Digraphs
Graphs, digraphs, and multidigraphs in GAP
1.9.0

6 September 2024

Jan De Beule

Julius Jonusas

James Mitchell

Wilf A. Wilson

Michael Young

Marina Anagnostopoulou-Merkouri

Finn Buck

Stuart Burrell

Graham Campbell

Raiyan Chowdhury

Reinis Cirpons

Ashley Clayton

Tom Conti-Leslie

Joseph Edwards

Luna Elliott

Isuru Fernando

Ewan Gilligan

Sebastian Gutsche

Samantha Harper

Max Horn

Christopher Jefferson

Olexandr Konovalov

Hyeokjun Kwon

Andrea Lee

Saffron McIver

Michael Orlitzky

Matthew Pancer

Markus Pfeiffer

Daniel Pointon

Lea Racine

Christopher Russell

Artur Schaefer

Isabella Scott

Kamran Sharma

Finn Smith

Ben Spiers

Maria Tsalakou

Meike Weiss

Murray Whyte

Fabian Zickgraf

Jan De Beule
Email: jdebeule@cage.ugent.be
Homepage: http://homepages.vub.ac.be/~jdbeule/
Address: Vrije Universiteit Brussel, Vakgroep Wiskunde, Pleinlaan 2, B - 1050 Brussels, Belgium

Julius Jonusas
Email: j.jonusas@gmail.com
Homepage: http://julius.jonusas.work

James Mitchell
Email: jdm3@st-andrews.ac.uk
Homepage: https://jdbm.me
Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Wilf A. Wilson
Email: gap@wilf-wilson.net
Homepage: https://wilf.me

Michael Young
Email: mct25@st-andrews.ac.uk
Homepage: https://mct25.host.cs.st-andrews.ac.uk
Address: Jack Cole Building, North Haugh, St Andrews, Fife, KY16 9SX, Scotland

Marina Anagnostopoulou-Merkouri

Email: mam49@st-andrews.ac.uk

Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Finn Buck

Email: finneganlbuck@gmail.com

Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Stuart Burrell

Email: stuartburrell1994@gmail.com

Homepage: https://stuartburrell.github.io

Reinis Cirpons

Email: rc234@st-andrews.ac.uk

Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Ashley Clayton

Email: ac323@st-andrews.ac.uk

Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Tom Conti-Leslie

Email: tom.contileslie@gmail.com

Homepage: https://tomcontileslie.com

Joseph Edwards

Email: jde1@st-andrews.ac.uk

Homepage: https://github.com/Joseph-Edwards

Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Luna Elliott

Email: TODO

Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Isuru Fernando

Email: isuruf@gmail.com

Ewan Gilligan

Email: eg207@st-andrews.ac.uk

Sebastian Gutsche

Email: gutsche@momo.math.rwth-aachen.de

Samantha Harper

Email: seh25@st-andrews.ac.uk

Max Horn

Email: horn@mathematik.uni-kl.de

Homepage: https://www.quendi.de/math

Address: Fachbereich Mathematik, TU Kaiserslautern, Gottlieb-Daimler-Straße 48, 67663 Kaiserslautern, Germany

Christopher Jefferson

Email: caj21@st-andrews.ac.uk

Homepage: https://caj.host.cs.st-andrews.ac.uk

Address: Jack Cole Building, North Haugh, St Andrews, Fife, KY16 9SX, Scotland

Olexandr Konovalov

Email: obk1@st-andrews.ac.uk

Homepage: https://olexandr-konovalov.github.io/

Address: Jack Cole Building, North Haugh, St Andrews, Fife, KY16 9SX, Scotland

Hyeokjun Kwon

Email: hk78@st-andrews.ac.uk

Andrea Lee

Email: ahwl1@st-andrews.ac.uk

Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Saffron McIver

Email: sm544@st-andrews.ac.uk

Michael Orlitzky

Email: michael@orlitzky.com

Homepage: https://michael.orlitzky.com/

Matthew Pancer

Email: mp322@st-andrews.ac.uk

Markus Pfeiffer

Email: markus.pfeiffer@morphism.de

Homepage: https://markusp.morphism.de/

Daniel Pointon

Email: dp211@st-andrews.ac.uk

Lea Racine

Email: lr217@st-andrews.ac.uk

Address: Jack Cole Building, North Haugh, St Andrews, Fife, KY16 9SX, Scotland

Artur Schaefer

Email: as305@st-and.ac.uk

Isabella Scott

Email: iscott@uchicago.edu

Kamran Sharma

Email: kks4@st-andrews.ac.uk

Address: Jack Cole Building, North Haugh, St Andrews, Fife, KY16 9SX, Scotland

Finn Smith

Email: fls3@st-andrews.ac.uk

Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Ben Spiers

Email: bspiers972@outlook.com

Maria Tsalakou

Email: mt200@st-andrews.ac.uk

Homepage: https://mariatsalakou.github.io/

Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Meike Weiss

Email: weiss@art.rwth-aachen.de

Homepage: https://bit.ly/4e6pUeP

Address: Chair of Algebra and Representation Theory, Pontdriesch 10-16, 52062 Aachen

Murray Whyte

Email: mw231@st-andrews.ac.uk

Address: Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Fabian Zickgraf

Email: f.zickgraf@dashdos.com

Abstract

The Digraphs package is a GAP package containing methods for graphs, digraphs, and multidigraphs.

Copyright

Acknowledgements

We would like to thank Christopher Jefferson for his help in including bliss in Digraphs. This package's methods for computing digraph homomorphisms are based on work by Max Neunhöffer, and independently Artur Schäfer.

Contents

Chapter 1

The Digraphs package

1.1 Introduction

This is the manual for version 1.9.0 of the Digraphs package. This package was developed at the University of St Andrews by:

- Jan De Beule,

- Julius Jonušas,

- James D. Mitchell,

- Maria Tsalakou,

- Wilf A. Wilson, and

- Michael C. Young.

Additional contributions were made by:

- Marina Anagnostopoulou-Merkouri,

- Finn Buck,

- Stuart Burrell,

- Reinis Cirpons,

- Tom Conti-Leslie,

- Luke Elliott,

- Ewan Gilligan,

- Max Horn,

- Christopher Jefferson,

- Markus Pfeiffer,

- Lea Racine,

- Christopher Russell,

- Finn Smith,

- Ben Spiers, and

- Murray Whyte.

The Digraphs package contains a variety of methods for efficiently creating and storing mutable and immutable digraphs and computing information about them. Full explanations of all the functions contained in the package are provided below.

If the GRAPE package is available, it will be loaded automatically. Digraphs created with the Digraphs package can be converted to GRAPE graphs with Graph (3.2.3), and conversely GRAPE graphs can be converted to Digraphs objects with Digraph (3.1.7). GRAPE is not required for Digraphs to run.

The bliss tool [JK07] is included in this package. It is an open-source tool for computing automorphism groups and canonical forms of graphs, written by Tommi Junttila and Petteri Kaski. Several of the methods in the Digraphs package rely on bliss . If the NautyTracesInterface package for GAP is available then it is also possible to use nauty [MP14] for computing automorphism groups and canonical forms in Digraphs. See Section 7.2 for more details.

The edge-addition-planarity-suite is also included in Digraphs; see [BM04], [Boy06], [BM06], and [Boy12] . The edge-addition-planarity-suite is an open-source implementation of the edge addition planar graph embedding algorithm and related algorithms by John M. Boyer. See Section 6.7 for more details.

From version 1.0.0 of this package, digraphs can be either mutable or immutable. Mutable digraphs can be changed in-place by many of the methods in the package, which avoids unnecessary copying. Immutable digraphs cannot be changed in-place, but their advantage is that the value of an attribute of an immutable digraph is only ever computed once. Mutable digraphs can be converted into immutable digraphs in-place using MakeImmutable (**Reference: MakeImmutable**). One of the motivations for introducing mutable digraphs in version 1.0.0 was that in practice the authors often wanted to create a digraph and immediately modify it (removing certain edges, loops, and so on). Before version 1.0.0, this involved copying the digraph several times, with each copy being discarded almost immediately. From version 1.0.0, this unnecessary copying can be eliminated by first creating a mutable digraph, then changing it in-place, and finally converting the mutable digraph to an immutable one in-place (if desirable).

1.1.1 Definitions

For the purposes of this package and its documentation, the following definitions apply:

A *digraph* $E = (E^0, E^1, r, s)$, also known as a *directed graph*, consists of a set of vertices E^0 and a set of edges E^1 together with functions $s, r : E^1 \rightarrow E^0$, called the *source* and *range*, respectively. The source and range of an edge is respectively the values of s, r at that edge. An edge is called a *loop* if its source and range are the same. A digraph is called a *multidigraph* if there exist two or more edges with the same source and the same range.

A *directed walk* on a digraph is a sequence of alternating vertices and edges $(v_1, e_1, v_2, e_2, ..., e_{n-1}, v_n)$ such that each edge e_i has source v_i and range v_{i+1}. A *directed path* is a directed walk where no vertex (and hence no edge) is repeated. A *directed circuit* is a directed walk where $v_1 = v_n$, and a *directed cycle* is a directed circuit where where no vertex is repeated, except for $v_1 = v_n$.

The *length* of a directed walk $(v_1, e_1, v_2, e_2, ..., e_{n-1}, v_n)$ is equal to $n-1$, the number of edges it contains. A directed walk (or path) $(v_1, e_1, v_2, e_2, ..., e_{n-1}, v_n)$ is sometimes called a directed walk (or path) *from vertex v_1 to vertex v_n*. A directed walk of zero length, i.e. a sequence (v) for some vertex v, is called *trivial*. A trivial directed walk is considered to be both a circuit and a cycle, as is the empty directed walk $()$. A *simple circuit* is another name for a non-trivial and non-empty directed cycle.

Chapter 2

Installing **Digraphs**

2.1   For those in a hurry

In this section we give a brief description of how to start using Digraphs.

It is assumed that you have a working copy of GAP with version number 4.10.0 or higher. The most up-to-date version of GAP and instructions on how to install it can be obtained from the main GAP webpage https://www.gap-system.org.

The following is a summary of the steps that should lead to a successful installation of Digraphs:

- ensure that the IO package version 4.5.1 or higher is available. IO must be compiled before Digraphs can be loaded.

- ensure that the orb package version 4.8.2 or higher is available. orb has better performance when compiled, but although compilation is recommended, it is not required to be compiled for Digraphs to be loaded.

- ensure that the datastructures package version 0.2.5 or higher is available.

