1 Introduction

In this assignment we will estimate the spread of oil on the shoreline of Santa Barbara to determine when to close a beach. We will use the health guideline where concentration of oil, $c_{limit} = .006$ or higher is deemed unsafe to bathe in. We will model this process through an advection-diffusion equation in two spatial dimensions, and we will assume the stretch of coast that we are interested in is straight. Therefore we must consider a domain $\Omega = [x_l; x_r]x[y_b; y_t]$ with a given velocity $\vec{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}$. The equation that satisfies the advection-diffusion equation

$$\frac{\partial c}{\partial t} + \vec{v} * \nabla c = D\Delta c + f,\ for\ all\ (x, y) \in \Omega \tag{1}$$

where D is the rate of diffusion of oil in water, with initial conditions

$$c(t_{start}, x, y) = 0; \tag{2}$$

and let us assume that the left, right, and top boundaries of domain Ω are far enough, so that the oil concentration stays zero at those boundaries, that is

$$c(t_{start}, x, y) = 0;\ if\ x = x_l,\ x = x_r,\ or\ y = y_t \tag{3}$$

The bottom boundary of the domain has to satisfy the no-flux condition

$$D\frac{\partial c}{\partial y} - v_y c = 0,\ if\ y = y_b$$

1.1 structure of the linear system

We will consider a rectangular domain $\Omega = [x_l; x_r]x[y_b; y_t]$

$$\begin{cases} PDE: & \frac{\partial c}{\partial t} + \vec{v} * \nabla c = D\Delta c + f,\ for\ all\ (x, y) \in \Omega \\ BC: & c(t, x, y) = c_{bc}(t, x, y),\ if\ x = x_l,\ x = x_r,\ or\ y = y_t \\ & D\frac{\partial c}{\partial y} - v_y c = 0,\ if\ y = y_b \\ IC: & c(t_{start}, x, y) = c_{start}(x, y),\ \ (x, y) \in \Omega \end{cases}$$

Let us first approximate the advection term $\vec{v} * \nabla c$ using the numerical solution at time $t = t_n$ and the diffusion term D $(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial x^2})$ at moment times $t = t_{n+1}$ we will also use the fact $v_x < 0$ and $v_y < 0$, and transferring all known terms to the right hand side we get

$$c^{n+1}_{i,j-1}(-\frac{D\Delta t}{\Delta y^2}) + c^{n+1}_{i-1,j}(-\frac{D\Delta t}{\Delta x^2}) + c^{n+1}_{i,j}(1 + \frac{2D\Delta t}{\Delta x^2} + \frac{2D\Delta t}{\Delta y^2}) + c^{n+1}_{i+1,j}(-\frac{D\Delta t}{\Delta x^2}) + c^{n+1}_{i,j+1}(-\frac{D\Delta t}{\Delta y^2}) = \tag{4}$$

$$c^n_{i,j} + \Delta t f(t_{n+1}, x_i, y_j) - v_x \Delta t \frac{c^n_{i+1,j} - c^n_{i,j}}{\Delta x} - v_y \Delta t \frac{c^n_{i,j+1} - c^n_{i,j}}{\Delta y}$$

This approximation will be valid for all internal grid points and the boundaries on the left,right and top will use the Dirichlet boundary condition. we must know apply equation (4) to the lower boundary, but we cannot apply it since it involves the numerical solution at point $(x_i, 0)$ which is outside the domain. To resolve this, we can approximate a "ghost point" node. We end up with the a valid approximation of advection-diffusion

$$c_{i-1,1}^{n+1}(-\frac{D\Delta t}{\Delta x^2}) + c_{i,1}^{n+1}(1 + \frac{2D\Delta t}{\Delta x^2} + \frac{2D\Delta t}{\Delta y^2} + \frac{2v_y\Delta t}{\Delta y}) + c_{i+1,1}^{n+1}(-\frac{D\Delta t}{\Delta x^2}) + c_{i,2}^{n+1}(-\frac{D\Delta t}{\Delta y^2}) = \quad (5)$$

$$c_{i,j}^n + \Delta t f(t_{n+1}, x_i, y_j) - v_x \Delta t \frac{c_{i+1,1}^n - c_{i,1}^n}{\Delta x} - v_y \Delta t \frac{c_{i,2}^n - c_{i,1}^n}{\Delta y} - \frac{2\Delta t}{\Delta y} g(t_{n+1}, x_i, y_1)$$

Our final result gives us the linear system

$$A \cdot \vec{c^{n+1}} = \vec{r}$$

2 Advection and Diffusion

we will use the advection and diffusion equation with the following:

Domain:

$$\Omega = [-1, 3]x[-1.5, 1.5]$$

diffusivity:

$$\lambda = 0.7$$

Velocity field:

$$v_x = -0.8$$

$$v_y = -0.4$$

Exact Solution:

$$T_{exact} = sin(x)cos(y)exp(-t)$$

Where initial conditions $T_{start}(x, y)$, boundary conditions $T_{bc}(t, x, y)$, and source term $f(t, x, y)$, and g(t,x,y) are calculated from the exact solution. We solve the heat equation where $t_{start} = 0$ and $t_{final} = 1$ for the grid resolutions $(N_x, N_y) = (20, 15), (40, 30), (80, 60)$ *and* $(160, 120)$ and a time-step of $\Delta t = 0.5\Delta x$.

