Differential Equation:

$$\frac{d^2y}{dx^2} + (a - 2q\cos(2x))y = 0 \tag{1}$$

For all q there are special values $a_n(q)$ and $b_n(q)$ such that the solutions are periodic with period π or 2π.

If $q = 0$ this equation then reduces to $y'' + ay = 0$ which has solutions $\cos(\sqrt{a}x)$ and $\sin(\sqrt{a}x)$. These solutions have period π or 2π if $a = n^2$. In this case we say $a(0) = a_n(0) = n^2 = b_n(0) = b(0)$.

In general $a_n(q) \neq b_n(q)$.

We also have:

$$a_0(q) < b_1(q) < a_1(q) < b_2(q) < \dots \tag{2}$$

When $q \neq 0$ we denote the periodic solutions to Mathieu's Equation

$$\mathrm{ce}_n(x,q),\ \mathrm{se}_n(x,q)$$

Function	Period	Symmetry
ce_{2r}	π	even
ce_{2r+1}	2π	even
se_{2r}	π	odd
se_{2r+1}	2π	odd

Fourier series

Given the previous properties we may expand the Mathieu functions as fourier series containing only either cos or sin terms and either only even or odd multiples.

$$\mathrm{ce}_{2r}(x,q) = \sum_{s=0}^{\infty} A_{2s}^{2r}(q)\cos(2sx) \tag{3}$$

$$\mathrm{ce}_{2r+1}(x,q) = \sum_{s=0}^{\infty} A_{2s+1}^{2r+1}(q)\cos((2s+1)x) \tag{4}$$

$$\mathrm{se}_{2r}(x,q) = \sum_{s=1}^{\infty} B_{2s}^{2r}(q)\sin(2sx) \tag{5}$$

$$\mathrm{se}_{2r+1}(x,q) = \sum_{s=0}^{\infty} B_{2s+1}^{2r+1}(q)\sin((2s+1)x) \tag{6}$$

Orthogonality

As the Mathieu equation is of Sturm-Liouville form, for fixed q the solutions ce_n and se_n must be orthogonal over $[0, 2\pi]$:

$$\int_0^{2\pi} \mathrm{ce}_m(x,q)\,\mathrm{ce}_n(x,q)dx = \pi\delta_{mn} \tag{7}$$

$$\int_0^{2\pi} \mathrm{se}_m(x,q)\,\mathrm{se}_n(x,q)dx = \pi\delta_{mn} \tag{8}$$

$$\int_0^{2\pi} \mathrm{ce}_m(x,q)\,\mathrm{se}_n(x,q)dx = 0 \tag{9}$$

Proof. Suppose we have two solutions to Mathieu's equation $y_1(x,q)$ and $y_2(x,q)$ with corresopnding eigenvalues λ_1, λ_2 where $\lambda_1 \neq \lambda_2$, then we have:

$$\frac{d^2y_1}{dx^2} + (\lambda_1 - 2q\cos(x))y_1 = 0 \tag{10}$$

$$\frac{d^2y_2}{dx^2} + (\lambda_2 - 2q\cos(x))y_2 = 0 \tag{11}$$

Multiply the first by y_2 and the second by y_1 and then subratct:

$$y_2\frac{d^2y_1}{dx^2} - y_1\frac{d^2y_2}{dx^2} = (\lambda_1 - \lambda_2)y_1y_2 \tag{12}$$

Integrate from 0 to 2π:

$$\int_0^{2\pi} \left(y_2\frac{d^2y_1}{dx^2} - y_1\frac{d^2y_2}{dx^2}\right)dx = \int_0^{2\pi}(\lambda_1 - \lambda_2)y_1y_2dx \tag{13}$$