- THIS STEP IS OPTIONAL: certain functions in Digraphs require the GRAPE package to be available; see Section 2.2.1 for full details. To use these functions make sure that the GRAPE package version 4.8.1 or higher is available. If GRAPE is not available, then Digraphs can be used as normal with the exception that the functions listed in Subsection 2.2.1 will not work.

- THIS STEP IS OPTIONAL: certain functions in Digraphs require the NautyTracesInterface package to be available. If you want to make use of these functions, please ensure that the NautyTracesInterface package version 0.2 or higher is available. If NautyTracesInterface is not available, then Digraphs can be used as normal with the exception that functions whose names contain "Nauty" will not work.

- download the package archive digraphs-1.9.0.tar.gz.tar.gz from the Digraphs package webpage.

- unzip and untar the file, this should create a directory called digraphs-1.9.0.tar.gz.

- locate the pkg directory of your GAP directory, which contains the directories lib, doc and so on. Move the directory digraphs-1.9.0.tar.gz into the pkg directory.

- it is necessary to compile the Digraphs package. Inside the `pkg/digraphs-1.9.0.tar.gz` directory, type

```
./configure
make
```

Further information about this step can be found in Section 2.3.

- start GAP in the usual way (i.e. type `gap` at the command line).

- type `LoadPackage("digraphs");`

If you want to check that the package is working correctly, you should run some of the tests described in Section 2.5.

2.1.1　Configuration options

In addition to the usual autoconf generated configuration flags, the following flags are provided.

Option	Meaning
`-enable-code-coverage`	enable code coverage support
`-enable-compile-warnings`	enable compiler warnings
`-enable-debug`	enable debug mode
`-with-external-bliss`	use external bliss
`-with-external-planarity`	use external edge-addition-planarity-suite
`-with-gaproot`	specify root of GAP installation
`-without-intrinsics`	do not use compiler intrinsics even if available

Table: Configuration flags

2.2　Optional package dependencies

The Digraphs package is written in GAP and C code and requires the IO package. The IO package is used to read and write transformations, partial permutations, and bipartitions to a file.

2.2.1　The Grape package

The GRAPE package must be available for the following operations to be available:

- `Graph` (3.2.3) with a digraph argument

- `AsGraph` (3.2.4) with a digraph argument

- `Digraph` (3.1.7) with a GRAPE graph argument

If GRAPE is not available, then Digraphs can be used as normal with the exception that the functions above will not work.

2.3 Compiling the kernel module

The Digraphs package has a GAP kernel component in C which should be compiled. This component contains certain low-level functions required by Digraphs.

It is not possible to use the Digraphs package without compiling it.

To compile the kernel component inside the `pkg/digraphs-1.9.0.tar.gz` directory, type

```
./configure
make
```

If you installed the package in another 'pkg' directory than the standard 'pkg' directory in your GAP installation, then you have to do two things. Firstly during compilation you have to use the option '–with-gaproot=PATH' of the 'configure' script where 'PATH' is a path to the main GAP root directory (if not given the default '../..' is assumed).

If you installed GAP on several architectures, you must execute the configure/make step for each of the architectures. You can either do this immediately after configuring and compiling GAP itself on this architecture, or alternatively set the environment variable 'CONFIGNAME' to the name of the configuration you used when compiling GAP before running './configure'. Note however that your compiler choice and flags (environment variables 'CC' and 'CFLAGS') need to be chosen to match the setup of the original GAP compilation. For example you have to specify 32-bit or 64-bit mode correctly!

2.4 Rebuilding the documentation

The Digraphs package comes complete with pdf, html, and text versions of the documentation. However, you might find it necessary, at some point, to rebuild the documentation. To rebuild the documentation, please use the function `DigraphsMakeDoc` (2.4.1).

2.4.1 DigraphsMakeDoc

▷ `DigraphsMakeDoc()` (function)

Returns: Nothing

This function should be called with no argument to compile the Digraphs documentation.

2.5 Testing your installation

In this section we describe how to test that Digraphs is working as intended. To test that Digraphs is installed correctly use `DigraphsTestInstall` (2.5.1) or for more extensive tests use `DigraphsTestStandard` (2.5.2).

If something goes wrong, then please review the instructions in Section 2.1 and ensure that Digraphs has been properly installed. If you continue having problems, please use the issue tracker to report the issues you are having.

2.5.1 DigraphsTestInstall

▷ DigraphsTestInstall() (function)

Returns: `true` or `false`.

This function can be called without arguments to test your installation of Digraphs is working correctly. These tests should take no more than a few seconds to complete. To test more comprehensively that Digraphs is working correctly, use `DigraphsTestStandard` (2.5.2).

2.5.2 DigraphsTestStandard

▷ DigraphsTestStandard() (function)

Returns: `true` or `false`.

This function can be called without arguments to test all of the methods included in Digraphs. These tests should take less than a minute to complete.

To quickly test that Digraphs is installed correctly use `DigraphsTestInstall` (2.5.1). For a more thorough test, use `DigraphsTestExtreme` (2.5.3).

2.5.3 DigraphsTestExtreme

▷ DigraphsTestExtreme() (function)

Returns: `true` or `false`.

This function should be called with no argument. It executes a series of very demanding tests, which measure the performance of a variety of functions on large examples. These tests take a long time to complete, at least several minutes.

For these tests to complete, the digraphs library `digraphs-lib` must be downloaded and placed in the `digraphs` directory in a subfolder named `digraphs-lib`. This library can be found on the Digraphs website.

Chapter 3

Creating digraphs

In this chapter we describe how to create digraphs.

3.1 Creating digraphs

3.1.1 IsDigraph

▷ IsDigraph   (Category)

Every digraph in Digraphs belongs to the category IsDigraph. Some basic attributes and operations for digraphs are DigraphVertices (5.1.1), DigraphEdges (5.1.3), and OutNeighbours (5.2.6).

3.1.2 IsMutableDigraph

▷ IsMutableDigraph   (Category)

IsMutableDigraph is a synonym for IsDigraph (3.1.1) and IsMutable (**Reference: IsMutable**). A mutable digraph may be changed in-place by methods in the Digraphs package, and is not attribute-storing – see IsAttributeStoringRep (**Reference: IsAttributeStoringRep**).

A mutable digraph may be converted into an immutable attribute-storing digraph by calling MakeImmutable (**Reference: MakeImmutable**) on the digraph.

3.1.3 IsImmutableDigraph

▷ IsImmutableDigraph   (Category)

IsImmutableDigraph is a subcategory of IsDigraph (3.1.1). Digraphs that lie in IsImmutableDigraph are immutable and attribute-storing. In particular, they lie in IsAttributeStoringRep (**Reference: IsAttributeStoringRep**).

A mutable digraph may be converted to an immutable digraph that lies in the category IsImmutableDigraph by calling MakeImmutable (**Reference: MakeImmutable**) on the digraph.

The operation DigraphMutableCopy (3.3.1) can be used to construct a mutable copy of an immutable digraph. It is however not possible to convert an immutable digraph into a mutable digraph in-place.

3.1.4 IsCayleyDigraph

▷ IsCayleyDigraph   (Category)

IsCayleyDigraph is a subcategory of IsDigraph. Digraphs that are Cayley digraphs of a group and that are constructed by the operation CayleyDigraph (3.1.12) are constructed in this category, and are always immutable.

3.1.5 IsDigraphWithAdjacencyFunction

▷ IsDigraphWithAdjacencyFunction   (Category)

IsDigraphWithAdjacencyFunction is a subcategory of IsDigraph. Digraphs that are *created* using an adjacency function are constructed in this category.

3.1.6 DigraphByOutNeighboursType

▷ DigraphByOutNeighboursType   (global variable)
▷ DigraphFamily   (family)

The type of all digraphs is DigraphByOutNeighboursType. The family of all digraphs is DigraphFamily.

3.1.7 Digraph

▷ Digraph(*[filt,]obj[, source, range]*)   (operation)
▷ Digraph(*[filt,]list, func*)   (operation)
▷ Digraph(*[filt,]G, list, act, adj*)   (operation)
    Returns: A digraph.

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

for a list (i.e. an adjacency list)

    if *obj* is a list of lists of positive integers in the range from 1 to Length(*obj*), then this function returns the digraph with vertices $E^0 =$ [1 .. Length(*obj*)], and edges corresponding to the entries of *obj*.

    More precisely, there is an edge from vertex i to j if and only if j is in *obj*[i]; the source of this edge is i and the range is j. If j occurs in *obj*[i] with multiplicity k, then there are k edges from i to j.

for three lists

    if *obj* is a duplicate-free list, and *source* and *range* are lists of equal length consisting of positive integers in the list [1 .. Length(*obj*)], then this function returns a digraph with vertices $E^0 =$ [1 .. Length(*obj*)], and Length(*source*) edges. For each i in [1

.. Length(*source*)] there exists an edge with source vertex source[i] and range vertex range[i]. See DigraphSource (5.2.5) and DigraphRange (5.2.5).

The vertices of the digraph will be labelled by the elements of *obj*.

for an integer, and two lists

if *obj* is an integer, and *source* and *range* are lists of equal length consisting of positive integers in the list [1 .. *obj*], then this function returns a digraph with vertices $E^0 = $ [1 .. *obj*], and Length(*source*) edges. For each i in [1 .. Length(*source*)] there exists an edge with source vertex source[i] and range vertex range[i]. See DigraphSource (5.2.5) and DigraphRange (5.2.5).

for a list and a function

if *list* is a list and *func* is a function taking 2 arguments that are elements of *list*, and *func* returns true or false, then this operation creates a digraph with vertices [1 .. Length(*list*)] and an edge from vertex i to vertex j if and only if *func*(*list*[i], *list*[j]) returns true.

for a group, a list, and two functions

The arguments will be G, list, act, adj.

Let G be a group acting on the objects in *list* via the action act, and let adj be a function taking two objects from *list* as arguments and returning true or false. The function adj will describe the adjacency between objects from *list*, which is invariant under the action of G. This variant of the constructor returns a digraph with vertices the objects of *list* and directed edges [x, y] when f(x, y) is true.

The action of the group G on the objects in *list* is stored in the attribute DigraphGroup (7.2.10), and is used to speed up operations like DigraphDiameter (5.4.1).

for a Grape package graph

if *obj* is a GRAPE package graph (i.e. a record for which the function IsGraph returns true), then this function returns a digraph isomorphic to *obj*.

for a binary relation

if *obj* is a binary relation on the points [1 .. n] for some positive integer *n*, then this function returns the digraph defined by *obj*. Specifically, this function returns a digraph which has *n* vertices, and which has an edge with source i and range j if and only if [i,j] is a pair in the binary relation *obj*.

for a string naming a digraph

if *obj* is a non-empty string, then this function returns the digraph that has name *obj*. Digraphs comes with a database containing a few hundred common digraph names that can be loaded in this way. Valid names include "folkman", "diamond" and "brinkmann". If the name is commonly followed by the word "graph", then it is called without writing "graph" at the end. You can explore the available graph names using ListNamedDigraphs (3.1.13). Digraph names are case and whitespace insensitive.

