GP49 - THERMAL WAVES IN A BAR

01/02/2021

REMOTE

* numbers (#) represent equations in the lab script

4 THEORETICAL ANALYSIS

Thermal conductivity $K_{copper} = 401\,Wm^{-1}K^{-1}$

Heat conduction equation $\Rightarrow P(x) = -KA\left(\frac{\partial T}{\partial x}\right)$ (1)

$\Rightarrow P(x+\delta x) = P(x) + \partial P/\partial x \, \delta x$ (2)

$\Rightarrow \frac{\partial P}{\partial x} = -KA\frac{\partial^2 T}{\partial x^2}$ (3)

ρ = density

Heat balance equation $\Rightarrow C\rho A\delta x\left(\frac{\partial T}{\partial t}\right) = -\delta P = KA\frac{\partial^2 T}{\partial x^2}\delta x$ (4) C = specific heat capacity

$\Rightarrow \boxed{\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}}$ (5) D = thermal diffusivity $= K/\rho C$

(@25°C)

For copper $\rightarrow$ $C_c = 385 \pm 1.0\,Jkg^{-1}K^{-1}$ $\rho_c = 8960.0 \pm 10.0\,kgm^{-3}$

$D_{copper} = K_c/C_c\rho_c$ $\sigma_D^2 = \left(\frac{\partial D}{\partial K}\right)^2\sigma_K^2 + \left(\frac{\partial D}{\partial \rho}\right)^2\sigma_\rho^2 + \left(\frac{\partial D}{\partial C}\right)^2\sigma_C^2$

$\boxed{D_{copper} = 1.16\times10^{-4}\,m^2s^{-1}\ (3.s.f)}$
$\boxed{\sigma_D = 3.286\times10^{-7}\ (4.s.f)}$

(assuming $\sigma_K = 0$)

UNITS

$\frac{Wm^{-1}K^{-1}}{kgm^{-3}Jkg^{-1}K^{-1}} = \frac{Wm^2}{J} = m^2s^{-1}$

4.1

The solution to this equation (5) is:

$T(x,t) = T_0 e^{-\alpha x} e^{i(wt-kx+\phi)}$ (6) (ϕ is chosen to satisfy initial conditions)

where: $\alpha = k = (w/2D)^{1/2}$ (10) (α is attenuation coefficient)

Phase velocity: $v_p = w/k$ (13)

$v_p = (2Dw)^{1/2} = (4D\pi f)^{1/2}$ $\alpha = (\pi f/D)^{1/2}$ $\Rightarrow \boxed{(v_p),(\alpha) \propto (f^{1/2})}$ proportional

$k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k} = 2\left(\frac{\pi D}{f}\right)^{1/2} = 2\sqrt{\pi D\tau}$

$\sigma_\lambda^2 = \left(\frac{\partial\lambda}{\partial D}\right)^2\sigma_D^2$ (assuming $\sigma_\tau = 0$)

$\boxed{\lambda = 0.296\,m\ (3.s.f) \qquad \sigma_\lambda = 4.19\times10^{-4}\,m\ (3.s.f)}$

$A \propto e^{-\alpha x}$. Let $A_0 = e^{-\alpha x_0}$, $\frac{A_0}{e} = e^{-\alpha(x+\Delta x)}$ $\Rightarrow \frac{1}{e} = e^{-\alpha\Delta x}$

$\Rightarrow \Delta x = 1/\alpha = \lambda/2\pi$

$\Rightarrow \boxed{\Delta x = 4.71\times10^{-2}\,m \qquad \sigma_{\Delta x} = 6.67\times10^{-5}\,m\ (3.s.f) \qquad \text{[SKIN DEPTH]}}$

4.2

We are now considering heat lost to the surroundings from the sides :

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2} - L(T - T_s) \quad (15)$$

L = heat loss constant
T_s = const surrounding temperature

This has solutions : $\theta = \theta_0 e^{-\alpha x} e^{i(\omega t - kx)}$, $\theta = T - T_s$ (16)

where : $(\alpha^2 - k^2) = L/D$ and $\alpha k = \omega/2D$ (18)

$$\alpha^2 = \frac{1}{2D}\left(L + \sqrt{L^2+\omega^2}\right) \quad (19)$$

(+ve root so no -ve in)

PHASE VELOCITY : $v_p = \omega/k \Rightarrow \boxed{v_p = \sqrt{2D(L+\sqrt{L^2+\omega^2})}}$

GROUP VELOCITY : $v_g = \frac{d\omega}{dk}$.

$$k^2 = \alpha^2 - L/D = \frac{1}{2D}\left(-L + \sqrt{L^2+\omega^2}\right)$$

$$2k \frac{dk}{d\omega} = \frac{\omega}{2D}(L^2+\omega^2)^{-1/2}$$

$$\Rightarrow \boxed{v_g = \frac{4kD\sqrt{L^2+\omega^2}}{\omega}}$$

$$v_p/v_g = \frac{(2D(L+(L^2+\omega^2)^{1/2}))^{1/2}}{4kD(L^2+\omega^2)^{1/2}\omega^{-1}} = \frac{L+(L^2+\omega^2)^{1/2}}{2(L^2+\omega^2)^{1/2}} < 1 \quad \forall \omega \neq 0$$

$\Rightarrow v_p/v_g < 1 \Rightarrow \boxed{v_p < v_g}$ where as in a light wave $v_g > v_p$

In a vacuum $v_p v_g = c^2$

In a good conductor $k = \sqrt{\frac{\mu\omega\sigma}{2}} \Rightarrow v_p = \sqrt{\frac{2\omega}{\mu\sigma}}$ $v_g = 2\sqrt{\frac{2\omega}{\mu\sigma}}$

$$\Rightarrow v_p v_g = \frac{4\omega}{\mu\sigma}$$

From above $v_p v_g = 4D\sqrt{L^2+\omega^2}$

For $L=0$ and $D = \frac{1}{\mu\sigma}$ our results are the same

UNDERGROUND PIPE

$T_{pipe} = T_0 + T_A \sin(\omega t - \alpha x) e^{-\alpha x}$

For yearly fluctuations of :

$T_0 \approx 10°C$ $T_A \approx 15°C$ (met office)

