ME 488: Design of Experiments

Lecture 6: Other Designs, Centerpoints, and RSM

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Topics

- Plackett-Burman Designs
- Adding Centerpoints
- Choosing a Design
- Response Surface Methods

Screening: Reducing the number of Factors

In some domains of mechanical engineering, it is very common to have large number of factors that could influence the response. Even a small number, still requiring dimensional orthogonality, can grow to become a large number. BFFEs are horrible at screening.

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Screening

Reducing the number of factors necessary by examining first order effects, with initial disregard to interaction effects

There are many methods of doing this, one common one is the **Plackett-Burman** design.

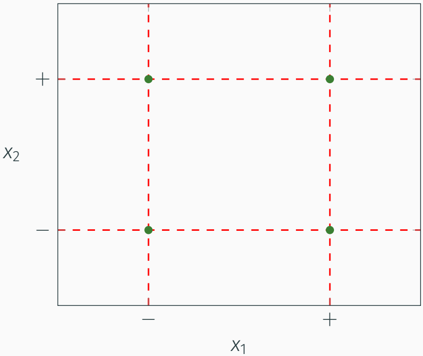
Screening K factors in K+1 runs/replicate!!!

- Current designs exist for $(K+1) \bmod 4 = 0$.
- Can use dummy factors if $(K+1) \bmod 4 \neq 0$.
- These are R_{III} designs, but
- No defining relationship due to the special way the aliasing occurs, the interactions are saturated and not exactly equal to a main effect.
- However, very efficient at quickly isolating large main effects in the absence of negligible interaction effects.

Example PB Design for $k = 11$, 12 runs/replicate

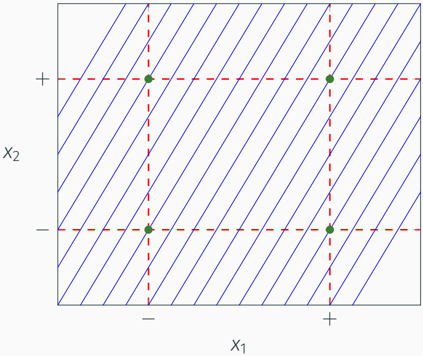
```
> pb(nruns=12, nfactors=11, randomize=FALSE)
    A  B  C  D  E  F  G  H  J  K  L
1   1  1 -1  1  1  1 -1 -1 -1  1 -1
2  -1  1  1 -1  1  1  1 -1 -1 -1  1
3   1 -1  1  1 -1  1  1  1 -1 -1 -1
4  -1  1 -1  1  1 -1  1  1  1 -1 -1
5  -1 -1  1 -1  1  1 -1  1  1  1 -1
6  -1 -1 -1  1 -1  1  1 -1  1  1  1
7   1 -1 -1 -1  1 -1  1  1 -1  1  1
8   1  1 -1 -1 -1  1 -1  1  1 -1  1
9   1  1  1 -1 -1 -1  1 -1  1  1 -1
10 -1  1  1  1 -1 -1 -1  1 -1  1  1
11  1 -1  1  1  1 -1 -1 -1  1 -1  1
12 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1 -1
class=design, type= pb
```

Traditional BFFE Design ($k = 2$)



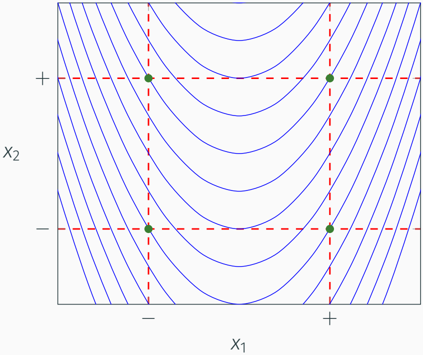
Traditional BFFE Design ($k = 2$)

For numerical and ordinal coded values this makes an assumption that the relationships are linear in nature



Traditional BFFE Design ($k = 2$)

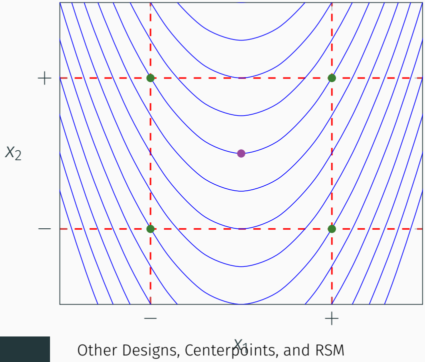
What happens when that assumption is wrong?



