \in [a,b]\}$.

There are two important properties of finite dimensional real or complex vector spaces V (i.e., $F = \mathbb{R}$ or $\mathbb{C}$) that do not necessarily hold for infinite dimensional vector spaces. First all norms are equivalent in the sense that, given two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ on a finite dimensional vector space V, there exist c_m, c_M such that (e.g., [10, Corollary 5.4.5, p.327])

$$c_m \|x\|_\alpha \;\leq\; \|x\|_\beta \;\leq\; c_M \|x\|_\alpha, \qquad x \in V \tag{25.18}$$

This means that if a sequence $\{x_i\} \subseteq V$ converges in some norm, it converges in all norms. For l_p norms the best bounds are [10, Problem 5.4.P3, p.333]: for $1 \leq p_1 < p_2 < \infty$,

$$\|x\|_{p_2} \;\leq\; \|x\|_{p_1} \;\leq\; n^{\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|x\|_{p_2}$$

[2]Let the singular value decompositions of B and C be $B = V_1\Sigma_1 W_1^{\mathsf{H}}$ and $C = V_2 \Sigma W_2^{\mathsf{H}}$. We can write

$$A \;=\; \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} W_1^{\mathsf{H}} & 0 \\ 0 & W_2^{\mathsf{H}} \end{bmatrix}$$

However $\left(VMW^{\mathsf{H}}\right)^\dagger = WM^\dagger V^{\mathsf{H}}$ only if V and W are unitary [258, Lemma 1]. The matrix $[V_1 \; V_2]$ is not unitary.

For example $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$, $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$, $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$ (see Figure 25.5). In contrast, for an infinite dimensional vector space such as $C[a,b]$, a sequence $\{f_k\}$ of functions in $C[a,b]$ may converge under the L_1 norm, remains bounded under L_2 norm, but diverge under the L_∞ norm (unbounded $\|f_k\|_\infty$).

Second a sequence $\{x_i\} \subseteq V$ converges to a vector in a finite dimensional vector space V if and only if it is a *Cauchy sequence*, i.e., for any $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$ such that $\|x_i - x_j\| \le \varepsilon$ for any $i, j \ge N(\varepsilon)$. A normed linear space V is said to be *complete* with respect to its norm $\|\cdot\|$ if every sequence in V that is a Cauchy sequence with respect to $\|\cdot\|$ converges to a point in V. Therefore all finite dimensional real or complex vector spaces are complete with respect to any norm, but infinite dimensional normed vector spaces, such as $C[a,b]$ with the L_1 norm, may not be complete.

Definition 25.5 (Inner product space)**.** Let V be a (finite or infinite dimensional) vector space over the field F with $F = \mathbb{R}$ or $\mathbb{C}$. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is an *inner product* if, for all $x, y, z \in V$ and all $c \in F$,

1. *Positivity*: $\langle x, x \rangle \ge 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.
2. *Additivity*: $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.
3. *Homogeneity*: $\langle cx, y \rangle = c\langle x, y \rangle$.
4. *Hermitian property*: $\langle x, y \rangle = \overline{\langle x, y \rangle}$.

where $\overline{a}$ denotes the complex conjugate of $a \in F$. The real or complex vector space together with an inner product $(V, \langle \cdot, \cdot \rangle)$ is called an *inner product space*. □

Note that regardless of $F = \mathbb{R}$ or $\mathbb{C}$, a norm in Definition 25.4 takes value in $\mathbb{R}$ whereas an inner product in Definition 25.5 takes value in F. Implicit in the nonnegativity property is that, while $\langle x, y \rangle \in F$, $\langle x, x \rangle \in \mathbb{R}$. The function defined on $\mathbb{C}^n$ by $\langle x, y \rangle := x^{\mathsf{H}} y \in F := \mathbb{C}$ is an inner product called the Euclidean inner product. Let $M \in \mathbb{F}^{n \times n}$ be a positive definite matrix and define the function $\langle x, y \rangle_M := y^{\mathsf{H}} M x$. Then $\langle \cdot, \cdot \rangle_M$ is also an inner product.

If $\langle \cdot, \cdot \rangle$ is an inner product on a real or complex vector space V, then the function $\|\cdot\| : V \to [0, \infty)$ defined by $\|x\| := \langle x, x \rangle^{1/2}$ is a norm on V. Such a norm is said to be *derived from an inner product*. The Euclidean norm $\|\cdot\|_2$ is a norm derived from the Euclidean inner product. An inner product space is therefore also a normed linear space with its derived norm. Not all norms are derived from an inner product, e.g., $\|\cdot\|_1$, $\|\cdot\|_\infty$ are not derived norms.

Inner products are defined for infinite dimensional vector spaces as well. For example an inner product on the vector space $C[a,b]$ of all continuous real or complex-valued functions on the real interval $[a,b]$ is

$$\langle f, g \rangle \; := \; \int_a^b f(t)\overline{g(t)}dt, \qquad f, g \in C[a,b]$$

The L_2 norm $\|f\|_2 := \sqrt{\int_a^b |f(t)|^2 dt}$ defined above is derived from the inner product $\langle f, f \rangle$.

25.1.7.2 Cauchy-Schwartz inequality, Hölder's inequality, dual norm

We now present an extremely useful inequality, the Cauchy-Schwarz inequality, and two generalizations.

Cauchy-Schwartz and Hölder's inequalities. The Cauchy-Schwarz inequality is an important property of all inner products on any finite or infinite dimensional vector space. The inequality holds regardless of whether the norm on the vector space is derived from the inner product. Hence $\langle x,x\rangle$, $\langle y,y\rangle$ on the right-hand side of (25.19) may not be the squared norms on V.

Theorem 25.19 (Cauchy-Schwarz inequality)**.** Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space over a field F with $F=\mathbb{R}$ or $\mathbb{C}$. Then

$$|\langle x,y\rangle|^2 \;\le\; \langle x,x\rangle\,\langle y,y\rangle, \qquad\qquad x,y\in V \tag{25.19}$$

with equality if and only if $x=ay$ for some $a\in F$ (i.e., x and y are linearly dependent).

Proof. To prove the Cauchy-Schwarz inequality suppose without loss of generality $y\neq 0$ (the inequality holds if $x=y=0$). Let $z:=\langle y,y\rangle x-\langle x,y\rangle y$. Then, since $\langle a_1u_1+a_2u_2, b_1v_1+b_2v_2\rangle = a_1\overline{b}_1\langle u_1,v_1\rangle + a_1\overline{b}_2\langle u_1,v_2\rangle + a_2\overline{b}_1\langle u_2,v_1\rangle + a_2\overline{b}_2\langle u_2,v_2\rangle$,

$$\begin{aligned} 0 \;\le\; \langle z,z\rangle \;&=\; \langle\langle y,y\rangle x-\langle x,y\rangle y,\; \langle y,y\rangle x-\langle x,y\rangle y\rangle \\ &=\; \langle y,y\rangle^2\langle x,x\rangle \;-\; \langle x,y\rangle\overline{\langle y,y\rangle}\langle y,x\rangle \;=\; \langle y,y\rangle\left(\langle x,x\rangle\langle y,y\rangle \;-\; |\langle x,y\rangle|^2\right) \end{aligned}$$

which implies the inequality since $\langle y,y\rangle>0$. □

Cauchy-Schwarz inequality has numerous applications. One example is the following bounds on samples in terms of their sample mean and standard deviation. Let $x_1,\dots,x_n$ be n given real numbers with sample mean μ and sample standard deviation σ defined by:

$$\mu \;:=\; \frac{1}{n}\sum_i x_i, \qquad\qquad \sigma \;:=\; \left(\frac{1}{n}\sum_i (x_i-\mu)^2\right)^{1/2}$$

It can then be shown that (Exercise 25.18)

$$\mu \,-\, \sigma\sqrt{n-1} \;\le\; x_i \;\le\; \mu+\sigma\sqrt{n-1}, \qquad\qquad i=1,\dots,n$$

with equality for some i if and only if $x_p=x_q$ for all $p,q\neq i$.

Hölder's inequalities. A generalization of the Cauchy-Schwarz inequality is Hölder's inequality. Hölder's inequality holds for general L^p spaces (the vector space of measurable functions f for which its L_p norm is finite), but we will restrict ourselves to $V=\mathbb{R}^n$ or $\mathbb{C}^n$ with l_p norms.

Theorem 25.20 (Hölder's inequality)**.** Consider the vector space $V = F^n$ with $F = \mathbb{R}$ or $\mathbb{C}$ with l_p norms, $p \in [1, \infty]$. Then for any $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (with the interpretation that if $p = 1$ then $q = \infty$)

$$\sum_{i=1}^{n} |x_i y_i| \ \leq \ \|x\|_p \, \|y\|_q \,, \qquad\qquad x,\ y \in V \tag{25.20}$$

with equality if and only if $x^p := (x_i^p, i = 1, \ldots, n)$ and $y^q := (y_i^q, i = 1, \ldots, n)$ are linearly dependent, i.e., $x^p = a y^q$ for some scalar $a \in F$.

The theorem can be proved by applying the following property to the convex function $f(x) = x^p$ for $p > 1$: for all $\alpha_i \geq 0$, $\sum_{i=1}^n \alpha_i = 1$, for all x_i,

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \ \leq \ \sum_{i=1}^{n} \alpha_i f(x_i)$$

Setting $p = q = 2$ leads to the Cauchy-Schwarz inequality

$$\left|x^{\mathsf{H}} y\right| \ \leq \ \sum_{i=1}^{n} |x_i y_i| \ \leq \ \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2} \ = \ \|x\|_2 \, \|y\|_2, \qquad x,\ y \in V$$

with equality if and only if the vectors x and y are linearly dependent ($x^p = a y^q \Leftrightarrow x = a^{1/p} y^{q/p}$). Note that this inequality is weaker than Hölder's inequality, though the Cauchy-Schwarz inequality holds for general inner products on arbitrary vector spaces with arbitrary norms.

Dual norm. Another generalization of the Cauchy-Schwarz inequality holds with dual norm, as we define now. Consider any norm $\|\cdot\|$ on the vector space $V = F^n$ with $F = \mathbb{R}$ or $\mathbb{C}$. Define its *dual norm* $\|\cdot\|_*$ by: for any $x \in F^n$

$$\|x\|_* \ := \ \max_{y: \|y\|=1} \mathrm{Re}\, x^{\mathsf{H}} y \ = \ \max_{y: \|y\|=1} \left|x^{\mathsf{H}} y\right| \tag{25.21}$$

The maximization is attained since inner product is continuous and the feasible set is compact. (If we think of x^{H} as an $1 \times n$ matrix then $\|x\|_*$ is the matrix norm induced by the general vector norm $\|\cdot\|$ on $\mathbb{F}^n$; see below.)

A very useful inequality is

$$\mathrm{Re}\, x^{\mathsf{H}} y \ \leq \ \left|x^{\mathsf{H}} y\right| \ \leq \ \|x\| \, \|y\|_* \qquad \forall x, y \in \mathbb{F}^n \tag{25.22}$$

which follows directly from the definition of the dual norm. It says that the absolute inner product of any two vectors are upper bounded by the product of the norm of one of the vectors and its dual norm of the other vector. For the Euclidean norm $\|\cdot\|_2$ this is the Cauchy-Schwarz inequality, but (25.22) holds for *any* norm. Comparing this with Hölder's inequality (25.20), the left-hand side of (25.22) is smaller than that of (25.20), $\left|x^{\mathsf{H}} y\right| \leq \sum_i |x_i y_i|$. The norms on the right-hand side of (25.22) are not restricted to l_p norms as those in (25.20) are. Indeed we now use Hölder's inequality to show that l_p and l_q norms are the dual of each other if $1/p + 1/q = 1$, and hence $\|x\|\,\|y\|_*$ reduces to the norms in Hölder's inequality if $\|\cdot\|$ is an l_p norm.

To simplify exposition we allow p, q with $1/p + 1/q = 1$ to take values in $[1, \infty]$ with the interpretation that if $p = 1$ then $q := \infty$.

Lemma 25.21. Let $p, q \in [1, \infty]$ and $1/p + 1/q = 1$. The l_p norm and the l_q norm are dual of each other.

Proof. We prove the case of $1 < p < \infty$; the case of $p = 1$ or $p = \infty$ follows a similar idea. Fix a pair $1 < p, q < \infty$ with $1/p + 1/q = 1$. Hölder's inequality implies, for all $x \in F^n$,

$$\|x\|_q \;\; \geq \;\; \max_{y:\|y\|_p=1} \sum_i |x_i y_i| \;\; \geq \;\; \max_{y:\|y\|_p=1} \left| x^{\mathsf{H}} y \right| \;\; = \;\; \|x\|_*$$

Therefore $\|x\|_q \geq \|x\|_*$, the dual norm of $\|\cdot\|_p$. To prove the reverse inequality we have from (25.22)

$$\|x\|_* \;\; \geq \;\; (\|y\|_p)^{-1} \left| x^{\mathsf{H}} y \right| \;\; = \;\; \left(\sum_i |y_i|^p \right)^{-1/p} \left| \sum_i \bar{x}_i y_i \right|, \qquad \forall y \in \mathbb{F}^n$$

Choose

$$y_i \;\; := \;\; |x_i|^{q/p} \, \frac{x_i}{|x_i|}$$

so that the inequality becomes (using $q = 1 + \frac{q}{p}$)

$$\|x\|_* \;\; \geq \;\; \left(\sum_i |x_i|^q \right)^{-1/p} \sum_i |x_i|^{1+q/p} \;\; = \;\; \left(\sum_i |x_i|^q \right)^{\frac{1}{q}} \;\; = \;\; \|x\|_q$$

Hence $\|x\|_* = \|x\|_q$ when $\|\cdot\| = \|\cdot\|_p$. □

In light of Lemma 25.21, examples of (25.22) include:

$$\begin{aligned}
\left| x^{\mathsf{H}} y \right| \;\; &\leq \;\; \|x\|_p \, \|y\|_q \qquad \left(p^{-1} + q^{-1} = 1 \right) \\
\left| x^{\mathsf{H}} y \right| \;\; &\leq \;\; \|x\|_2 \, \|y\|_2 \qquad (p = q = 2, \text{ Cauchy-Schwarz inequality}) \\
\|x\|_2^2 \;\; &\leq \;\; \|x\|_1 \, \|x\|_\infty \qquad (y := x, p = 1, q = \infty)
\end{aligned}$$

A crucial fact for the vector space $V = \mathbb{R}^n$ or $\mathbb{C}^n$ is that the dual of a dual norm is the original norm, i.e., $\|\cdot\|_{**} = \|\cdot\|$ for an arbitrary norm $\|\cdot\|$ on V (see [10, Theorem 5.5.9, p.338]). For the special case of l_p norms, this is implied by Lemma 25.21. Moreover the only l_p norm that is its own dual is the Euclidean norm $\|\cdot\|_2$ ([10, Theorem 5.4.17, p.331]). This fact and a remarkable property of dual norm specialized to $\mathbb{R}^n$ are used in Chapter 25.1.8 to prove a mean value theorem for vector-valued functions (Lemma 25.29). Specifically, for the vector space $V = \mathbb{R}^n$, it is shown in Chapter 25.1.8 that, given any $x \in \mathbb{R}^n$, there is a normalized $y_*(x) \in \mathbb{R}^n$ with $\|y_*(x)\|_* = 1$ such that the norm $\|x\|$ is attained by their inner product, $\|x\| = x^{\mathsf{T}} y_*(x)$. Similarly, there exists an $y(x)$ with $\|y(x)\| = 1$ such that $\|x\|_* = x^{\mathsf{T}} y(x)$. This is remarkable because it says that any norm $\|\cdot\|$ and its dual norm are always attained by the Euclidean inner product even if $\|\cdot\|$ may not be a derived norm, e.g., $\|\cdot\|_1$, $\|\cdot\|_\infty$.

