ME 488: Design of Experiments

Lecture 3: Full Factorials

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Fall 2017

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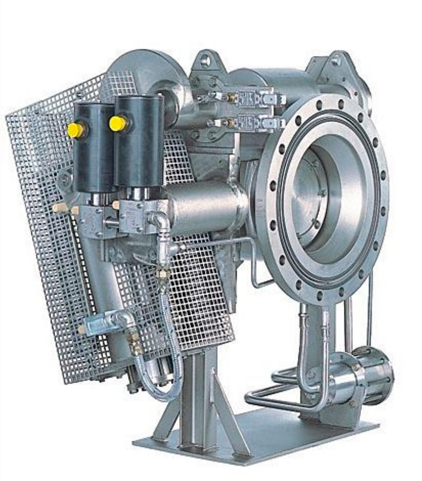
Nuclear Power Shut-Off Valves

Kühme DN 200 Butterfly Reactor Containment Shut-Off Valve



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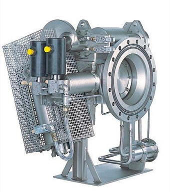


Optimization Concerns

Due to its use on a submarine, we want to minimize the acoustic operating characteristics. We have design options of:

- Nominal Diameter: Either 600 mm or 1200 mm
- Closing Springs: 1000N/m or 2000N/m*
- Seal Method: Metal-Metal or Metal-Elastomer*

Factorial Designs

- Two Treatment Variables A and B with a total number of levels
  of a and b respectively
- $a \times b$ factorial design: levels of both variables are selected in random order
- More general: $a \times b \times c \times \ldots$ factorial design
- If the same number of replicates in each cell the design is **Balanced**
- If all possible *combinations* of $a \times b \times c \times \ldots$ are included in design, the design is said to be **Full**

Balanced Full Factorial Design

Full Factorial Designs

$a \times b$ Factorial Design

- k Treatment Variables
- Main Effects $\binom{k}{1}$
- Two-Way Interactions $\binom{k}{2}$
- Three-Way Interactions $\binom{k}{3}$
- $\ldots$
- $df_{total} = a \times b \times c \times \cdots \times l_k \times (n-1)$

Most Common Design Setup

- k Factors, each with 2 Levels designated as $+$ and $-$
- 2^k refers to the number of unique cells or runs in each replicate of the design
- Easy to Analyze
- Good at the beginning of a study
- Only valid if effect is unidirectional

Sort of Trivial

- Each replicate consists of $2^1 = 2$ runs

Run	x_1
1	$-$
2	$+$

- Model : $y = \beta_0 + \beta_1 x_1$
- Idea is to see if a factor [x_1] matters
- Only checks at two different values of that factor

Back to our Valve..

- We want to see if the Nominal Diameter matters
- Each replicate consists of $2^1 = 2$ runs
- Design Matrix

Standard Order	*Run Order*	x_1
1	2	$-$
2	1	$+$

- Model : $y = \beta_0 + \beta_1 x_1 + \varepsilon_{error}$

Back to our Valve..

- Model Matrix [sort of pointless for 2^1 design]

Standard Order	*Run Order*	x_1
1	2	$-$
2	1	$+$

- Planning Matrix

Standard Order	*Run Order*	*Nominal Diameter*
1	2	600mm
2	1	1200mm

- We aren't going to fully analyze this, we will look at a more complex example later.

2^2 Design

Very Common Design

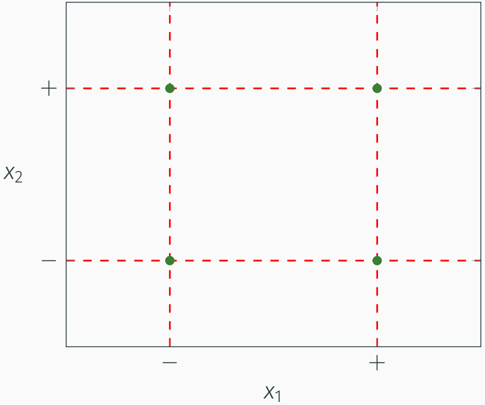
- Each replicate consists of $2^2 = 4$ runs

