1 Introduction

In this assignment my goal was to determine the optimal time to boil a potato. We would consider the potato to be rectangle in two spatial dimensions. The simulation we will run involves placing a potato of size 4cm by 5cm where $t_start = 0$ and the water and room temperature are at $T_{room} = 20°$. The temperature of the water will rise from 20° to 100° in 60 seconds, and then stay at 100° afterwards. At $T_cooking = 65°$ the potato will begin cooking at this point we will assume it will take 300 seconds for the potato to fully cook.

The temperature $T = T(t, x, y)$ inside the potato satisfies the heat equation

$$\frac{\partial T}{\partial t} = \lambda \Delta T,\ (x, y) \in \Omega \tag{1}$$

with boundary conditions

$$T(t, x, y) = T_{water}(t),\ (x, y) \in \partial\Omega \tag{2}$$

and initial conditions

$$T(0, x, y) = T_{room},\ (x, y) \in \Omega \tag{3}$$

where λ is the thermal diffusivity of the potato.

1.1 structure of the linear system

We will consider a rectangular domain $\Omega = [x_l; x_r]x[y_b; y_t]$

$$\begin{cases} \frac{\partial T}{\partial t} = \lambda \Delta T + f, & (x, y) \in \Omega \\ T(t, x, y) = T_{bc}(t, x, y), & (x, y) \in \partial\Omega \\ T(t_{start}, x, y) = T_{start}(x, y), & (x, y) \in \Omega \end{cases}$$

To find the numerical solution we will discretize our domain into N_x and N_y points. at which we can approximate the first equation in the piecewise equation, which will give us

$$\frac{T^{n+1}_{i,j} - T^n_{i,j}}{\Delta t} = \lambda\left(\frac{T^{n+1}_{i+1,j} - 2T^{n+1}_{i,j} + T^{n+1}_{i-1,j}}{\Delta x^2} + \frac{T^{n+1}_{i,j+1} - 2T^{n+1}_{i,j} + T^{n+1}_{i,j-1}}{\Delta y^2}\right) + f^n_{i,j} \tag{4}$$

where $T^n_{i,j} = T(t_n, x_i, y_j)$ and Δx and Δy are the given space steps for x and y. Then it can be rewritten as

$$T^{n+1}_{i,j-1}\left(-\frac{\lambda\Delta t}{\Delta y^2}\right) + T^{n+1}_{i-1,j}\left(-\frac{\lambda\Delta t}{\Delta x^2}\right) + T^{n+1}_{i,j}\left(1 + 2\frac{\lambda\Delta t}{\Delta x^2} + 2\frac{D\Delta t}{\Delta y^2}\right) + T^{n+1}_{i+1,j}\left(-\frac{\lambda\Delta t}{\Delta x^2}\right) + T^{n+1}_{i,j+1}\left(-\frac{\lambda\Delta t}{\Delta y^2}\right) = \tag{5}$$

$$T^n_{i,j} + \Delta t f(t_{n+1}, x_i, y_i)$$

this approximation (5) gives us the value for all interior nodes and while boundary nodes can be solved with the boundary condition, as a result we end up with the linear system

$$A \cdot \vec{T}^{n+1} = \vec{r}$$

2 Implicit Scheme

We will test the implicit scheme using the following:

Domain:

$$\Omega = [-1, 1]x[-0.5, 1.7]$$

Thermal diffusivity:

$$\lambda = 0.75$$

Exact Solution:

$$T_{exact} = sin(x)cos(y)exp(-t)$$

Where initial conditions $T_{start}(x, y)$, boundary conditions $T_{bc}(t, x, y)$, and source term $f(t, x, y)$ are calculated from the exact solution. We solve the heat equation where $t_{start} = 0$ and $t_{final} = 1$ for the grid resolutions $(N_x, N_y) = (25, 30), (50, 60)$ *and* $(100, 120)$ and a time-step of $\Delta t = 0.5\Delta x$.

2.1 Results

(N_x, N_y)	(25,30)	(50,60)	(100,120)
max error	0.000583961	0.000278453	0.000136207
Order k	0	1.06844	1.03163

Our results for calculating the numerical solution for the implicit schemes tells us that the order of accuracy of the implicit scheme is order of accuracy 1.

3 Cooking a Potato

The conditions we used to boil the potato are:

Domain:

$$\Omega = [-2, 2]x[-2.5, 2.5]$$

Thermal diffusivity:

$$\lambda = 1.5\ x\ 10^{-3} cm^2/s$$

initial conditions:

$$T_{start}(x, y) = 20°C$$

Boundary conditions:

$$T_{bc}(t, x, y) = min(20 + 80\frac{t}{60}, 100)°C$$

source term:

$$f(t, x, y) = 0$$

We solve the heat equation from $t_{start} = 0s$ to $t_{final} = 1500s$ using the grid $N_x = 80$ and $N_y = 100$ with a time-step of $\Delta t = 5s$

