MAGIC SQUARES

A Magic Square is one which has a perfect sum and equal number of rows and columns.

8	1	6
3	5	7
4	9	2

In this case, all rows, columns, and longest diagonals have the same sum, 15. So, the magic number (this sum) is 15. Note that 15 is divisible by 3 (the number of rows in the magic square, $\boldsymbol{N}$). These basics are true for all magic squares.

ROWS:

1. $8+1+6 = 15$
2. $3+5+7 = 15$
3. $4+9+2 = 15$

COLUMNS:

1. $8+3+4=15$
2. $1+5+9=15$
3. $6+7+2=15$

LONGEST DIAGONALS:

(These are always only 2)

1. $4+5+6 = 15$
2. $8+5+2 = 15$

Here are some side by side odd numbered perfect Magic squares for comparison and reference later.

3x3

8	1	6
3	5	7
4	9	2

5x5

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

7x7

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

First, before I get to how these squares are formed, the above examples are called *'Perfect Magic Squares'*. In other words, there are multiple formations of Magic Squares, but according to the ancient Siamese people, it was found that in the development of these squares, the fastest paths always had the same pattern. Magic squares that followed this path were then called Perfect magic squares. For example, the squares below are all magic squares. However, the first one is the only perfect Magic Square.

8	1	6
3	5	7
4	9	2

6	7	2
1	5	9
8	3	4

112	119	114
117	115	113
116	111	118

Now for the details:

All perfect magic squares (with $\boldsymbol{N}$ number of rows where $\boldsymbol{N}$ is any odd number greater than 1) follow the same pattern. It's known as the ***Siamese method***.

You will notice that in a Perfect Magic square, if the first number is $\boldsymbol{F}$, and the last is $\boldsymbol{L}$, then:

$$\boldsymbol{L - F = N^2 - 1} \ (\boldsymbol{where\ N\ is\ the\ number\ of\ rows\ in\ the\ Magic\ Square}) .$$

In other words, for any perfect magic square, $\boldsymbol{L = N^2}$ and $\boldsymbol{F = 1}$.

How it works:

The method works starting with the box in the center of the top row. The numbers you place in the Magic square increase by one each time, following a sequence with a common difference of 1.

The Siamese method follows that when you're at any position of the magic square and are looking to continue the sequence, ***you move one square right, and one square up***. If the box you land on is occupied, you stand on it, ***move one box left, and 2 boxes down*** (Alternatively, stay in the box of the number you have just written and fill the box below it with the next number of the sequence. It's the same thing). Make sense? Hopefully, it does.

However, there are several scenarios where there is simply no space left to move. What then? For that, visit the summary section of this document. A detailed methodology is given (but first, let's try and fully understand the dynamics of the Perfect Magic Square).

This will all make sense in the patterns in the summary section a bit later. For now, here are some mathematical principles governing Perfect Magic Squares.

Constants to be used later:

$\boldsymbol{N}, number\ of rows\ or\ columns\ in\ the\ Magic\ square.$ $\boldsymbol{M}, Magic\ Number\ of\ the\ Magic\ square$

$\boldsymbol{F}, First\ number\ in\ the\ Magic\ Square's\ sequence, aka\ the\ smallest\ number\ of\ the\ grid$

$\boldsymbol{L}, Last\ number\ in\ the\ Magic\ Square's\ sequence, aka\ the\ largest\ number\ of\ the\ grid$

$\boldsymbol{K}, Middle\ number\ in\ the\ Magic\ Square's\ sequence, always\ in\ the\ center\ of\ the\ grid$

For perfect Magic squares (With $\boldsymbol{N > 1}$ and $\boldsymbol{N}$ $is\ an\ odd\ positive\ integer$) the table below is quite useful:

Number of squares in Magic Square	First number of sequence (F)	Last number of sequence (L)	Middle Number/ Median Number (K)	Magic Number / Perfect sum (M)
N^2	$F = 1$	$L = N^2$	$K = \dfrac{N^2+1}{2}$	$M = \dfrac{N(N^2+1)}{2}$

Here is the summary section of all odd numbered Magic squares. This 7x7 grid is for illustration.