Fluctuations $= T_A e^{-\alpha x}$

$\Rightarrow 0°C < 10°C - 15°C\, e^{-\alpha x}$

$\frac{10}{15} < e^{-\alpha x} \Rightarrow \boxed{x > 0.834 \text{ m } (3.s.f)}$

For damp soil

In UK $T_{min} = -10°C = 263K$

Period = 1 year

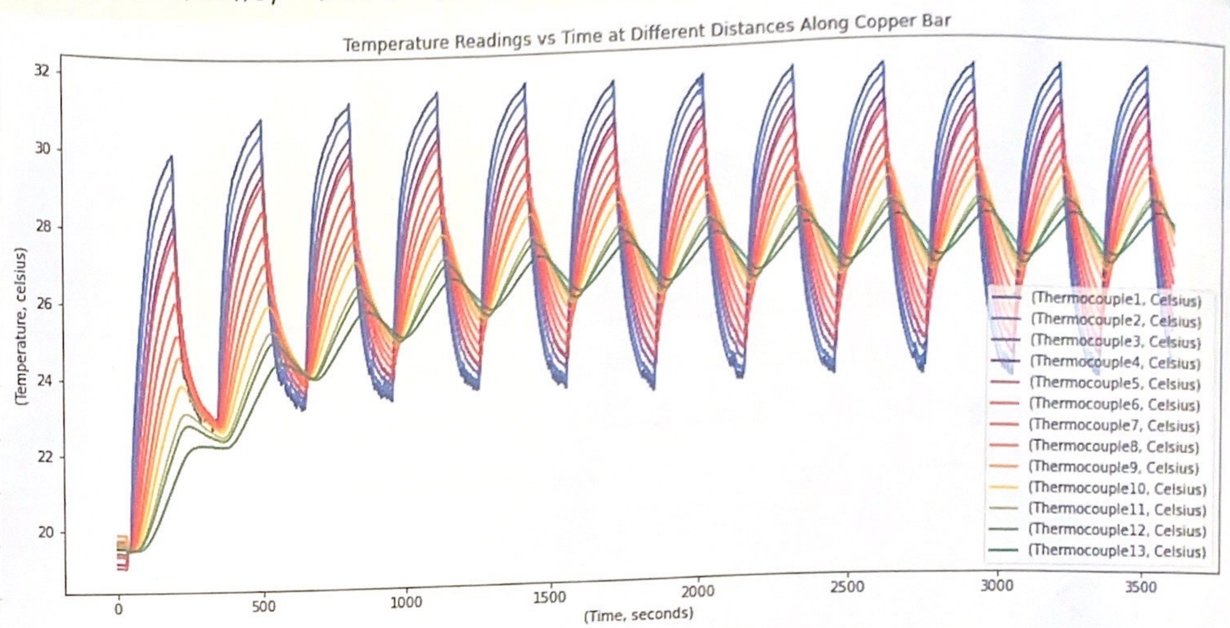
$K \approx 1.0 \text{ Wm}^{-1}\text{K}^{-1}$

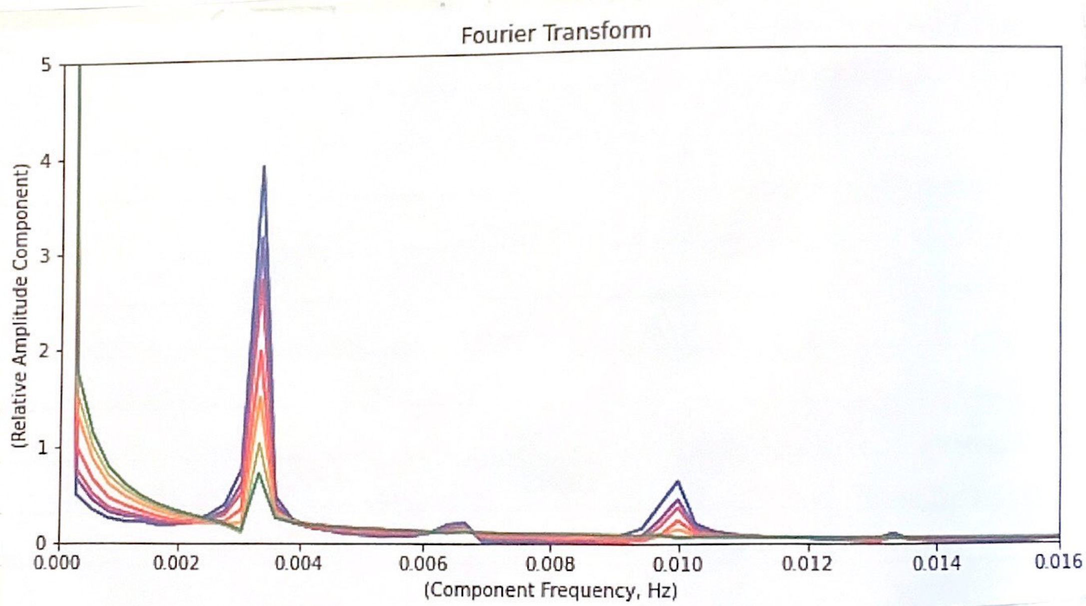
$\rho \approx 1600 \text{ kg m}^{-3}$

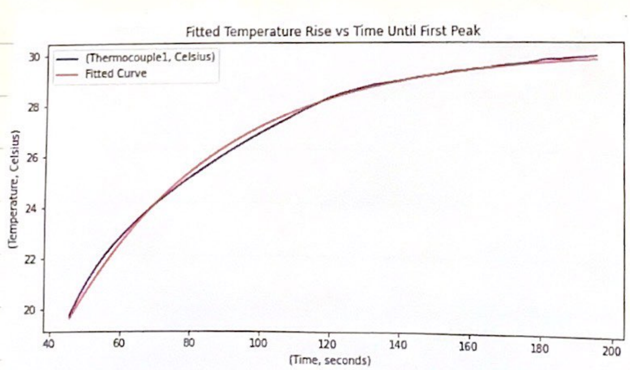
$C \approx 1480 \text{ J kg}^{-1}\text{K}^{-1}$

