Exam Notes for MDM4UI

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Abstract

Notes for all topics taught in MDM4UI (Data Management) from the winter term of the 2016-2017 school year at GCI. Please refer to the appendix for a walk-through of example questions that will probably be on the exam. Sorry for the formatting. I'm not quite done yet.

1. Permutations

Permutations are the arrangements of objects in a set order. For instance, how many ways can you rearrange the letters in the word "APPLE", or how many ways can you bring 4 people out of 20 friends to a party.

1.1. Factorial Notation

Often in permutations, we need to multiply descending consecutive numbers. Since these numbers are descending and not ascending, we can safely assume that **every factorial must be positive**. For instance, $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ can often be simplified and written as **5!**, or 5 factorial.
The general rule for factorial notation is:

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

The exception to this rule is 0!, which in this case would be $0! = 1$, which would follow the same thinking as $n^0 = 0$.

1.2. Rule of Product and Rule of Sum

How many different results can one get from flipping a coin 3 times? You have 2 possibilities, heads or tails, and you flip it 3 times. Essentially, you have three groups of two, or $2 \cdot 3$ different results, which turns out to be 6 different possible results. This is called the *Fundamental Rule of Product* or the *Counting Principle*

Definition 1. *The Rule of Product states that if the 1st action can be performed in n ways, and the 2nd action can be performed in m ways. They can be performed together in $n \cdot m$ ways.*

What happens if the actions cannot be performed together? Such as which car to buy? In that case we have another rule, known as the *Rule of Sum.*

Definition 2. *The Rule of Sum states that if the 1st action can be performed in n ways, and the 2nd action can be performed in m ways, and these actions cannot occur together, then there are $n + m$ ways for either of the actions to occur.*

1.3. N-Arrangements

Let's say you have 5 people, and you need to arrange them in a line. You put one person in the first spot, and now you have 4 left. Repeat this until you are left with one person, which fits into the last spot. You can express this mathematically by writing out $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or 5 factorial.
This is quite similar to the Rule of Product as:

1. Each placement is a choice
2. The number of choices is reduced with each previous action.

A permutation of n objects is an arrangement of the objects in a **definite order**. This can be expressed with permutation notation, shown below where P(n,n) equals the number of ways to permute n objects.

$$P(n, n) = n!$$

This equation will only work when the number of spaces to occupy and the number of objects are equal. In other cases we'll have to use an r-arrangement.

1.4. R-Arrangements

If you have more objects than places you can put the objects into, then we can use something called an r-arrangement. An r-arrangement is a permutation of n objects taken r at a time, as is shown below.

$$P(n, r) = \frac{n!}{(n-r)!}$$

Where $P(n, r)$ equals the total number of arrangements you have, divided by the number of unused objects. This is most commonly displayed on calculators as nPr. For example you have 4 subjects, but you can only study two at a time. The equation for that would be $\frac{4!}{(4-2)!} = 12$ combinations.

1.5. Permutations with Repeating Elements

All previous examples will ONLY work with non-repeating elements, such as the letters in the word "MONTREAL", or 5 different books. They will definitely NOT work for questions with repeating elements, such as the word "FIJI". For all repeating elements we'll have to follow another rule shown below.

Definition 3. *In general, the number of arrangements of* n *objects of which* a *of one kind are alike, and* b *of one kind are alike and so forth is given by the expression below, Where the number of permutations can be found by dividing the total number of possible combinations by the repeating elements.*

$$\frac{n!}{a!b!c!\cdots}$$

1.6. Problem Solving with Permutations

There are 3 methods for solving problems with permutations. They are called: the Indirect Method, the Case Method, and Circular Arrangements.

1.6.1. Indirect Method

The indirect method involves taking all possibilities and subtracting those which are not wanted. For instance, say you have 3 people that need to sit at a table, but 2 of them don't want to sit beside each other. There is a step-by-step process that you can use for this question that is explained below.

1. Find all possible permutations (ex. P(3,3)),
2. Find all unwanted permutations (in this case when they are together),
3. Subtract the unwanted permutations from the wanted permutations.

This can be written as:

$$\textbf{Wanted Outcomes} = \text{Total Outcomes} - \text{Unwanted Outcomes}$$

1.6.2. Case Method

The Case Method involves breaking down the equation into manageable parts, then adding them to get the final solution. There is a step-by-step example problem in Section 1 of the appendix, along with an explanation of why, and how you manage the "breaking into parts" aspect.

1.6.3. Circular Arrangements

Circular arrangements are no longer a part of the curriculum, and as such they will not be tested on, but they are still quite useful to know in niche cases. If you have seven people seated at a table, and they all move to their rights, their positions may have moved, but their order remains the same, like a circle. Keep in mind that if there is some fixed point, the question just ends into a typical linear arrangement, with the equation simplifying to $1 \cdot (n-1)!$.