Run	x_1	x_2
1	$-$	$-$
2	$-$	$+$
3	$+$	$-$
4	$+$	$+$

- Model Main Effects Only: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon_{error}$
- Interaction Effects:
  $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_{12} + \varepsilon_{error};\ x_{12} = x_1 x_2$
- Still only checks at two different values of that factor
- But allows investigation of interaction effects

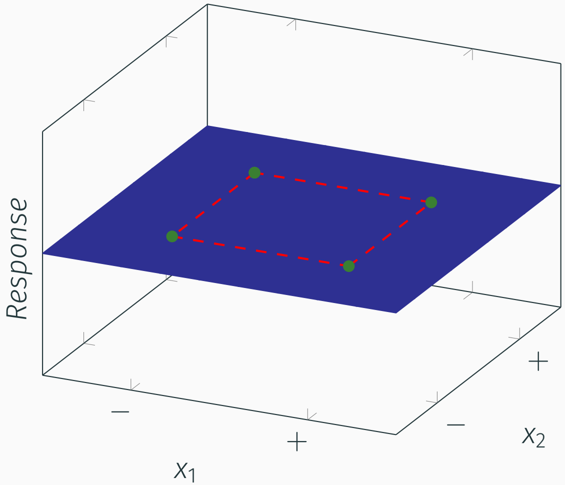
2^2 Design Graphic

2^2 Designs are visualized as a square

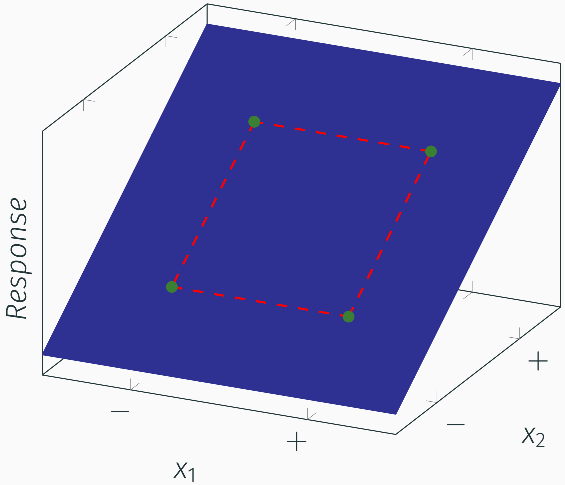


These are four 'test' points seeking the slope of the response

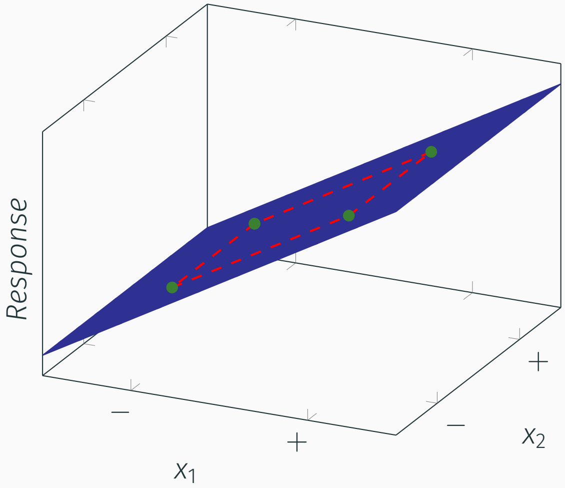
2^2 when neither x_1 or x_2 matter



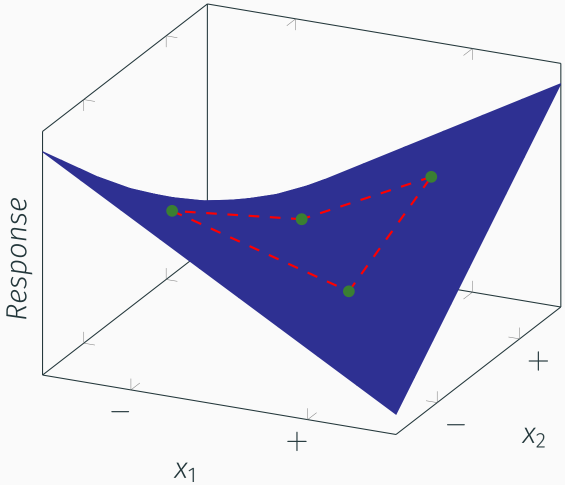
2^2 when x_1 doesn't matter



2^2 when x_2 doesn't matter



2^2 when BOTH matter



Degrees of Freedom Run Out...

