Solution of the Stochastic Differential Equation for stock prices

Given the stochastic differential equation (SDE) for asset prices:

$$dS = aSdt + bSdW \tag{1}$$

To solve for $S(t)$, we define a new function $G(S,t) = \ln(S)$. According to Itô's Lemma, the differential dG for a stochastic process is given by:

$$dG = \left(\frac{\partial G}{\partial t} + aS\frac{\partial G}{\partial S} + \frac{1}{2}(bS)^2\frac{\partial^2 G}{\partial S^2}\right)dt + bS\frac{\partial G}{\partial S}dW \tag{2}$$

1. Calculate Partial Derivatives

For $G = \ln(S)$, the partial derivatives are:

- $\frac{\partial G}{\partial t} = 0$
- $\frac{\partial G}{\partial S} = \frac{1}{S}$
- $\frac{\partial^2 G}{\partial S^2} = -\frac{1}{S^2}$

2. Substitute into Itô's Lemma

Substituting these derivatives into the dG formula:

$$d(\ln S) = \left(0 + aS\left(\frac{1}{S}\right) + \frac{1}{2}b^2S^2\left(-\frac{1}{S^2}\right)\right)dt + bS\left(\frac{1}{S}\right)dW \tag{3}$$

3. Simplify the Expression

Cancelling the S terms yields:

$$d(\ln S) = \left(a - \frac{1}{2}b^2\right)dt + bdW \tag{4}$$

4. Integrate Both Sides

Integrating from time 0 to t:

$$\int_0^t d(\ln S) = \int_0^t \left(a - \frac{1}{2}b^2\right)dt' + \int_0^t bdW(t') \tag{5}$$

Evaluating the integrals:

$$\ln S(t) - \ln S(0) = \left(a - \frac{1}{2}b^2\right)t + b(W(t) - W(0)) \tag{6}$$

5. Final Analytical Solution

Since $W(0) = 0$, we have:

$$\ln\left(\frac{S(t)}{S(0)}\right) = \left(a - \frac{1}{2}b^2\right)t + bW(t) \tag{7}$$

Taking the exponential of both sides results in the analytical solution:

$$S(t) = S(0)\exp\left(\left(a - \frac{1}{2}b^2\right)t + bW(t)\right) \tag{8}$$

This is the analytic solution used in the paper.