25.1.7.3 Matrix norms

This subsection mostly follows [10, Chapter 5.6]. The set $M_{m,n} := M_{m,n}(\mathbb{C})$ of all $m \times n$ complex matrices is a vector space whether we view an element $A \in M_{mn}$ as a vector in $V = \mathbb{C}^{mn}$ over field $F = \mathbb{C}$ or $\mathbb{R}$ or an array of numbers in $V = \mathbb{C}^{m\times n}$ over $F = \mathbb{C}$ or $\mathbb{R}$. A matrix norm on M_{mn} therefore follows the same definition as in Definition 25.4.

Definition 25.6 (Matrix norm)**.** A function $\|\cdot\| : M_{m,n} \to \mathbb{R}$ is a *matrix norm*, or simply a *norm*, if, for all complex matrices $A, B \in M_{m,n}$, $c \in \mathbb{C}$,

1. *Positivity*: $\|A\| \geq 0$ and $\|A\| = 0$ if and only if $A = 0$.
2. *Homogeneity*: $\|cA\| = |c|\,\|A\|$.
3. *Triangular inequality*: $\|A+B\| \leq \|A\| + \|B\|$.

□

A key difference between the vector spaces $\mathbb{C}^{mn}$ and $\mathbb{C}^{m\times n}$ is that matrix multiplication is defined for elements A, B of $\mathbb{C}^{m\times n}$. We would therefore like to estimate the 'size' of a matrix product AB in terms of the 'sizes' of A and B. This is done by matrix norms $\|\cdot\|$ that also satisfies a fourth property:

4. *Submultiplicativity*: $\|AB\| \leq \|A\|\,\|B\|$ when A and B have compatible sizes (e.g., $m = n$) and the norms are properly defined for AB, A and B.

Not all matrix norms are submultiplicative. Some authors include submultiplicativity in the definition of matrix norm when restricted to square matrices $(m = n)$, e.g., [10, Chapter 5.6]. In the following we first discuss a special class of matrix norms, called induced norms, that are not only submultiplicative, but also have a certain minimality property. Then we discuss vector norms that are l_p norms on the vector space C^{n^2}. They may or may not be submultiplicative. See Figure 25.6.

Figure 25.6: Matrix norms.

Induced norms. A widely used matrix norm $\|\cdot\|_{m,n}$ on $M_{m,n}(\mathbb{C})$ is an *induced norm*, induced by any vector norms $\|\cdot\|_n$ and $\|\cdot\|_m$ on C^n and C^m respectively, defined by: for $A \in M_{m,n}$,

$$\|A\|_{m,n} \;:=\; \max_{x:\|x\|_n=1} \|Ax\|_m \;=\; \max_{x:x\neq 0} \frac{\|Ax\|_m}{\|x\|_n} \tag{25.23}$$

It is sometimes called an *operator norm*. Every induced norm is submultiplicative: for $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$ with arbitrary norms $\|\cdot\|_m$, $\|\cdot\|_n$, $\|\cdot\|_k$ on $\mathbb{C}^m$, $\mathbb{C}^n$, $\mathbb{C}^k$ respectively,

$$\|AB\|_{m,k} \;=\; \max_{\substack{x:x\neq 0\\ \|Bx\|\neq 0}} \frac{\|ABx\|_m}{\|x\|_k} \;=\; \max_{\substack{x:x\neq 0\\ \|Bx\|\neq 0}} \frac{\|ABx\|_m}{\|Bx\|_n}\frac{\|Bx\|_n}{\|x\|_k} \;\leq\; \max_{y:y\neq 0}\frac{\|Ay\|_m}{\|y\|_n}\max_{x:x\neq 0}\frac{\|Bx\|_n}{\|x\|_k} \;=\; \|A\|_{m,n}\,\|B\|_{n,k}$$

It also satisfies the additional properties:

1. $\|I\|_{m,n} = 1$ for the identity matrix I.

2. $\|Ax\|_m \le \|A\|_{m,n}\|x\|_n$ for any $A \in \mathbb{C}^{m\times n}$ and any $x \in \mathbb{C}^n$ (follows from submultiplicativity).

3. $\|A\|_{m,n} = \max\{|y^{\mathsf{H}}Ax| : \|x\| = \|y\|_* = 1,\ x \in \mathbb{C}^n, y \in \mathbb{C}^m\}$.

Examples of induced norms on $M_{m,n}$ are norms induced by the l_p norm on both $\mathbb{C}^n$ and $\mathbb{C}^m$:

$$\|A\|_p \ := \ \max_{x:\|x\|_p=1} \|Ax\|_p \ = \ \max_{x:x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

Theorem 25.22. Let $A \in M_{m,n}$ a $m\times n$ complex matrix. Then the induced norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ satisfy:

1. *Max column sum* (induced by l_1 norm): $\|A\|_1 = \max_j \sum_i |A_{ij}|$.

2. *Max row sum* (induced by l_∞ norm):$\|A\|_\infty = \max_i \sum_j |A_{ij}|$.

3. *Spectral norm* (induced by l_2 norm): $\|A\|_2 = \sigma_{\max}(A) = \sqrt{\lambda_{\max}\left(A^{\mathsf{H}}A\right)}$ where $\sigma_{\max}(A)$ is the largest singular value of A and $\lambda_{\max}\left(A^{\mathsf{H}}A\right) \ge 0$ is the largest eigenvalue of the positive semidefinite matrix $A^{\mathsf{H}}A$.

4. If A is square and nonsingular then $\|A^{-1}\|_2 = 1/\sigma_{\min}(A)$, the reciprocal of the smallest singular value of A.

5. $\|A^{\mathsf{H}}A\|_2 \ = \ \|AA^{\mathsf{H}}\|_2 \ = \ \|A\|_2^2$.

6. $\|A\|_2 \ = \ \max\{|y^{\mathsf{H}}Ax| : \|x\|_2 = \|y\|_2 = 1,\ x \in \mathbb{C}^n, y \in \mathbb{C}^m\}$.

A norm $\|\cdot\|$ is *unitarily invariant* if $\|A\| = \|UAV\|$ for all $A \in M_n$ and for all unitary matrices $U,V \in M_n$. It is *self-adjoint* if $\|A\| = \|A^{\mathsf{H}}\|$ for all $A \in M_n$. The following result shows that the spectral norm is the only induced norm that is unitarily invariant and self-adjoint [10, Theorems 5.6.34, 5.6.35].

Lemma 25.23. Let $\|\cdot\|$ be a submultiplicative matrix norm on M_n. The following are equivalent:

1. $\|\cdot\|$ is the spectral norm.

2. $\|\cdot\|$ is an induced norm that is unitarily invariant, i.e., $\|A\| = \|UAV\|$ for all $A \in M_n$ and for all unitary matrices $U,V \in M_n$.

3. $\|\cdot\|$ is an induced norm that is self-adjoint, i.e., $\|A\| = \|A^{\mathsf{H}}\|$ for all $A \in M_n$.

Other matrix norms. We can also view a complex matrix $A \in M_{m,n}$ as a vector in $\mathbb{C}^{mn}$ and treat the l_p norms on $\mathbb{C}^{mn}$ as matrix norms on $M_{m,n}$. We sometimes refer these norms as *vector norms* on $M_{m,n}$. Examples include

- l_1 *norm*: $\|A\|_{\text{sum}} := \sum_{i,j} |A_{ij}|$.
- l_2 *or Frobenius norm*: $\|A\|_F := \left(\sum_{i,j} |A_{ij}|^2\right)^{1/2}$.
- l_∞ *norm*: $\|A\|_{\max} := \max_{i,j} |A_{ij}|$.

The *Frobenius inner product* on complex matrices in $M_{m,n}$ is defined to be

$$\langle A,B\rangle_F \;:=\; \operatorname{tr} B^{\mathsf{H}}A \;=\; \sum_{i=1}^{m}\sum_{j=1}^{n} \overline{B}_{ij}A_{ij}$$

It is simply the Euclidean inner product when we view a matrix $A \in M_{m,n}$ as a vector in $\mathbb{C}^{mn}$. The Forbenius norm is then derived from the Frobenius inner product, $\|A\|_F := \sqrt{\langle A,A\rangle_F}$.

They satisfy the following properties

Theorem 25.24. Let $A \in M_n$ be a $n \times n$ complex matrix.

1. $\|\cdot\|_{\text{sum}}$ and $\|\cdot\|_F$ are submultiplicative matrix norms, but $\|\cdot\|_{\max}$ is a matrix norm that is not submultiplicative.
2. The Frobenius norm is given by
$$\|A\|_F \;=\; \left|\operatorname{tr}\left(AA^{\mathsf{H}}\right)\right|^{1/2} \;=\; \sqrt{\sum_i \sigma_i^2(A)} \;=\; \sqrt{\sum_i \lambda_i\left(AA^{\mathsf{H}}\right)}$$
where $\sigma_i(A)$ denote the singular values of A and $\lambda_i(AA^{\mathsf{H}})$ denote the eigenvalues of the positive semidefinite matrix AA^{H}.
3. $\|A\|_F = \|A^{\mathsf{H}}\|_F = \|UAV\|_F$ for any unitary matrices $U,V \in M_n$ (unitarily invariant).

Hence while the spectral norm $\|\cdot\|_2$ is the only unitarily invariant and the only self-adjoint induced norm (Lemma 25.23), the Frobenius norm $\|\cdot\|_F$ is a unitarily invariant and self-adjoint norm that is not induced by a vector norm on $\mathbb{C}^n$.

Since M_n is a finite dimensional vector space over field $F = \mathbb{C}$ or $\mathbb{R}$, all matrix norms, whether or not they are submultiplicative, are equivalent in the sense of (25.18) and therefore have the same convergence sequences. In particular a matrix norm that is not submultiplicative is equivalent to every submultiplicative matrix norm, and vice versa. Moreover any vector norm on M_n becomes a submultiplicative matrix norm when scaled up sufficiently [10, Theorems 5.7.8, 5.7.11, pp. 372].

Lemma 25.25. 1. Given any matrix norm $N(\cdot)$ (e.g., a vector norm) on M_n and any submultiplicative matrix norm $\|\cdot\|$ on M_n, there exists finite positive constants c_m, c_M such that

$$c_m\|A\| \;\le\; N(A) \;\le\; c_M\|A\|, \qquad\qquad A \in M_n \tag{25.24}$$

2. Let $N(\cdot)$ be a vector norm on M_n and $c(N) := \max_{N(A)=1=N(B)} N(AB)$. Then $\gamma N(\cdot)$ is a submultiplicative matrix norm on M_n if and only if $\gamma \ge c(N)$

Spectral radius, matrix norm and convergence. Induced norms have a certain minimality property among matrix norms. This can be useful, e.g., in analyzing iterative algorithms of the form $x(t+1) = Ax(t)$. We now describe the relationship between the spectral radius $\rho(A)$ of a matrix A, its matrix $\|A\|$, and convergence properties of A^k and $\sum_{j\leq k} A^j$.

Theorem 25.26 (Spectral radius, singular values, norms)**.** Let $\|\cdot\|$ be a submultiplicative matrix norm on M_n and $A \in M_n$. Let λ_i and σ_i be the eigenvalues and singular values of A respectively with

$$|\lambda_1| \;\geq\; \cdots \;\geq\; |\lambda_n|, \qquad\qquad \sigma_1 \;\geq\; \cdots \;\geq\; \sigma_n$$

Let $\rho(A) := |\lambda_1|$ denote the spectral radius of A.

1. $|\lambda_1| \leq \sigma_1$ and $|\lambda_n| \geq \sigma_n > 0$, i.e., $|\lambda_i| \in [\sigma_n, \sigma_1]$.
2. For all i, $1/\|A^{-1}\| \leq |\lambda_i| \leq \rho(A) \leq \|A\|$ if A is nonsingular.
3. Given any $\varepsilon > 0$ there is a submultiplicative matrix norm $\|\cdot\|$ such that $\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon$. Moreover

$$\rho(A) \;=\; \inf\{\|A\| \,:\, \|\cdot\| \text{ is an induced norm}\}$$

In Theorem 25.26, 1 is proved in [10, Theorem 5.6.9], 2 follows from 1 by taking $\|\cdot\|$ to be the spectral norm, and 3 is proved in [10, Lemma 5.6.10, p.347]. See Exercise 25.22 for details.

As mentioned above M_n is a finite dimensional vector space over field $F = \mathbb{C}$ or $\mathbb{R}$, convergence of matrices is defined in the same way as the convergence of elements in any normed vector space $(V, \|\cdot\|)$, i.e., a sequence $\{x_k\} \subseteq V$ converges to a limit $x \in V$ if $\|x_k - x\| \to 0$ as $k \to \infty$.

Definition 25.7 (Matrix convergence)**.** We say a sequence $\{A^k\} \subseteq M_n$ (or a power series $\{\sum_{j\leq k} A^j\} \subseteq M_n$) *converges* if there exists a matrix $A \in M_n$ such that $A^k \to A$ (or $\sum_{j\leq k} A^j \to A$) as $k \to \infty$ with respect to the underlying matrix norm $\|\cdot\|$, i.e., if $\lim_{k\to\infty} \|A^k - A\| = 0$ (or $\lim_{k\to\infty} \|\sum_{j\leq k} A^j - A\| = 0$).

All matrix norms, whether or not they are submultiplicative, are norms on M_n and therefore equivalent in the sense of (25.18). Hence if A^k converges under a norm, it converges under all norms.

Theorem 25.27 (Sequence convergence)**.** Let $\|\cdot\|$ be a submultiplicative matrix norm on M_n and $A \in M_n$. Let $\rho(A)$ denote the spectral radius of A.