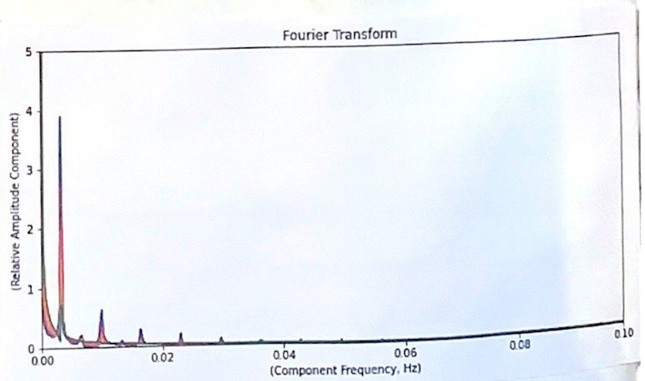
$\Rightarrow \alpha = 0.486$ (3.s.f)

FOR PRELIMINARY DATA OF SQUARE WAVE, PERIOD = 10 mins









$\alpha = 2.30 \times 10^{-2} \pm 1.66 \times 10^{-8}$ (3.s.f)

For $A(1 - e^{-\alpha t})$

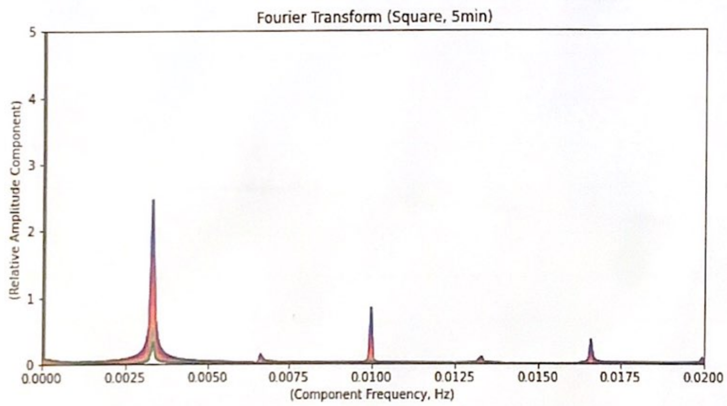
(zoomed out graph of above)

5.2

For preliminary data of a square wave with period 10 minutes:
The thermocouples $1 \rightarrow 13$ represent distances along the bar, with 1 being the closest to the heat source.
For the higher thermocouples the wave approaches a sine wave.
As shown in the fourier transform, by thermocouple 8, higher frequencies have decayed, leaving the fundamental sine wave.

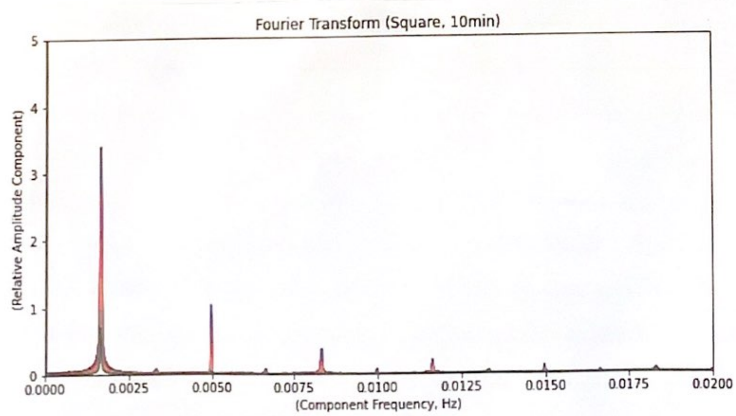
Fitting $A(1-e^{-\alpha t})$ to the rise gives an α of 2.30×10^{-2} (3.s.f)

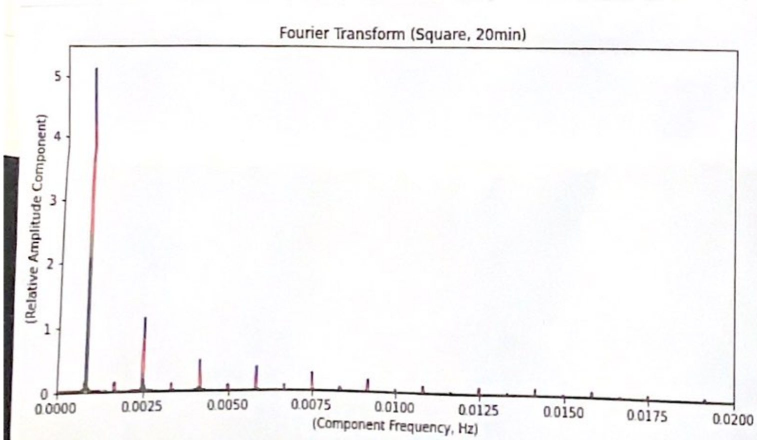
COMPARING TIME PERIODS OF SQUARE WAVES

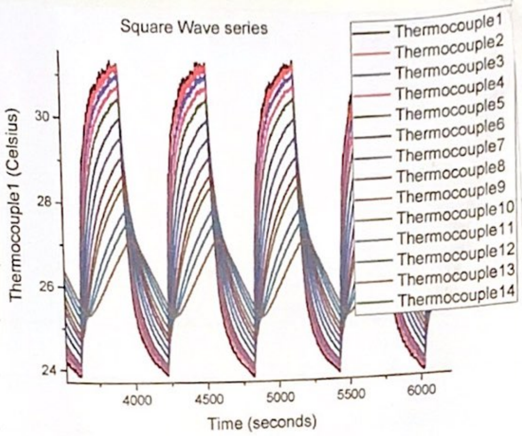


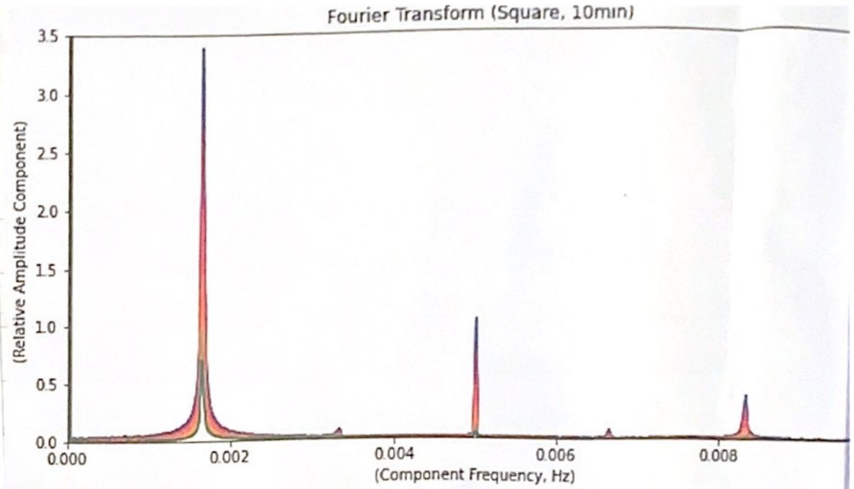
As period increases:

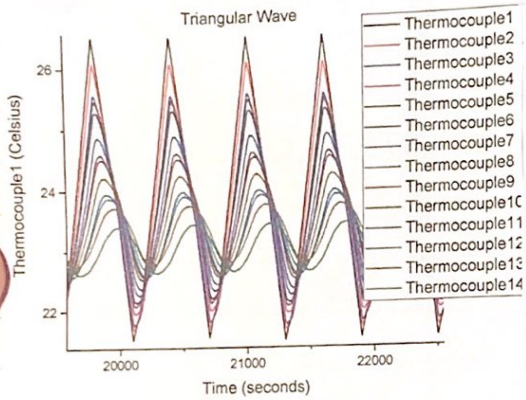
- Peaks get sharper
- Amplitude of first harmonic is higher
- Amplitudes decay faster
- Peaks get closer together

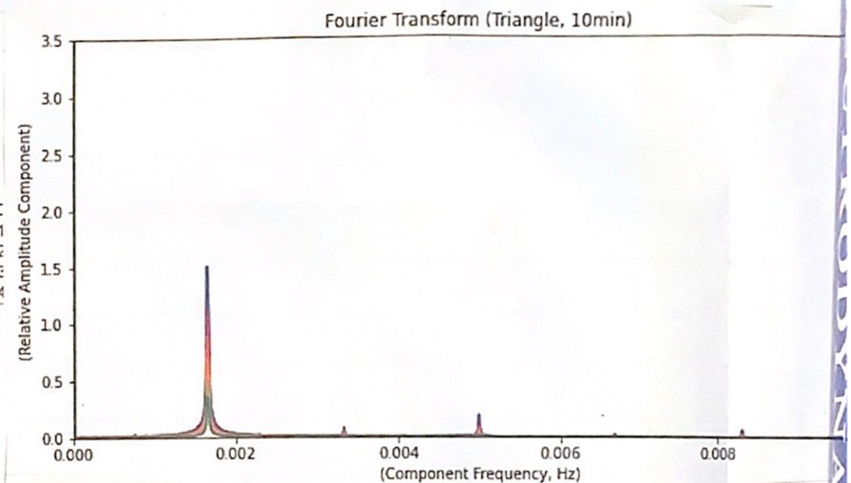


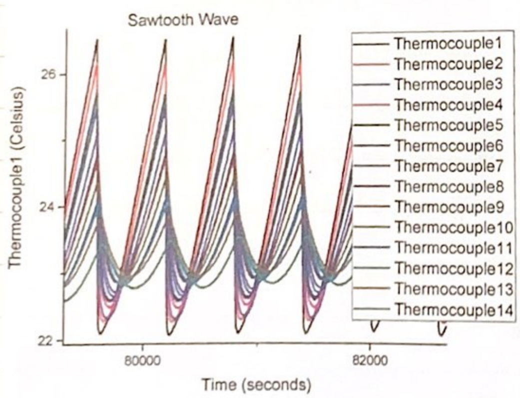


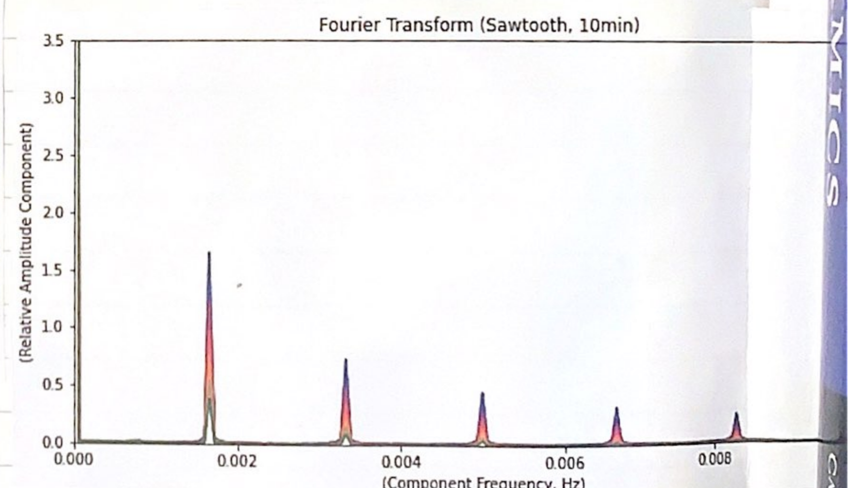






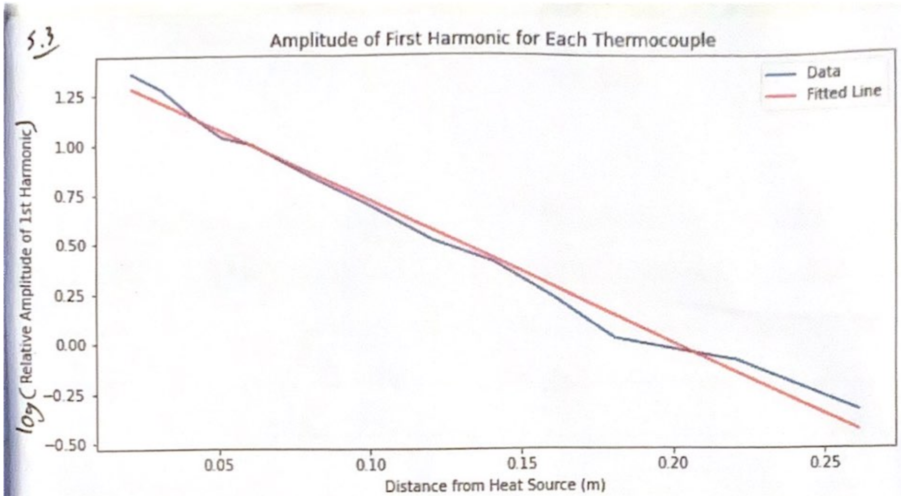


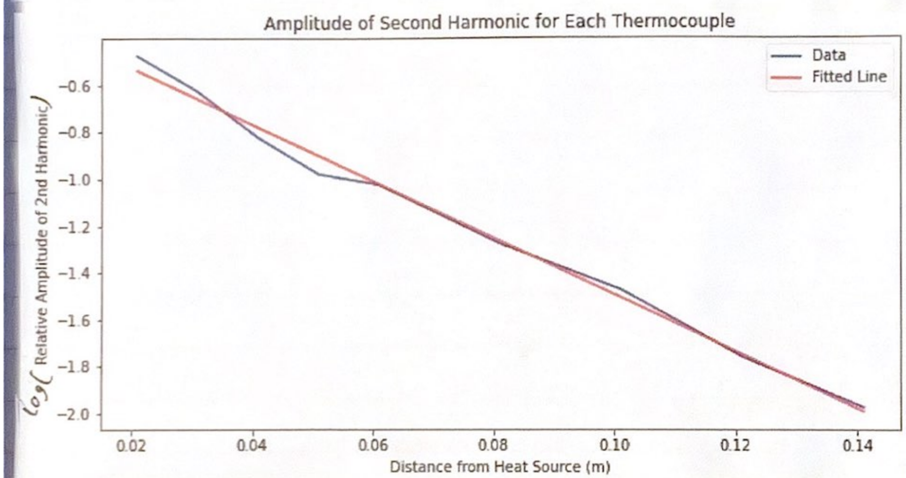


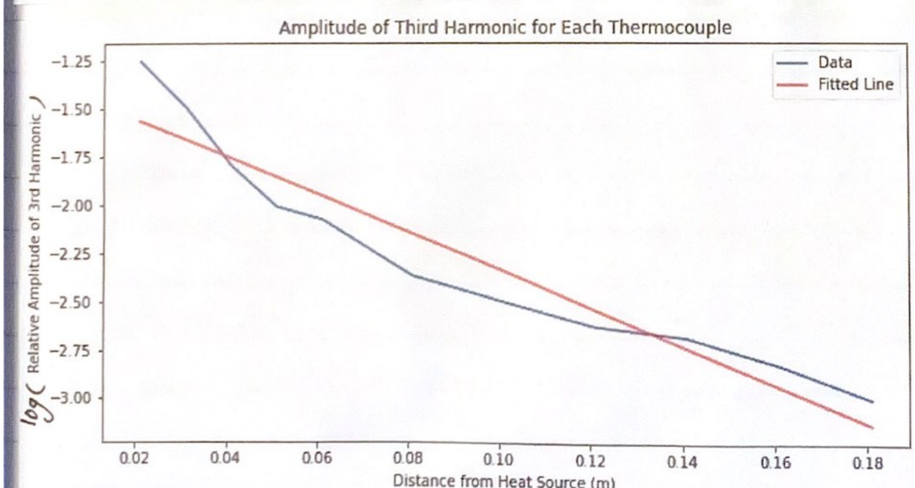


Square : Much higher amplitude of fundamental harmonic than triangle / sawtooth
Triangle : Higher harmonics decay much faster than square / sawtooth
Sawtooth : Higher harmonics have a bigger contribution than square / triangle

5.3







PRELIMINARY DATA

$\alpha = 7.16 \pm 0.0596$ (2dp)

$f = 0.00333$

$D = 2.04 \times 10^{-4}$

$\sigma_D = 3.40 \times 10^{-6}$

$K = 704\ Wm^{-1}K^{-1}$

