Power System Analysis

Chapter 2 Transmission line models

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Outline

1. Line characteristics
2. Line models

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1. Line characteristics
    - Resistance r and conductance g
    - Series inductance l
    - Shunt capacitance c
    - Balanced three-phase lines
2. Line models

Three-phase line

Alternating currents in conductors line interact electromagnetically

Interactions couple voltages & currents across phases

In balanced operation, phases behave as if they are decoupled

In each phase, line is characterized by

- series impedance / meter $z := r + i\omega l$ Ω/m $r > 0,\ l > 0$
- shunt admittance / meter to neutral $y := g + i\omega c$ Ω^{-1}/m $g \geq 0,\ c > 0$

Assumptions

$$i_1(t) + \cdots + i_n(t) = 0 \quad \text{for all } t$$

$$q_1(t) + \cdots + q_n(t) = 0 \quad \text{for all } t$$

Line characteristics

Series resistance r and shunt conductance g

Series resistance r depends on

- Temperature and cross-sectional area of the conductor (this is called the dc resistance)
- AC frequency (this is called the ac resistance and defined to be $P_{\text{loss}}/|I|^2$)

Shunt conductance g accounts for real power loss between conductors or conductors and ground

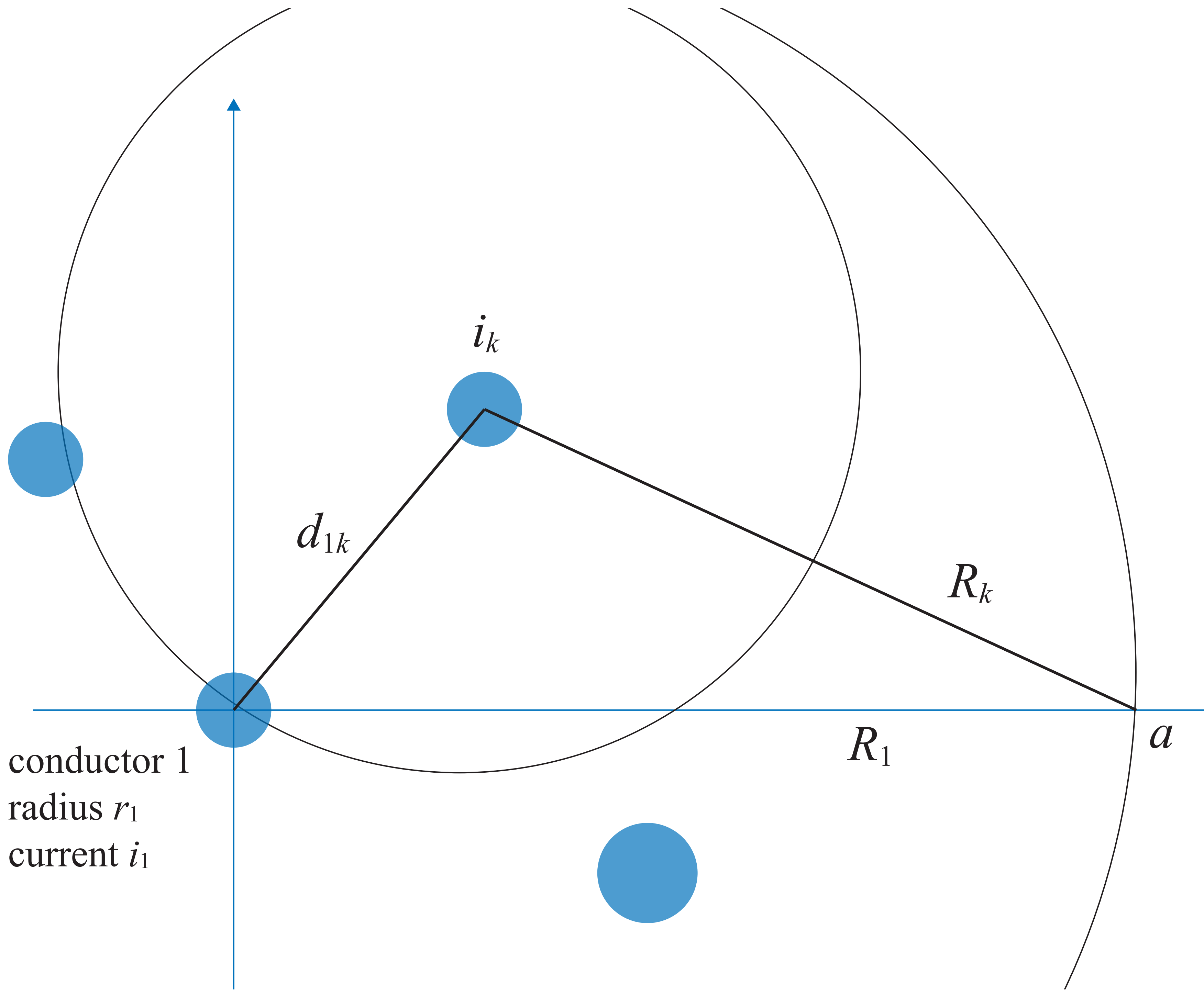
- Due to leakage currents at insulators, depending on environment such as moisture level
- Due to corona when a strong electric field at conductor surface ionizes the air, causing it to conduct, depending on meteorological conditions such as rain
- Power loss due to shunt conductance g is typically much smaller than $r|I|^2$; hence g is often assumed zero in transmission line models

Line characteristics

Series inductance L

Total flux linkages λ_k of conductor k depends on currents $i_{k'}$ in all conductors k'

$$\lambda_k = \underbrace{\left(\frac{\mu_0}{2\pi}\ln\frac{1}{r'_k}\right)}_{\text{self inductance } L_{kk} \text{ henrys/m}} i_k + \sum_{k'\neq k}\underbrace{\left(\frac{\mu_0}{2\pi}\ln\frac{1}{d_{kk'}}\right)}_{\text{mutual inductances } L_{kk'} \text{ henrys/m}} i_{k'}$$