1. If $\|A\| < 1$ then $\lim_{k\to\infty} A^k = 0$, i.e., $\left|[A^k]_{ij}\right| \to 0$ as $k \to \infty$ for all i, j.
2. $\rho(A) < 1$ if and only if $\lim_{k\to\infty} A^k = 0$.
3. *Gelfand formula*: $\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$.

In Theorem 25.27, 1 is proved in [10, Lemma 5.6.11] and uses the fact that if A^k converges then it converges under the vector norm $\|A\|_{\max} := \max_{i,j} |A_{ij}|$, and 2 is proved in [10, Lemma 5.6.12] and says that, unlike $\|A\| < 1$, $\rho(A) < 1$ is both necessary and sufficient for the convergence of $\lim_{k\to\infty} A^k$. Theorem

25.27.3 holds not only for multiplicative matrix norms, but also for any matrix norm, including vector norms [10, Corollary 5.6.14, Theorem 5.7.10]. It follows from the fact that, under a submultiplicative matrix norm, $\tilde{A} := (\rho(A)+\varepsilon)^{-1}A$ has spectral radius strictly less than 1 and converges for any $\varepsilon > 0$, implying that $\|A^k\|^{1/k} \leq \rho(A)+\varepsilon$ for sufficiently large k. On the other hand $\rho(A) \leq \|A^k\|^{1/k}$ and hence $\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$. Extension to norms that are no submultiplicative makes use of (25.24).

Remark 25.3. We often want to establish $\|A\| < 1$ for some matrix norm in order to prove convergence of sequences or power series of A. We are therefore interested in a *minimal matrix norm* $\|\cdot\|$, i.e., a submultiplicative norm on M_n such that the only submultiplicative norm $N(\cdot)$ on M_n with $N(A) \leq \|A\|$ for all $A \in M_n$ is $N(\cdot) = \|\cdot\|$. It can be shown that a submultiplicative matrix norm on M_n is minimal if and only if it is an induced norm [10, Theorem 5.6.32, p.356]. □

The sum $S_k := \sum_{j=0}^{k} a_j$ of a finitely many complex numbers $a_j \in \mathbb{C}$ does not depend on the order in which a_j are summed. An infinite series $S := \lim_{k\to\infty} S_k = \sum_{j=0}^{\infty} a_j$ may, e.g., $S := 1-1+1-1+\cdots$ where the partial sums S_k oscillate between 1 and -1. This motivates a stronger notion of convergence. Specifically an infinite sum $\sum_{j=0}^{\infty} a_j$ of complex numbers $a_j \in \mathbb{C}$ is said to *converge absolutely* if $\lim_{k\to\infty} \sum_{j=0}^{k} |a_j| = a$ for some real number $a \in \mathbb{R}$.

Definition 25.8 (Series convergence)**.** Considered a norm vector space $(M_n, \|\cdot\|)$. We say a power series $\{\sum_{j\leq k} A^j\} \subseteq M_n$

1. *converges* if there exists a matrix $A \in M_n$ such that $\sum_{j\leq k} A^j \to A$ as $k \to \infty$, i.e., if $\lim_{k\to\infty} \|\sum_{j\leq k} A^j - A\| = 0$.

2. *converges absolutely* if there exists a matrix $A \in M_n$ such that $\sum_{j\leq k} A^j \to A)$ as $k \to \infty$ with respect to the underlying matrix norm $\|\cdot\|$, i.e., if $\lim_{k\to\infty} \|\sum_{j\leq k} A^j - A\| = 0$.

For a complex power series $S(z) := \lim_{k\to\infty} \sum_{j=0}^{k} a_j z^j$, it is known that there is a *radius of convergence* $R \geq 0$, possibly ∞, such that the power series converges absolutely for $|z| < R$, diverges if $|z| > R$, and may converge or diverge if $|z| = R$. For any complex $n \times n$ matrix $A \in M_n$ and any submultiplicative matrix norm $\|\cdot\|$ we have

$$\left\| \sum_k a_k A^k \right\| \;\leq\; \sum_k |a_k| \, \|A^k\| \;\leq\; \sum_k |a_k| \, \|A\|^k$$

where the first inequality is due to the triangular inequality and the second due to submultiplicativity. This means that a matrix power series $\sum_{k=0}^{\infty} a_k A^k$ converges absolutely if there exists a matrix norm $\|\cdot\|$ such that $\|A\| < R$, the radius of convergence for $\sum_k a_k z^k$, i.e., see Exercise 25.24. Such a norm exists if and only if $\rho(A) < R$ because, given any $\varepsilon > 0$, there exists a (submultiplicative) matrix norm $\|\cdot\|$ with $\rho(A) \leq \|A\| \leq \rho(A)+\varepsilon$ [10, Lemma 5.6.10, p.347]. This fact and some corollaries are summarized in the next result [10, pp.350-351].

Theorem 25.28 (Series convergence)**.** Let $A \in M_n$.

1. Let R be the radius of convergence of a scalar power series $\sum_{k=0}^{\infty} a_k z^k$. The matrix power series $\sum_{k=0}^{\infty} a_k A^k$ converges if $\rho(A) < R$, which holds if there exists a multiplicative matrix norm $\|\cdot\|$ on M_n such that $\|A\| < R$.

Let $\|\cdot\|$ be a submultiplicative matrix norm on M_n.

2. If $\|I-A\| < 1$ then A is nonsingular and

$$A^{-1} = \sum_{k=0}^{\infty}(I-A)^k$$

3. If $\|A\| < 1$ then $I-A$ is nonsingular and

$$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$$

4. If $\|I\| = 1$ (e.g., if $\|\cdot\|$ is an induced norm) and $\|A\| < 1$ then

$$\frac{1}{1+\|A\|} \leq \|(I-A)^{-1}\| \leq \frac{1}{1-\|A\|}$$

The theorem is proved in Exercise **??**.

25.1.8 Mean value theorems

When restricted to the vector space $\mathbb{R}^n$ endowed with any norm $\|\cdot\|$, the definition of dual norm $\|\cdot\|_*$ in (25.21) reduces to: for any $x \in \mathbb{R}^n$,

$$\|x\|_* := \max_{y:\|y\|=1} x^{\mathsf{T}} y = \max_{y:\|y\|=1} \left|x^{\mathsf{T}} y\right| \tag{25.25}$$

The maximization is attained since inner product is continuous and the feasible set is compact. Hence there is a normalized $y(x) \in \mathbb{R}^n$ that satisfies

$$x^{\mathsf{T}} y(x) = \|x\|_* \quad \text{and} \quad \|y(x)\| = 1 \tag{25.26a}$$

Recall a crucial fact that, for the vector space $V = \mathbb{R}^n$ or $\mathbb{C}^n$, the dual of a dual norm is the original norm, i.e., $\|\cdot\|_{**} = \|\cdot\|$ for an arbitrary norm $\|\cdot\|$ on V (see [10, Theorem 5.5.9, p.338]). Therefore, given any $x \in \mathbb{R}^n$, there exists an $y_*(x) \in \mathbb{R}^n$ such that

$$x^T y_*(x) = \|x\| \quad \text{and} \quad \|y_*(x)\|_* = 1 \tag{25.26b}$$

because

$$\|x\| = \|x\|_{**} = \max_{y:\|y\|_*=1} x^T y = x^T y_*(x)$$

where $y_*(x)$ is a maximizer (which clearly exists).[3] Remarkably, for $\mathbb{R}^n$, (25.26) says that both the norm and its dual norm of any vector can be attained by the inner product of the vector with another vector, for any norm that may not be derived from an inner product, e.g., $\|\cdot\|_1$, $\|\cdot\|_\infty$.

[3]For the p-norm the dual is the q-norm with $p^{-1}+q^{-1}=1$ (see Lemma 25.21) and

$$(y(x))_i := \frac{x_i^{p-1}}{\|x\|_p^{p-1}} \operatorname{sign}((x_i)^p)$$

so that $x^T y(x) = \|x\|_p$ and $\|y(x)\|_q = 1$.

We now use (25.22) and (25.26b) to prove the mean value theorem for vector-valued functions.

Lemma 25.29. Consider any differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$. Given any x, y, w in $\mathbb{R}^n$ we have

$$
\begin{aligned}
w^T (f(y) - f(x)) &= w^T \frac{\partial f}{\partial x}(z)\,(y-x) && (25.27a)\\
\|f(y) - f(x)\| &\le \left\| \frac{\partial f}{\partial x}(z) \right\| \|y-x\| && (25.27b)
\end{aligned}
$$

where $z := \alpha x + (1-\alpha)y$ for some $\alpha \in [0,1]$, $\|\cdot\|$ is any norm, and for matrix, it denotes the induced norm.

Proof of Lemma 25.29. Fix any x, y, w in $\mathbb{R}^n$. Let $z(\alpha) := (1-\alpha)x + \alpha y$ for $\alpha \in [0,1]$ so that $z(0) = x$ and $z(1) = y$, and $z(\alpha)$ traces the straight path from x to y. Define the function

$$g(\alpha) \; := \; g_w(\alpha) \;\; := \;\; w^T f(z(\alpha))$$

as a function of $\alpha \in [0,1]$. Since g is from $\mathbb{R}$ to $\mathbb{R}$ the standard mean value theorem implies that

$$g(1) - g(0) \;\; = \;\; g'(\beta)$$

for some $\beta \in [0,1]$ that depends on w. Since $g(0) = w^T f(x)$ and $g(1) = w^T f(y)$ this becomes (using chain rule)

$$w^T (f(y) - f(x)) \;\; = \;\; w^T \frac{\partial f}{\partial x}(z(\beta))\,(y-x)$$

proving (25.27a).

To prove (25.27b), use (25.26b) to choose $w \in \mathbb{R}^n$ such that[4]

$$w^T (f(y) - f(x)) \; = \; \|f(y) - f(x)\| \quad \text{and} \quad \|w\|_* = 1$$

Substituting this w into (25.27a) yields

$$
\begin{aligned}
\|f(y) - f(x)\| &= w^T (f(y) - f(x)) \; = \; w^T \frac{\partial f}{\partial x}(z(\beta))\,(y-x)\\
&\le \|w\|_* \cdot \left\| \frac{\partial f}{\partial x}(z(\beta))(x-y) \right\|\\
&\le \left\| \frac{\partial f}{\partial x}(z(\beta)) \right\| \cdot \|x-y\|
\end{aligned}
$$

proving (25.27b). In the above, the first inequality follows from (25.22) and the second inequality follows from the definition of the induced norm of $\frac{\partial f}{\partial x}$. This completes the proof of Lemma 25.29. □

---

[4] If the norm $\|\cdot\|$ is Euclidean then the argument below simplifies to: setting $w := f(y) - f(x)$ in (25.27a) yields

$$
\begin{aligned}
\|f(y) - f(x)\|_2^2 &= (f(y) - f(x))^T \frac{\partial f}{\partial x}(z(\beta))\,(y-x)\\
&\le \|f(y) - f(x)\|_2 \cdot \left\| \frac{\partial f}{\partial x}(z(\beta)) \right\|_2 \|y-x\|_2
\end{aligned}
$$

proving (25.27b). This is done in [259].

25.2 Algebraic graph theory

Consider a directed graph $G = (\overline{N}, E)$ with $|\overline{N}| = N+1$ and $|E| = M$ with an arbitrary orientation. Let $\overline{C}$ denote the $(N+1)\times M$ incidence matrix defined by:

$$\overline{C}_{jl} \;=\; \begin{cases} 1 & \text{if } l = j \to k \text{ for some bus } k \\ -1 & \text{if } l = i \to j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

Let the $N\times M$ matrix C denote the reduced incidence matrix of G obtained from $\overline{C}$ by removing its first row. If G has c connected components, then rank $\overline{C} = N+1-c$. In particular if G is connected then rank $\overline{C} = N$. Indeed $\overline{C}$ can be written as a block diagonal matrix with the kth diagonal block C_k being the incident matrix of the kth connected component that has n_k vertices. It can be proved that rank $C_k = n_k - 1$.

We take $\mathbb{R}^{N+1}$ as the *node space* of G and it has a simple structure. The null space $\text{null}\left(\overline{C}^T\right)$ consists of all $\overline{\theta} \in \mathbb{R}^{N+1}$ such that $\overline{C}^T\overline{\theta} = 0$. This implies that $\overline{\theta}_i = \overline{\theta}_j$ if $(i,j)\in E$ is a link, i.e., a vector $\overline{\theta}$ is in $\text{null}(\overline{C})$ if and only if $\overline{\theta}_i$ takes the same value at every node in the same connected component. In particular, if G is connected, then $\text{null}\left(\overline{C}^T\right)$ is $\text{span}(\mathbf{1})$ and therefore its orthogonal complement $\text{range}(\overline{C})$ has dimension N and consists of all vectors $p \in \mathbb{R}^{N+1}$ such that $\mathbf{1}^T p = 0$. See Figure 25.7.

We take $\mathbb{R}^M$ as the *edge space* of G.[5] Since rank $\overline{C}^T = \text{rank}\,\overline{C} = N$ for a connected G, $\text{null}(\overline{C}) = M - N$; see Figure 25.7. A *cycle* in G is a set of edges in E that forms a cycle subgraph. Given a cycle σ in G, pick an orientation for σ, say, clockwise. Define the indicator function (vector) $z(\sigma)$ as

$$z_l(\sigma) \;=\; \begin{cases} +1 & \text{if edge } l \text{ is in } \sigma \text{ and has the same orientation as } \sigma \\ -1 & \text{if edge } l \text{ is in } \sigma \text{ and has the opposite orientation as } \sigma \\ 0 & \text{otherwise} \end{cases}$$

Partition $\overline{N}$ into two nonempty disjoint subsets N_1 and N_2. A *cut* in G is a set of edges in E each of which has one endpoint in N_1 and the other endpoint in N_2. Given a cut κ in G, pick an orientation, say, from N_1 to N_2. Define the indicator function (vector) $z(\kappa)$ as

$$z_l(\kappa) \;=\; \begin{cases} +1 & \text{if edge } l \text{ is in } \kappa \text{ and has the same orientation as } \kappa \\ -1 & \text{if edge } l \text{ is in } \kappa \text{ and has the opposite orientation as } \kappa \\ 0 & \text{otherwise} \end{cases}$$

Both the vectors $z(\sigma)$ and $z(\kappa)$ are in $\{0,1,-1\}^M$. Given a partition of $\overline{N}$ into N_1 and N_2, the indicator function $z(\kappa)$ of the cut can be expressed as

$$z(\kappa) \;:=\; \pm\frac{1}{2}\left(\sum_{i\in N_1} c_i - \sum_{i\in N_2} c_i\right)$$

[5] All results in this section extend to the case where the edge space is $\mathbb{C}^M$ instead.