Everything is good, except when you want to detect interactions...

- $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_{12} + \varepsilon_{error}$
- $df_{x_1} = K_{x_1} - 1 = 2 - 1 = 1$
- $df_{x_2} = K_{x_2} - 1 = 2 - 1 = 1$
- $df_{x_{12}} = (K_{x_1} - 1)(K_{x_2} - 1) = (2 - 1)(2 - 1) = 1$
- $df_{total} = K_{x_1} K_{x_2} - 1 = (2)(2) - 1 = 3$; For $n_i = 1$

But that means

$$df_{error} = df_{total} - df_{x_1} - df_{x_2} - df_{x_{12}} = 3 - 1 - 1 - 1 = 0$$

So how do we gain df_{error} to look at interactions?

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But that means

$$df_{error} = df_{total} - df_{x_1} - df_{x_2} - df_{x_{12}} = 3 - 1 - 1 - 1 = 0$$

So how do we gain df_{error} to look at interactions?
Increase n_i, AKA **Do more replicates!**

Just to make it clear...

When doing a 2^2 design:

- There are TWO Factors [e.g. Temp, Press, Viscosity, etc]
- There are TWO Levels of Each Factor [e.g. $+$, and $-$, but means 'On'/'Off', '24.3mm'/'12mm', etc.]
- Each Replicate needs $2^2 = 4$ Runs [measurements] taken
- You can model it with or without df_{error}
- You can model it with or without interactions
- If you run only one Replicate, then you have no df_{error} for an Interaction Model

2^2 Design Review

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Now, let's look at a 2^3 design before we get into examples, since this one is a bit less trivial

Now we are getting somewhere

- Each replicate consists of $2^3 = 8$ runs

Run	x_1	x_2	x_3
1	$-$	$-$	$-$
2	$-$	$-$	$+$
3	$-$	$+$	$-$
4	$-$	$+$	$+$
5	$+$	$-$	$-$
6	$+$	$-$	$+$
7	$+$	$+$	$-$
8	$+$	$+$	$+$

- Still only checks at TWO different values of each factor

Possible Models

- Model Main Effects Only:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon_{error}$$

- Two Way Interaction Effects:

$$\begin{gathered} y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \\ + \beta_{12} x_{12} + \beta_{13} x_{13} + \beta_{23} x_{23} + \varepsilon_{error} \\ ;\, x_{12} = x_1 x_2;\, x_{13} = x_1 x_3;\, x_{23} = x_2 x_3 \end{gathered}$$

- Three Way Interaction Effects:

$$\begin{gathered} \text{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \\ + \beta_{12} x_{12} + \beta_{13} x_{13} + \beta_{23} x_{23} \\ + \beta_{123} x_{123} + \varepsilon_{error} \\ ;\, x_{12} = x_1 x_2;\, x_{13} = x_1 x_3;\, x_{23} = x_2 x_3;\, x_{123} = x_1 x_2 x_3 \end{gathered}$$

Quick Check

Quick check: What does β_0 represent?

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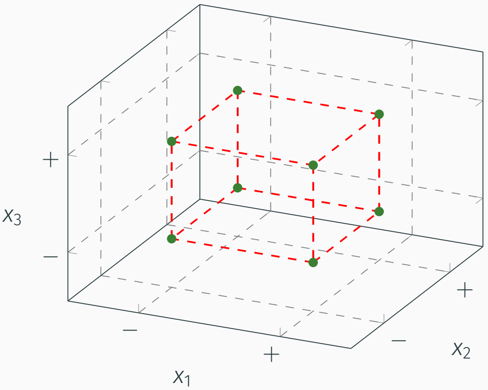
$$\beta_0 = \bar{y}$$

Its the 'Average' model [i.e. none of the factors matter]