2. Combinations

2.1. What are Combinations?

A Combination is a way of selecting objects (permuting) from a group, without the order being important. In essence, it is orderless permutations. Let ABC be a permutation. With permutations we asserted that ABC and BAC are different, as they are ordered differently. In combinations, we assert that ABC and BAC are the same, as they contain the same objects, just in a different order.

2.2. Set Theory

Set Theory involves the creation and manipulation of sets and their properties. A set is a collection of distinct objects in which their order does not matter. The objects in this set are called elements. If a set does not have any elements in it, we refer to it as a *null set*, which is shown as $\emptyset$ or $(\emptyset)$. This null set is a subset of everything. If we want to describe a set containing elements, we must follow the example below.

$$X = \{a, b, c, d, e, f, \cdots, y, z\}$$

If we wish to describe the number of elements in the set, we can use $n(X)$, where X represents the set, and n returns the number of elements in that set. In this case $n(X) = 26$, as set X contains all the letters of the alphabet. Sets are referred to differently when compared to other sets, as defined below.

Definition 4. *Special Properties of Sets when Compared to Other Sets*

1. *If two sets have no elements in common they are called disjoint sets.*
2. *If two sets have all elements in common they are equal.*
3. *If all of elements of A are also in B then A is a subset of B* $(A \subseteq B)$

2.2.1. Universal Set and Complement Sets

The set of all elements being considered is called the*universal set* and is always denoted by S. When referring to a set of all elements that are in the universal set, but **not in set A**, we call this the complement of A or A'. For example:

$$S = \{1, 2, 3, 4\}, A = \{1, 3\}, A' = \{2, 4\}$$

2.2.2. Unions and Intersections

For two sets of A and B, the **union** of A and B or $A \cup B$, is the set of all elements in A or B, not including duplicates.

$$S = \{1, 2, 3, 4\}, A = \{1, 3\}, B = \{1, 2\}, A \cup B = \{1, 2, 3\}$$

For two sets of A and B the **intersection** of A and B, $A \cap B$, is the set of all the elements that are in both set A and set B.

$$S = \{1, 2, 3, 4\}, A = \{1, 3\}, B = \{1, 2\}, A \cap B = \{1\}$$

If you are ever having trouble remembering the difference between the symbols representing the union and the intersection, remember that a union looks like a cup, and intersection looks like a cap.

2.2.3. Subsets

Subsets, are sets that are also in another set. To find every possible subset (including null set) we need to use the equation:

$$2^{n(X)}$$

In this case, n(X) is the number of elements in Set X. If a question comes up, where we need the number of subsets of a given set with the length y, of set x, we'd simply use $\binom{n(x)}{y}$

2.3. Introduction to Combinations

A combination is a selection of r objects, from n distinct objects *without regard for order.* The equation for solving combinations is written below.

$$C(n, r) = \frac{n!}{(n - r)! \cdot r!}$$

This equations finds all combinations by dividing number of objects, by the unused objects $(n-r)!$ and the order of the selected objects. This can be further simplified to:

$$C(n,r) = \frac{P(n,r)}{r!}$$

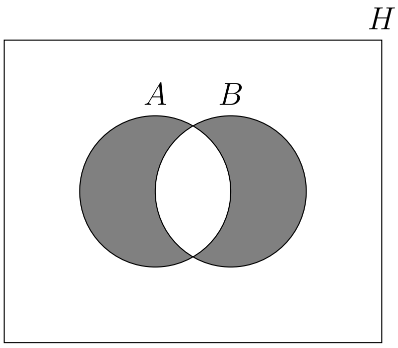
If order does not matter, the process is called a *Combination*, whereas when order does matter, it is called a *Permutation*. On your calculator, the formulae for combinations will be shown as **nCr**. If written out by hand, we will use the form:

$$\binom{n}{r} = \frac{n!}{(n-r)! \cdot n!}$$

Or simply $\binom{n}{r}$.

2.4. Inclusion—Exclusion Principle

The Inclusion-exclusion principle is a method of obtaining the number of elements in finite sets. Typically used when asked for the number of combinations of two things, where there is overlap between the two.
Let A represent Set A, and let B represent Set B.



This diagram perfectly represents the overlap between the two sets. If you caught it earlier, what we're actually finding is $A \cup B$. The Inclusion—Exclusion Principle is merely a way of finding it algebraically. The image above represents the equation below.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

3. Modelling with Matrices

3.1. Intro to Matrices

A matrix is a rectangular array of numbers that is used to organize data. If you are used to computer programming, you will recognize them as 2D arrays. If you are good with set notation, they are in essence, a set of sets. Matrices follow the format below:

$$X = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

Matrices have various properties, and unique terms created to describe them. For instance, looking at the matrix above, we can safely say that a in an entry, as defined below.