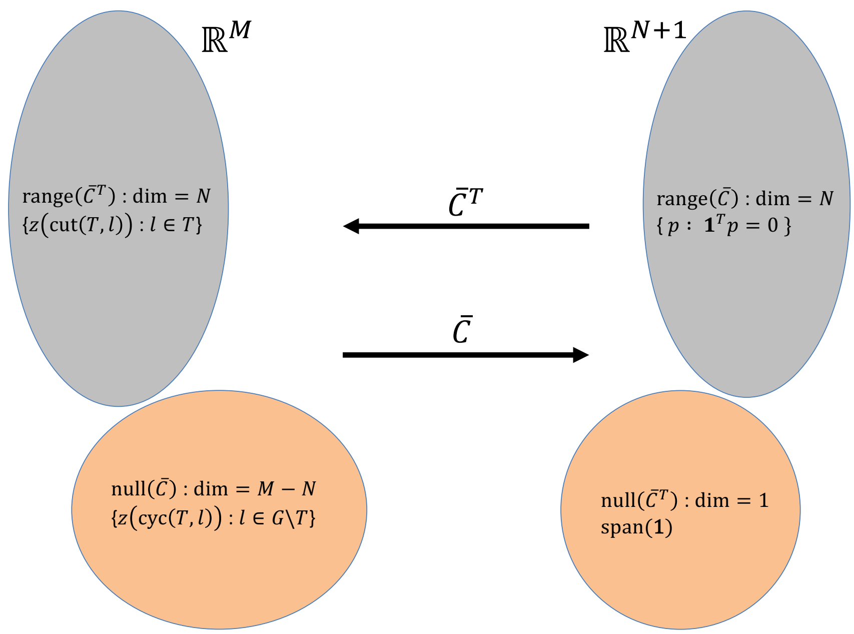


Figure 25.7: The edge space of a connected graph G is $\mathbb{R}^M = \text{null}(\overline{C}) \oplus \text{range}\left(\overline{C}^T\right)$. The cycle subspace $\text{null}(\overline{C})$ has dimension $M - N$ with a basis $\{z(\text{cyc}(T,l)) : l \in G \backslash T\}$, and the cut subspace range $\left(\overline{C}^T\right)$ has dimension N with a basis $\{z(\text{cut}(T,l)) : l \in T\}$. The vertex space is $\mathbb{R}^{N+1} = \text{null}\left(\overline{C}^T\right) \oplus \text{range}(\overline{C})$ with dimension N and 1.

where c_i are the ith rows of $\overline{C}$. This means that $z(\kappa)$ is in the range space of $\overline{C}^T$, and hence is orthogonal to the kernal of $\overline{C}$, i.e., if $\overline{C}\tilde{z} = 0$ then $z^T(\kappa)\tilde{z} = 0$. Call the null space of $\overline{C}$ the *cycle subspace* of G and its orthogonal complement the *cut subspace* of G; see Figure 25.7.

Fix any spanning tree T of G. For each edge l of G not in T, there is a unique cycle consisting of l and only edges in T; denote this cycle by $\mathrm{cyc}(T,l)$. For each edge l of T, there is a unique cut consisting of l and only edges not in T; denotes this cut by $\mathrm{cut}(T,l)$. Give $\mathrm{cyc}(T,l)$ and $\mathrm{cut}(T,l)$ the orientations that coincide with the orientation of l in G. These definitions are illustrated in Figure 25.8. The following

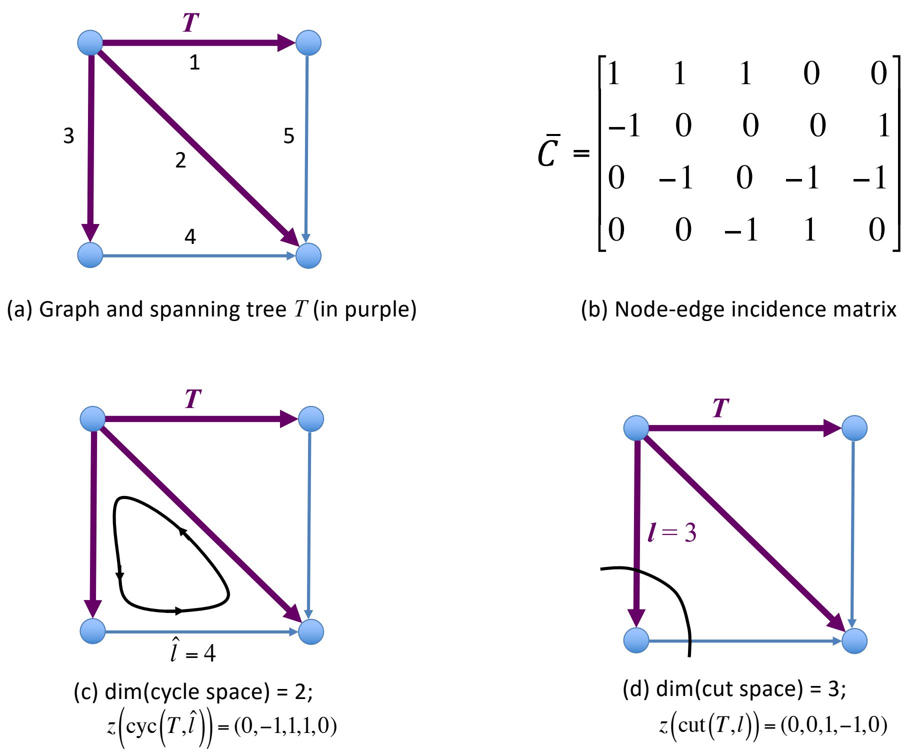


Figure 25.8: (a) A connected graph with a spanning tree T. (b) 4×5 incidence matrix $\overline{C}$. (c) The cycle subspace $\mathrm{null}(\overline{C})$ with dimension 2, one for each edge not in the spanning tree T, and an example basis vector $z(\mathrm{cyc}(T,l)) \in \mathrm{null}(\overline{C})$. (d) The cut subspace $\mathrm{range}(\overline{C}^T)$ with dimension 3, one for each edge in T, and an example basis vector $z(\mathrm{cut}(T,l)) \in \mathrm{range}(\overline{C}^T)$. The examples $z(\mathrm{cyc}(T,l))$ and $z(\mathrm{cut}(T,l))$ are indeed orthogonal.

properties of the edge space of G are illustrated in Figure 25.7.

Theorem 25.30 (Edge space $\mathbb{R}^M$ of G)**.**

1. The cycle subspace $\mathrm{null}(\overline{C})$ is a vector space of dimension $M-N$; $z(\sigma) \in \mathrm{null}(\overline{C})$ for any cycle σ.
2. Given a spanning tree T, the set $\{z(\mathrm{cyc}(T,l)) : l \in G\setminus T\}$ forms a basis that spans $\mathrm{null}(\overline{C})$.
3. The cut subspace $\mathrm{range}\left(\overline{C}^T\right)$ is a vector space of dimension N; $z(\kappa) \in \mathrm{range}\left(\overline{C}^T\right)$ for any cut κ.
4. Given a spanning tree T, the set $\{z(\mathrm{cut}(T,l)) : l \in T\}$ forms a basis that spans $\mathrm{range}\left(\overline{C}^T\right)$.

5. The edge space of G is the orthogonal direct sum of its cycle subspace and cut subspace, i.e., $\mathbb{R}^M = \text{null}(\overline{C}) \oplus \text{range}\left(\overline{C}^T\right)$ and $z_\sigma^T z_\kappa = 0$ for any $z_\sigma \in \text{null}(\overline{C})$ and $z_\kappa \in \text{range}\left(\overline{C}^T\right)$.

Theorem 25.31. 1. (Poincaré 1901) Any square submatrix of the incidence matrix $\overline{C}$ of a graph G has determinant equal to $0, +1$, or -1.

2. Let $F \subseteq E$ with $|F| = N$. Let C_F be an $N \times N$ submatrix of $\overline{C}$, consisting of the intersection of those N columns of $\overline{C}$ corresponding to the N edges in F and any N rows of $\overline{C}$. Then C_F is invertible if and only if the subgraph induced by F is a spanning tree of G.

3. (Inverse of C_T) Let T be a spanning tree of G. Let C_T denote the corresponding $N \times N$ submatrix. Then $\left[C_T^{-1}\right]_{li} = \pm 1$ if edge l is in the unique path in T joining node i and the reference node 0 corresponding to the row excluded from C_T. Otherwise $\left[C_T^{-1}\right]_{li} = 0$.

A basis for the cycle subspace $\text{null}(\overline{C})$ and that of the cut subspaces $\text{range}\left(\overline{C}^T\right)$ can be explicitly determined in terms of the incidence matrix $\overline{C}$, as follows. Partition $\overline{C}$ such that columns $1, \dots, N$ are the edges of a spanning tree T of G. Partition $\overline{C}$ as (node 0 is the reference bus):

$$\overline{C} \quad = \quad \begin{bmatrix} C_T & C_{-T} \\ d_{0T} & d_{-0T} \end{bmatrix} \tag{25.28a}$$

By Theorem 25.31, C_T is invertible and its N rows form a basis since T is a spanning tree of G. Let Z_σ denote the $M \times (M-N)$ matrix whose columns are the basis $\{z(\text{cyc}(T,l)) \mid l \in G \backslash T\}$ of the cycle subspace $\text{null}(\overline{C})$, written as (possibly after rearranging the columns):

$$Z_\sigma \quad = \quad \begin{bmatrix} Z_T \\ I_{M-N} \end{bmatrix} \tag{25.28b}$$

The lower submatrix of Z_σ is I_{M-N} because these rows correspond to edges not in the spanning tree T and the orientations of the cycles have been chosen so that they coincide with the orientation of these edges. By the definition of Z_σ we have the important topological relation $\overline{C} Z_\sigma = 0$. Using (25.28) we have

$$Z_T \quad = \quad -C_T^{-1} C_{-T}$$

From Theorem 25.31.3, each column of Z_T corresponds to a directed edge $i \to j$ not in the spanning tree T, and its nonzero entries correspond to edges on the unique path between node i and node j in T. Hence a basis for the cycle subspace is given by the columns of

$$Z_\sigma \quad = \quad \begin{bmatrix} -C_T^{-1} C_{-T} \\ I_{M-N} \end{bmatrix} \tag{25.29a}$$

Note that Theorem 25.31 implies that C_T^{-1} has integral entries, so Z also has integral entries. Similarly, we can explicitly determine the cut matrix. Let Z_κ denote the $M \times N$ matrix whose columns are the basis $\{z(\text{cut}(T,l)) \mid l \in T\}$ of the cut subspace $\text{range}\left(\overline{C}^T\right)$, written as (possibly after rearranging the columns):

$$Z_\kappa \quad = \quad \begin{bmatrix} I_N \\ Z_{-T} \end{bmatrix}$$

Since every column of K_κ belongs to the orthogonal complement of null$(\overline{C})$, we have $Z_\sigma^T K_\kappa = 0$. Hence

$$Z_{-T} \;=\; C_{-T}^T C_T^{-T}$$

where $M^{-T} := (M^{-1})^T = (M^T)^{-1}$ for any invertible matrix M and the basis for the cut space is

$$Z_\kappa \;=\; \begin{bmatrix} I_N \\ C_{-T}^T C_T^{-T} \end{bmatrix} \tag{25.29b}$$

Since $Z_\sigma = -C_T^{-1} C_{-T}$ and $Z_\kappa = C_{-T}^T C_T^{-T}$ we have $Z_\sigma^T + Z_\kappa = 0_{(M-N)\times N}$. This implies for $l \in T$ and $\hat{l} \in G \backslash T$ that

$$l \in \mathrm{cyc}(T, \hat{l}) \quad \Leftrightarrow \quad \hat{l} \in \mathrm{cut}(T, l)$$

Example 25.5. For the graph in Figure 25.8 we have

$$Z_\sigma \;=\; \left[\begin{array}{cc} 0 & 1 \\ -1 & -1 \\ 1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \quad \text{and} \quad Z_\kappa \;=\; \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & -1 \\ -1 & 1 & 0 \end{array}\right]$$

One can verify that, indeed, $Z_\sigma^T + Z_\kappa = 0$.

This structure can be used to understand loop flows in the DC power flow model. We call a line flow vector P a *loop flow* if it satisfies power balance with zero injections, i.e., $\overline{C}P = 0$. Hence P_σ is a loop flow if and only if it is in the cycle subspace null$(\overline{C})$ of G, i.e., $P_\sigma = Z_\sigma \alpha$ for some vector $\alpha \in \mathbb{R}^{M-N}$. Given any balanced injection vector p with $\sum_j p_j = 0$, the line flows P that satisfy $p = \overline{C}P$ are not unique. If P satisfies $p = \overline{C}P$, so does $P + P_\sigma$ for any loop flow P_σ. See Remark **??**.

A matrix is called *totally unimodular* if any square submtrix has determinant equal to $0, +1$, or -1. Hence Theorem 25.31.1 implies that the incidence matrix D of any *directed* graph G is totally unimodular.

Theorem 25.32. Given any (directed) graph G,

1. D is totally unimodular.
2. If A is a totally unimodular matrix and b is an integral vector, then, for any c, the solution of the linear program

$$\min_x \; c^T x \qquad \text{subject to } Ax \le b$$

   has an optimal solution which is integral, provided a finite solution exists.

The significance of the theorem is that many optimization problem on graphs have LP formulations where A is the incidence matrix or its variant, e.g. max flow, shortest path problems.

Reduced incidence matrix of tree. Consider the $(n+1) \times m$ incidence matrix C of a (connected) tree graph defined by

$$C_{jl} = \begin{cases} 1 & \text{if } l = j \to k \text{ for some bus } k \\ -1 & \text{if } l = i \to j \text{ for some bus } i \\ 0 & \text{otherwise} \end{cases}$$

Denote by c_0^T the first row of C corresponding to node 0 and by $\hat{C}$ the $n \times n$ submatrix consisting of the remaining rows of C so that

$$C =: \begin{bmatrix} - \; c_0^T \; - \\ \hat{C} \end{bmatrix}$$

Then the $\hat{C}$ is full rank and its inverse $\hat{C}^{-1}$ is given by

$$\left[\hat{C}^{-1}\right]_{lj} = \begin{cases} -1 & l \in \mathsf{P}_j \\ 1 & -l \in \mathsf{P}_j \\ 0 & \text{otherwise} \end{cases}$$

Here "$l \in \mathsf{P}_j$" means a directed line l that points away from bus 0 and is in the unique path P_j from bus 0 to bus j, and "$-l \in \mathsf{P}_j$" means a directed line l in P_j that points towards bus 0. Moreover $\hat{C}^{-T} c_0 = -\mathbf{1}$. See Exercise 5.9.

25.3 Nonsmooth convex optimization

Materials in this section follow [76, Chapter 5], [77].