B	B	B	**<u>B (F) first</u>**	B	B	**<u>Z</u>**
✔						A
✔						A
✔			**<u>K middle</u>**			A
✔						A
<u>Z</u>						A
✔	✔	✔	**<u>(L) last</u>**	✔	✔	A

The <u>SIAMESE METHOD</u> is used in the situation above (odd number of rows greater than 1).

Here's a 3x3 for comparison. The boxes from the 7x7 above still apply and still follow the rules.

B	**<u>B (F) first</u>**	**<u>Z</u>**
<u>Z</u>	**<u>K middle</u>**	A
✔	**<u>(L) last</u>**	A

<u>RULES OF THE SIAMESE METHOD (for odd values of *N* greater than 1):</u>

✔	When in this box, your next box is always 1 up, 1 right (North East).
<u>(L) last</u>	Only filled at the end of the sequence. When filed, it is a guarantee that your sequence has ended successfully. For perfect Magic Squares, it is always N^2.
	These boxes follow the same rule as the boxes with ✔ with only one exception. If the box you are to fill next (North East) is full, stay on the square of the number you've just written and occupy the box below it (South).
A	Next box is always one row up, in the first column.
<u>Z</u>	These 2 boxes are special in all odd numbered grids because they are always: **1.** The top right corner of the Magic square (1st row, last column), no matter the value of *N* (odd numbers greater than 1) **2.** 2nd last row, 1st column. In the sequence, the next box is always one row down, in the same column.
B	Next box is always one column to the right, in the last row.

These rules will always apply for all Magic squares with *N* as an odd number greater than 1.

Now that you know the rules, it's time to apply them. Remember, all perfect Magic squares have their first number as 1. So, go ahead, grab a piece of paper, and draw out your grid. Follow the rules in the summary above and for starters, begin with a 3x3 grid and work your way upwards as you gain the confidence. Compare your results with the grids in the first page.

Once you are comfortable with the Siamese method, you are finally ready to play around with the values in the Magic Squares.

Odd numbered Magic Squares that aren't perfect:

A magic square can still be made with different numbers, not just those between **1 and N^2.** This can be achieved only if you know 2 things.

1. The value of N.
2. The Magic number, M (This number must be divisible by N at all costs, otherwise it won't work)

You can then manipulate the values of the Magic square as desired. With M known, work your way backwards.

Note: Remember, M can be any Positive integer greater than the perfect sum of the perfect magic square with the same number of sides N . In other words, $M \geq \frac{N(N^2+1)}{2}$.

For example, for a 3x3, $M \geq 15.$

As you can see, M is divisible by N and satisfies the rule $M \geq \frac{N(N^2+1)}{2}$**.** Using M, follow these steps.

1. Divide M by the number of rows. ($\frac{M}{N}$)
2. Your quotient will give you the number in the middle of the grid, K. ($\frac{M}{N} = K$)
3. Using a constant for the Magic square in question (Let's call it θ), follow the next instructions carefully.

Step 1:

For a 3x3 as an example, with $F = 1$, the constant $\theta = 4.$ $M \geq 15.$

For a 5x5 as an example, with $F = 1$, the constant $\theta = 6.$ $M \geq 65$

For a 7x7 as an example, with $F = 1$, the constant $\theta = 8.$ $M \geq 175$

NB: You can notice that $\theta = N + 1.$

Follow through below. For context, each square is to be mapped out in the format given in the first grid. So, rows are labelled with A, B and C, while columns are labelled 1,2 and 3.

A1	A2	A3
B1	B2	B3
C1	C2	C3

Grid 1

8	1	6
3	5	7
4	9	2

Grid 2

118	111	116
113	115	117
114	119	112

Notice, in both Grids 1 and 2, starting from the top row center (A2), they all follow the same sequence, with a common difference of 1. The pattern then goes the same way. A2, C3, B1, C1, B2, A3, B3, A1 and finally C2.