Note that any undirected graphs in the database are stored as symmetric digraphs, so the resulting digraph will have twice as many edges as its undirected counterpart.

```
                              ─────── Example ───────
 gap> gr := Digraph([
 > [2, 5, 8, 10], [2, 3, 4, 2, 5, 6, 8, 9, 10], [1],
 > [3, 5, 7, 8, 10], [2, 5, 7], [3, 6, 7, 9, 10], [1, 4],
 > [1, 5, 9], [1, 2, 7, 8], [3, 5]]);
 <immutable multidigraph with 10 vertices, 38 edges>
 gap> gr := Digraph(["a", "b", "c"], ["a"], ["b"]);
 <immutable digraph with 3 vertices, 1 edge>
 gap> gr := Digraph(5, [1, 2, 2, 4, 1, 1], [2, 3, 5, 5, 1, 1]);
 <immutable multidigraph with 5 vertices, 6 edges>
 gap> Petersen := Graph(SymmetricGroup(5), [[1, 2]], OnSets,
 > function(x, y) return Intersection(x, y) = []; end);;
 gap> Digraph(Petersen);
 <immutable digraph with 10 vertices, 30 edges>
 gap> gr := Digraph([1 .. 10], ReturnTrue);
 <immutable digraph with 10 vertices, 100 edges>
 gap> Digraph("Diamond");
 <immutable digraph with 4 vertices, 10 edges>
```

The next example illustrates the uses of the fourth and fifth variants of this constructor. The resulting digraph is a strongly regular graph, and it is actually the point graph of the van Lint-Schrijver partial geometry, [vLS81]. The algebraic description is taken from the seminal paper of Calderbank and Kantor [CK86].

```
                              ─────── Example ───────
 gap> f := GF(3 ^ 4);
 GF(3^4)
 gap> gamma := First(f, x -> Order(x) = 5);
 Z(3^4)^64
 gap> L := Union([Zero(f)], List(Group(gamma)));
 [ 0*Z(3), Z(3)^0, Z(3^4)^16, Z(3^4)^32, Z(3^4)^48, Z(3^4)^64 ]
 gap> omega := Union(List(L, x -> List(Difference(L, [x]), y -> x - y)));
 [ Z(3)^0, Z(3), Z(3^4)^5, Z(3^4)^7, Z(3^4)^8, Z(3^4)^13, Z(3^4)^15,
   Z(3^4)^16, Z(3^4)^21, Z(3^4)^23, Z(3^4)^24, Z(3^4)^29, Z(3^4)^31,
   Z(3^4)^32, Z(3^4)^37, Z(3^4)^39, Z(3^4)^45, Z(3^4)^47, Z(3^4)^48,
   Z(3^4)^53, Z(3^4)^55, Z(3^4)^56, Z(3^4)^61, Z(3^4)^63, Z(3^4)^64,
   Z(3^4)^69, Z(3^4)^71, Z(3^4)^72, Z(3^4)^77, Z(3^4)^79 ]
 gap> adj := function(x, y)
 >   return x - y in omega;
 > end;
 function( x, y ) ... end
 gap> digraph := Digraph(AsList(f), adj);
 <immutable digraph with 81 vertices, 2430 edges>
 gap> group := Group(Z(3));;
 gap> act := \*;
 <Operation "*">
 gap> digraph := Digraph(group, List(f), act, adj);
 <immutable digraph with 81 vertices, 2430 edges>
```

3.1.8 DigraphByAdjacencyMatrix

▷ DigraphByAdjacencyMatrix([*filt,*]*list*)   (operation)
   Returns: A digraph.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

If `list` is the adjacency matrix of a digraph in the sense of AdjacencyMatrix (5.2.1), then this operation returns the digraph which is defined by `list`.

Alternatively, if `list` is a square boolean matrix, then this operation returns the digraph with Length(`list`) vertices which has the edge [i,j] if and only if `list`[i][j] is true.

```
───────────────────── Example ─────────────────────
gap> DigraphByAdjacencyMatrix([
> [0, 1, 0, 2, 0],
> [1, 1, 1, 0, 1],
> [0, 3, 2, 1, 1],
> [0, 0, 1, 0, 1],
> [2, 0, 0, 0, 0]]);
<immutable multidigraph with 5 vertices, 18 edges>
gap> D := DigraphByAdjacencyMatrix([
> [true, false, true],
> [false, false, true],
> [false, true, false]]);
<immutable digraph with 3 vertices, 4 edges>
gap> OutNeighbours(D);
[ [ 1, 3 ], [ 3 ], [ 2 ] ]
gap> D := DigraphByAdjacencyMatrix(IsMutableDigraph,
> [[true, false, true],
>  [false, false, true],
>  [false, true, false]]);
<mutable digraph with 3 vertices, 4 edges>
```

3.1.9 DigraphByEdges

▷ DigraphByEdges(*[filt,]list[, n]*)   (operation)

    Returns: A digraph.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

If `list` is list of pairs of positive integers, then this function returns the digraph with the minimum number of vertices m such that its list equal `list`.

If the optional second argument `n` is a positive integer with n >= m (with m defined as above), then this function returns the digraph with `n` vertices and list `list`.

See DigraphEdges (5.1.3).

```
───────────────────── Example ─────────────────────
gap> DigraphByEdges(
> [[1, 3], [2, 1], [2, 3], [2, 5], [3, 6],
>  [4, 6], [5, 2], [5, 4], [5, 6], [6, 6]]);
<immutable digraph with 6 vertices, 10 edges>
```

```
gap> DigraphByEdges(
> [[1, 3], [2, 1], [2, 3], [2, 5], [3, 6],
>  [4, 6], [5, 2], [5, 4], [5, 6], [6, 6]], 12);
<immutable digraph with 12 vertices, 10 edges>
gap> DigraphByEdges(IsMutableDigraph,
> [[1, 3], [2, 1], [2, 3], [2, 5], [3, 6],
>  [4, 6], [5, 2], [5, 4], [5, 6], [6, 6]], 12);
<mutable digraph with 12 vertices, 10 edges>
```

3.1.10 EdgeOrbitsDigraph

▷ EdgeOrbitsDigraph(`G, edges[, n]`)   (operation)

    Returns: An immutable digraph.

    If `G` is a permutation group, `edges` is an edge or list of edges, and `n` is a non-negative integer such that `G` fixes [1 .. n] setwise, then this operation returns an immutable digraph with `n` vertices and the union of the orbits of the edges in `edges` under the action of the permutation group `G`. An edge in this context is simply a pair of positive integers.

    If the optional third argument `n` is not present, then the largest moved point of the permutation group `G` is used by default.

```
_____ Example _____
gap> digraph := EdgeOrbitsDigraph(Group((1, 3), (1, 2)(3, 4)),
>                                  [[1, 2], [4, 5]], 5);
<immutable digraph with 5 vertices, 12 edges>
gap> OutNeighbours(digraph);
[ [ 2, 4, 5 ], [ 1, 3, 5 ], [ 2, 4, 5 ], [ 1, 3, 5 ], [  ] ]
gap> RepresentativeOutNeighbours(digraph);
[ [ 2, 4, 5 ], [  ] ]
```

3.1.11 DigraphByInNeighbours

▷ DigraphByInNeighbours(`[filt, ]list`)   (operation)
▷ DigraphByInNeighbors(`[filt, ]list`)   (operation)

    Returns: A digraph.

    If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

    If `list` is a list of lists of positive integers list the range [1 .. Length(`list`)], then this function returns the digraph with vertices $E^0 = $ [1 .. Length(`list`)], and edges corresponding to the entries of `list`. More precisely, there is an edge with source vertex i and range vertex j if i is in the list `list`[j].

    If i occurs in the list `list`[j] with multiplicity k, then there are k multiple edges from i to j.

    See InNeighbours (5.2.7).

```
_____ Example _____
gap> D := DigraphByInNeighbours([
> [2, 5, 8, 10], [2, 3, 4, 5, 6, 8, 9, 10],
> [1], [3, 5, 7, 8, 10], [2, 5, 7], [3, 6, 7, 9, 10], [1, 4],
```

```
> [1, 5, 9], [1, 2, 7, 8], [3, 5]]);
<immutable digraph with 10 vertices, 37 edges>
gap> D := DigraphByInNeighbours([[2, 3, 2], [1], [1, 2, 3]]);
<immutable multidigraph with 3 vertices, 7 edges>
gap> D := DigraphByInNeighbours(IsMutableDigraph,
>                               [[2, 3, 2], [1], [1, 2, 3]]);
<mutable multidigraph with 3 vertices, 7 edges>
```

3.1.12 CayleyDigraph

▷ CayleyDigraph(*G*[, *gens*])   (operation)

Returns: An immutable digraph.

Let *G* be any group and let *gens* be a list of elements of *G*. This operation returns an immutable digraph that corresponds to the Cayley graph of *G* with respect to *gens*.

The vertices of the digraph correspond to the elements of *G*, in the order given by AsList(*G*). There exists an edge from vertex u to vertex v if and only if there exists a generator g in *gens* such that AsList(*G*)[u] * g = AsList(*G*)[v].

The labels of the vertices u, v, and the edge [u, v] are the corresponding elements AsList(*G*)[u], AsList(*G*)[v], and generator g, respectively; see DigraphVertexLabel (5.1.11) and DigraphEdgeLabel (5.1.13).

If the optional second argument *gens* is not present, then the generators of *G* are used by default.

The digraph created by this operation belongs to the category IsCayleyDigraph (3.1.4), the group *G* can be recovered from the digraph using GroupOfCayleyDigraph (5.5.1), and the generators *gens* can be obtained using GeneratorsOfCayleyDigraph (5.5.2).

Note that this function can only return an immutable digraph.

```
———————————————————————— Example ————————————————————————
gap> G := DihedralGroup(8);
<pc group of size 8 with 3 generators>
gap> CayleyDigraph(G);
<immutable digraph with 8 vertices, 24 edges>
gap> G := DihedralGroup(IsPermGroup, 8);
Group([ (1,2,3,4), (2,4) ])
gap> CayleyDigraph(G);
<immutable digraph with 8 vertices, 16 edges>
gap> digraph := CayleyDigraph(G, [()]);
<immutable digraph with 8 vertices, 8 edges>
gap> GroupOfCayleyDigraph(digraph) = G;
true
gap> GeneratorsOfCayleyDigraph(digraph);
[ () ]
```

3.1.13 ListNamedDigraphs

▷ ListNamedDigraphs(*s*[, *level*])   (operation)

Returns: A list of strings representing digraph names.