25.3.1 Polyhedral sets and extreme points

We follow [76, Chapter 2] and define a *polyhedral set* $X \subseteq \mathbb{R}^n$ as a nonempty set specified by a finite number of affine inequalities:

$$X := \{x \in \mathbb{R}^n : Ax \le b\}$$

for a given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Hence a polyhedral set is nonempty closed and convex. An important characterization of a polyhedral set is the following result e.g. [76, Proposition 2.3.3, p.106].

Theorem 25.33 (Minkowski-Weyl representation)**.** A set $X \subseteq \mathbb{R}^n$ is polyhedral if and only if there is a finite set $\{v_1, \dots, v_m\}$ and a finitely generated cone $K := \text{cone}(a_1, \dots, a_k)$ such that

$$X = \text{conv}(v_1, \dots, v_m) + \text{cone}(a_1, \dots, a_k)$$

i.e.

$$X = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{m} \alpha_i v_i + y, \; \alpha_i \ge 0, \sum_i \alpha_i = 1, \; y \in K \right\}$$

□

Given a nonempty convex set $X \subseteq \mathbb{R}^n$ a vector $x \in X$ is an *extreme point* if there does not exist $y \neq x$, $z \neq x$, and $\alpha \in (0,1)$ such that $x = \alpha z + (1-\alpha)y$, or equivalently, if x is not a convex combination of other vectors in X that are distinct from x. Several facts are useful. An interior point cannot be an extreme point and an open set has no extreme points. A cone may have at most one extreme point, the origin. A polyhedral set has at most finitely many extreme points, and the minimum of a linear program is attained at an extreme point of its polyhedral feasible set. A polyhedral set may not possess any extreme points e.g. $X = \{(x_1,x_2) : x_1 = x_2\}$. The following result from [76, Propositions 2.1.5 and 2.1.3, p.98] provides an exact characterization of the existence of extreme points for polyhedral sets.

Lemma 25.34.

1. Let $X := \{x \in \mathbb{R}^n : Ax \leq b\}$ for some $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ be a polyhedral set. Then X has an extreme point if and only if A has n linearly independent rows, i.e., rank $A = n$.
2. Let $X \subseteq \mathbb{R}^n$ be a closed convex set. If for some $A \in \mathbb{R}^{m\times n}$ of rank n and $b \in \mathbb{R}^m$ we have $Ax \leq b$ for all $x \in X$. Then X has at least one extreme point.

□

A convex set that is compact is the convex hull of its extreme points; see e.g. [77, Theorem 2.3.4, p.111]. Carathéodory theorem then implies that every vector is a convex combination of at most $n+1$ extreme points. These constituent extreme points, however, may be different for different vectors.

Lemma 25.35. Let $X \subseteq \mathbb{R}^n$ be convex and compact. Then

1. $X = \text{conv}\{\text{extreme points of } X\}$.
2. If $x \in X$ then $x = \sum_{i=1}^{n+1} \alpha_i v_i$ where $\alpha_i \in [0,1]$ and $\sum_i \alpha_i = 1$, and v_i are extreme points of X.

□

25.3.2 Semidefinite programs

Semidefinite program (SDP) $K := \mathbb{S}^m_+$. The set $\mathbb{S}^m$ of $m \times m$ Hermitian matrices is a linear space over the field $\mathbb{R}$. It is partitioned into the set $\mathbb{S}^m_+$ of positive semidefinite matrices, the set $\mathbb{S}^m_-$ of negative semidefinite with $\mathbb{S}^m_+ \cap \mathbb{S}^m_-$ being the zero matrix, and the set of non-definite Hermitian matrices (those with both positive and negative eigenvalues). Both $\mathbb{S}^m_+$ and $\mathbb{S}^m_-$ are self-dual proper cones. They are also polar cones of each other.

Let $K = \mathbb{S}^m_+$ be the cone of psd matrices. The affine mapping $f : \mathbb{R}^n \to \mathbb{S}^m$ in (**??**) is of the form $f(x) = -(F_0 + x_1F_1 + \cdots + x_nF_n)$ where F_i are $m \times m$ Hermitian matrices. The conic program (**??**) then becomes (cf. linear program above):

$$\min_{x\in\mathbb{R}^n} \; c^T x \quad \text{s.t.} \quad a_0 + Ax = 0, \;\; F_0 + x_1F_1 + \cdots + x_nF_n \preceq 0 \tag{25.30a}$$

where $a_0 \in \mathbb{R}^k$ and $A \in \mathbb{R}^{k\times n}$. Let $\mu \in \mathbb{R}^k$ and $Z \in K^* = K$ denote dual variables. The Lagrangian is

$$L(x,\mu,Z) \;:=\; c^T x \,+\, \mu^T(a_0 + Ax) \,+\, Z \cdot \left(F_0 + \sum_{i=1}^n x_iF_i\right) \tag{25.30b}$$

The dual function is

$$q(\mu,Z) \;:=\; \min_{x\in\mathbb{R}^n} L(x,\mu,Z) \;=\; \begin{cases} \mu^T a_0 \,+\, Z\cdot F_0 & \text{if } \; c_i + \sum_j A_{ji}\mu_j + Z\cdot F_i = 0, \;\; i=1,\dots,n \\ -\infty & \text{otherwise} \end{cases}$$

and hence the dual problem is

$$\max_{\mu\in\mathbb{R}^k, Z\in K} \; a_0^T\mu \,+\, \mathrm{tr}(F_0Z) \quad \text{s.t.} \quad c_i + \sum_j A_{ji}\mu_j + \mathrm{tr}(ZF_i) = 0, \;\; i=1,\dots,n \tag{25.30c}$$

The primal problem (25.30a) is *strictly feasible* if there exists $\bar{x}$ such that $f(x)\in\mathrm{int}(K)$ and $a_0 + A\bar{x} = 0$. Similarly the dual problem (25.30c) is *strictly feasible* if there exist $\bar{Z}\in\mathrm{int}(K)$ and an unconstrained vector $\bar{\mu}$ such that $c_i + F_i\cdot\bar{Z} + \sum_j A_{ji}\bar{\mu}_j = 0, \;\; i=1,\dots,n$. This is the Slater condition for semidefinite programs.

Duality and optimality conditions for linear programs extend directly to semidefinite programs.

Theorem 25.36 (Saddlepoint and KKT theorem)**.** Let p^* be the optimal value of the primal problem (25.30a) and q^* be that of the dual problem (25.30c). Then $q^*\le p^*$. Suppose one of the problems (25.30a) and (25.30c) is strictly feasible and bounded, then the optimal value of the other problem is bounded and attained.

1. *Saddlepoint condition:* Moreover $q^* = p^*$ and $(\bar{x},\bar{\mu},\bar{Z})$ is a primal-dual optimal solution if and only if it is a saddle point of the Lagrangian (25.30b), i.e.,

$$\bar{x} = \arg\min_x \; L(x,\bar{\mu},\bar{Z}) \quad \text{and} \quad (\bar{\mu},\bar{Z}) = \arg\max_{\mu\in\mathbb{R}^k, Z\in K} L(\bar{x},\mu,Z)$$

2. *KKT condition:* The saddle point condition is equivalent to:
   - *Primal feasibility*: $a_0 + A\bar{x} = 0$, $f(\bar{x})\succeq 0$.
   - *Dual feasibility*: $\bar{Z}\succeq 0$, $c_i + \mathrm{tr}(F_i\bar{Z}) + \sum_j A_{ji}\bar{\mu}_j = 0$, $i=1,\dots,n$.
   - *Complementary slackness*: $\bar{Z}\cdot f(\bar{x}) = \mathrm{tr}\left(\bar{Z}\left(F_0 + \sum_i F_i\right)\right) = 0$.

We often use the following form of semidefinite programs with inequality constraints:

$$\max_{Z\in K} \; \mathrm{tr}(A_0Z) \quad \text{s.t.} \quad \mathrm{tr}(A_iZ)\le c_i, \;\; i=1,\dots,n \tag{25.31a}$$

where $K := \mathbb{S}_+^m$. It can be converted into an SDP with equality constraints, as in (25.30c) with $a_0 := 0$ and $A := 0$, by introducing slack variables. Its dual is (Exercise 25.27):

$$\min_{x\in\mathbb{R}_-^n} \; -c^Tx \quad \text{s.t.} \quad A_0 + \sum_i x_iA_i \;\preceq\; 0 \tag{25.31b}$$

and the KKT condition is: $(\bar{Z},\bar{x})$ is primal dual optimal if and only if

- *Primal feasibility*: $\bar{Z}\in K$, $\mathrm{tr}(A_i\bar{Z})\le c_i, i=1,\dots,n$.
- *Dual feasibility*: $A_0 + \sum_i \bar{x}_iA_i \preceq 0$, $\bar{x}\in\mathbb{R}_-^n$.
- *Complementary slackness*:

$$\bar{Z}\cdot\left(A_0 \,+\, \sum_i \bar{x}_iA_i\right) \;=\; 0, \qquad \sum_i \bar{x}_i\left(c_i - \mathrm{tr}\left(A_i\bar{Z}\right)\right) \;=\; 0$$

25.4 Semidefinite relaxations

25.4.1 Graph, partial matrix and completion

Consider a graph $G = (N, E)$ with $N := \{1, \ldots, n\}$. G can either be undirected or directed with an arbitrary orientation. Two nodes j and k are *adjacent* if $j \sim k \in E$. A *complete* graph is one where every pair of nodes is adjacent. A subgraph of G is a graph $F = (N', E')$ with $N' \subseteq N$ and $E' \subseteq E$. A *clique* of G is a complete subgraph of G. A *maximal clique* of G is a clique that is not a subgraph of another clique of G.

By a *path* connecting nodes j and k we mean either a set of *distinct* nodes $(j, n_1, \ldots, n_i, k)$ such that $(j \sim n_1), (n_1 \sim n_2), \ldots, (n_i \sim k)$ are edges in E or this set of edges, depending on the context. A *cycle* $(n_1, \ldots, n_i)$ is a path such that $(n_1 \sim n_2), \ldots, (n_i \sim n_1)$ are edges in E. By convention we exclude a pair of adjacent nodes (j, k) as a cycle. G is *connected* if there is a path between every pair of nodes. G is *k-vertex connected* or *k-connected*, $k = 1, \ldots, n$, if it remains connected after removing fewer than k nodes. G is *k-edge-connected*, $k = 1, \ldots, n$, if it remains connected after removing fewer than k edges. Hence if G is k-connected (k-edge-connected) then it is j-connected (j-edge-connected), $j \leq k$. A *connected component* of G is a subgraph of G that is connected.

A cycle in G that has no chord (an edge connecting two nodes that are non-adjacent in the cycle) is called a *minimal cycle*. G is *chordal* if all its minimal cycles are of length 3 (recall that an edge (j, k) is not considered a cycle). A *chordal extension* of G is a chordal graph on the same set of nodes as G that contains G as a subgraph. Every graph has a chordal extension; e.g. the complete graph on the same set of nodes is a trivial chordal extension.

Fix a graph $G = (N, E)$ with $N := \{1, \ldots, n\}$ and $E \subseteq N \times N$. For our purposes here we assume G is undirected so that $(j, k) \in E$ if and only if $(k, j) \in E$. Suppose the matrices C_l in (14.3), $l = 0, \ldots, L$, are all defined on G, i.e., for all l, $[C_l]_{jk} = 0$ if $(j, k) \notin E$. Then given any $n \times n$ matrix X, tr $C_l X =$ tr $C_l X_G$ where X_G is the submatrix of X defined by G. Conversely, given a partial matrix X_G that satisfies (14.3b), *any* completion X of X_G satisfies (14.3b). Even though both the objective function (14.3a) and the constraints (14.3b) depend only on the partial matrix X_G, the constraint $X \succeq 0$ in (14.3c) depends also on entries not in X_G. Indeed the number of complex variables in X is n^2 while the number of complex variables in X_G is only $n + 2|E|$, which is much smaller than n^2 if G is large but sparse. Hence instead of solving for a full psd matrix X directly as in SDP (14.3) we would like to compute a partial matrix X_G that has a psd completion X that satisfies (14.3b)–(14.3c). If the completion X is rank-1 then it also solves the problem (14.2) and hence yields a solution to the original QCQP (13.10) through spectral decomposition of X. Theorem 14.1 provides an exact characterization of when this is possible.

To solve the QCQP (13.10), Theorem 14.1 suggests the following strategy that exploits the sparsity of graph G: instead of solving SDP (14.3) for a psd matrix $X^{\text{opt}} \in \mathbb{S}^n_+$, solve for a psd partial matrix X_F^{opt} defined on a chordal extension F of G. If the solution X_F^{opt} turns out to be rank-1 as well then an optimal solution x^{opt} of QCQP (13.10) can be recovered from X_F^{opt} (see Section **??**).

Two questions naturally arise in this approach: (i) How to formulate a semidefinite relaxation based on a given a chordal extension F of G? (ii) How to choose a good chordal extension F of G so that the resulting relaxation can be solved efficiently? We next illustrate the issues involved in these two questions through an example. See [117, 118] for more details.

25.4.2 Chordal relaxation

Fix a graph $G = (N, E)$. Let $F = (N, E')$ be a chordal extension of G with $E' \supseteq E$. Let $q_1, \ldots, q_K$ be the set of maximal cliques of F and $X(q_k), k = 1, \ldots, K$, be the set of principal submatrices of X defined on these cliques. Consider the following problem where the optimization variable is the Hermitian partial matrix $W_F \in \mathbb{C}^{n+2|E'|}$ defined on the chordal extension F:

$$\min_{X_F = X_F^H} \quad \text{tr}\, C_0 X_G \tag{25.32a}$$

$$\text{subject to} \quad \text{tr}\, C_l X_G \leq b_l, \;\; l = 1, \ldots, L \tag{25.32b}$$

$$X_F(q_k) \succeq 0, \;\; k = 1, \ldots, K \tag{25.32c}$$

We call this problem a *chordal relaxation* of QCQP (13.10). Recall that we assume C_l, $l = 0, \ldots, L$, are all defined on G, i.e., $[C_l]_{jk} = 0$ if $(j,k) \notin E$. This implies that $\text{tr} C_l X = \text{tr} C_l X_F = \text{tr} C_l X_G$. Then chordal relaxation (25.32) is equivalent to SDP (14.3) in the sense that given any feasible solution X_F of (25.32), there is a psd completion X that is feasible for (14.3) and has the same cost, and vice versa. This is a consequence of [116, Theorem 7] that says every psd partial matrix has a psd completion if and only if the underlying graph is chordal. See also Theorem 14.4 and Corollary 14.5.