2^3 Design Graphic

2^3 Designs are visualized as a cube



These are eight 'test' points seeking the response surface

Degrees of Freedom Problem Again

Source		*df*	$df_{n=1}$	$df_{n=2}$	df_n
A	x_1	$K_{x_1} - 1$	1	1	1
B	x_2	$K_{x_2} - 1$	1	1	1
C	x_3	$K_{x_3} - 1$	1	1	1
AB	x_{12}	$(K_{x_1} - 1)(K_{x_2} - 1)$	1	1	1
AC	x_{13}	$(K_{x_1} - 1)(K_{x_3} - 1)$	1	1	1
BC	x_{23}	$(K_{x_2} - 1)(K_{x_3} - 1)$	1	1	1
ABC	x_{123}	$(K_{x_1} - 1)(K_{x_2} - 1)(K_{x_3} - 1)$	1	1	1
Error	ε_{error}	$K_{x_1} K_{x_2} K_{x_3} (n - 1)$	0	8	$8(n - 1)$
Total		$K_{x_1} K_{x_2} K_{x_3} n - 1$	7	15	$8n - 1$

So how do we gain df_{error} to allow us to look at interactions?

Degrees of Freedom Problem Again

- Yes, we can increase n_i, AKA Do more Replicates

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  - Make no Engineering sense
  - or are Statistically insignificant

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  - Make no Engineering sense
  - or are Statistically insignificant

But we have to be careful and not cut too much!

- Take out interactions of little value
- Don't let regression model suffer too much

Measuring a Model

- Measure of Usefulness of a Regression Model is expressed in a term called **Coefficient of Determination**
- But most people call it $r^2 = \frac{SS_{model}}{SS_{total}}$
- The ratio of the variation the model can explain to the amount of total variation in the data.
- Usually easier to calculate from an ANOVA by:
- $r^2 = 1 - \frac{SS_{error}}{SS_{total}}$

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- Range of r^2 is? $\leq r^2 \leq$
- $r^2 = 1$: Means what? and $r^2 = 0$: Means what?

2^3 Design

But we are dealing with a bunch of independent factors and interactions so...

- We use an **Adjusted Coefficient of Determination**
- But most people call it r^2_{adj}
- $r^2_{adj} = 1 - \frac{df_{total}}{df_{error}} \frac{SS_{error}}{SS_{total}}$
- $r^2_{adj} < r^2$
- r^2_{adj} is safer to use in application

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- r^2_{adj} is safer to use in application

As for those pesky coefficients...without boring you with the needless details

- We just want to be confident that they are not 0. Why?

Use Engineering Sense

Occam's Razor: Simplest model is 'probably' best

- Throw out terms until r^2_{adj} doesn't degrade
- Use Engineering Sense and Principles
  1. Hierarchical Ordering Principle
     - Lower order effects are likely more important than higher
     - Effects of the same order are likely equally important
  2. Effect Sparsity Principle
     - Number of important effects in a factorial experiment is small
     - True model is most often linear.
  3. Effect Heredity Principle
     - For an interaction to be significant, usually at least one of its parent factors should be significant.

Back to our valve...

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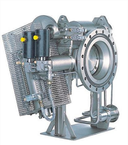

Optimization Concerns

Due to its use on a submarine, we want to minimize the acoustic operating characteristics. We have design options of:

- Nominal Diameter: Either 600 mm or 1200 mm
- Closing Springs: 1000N/m or 2000N/m
- Seal Manufacture: Metal-Metal or Metal-Elastomer

Setup

- There are THREE Factors [$K = 3$]
  - x_1: Nominal Diameter
  - x_2: Closing Springs Constant
  - x_3: Seal Manufacture
- There are TWO Levels to each factor
  - x_1: Nominal Diameter [$-$:600mm; $+$:1200mm]
  - x_2: Closing Springs Constant [$-$:1000N/m; $+$:2000N/m]
  - x_3: Seal Manufacture [$-$: M-M; $+$: M-E]
- So this is a 2^3 Design
- We are going to test all combinations with equal number of samples per combination, so it is a?
- Each replicate consists of $2^3 = 8$ runs

Design Matrix

Standard Order	*Run Order*	x_1	x_2	x_3
1	6	$-$	$-$	$-$
2	8	$-$	$-$	$+$
3	5	$-$	$+$	$-$
4	2	$-$	$+$	$+$
5	4	$+$	$-$	$-$
6	3	$+$	$-$	$+$
7	7	$+$	$+$	$-$
8	1	$+$	$+$	$+$