Line characteristics

Series inductance l

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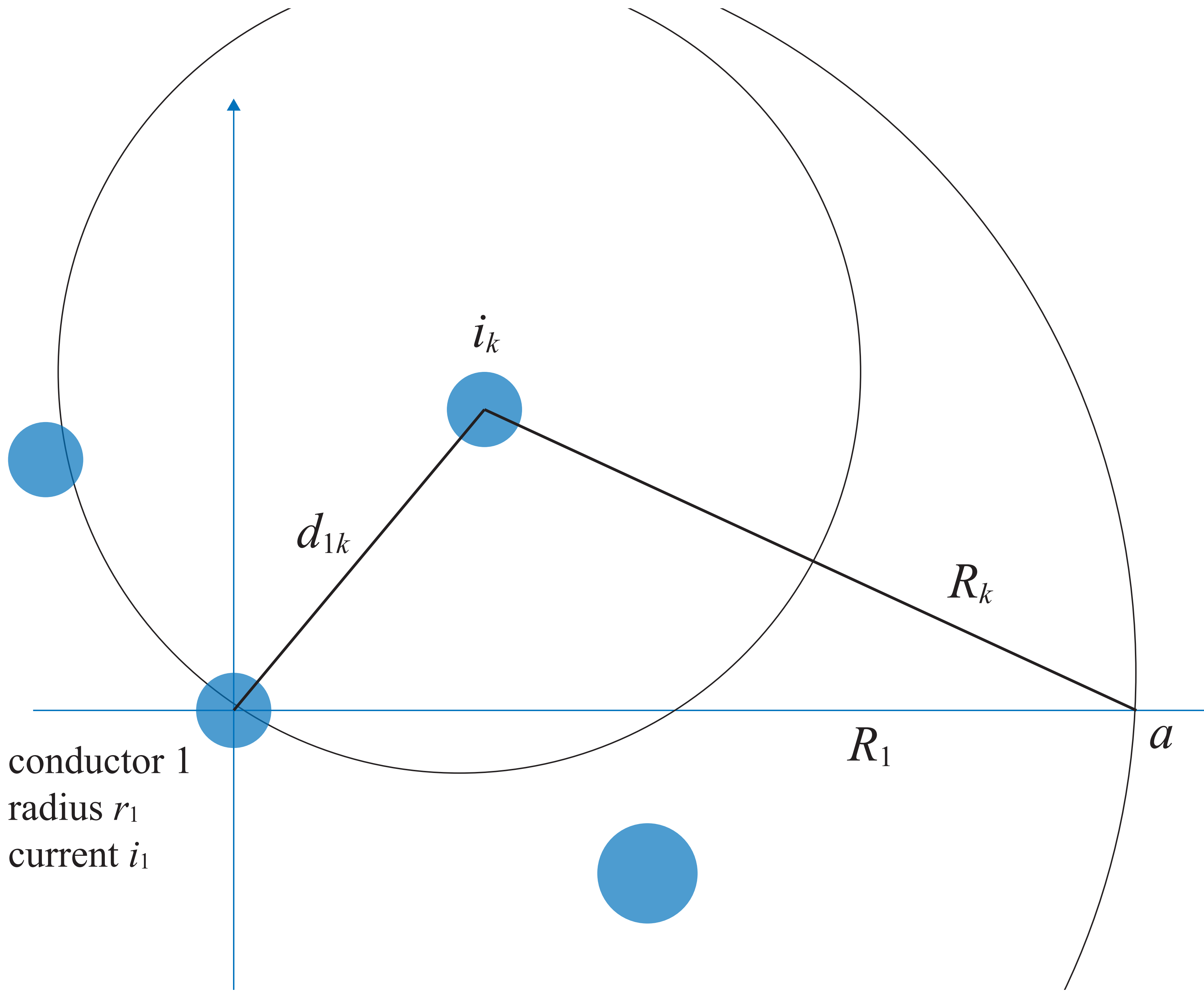
$$\lambda_k = \left(\frac{\mu_0}{2\pi}\ln\frac{1}{r'_k}\right) i_k + \sum_{k'\neq k}\left(\frac{\mu_0}{2\pi}\ln\frac{1}{d_{kk'}}\right) i_{k'}$$



In vector form: $\lambda = Li$

Faraday's law: $v(t) = \dfrac{d}{dt}\lambda(t) = L\dfrac{d}{dt}i(t)$

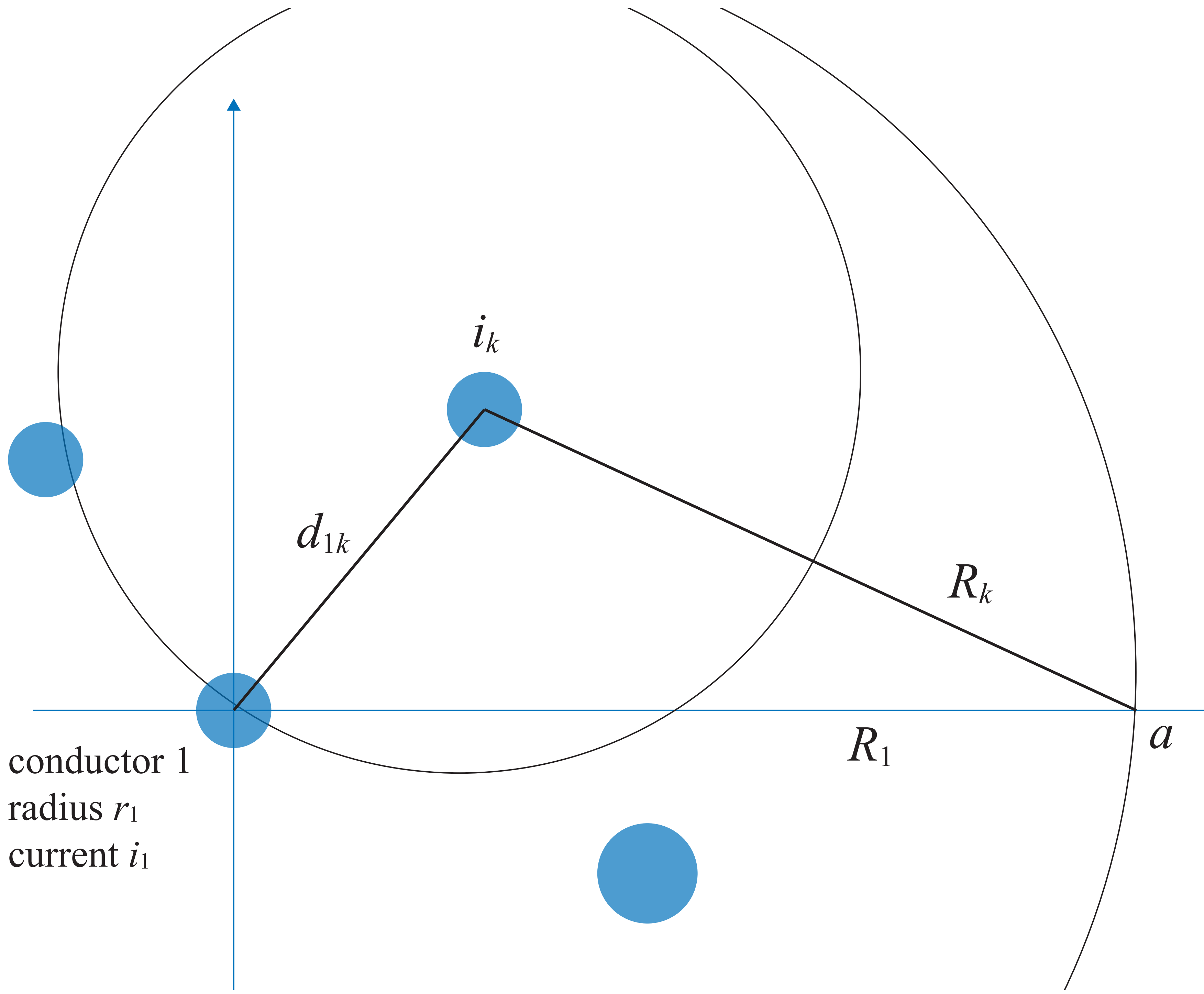
voltage drop along conductor

Line characteristics

Shunt capacitance c

Voltage on surface of conductor k relative to reference:

$$v_k = \left(\frac{1}{2\pi\epsilon}\ln\frac{1}{r_k}\right)q_k + \sum_{k'\neq k}\left(\frac{1}{2\pi\epsilon}\ln\frac{1}{d_{kk'}}\right)q_{k'}$$