The first step in constructing the chordal relaxation (25.32) is to list all the maximal cliques q_k. Even though listing all maximal cliques of a general graph is NP-hard it can be done efficiently for a chordal graph. This is because a graph is chordal if and only if it has a perfect elimination ordering [260] and computing this ordering takes linear time in the number of nodes and edges [261]. Given a perfect elimination ordering all maximal cliques q_k can be enumerated and $X_F(q_k)$ constructed efficiently [117]. For optimal power flow problems the computation depends only on the topology of the power network, not on operational data, and therefore can be done offline.

We now show that (25.32) is indeed an SDP by converting it into the standard form (14.3) with the introduction of auxiliary variables, following the procedure described in [117]. This conversion also illustrates the difficulty in choosing a good chordal extension F (see Remark 25.4 below).

The (fully specified) matrices $X_F(q_k)$ in (25.32c) can be treated as principal submatrices of an $n \times n$ matrix X. They may not however be integrated directly into a common $n \times n$ matrix variable X because different $X_F(q_k)$ may share entries. We now explain the issue and its resolution using the example in Figure 1. They are the same in the general case with more cumbersome notations; see [117, 118]. Suppose we have chosen the chordal extension F in Figure 25.9(b) with two overlapping cliques q_1 and q_2 as explained in the caption of the figure. To decouple the two matrices $X_F(q_1)$ and $X_F(q_2)$, define the 3×3 matrix

$$X'(q_1) \;\; := \;\; \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & u_{22} & u_{23} \\ x_{31} & u_{32} & u_{33} \end{bmatrix}$$

where the decoupling variables u_{jk} are constrained to be:

$$u_{jk} \;\; = \;\; x_{jk} \quad \text{for } j,k = 2,3 \tag{25.33}$$

The constraints (25.32c) are replaced by

$$X'_F(q_1) \succeq 0 \quad \text{and} \quad X_F(q_2) \succeq 0 \tag{25.34}$$

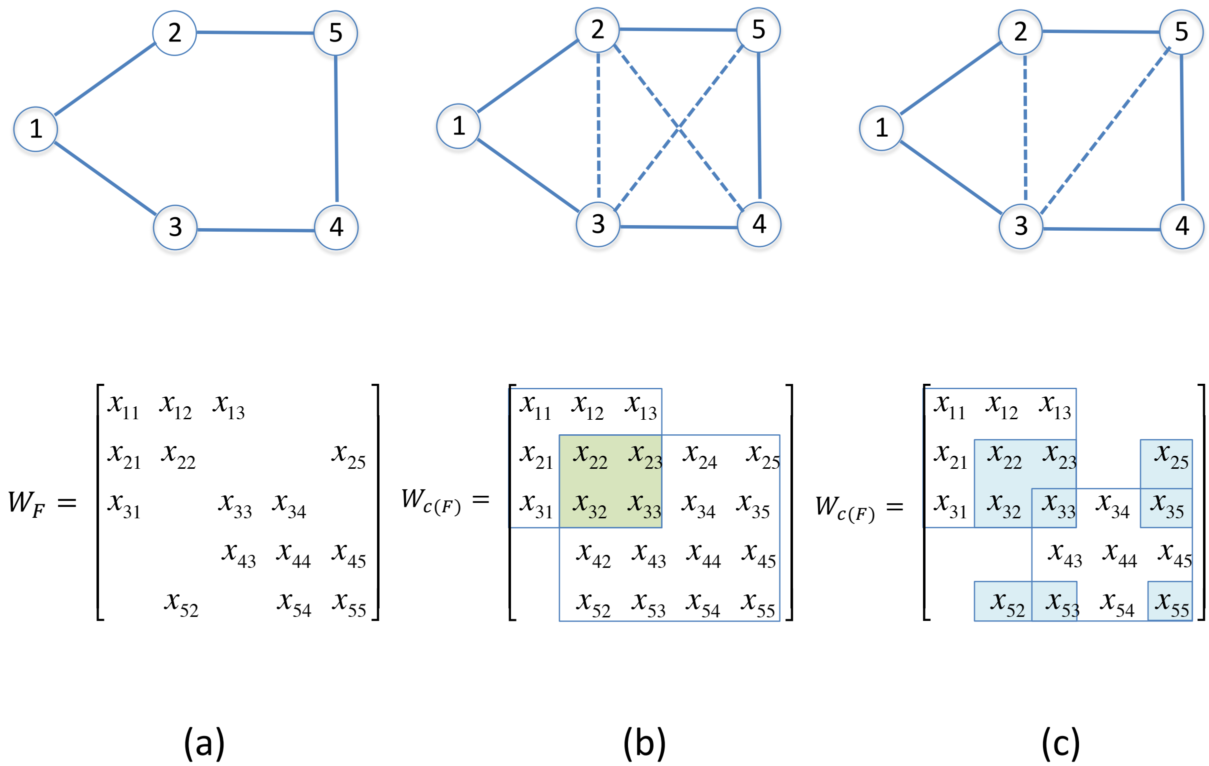


Figure 25.9: Chordal extensions of G. (a) Graph G and the partial matrix X_G. (b) A chordal extension F and its X_F that have 2 maximal cliques, $q_1 := (1,2,3)$ and $q_2 := (2,3,4,5)$. These cliques share two nodes, 2 and 3. The corresponding $X_F(q_1)$ and $X_F(q_2)$ are outlined in X_F with the overlapping entries shaded in green. The chordal relaxation based on this F requires 4 decoupling variables u_{jk}. (c) Another chordal extension F and its X_F that have 3 maximal cliques, outlined and shaded in blue in X_F. The chordal relaxation based on this F requires 8 decoupling variables u_{jk}.

Define the 7×7 block-diagonal matrix

$$X' \quad := \quad \begin{bmatrix} X'_F(q_1) & 0 \\ 0 & X_F(q_2) \end{bmatrix}$$

Then the chordal relaxation (25.32) can be written in the standard form (14.3) in terms of these 7×7 block-diagonal Hermitian matrices:

$$\min_{X' \in \mathbb{S}^7} \quad \text{tr}\, C'_0 X' \tag{25.35a}$$

$$\text{subject to} \quad \text{tr}\, C'_l X' \le b_l, \;\; l = 1, \ldots, L \tag{25.35b}$$

$$\text{tr}\, C'_r X' = 0, \;\; r = 1,2,3,4 \tag{25.35c}$$

$$X' \succeq 0 \tag{25.35d}$$

for appropriate choices of C'_l, $l = 0, \ldots, L$. The constraint $X' \succeq 0$ in (25.35d) is equivalent to the requirement (25.34) on its submatrices and C'_r in (25.35c) is chosen to enforce the requirement (25.33). Hence the chordal relaxation (25.32) is indeed an SDP.

Remark 25.4. There are two conflicting factors in choosing a good chordal extension F. First an F that contains fewer number of maximal cliques q generally involves larger cliques, leading to larger submatrices $X_F(q)$; for example the complete graph F has a single maximal clique but the corresponding $X_F(q) = X$ has n^2 entries and the chordal relaxation (25.32) offers no computational advantage over solving (in fact it is exactly) the original SDP (14.3). This argues for a chordal extension F with smaller, possibly more, maximal cliques q. Second, however, having more maximal cliques q tends to require more decoupling variables u_{jk}. Every decoupling variable u_{jk} introduces an extra equality constraint in (25.35c), thus increasing the required computational effort. For instance the transformed problem based on the chordal extension in Figure 25.9(b) involves 2 maximal cliques of sizes 3 and 4, and 4 additional equality constraints in (25.35c). The transformed problem based on the chordal extension in Figure 25.9(c), on the other hand, requires 3 maximal cliques each of size 3, and 8 additional equality constraints.

In summary even though the ambient dimension of the new variable X' is generally larger than that of the original $n \times n$ matrix variable X (7×7 as opposed to 5×5 for the example in Figure 25.9(b)), the chordal relaxation (25.32) can typically be solved much more efficiently than SDP (14.3) if G is large and sparse; for OPF examples, see [140, 32]. Choosing a good chordal extension F of G is important but nontrivial. See [117, 118] for methods to compute efficient chordal extensions and sparse SDP solutions.

25.5 Envelop theorems

This subsection collects several variants of envelope theorems, taken from [26, Proposition A.43, p.649], [262], and [185, Theorems 1, 2, 3].

The following saddlepoint envelope theorem is from [262, Theorem 298]. It makes mild assumptions, e.g., does not need convexity or differentiability (except differentiability in parameter p), and unifies several variants.

Theorem 25.37 (Saddlepoint envelope Theorem [262])**.** Let X and Y be metric spaces and $P \subseteq \mathbb{R}^n$ be an open set. Let $L : X \times Y \times P \ \to\ \mathbb{R}$. For each $p \in P$, let $(x^*(p), y^*(p)) \in X \times Y$ be a saddle point of L, i.e.,

$$L(x^*, y; p) \ \leq\ L(x^*(p), y^*(p); p) \ \leq\ L(x, y^*(p); p), \qquad x \in X,\ y \in Y \tag{25.36}$$

and define the *value function* as

$$V(p) \ :=\ L(x^*(p), y^*(p); p)$$

Suppose:

1. $x^*(p)$ and $y^*(p)$ are continuous functions (in particular, this assumes that there is a unique saddle point $(x^*(p), y^*(p))$ for each $p \in P$).
2. $\frac{\partial L}{\partial p}(x, y; p)$ exists and is jointly continuous on $X \times Y \times P$.

Then V is continuously differentiable and

$$\nabla V(p) \ =\ \nabla_p L(x^*(p), y^*(p); p)$$

i.e., $\frac{\partial V}{\partial p_i}(p) = \frac{\partial L}{\partial p_i}(x, y; p)$ evaluated at $(x, y) = (x^*(p), y^*(p))$.

Proof. We will prove that the directional derivative of V at each $p \in P$ in each direction $h \in \mathbb{R}^n$:

$$dV(p; h) \ :=\ \lim_{t \downarrow 0} \frac{V(p + th) - V(p)}{t}$$

exists[6] and equals $\frac{\partial V}{\partial p}(p) \cdot h$. This is equivalent to the differentiability of f. Moreover we will show that $\nabla V(p)$ is continuous on P.

Let $h \in \mathbb{R}^n$ be such that $[p, p+h] \subset P$ where $[p, p+h] := \{p + th : 0 \leq t \leq 1\}$ (such h always exists since P is open). By definition we have

$$V(p+h) - V(p) \ =\ L(x^*(p+h), y^*(p+h); p+h) \ -\ L(x^*(p), y^*(p); p)$$

The saddlepoint property (25.36) then implies the inequalities in the following:

$$\begin{aligned}
V(p+h) - V(p) \ &=\ \underbrace{L(x^*(p+h), y^*(p+h); p+h) \ -\ L(x^*(p+h), y^*(p); p+h)}_{\geq 0} && (25.37a)\\
&\quad +\ L(x^*(p+h), y^*(p); p+h) \ -\ L(x^*(p+h), y^*(p); p) && (25.37b)\\
&\quad +\ \underbrace{L(x^*(p+h), y^*(p); p) \ -\ L(x^*(p), y^*(p); p)}_{\geq 0} && (25.37c)
\end{aligned}$$

Since $L(x, y; p)$ is differentiable with respect to p for each (x, y), we can apply the mean value theorem to (25.37b) to get

$$V(p+h) - V(p) \ \geq\ \frac{\partial L}{\partial p}(x^*(p+h), y^*(p); p_1(h)) \cdot h$$

[6] Since $V(p)$ is not assumed to be convex, the limit in the definition of $dV(p; h)$ may not exist.

for some $p_1(h) \in [p, p+h]$. Similarly we have

$$\begin{aligned} V(p+h) - V(p) &= \underbrace{L(x^*(p+h), y^*(p+h); p+h) \; - \; L(x^*(p), y^*(p+h); p+h)}_{\leq 0} \\ & \quad + \; L(x^*(p), y^*(p+h); p+h) \; - \; L(x^*(p), y^*(p+h); p) \\ & \quad + \; \underbrace{L(x^*(p), y^*(p+h); p) \; - \; L(x^*(p), y^*(p); p)}_{\leq 0} \\ &\leq \frac{\partial L}{\partial p}(x^*(p), y^*(p+h); p_2(h)) \cdot h \end{aligned}$$

for some $p_2(h) \in [p, p+h]$. Combining, and replacing h by th, we have

$$\frac{\partial L}{\partial p}(x^*(p+th), y^*(p); p_1(th)) \cdot th \;\leq\; V(p+th) - V(p) \;\leq\; \frac{\partial L}{\partial p}(x^*(p), y^*(p+th); p_2(th)) \cdot th$$

Hence

$$\frac{\partial L}{\partial p}(x^*(p+th), y^*(p); p_1(th)) \cdot h \;\leq\; \tfrac{V(p+th)-V(p)}{t} \;\leq\; \frac{\partial L}{\partial p}(x^*(p), y^*(p+th); p_2(th)) \cdot h$$

Taking $t \downarrow 0$ and using the continuity of $\frac{\partial L}{\partial p}$ we get

$$dV(p;h) \;=\; \frac{\partial L}{\partial p}(x^*(p), y^*(p); p) \cdot h$$

for all $p \in P$ and all $h \in \mathbb{R}^n$. Hence

$$\frac{\partial V}{\partial p}(p) \;=\; \frac{\partial L}{\partial p}(x^*(p), y^*(p); p)$$

exists. Moreover it is continuous since $\frac{\partial L}{\partial p}$ is continuous. $\square$

Remark 25.5. It is important that the feasible sets (X, Y) are independent of p. The saddlepoint property (25.36) can still hold if the feasible sets (X_p, Y_p) depend on p, i.e., for all $p \in P$,

$$L(x^*(p), y; p) \;\leq\; L(x^*(p), y^*(p); p) \;\leq\; L(x, y^*(p); p), \qquad x \in X_p, \; y \in Y_p$$

Yet the conclusion of Theorem 25.38 in general does not hold. This is because the inequalities in (25.37a) and (25.37c) rely on inequalities of the form:

$$L(x^*(p), y^*(p); p) \;\leq\; L(x^*(q), y^*(p); p)$$

which may not hold if $x^*(q)$ is in $X_q \setminus X_p$. This inequality will hold if $x^*(p) \in X_q$ for all $p, q \in P$, i.e., even if X_p and X_q are different, every optimal point $x^*(p)$ is feasible for every $q \in P$. See Exercise 25.28. $\square$

An important implication of Remark 25.5 is that in a two-stage stochastic program with recourse, since the feasible set for the second-stage problem usually depends on the first-stage decision x_1, the differentiability of the value function or recourse function $\overline{F}(x_1)$ in (21.11) generally does not follow directly from envelope theorems.

The following version is the classical envelope theorem. The key condition is that the first-order stationarity condition hold with equality, which is the reason for X to be open so that the optimal point $x^*(p)$ is in the interior of X. Note that convexity is not assumed since the proof only needs the necessity of the stationarity condition.