Planning Matrix

Standard Order	*Run Order*	*Nom. Dia.*	*Spring Const.*	*Seal*
1	6	600mm	1000N/m	M-M
2	8	600mm	1000N/m	M-E
3	5	600mm	2000N/m	M-M
4	2	600mm	2000N/m	M-E
5	4	1200mm	1000N/m	M-M
6	3	1200mm	1000N/m	M-E
7	7	1200mm	2000N/m	M-M
8	1	1200mm	2000N/m	M-E

Model Matrix

Standard Order	*Run Order*	x_1	x_2	x_3	x_{12}	x_{13}	x_{23}	x_{123}
1	6	$-$	$-$	$-$	$+$	$+$	$+$	$-$
2	8	$-$	$-$	$+$	$+$	$-$	$-$	$+$
3	5	$-$	$+$	$-$	$-$	$+$	$-$	$+$
4	2	$-$	$+$	$+$	$-$	$-$	$+$	$-$
5	4	$+$	$-$	$-$	$-$	$-$	$+$	$+$
6	3	$+$	$-$	$+$	$-$	$+$	$-$	$-$
7	7	$+$	$+$	$-$	$+$	$-$	$-$	$-$
8	1	$+$	$+$	$+$	$+$	$+$	$+$	$+$

Collecting Data and putting it into Model Matrix

Due to the cost and radiation exposure concerns only one replicate was done

Standard Order	y	x_1 A	x_2 B	x_3 C	x_{12} AB	x_{13} AC	x_{23} BC	x_{123} ABC
1	68	$-$	$-$	$-$	$+$	$+$	$+$	$-$
2	87	$-$	$-$	$+$	$+$	$-$	$-$	$+$
3	64	$-$	$+$	$-$	$-$	$+$	$-$	$+$
4	57	$-$	$+$	$+$	$-$	$-$	$+$	$-$
5	91	$+$	$-$	$-$	$-$	$-$	$+$	$+$
6	131	$+$	$-$	$+$	$-$	$+$	$-$	$-$
7	85	$+$	$+$	$-$	$+$	$-$	$-$	$-$
8	123	$+$	$+$	$+$	$+$	$+$	$+$	$+$

Possible Models

- Model Main Effects Only:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon_{error}$$

- Two Way Interaction Effects:

$$\begin{gathered} y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \\ +\beta_{12} x_{12} + \beta_{13} x_{13} + \beta_{23} x_{23} + \varepsilon_{error} \\ ;\, x_{12} = x_1 x_2;\, x_{13} = x_1 x_3;\, x_{23} = x_2 x_3 \end{gathered}$$

- ***Three Way Interaction Effects***:

$$\begin{gathered} y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \\ +\beta_{12} x_{12} + \beta_{13} x_{13} + \beta_{23} x_{23} \\ +\beta_{123} x_{123} + \varepsilon_{error} \\ ;\, x_{12} = x_1 x_2;\, x_{13} = x_1 x_3;\, x_{23} = x_2 x_3;\, x_{123} = x_1 x_2 x_3 \end{gathered}$$

2^3 Example: Full walk-through with R

Put the data into a CSV File and bring it into R

```
> myData <- read.csv("ME488_Lecture_Week3_DATA.csv")
> myData # Confirm Data is loaded
```

Output

```
    I Aco_Res  A  B  C
1 1      91  1 -1 -1
2 2     123  1  1  1
3 3      68 -1 -1 -1
4 4     131  1 -1  1
5 5      85  1  1 -1
6 6      87 -1 -1  1
7 7      64 -1  1 -1
8 8      57 -1  1  1
```

Run a Regression Model and Look at the Summary

```
> m<-lm(Aco_Res~A*B*C, data=myData)
> summary(m)
```