Definition 5. *Entry: An entry refers to a number in a matrix.*

How did we figure out the number of entries in the matrix? Simple, we can use the matrix's *dimensions* to figure that out.

Definition 6. *Dimensions: These refer to the size of the matrix, described by the number of rows and columns. The formula below is used to solve for dimensions, where D is the result, R is the rows, and C is the columns.*

$$D = R \cdot C$$

Matrix's entries are numbered in a way so that it is easy to refer to a specific entry. Capital letters are used for the names of matrices, and lowercase characters are used for the names of entries. Shown below are the entries named, with numbers instead of letters. The first character represents the y value, whereas the second represents the x value. In the equation below, a_{13} represents the entry in row 1, column 3.

$$X = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

3.2. Variations and Classifications of Matrices

Matrices can be transformed, and compared to each other, much like sets. The definitions of the variations and classifications of the matrices are listed below.

Definition 7. *Transpose Matrix: A transpose matrix is a matrix obtained by interchanging the rows and columns. This is written as X^+. Shown below is Matrix W, and the transposed equivalent.*

$$W = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} - > X^+ = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

Definition 8. *Column Matrix: A Column Matrix is a matrix that has only one x value. Instead of rows and columns, it is merely a column. For instance, Matrix X in this case would be a Column Matrix.*

$$X = \begin{bmatrix} a & \\ b & \\ c & \end{bmatrix}$$

Definition 9. *Row Matrix: A Row Matrix is a matrix that has only one y value. It is the transpose of the Column Matrix For instance, Matrix Y in this case would be a Row Matrix.*

$$Y = \begin{bmatrix} a & b & c \end{bmatrix}$$

Definition 10. *Square Matrix: A Square Matrix is a matrix in which the rows and columns are equal. For instance, Matrix Z below is a Square Matrix.*

$$Z = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

3.3. Basic Operations with Matrices

Matrices may only be added or subtracted when they have the same dimensions. Adding matrices is done by adding entry by entry to the other matrix. A_{12} will be added with B_{12} and so forth. Shown below is a valid example of adding with matrices in which the entries are labeled to show proper addition:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} (a+b)_{11} & (a+b)_{12} & (a+b)_{13} \\ (a+b)_{21} & (a+b)_{22} & (a+b)_{23} \end{bmatrix}$$

They may also be multiplied by a coefficient, by multiplying the coefficient with every entry in the matrix. Shown below is a proper solution in which a matrix is multiplied by a coefficient, with entries labeled:

$$x \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \end{bmatrix} = \begin{bmatrix} xy_{11} & xy_{12} & xy_{13} \\ xy_{21} & xy_{22} & xy_{23} \end{bmatrix}$$

3.4. Multiplying Matrices Together

Matrices may also be multiplied with other matrices, yet to do so we must follow certain rules. Let's say we have Matrix A and Matrix B as shown below. To multiply, we need to make sure the *inner dimensions* are the same. We know that Matrix A's dimensions are $3 \cdot 2$, whereas for Matrix B, they are $2 \cdot 3$. Since we know this, we can find the *inner dimensions*. The *inner dimensions*, equals the columns of Matrix A and the rows of Matrix B. In this case, that would work out to be 3 and 3. Since they are the same, we are allowed to multiply these matrices.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} B = \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$$

Next, the rows of Matrix A times the columns of Matrix B, also known as the *outer dimensions*, will give us the dimensions of the resultant matrix. Since Matrix A has two rows, and Matrix B has two columns, our resultant Matrix will possess the dimensions $2 \cdot 2$. Shown below is the updated equation.

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \cdot \begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

Multiplying them is a bit strange at first, because to do it, we multiply row by row. Starting at the first row in Matrix A, we can see we have a_{11}, b_{12}, c_{13} as our values, with their locations being labeled for convenience. We must multiply these values each column in Matrix B, starting with the values u_{11}, w_{21}, y_{31}. To find entry (1,1) we can use the formula shown below, where E represents the entry, r represents a row in Matrix A, and c represents a column in Matrix B.

$$E_{11} = r_{11} \cdot c_{11} + r_{12} \cdot c_{21} + r_{13} \cdot c_{31}$$

If you want to do it just by eye, you can multiply A_{11} by B_{11}, A_{12} by B_{21}, and A_{13} by B_{31}, and add them all up. It will seem totally awful at first, but after you do a couple it'll all work out fine.

3.5. Identity Matrices

An Identity Matrix is a $n \cdot n$ size matrix in which every number along the main diagonal is a 1, and the rest are zeros. Shown below is Matrix I, a $3 \cdot 3$ Identity Matrix.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix is very special, as any square matrix multiplied by its inverse will equal an Identity Matrix. We can further formalize this assumption, by referring to a given matrix as Matrix A.

$$A \cdot A^{-1} = I$$

Its too bad we can't divide matrices, how can we ever solve for I now?