2.1 Results

(N_x, N_y)	(20,15)	(40,30)	(80,60)	(160,120)
max error	0.0240409	0.0119852	0.00598461	0.00299154
Order k	0	1.00424	1.00193	1.00037

Our results for calculating the numerical solution for the advection and diffusion equation tells us that the order of accuracy of the implicit scheme is order of accuracy 1.

3 Oil Spill

The conditions we used to determine the advection and diffusion of the oil spill:

Domain:

$$\Omega = [0, 12]x[0, 3]$$

diffusivity:

$$D = 0.2$$

initial conditions:

$$c_{start}(x, y) = 0$$

Boundary conditions:

$$c_{bc}(t, x, y) = 0$$
$$g(t, x, y) = 0$$

source term:

$$\begin{cases} \frac{1}{2}(1 - tanh(\frac{\sqrt{(x-x_s)^2+y^2}-r_s}{\epsilon})), & ift < 0.5 \\ 0, & if \;\; t > 0.5 \end{cases}$$

where $x_s = 10$, $r_s = 0.1$, and $\epsilon = 0.1$
We solve the heat equation from $t_{start} = 0$ to $t_{final} = 10$ using the grid $N_x = 160$ and $N_y = 40$ with a time-step of $\Delta t = 0.1$

3.1 Results

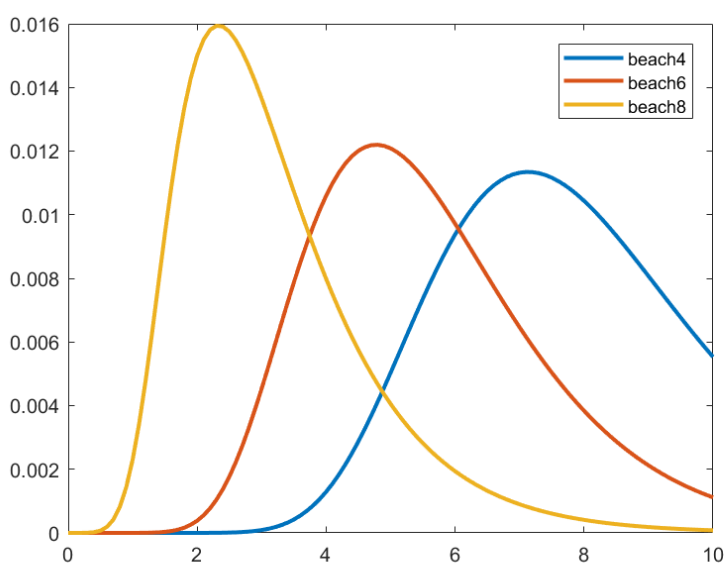

Figure 1: where x-axis is in time and y-axis is the concentration of oil

The graph shows each of the beaches at (x,y) = (4,0), (6,0), and (8,0), and the concentration the beach is at from $t_{start} = 0$ to $t_{final} = 10$. From the graph we conclude that when t= 1.3 the concentration of beach (8,0) is over the health limit of $c_{limit} = .006$, at t = 3.3 beach (6,0) is not safe to enter, and lastly at t = 5.4 beach (4,0) is not safe to bathe in.

3.2 Snapshots

These are the oil concentration snapshots taken of the oil spill at t=1,4,7.

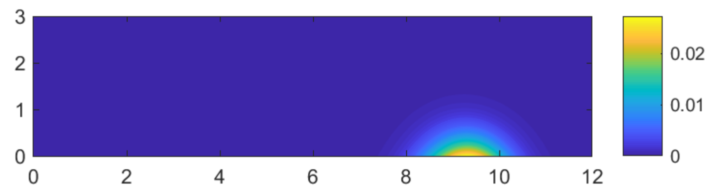

Figure 2: t=1

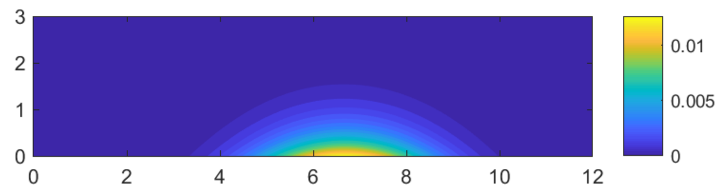

Figure 3: t=4

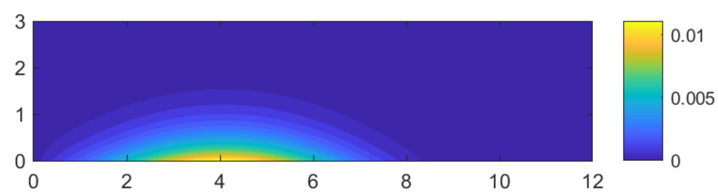

Figure 4: t=7