$\sigma_K = 11.9\ Wm^{-1}K^{-1}$

$\alpha = 12.09 \pm 0.152$ (2.d.p)

$f = 0.01$

$D = 2.15 \times 10^{-4}$

$\sigma_D = 5.40 \times 10^{-6}$

$K = 738\ Wm^{-1}K^{-1}$

$\sigma_K = 18.7\ Wm^{-1}K^{-1}$

$\alpha = 9.78 \pm 1.028$ (2.d.p)

$f = 0.0164$

$D = 5.39 \times 10^{-4}\ m^2s^{-1}$

$\sigma_D = 1.13 \times 10^{-4}\ m^2s^{-1}$

$\boxed{\text{Massive error} \Rightarrow \text{outlier}}$

$K = 1859\ Wm^{-1}K^{-1}$

$\sigma_K = 390\ Wm^{-1}K^{-1}$ (3.s.f)

Gradient of graph $= -\alpha$

$D = \frac{2\pi f}{2\alpha^2} = \frac{\pi f}{\alpha^2}$ $\quad \sigma_D^2 = \left(\frac{2\pi f}{\alpha^3}\right)^2 \sigma_\alpha^2$

$K = D\rho C$ $\quad \sigma_K^2 = (\rho C)^2\sigma_D^2 + (D\rho)^2\sigma_C^2 + (DC)^2\sigma_\rho^2$

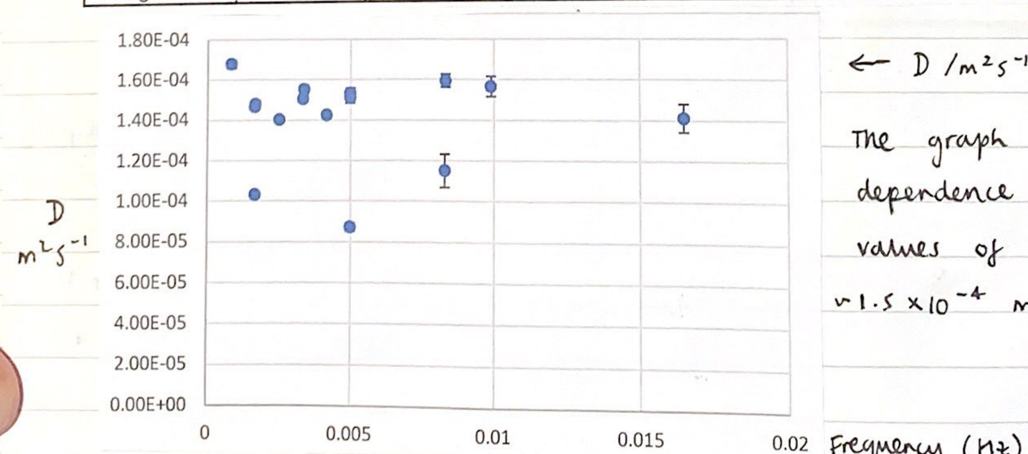
On next page have calculated for overnight square, triangle, sawtooth