Step 2:

Look at just the middle column. Notice that from top to bottom, numbers are increasing by $\boldsymbol{\theta}$**.** Pretty cool right! With this knowledge, if you have the value of $\boldsymbol{M}$, you can get the value of the middle number $\boldsymbol{K}$. After that, the rest of the numbers in the middle column can be calculated by either:

1. Adding $\boldsymbol{\theta}$ to $\boldsymbol{K}$ for numbers below the box containing $\boldsymbol{K}$**,** that is, $\boldsymbol{K+\ \theta}$
2. Or subtracting $\boldsymbol{\theta}$ from $\boldsymbol{K}$ for numbers above the box containing $\boldsymbol{K}$**,** that is**,** $\boldsymbol{K-\ \theta}$

In the grid examples above, for example, $\boldsymbol{A2=B2-\theta}$ and $\boldsymbol{C2=B2+\theta}$

For any larger Magic Squares, the same rules hold. **<u>Example 1</u>** is the 5x5 below.

<u>Grid Map.</u> **<u>5x5 Grid.</u>**

A1	A2	A3	A4	A5		17	24	1	8	15		$\boldsymbol{A3=B3-\theta}$
B1	B2	B3	B4	B5		23	5	7	14	16		$\boldsymbol{B3=C3-\theta}$
C1	C2	C3	C4	C5	**<u>Separation here</u>**	4	6	13	20	22	**<u>Separation here</u>**	$\boldsymbol{C3=K}$
D1	D2	D3	D4	D5		10	12	19	21	3		$\boldsymbol{D3=C3+\theta}$
E1	E2	E3	E4	E5		11	18	25	2	9		$\boldsymbol{E3=D3+\theta}$

Example 2:

Here, the following is known:

$\boldsymbol{N = 5, M = 1105, \theta = N + 1 = 6}$ **and** $\boldsymbol{K = \frac{M}{N} = 221.}$

Grid Map. **5x5 Grid.**

A1	A2	A3	A4	A5		225	232	209	216	223		$\boldsymbol{A3 = B3 - \theta}$
B1	B2	B3	B4	B5		231	213	215	222	224		$\boldsymbol{B3 = C3 - \theta}$
C1	C2	C3	C4	C5	**Separation here**	212	214	221	228	230	**Separation here**	$\boldsymbol{C3 = K}$
D1	D2	D3	D4	D5		218	220	227	229	211		$\boldsymbol{D3 = C3 + \theta}$
E1	E2	E3	E4	E5		219	226	233	210	217		$\boldsymbol{E3 = D3 + \theta}$

Exercise:

1. Build a 7x7 grid with $\boldsymbol{M = 903}$.
2. Build a 9x9 grid with $\boldsymbol{M = 369}$.
3. Build a 11x11 grid with $\boldsymbol{M = 770}$.

Solutions:

7x7: $\boldsymbol{M = 903}$.

134	143	152	105	114	123	132
142	151	111	113	122	131	133
150	110	112	121	130	139	141
109	118	120	129	138	140	149
117	119	128	137	146	148	108
125	127	136	145	147	107	116
126	135	144	153	106	115	124

9x9: $M = 369.$

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

11x11: $M = 770.$

77	90	103	116	129	10	23	36	49	62	75
89	102	115	128	20	22	35	48	61	74	76
101	114	127	19	21	34	47	60	73	86	88
113	126	18	31	33	46	59	72	85	87	100
125	17	30	32	45	58	71	84	97	99	112
16	29	42	44	57	70	83	96	98	111	124
28	41	43	56	69	82	95	108	110	123	15
40	53	55	68	81	94	107	109	122	14	27
52	54	67	80	93	106	119	121	13	26	39
64	66	79	92	105	118	120	12	25	38	51
65	78	91	104	117	130	11	24	37	50	63

<u>Conclusion</u>

With all this, the possibilities are endless. For programmers, a good project might be coding to generate Magic Squares of such a nature to infinite magnitudes. A hint could be to use a user input as $\boldsymbol{M}$ **or** $\boldsymbol{N}$ **or even both.** I have succeeded in this endeavor and you can access the C++ code in a text document HERE. Enjoy.

_***Cj***