This function searches through the list of names that are currently in the Digraphs database of named digraphs. The first argument *s* should be a partially completed string; this function returns all completions str of the string *s* such that Digraph(str) will successfully return a digraph.

The optional second argument `level` controls the flexibility of the search. If `level` = 1, then only strings beginning exactly with `s` are returned. If `level` = 2, then all names containing `s` as a substring are returned. If `level` = 3, then once again a substring search is carried out, but characters that are not alphanumeric are ignored in the search.

If `level` is not specified, it is set by default to equal 2.

The search is always case and whitespace insensitive, and this is also the case when applying `Digraph` (3.1.7) to a string.

3.2 Changing representations

3.2.1 AsBinaryRelation

▷ AsBinaryRelation(`digraph`)   (operation)

  Returns: A binary relation.

  If `digraph` is a digraph with a positive number of vertices *n*, and no multiple edges, then this operation returns a binary relation on the points `[1..n]`. The pair `[i,j]` is in the binary relation if and only if `[i,j]` is an edge in `digraph`.

```
_____ Example _____
gap> D := Digraph([[3, 2], [1, 2], [2], [3, 4]]);
<immutable digraph with 4 vertices, 7 edges>
gap> AsBinaryRelation(D);
Binary Relation on 4 points
```

3.2.2 AsDigraph (for a binary relation)

▷ AsDigraph(`[filt, ]f[, n]`)   (operation)

  Returns: A digraph, or `fail`.

  If `f` is a binary relation represented as one of the following in GAP:

a transformation
  satisfying `IsTransformation` (**Reference: IsTransformation**);

a permutation
  satisfying `IsPerm` (**Reference: IsPerm**);

a partial perm
  satisfying `IsPartialPerm` (**Reference: IsPartialPerm**);

a binary relation
  satisfying `IsBinaryRelation` (**Reference: IsBinaryRelation**);

and `n` is a non-negative integer, then `AsDigraph` attempts to construct a digraph with `n` vertices whose edges are determined by `f`.

The digraph returned by `AsDigraph` has for each vertex `v` in `[1 .. n]`, an edge with source `v` and range `v ^ f`. If `v ^ f` is greater than `n` for any `v`, then `fail` is returned.

If the optional second argument `n` is not supplied, then the degree of the transformation `f`, the largest moved point of the permutation `f`, the maximum of the degree and the codegree of the partial perm `f`, or as applicable, is used by default.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
 _____ Example _____
  gap> f := Transformation([4, 3, 3, 1, 7, 9, 10, 4, 2, 3]);
  Transformation( [ 4, 3, 3, 1, 7, 9, 10, 4, 2, 3 ] )
  gap> AsDigraph(f);
  <immutable functional digraph with 10 vertices>
  gap> AsDigraph(f, 4);
  <immutable functional digraph with 4 vertices>
  gap> AsDigraph(f, 5);
  fail
  gap> AsDigraph((1, 2, 3, 4)) = CycleDigraph(4);
  true
  gap> D := AsDigraph(IsMutableDigraph, (1, 3)(2, 4), 5);
  <mutable digraph with 5 vertices, 5 edges>
  gap> DigraphEdges(D);
  [ [ 1, 3 ], [ 2, 4 ], [ 3, 1 ], [ 4, 2 ], [ 5, 5 ] ]
  gap> b := BinaryRelationOnPoints(
  > [[3], [1, 3, 5], [1], [1, 2, 4], [2, 3, 5]]);
  Binary Relation on 5 points
  gap> D := AsDigraph(b);
  <immutable digraph with 5 vertices, 11 edges>
```

3.2.3 Graph

▷ Graph(*digraph*)   (operation)

Returns: A GRAPE package graph.

If *digraph* is a mutable or immutable digraph without multiple edges, then this operation returns a GRAPE package graph that is isomorphic to *digraph*.

If *digraph* is a multidigraph, then since GRAPE does not support multiple edges, the multiple edges will be reduced to a single edge in the result. In order words, for a multidigraph this operation will return the same as `Graph(DigraphRemoveAllMultipleEdges(digraph))`.

```
 _____ Example _____
  gap> Petersen := Graph(SymmetricGroup(5), [[1, 2]], OnSets,
  > function(x, y) return Intersection(x, y) = []; end);;
  gap> Display(Petersen);
  rec(
    adjacencies := [ [ 3, 5, 8 ] ],
    group :=
     Group( [ ( 1, 2, 3, 5, 7)( 4, 6, 8, 9,10), ( 2, 4)( 6, 9)( 7,10)
       ] ),
    isGraph := true,
    names := [ [ 1, 2 ], [ 2, 3 ], [ 3, 4 ], [ 1, 3 ], [ 4, 5 ],
        [ 2, 4 ], [ 1, 5 ], [ 3, 5 ], [ 1, 4 ], [ 2, 5 ] ],
    order := 10,
    representatives := [ 1 ],
    schreierVector := [ -1, 1, 1, 2, 1, 1, 1, 1, 2, 2 ] )
  gap> Digraph(Petersen);
```

```
<immutable digraph with 10 vertices, 30 edges>
gap> Graph(last) = Petersen;
true
```

3.2.4 AsGraph

▷ AsGraph(*digraph*)   (attribute)

    Returns: A GRAPE package graph.

If *digraph* is a digraph, then this method returns the same as Graph (3.2.3), except that if *digraph* is immutable, then the result will be stored as a mutable attribute of *digraph*. In this latter case, when AsGraph(*digraph*) is called subsequently, the same GAP object will be returned as before.

```
———————————————————————— Example ————————————————————————
gap> D := Digraph([[1, 2], [3], []]);
<immutable digraph with 3 vertices, 3 edges>
gap> G := AsGraph(D);
rec( adjacencies := [ [ 1, 2 ], [ 3 ], [  ] ], group := Group(()),
  isGraph := true, names := [ 1 .. 3 ], order := 3,
  representatives := [ 1, 2, 3 ], schreierVector := [ -1, -2, -3 ] )
```

3.2.5 AsTransformation

▷ AsTransformation(*digraph*)   (attribute)

    Returns: A transformation, or fail

If *digraph* is a functional digraph, then AsTransformation returns the transformation which is defined by *digraph*. See IsFunctionalDigraph (6.2.9). Otherwise, AsTransformation(*digraph*) returns fail.

If *digraph* is a functional digraph with n vertices, then AsTransformation(*digraph*) will return the transformation f of degree at most n where for each $1 \leq i \leq n$, i ^ f is equal to the unique out-neighbour of vertex i in *digraph*.

```
———————————————————————— Example ————————————————————————
gap> D := Digraph([[1], [3], [2]]);
<immutable digraph with 3 vertices, 3 edges>
gap> AsTransformation(D);
Transformation( [ 1, 3, 2 ] )
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> AsTransformation(D);
Transformation( [ 2, 3, 1 ] )
gap> AsPermutation(last);
(1,2,3)
gap> D := Digraph([[2, 3], [], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> AsTransformation(D);
fail
```

3.3 New digraphs from old

3.3.1 DigraphImmutableCopy

▷ DigraphImmutableCopy(*digraph*)   (operation)
▷ DigraphMutableCopy(*digraph*)   (operation)
▷ DigraphCopySameMutability(*digraph*)   (operation)
▷ DigraphCopy(*digraph*)   (operation)
    Returns: A digraph.

Each of these operations returns a new copy of *digraph*, of the appropriate mutability, retaining none of the attributes or properties of *digraph*.

DigraphCopy is a synonym for DigraphCopySameMutability.

```
 _____ Example _____
 gap> D := CycleDigraph(10);
 <immutable cycle digraph with 10 vertices>
 gap> DigraphCopy(D) = D;
 true
 gap> IsIdenticalObj(DigraphCopy(D), D);
 false
 gap> DigraphMutableCopy(D);
 <mutable digraph with 10 vertices, 10 edges>
```

3.3.2 DigraphImmutableCopyIfImmutable

▷ DigraphImmutableCopyIfImmutable(*digraph*)   (operation)
▷ DigraphImmutableCopyIfMutable(*digraph*)   (operation)
▷ DigraphMutableCopyIfMutable(*digraph*)   (operation)
▷ DigraphMutableCopyIfImmutable(*digraph*)   (operation)
    Returns: A digraph.

Each of these operations returns either the original argument *digraph*, or a new copy of *digraph* of the appropriate mutability, according to the mutability of *digraph*.

```
 _____ Example _____
 gap> C := CycleDigraph(10);
 <immutable cycle digraph with 10 vertices>
 gap> D := DigraphImmutableCopyIfImmutable(C);
 <immutable digraph with 10 vertices, 10 edges>
 gap> IsIdenticalObj(C, D);
 false
 gap> C = D;
 true
 gap> D := DigraphImmutableCopyIfMutable(C);
 <immutable cycle digraph with 10 vertices>
 gap> IsIdenticalObj(C, D);
 true
 gap> C = D;
 true
 gap> D := DigraphMutableCopyIfMutable(C);
 <immutable cycle digraph with 10 vertices>
 gap> IsMutableDigraph(D);
 false
```

```
gap> D := DigraphMutableCopyIfImmutable(C);
<mutable digraph with 10 vertices, 10 edges>
gap> IsMutableDigraph(D);
true
gap> C := CycleDigraph(IsMutableDigraph, 10);
<mutable digraph with 10 vertices, 10 edges>
gap> D := DigraphImmutableCopyIfImmutable(C);
<mutable digraph with 10 vertices, 10 edges>
gap> IsIdenticalObj(C, D);
true
gap> C = D;
true
gap> D := DigraphImmutableCopyIfMutable(C);
<immutable digraph with 10 vertices, 10 edges>
gap> IsIdenticalObj(C, D);
false
gap> C = D;
true
gap> D := DigraphMutableCopyIfMutable(C);
<mutable digraph with 10 vertices, 10 edges>
gap> IsMutableDigraph(D);
true
gap> D := DigraphMutableCopyIfImmutable(C);
<mutable digraph with 10 vertices, 10 edges>
gap> IsIdenticalObj(C, D);
true
gap> IsMutableDigraph(D);
true
```

3.3.3 InducedSubdigraph

▷ InducedSubdigraph(`digraph, verts`)   (operation)

   Returns: A digraph.

   If `digraph` is a digraph, and `verts` is a subset of the vertices of `digraph`, then this operation returns a digraph constructed from `digraph` by retaining precisely those vertices in `verts`, and those edges whose source and range vertices are both contained in `verts`.

   The vertices of the induced subdigraph are [1..Length(verts)] but the original vertex labels can be accessed via DigraphVertexLabels (5.1.12).

   If `digraph` belongs to IsMutableDigraph (3.1.2), then `digraph` is modified in place. If `digraph` belongs to IsImmutableDigraph (3.1.3), a new immutable digraph containing the appropriate vertices and edges is returned.