3.6. Finding Inverse Matrices

Finding the inverse of large matrices is one of the most mathematically intensive processes out there. Good thing we'll only be finding the inverse of a $2 \cdot 2$ matrix instead.

4. Probability

4.1. What is probability?

Probability is the measurement of the chance of an event occurring. If you've ever checked the weather and seen that it's a "20% chance of rain", you've seen probability in action. To start, we'll need to define some terms.

Definition 11. *Outcomes: A possible result, typically referencing all of the possible results from the situation.*

Definition 12. *Events: These typically refer to desired outcomes*

Typically when finding probability, we'll divide all possible outcomes from our desired outcomes. Thinking in terms of set notation, we'll refer to all of our desired outcomes as n(A), where A is our desired outcome. n(S) will refer to all of the possible outcomes, and P(A) referring to the likelihood of A occurring. To generalize, we'll use the equation below.

$$P(A) = \frac{n(A)}{n(S)}$$

Since the number of desired outcomes will never be greater than the number of total outcomes, we'll always have a decimal between 0 (0%) and 1 (100%), which we can convert to a percentage (ex. $\frac{1}{4} = 0.25 = 25\%$), though typically we'll simply leave it as a reduced version of the fraction.

4.2. Mutually Exclusive Events and Non-Mutually Exclusive Events

Mutually Exclusive events are events that have zero overlap with other events. Recall the Venn Diagram of Sets in the Combinations section, and assume A and B are both events. To find the probability of either of them happening, we'll have to remember the equation:

$$n(A \cup B) = n(A) + n(B)$$

This equation finds all the ways A or B can happen, with zero overlap. The problem with this equation is that it only finds the number of wanted outcomes. Remember, to find the probability of something, we find the wanted outcomes, and divide it by all of the possible outcomes. After this, our equation will look like this:

$$\frac{n(A \cup B)}{n(S)} = \frac{n(A) + n(B)}{n(S)}$$

Which reduces to:

$$P(A \cup B) = P(A) + P(B)$$

For non-mutually exclusive events, we must instead use a different equation, more accurate to the one we've discussed in the Sets Unit. Following the same logic we used earlier, our equation works out to:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Where $P(A \cap B)$ equals the probability of both of those events occurring.

4.3. Independent Probability

When two events occur at the same time, or one after another with no effect on each other, they are called Independent Events. If you roll a die, then select a card, this would be independent, as the outcomes don't affect each other. So solve an Independent Probability question we'll have to think in terms of sets, so the probability of A and B, can be solved with the equation below.

$$P(A \cap B) = P(A) \cap (B)$$

But what happens if we need to find the probability of (A or B)? To find that we'll need to use another equation.

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

4.4. *Markov Chains*

Markov Chains are ways of predicting probability after successive iterations. Suppose that Product A has a 70% chance of repurchasing, and Product B has a 70% chance of repurchasing, how would we model this? We'd use a Markov Chain. Shown below is our probability matrix.

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

Now, we have to find the initial probability of something, which we call a **transition matrix**. We'll represent this in the form of how many people started buying A or B, (shown below). Since this is the initial time, we'll refer to it as S_0, though you may also hear it called a **probability vector**.

$$S_0 = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}$$

To find the next probability vector, or S_1, we'll have to multiply the transition matrix by the probability vector.

$$\begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \cdot \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.54 & 0.46 \end{bmatrix} = S_1$$

Much like rolling a die, the more times you repeat this process, the closer you are to the actual results. Though what if there was an easier way to do this?

4.5. *Steady State Vectors*

In Markov Chains, the probability vectors will eventually stop changing. A probability vector that remains unchanged upon multiplication is called the **steady state vector**. This vector will represent the long term trend of the event. To find this without repeating the Markov Chain multiplication, we must think of the problem in terms of a system of equations.

$$\begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}$$

To have a systems of equations from this, we must first assert that $a+b=1$, and find the other by multiplying for one of the values. In this case, we can see that $0.7a + 0.3b = a$. Now solve this systems of equations like you have done in Grade 9.

4.6. Odds

Have you ever heard the (seemingly wrong) Roll Up The Rim odds? They claim that 1 in 5 people are winners! But what does this mean? Odds are the degree of confidence that someone has that an event will occur. To find the odds of something occurring, we must use the formula below.

$$P(A) : P(A)'$$

This equation represents a ratio of the probability of A happening, against the probability of A **not** happening. Likewise with the odds of something **not occuring**, we can use the reverse of this formula.

$$P(A)' : P(A)$$