We have:

$$y_2\frac{d^2y_1}{dx^2} - y_1\frac{d^2y_2}{dx^2} = y_2\frac{d^2y_1}{dx^2} + \frac{dy_1}{dx}\frac{dy_2}{dx} - \frac{dy_1}{dx}\frac{dy_2}{dx} - y_1\frac{d^2y_2}{dx^2} \tag{14}$$

$$= \frac{d}{dx}\left(y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right) \tag{15}$$

Hence:

$$\left[y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx}\right]_0^{2\pi} = (\lambda_1 - \lambda_2)\int_0^{2\pi} y_1y_2dx \tag{16}$$

However, we know that y_1 and y_2 (and hence $\frac{dy_1}{dx}$, $\frac{dy_1}{dx}$) are periodic with period 2π (or π makes no difference) so the LHS is identically zero:

$$0 = (\lambda_1 - \lambda_2)\int_0^{2\pi} y_1y_2dx \tag{17}$$

As $\lambda_1 - \lambda_2 \neq 0$ we must have:

$$\int_0^{2\pi} y_1y_2dx = 0 \tag{18}$$

If $\lambda_1 = \lambda_2$ then $y_1 = y_2$ so

$$\int_0^{2\pi} y_1^2dx \geq 0 \tag{19}$$

Which gives us the freedom to choose this value equal to π. □

Using the orthogonality of the funuctions we can find a suitable normalisation for the fouurier coefficients.

Using the sin and cos orthogonality rules:

$$\int_0^{2\pi} \cos(mx)\cos(nx)dx = \begin{cases} 2\pi, & \text{if } m = n = 0 \\ \pi\delta_{mn}, & \text{otherwise} \end{cases} \tag{20}$$

$$\int_0^{2\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn} \tag{21}$$

We can derive the following:

$$2\left(A_0^{2r}\right)^2 + \sum_{s=1}^{\infty}\left(A_{2s}^{2r}\right)^2 = 1 \tag{22}$$

$$\sum_{s=0}^{\infty}\left(A_{2s+1}^{2r+1}\right)^2 = 1 \tag{23}$$

$$\sum_{s=1}^{\infty}\left(B_{2s}^{2r}\right)^2 = 1 \tag{24}$$

$$\sum_{s=0}^{\infty}\left(B_{2s+1}^{2r+1}\right)^2 = 1 \tag{25}$$

Matrix Formulation

From the differential equation and using the fourier expansions of the Mathieu functions we can derive an infinite matrix formulation of the problem. From this one can calculate the eigenvalue a_n, b_n, as well as the coefficients.

$\mathrm{ce}_{2r}(x, q)$

$$\begin{aligned}
0 &= \frac{d^2\,\mathrm{ce}_{2r}}{dx^2} + (a_{2n} - 2q\cos(2x))\,\mathrm{ce}_{2n} && (26)\\
&= \frac{d^2}{dx^2}\left(\sum_{s=0}^{\infty} A_{2s}^{2r}(q)\cos(2sx)\right) + (a_{2n} - 2q\cos(2x))\sum_{s=0}^{\infty} A_{2s}^{2r}(q)\cos(2sx) && (27)\\
&= -\sum_{s=0}^{\infty} A_{2s}^{2r}(q)(2s)^2\cos(2sx) + a_{2n}\sum_{s=0}^{\infty} A_{2s}^{2r}(q)\cos(2sx) - 2q\cos(2x)\sum_{s=0}^{\infty} A_{2s}^{2r}(q)\cos(2sx) && (28)
\end{aligned}$$

Using

$$\cos(2x)\cos(2nx) = \frac{1}{2}\left[\cos(2(n+1)x) + \cos(2(n-1)x)\right] \tag{29}$$

We can equate the coefficients to get:

$$
\begin{aligned}
aA_0 - qA_2 =& 0 \qquad (30)\\
-2qA_0 + (a-4)A_2 - qA_4 =& 0 \qquad (31)\\
-qA_{2k-2} + (a-(2k)^2)A_{2k} - qA_{2k+2} =& 0 \qquad (32)
\end{aligned}
$$