```
━━━━━━━━━━━━━━━━━━━━━━━━━ Example ━━━━━━━━━━━━━━━━━━━━━━━━━
gap> D := Digraph([[1, 1, 2, 3, 4, 4], [1, 3, 4], [3, 1], [1, 1]]);
<immutable multidigraph with 4 vertices, 13 edges>
gap> InducedSubdigraph(D, [1, 3, 4]);
<immutable multidigraph with 3 vertices, 9 edges>
gap> DigraphVertices(last);
[ 1 .. 3 ]
gap> D := DigraphMutableCopy(D);
<mutable multidigraph with 4 vertices, 13 edges>
gap> new := InducedSubdigraph(D, [1, 3, 4]);
```

```
<mutable multidigraph with 3 vertices, 9 edges>
gap> D = new;
true
```

3.3.4 ReducedDigraph

▷ ReducedDigraph(*digraph*)   (operation)
▷ ReducedDigraphAttr(*digraph*)   (attribute)
    Returns: A digraph.

This function returns a digraph isomorphic to the subdigraph of *digraph* induced by the set of non-isolated vertices, i.e. the set of those vertices of *digraph* which are the source or range of some edge in *digraph*. See InducedSubdigraph (3.3.3).

The ordering of the remaining vertices of *digraph* is preserved, as are the labels of the remaining vertices and edges; see DigraphVertexLabels (5.1.12) and DigraphEdgeLabels (5.1.14). This can allow one to match a vertex in the reduced digraph to the corresponding vertex in *digraph*.

If *digraph* is immutable, then a new immutable digraph is returned. Otherwise, the isolated vertices of the mutable digraph *digraph* are removed in-place.

```
 ──────────────────────────── Example ────────────────────────────
 gap> D := Digraph([[1, 2], [], [], [1, 4], []]);
 <immutable digraph with 5 vertices, 4 edges>
 gap> R := ReducedDigraph(D);
 <immutable digraph with 3 vertices, 4 edges>
 gap> OutNeighbours(R);
 [ [ 1, 2 ], [  ], [ 1, 3 ] ]
 gap> DigraphEdges(D);
 [ [ 1, 1 ], [ 1, 2 ], [ 4, 1 ], [ 4, 4 ] ]
 gap> DigraphEdges(R);
 [ [ 1, 1 ], [ 1, 2 ], [ 3, 1 ], [ 3, 3 ] ]
 gap> DigraphVertexLabel(R, 3);
 4
 gap> DigraphVertexLabel(R, 2);
 2
 gap> D := Digraph(IsMutableDigraph, [[], [3], [2]]);
 <mutable digraph with 3 vertices, 2 edges>
 gap> ReducedDigraph(D);
 <mutable digraph with 2 vertices, 2 edges>
 gap> D;
 <mutable digraph with 2 vertices, 2 edges>
```

3.3.5 MaximalSymmetricSubdigraph

▷ MaximalSymmetricSubdigraph(*digraph*)   (operation)
▷ MaximalSymmetricSubdigraphAttr(*digraph*)   (attribute)
▷ MaximalSymmetricSubdigraphWithoutLoops(*digraph*)   (operation)
▷ MaximalSymmetricSubdigraphWithoutLoopsAttr(*digraph*)   (attribute)
    Returns: A digraph.

If *digraph* is a digraph, then MaximalSymmetricSubdigraph returns a symmetric digraph without multiple edges which has the same vertex set as *digraph*, and whose edge list is formed from

digraph by ignoring the multiplicity of edges, and by ignoring edges [u,v] for which there does not exist an edge [v,u].

The digraph returned by MaximalSymmetricSubdigraphWithoutLoops is the same, except that loops are removed.

If *digraph* is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph *digraph* is changed in-place into such a digraph described above.

See IsSymmetricDigraph (6.2.14), IsMultiDigraph (6.2.11), and DigraphHasLoops (6.2.1) for more information.

```
—————————————————— Example ——————————————————
gap> D := Digraph([[2, 2], [1, 3], [4], [3, 1]]);
<immutable multidigraph with 4 vertices, 7 edges>
gap> not IsSymmetricDigraph(D) and IsMultiDigraph(D);
true
gap> OutNeighbours(D);
[ [ 2, 2 ], [ 1, 3 ], [ 4 ], [ 3, 1 ] ]
gap> S := MaximalSymmetricSubdigraph(D);
<immutable symmetric digraph with 4 vertices, 4 edges>
gap> IsSymmetricDigraph(S) and not IsMultiDigraph(S);
true
gap> OutNeighbours(S);
[ [ 2 ], [ 1 ], [ 4 ], [ 3 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> MaximalSymmetricSubdigraph(D);
<mutable empty digraph with 3 vertices>
gap> D;
<mutable empty digraph with 3 vertices>
```

3.3.6 MaximalAntiSymmetricSubdigraph

▷ MaximalAntiSymmetricSubdigraph(*digraph*)   (operation)
▷ MaximalAntiSymmetricSubdigraphAttr(*digraph*)   (attribute)

Returns: A digraph.

If *digraph* is a digraph, then MaximalAntiSymmetricSubdigraph returns an anti-symmetric subdigraph of *digraph* formed by retaining the vertices of *digraph*, discarding any duplicate edges, and discarding any edge [i,j] of *digraph* where i > j and the reverse edge [j,i] is an edge of *digraph*. In other words, for every symmetric pair of edges [i,j] and [j,i] in *digraph*, where i and j are distinct, it discards the edge $[\max(i,j), \min(i,j)]$.

If *digraph* is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph *digraph* is changed in-place.

See IsAntisymmetricDigraph (6.2.2) for more information.

```
—————————————————— Example ——————————————————
gap> D := Digraph([[2, 2], [1, 3], [4], [3, 1]]);
<immutable multidigraph with 4 vertices, 7 edges>
gap> not IsAntiSymmetricDigraph(D) and IsMultiDigraph(D);
true
gap> OutNeighbours(D);
[ [ 2, 2 ], [ 1, 3 ], [ 4 ], [ 3, 1 ] ]
gap> D := MaximalAntiSymmetricSubdigraph(D);
<immutable antisymmetric digraph with 4 vertices, 4 edges>
```

```
gap> IsAntiSymmetricDigraph(D) and not IsMultiDigraph(D);
true
gap> OutNeighbours(D);
[ [ 2 ], [ 3 ], [ 4 ], [ 1 ] ]
gap> D := Digraph(IsMutableDigraph, [[2], [1]]);
<mutable digraph with 2 vertices, 2 edges>
gap> MaximalAntiSymmetricSubdigraph(D);
<mutable digraph with 2 vertices, 1 edge>
gap> D;
<mutable digraph with 2 vertices, 1 edge>
```

3.3.7 UndirectedSpanningForest

▷ UndirectedSpanningForest(*digraph*)   (operation)
▷ UndirectedSpanningForestAttr(*digraph*)   (attribute)
▷ UndirectedSpanningTree(*digraph*)   (operation)
▷ UndirectedSpanningTreeAttr(*digraph*)   (attribute)

Returns: A digraph, or fail.

If *digraph* is a digraph with at least one vertex, then UndirectedSpanningForest returns an undirected spanning forest of *digraph*, otherwise this attribute returns fail. See IsUndirectedSpanningForest (4.1.2) for the definition of an undirected spanning forest.

If *digraph* is a digraph with at least one vertex and whose MaximalSymmetricSubdigraph (3.3.5) is connected (see IsConnectedDigraph (6.6.3)), then UndirectedSpanningTree returns an undirected spanning tree of *digraph*, otherwise this attribute returns fail. See IsUndirectedSpanningTree (4.1.2) for the definition of an undirected spanning tree.

If *digraph* is immutable, then an immutable digraph is returned. Otherwise, the mutable digraph *digraph* is changed in-place into an undirected spanning tree of *digraph*.

Note that for an immutable digraph that has known undirected spanning tree, the attribute UndirectedSpanningTree returns the same digraph as the attribute UndirectedSpanningForest.

```
———————————————————— Example ————————————————————
gap> D := Digraph([[1, 2, 1, 3], [1], [4], [3, 4, 3]]);
<immutable multidigraph with 4 vertices, 9 edges>
gap> UndirectedSpanningTree(D);
fail
gap> forest := UndirectedSpanningForest(D);
<immutable undirected forest digraph with 4 vertices, 4 edges>
gap> OutNeighbours(forest);
[ [ 2 ], [ 1 ], [ 4 ], [ 3 ] ]
gap> IsUndirectedSpanningForest(D, forest);
true
gap> DigraphConnectedComponents(forest).comps;
[ [ 1, 2 ], [ 3, 4 ] ]
gap> DigraphConnectedComponents(MaximalSymmetricSubdigraph(D)).comps;
[ [ 1, 2 ], [ 3, 4 ] ]
gap> UndirectedSpanningForest(MaximalSymmetricSubdigraph(D))
> = forest;
true
gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>
gap> tree := UndirectedSpanningTree(D);
```

```
<immutable undirected tree digraph with 4 vertices>
gap> IsUndirectedSpanningTree(D, tree);
true
gap> tree = UndirectedSpanningForest(D);
true
gap> UndirectedSpanningForest(EmptyDigraph(0));
fail
gap> D := PetersenGraph(IsMutableDigraph);
<mutable digraph with 10 vertices, 30 edges>
gap> UndirectedSpanningTree(D);
<mutable digraph with 10 vertices, 18 edges>
gap> D;
<mutable digraph with 10 vertices, 18 edges>
```

3.3.8 DigraphShortestPathSpanningTree

▷ DigraphShortestPathSpanningTree(`digraph, v`)   (operation)

Returns: A digraph, or `fail`.

If `v` is a vertex in `digraph` and every other vertex of `digraph` is reachable from `v`, then this operation returns the shortest path spanning tree of `digraph` rooted at `v`. If there exist vertices in `digraph` (other than `v`) that are not reachable from `v`, then DigraphShortestPathSpanningTree returns `fail`. See IsReachable (5.4.19).

The *shortest path spanning tree of `digraph` rooted at `v`* is a subdigraph of `digraph` that is a directed tree, with unique source vertex `v`, and where for each other vertex `u` in `digraph`, the unique directed path from `v` to `u` in the tree is the lexicographically-least shortest directed path from `v` to `u` in `digraph`.

See IsDirectedTree (6.6.8), DigraphSources (5.1.9), and DigraphShortestPath (5.4.23).

If `digraph` belongs to IsMutableDigraph (3.1.2), then `digraph` is modified in place. If `digraph` belongs to IsImmutableDigraph (3.1.3), then the spanning tree is a new immutable digraph.