Theorem 25.38 (Envelope Theorem [262])**.** Let $X \subseteq \mathbb{R}^N$ and $P \subseteq \mathbb{R}^L$ be open sets. Consider the constrained optimization for each $p \in P$:

$$\min_{x \in X} \; f(x,p) \quad \text{s.t.} \quad g(x,p) = 0$$

with the associated Lagrange multiplier $y \in \mathbb{R}^M$, where $f : X \times P \to \mathbb{R}$ and $g := (g_1, \ldots, g_M) : X \times P \to \mathbb{R}^M$. Let $x^*(p)$ denote an optimal solution and $V(p) := f(x^*(p), p)$ the optimal value. Define the Lagrangian

$$L(x,y;p) \;\; := \;\; f(x,p) \;+\; y^T g(x,p)$$

Suppose

1. $f, g_1, \ldots, g_M$ are continuously differentiable on $X \times P$.

2. The conclusion of the Lagrange Multiplier Theorem holds for each $p \in P$: there exist $y^*(p) \in \mathbb{R}^M$ such that the first-order stationarity condition holds with equality:

$$\frac{\partial L}{\partial x}(x^*(p), y^*(p); p) \;\; = \;\; \frac{\partial f}{\partial x}(x^*(p), p) \;+\; (y^*(p))^T \, \frac{\partial g}{\partial x}(x^*(p), p) \;\; = \;\; 0$$

3. $x^*(p)$ and $y^*(p)$ are continuously differentiable functions (in particular, this assumes that the optimal primal and dual solutions exist and are unique).

Then $V(p)$ is continuously differentiable and

$$\frac{\partial V}{\partial p}(p) \;\; = \;\; \frac{\partial L}{\partial p}(x^*(p), y^*(p); p) \;\; = \;\; \frac{\partial f}{\partial p}(x^*(p), p) \;+\; (y^*(p))^T \, \frac{\partial g}{\partial p}(x^*(p), p)$$

The theorem can be proved by appealing to Theorem 25.38 but a direct proof is simpler.

Proof. $V(p)$ is continuously differentiable since $f(p)$ and $x^*(p)$ are. Since $x^*(p)$ satisfies $g(x^*(p), p) = 0$ we have

$$V(p) \;\; = \;\; L(x^*(p), y^*(p); p) \;\; = \;\; f(x^*(p), p) \;+\; \sum_m y_m^*(p) \, g_m(x^*(p), p)$$

Differentiability assumptions yield

$$
\begin{aligned}
\frac{\partial V}{\partial p_l}(p) &= \sum_n \frac{\partial f}{\partial x_n}(x^*(p),p)\cdot\frac{\partial x_n^*}{\partial p_l}(x^*(p),p) + \frac{\partial f}{\partial p_l}(x^*(p),p) + \sum_m \frac{\partial y_m^*}{\partial p_l}(p)\cdot g_m(x^*(p),p) \\
&\quad + \sum_m y_m^*(p)\left(\sum_n \frac{\partial g_m}{\partial x_n}(x^*(p),p)\cdot\frac{\partial x_n^*}{\partial p_l}(x^*(p),p) + \frac{\partial g_m}{\partial p_l}(x^*(p),p)\right) \\
&= \underbrace{\sum_m \frac{\partial y_m^*}{\partial p_l}(p)\cdot g_m(x^*(p),p)}_{=0 \quad \because\, g_m(x^*(p),p)=0} + \sum_n \underbrace{\left(\frac{\partial f}{\partial x_n}(x^*(p),p) + \sum_m y_m^*(p)\frac{\partial g_m}{\partial x_n}(x^*(p),p)\right)}_{=0 \quad \because\, \text{stationarity}}\cdot\frac{\partial x_n^*}{\partial p_l}(x^*(p),p) \\
&\quad + \frac{\partial f}{\partial p_l}(x^*(p),p) + \sum_m y_m^*(p)\cdot\frac{\partial g_m}{\partial p_l}(x^*(p),p)
\end{aligned}
$$

Hence

$$\frac{\partial V}{\partial p}(p) \;=\; \frac{\partial f}{\partial p}(x^*(p),p) + (y^*(p))^T\,\frac{\partial g}{\partial p}(x^*(p),p)$$

as desired. □

Remark 25.6. It is important that the set X is open so that the first-order stationarity condition holds with equality. If the feasible set X_p depends on p, then either X_p is assumed open or $x^*(p)$ is in the interior of X_p. This means that if the constraint $x \in X_p$ is represented by $h(x,p) \le 0$, the corresponding Lagrange multipliers will be zero at optimality so that the stationarity condition and the conclusion of the theorem will remain unchanged.

The following result is taken from [26, Proposition A.43, p.649].

Theorem 25.39 (Danskin's Theorem)**.** Let $X \subseteq \mathbb{R}^n$ and $f : X \times \mathbb{R}^m \to \mathbb{R}$ be a continuous function. Suppose $f(x,p)$ is convex in p for every $x \in X$. Let

$$V(p) \;:=\; \sup_{x\in X} f(x,p)$$

1. Suppose X is compact so that a maximizer $x^*(p)$ always exists with $V(p) = f(x^*(p),p)$. Let the set of maximizers be

$$X^*(p) \;:=\; \{x \in X : V(p) = f(x,p)\}$$

   (a) The function $V : \mathbb{R}^m \to \mathbb{R}$ is convex and has directional derivative $dV(p;h)$ at p in the direction of $h \in \mathbb{R}^m$ given by:

$$dV(p;h) \;:=\; \lim_{t\downarrow 0}\frac{V(p+th)-V(p)}{t} \;=\; \max_{x\in X^*(p)} df(x,h;p)$$

   where $df(x,h;p) := \lim_{t\downarrow 0}\frac{f(x+th,p)-f(x,p)}{t}$ is the directional derivative of the function $f(\cdot,p)$.

(b) If $X^*(p) = \{x^*(p)\}$ is a singleton and $f(x^*(p),\cdot)$ is differentiable in its second argument at p, then $V(p)$ is differentiable at p and

$$\nabla V(p) \;=\; \nabla_p f(x^*(p),p) \;=\; \left(\frac{\partial f}{\partial p_j}(x^*(p),p),\ j=1,\dots,m\right)$$

2. The conclusions of 1 hold if, instead of assuming X is compact, we assume that

- $X^*(p)$ is nonempty for every $p \in \mathbb{R}^m$; and
- For every sequence $\{p_k\}$ converging to some p, there exists a bounded sequence $\{x_k^*\}$ of maximizes $x_k^* \in X^*(p)$ for all k (so that $\{x_k^*\}$ has a convergent subsequence).

Remark 25.7. As for Theorem 25.38, it is important that the feasible set X does not depend on p, for the same reason discussed in Remark 25.5.

Theorem 25.39 guarantees the existence of directional derivative of $V(p)$ if f is jointly continuous in (x,p) and convex in p for every $x \in X$. Differentiability of V however needs uniqueness of the maximizer $x^*(p)$ and differentiability of $f(x^*(p),\cdot)$ at p. See [185, Theorems 1 and 2] for Envelope Theorems that allow nonunique maximizer $x^*(p)$ but requires an upper bound on $|\partial f(x,p)/\partial p_i|$ uniformly in p_i. The formulation in [185, Theorems 1 and 2] also assumes that the feasible set X is independent of p.

Remark 25.8. Consider a real-valued function $f : X \times \mathbb{R}^m \to \mathbb{R}$ and

$$g_1(y) \;:=\; \sup_{x\in X}\, f(x,y), \qquad\qquad g_2(y) \;:=\; \inf_{x\in X}\, f(x,y)$$

where X is an arbitrary subset of $\mathbb{R}^n$.

Taking supremum. If f is convex in y for every $x \in X$ then $g_1(y)$ is convex in y as Theorem 25.39 shows. Moreover if $f(x,\cdot)$ is closed for each $x \in X$ then $g_1(\cdot)$ is closed as well ([76, Proposition 1.1.6, p.13]). This is the situation e.g. when f is the Lagrangian function of a constrained optimization.

Taking partial minimization. If $f(x,y)$ is *jointly* convex in (x,y) instead (this is not the case with Lagrangian functions) then $g_2(y)$ is convex ([76, Proposition 3.3.1, p.122]). Moreover the epigraph $\text{epi}(g_2(y)) := \{(y,z) : z \geq g_2(y), y \in \mathbb{R}^m\}$ is essentially the projection of $\text{epi}(f) := \{(x,y,z) : z \geq f(x,y), x \in X, y \in \mathbb{R}^m\}$ on the space of (y,z), except possibly for somme boundary points y when the infimum over $x \in X$ is not attained in which case $(y, g_2(y))$ are missing. Precisely

$$P(\text{epi}(f)) \;\subseteq\; \text{epi}(g_2) \;\subseteq\; \text{cl}\,P(\text{epi}(f))$$

where the projection P is defined by $P(S) := \{(y,z) : (x,y,z) \in S\}$ for any subset $S \subseteq X \times \mathbb{R}^m \times \mathbb{R}$. □

25.6 Bibliographical notes

There are many excellent texts on linear algebra. Most of the materials in Chapter 25.1.5 can be found in [257, Chapter 7.3] for singular value decomposition and properties of singular values, in [257, Chapters 2.5, 4.1] for spectral theorems for normal and Hermitian matrices, and [257, Chapter 4.4.] for complex symmetric matrices. The basic notions of algebraic graph theory in Chapter 25.2 mostly follow [263].

There are many classic texts on nonsmooth convex analysis and optimization (e.g. Rockafellar, Clarke, ...). The materials in Section 25.3 mostly follow [76, Chapter 5], [77]. Books on nonsmooth analysis include [78, 77, 79] with [78] focuses more on control theory for applications of nonsmooth analysis and [77, 79] more on nonsmooth convex optimization. The emphasis of [77] is on $\mathbb{R}^n$ whereas that of [79] is on infinite dimensional vector spaces.

25.7 Problems

Chapters 25.1.2–25.1.5.

Exercise 25.1 (Matrix sum and product)**.** Let $A, B \in \mathbb{C}^{n\times n}$.

1. Show that if A, B are nonsingular then AB is nonsingular but $A+B$ can be singular.

2. Show that if $A \succ 0$ and $B \succ 0$ then $A+B \succ 0$ but AB may not be positive definite. Show that if $AB = BA$ or if A and B have the same set of eigenvectors then $AB \succ 0$. Give an example of $A \succ 0$ and $B \succ 0$ that share the same set of eigenvectors and hence $AB \succ 0$. (Hint: $AB = BA$ if and only if A and B are simultaneously diagonalizable.)

3. $AB = BA$ is only sufficient for $AB \succ 0$. Suppose $A \succ 0$, $B \succ 0$, and $AB \neq BA$. Give an example of A, B where $AB \succ 0$ and another example where $AB \not\succ 0$.

Exercise 25.2 (Quadratic)**.** Let $M = A + \mathbf{i}B$ where $A, B \in \mathbb{R}^{n\times n}$ and $\alpha = \rho + \mathbf{i}\varepsilon$ where $\rho, \varepsilon \in \mathbb{R}^n$. Show that, if M is (complex) symmetric, then

$$\alpha^{\mathsf{H}} M \alpha \;=\; (\rho^{\mathsf{T}} A\rho + \varepsilon^{\mathsf{T}} A\varepsilon) \;+\; \mathbf{i}(\rho^{\mathsf{T}} B\rho + \varepsilon^{\mathsf{T}} B\varepsilon)$$

Show that, if M is (complex) symmetric, then

1. If $A \succ 0$ then M^{-1} exists and $\mathrm{Re}(M^{-1}) \succ 0$.

2. If $B \succ 0$ then M^{-1} exists and $\mathrm{Im}(M^{-1}) \prec 0$.

Exercise 25.3 (Schur complement)**.** Let $M \in \mathbb{C}^{n\times n}$ and partition it into blocks:

$$M = \begin{bmatrix} A & B \\ D & C \end{bmatrix}$$

such that $A \in \mathbb{C}^{(n-k)\times(n-k)}$, $k < n$, and the other submatrices are of appropriate dimensions. If M and A are invertible then

$$M^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(M/A)^{-1}DA^{-1} & -A^{-1}B(M/A)^{-1} \\ -(M/A)^{-1}DA^{-1} & (M/A)^{-1} \end{bmatrix}$$

where $M/A := C - DA^{-1}B$ is the Schur complement of A of matrix M.

Exercise 25.4 (Push-through identities)**.** Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times k}$ and $C \in \mathbb{C}^{k\times n}$. Then

1. $(I_n + BC)^{-1}B = B(I_k + CB)^{-1}$ provided the inverses exist.
2. $(A + BC)^{-1}B = B(A + CB)^{-1}$ provided $n = k$, $AB = BA$ and the inverses exist.

Note that when $k \ll n$, $I_k + CB$ can be much easier to invert than $I_n + BC$.

Exercise 25.5. Find the singular value decomposition, pseudo-inverse $A^\dagger$, null(A), range(A), null(A^T) and range(A^T) of the following:

1. $A = \begin{bmatrix} a \\ b \end{bmatrix}$.
2. $A = [1 \;\; 2]$.
3. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.
4. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Discuss the existence and uniqueness of solutions to $Ax = b$ given b.

Exercise 25.6. Consider $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$. Let $B := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $C := \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ so that $A = [B \;\; C]$. Show that $A^\dagger = A^{-1} \neq \begin{bmatrix} B^\dagger \\ C^\dagger \end{bmatrix}$.

Exercise 25.7 (Singular value decomposition)**.** On the uniqueness of the unitary matrix W in Theorem 25.8, suppose $m \leq n$ but $\text{rank}(A) =: r < m$. For a given V given in Theorem 25.8, show that W defined by $W^* := \Sigma^\dagger V^* A$ generally does not satisfy the singular value decomposition (25.10). Here $\Sigma^\dagger$ is obtained from Σ by replacing its positive singular values σ_i by $1/\sigma_i$ and taking the transpose.

Exercise 25.8 (Singular value decomposition)**.** Let $x \in \mathbb{C}^n$ be an $n \times 1$ matrix. Compute a singular value decomposition of x.

Exercise 25.9 (SVD and unitary diagonalization)**.** Prove Theorem 25.13.

Chapter 25.1.6.

Exercise 25.10 (Pseudo-inverse of A)**.** Consider a matrix $A \in \mathbb{C}^{m\times n}$ as a mapping $A : \mathbb{C}^n \to \mathbb{C}^m$ and its Hermitian transpose $A^* : \mathbb{C}^m \to \mathbb{C}^n$. Show that the mapping A restricted from $\text{range}(A^*)$ to $\text{range}(A)$ is surjective and injective. This means that an inverse, denoted $A^\dagger : \text{range}(A) \to \text{range}(A^*)$, always exists for any matrix A.