Traditional BFFE Design ($k = 2$)

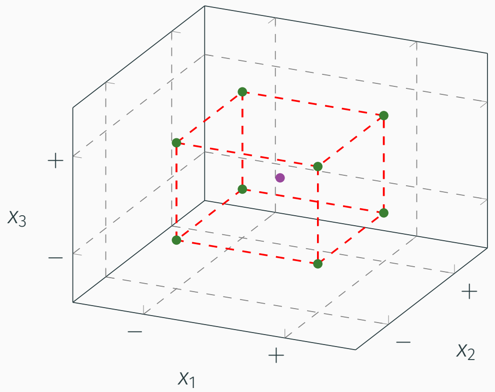
Add a few center points ($X_1 = 0, X_2 = 0$) to the beginning, end and evenly throughout the experiment. Rule of thumb is to add 3 to 5 of them in an experiment.

Bonus is this gives us DoF_{error} without having to do extra replicates! But only works for numerical ordinal factors.



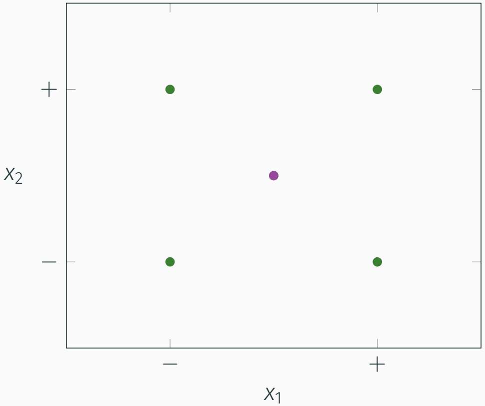
Traditional BFFE Design ($k = 3$)

Same idea in higher order models



The Idea behind BCC

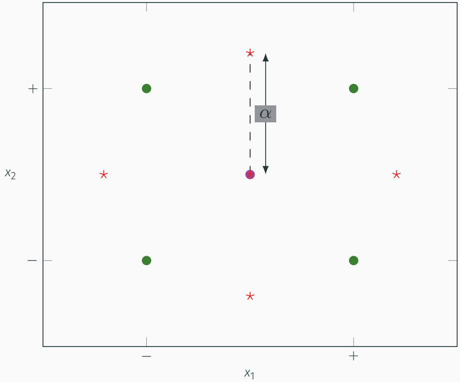
Start with a FFE or FrFE with center points



The Idea behind BCC

Then add **star** points at a distance of α.

[I know sorry, they just can't seem to use other letters]

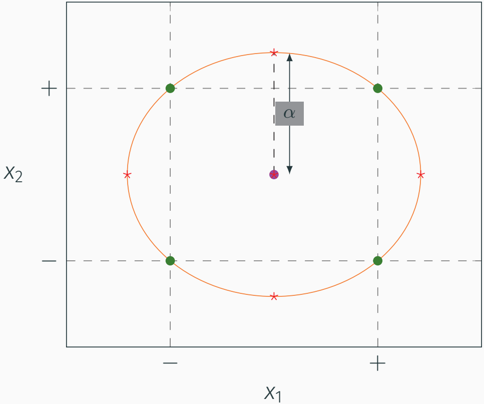


Rotatabilty

A design is rotatable if the variance of the predicted response at any point x depends only on the **distance** of x from the design center point, as opposed to its **direction**. A design with this property can be rotated around its center point without changing the prediction variance at x.