Similarly for the other functions:
ce_{2r+1}

$$
\begin{aligned}
(a-1-q)A_1 - qA_3 =& 0 \qquad (33)\\
-qA_{2k-1} + (a-(2k+1)^2)A_{2k+1} - qA_{2k+3} =& 0 \qquad (34)
\end{aligned}
$$

se_{2r}

$$
\begin{aligned}
(b-4)B_2 - qB_4 =& 0 \qquad (35)\\
-qB_{2k-2} + (b-(2k)^2)B_{2k} - qB_{2k+2} =& 0 \qquad (36)
\end{aligned}
$$

se_{2r+1}

$$
\begin{aligned}
(b-1+q)B_1 - qB_3 =& 0 \qquad (37)\\
-qB_{2k-1} + (b-(2k+1)^2)B_{2k+1} - qB_{2k+3} =& 0 \qquad (38)
\end{aligned}
$$

$$
\begin{pmatrix} 0 & \sqrt{2}q & 0 & 0 & \dots \\ \sqrt{2}q & 4 & q & 0 & \dots \\ 0 & q & 16 & q & \dots \\ 0 & 0 & q & 36 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \sqrt{2}A_0^{2r} \\ A_2^{2r} \\ A_4^{2r} \\ A_6^{2r} \\ \vdots \end{pmatrix} = a_{2r}(q) \begin{pmatrix} \sqrt{2}A_0^{2r} \\ A_2^{2r} \\ A_4^{2r} \\ A_6^{2r} \\ \vdots \end{pmatrix} \qquad (39)
$$

$$
\begin{pmatrix} 1+q & q & 0 & 0 & \dots \\ q & 9 & q & 0 & \dots \\ 0 & q & 25 & q & \dots \\ 0 & 0 & q & 49 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} A_1^{2r+1} \\ A_3^{2r+1} \\ A_5^{2r+1} \\ A_7^{2r+1} \\ \vdots \end{pmatrix} = a_{2r+1}(q) \begin{pmatrix} A_1^{2r+1} \\ A_3^{2r+1} \\ A_5^{2r+1} \\ A_7^{2r+1} \\ \vdots \end{pmatrix} \qquad (40)
$$

$$
\begin{pmatrix} 4 & q & 0 & 0 & \dots \\ q & 16 & q & 0 & \dots \\ 0 & q & 36 & q & \dots \\ 0 & 0 & q & 64 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} B_2^{2r} \\ B_4^{2r} \\ B_6^{2r} \\ B_8^{2r} \\ \vdots \end{pmatrix} = b_{2r}(q) \begin{pmatrix} B_2^{2r} \\ B_4^{2r} \\ B_6^{2r} \\ B_8^{2r} \\ \vdots \end{pmatrix} \qquad (41)
$$

$$
\begin{pmatrix}
1-q & q & 0 & 0 & \dots \\
q & 9 & q & 0 & \dots \\
0 & q & 25 & q & \dots \\
0 & 0 & q & 49 & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
B_1^{2r+1} \\ B_3^{2r+1} \\ B_5^{2r+1} \\ B_7^{2r+1} \\ \vdots
\end{pmatrix}
= b_{2r+1}(q)
\begin{pmatrix}
B_1^{2r+1} \\ B_3^{2r+1} \\ B_5^{2r+1} \\ B_7^{2r+1} \\ \vdots
\end{pmatrix}
\tag{42}
$$

https://www.biblioteca.org.ar/libros/90028.pdf

We can also expand $\cos(x)$ and $\sin(x)$ in terms of $\mathrm{ce}_n(x,q)$ and $\mathrm{se}_n(x,q)$:

$$\cos(2sx) = \sum_{r=0}^{\infty} \overline{A}_{2r}^{2s}(q)\,\mathrm{ce}_{2r}(x,q) \tag{43}$$

$$\cos((2s+1)x) = \sum_{r=0}^{\infty} \overline{A}_{2r+1}^{2s+1}(q)\,\mathrm{ce}_{2r+1}(x,q) \tag{44}$$

$$\sin(2sx) = \sum_{r=0}^{\infty} \overline{B}_{2r}^{2s}(q)\,\mathrm{se}_{2r}(x,q) \tag{45}$$

$$\sin((2s+1)x) = \sum_{r=0}^{\infty} \overline{B}_{2r+1}^{2s+1}(q)\,\mathrm{se}_{2r+1}(x,q) \tag{46}$$