OVERNIGHT DATA

5.3

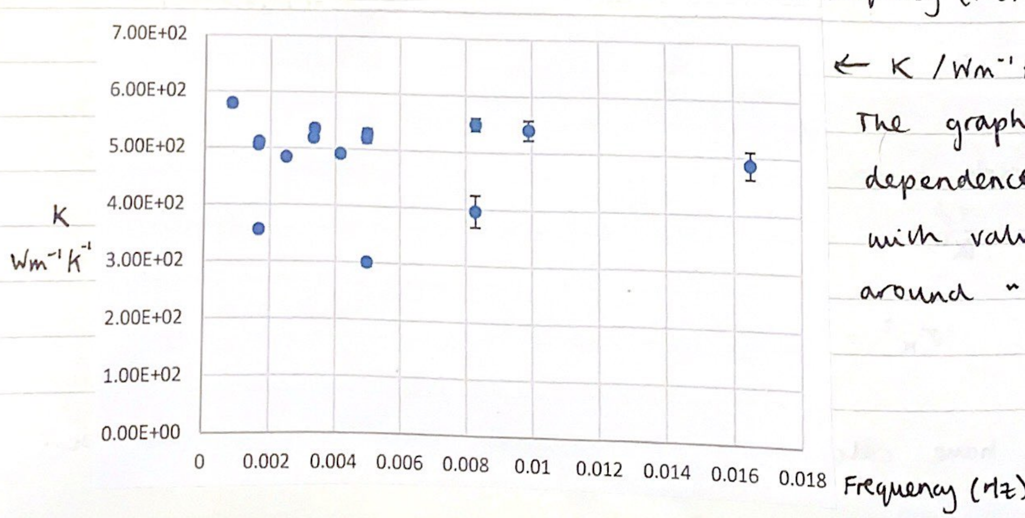
Below are values of the diffusivity and thermal conductivity, calculated as on the previous page from similar graphs for the overnight data.

Input Wave	Harmonic	f /Hz	α /m^{-1}	σ_α /m^{-1}	D /$m^2 s^{-1}$	σ_D /$m^2 s^{-1}$	K /$Wm^{-1}K^{-1}$	σ_K /$Wm^{-1}K^{-1}$
Square 5	0	0.00336	8.245	0.0536	1.55E-04	2.01889E-06	5.36E+02	7.13E+00
Square 5	1	0.00991	14.099	0.2241	1.57E-04	4.97887E-06	5.40E+02	1.72E+01
Square 5	2	0.0165	19.167	0.469	1.41E-04	6.90517E-06	4.87E+02	2.39E+01
Square 10	0	0.00167	7.1317	0.0481	1.03E-04	1.39143E-06	3.56E+02	4.90E+00
Square 10	1	0.00498	13.402	0.1492	8.71E-05	1.93941E-06	3.00E+02	6.74E+00
Square 10	2	0.0083	15.059	0.5345	1.15E-04	8.16239E-06	3.97E+02	2.82E+01
Square 20	0	0.00083	3.942	0.0246	1.68E-04	2.09432E-06	5.79E+02	7.41E+00
Square 20	1	0.0025	7.479	0.0304	1.40E-04	1.14147E-06	4.84E+02	4.17E+00
Square 20	2	0.00416	9.578	0.0566	1.42E-04	1.6837E-06	4.91E+02	5.97E+00
Sawtooth 10	0	0.00165	5.946	0.0293	1.47E-04	1.44496E-06	5.06E+02	5.19E+00
Sawtooth 10	1	0.00333	8.335	0.0486	1.51E-04	1.75608E-06	5.19E+02	6.23E+00
Sawtooth 10	2	0.00499	10.113	0.0806	1.53E-04	2.44329E-06	5.29E+02	8.56E+00
Triangle 10	0	0.00167	5.952	0.0289	1.48E-04	1.43815E-06	5.11E+02	5.17E+00
Triangle 10	1	0.00498	10.18	0.0898	1.51E-04	2.66344E-06	5.21E+02	9.30E+00
Triangle 10	2	0.00832	12.807	0.1294	1.59E-04	3.22029E-06	5.50E+02	1.12E+01