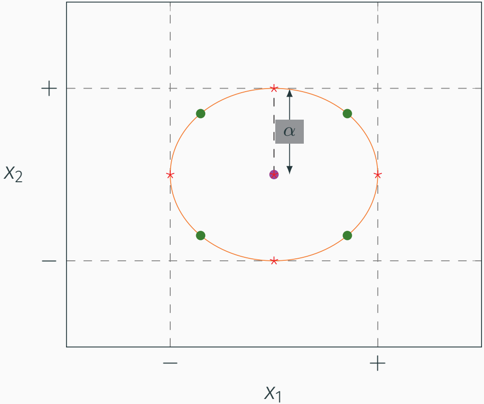
Central Composite Circumscribed

Three versions, two of which are rotatable, the first one is CCC ($\alpha = \sqrt{2}$).



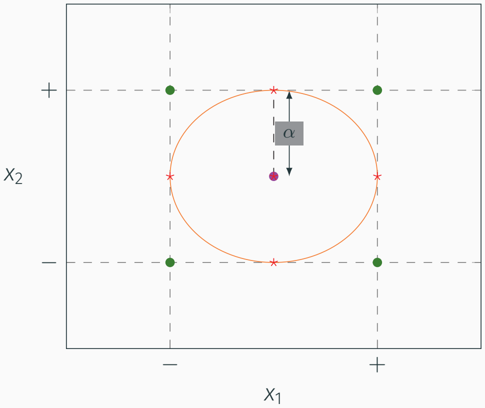
Central Composite Inscribed

Three versions, two of which are rotatable, the second one is CCI ($\alpha = 1$).



Central Composite Faced

The last one, CCF is not rotatable, but easy to accomplish. ($\alpha = 1$).



Determining α

To maintain rotatabilty, (one) α is determined by:

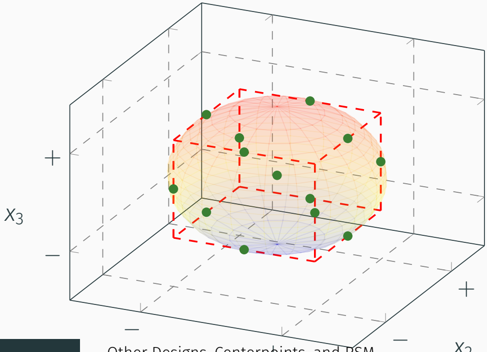
$$\alpha = \sqrt[4]{\text{number of factorial runs}}$$

Examples

$$2^3 \rightarrow \alpha = \sqrt[4]{2^3} = 1.682$$
$$2^4 \rightarrow \alpha = \sqrt[4]{2^4} = 2.000$$
$$2^5 \rightarrow \alpha = \sqrt[4]{2^5} = 2.378$$
$$2^{5-1} \rightarrow \alpha = \sqrt[4]{2^4} = 2.000$$
$$2^6 \rightarrow \alpha = \sqrt[4]{2^6} = 2.828$$
$$2^{6-1} \rightarrow \alpha = \sqrt[4]{2^5} = 2.378$$

Box-Behnken Design

BBD is a special design for RSM, but require 3 levels for each factor and are more advanced. For this class, it is only important to know that they are considered rotatable and, with the exception of the central point, the points sit on the midpoints of the cube edges



CCC

Provides high quality predictions over the entire design space, but require factor settings outside the range of the factors in the factorial part. Requires 5 levels per factor.

CCI

Uses only points within the factor ranges originally specified, but do not provide the same high quality prediction over the entire space compared to CCC. Requires 5 levels per factor.

CCF

Provides relatively high quality predictions over the entire design space and do not require using points outside the original factor range. However, they give poor precision for estimating pure quadratic coefficients. Requires 3 levels per factor.

BB

Require fewer treatment combinations than CC(C/F/I) involving 3 or 4 factors. Rotable (or nearly so) but contains regions of poor prediction quality like the CCI. Its ”missing corners” may be useful when the experimenter should avoid combined factor extremes. Requires 3 levels per factor

Response Surface Methods

Response Surface Methods

- If we have planar response surface [validated by centerpoint inclusion] and our levels chosen spanned the parameter limits, there is no need to explore for optimality, it is located quickly from our standard regression model
- But if those conditions are not met, we need a method to efficiently seek the optimal parameters.