```
─────────────────────── Example ───────────────────────
gap> D := Digraph([[1, 2], [3], [2, 4], [1], [2, 4]]);
<immutable digraph with 5 vertices, 8 edges>
gap> ForAll([2 .. 5], v -> IsReachable(D, 1, v));
false
gap> DigraphShortestPathSpanningTree(D, 1);
fail
gap> tree := DigraphShortestPathSpanningTree(D, 5);
<immutable directed tree digraph with 5 vertices>
gap> OutNeighbours(tree);
[ [  ], [ 3 ], [  ], [ 1 ], [ 2, 4 ] ]
gap> ForAll(DigraphVertices(D), v ->
> DigraphShortestPath(D, 5, v) = DigraphPath(tree, 5, v));
true
```

3.3.9 QuotientDigraph

▷ QuotientDigraph(`digraph, p`)   (operation)

Returns: A digraph.

If `digraph` is a digraph, and `p` is a partition of the vertices of `digraph`, then this operation returns a digraph constructed by amalgamating all vertices of `digraph` which lie in the same part of `p`.

A partition of the vertices of `digraph` is a list of non-empty disjoint lists, such that the union of all the sub-lists is equal to vertex set of `digraph`. In particular, each vertex must appear in precisely one sub-list.

The vertices of `digraph` in part i of `p` will become vertex i in the quotient, and there exists some edge in `digraph` with source in part i and range in part j if and only if there is an edge from i to j in the quotient. In particular, this means that the quotient of a digraph has no multiple edges. which was a change introduced in version 1.0.0 of the Digraphs package.

If `digraph` belongs to IsMutableDigraph (3.1.2), then `digraph` is modified in place. If `digraph` belongs to IsImmutableDigraph (3.1.3), a new immutable digraph with the above properties is returned.

```
                       _____ Example _____
  gap> D := Digraph([[2, 1], [4], [1], [1, 3, 4]]);
  <immutable digraph with 4 vertices, 7 edges>
  gap> DigraphVertices(D);
  [ 1 .. 4 ]
  gap> DigraphEdges(D);
  [ [ 1, 2 ], [ 1, 1 ], [ 2, 4 ], [ 3, 1 ], [ 4, 1 ], [ 4, 3 ],
    [ 4, 4 ] ]
  gap> p := [[1], [2, 4], [3]];
  [ [ 1 ], [ 2, 4 ], [ 3 ] ]
  gap> quo := QuotientDigraph(D, p);
  <immutable digraph with 3 vertices, 6 edges>
  gap> DigraphVertices(quo);
  [ 1 .. 3 ]
  gap> DigraphEdges(quo);
  [ [ 1, 1 ], [ 1, 2 ], [ 2, 1 ], [ 2, 2 ], [ 2, 3 ], [ 3, 1 ] ]
  gap> QuotientDigraph(EmptyDigraph(0), []);
  <immutable empty digraph with 0 vertices>
```

3.3.10 DigraphReverse

▷ DigraphReverse(`digraph`)   (operation)
▷ DigraphReverseAttr(`digraph`)   (attribute)
   Returns: A digraph.

The reverse of a digraph is the digraph formed by reversing the orientation of each of its edges, i.e. for every edge [i, j] of a digraph, the reverse contains the corresponding edge [j, i].

DigraphReverse returns the reverse of the digraph `digraph`. If `digraph` is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph `digraph` is changed in-place into its reverse.

```
                       _____ Example _____
  gap> D := Digraph([[3], [1, 3, 5], [1], [1, 2, 4], [2, 3, 5]]);
  <immutable digraph with 5 vertices, 11 edges>
  gap> DigraphReverse(D);
  <immutable digraph with 5 vertices, 11 edges>
  gap> OutNeighbours(last);
  [ [ 2, 3, 4 ], [ 4, 5 ], [ 1, 2, 5 ], [ 4 ], [ 2, 5 ] ]
  gap> D := Digraph([[2, 4], [1], [4], [3, 4]]);
  <immutable digraph with 4 vertices, 6 edges>
```

```
gap> DigraphEdges(D);
[ [ 1, 2 ], [ 1, 4 ], [ 2, 1 ], [ 3, 4 ], [ 4, 3 ], [ 4, 4 ] ]
gap> DigraphEdges(DigraphReverse(D));
[ [ 1, 2 ], [ 2, 1 ], [ 3, 4 ], [ 4, 1 ], [ 4, 3 ], [ 4, 4 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> OutNeighbours(D);
[ [ 2 ], [ 3 ], [ 1 ] ]
gap> DigraphReverse(D);
<mutable digraph with 3 vertices, 3 edges>
gap> OutNeighbours(D);
[ [ 3 ], [ 1 ], [ 2 ] ]
```

3.3.11 DigraphDual

▷ DigraphDual(*digraph*)   (operation)
▷ DigraphDualAttr(*digraph*)   (attribute)
    Returns: A digraph.

The *dual* of *digraph* has the same vertices as *digraph*, and there is an edge in the dual from i to j whenever there is no edge from i to j in *digraph*. The *dual* is sometimes called the *complement*.

DigraphDual returns the dual of the digraph *digraph*. If *digraph* is an immutable digraph, then a new immutable digraph is returned. Otherwise, the mutable digraph *digraph* is changed in-place into its dual.

```
                                __ Example __
gap> D := Digraph([[2, 3], [], [4, 6], [5], [],
> [7, 8, 9], [], [], []]);
<immutable digraph with 9 vertices, 8 edges>
gap> DigraphDual(D);
<immutable digraph with 9 vertices, 73 edges>
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphDual(D);
<mutable digraph with 3 vertices, 6 edges>
gap> D;
<mutable digraph with 3 vertices, 6 edges>
```

3.3.12 DigraphSymmetricClosure

▷ DigraphSymmetricClosure(*digraph*)   (operation)
▷ DigraphSymmetricClosureAttr(*digraph*)   (attribute)
    Returns: A digraph.

If *digraph* is a digraph, then this attribute gives the minimal symmetric digraph which has the same vertices and contains all the edges of *digraph*.

A digraph is *symmetric* if its adjacency matrix AdjacencyMatrix (5.2.1) is symmetric. For a digraph with multiple edges this means that there are the same number of edges from a vertex u to a vertex v as there are from v to u; see IsSymmetricDigraph (6.2.14).

If *digraph* is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph *digraph* is changed in-place into its symmetric closure.

```
───────────────────────── Example ─────────────────────────
  gap> D := Digraph([[1, 2, 3], [2, 4], [1], [3, 4]]);
  <immutable digraph with 4 vertices, 8 edges>
  gap> D := DigraphSymmetricClosure(D);
  <immutable symmetric digraph with 4 vertices, 11 edges>
  gap> IsSymmetricDigraph(D);
  true
  gap> List(OutNeighbours(D), AsSet);
  [ [ 1, 2, 3 ], [ 1, 2, 4 ], [ 1, 4 ], [ 2, 3, 4 ] ]
  gap> D := Digraph([[2, 2], [1]]);
  <immutable multidigraph with 2 vertices, 3 edges>
  gap> D := DigraphSymmetricClosure(D);
  <immutable symmetric multidigraph with 2 vertices, 4 edges>
  gap> OutNeighbours(D);
  [ [ 2, 2 ], [ 1, 1 ] ]
  gap> D := CycleDigraph(IsMutableDigraph, 3);
  <mutable digraph with 3 vertices, 3 edges>
  gap> DigraphSymmetricClosure(D);
  <mutable digraph with 3 vertices, 6 edges>
  gap> D;
  <mutable digraph with 3 vertices, 6 edges>
```

3.3.13 DigraphTransitiveClosure

▷ DigraphTransitiveClosure(*digraph*)   (operation)
▷ DigraphTransitiveClosureAttr(*digraph*)   (attribute)
▷ DigraphReflexiveTransitiveClosure(*digraph*)   (operation)
▷ DigraphReflexiveTransitiveClosureAttr(*digraph*)   (attribute)

   Returns: A digraph.

   If *digraph* is a digraph with no multiple edges, then these attributes return the (reflexive) transitive closure of *digraph*.

   A digraph is *reflexive* if it has a loop at every vertex, and it is *transitive* if whenever [i,j] and [j,k] are edges of *digraph*, [i,k] is also an edge. The *(reflexive) transitive closure* of a digraph *digraph* is the least (reflexive and) transitive digraph containing *digraph*.

   If *digraph* is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph *digraph* is changed in-place into its (reflexive) transitive closure.

   Let *n* be the number of vertices of *digraph*, and let *m* be the number of edges. For an arbitrary digraph, these attributes will use a version of the Floyd-Warshall algorithm, with complexity $O(n^3)$. However, for a topologically sortable digraph [see DigraphTopologicalSort (5.1.10)], these attributes will use methods with complexity $O(m + n + m \cdot n)$ when this is faster.

```
───────────────────────── Example ─────────────────────────
  gap> D := DigraphFromDiSparse6String(".H'eOWR'Ul^");
  <immutable digraph with 9 vertices, 8 edges>
  gap> IsReflexiveDigraph(D) or IsTransitiveDigraph(D);
  false
  gap> OutNeighbours(D);
  [ [ 4, 6 ], [ 1, 3 ], [ ], [ 5 ], [ ], [ 7, 8, 9 ], [ ], [ ],
    [ ] ]
  gap> T := DigraphTransitiveClosure(D);
  <immutable transitive digraph with 9 vertices, 18 edges>
```

```
gap> OutNeighbours(T);
[ [ 4, 6, 5, 7, 8, 9 ], [ 1, 3, 4, 5, 6, 7, 8, 9 ], [ ], [ 5 ],
  [ ], [ 7, 8, 9 ], [ ], [ ], [ ] ]
gap> RT := DigraphReflexiveTransitiveClosure(D);
<immutable preorder digraph with 9 vertices, 27 edges>
gap> OutNeighbours(RT);
[ [ 4, 6, 5, 7, 8, 9, 1 ], [ 1, 3, 4, 5, 6, 7, 8, 9, 2 ], [ 3 ],
  [ 5, 4 ], [ 5 ], [ 7, 8, 9, 6 ], [ 7 ], [ 8 ], [ 9 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphReflexiveTransitiveClosure(D);
<mutable digraph with 3 vertices, 9 edges>
gap> D;
<mutable digraph with 3 vertices, 9 edges>
```

3.3.14 DigraphTransitiveReduction

▷ DigraphTransitiveReduction(*digraph*)   (operation)
▷ DigraphTransitiveReductionAttr(*digraph*)   (attribute)
▷ DigraphReflexiveTransitiveReduction(*digraph*)   (operation)
▷ DigraphReflexiveTransitiveReductionAttr(*digraph*)   (attribute)
     Returns: A digraph.

If *digraph* is a topologically sortable digraph [see DigraphTopologicalSort (5.1.10)] with no multiple edges, then these operations return the (reflexive) transitive reduction of *digraph*.

The (reflexive) transitive reduction of such a digraph is the unique least subgraph such that the (reflexive) transitive closure of the subgraph is equal to the (reflexive) transitive closure of *digraph* [see DigraphReflexiveTransitiveClosure (3.3.13)]. In order words, it is the least subgraph of *digraph* which retains the same reachability as *digraph*.

If *digraph* is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph *digraph* is changed in-place into its (reflexive) transitive reduction.

Let *n* be the number of vertices of an arbitrary digraph, and let *m* be the number of edges. Then these operations use methods with complexity $O(m + n + m \cdot n)$.