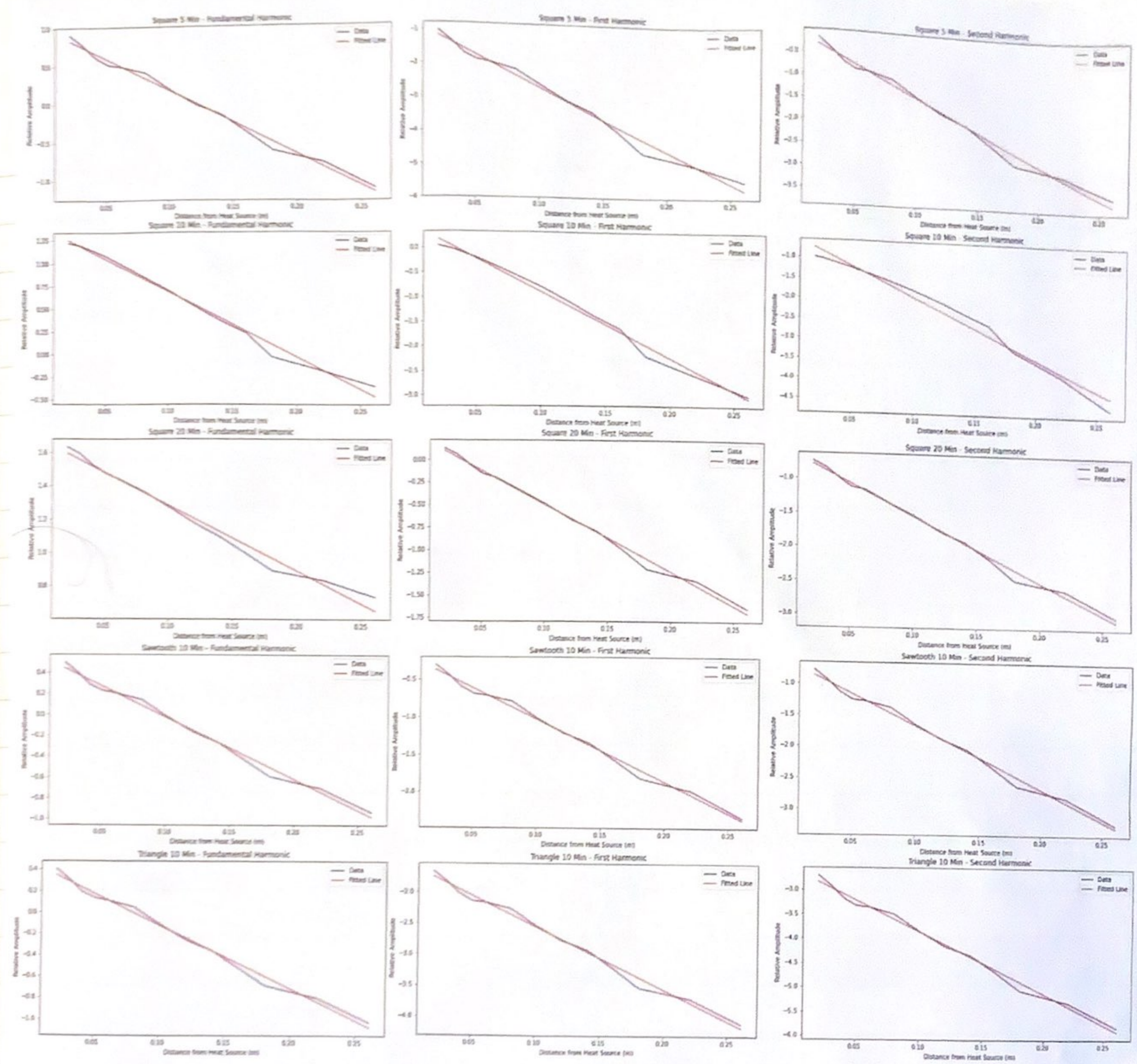
← D /$m^2 s^{-1}$ vs frequency

The graph shows no clear dependence on frequency with values of D grouped around $\sim 1.5 \times 10^{-4}$ $m^2 s^{-1}$



← K /$Wm^{-1}K^{-1}$ vs frequency

The graph shows no clear dependence on frequency with values of K grouped around ~ 500 $Wm^{-1}K^{-1}$



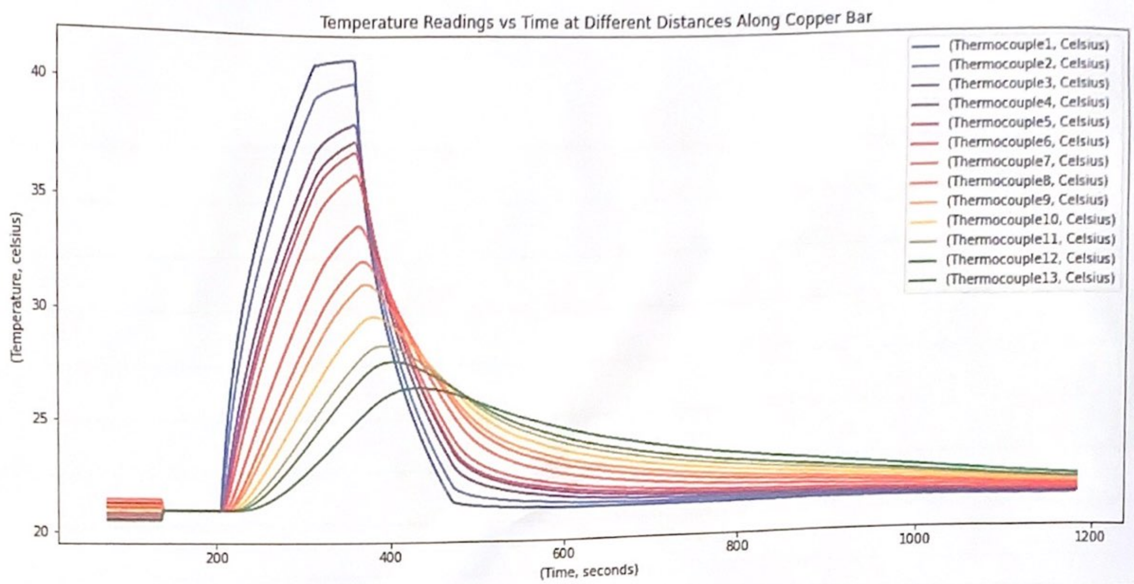
Linear fit to log (Amplitude) against the thermocouple distances.

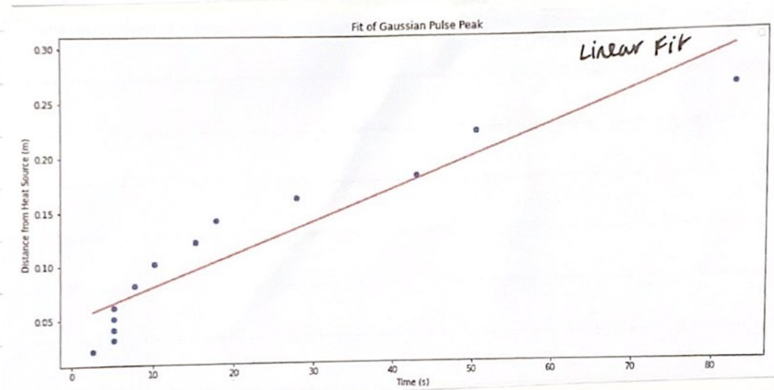
The gradient gives $m = -\alpha$

$$\bar{D} = 1.41 \times 10^{-4} \ m^2 s^{-1} \qquad \bar{K} = 487 \ W m^{-1} K^{-1}$$

$$\sigma_{\bar{D}} = 3.54 \times 10^{-6} \ m^2 s^{-1} \qquad \sigma_{\bar{K}} = 1.23 \ W m^{-1} K^{-1}$$