Response Surface Methods

- If we have planar response surface [validated by centerpoint inclusion] and our levels chosen spanned the parameter limits, there is no need to explore for optimality, it is located quickly from our standard regression model
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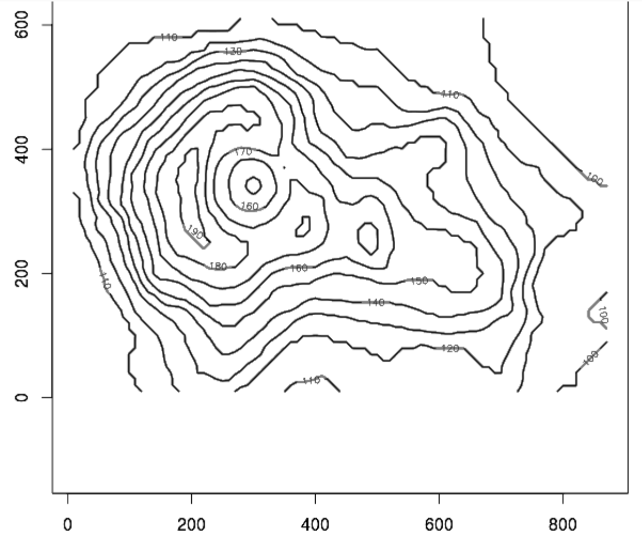
Why not just go to 3^k models?

- Sounds like a plan, but...
- Number of runs/replicate blows up in your face
- Aliasing determination is a calculative nightmare.


110
130
110
170
100
160
190
180
160
150
140
110
100
120
100
110
0
200
400
600
800
0
200
400
600

In this example scenario [1], we are attempting to locate the optimal response of a system given we can control two factors f_1 and f_2.

First Step: 2^2 **Full Factorial performed with 2 centerpoints**

f_1	f_2	x_1	x_2	x_1x_2	y
200	175	+1	+1	+1	75.9
200	125	+1	−1	−1	82.1
100	175	−1	+1	−1	70.1
100	125	−1	−1	+1	69.7
150	150	0	0	0	75.6
150	150	0	0	0	76.2

[1] From Understanding Industrial Designed Experiments. Schmidt.

Evaluate Regression Model of the FFE

Using regression and an $\alpha = 0.10$ [the Type I error..sigh]

```
lm(formula = y ~ x1 * x2 + I(x1^2), data = test)
...
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  75.9000     0.3000 253.000  0.00252 **
x1            4.5500     0.2121  21.449  0.02966 *
x2           -1.4500     0.2121  -6.835  0.09248 .
I(x1^2)      -1.4500     0.3674  -3.946  0.15799
x1:x2        -1.6500     0.2121  -7.778  0.08140 .
...
Residual standard error: 0.4243 on 1 degrees of freedom
Multiple R-squared:  0.9983,  Adjusted R-squared:  0.9914
F-statistic: 145.7 on 4 and 1 DF,  p-value: 0.06204
```

$$\hat{y} = 75.9 + 4.55x_1 - 1.45x_2 - 1.45{x_1}^2 - 1.65x_1x_2$$

Next Steps

Since the x^2, or quadratic factor, is insignificant (p=0.158), the optimum lies outside of this experimental region, so where do we go next?

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Follow the Steepest Ascent

$$\vec{g} = \nabla\hat{y} = \left(\frac{\partial\hat{y}}{\partial x_1}, \frac{\partial\hat{y}}{\partial x_2}\right)\bigg|_{x_1,x_2=(0,0)}$$

$$\vec{g} = (4.55, -1.45)$$

Scaled Ascent

But to be reasonable with our ascent, scale the $\vec{g}$ on the smallest absolute value of the components

$$\vec{g} = (4.55, -1.45)$$
$$\vec{g_{scaled}} = \left(\frac{4.55}{|-1.45|}, \frac{-1.45}{|-1.45|}\right)$$
$$\vec{g_{scaled}} = (3.14, -1)$$

Climb the Ascent

Now do a few experiments along the ascent, until it drops at least twice

x_1	x_2	y
$+3.14$	-1	91.0
$+6.28$	-2	89.4
$+9.42$	-3	77.0

Looks like the first one should be our new centerpoint, the others overshoot the optimum.

