Stellar Dynamics Coursework

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Question 3

The distribution function is:

$$f = \begin{cases} C\left(\frac{A}{r} - \frac{v^2}{2}\right) & \text{for } v < \sqrt{\frac{2A}{r}} \\ 0 & \text{for } v > \sqrt{\frac{2A}{r}} \end{cases} \tag{1}$$

(a) <u>What is the number density profile of the test particles, $n(r)$?</u>

$$n(r) = \int f(x, v)\, \mathrm{d}^3\mathbf{v} \tag{2}$$

Change of coordinates: $\mathrm{d}^3\mathbf{v} = v^2 \sin^2(v_\theta)\, \mathrm{d}v \mathrm{d}v_\theta \mathrm{d}v_\phi$.

$$\Rightarrow n(r) = \int \mathrm{d}v_\theta \mathrm{d}v_\phi \int C\left(\frac{A}{r} - \frac{v^2}{2}\right) \mathrm{d}v \tag{3}$$

$$= 4\pi C \int_0^{\sqrt{\frac{2A}{r}}} \frac{Av^2}{r} - \frac{v^4}{2}\, \mathrm{d}v \tag{4}$$

$$= 4\pi C \left[\frac{Av^3}{3r} - \frac{v^5}{10}\right]_0^{\sqrt{\frac{2A}{r}}} \tag{5}$$

$$= 4\pi C \left[\frac{2^{3/2}}{3}\left(\frac{A}{r}\right)^{5/2} - \frac{2^{3/2}}{5}\left(\frac{A}{r}\right)^{5/2}\right] \tag{6}$$

$$= \frac{16\pi C\sqrt{2}}{15}\left(\frac{A}{r}\right)^{5/2} \tag{7}$$

(b) <u>Is the velocity dispersion tensor of the test particles isotropic?</u>

The distribution function is isotropic and so $u_i = \langle v_i \rangle$ has the same value for $i = r, \theta, \phi$. Hence the velocity dispersion tensor will also be isotropic.

(c) <u>If this distribution function is a solution to the CBE what is the gravitational potential?</u>

Solves CBE if $\frac{\mathrm{d}f}{\mathrm{d}t} = 0$.

$$0 = \frac{\mathrm{d}f}{\mathrm{d}t} = C\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A}{r}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{v^2}{2}\right)\right] \tag{8}$$

$$= C\left[-\frac{A}{r^2}\frac{\mathrm{d}r}{\mathrm{d}t} - v\frac{\mathrm{d}v}{\mathrm{d}t}\right] \tag{9}$$

$$= C\left[-\frac{Av}{r^2} - va\right] \tag{10}$$

$$\Rightarrow \frac{A}{r^2} = a \tag{11}$$

Since $a = -\nabla\phi$:

$$\phi = \int \frac{A}{r^2}\,\mathrm{d}r \tag{12}$$
$$= -\frac{A}{r} \tag{13}$$

Question 5

Plummer density profile is:

$$\rho(r) = \frac{3M}{4\pi}\frac{a^2}{(r^2+a^2)^{5/2}} \tag{14}$$

(a) Show that the associated gravitational potential is $\phi(r) = \frac{GM}{\sqrt{r^2+a^2}}$.

The Poisson equation is $\Delta\phi = 4\pi G\rho$. If the potential is as given substituting it into the above will yield the density profile (14). (Easier than integrating!)
N.B. I have changed the sign of the potential to use the convention $\phi(r) < 0\ \forall\ r \neq \infty$.

The Plummer density profile is spherically symmetric, so in spherical coordinates:

$$\Delta\phi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) \tag{15}$$
$$= \frac{1}{r^2}\left(2r\frac{\mathrm{d}\phi}{\mathrm{d}r} + r^2\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2}\right) \tag{16}$$

We have:

$$\frac{\mathrm{d}\phi}{\mathrm{d}r} = GM\frac{r}{(r^2+a^2)^{3/2}} \tag{17}$$
$$\text{and } \frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} = GM\frac{a^2-2r^2}{(r^2+a^2)^{5/2}} \tag{18}$$

Hence:

$$\Delta\phi = \frac{1}{r^2}\left[GMr^2\frac{1}{(r^2+a^2)^{3/2}} + 2GMr^2\frac{a^2-2r^2}{(r^2+a^2)^{5/2}}\right] \tag{19}$$
$$= \frac{3GMa^2}{(r^2+a^2)^{5/2}} \tag{20}$$
$$\Rightarrow \frac{\Delta\phi}{4\pi G} = \frac{3M}{4\pi}\frac{a^2}{(r^2+a^2)^{5/2}} \tag{21}$$
$$= \rho(r) \tag{22}$$

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(b) Hence show the desnity and potential imply the ratio $\rho \propto \phi^5$.