2^3 Example: Full walk-through with R

Output

```
Call:
lm(formula = Aco_Res ~ A * B * C, data = myData)

Residuals:
ALL 8 residuals are 0: no residual degrees of freedom!

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    88.25         NA      NA       NA
A              19.25         NA      NA       NA
B              -6.00         NA      NA       NA
C              11.25         NA      NA       NA
A:B             2.50         NA      NA       NA
A:C             8.25         NA      NA       NA
B:C            -3.50         NA      NA       NA
A:B:C           3.00         NA      NA       NA

Residual standard error: NaN on 0 degrees of freedom
Multiple R-squared:      1,   Adjusted R-squared:    NaN
F-statistic:   NaN on 7 and 0 DF,  p-value: NA
```

Model at this point

- Three Way Interaction Effects:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$
$$+\beta_{12} x_{12} + \beta_{13} x_{13} + \beta_{23} x_{23}$$
$$+\beta_{123} x_{123} + \varepsilon_{error}$$
$$; x_{12} = x_1 x_2; x_{13} = x_1 x_3; x_{23} = x_2 x_3; x_{123} = x_1 x_2 x_3$$

$$y = 88.25 + 19.25x_1 - 6.00x_2 + 11.25x_3 + 2.50x_{12} + 8.25x_{13} - 3.50x_{23} + 3.00x_{123} + \varepsilon_{error}$$

Supposedly everything matters, but no confidence in anything!

Downgrade the Model

- Drop the Three Way Interaction Effect

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3$$
$$+\beta_{12} x_{12} + \beta_{13} x_{13} + \beta_{23} x_{23} + \varepsilon_{error}$$

- And run it again

```
> m<-lm(Aco_Res~A+B+C+A*B+A*C+B*C, data=myData)
> summary(m)
```

Output

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    88.25       3.00  29.417   0.0216 *
A              19.25       3.00   6.417   0.0984 .
B              -6.00       3.00  -2.000   0.2952
C              11.25       3.00   3.750   0.1659
A:B             2.50       3.00   0.833   0.5577
A:C             8.25       3.00   2.750   0.2220
B:C            -3.50       3.00  -1.167   0.4511
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 8.485 on 1 degrees of freedom
Multiple R-squared:  0.9857,   Adjusted R-squared:  0.8998
F-statistic: 11.48 on 6 and 1 DF,  p-value: 0.2222
```

Issues

- The F is above 8 but we have to be careful due to low sample size

```
F-statistic: 11.48 on 6 and 1 DF,  p-value: 0.2222
```

- A few coefficients p-values are horrible, A:B [Nominal Diameter interacting with Spring Constants] is the worst

```
A:B              2.50       3.00   0.833   0.5577
```

- We still have room to move in r^2_{adj}

```
Multiple R-squared:  0.9857,   Adjusted R-squared:  0.8998
```

So lets get rid of A:B and see what happens

```
> m<-lm(Aco_Res~A+B+C+A*C+B*C, data=myData)
> summary(m)
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   88.250      2.761  31.959 0.000978 ***
A             19.250      2.761   6.971 0.019963 *
B             -6.000      2.761  -2.173 0.161884
C             11.250      2.761   4.074 0.055297 .
A:C            8.250      2.761   2.988 0.096145 .
B:C           -3.500      2.761  -1.268 0.332576
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 7.81 on 2 degrees of freedom
Multiple R-squared:  0.9757,   Adjusted R-squared:  0.9151
F-statistic: 16.09 on 5 and 2 DF,  p-value: 0.05954
```

2^3 Example: Full walk-through with R

You can keep refining..and will do in homework!

But before we go, remember we are trying to see if the confidence interval on the coefficients contain zero.

```
> confint(m, level=0.95)

                  2.5 %     97.5 %
(Intercept)  76.3689118 100.131088
A             7.3689118  31.131088
B           -17.8810882   5.881088
C            -0.6310882  23.131088
A:C          -3.6310882  20.131088
B:C         -15.3810882   8.381088
```

We are pretty sure A [Nominal Diameter] matters, and it looks like C [Seal Method] is getting close. Getting a sense that B [Spring Constants] don't really matter.

But...

- At the end we started dropping terms, which means we wasting resources by over-collecting data
- Is there a way to capitalize on this, if we know ahead of time that some interactions are not important?
- Yes, **Fractional Factorial Designs**

Things to help you succeed

- Design, Planning, and Model Matrices
- Relationship between Factors, x_i and ABC notation
- Interpreting a summary of a regression in R