Line characteristics

Shunt capacitance c

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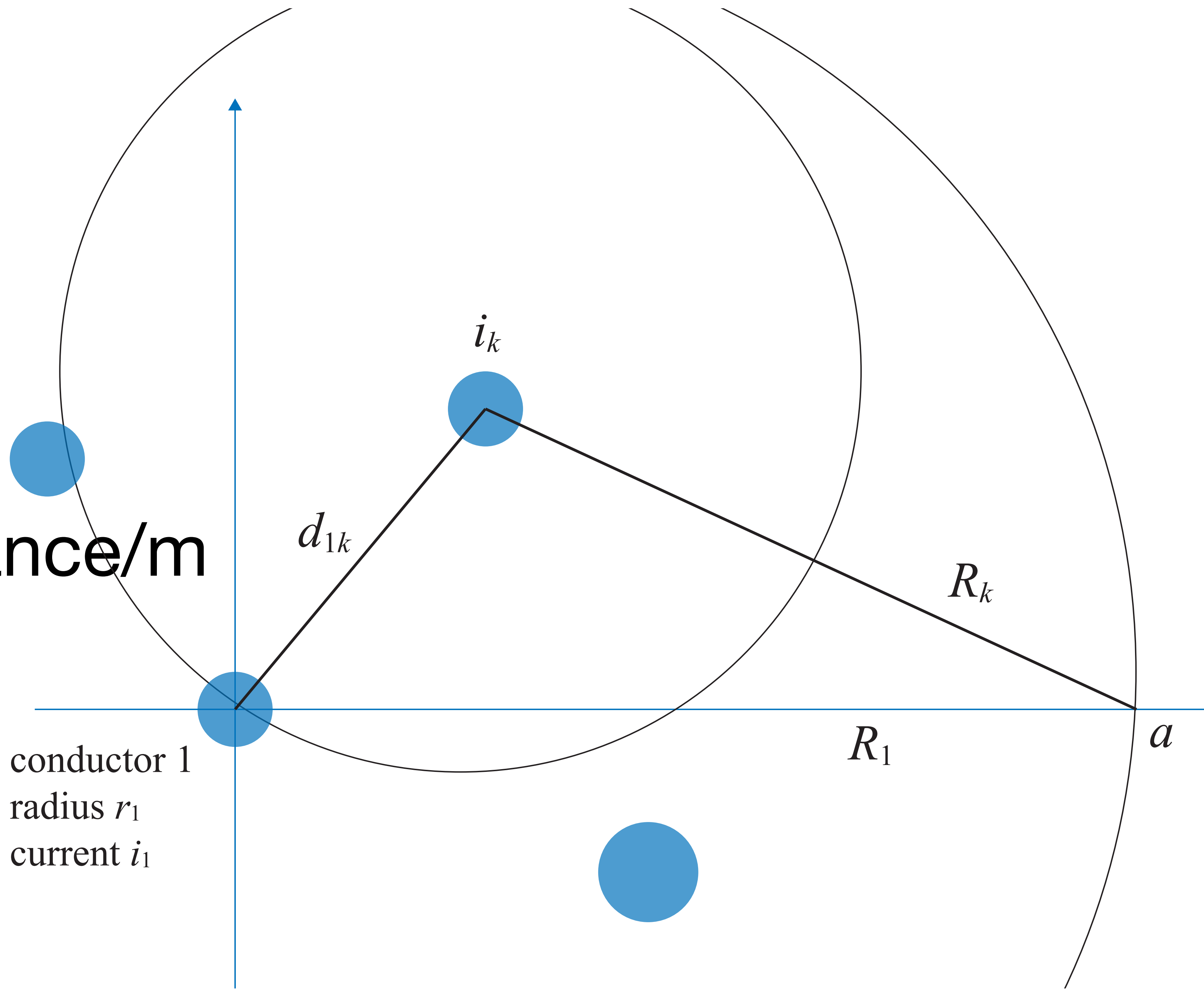
In vector form: $\quad v \;=\; F\,q$

Let $C := F^{-1}$. $\quad C_{kk}$: self capacitance/m, $C_{kk'}$: mutual capacitance/m

Therefore: $\quad i(t) \;=\; C\frac{d}{dt}v(t)$

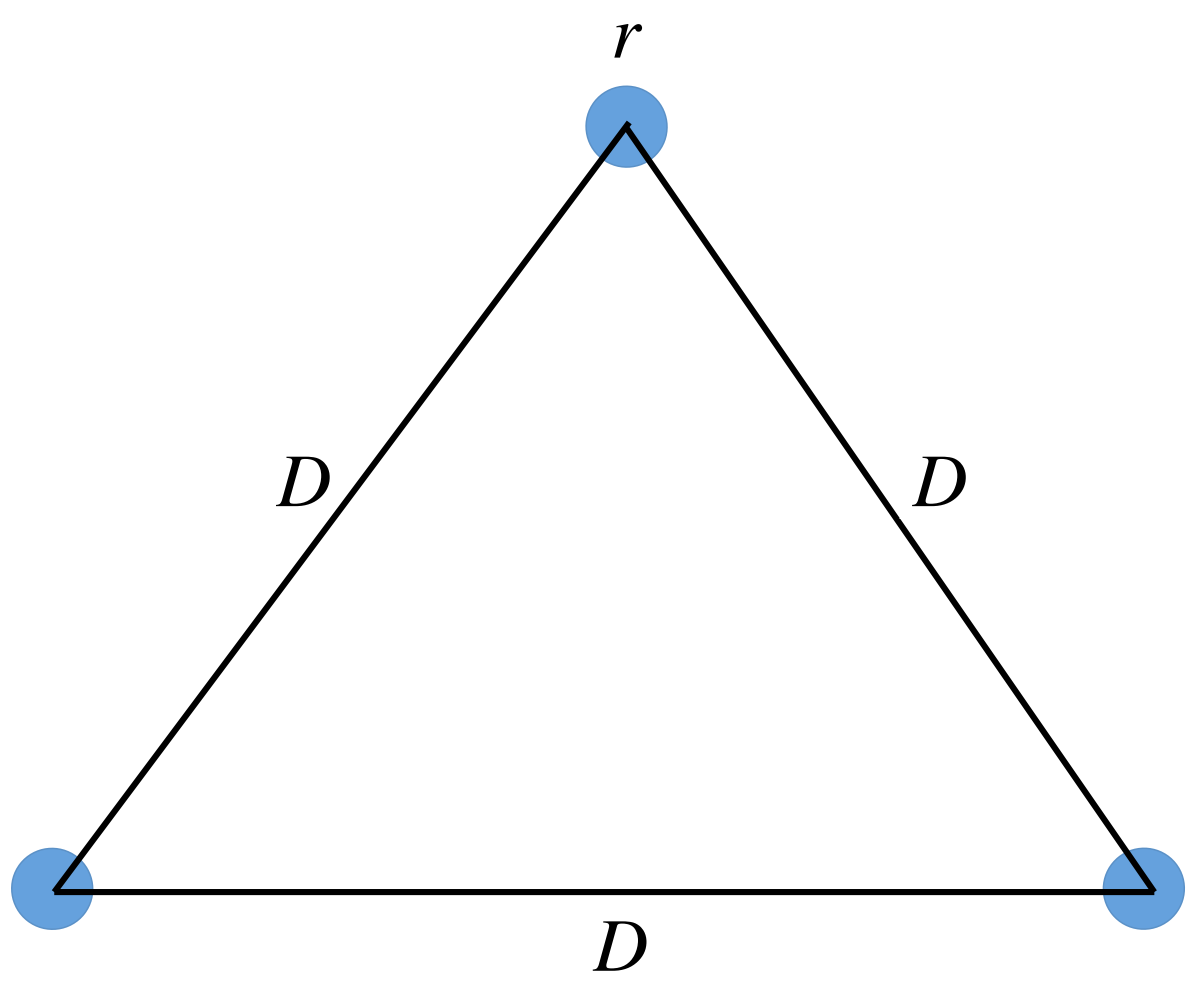


Line characteristics

Balanced three-phase line

Assumptions:

1. Conductors equally spaced at D with equal radii r
2. $i_1(t) + \cdots + i_n(t) \;=\; 0$ for all t
3. $q_1(t) + \cdots + q_n(t) \;=\; 0$ for all t



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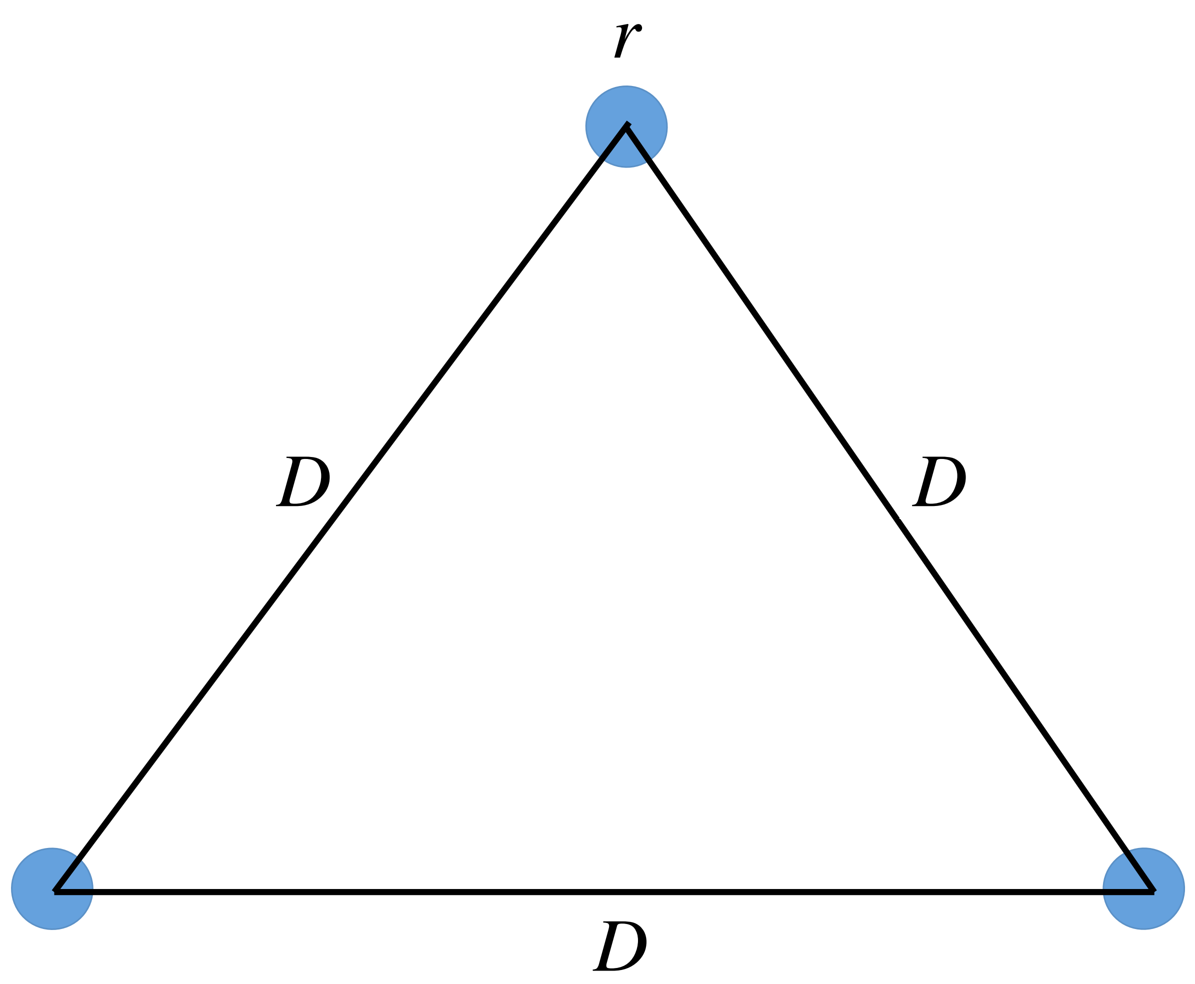

Phases are decoupled (vars of conductor k independent of vars of $k' \neq k$)

$$\lambda_k = \underbrace{\left(\frac{\mu_0}{2\pi} \ln \frac{D}{r'}\right)}_{\text{inductance } l \ \text{(H/m)}} i_k \qquad\qquad v_k = \underbrace{\left(\frac{1}{2\pi\epsilon} \ln \frac{D}{r}\right)}_{(\,\cdot\,)^{-1}\text{: capacitance } c \ \text{(F/m)}} q_k$$

Per-phase line characteristics (balanced)

$$z := r + i\omega l, \quad r > 0, l > 0$$

$$y := g + i\omega c, \quad g \geq 0, c > 0$$

Outline

1. Line characteristics
2. Line models
    - Transmission matrix
    - Π circuit model
    - Real and reactive line losses
    - Special cases: lossless line, short lossless line

Balanced three-phase line

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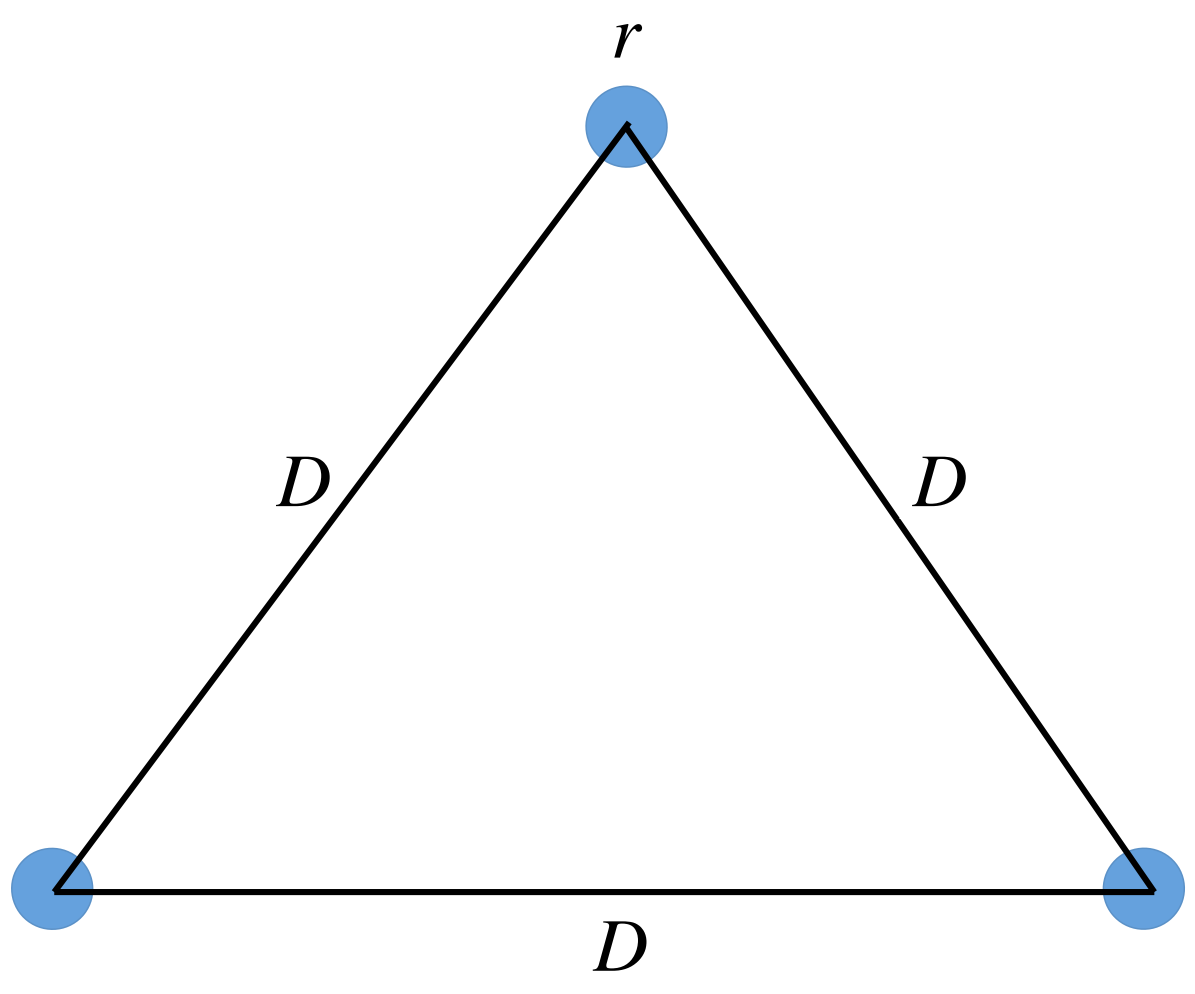