3.1 Results

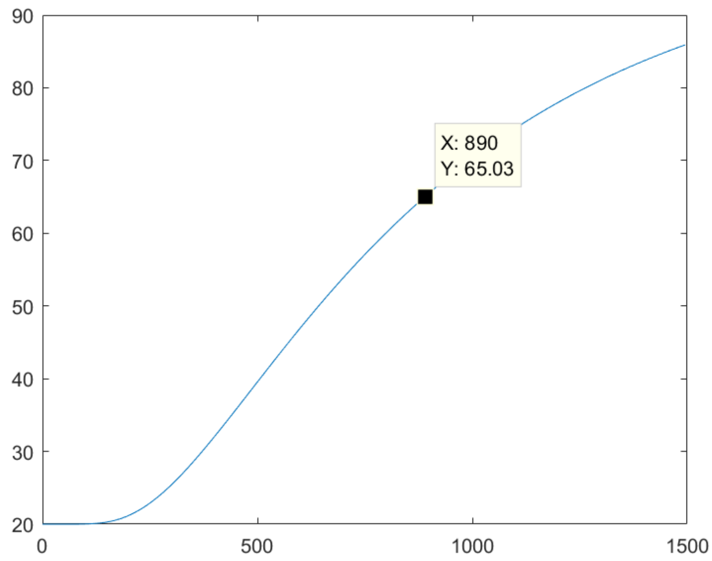

Figure 1: where the x-axis is in time and y-axis is the temperature

The time it takes for the potato to reach an internal temperature of $65°C$ at the center of the potato ((x,y) = (0,0)). Since the current time is at 890s then the potato will be fully cooked at 1190s or approximately 20mins.

3.2 Snapshots

These are the temperature distribution snapshots taken of the potato cooking at t=0,200,400,600s.

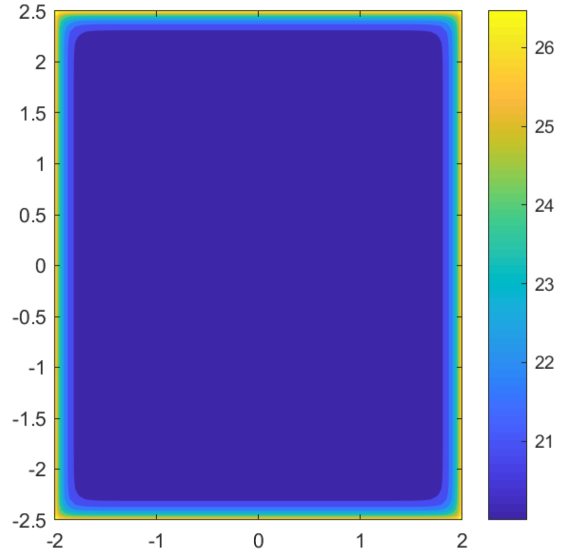

Figure 2: t=0s

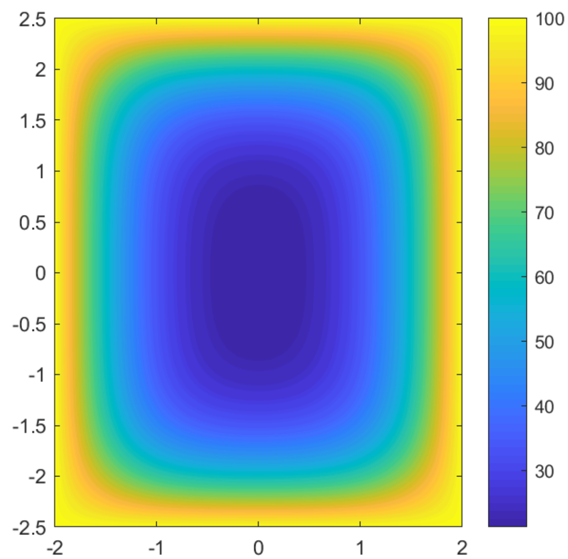

Figure 3: t=200s

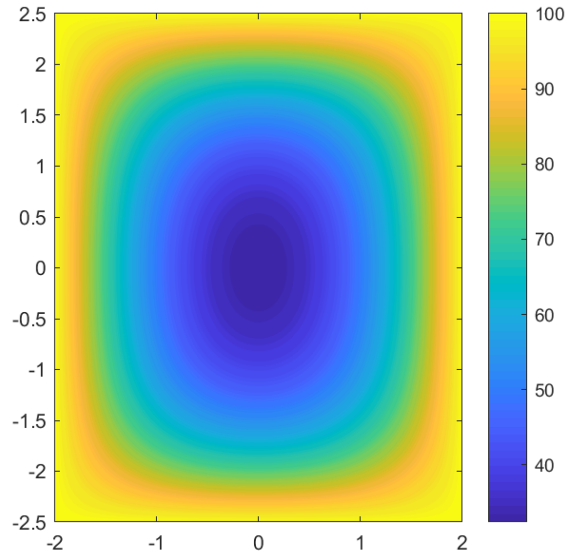

Figure 4: t=400s

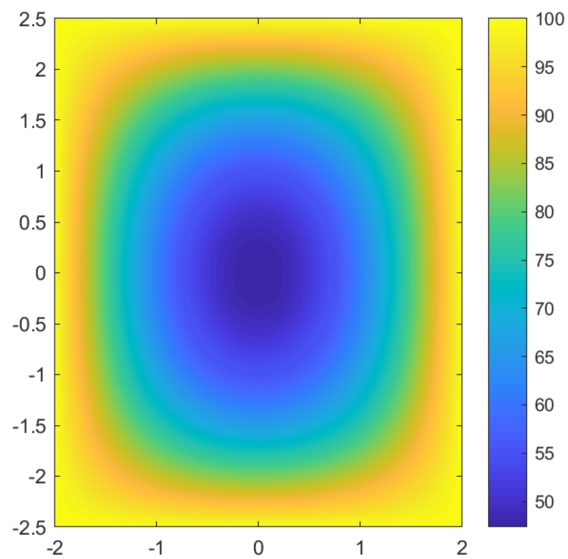

Figure 5: t=600s