```
──────────────── Example ────────────────
gap> D := Digraph([[1, 2, 3], [3], [3]]);;
gap> DigraphHasLoops(D);
true
gap> D1 := DigraphReflexiveTransitiveReduction(D);
<immutable digraph with 3 vertices, 2 edges>
gap> DigraphHasLoops(D1);
false
gap> OutNeighbours(D1);
[ [ 2 ], [ 3 ], [ ] ]
gap> D2 := DigraphTransitiveReduction(D);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphHasLoops(D2);
true
gap> OutNeighbours(D2);
[ [ 2, 1 ], [ 3 ], [ 3 ] ]
gap> DigraphReflexiveTransitiveClosure(D)
>  = DigraphReflexiveTransitiveClosure(D1);
```

```
true
gap> DigraphTransitiveClosure(D)
>  = DigraphTransitiveClosure(D2);
true
gap> D := Digraph(IsMutableDigraph, [[1], [1], [1, 2, 3]]);
<mutable digraph with 3 vertices, 5 edges>
gap> DigraphReflexiveTransitiveReduction(D);
<mutable digraph with 3 vertices, 2 edges>
gap> D;
<mutable digraph with 3 vertices, 2 edges>
```

3.3.15   DigraphAddVertex

▷ DigraphAddVertex(`digraph[, label]`)   (operation)
    Returns: A digraph.

The operation returns a digraph constructed from `digraph` by adding a single new vertex, and no new edges.

If the optional second argument `label` is a GAP object, then the new vertex will be labelled `label`.

If `digraph` belongs to IsMutableDigraph (3.1.2), then the vertex is added directly to `digraph`. If `digraph` belongs to IsImmutableDigraph (3.1.3), an immutable copy of `digraph` with the additional vertex is returned.

```
_____ Example _____
gap> D := CompleteDigraph(3);
<immutable complete digraph with 3 vertices>
gap> new := DigraphAddVertex(D);
<immutable digraph with 4 vertices, 6 edges>
gap> D = new;
false
gap> DigraphVertices(new);
[ 1 .. 4 ]
gap> new := DigraphAddVertex(D, Group([(1, 2)]));
<immutable digraph with 4 vertices, 6 edges>
gap> DigraphVertexLabels(new);
[ 1, 2, 3, Group([ (1,2) ]) ]
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 2, 3);
<mutable digraph with 5 vertices, 12 edges>
gap> new := DigraphAddVertex(D);
<mutable digraph with 6 vertices, 12 edges>
gap> D = new;
true
```

3.3.16   DigraphAddVertices (for a digraph and an integer)

▷ DigraphAddVertices(`digraph, m`)   (operation)
▷ DigraphAddVertices(`digraph, labels`)   (operation)
    Returns: A digraph.

For a non-negative integer `m`, this operation returns a digraph constructed from `digraph` by adding `m` new vertices.

Otherwise, if `labels` is a list consisting of k GAP objects, then this operation returns a digraph constructed from `digraph` by adding k new vertices, which are labelled according to this list.

If `digraph` belongs to IsMutableDigraph (3.1.2), then the vertices are added directly to `digraph`, which is changed in-place. If `digraph` belongs to IsImmutableDigraph (3.1.3), then `digraph` itself is returned if no vertices are added (i.e. `m=0` or `labels` is empty), otherwise the result is a new immutable digraph.

```
─────────────────────────── Example ───────────────────────────
gap> D := CompleteDigraph(3);
<immutable complete digraph with 3 vertices>
gap> new := DigraphAddVertices(D, 3);
<immutable digraph with 6 vertices, 6 edges>
gap> DigraphVertices(new);
[ 1 .. 6 ]
gap> new := DigraphAddVertices(D, [Group([(1, 2)]), "d"]);
<immutable digraph with 5 vertices, 6 edges>
gap> DigraphVertexLabels(new);
[ 1, 2, 3, Group([ (1,2) ]), "d" ]
gap> DigraphAddVertices(D, 0) = D;
true
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 2, 3);
<mutable digraph with 5 vertices, 12 edges>
gap> new := DigraphAddVertices(D, 4);
<mutable digraph with 9 vertices, 12 edges>
gap> D = new;
true
```

3.3.17 DigraphAddEdge (for a digraph and an edge)

▷ DigraphAddEdge(`digraph, edge`) (operation)
▷ DigraphAddEdge(`digraph, src, ran`) (operation)

Returns: A digraph.

If `edge` is a pair of vertices of `digraph`, or `src` and `ran` are vertices of `digraph`, then this operation returns a digraph constructed from `digraph` by adding a new edge with source `edge`[1] [`src`] and range `edge`[2] [`ran`].

If `digraph` belongs to IsMutableDigraph (3.1.2), then the edge is added directly to `digraph`. If `digraph` belongs to IsImmutableDigraph (3.1.3), then an immutable copy of `digraph` with the additional edge is returned.

```
─────────────────────────── Example ───────────────────────────
gap> D1 := Digraph([[2], [3], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> DigraphEdges(D1);
[ [ 1, 2 ], [ 2, 3 ] ]
gap> D2 := DigraphAddEdge(D1, [3, 1]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphEdges(D2);
[ [ 1, 2 ], [ 2, 3 ], [ 3, 1 ] ]
gap> D3 := DigraphAddEdge(D2, [2, 3]);
<immutable multidigraph with 3 vertices, 4 edges>
gap> DigraphEdges(D3);
[ [ 1, 2 ], [ 2, 3 ], [ 2, 3 ], [ 3, 1 ] ]
```

```
gap> D := CycleDigraph(IsMutableDigraph, 4);
<mutable digraph with 4 vertices, 4 edges>
gap> new := DigraphAddEdge(D, [1, 3]);
<mutable digraph with 4 vertices, 5 edges>
gap> DigraphEdges(new);
[ [ 1, 2 ], [ 1, 3 ], [ 2, 3 ], [ 3, 4 ], [ 4, 1 ] ]
gap> D = new;
true
```

3.3.18 DigraphAddEdgeOrbit

▷ DigraphAddEdgeOrbit(`digraph, edge`)   (operation)

    Returns: A new digraph.

    This operation returns a new digraph with the same vertices and edges as `digraph` and with additional edges consisting of the orbit of the edge `edge` under the action of the `DigraphGroup` (7.2.10) of `digraph`. If `edge` is already an edge in `digraph`, then `digraph` is returned unchanged. The argument `digraph` must be an immutable digraph.

    An edge is simply a pair of vertices of `digraph`.

```
────────────────────────── Example ──────────────────────────
gap> gr1 := CayleyDigraph(DihedralGroup(8));
<immutable digraph with 8 vertices, 24 edges>
gap> gr2 := DigraphAddEdgeOrbit(gr1, [1, 8]);
<immutable digraph with 8 vertices, 32 edges>
gap> DigraphEdges(gr1);
[ [ 1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 2, 1 ], [ 2, 5 ], [ 2, 6 ],
  [ 3, 8 ], [ 3, 4 ], [ 3, 7 ], [ 4, 6 ], [ 4, 7 ], [ 4, 1 ],
  [ 5, 7 ], [ 5, 6 ], [ 5, 8 ], [ 6, 4 ], [ 6, 8 ], [ 6, 2 ],
  [ 7, 5 ], [ 7, 1 ], [ 7, 3 ], [ 8, 3 ], [ 8, 2 ], [ 8, 5 ] ]
gap> DigraphEdges(gr2);
[ [ 1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 1, 8 ], [ 2, 1 ], [ 2, 5 ],
  [ 2, 6 ], [ 2, 7 ], [ 3, 8 ], [ 3, 4 ], [ 3, 7 ], [ 3, 6 ],
  [ 4, 6 ], [ 4, 7 ], [ 4, 1 ], [ 4, 5 ], [ 5, 7 ], [ 5, 6 ],
  [ 5, 8 ], [ 5, 4 ], [ 6, 4 ], [ 6, 8 ], [ 6, 2 ], [ 6, 3 ],
  [ 7, 5 ], [ 7, 1 ], [ 7, 3 ], [ 7, 2 ], [ 8, 3 ], [ 8, 2 ],
  [ 8, 5 ], [ 8, 1 ] ]
gap> gr3 := DigraphRemoveEdgeOrbit(gr2, [1, 8]);
<immutable digraph with 8 vertices, 24 edges>
gap> gr3 = gr1;
true
```

3.3.19 DigraphAddEdges

▷ DigraphAddEdges(`digraph, edges`)   (operation)

    Returns: A digraph.

    If `edges` is a (possibly empty) list of pairs of vertices of `digraph`, then this operation returns a digraph constructed from `digraph` by adding the edges specified by `edges`. More precisely, for every edge in `edges`, a new edge will be added with source edge[1] and range edges[2].

    If an edge is included in `edges` with multiplicity k, then it will be added k times. If `digraph` belongs to IsMutableDigraph (3.1.2), then the edges are added directly to `digraph`. If `digraph` belongs to IsImmutableDigraph (3.1.3), then the result is returned as an immutable digraph.