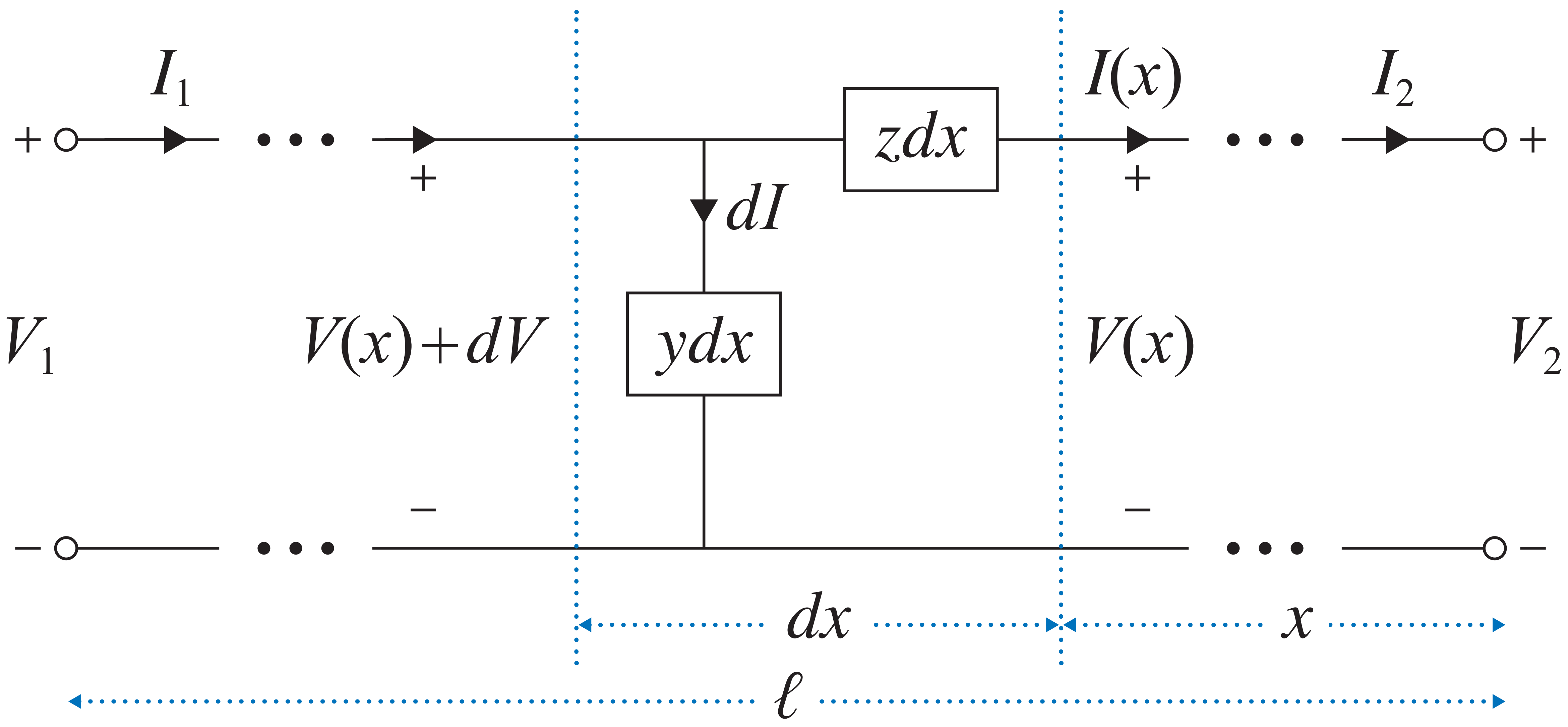
Per-phase line characteristics

- series impedance / meter $\quad z \;:=\; r + i\omega l \quad \Omega/\text{m} \quad r > 0,\ l > 0$
- shunt admittance / meter to neutral $\quad y \;:=\; g + i\omega c \quad \Omega^{-1}/\text{m} \quad g \geq 0,\ c > 0$

Next: use line parameter (z, y) to model end-to-end behavior of per-phase line (transmission matrix)

Transmission matrix

Distributed element model



$$dV = zI(x)\ dx$$

$$dI = (V(x) + dV)y\ dx \approx yV(x)\ dx$$

ODE:

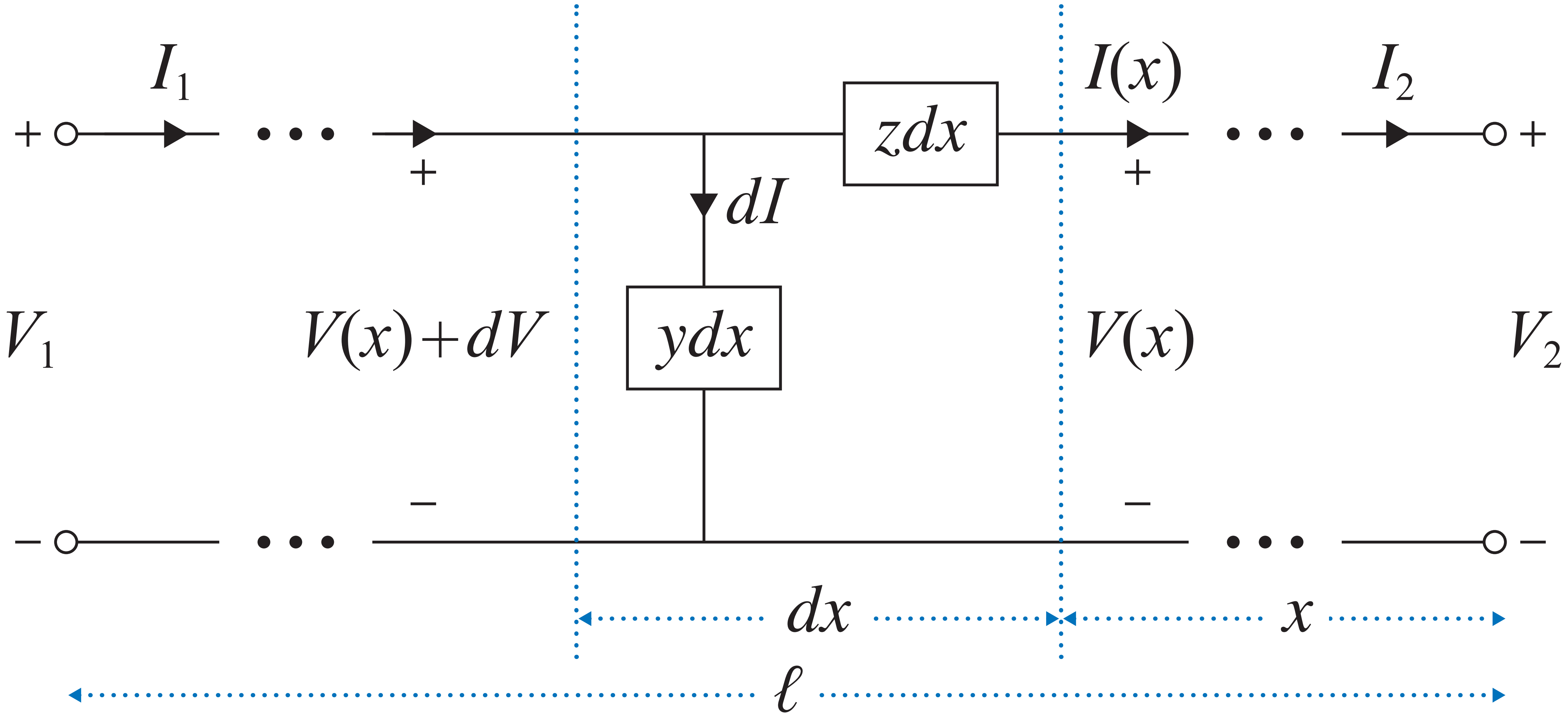
$$\frac{d}{dx}\begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} 0 & z \\ y & 0 \end{bmatrix}\begin{bmatrix} V \\ I \end{bmatrix}$$

boundary cond:

$$V(0) = V_2,\ I(0) = I_2$$

Transmission matrix

Distributed element model



$$\begin{bmatrix} V(x) \\ I(x) \end{bmatrix} = U \begin{bmatrix} e^{\gamma x} & 0 \\ 0 & e^{-\gamma x} \end{bmatrix} U^{-1} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

$$U := \begin{bmatrix} Z_c & -Z_c \\ 1 & 1 \end{bmatrix}, \quad U^{-1} := \frac{1}{2Z_c} \begin{bmatrix} 1 & Z_c \\ -1 & Z_c \end{bmatrix}$$

characteristic impedance $Z_c := \sqrt{\frac{z}{y}} \quad \Omega/m$

propagation constant $\gamma := \sqrt{zy} \quad m^{-1}$

Transmission matrix

Distributed element model

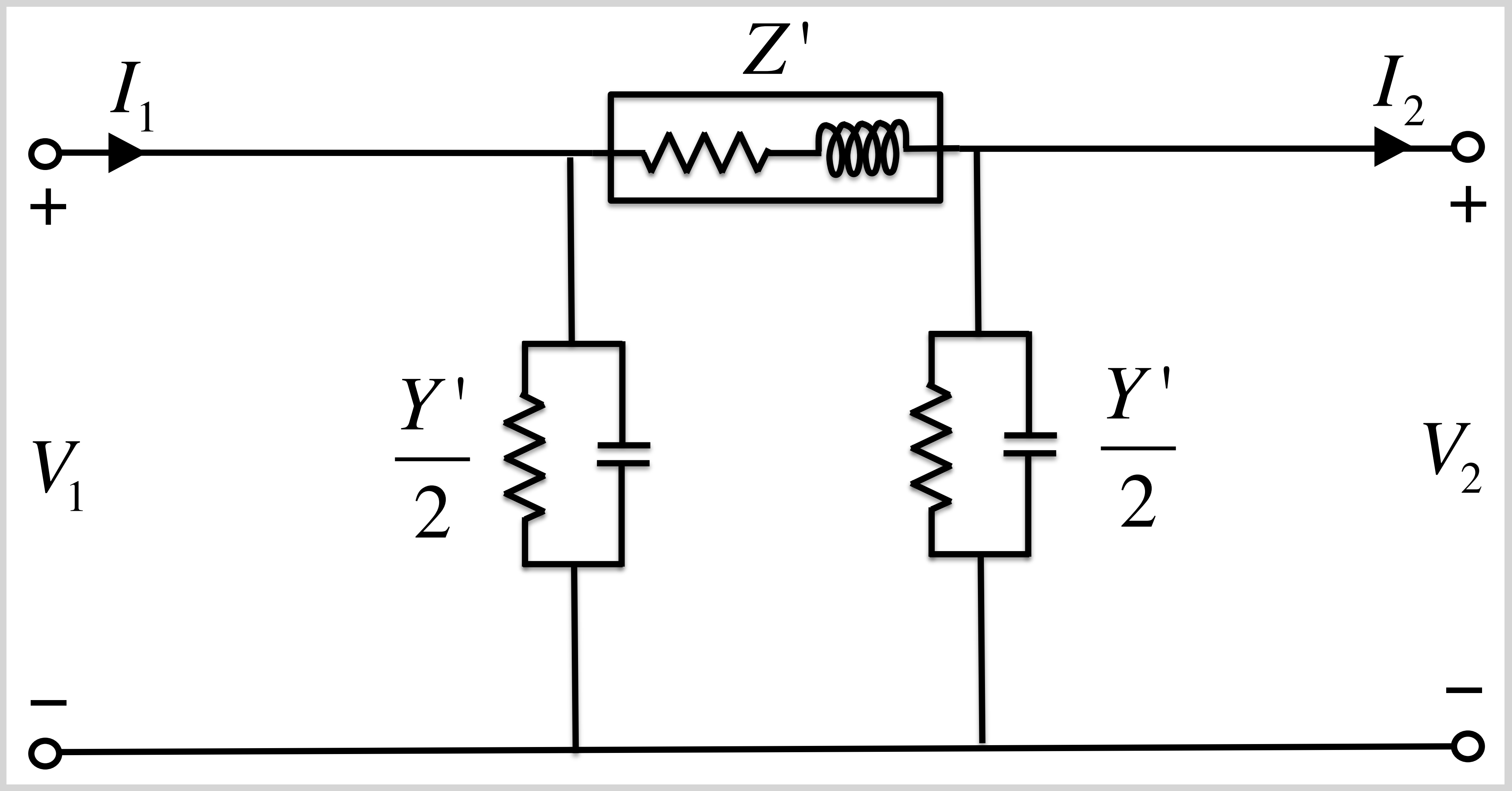
Transmission matrix maps receiving-end (V_2, I_2) to sending-end (V_1, I_1)

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh(\gamma\ell) & Z_c \sinh(\gamma\ell) \\ Z_c^{-1} \sinh(\gamma\ell) & \cosh(\gamma\ell) \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

characteristic impedance $Z_c \ := \ \sqrt{\frac{z}{y}} \quad \Omega/m$

propagation constant $\gamma \ := \sqrt{zy} \quad m^{-1}$

Π circuit model
Lumped element model



Transmission matrix

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} 1 + Z'Y'/2 & Z' \\ Y'(1 + Z'Y'/4) & 1 + Z'Y'/2 \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}$$

Admittance matrix

$$\begin{bmatrix} I_1 \\ -I_2 \end{bmatrix} = \begin{bmatrix} Z'^{-1} + Y'/2 & -Z'^{-1} \\ -Z'^{-1} & Z'^{-1} + Y'/2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