5.4 SINGLE PULSE



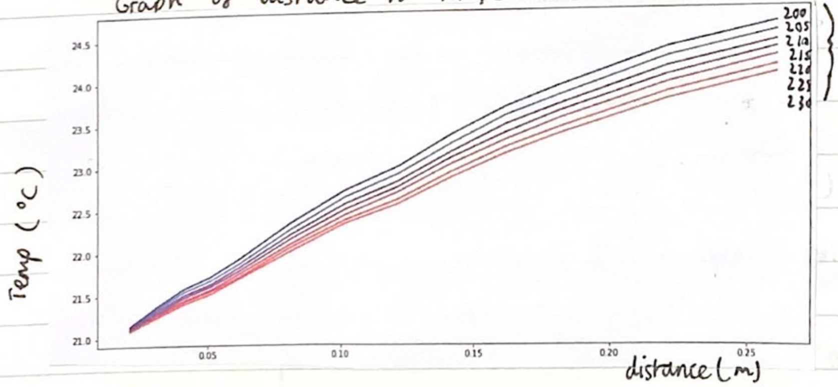


Plot of position vs time @ peaks → gradient gives velocity

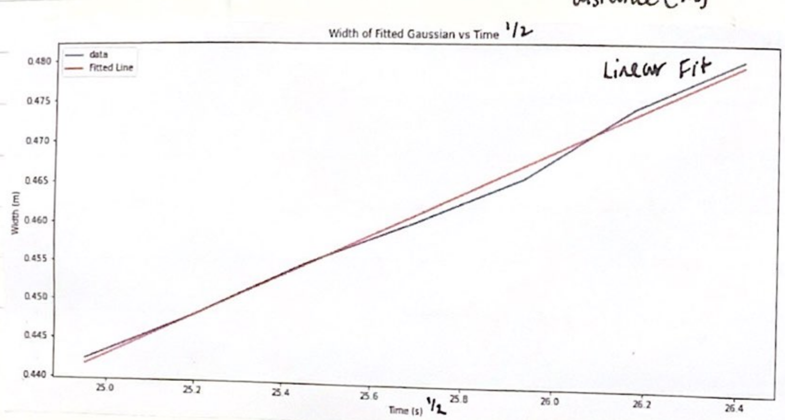
$v = 2.98 \times 10^{-3}\ ms^{-1}$ (3.s.f)

$\sigma_v = 1.13 \times 10^{-7}\ ms^{-1}$ (3.s.f)

Graph of distance vs temperature at different times (used for points in graph below)



Gaussian fits were then fitted to these graphs, and plotted below, showing width against $(\text{time})^{1/2}$



gradient :

2.596×10^{-2}

$\sigma = 5.905 \times 10^{-7}$

PROOF THAT A GAUSSIAN IS A SOLUTION TO $\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$ (22)

ANSATZ $T = \frac{T_0}{\sqrt{\alpha t}} e^{-x^2/\beta t}$ (21)

$$\frac{\partial T}{\partial t} = e^{-x^2/\beta t} \frac{x^2}{\beta t^2} \frac{T_0}{\sqrt{\alpha t}} - \frac{1}{2} \frac{T_0}{\sqrt{\alpha t^3}} e^{-x^2/\beta t} = e^{-x^2/\beta t} T_0 \left(\frac{x^2}{\beta \sqrt{\alpha t^5}} - \frac{1}{2\sqrt{\alpha t^3}} \right)$$

$$D \frac{\partial^2 T}{\partial x^2} = \frac{D T_0}{\sqrt{\alpha t}} \left[\frac{\partial}{\partial x} e^{-x^2/\beta t} \left(-\frac{2x}{\beta t} \right) \right] = \frac{D T_0}{\sqrt{\alpha t}} \left[e^{-x^2/\beta t} \left(-\frac{2x}{\beta t} \right)^2 - e^{-x^2/\beta t} \left(\frac{2}{\beta t} \right) \right]$$

$$= e^{-x^2/\beta t} T_0 \left(\frac{4x^2 D}{\beta^2 \sqrt{\alpha t^5}} - \frac{2D}{\beta \sqrt{\alpha t^3}} \right)$$

If ansatz is true: $\frac{4x^2 D}{\beta^2 \sqrt{\alpha t^5}} - \frac{2D}{\beta \sqrt{\alpha t^3}} = \frac{x^2}{\beta \sqrt{\alpha t^5}} - \frac{1}{2\sqrt{\alpha t^3}}$

$$\Rightarrow \left. \begin{aligned} \frac{4D}{\beta^2} &= \frac{1}{\beta} \\ 2D/\beta &= 1/2 \end{aligned} \right\} \Rightarrow 4D = \beta$$

$\Rightarrow T = \frac{T_0}{\sqrt{\alpha t}} e^{-x^2/4Dt} \Rightarrow$ Gaussian is a solution

$$\frac{\Delta \sigma}{\Delta t} = \text{gradient} = 2\sqrt{D} \Rightarrow \boxed{D = 1.685 \times 10^{-4} \ m^2 s^{-1}}$$

$$\sigma_D^2 = \left(\frac{\partial D}{\partial m} \right)^2 \sigma_m^2 \Rightarrow \boxed{\sigma_D = 7.66 \times 10^{-9} \ m^2 s^{-1}}$$

$$\Rightarrow \boxed{K = 581 \ Wm^{-1}K^{-1} \ (3.s.f)}$$

$$\boxed{\sigma_K = 1.64 \ Wm^{-1}K^{-1} \ (3.s.f)}$$

CONCLUSION

Every value of D and K are consistent, suggesting the theory is correct, however our errors don't overlap so the experiment is not conclusive enought to find an official value of D or K. We have found that thermal pulses travel down the bar in the form of a gaussian wave packet travelling down the bar. This is consistent with the theory.