Our new experiment, centered at our new location

Setting up a CCC

f_1	f_2	x_1	x_2	y
357	100	4.14	−2.0	87.0
357	150	4.14	0.0	83.1
257	100	2.14	−2.0	75.7
257	150	2.14	0.0	86.9
307	125	3.14	−1.0	91.0
307	125	3.14	−1.0	90.1

Example of RSM

Convert to a new set to make things a bit easier to build

$$w_i = 2\left(\frac{f_i - \bar{f}_i}{f_{i_{max}} - f_{i_{min}}}\right)$$

Setting up a CCC

f_1	f_2	w_1	w_2	y
357	100	$+1$	-1	87.0
357	150	$+1$	$+1$	83.1
257	100	-1	-1	75.7
257	150	-1	$+1$	86.9
307	125	0	0	91.0
307	125	0	0	90.1

Run a new regression

```
lm(formula = y ~ w1 * w2 + I(w1^2), data = test)
...
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  90.5500     0.4500 201.222  0.00316 **
w1            1.8750     0.3182   5.893  0.10702
w2            1.8250     0.3182   5.735  0.10989
I(w1^2)      -7.3750     0.5511 -13.381  0.04749 *
w1:w2        -3.7750     0.3182 -11.864  0.05353 .
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘’. 0.1 ‘’  1

Residual standard error: 0.6364 on 1 degrees of freedom
Multiple R-squared:  0.9974,  Adjusted R-squared:  0.9871
F-statistic: 96.86 on 4 and 1 DF,  p-value: 0.07604
```

Since the w^2, or quadratic factor, is now significant (p=0.04749), inside our experimental region, so we need

Extra points and data

$$\alpha = \sqrt[4]{n_F} = \sqrt[4]{2^2} = \sqrt{2} = 1.414$$

w_1	w_2	y
$+1$	-1	87.0
$+1$	$+1$	83.1
-1	-1	75.7
-1	$+1$	86.9
0	0	91.0
0	0	90.1
$+1.414$	0	85.4
-1.414	0	77.6
0	$+1.414$	80.5
0	-1.414	77.4

Run a final regression [spare the details]

$$\hat{y} = 90.549 + 2.316w_1 - 3.775w_1w_2 - 3.787{w_1}^2 - 5.063{w_2}^2$$

Find Stationary Point

$$\begin{aligned}
\frac{\partial \hat{y}}{\partial w_1} &= 2.316 \qquad -3.775w_2 - 7.574w_1 &&= 0 \\
\frac{\partial \hat{y}}{\partial w_2} &= 0 \qquad -3.775w_1 \qquad -10.126w_2 &&= 0
\end{aligned}$$

$$\therefore (w_1, w_2)_{stationary} = (0.375, -0.139)$$

$$(f_1, f_2) = (325.75, 116.5)$$

Check what kind of stationary point is this

$$b_{11} = \text{the coefficient of } {w_1}^2$$
$$b_{22} = \text{the coefficient of } {w_2}^2$$
$$b_{12} = \text{the coefficient of } w_1 w_2$$
$$B = \begin{bmatrix} b_{11} & \frac{b_{12}}{2} \\ \frac{b_{12}}{2} & b_{22} \end{bmatrix}$$
$$B\vec{v} = \lambda\vec{v}$$
$$\lambda = \{-6.417, -2.433\}$$

Both eigenvalues are negative, so the stationary point is a maximum.

Eigenvalue Summary [for those who have forgotten]

- All Positive = Minimum
- All Negative = Maximum
- Some of Each = Saddle Point
- All one sign but some close to 0 = Rising or Falling Ridge

Review the Process