Long line ($\ell > 150$ miles) : use Z' and Y'

Medium line ($50 < \ell < 150$ miles) : use $Z = z\ell$ and $Y = i\omega C$

Short line ($\ell < 50$ miles) : use $Z = z\ell$ and $Y = 0$

Line current loss

Sending-end current

$$I_{12} = \frac{1}{Z'}(V_1 - V_2) + \frac{Y'}{2}V_1$$

$$I_{21} = \frac{1}{Z'}(V_2 - V_1) + \frac{Y'}{2}V_2$$

$$\left(I_{12} = I_1,\ I_{21} = -I_2\right)$$

Line current loss

Sending-end current

$$I_{12} = \frac{1}{Z'}(V_1 - V_2) + \frac{Y'}{2} V_1$$

$$I_{21} = \frac{1}{Z'}(V_2 - V_1) + \frac{Y'}{2} V_2$$

$$\left(I_{12} = I_1,\ I_{21} = -I_2\right)$$

Line current loss

$$I_{12} + I_{21} = \frac{Y'}{2}\left(V_1 + V_2\right)$$

If $Y' = 0$ then $I_{12} = -I_{21}$ sending current = receiving current

Line power loss

Sending-end power

$$S_{12} := V_1 \bar{I}_{12} = \frac{1}{\bar{Z}'}\left(|V_1|^2 - V_1\bar{V}_2\right) + \frac{\bar{Y}'}{2}|V_1|^2$$

$$S_{21} := V_2 \bar{I}_{21} = \frac{1}{\bar{Z}'}\left(|V_2|^2 - V_2\bar{V}_1\right) + \frac{\bar{Y}'}{2}|V_2|^2$$

Real and reactive power losses

$$S_{12} + S_{21} = Z'|I_{12}^s|^2 + \frac{\bar{Y}'}{2}\left(|V_1|^2 + |V_2|^2\right)$$

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2. Line models
    - Transmission matrix
    - Π circuit model
    - Real and reactive line losses
    - Special cases: lossless line, short lossless line

Special cases

Per-phase transmission line

general per-phase line

$z := r + i\omega l, \quad r > 0, l > 0$

$y := g + i\omega c, \quad g \geq 0, c > 0$

lossless line

$r = g := 0$

short lossless line

$r := 0, y := 0$

Lossless line: $r = g := 0$

Characteristic impedance is real

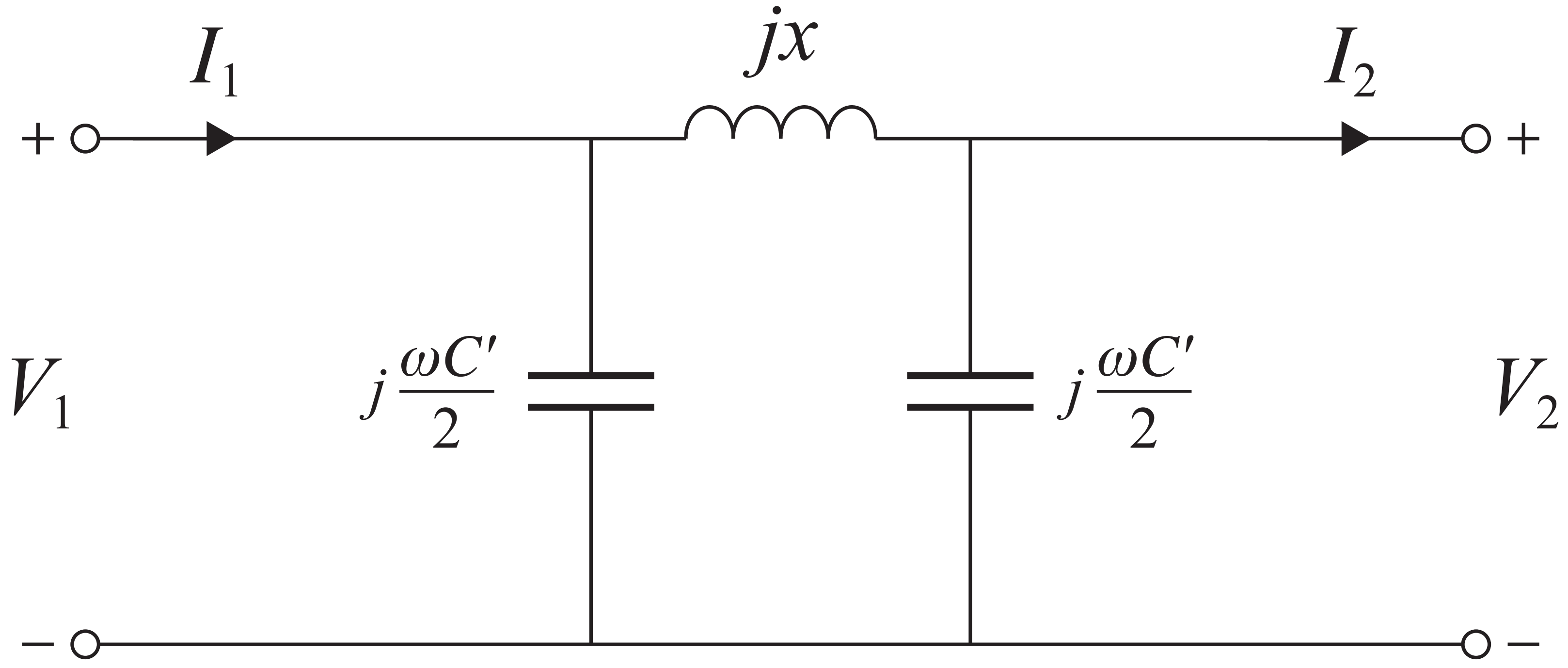
$$Z_c = \sqrt{\frac{z}{y}} = \sqrt{\frac{i\omega l}{i\omega c}} = \sqrt{\frac{l}{c}} \quad \Omega$$

Propagation constant is imaginary

$$\gamma = \sqrt{zy} = \sqrt{(i\omega l)(i\omega c)} = i\omega\sqrt{lc} \quad m^{-1} \qquad \beta := \omega\sqrt{lc}$$

Π circuit model: both series impedance and shunt admittance are reactive:

$$Z' = i\, Z_c \sin(\beta\ell) \quad \Omega, \qquad \frac{Y'}{2} = i\frac{\omega c\ell}{2}\,\frac{\tan(\beta\ell/2)}{\beta\ell/2} \quad \Omega^{-1}$$