We have $\rho \propto r^{-5/2}$ and $\phi \propto r^{-1/2}$. Hence $\rho \propto \phi^5$.

(c) Use this to show that the Plummer model has a distribution function $f(E) \propto (-E)^{7/2}$.

The "Jeans problem" involves solving

$$f(E) = -\frac{1}{2\sqrt{2}\pi^2}\frac{\mathrm{d}}{\mathrm{d}E}\int_0^E \frac{\mathrm{d}\rho}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\sqrt{\phi-E}} \tag{23}$$

Neglecting numerical factors, and changing variables to $\varepsilon = -E$ and $\Psi = -\phi$ such that $[\varepsilon,\ \Psi] > 0$ always:

$$f(\varepsilon) \propto \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_0^\varepsilon \frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\frac{\mathrm{d}\Psi}{\sqrt{\varepsilon-\Psi}} \tag{24}$$

Integrating by parts, with $u = \frac{\mathrm{d}\rho}{\mathrm{d}\Psi}$ and $\mathrm{d}v = (\varepsilon - \Psi)^{1/2}$:

$$f(\varepsilon) \propto \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[-2\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\sqrt{\varepsilon-\Psi}\,\Bigg|_0^{\varepsilon} + 2\int_0^{\varepsilon}\sqrt{\varepsilon-\Psi}\frac{\mathrm{d}^2\rho}{\mathrm{d}\Psi^2\mathrm{d}\Psi}\right] \tag{25}$$

$$\propto \frac{1}{\sqrt{\varepsilon}}\frac{\mathrm{d}\rho}{\mathrm{d}\Psi}\bigg|_{\Psi=0} + \int_0^{\varepsilon}\frac{\mathrm{d}^2\rho}{\mathrm{d}\Psi^2}\frac{\mathrm{d}\Psi}{\sqrt{\varepsilon-\Psi}} \tag{26}$$

Since $\rho \propto \phi^5$, $\frac{\mathrm{d}\rho}{\mathrm{d}\Psi} \propto \Psi^4$ and $\frac{\mathrm{d}^2\rho}{\mathrm{d}\Psi^2} \propto \Psi^3$.

$$\Rightarrow f(\varepsilon) \propto \frac{1}{\sqrt{\varepsilon}}\left(\Psi^4\right)\big|_{\Psi=0} + \int_0^{\varepsilon}\frac{\Psi^3\,\mathrm{d}\Psi}{\sqrt{\varepsilon-\Psi}} \tag{27}$$

The first term vanishes; substituting $x = \varepsilon - \Psi$ to perform the integral in the second term:

$$f(\varepsilon) \propto -\int_{x=\varepsilon}^{x=0}\frac{(\varepsilon-x)^3}{\sqrt{x}}\,\mathrm{d}x \tag{28}$$

$$\propto -\int_{\varepsilon}^{0}\varepsilon^3x^{-1/2} - 3\varepsilon^2x^{1/2} + 3\varepsilon x^{3/2} - x^{5/2}\,\mathrm{d}x \tag{29}$$

$$\propto -\left[2\varepsilon^3x^{1/2} - 2\varepsilon^2x^{3/2} + \frac{6}{5}\varepsilon x^{5/2} - \frac{2}{7}x^{7/2}\right]_{\varepsilon}^{0} \tag{30}$$

$$\propto 2\varepsilon^{7/2} + 2\varepsilon^{7/2} + \frac{6}{5}\varepsilon^{7/2} - \frac{2}{7}\varepsilon^{7/2} \tag{31}$$

$$\propto \varepsilon^{7/2} \tag{32}$$

$$\Rightarrow f(E) \propto (-E)^{7/2} \tag{33}$$

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