By fouriers stuff we find:

$$A_{2s}^{2r}(q) = \overline{A}_{2r}^{2s}(q) \tag{47}$$

Except:

$$2A_0^{2r}(q) = \overline{A}_{2r}^{0}(q) \tag{48}$$

$$A_{2s+1}^{2r+1}(q) = \overline{A}_{2r+1}^{2s+1}(q) \tag{49}$$

$$B_{2s}^{2r}(q) = \overline{B}_{2r}^{2s}(q) \tag{50}$$

$$B_{2s+1}^{2r+1}(q) = \overline{B}_{2r+1}^{2s+1}(q) \tag{51}$$

Differential Equation modified:

$$\frac{d^2y}{dx^2} - (a - 2q\cosh(2x))y = 0 \tag{52}$$

This equation relates to that of the angular mathieu functions via $x \to ix$ and hence the corresponding solutions are:

$$\mathrm{ce}_n(ix,q) \tag{53}$$

$$-i\,\mathrm{se}_n(ix,q) \tag{54}$$

Provided that $a = a_n(q)$ or $a = b_n(q)$.
We can write these solutions as:

$$\mathrm{Ce}_n(x,q) = \mathrm{ce}_n(ix,q) \tag{55}$$

$$\mathrm{Se}_n(x,q) = -i\,\mathrm{se}_n(ix,q) \tag{56}$$

Using these we can rewrite the fourier expansions as:

$$\mathrm{Ce}_{2r}(x,q) = \sum_{s=0}^{\infty} A_{2s}^{2r}(q)\cosh(2sx) \tag{57}$$

$$\mathrm{Ce}_{2r+1}(x,q) = \sum_{s=0}^{\infty} A_{2s+1}^{2r+1}(q)\cosh((2s+1)x) \tag{58}$$

$$\mathrm{Se}_{2r}(x,q) = \sum_{s=1}^{\infty} B_{2s}^{2r}(q)\sinh(2sx) \tag{59}$$

$$\mathrm{Se}_{2r+1}(x,q) = \sum_{s=0}^{\infty} B_{2s+1}^{2r+1}(q)\sinh((2s+1)x) \tag{60}$$

However this isn't the best formulation of these functions as cosh and sinh grom very rapidly. Instead we should expand these in terms of bessel functions.
We know that the bessel functions are orthonormal in two ways:

$$\int_0^1 J_n(\alpha_{ni}x)J_n(\alpha_{nj}x)dx = \delta_{ij}\frac{J_n'(\alpha_{ni})}{2} \tag{61}$$

$$\int_0^{\infty} J_n(x)J_m(x)\frac{dx}{x} = \delta_{nm}\frac{1}{2n} \tag{62}$$

$$\int_{-\infty}^{\infty} J_n(e^u)J_m(e^u)du = \delta_{nm}\frac{1}{2n} \tag{63}$$

$$\int_0^{\infty} J_n(2\sqrt{q}\sinh(v))J_m(2\sqrt{q}\sinh(v))\coth(v)dv = \delta_{nm}\frac{1}{2n} \tag{64}$$

GETS FUNNY AT $m = n = 0$
Using the second one as a basis for $L^2[0,\infty]$ we can expand the modified mathieu functions as a sum of Bessel functions:
Let y

$$\mathrm{Ce}_n(x) = \sum_0^{\infty} c_n J_n\left(2\sqrt{q}\sinh(x)\right) \tag{65}$$