Exercise 25.11 (Pseudo-inverse of A)**.** For the mapping A in Exercise 25.10, show that $A^\dagger = W\Sigma^\dagger V^*$, i.e., A and $A^\dagger$ are inverse of each other when restricted to $\text{range}(A^*)$ and $\text{range}(A)$.

Exercise 25.12 (Pseudo-inverse of A)**.** Consider a matrix $A \in \mathbb{C}^{m\times n}$ with rank $A = r \leq \min\{m,n\}$. Let $A = V\Sigma W^*$ be its singular value decomposition and $A^\dagger = W\Sigma^\dagger V^*$ be its pseudo-inverse. Prove (Corollary 25.17.3): If $r = n \leq m$ then $A^\dagger = (A^*A)^{-1}A^*$.

Exercise 25.13 (Pseudo-inverse of A)**.** Consider a matrix $A \in \mathbb{C}^{m\times n}$ with rank $A = r \leq \min\{m,n\}$. Instead of using the formula $A^\dagger = W\Sigma^\dagger V^*$, use the fact that $A^\dagger$ and A are inverse of each other when restricted to $\text{range}(A^*)$ and $\text{range}(A)$ to prove:

1. If $r = m \leq n$ then $A^\dagger = A^*(AA^*)^{-1}$.
2. If $r = n \leq m$ then $A^\dagger = (A^*A)^{-1}A^*$.

Exercise 25.14 (Pseudo-inverse and norm minimization)**.** Consider a matrix $A \in \mathbb{R}^{m\times n}$ with rank $A = m \leq n$. Show that the pseudo-inverse solution $A^\dagger b$ of $Ax = b$ is the optimal solution of the quadratic program

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|x\|_2^2 \quad \text{s.t.} \quad Ax = b$$

Optimization problems often have multiple equivalent formulations that involve different variables and constraints. The next two exercises explore the relationship between these equivalent constraints and their Lagrange multipliers when the constraints are affine. See also Exercise 20.3 on equivalent formulations of economic dispatch with reduced model.

Exercise 25.15 (Equivalent constraints)**.** Consider the equations $A_1x = b_1$ and $A_2x = b_2$ with $x \in \mathbb{R}^n$, $A_1 \in \mathbb{R}^{m\times n}$, $A_2 \in \mathbb{R}^{k\times n}$, $b_1 \in \mathbb{R}^m$, $b_2 \in \mathbb{R}^k$, and m may not be equal to k. Suppose

- *Feasibility*: $b_1 \in \text{range}(A_1)$ and $b_2 \in \text{range}(A_2)$ so solutions for these equations always exist.
- *Equivalence*: x satisfies $A_1x = b_1$ if and only if it satisfies $A_2x = b_2$.

Remark 25.2 implies that the solution set of $A_1x = b_1$ is given by

$$X_1 \;:=\; \{x : x \;=\; A_1^\dagger b_1 + w_1,\; w_1 \in \text{null}(A_1)\}$$

and the solution set of $A_2x = b_2$ is given by

$$X_2 \;:=\; \{x : x \;=\; A_2^\dagger b_2 + w_2,\; w_2 \in \text{null}(A_2)\}$$

Show that there is a bijection between X_1 and X_2.

Exercise 25.16 (Equivalent constraints)**.** Consider the setup in Exercise 25.15 and the equivalent problems

$$\min_x \; f(x) \quad \text{subject to} \quad A_1x = b_1 \qquad [\lambda_1] \tag{25.38}$$

$$\min_x \; f(x) \quad \text{subject to} \quad A_2x = b_2 \qquad [\lambda_2] \tag{25.39}$$

with Lagrange multipliers λ_1, λ_2 respectively. Suppose f is differentiable (not necessarily convex). Let (x^*, λ_1^*) be a primal-dual optimal point with zero duality gap for (25.38) and (x^*, λ_2^*) be a primal-dual optimal point with zero duality gap for (25.39). Show that $A_1^T\lambda_1^* = A_2^T\lambda_2^*$.

Chapter 25.1.7.

Exercise 25.17 (Euclidean norm)**.** Show that the Euclidean norm $\|\cdot\|_2$ on $\mathbb{C}^n$ is the only unitarily invariant norm with $\|e_i\| = 1$. Positive scalar multiples of Euclidean norms are also unitarily invariant with $\|e_i\|$ not necessarily 1.

Exercise 25.18 (Cauchy-Schwarz inequality)**.** Let $x_1, \ldots, x_n$ be n given real numbers with sample mean μ and sample standard deviation σ defined by:

$$\mu \;:=\; \frac{1}{n}\sum_i x_i, \qquad\qquad \sigma \;:=\; \left(\frac{1}{n}\sum_i (x_i - \mu)^2\right)^{1/2}$$

It can then be shown that (Exercise 25.18)

$$\mu - \sigma\sqrt{n-1} \;\leq\; x_i \;\leq\; \mu + \sigma\sqrt{n-1}, \qquad\qquad i = 1, \dots, n$$

with equality for some i if and only if $x_p = x_q$ for all $p, q \neq i$.

Exercise 25.19 (Hölder's inequality)**.** Prove Theorem 25.20 on the vector space $V = C^n$ or $\mathbb{R}^n$ with l_p norms (Hölder's inequality): For any $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$

$$\sum_{i=1}^{n} |x_i y_i| \;\leq\; \|x\|_p \, \|y\|_q \,, \qquad\qquad x,\, y \in V$$

with equality if and only if $x^p := (x_i^p, i = 1, \dots, n)$ and $y^q := (y_i^q, i = 1, \dots, n)$ are linearly dependent, i.e., $x^p = a y^q$ for some scalar $a \in F$.

Exercise 25.20 (Induced norms)**.** Let $A \in M_{m,n}$ be a $m \times n$ complex matrix. Prove Theorem 25.22:

1. *Max column sum* (induced by l_1 norm): $\|A\|_1 = \max_j \sum_i |A_{ij}|$.
2. *Max row sum* (induced by l_∞ norm):$\|A\|_\infty = \max_i \sum_j |A_{ij}|$.
3. *Spectral norm* (induced by l_2 norm): $\|A\|_2 = \sigma_{\max}(A) = \sqrt{\lambda_{\max}\left(A^{\mathsf{H}}A\right)}$ where $\sigma_{\max}(A)$ is the largest singular value of A and $\lambda_{\max}\left(A^{\mathsf{H}}A\right) \geq 0$ is the largest eigenvalue of the positive semidefinite matrix $A^{\mathsf{H}}A$.
4. If A is square and nonsingular then $\|A^{-1}\|_2 = 1/\sigma_{\min}(A)$, the reciprocal of the smallest singular value of A.
5. $\|A^{\mathsf{H}}A\|_2 \;=\; \|AA^{\mathsf{H}}\|_2 \;=\; \|A\|_2^2$.
6. $\|A\|_2 \;=\; \max\{|y^{\mathsf{H}}Ax| \,:\, \|x\|_2 = \|y\|_2 = 1,\, x \in \mathbb{C}^n, y \in \mathbb{C}^m\}$.

Exercise 25.21 (Vector norms on matrices)**.** Prove Theorem 25.24: Let $A \in M_n$ be a $n \times n$ complex matrix.

1. $\|\cdot\|_{\text{sum}}$ and $\|\cdot\|_F$ are submultiplicative matrix norms, but $\|\cdot\|_{\max}$ is a matrix norm that is not submultiplicative.
2. The Frobenius norm is given by

$$\|A\|_F \;=\; \left|\operatorname{tr}\left(AA^{\mathsf{H}}\right)\right|^{1/2} \;=\; \sqrt{\sum_i \sigma_i^2(A)} \;=\; \sqrt{\sum_i \lambda_i\left(AA^{\mathsf{H}}\right)}$$

   where $\sigma_i(A)$ denote the singular values of A and $\lambda_i(AA^{\mathsf{H}})$ denote the eigenvalues of the positive semidefinite matrix AA^{H}.

3. $\|A\|_F = \|A^{\mathsf{H}}\|_F = \|UAV\|_F$ for any unitary matrices $U, V \in M_n$ (unitarily invariant).

Exercise 25.22 (Spectral radius, singular values, norms)**.** Let $A \in M_n$. Let $\|\cdot\|$ be a submultiplicative matrix norm on M_n and $A \in M_n$. Let λ_i and σ_i be the eigenvalues and singular values of A respectively with

$$|\lambda_1| \ \geq \ \cdots \ \geq \ |\lambda_n|, \qquad\qquad \sigma_1 \ \geq \ \cdots \ \geq \ \sigma_n$$

Let $\rho(A) := |\lambda_1|$ denote the spectral radius of A. Prove Theorem 25.26:

1. $|\lambda_1| \leq \sigma_1$ and $|\lambda_n| \geq \sigma_n > 0$, i.e., $|\lambda_i| \in [\sigma_n, \sigma_1]$.
2. For all i, $1/\|A^{-1}\| \leq |\lambda_i| \leq \rho(A) \leq \|A\|$ if A is nonsingular.
3. Given any $\varepsilon > 0$ there is a submultiplicative matrix norm $\|\cdot\|$ such that $\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon$. Moreover

$$\rho(A) \ = \ \inf\{\|A\| \, : \, \|\cdot\| \text{ is an induced norm}\}$$

Exercise 25.23 (Sequence convergence)**.** Let $\|\cdot\|$ be a submultiplicative matrix norm on M_n and $A \in M_n$. Let $\rho(A)$ denote the spectral radius of A. Prove Theorem 25.27:

1. If $\|A\| < 1$ then $\lim_{k\to\infty} A^k = 0$, i.e., $\left|[A^k]_{ij}\right| \to 0$ as $k \to \infty$ for all i, j.
2. $\rho(A) < 1$ if and only if $\lim_{k\to\infty} A^k = 0$.
3. *Gelfand formula*: $\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$.

Exercise 25.24 (Series convergence)**.** Suppose there exists a matrix norm $\|\cdot\|$ such that $\|A\| < R$ where R is the radius of convergence for the power series $\sum_k a_k z^k$. Show that the matrix power series $\sum_k a_k A^k$ converges absolutely, i.e., $\lim_{k\to\infty} |a_k| \|A^k\|$

Chapter 25.3.

Exercise 25.25 (Nonnegative cone $\mathbb{R}^m$)**.** For $y^*, \lambda^* \in \mathbb{R}^m$ with $\lambda^* \geq 0$, prove:

$$y^* \leq 0, \ \ (\lambda^*)^T y^* = 0 \quad \Leftrightarrow \quad y^* \ \in \ N_{\mathbb{R}^m_+}(\lambda^*)$$

Exercise 25.26 (quadratic program)**.** Consider $\min_{x\in\mathbb{R}^n} f(x)$ where

$$f(x) \;:=\; x^T A^T A x \,+\, 2b^T x \,+\, c$$

$A \in \mathbb{R}^{m\times n}$ and $A^T A \succ 0$ is positive definite. Show that

1. *Completing square.*

$$f(x) \;=\; \|Ax+\tilde{b}\|_2^2 \,+\, \left(c - \|\tilde{b}\|_2^2\right)$$

   where $\tilde{b} := A\left(A^T A\right)^{-1} b$ and hence $\|\tilde{b}\|_2^2 = b^T \left(A^T A\right)^{-1} b$.

2. The unique minimizer $x^* = -\left(A^T A\right)^{-1} A^T b$ and the optimal value is $c - b^T\left(A^T A\right)^{-1} b$.

Exercise 25.27 (SDP)**.** Consider the following semidefinite program with inequality constraints:

$$\max_{Z\in K}\ \operatorname{tr}(A_0 Z) \quad \text{s.t.} \quad \operatorname{tr}(A_i Z) \,\le\, c_i, \;\; i = 1,\ldots,n \tag{25.40}$$

where $K := \mathbb{S}_+^m$ is the set of $m\times m$ psd matrices.

1. Convert it into an equivalent SDP with equality constraints similar to problem (25.30c).
2. Derive its dual problem:

$$\min_{x\in\mathbb{R}_-^n}\ -c^T x \quad \text{s.t.} \quad A_0 + \sum_i x_i A_i \,\preceq\, 0 \tag{25.41}$$

3. Show that $(\bar{Z},\bar{x})$ is primal dual optimal if and only if
   - *Primal feasibility*: $\bar{Z} \in K$, $\operatorname{tr}(A_i\bar{Z}) \le c_i, i = 1,\ldots,n$.
   - *Dual feasibility*: $A_0 + \sum_i \bar{x}_i A_i \preceq 0$, $\bar{x} \in \mathbb{R}_-^n$.
   - *Complementary slackness*:

$$\bar{Z}\cdot\left(A_0 \,+\, \sum_i \bar{x}_i A_i\right) \;=\; 0, \qquad \sum_i \bar{x}_i \left(c_i - \operatorname{tr}(A_i\bar{Z})\right) \;=\; 0$$

   (Hint: Use Theorem 25.36 or derive directly.)

The following problem studies Theorem 25.38 when the feasible set X_p depends on p. It shows that the theorem generally no longer holds.

Exercise 25.28 (Saddlepoint envelope theorem)**.** Consider the master problem:

$$\min_x\ f(x) := (x-p)^2 \quad \text{s. t.} \quad \frac{p}{4} \,\le\, x \,\le\, \frac{p}{2} \tag{25.42}$$

for $p \in P := (0,2)$. Clearly the unique minimizer $x^*(p) = p/2$. We study three ways to dualize the constraints, resulting in different Lagrangian functions, feasible sets, and saddlepoints.

1. Dualize both constraints with dual variables $y := (y_1, y_2) \geq 0$ and the Lagrangian

$$L(x,y;p) \quad := \quad f(x) \; + \; y_1 \left(\frac{p}{4} - x \right) \; + \; y_2 \left(x - \frac{p}{2} \right)$$

   Exhibit that Theorem 25.38 holds.

2. Consider the form of (25.42)

$$\min_{x \in X_p} \; f(x) := (x-p)^2 \quad \text{s. t.} \quad x \; \geq \; \frac{p}{4} \tag{25.43}$$

   with $X_p := \{x : x \leq p/2\}$, and Lagrangian

$$L_1(x, y_1; p) \quad := \quad f(x) \; + \; y_1 \left(\frac{p}{4} - x \right)$$

   Show that Theorem 25.38 does not hold because of the reason explained in Remark 25.5.

3. Consider the following form of (25.42)

$$\min_{x \in X_p} \; f(x) := (x-p)^2 \quad \text{s. t.} \quad x \; \leq \; \frac{p}{2} \tag{25.44}$$

   with $X_p := \{x : x \geq p/4\}$, and Lagrangian

$$L_2(x, y_2; p) \quad := \quad f(x) \; + \; y_2 \left(x - \frac{p}{2} \right)$$

   Show that Theorem 25.38 holds because $x^*(p) \in X_q$ for all $p, q \in P$.

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