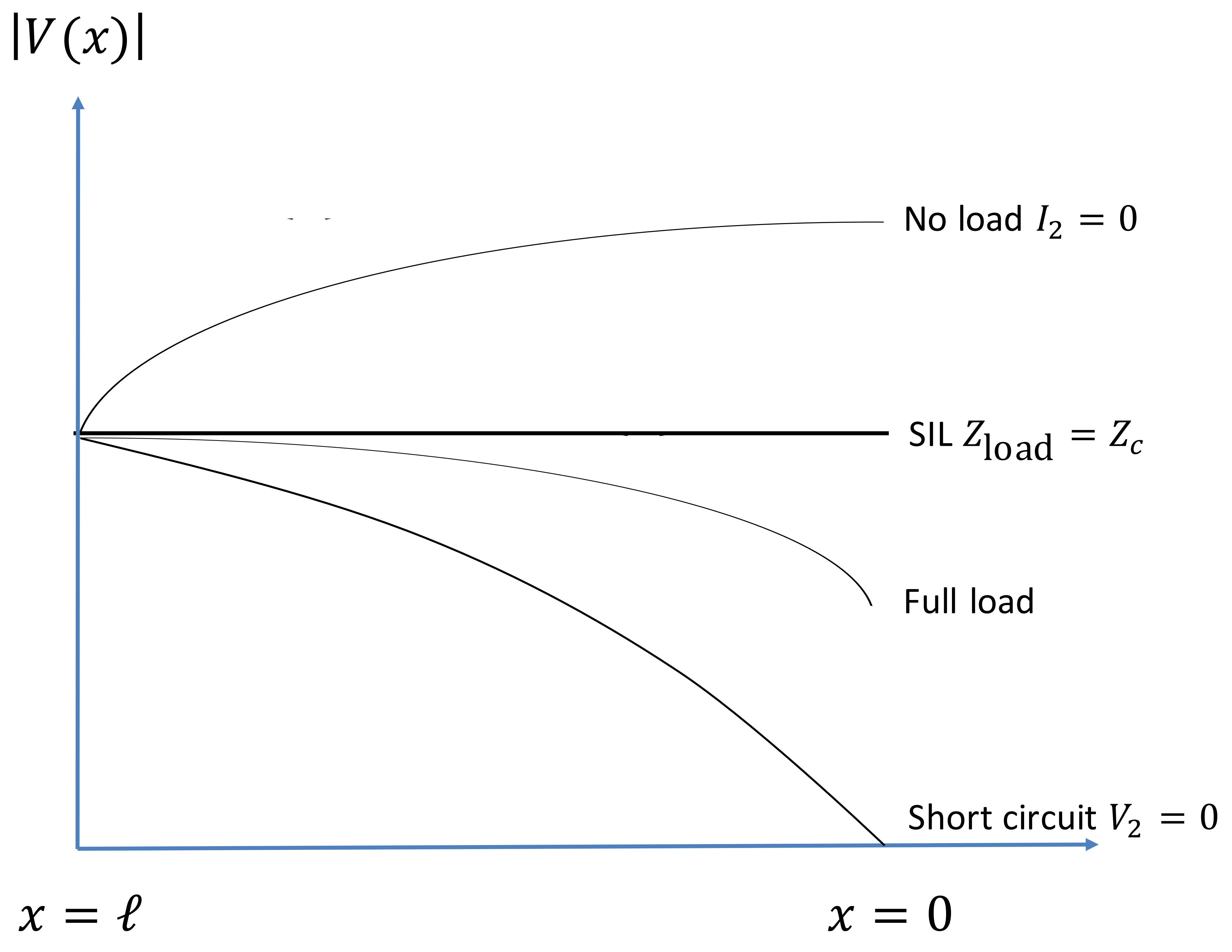


Lossless line: $r = g := 0$

Voltage along the line

$$V(x) = V_2 \cos(\beta x) + i Z_c I_2 \sin(\beta x)$$

$$\beta := \omega\sqrt{lc}$$



Generally voltage drops along the line towards load

Short lossless line: $r := 0, y := 0$

Sending-end power from i to j:

$$S_{ij} \;=\; V_i \bar{I}_{ij} \;=\; V_i \, \frac{\bar{V}_i - \bar{V}_j}{iX} \;=\; i\frac{1}{X}\left(|V_i|^2 - V_i \bar{V}_j \right)$$

Explore 3 implications

- How does load voltage magnitude $|V_2|$ depend on active load power P_{load} ?
- Decoupling: P mainly depends on θ, $|V|$ on Q
- Linear model: DC power flow

Short lossless line: $r := 0, y := 0$

Sending-end power from i to j:

$$S_{ij} \;=\; V_i \bar{I}_{ij} \;=\; V_i \,\frac{\bar{V}_i - \bar{V}_j}{iX} \;=\; i\frac{1}{X}\left(|V_i|^2 - V_i \bar{V}_j\right)$$

Receiving-end load power at bus 2:

$$-S_{21} = -V_2 \bar{I}_{21} \;= -i\frac{1}{X}\left(|V_2|^2 - V_2 \bar{V}_1\right)$$

Suppose: $-S_{21}$ supplies a load with load power $P_{\text{load}} + iQ_{\text{load}}$, i.e.,

$$-S_{21} = \; P_{\text{load}}(1 + i\tan\phi)$$

$\phi := \theta_{V_2} - \theta_{-I_{21}}$: load power factor angle

Short lossless line: $r := 0, y := 0$

Load voltage solution and collapse

How does load voltage $|V_2|$ depend on active load power P_{load} ?

$$-i\frac{1}{X}\left(|V_2|^2 - V_2\bar{V}_1\right) = P_{\text{load}}(1 + i\tan\phi)$$

Assume: $V_1 := 1\angle 0^\circ \Rightarrow \theta_{21} := \theta_2 - \theta_1 = \theta_2$

Then

- 2 real equations in $(|V_2|, \theta_2)$ with P_{load} as parameter
- Solve for load voltage $|V_2|$ given any P_{load}
- As load power P_{load} increases, solutions $|V_2|$ trace out a nose curve
- If P_{load} increases further, no real solutions for $|V_2|$ exists - voltage collapse

Short lossless line: $r := 0, y := 0$

Sending-end power from i to j:

$$S_{ij} \;=\; V_i \bar{I}_{ij} \;=\; V_i \, \frac{\bar{V}_i - \bar{V}_j}{iX} \;=\; i\frac{1}{X}\left(|V_i|^2 - V_i \bar{V}_j \right)$$

Hence

$$P_{12} = \frac{|V_1||V_2|}{X} \sin\theta_{12}$$

$$Q_{12} = \frac{1}{X}\left(|V_1|^2 - |V_1||V_2|\cos\theta_{12} \right)$$

$$Q_{21} = \frac{1}{X}\left(|V_2|^2 - |V_1||V_2|\cos\theta_{12} \right)$$

Short lossless line: $r := 0, y := 0$

Decoupling

1. P_{12} and $|V_i|$ are roughly decoupled, Q_{ij} and θ_{12} are roughly decoupled

$$\frac{\partial P_{12}}{\partial |V_i|} = \frac{|V_j|}{X} \sin\theta_{12} \approx 0 \qquad \frac{\partial Q_{ij}}{\partial \theta_{12}} = \frac{|V_1||V_2|}{X} \sin\theta_{12} \approx 0$$

2. P_{12} mainly depends on θ_{12} with rate

$$\frac{\partial P_{12}}{\partial \theta_{12}} = \frac{|V_1||V_2|}{X} \cos\theta_{12} \approx \frac{|V_1||V_2|}{X}$$

3. Voltage regulation

$$\frac{\partial Q_{12}}{\partial |V_2|} = -\frac{|V_1|}{X} \cos\theta_{12} < 0 \qquad \frac{\partial Q_{21}}{\partial |V_2|} = \frac{1}{X}\left(2|V_2| - |V_1|\cos\theta_{12}\right) > 0$$

To maintain high load voltage $|V_2|$:

decrease sending-end Q_{12}, increase load injection Q_{21}

Short lossless line: $r := 0, y := 0$

Linear model

DC power flow model: $R = 0$, fixed $|V_i|$, $\sin\theta_{12} \approx \theta_{12}$, ignore Q_{ij}

$$P_{ij} = \frac{|V_1||V_2|}{X}\,\theta_{12} \;=:\; b_{12}(\theta_1 - \theta_2)$$

- Widely used for electricity market and long-time planning applications
- Reasonable model for transmission system apps, not for distribution system apps where r is high