``` 
——————————————— Example ———————————————
gap> func := function(n)
>  local source, range, i;
>  source := [];
>  range  := [];
>  for i in [1 .. n - 2] do
>     Add(source, i);
>     Add(range, i + 1);
>  od;
>  return Digraph(n, source, range);
> end;;
gap> D := func(1024);
<immutable digraph with 1024 vertices, 1022 edges>
gap> new := DigraphAddEdges(D,
> [[1023, 1024], [1, 1024], [1023, 1024], [1024, 1]]);
<immutable multidigraph with 1024 vertices, 1026 edges>
gap> D = new;
false
gap> D2 := DigraphMutableCopy(func(1024));
<mutable digraph with 1024 vertices, 1022 edges>
gap> new := DigraphAddEdges(D2,
> [[1023, 1024], [1, 1024], [1023, 1024], [1024, 1]]);
<mutable multidigraph with 1024 vertices, 1026 edges>
gap> D2 = new;
true
```

3.3.20 DigraphRemoveVertex

▷ DigraphRemoveVertex(*digraph, v*)   (operation)

Returns: A digraph.

If *v* is a vertex of *digraph*, then this operation returns a digraph constructed from *digraph* by removing vertex *v*, along with any edge whose source or range vertex is *v*.

If *digraph* has n vertices, then the vertices of the returned digraph are [1..n-1], but the original labels can be accessed via DigraphVertexLabels (5.1.12).

If *digraph* belongs to IsMutableDigraph (3.1.2), then the vertex is removed directly from *digraph*. If *digraph* belongs to IsImmutableDigraph (3.1.3), an immutable copy of *digraph* without the vertex is returned.

```
——————————————— Example ———————————————
gap> D := Digraph(["a", "b", "c"],
>                 ["a", "a", "b", "c", "c"],
>                 ["b", "c", "a", "a", "c"]);
<immutable digraph with 3 vertices, 5 edges>
gap> DigraphVertexLabels(D);
[ "a", "b", "c" ]
gap> DigraphEdges(D);
[ [ 1, 2 ], [ 1, 3 ], [ 2, 1 ], [ 3, 1 ], [ 3, 3 ] ]
gap> new := DigraphRemoveVertex(D, 2);
<immutable digraph with 2 vertices, 3 edges>
gap> DigraphVertexLabels(new);
[ "a", "c" ]
gap> D := CycleDigraph(IsMutableDigraph, 5);
```

```
<mutable digraph with 5 vertices, 5 edges>
gap> new := DigraphRemoveVertex(D, 1);
<mutable digraph with 4 vertices, 3 edges>
gap> DigraphVertexLabels(D);
[ 2, 3, 4, 5 ]
gap> D = new;
true
```

3.3.21 DigraphRemoveVertices

▷ DigraphRemoveVertices(`digraph, verts`) (operation)

    Returns: A digraph.

If `verts` is a (possibly empty) duplicate-free list of vertices of `digraph`, then this operation returns a digraph constructed from `digraph` by removing every vertex in `verts`, along with any edge whose source or range vertex is in `verts`.

If `digraph` has n vertices, then the vertices of the new digraph are [1 .. n-Length(`verts`)], but the original labels can be accessed via DigraphVertexLabels (5.1.12).

If `digraph` belongs to IsMutableDigraph (3.1.2), then the vertices are removed directly from `digraph`. If `digraph` belongs to IsImmutableDigraph (3.1.3), an immutable copy of `digraph` without the vertices is returned.

```
_____ Example _____
gap> D := Digraph([[3], [1, 3, 5], [1], [1, 2, 4], [2, 3, 5]]);
<immutable digraph with 5 vertices, 11 edges>
gap> SetDigraphVertexLabels(D, ["a", "b", "c", "d", "e"]);
gap> new := DigraphRemoveVertices(D, [2, 4]);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphVertexLabels(new);
[ "a", "c", "e" ]
gap> D := CycleDigraph(IsMutableDigraph, 5);
<mutable digraph with 5 vertices, 5 edges>
gap> new := DigraphRemoveVertices(D, [1, 3]);
<mutable digraph with 3 vertices, 1 edge>
gap> DigraphVertexLabels(D);
[ 2, 4, 5 ]
gap> D = new;
true
```

3.3.22 DigraphRemoveEdge (for a digraph and an edge)

▷ DigraphRemoveEdge(`digraph, edge`) (operation)
▷ DigraphRemoveEdge(`digraph, src, ran`) (operation)

    Returns: A digraph.

If `digraph` is a digraph with no multiple edges and `edge` is a pair of vertices of `digraph`, or `src` and `ran` are vertices of `digraph`, then this operation returns a digraph constructed from `digraph` by removing the edge specified by `edge` or [`src, ran`].

If `digraph` belongs to IsMutableDigraph (3.1.2), then the edge is removed directly from `digraph`. If `digraph` belongs to IsImmutableDigraph (3.1.3), an immutable copy of `digraph` without the edge is returned.

Note that if `digraph` belongs to IsImmutableDigraph (3.1.3), then a new copy of `digraph` will be returned even if `edge` or `[src, ran]` does not define an edge of `digraph`.

```
─────────────────────────────── Example ───────────────────────────────
  gap> D := CycleDigraph(250000);
  <immutable cycle digraph with 250000 vertices>
  gap> D := DigraphRemoveEdge(D, [250000, 1]);
  <immutable digraph with 250000 vertices, 249999 edges>
  gap> new := DigraphRemoveEdge(D, [25000, 2]);;
  gap> new = D;
  true
  gap> IsIdenticalObj(new, D);
  false
  gap> D := DigraphMutableCopy(D);;
  gap> new := DigraphRemoveEdge(D, 2500, 2);;
  gap> IsIdenticalObj(new, D);
  true
```

3.3.23 DigraphRemoveEdgeOrbit

▷ DigraphRemoveEdgeOrbit(`digraph, edge`)   (operation)

    Returns: A new digraph.

This operation returns a new digraph with the same vertices as `digraph` and with the orbit of the edge `edge` (under the action of the DigraphGroup (7.2.10) of `digraph`) removed. If `edge` is not an edge in `digraph`, then `digraph` is returned unchanged. The argument `digraph` must be an immutable digraph.

An edge is simply a pair of vertices of `digraph`.

```
─────────────────────────────── Example ───────────────────────────────
  gap> gr1 := CayleyDigraph(DihedralGroup(8));
  <immutable digraph with 8 vertices, 24 edges>
  gap> gr2 := DigraphAddEdgeOrbit(gr1, [1, 8]);
  <immutable digraph with 8 vertices, 32 edges>
  gap> DigraphEdges(gr1);
  [ [ 1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 2, 1 ], [ 2, 5 ], [ 2, 6 ],
    [ 3, 8 ], [ 3, 4 ], [ 3, 7 ], [ 4, 6 ], [ 4, 7 ], [ 4, 1 ],
    [ 5, 7 ], [ 5, 6 ], [ 5, 8 ], [ 6, 4 ], [ 6, 8 ], [ 6, 2 ],
    [ 7, 5 ], [ 7, 1 ], [ 7, 3 ], [ 8, 3 ], [ 8, 2 ], [ 8, 5 ] ]
  gap> DigraphEdges(gr2);
  [ [ 1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 1, 8 ], [ 2, 1 ], [ 2, 5 ],
    [ 2, 6 ], [ 2, 7 ], [ 3, 8 ], [ 3, 4 ], [ 3, 7 ], [ 3, 6 ],
    [ 4, 6 ], [ 4, 7 ], [ 4, 1 ], [ 4, 5 ], [ 5, 7 ], [ 5, 6 ],
    [ 5, 8 ], [ 5, 4 ], [ 6, 4 ], [ 6, 8 ], [ 6, 2 ], [ 6, 3 ],
    [ 7, 5 ], [ 7, 1 ], [ 7, 3 ], [ 7, 2 ], [ 8, 3 ], [ 8, 2 ],
    [ 8, 5 ], [ 8, 1 ] ]
  gap> gr3 := DigraphRemoveEdgeOrbit(gr2, [1, 8]);
  <immutable digraph with 8 vertices, 24 edges>
  gap> gr3 = gr1;
  true
```

3.3.24 DigraphRemoveEdges

▷ DigraphRemoveEdges(`digraph, edges`) (operation)

Returns: A digraph.

If one of the following holds:

- `digraph` is a digraph with no multiple edges, and `edges` is a list of pairs of vertices of `digraph`, or

- `digraph` is a digraph and `edges` is an empty list

then this operation returns a digraph constructed from `digraph` by removing all of the edges specified by `edges` (see DigraphRemoveEdge (3.3.22)).

If `digraph` belongs to IsMutableDigraph (3.1.2), then the edge is removed directly from `digraph`. If `digraph` belongs to IsImmutableDigraph (3.1.3), the edge is removed from an immutable copy of `digraph` and this new digraph is returned.

Note that if `edges` is empty, then this operation will always return `digraph` rather than a copy. Also, if any element of `edges` is invalid (i.e. does not define an edge of `digraph`) then that element will simply be ignored.

```
 ──────────────── Example ────────────────
 gap> D := CycleDigraph(250000);
 <immutable cycle digraph with 250000 vertices>
 gap> D := DigraphRemoveEdges(D, [[250000, 1]]);
 <immutable digraph with 250000 vertices, 249999 edges>
 gap> D := DigraphMutableCopy(D);
 <mutable digraph with 250000 vertices, 249999 edges>
 gap> new := DigraphRemoveEdges(D, [[1, 2], [2, 3], [3, 100]]);
 <mutable digraph with 250000 vertices, 249997 edges>
 gap> new = D;
 true
```

3.3.25 DigraphRemoveLoops

▷ DigraphRemoveLoops(`digraph`) (operation)
▷ DigraphRemoveLoopsAttr(`digraph`) (attribute)

Returns: A digraph.

If `digraph` is a digraph, then this operation returns a digraph constructed from `digraph` by removing every loop. A loop is an edge with equal source and range.

If `digraph` is immutable, then a new immutable digraph is returned. Otherwise, the loops are removed from the mutable digraph `digraph` in-place.

```
 ──────────────── Example ────────────────
 gap> D := Digraph([[1, 2, 4], [1, 4], [3, 4], [1, 4, 5], [1, 5]]);
 <immutable digraph with 5 vertices, 12 edges>
 gap> DigraphRemoveLoops(D);
 <immutable digraph with 5 vertices, 8 edges>
 gap> D := Digraph(IsMutableDigraph, [[1, 2], [1]]);
 <mutable digraph with 2 vertices, 3 edges>
 gap> DigraphRemoveLoops(D);
 <mutable digraph with 2 vertices, 2 edges>
 gap> D;
 <mutable digraph with 2 vertices, 2 edges>
```

3.3.26 DigraphRemoveAllMultipleEdges

▷ DigraphRemoveAllMultipleEdges(*digraph*)   (operation)
▷ DigraphRemoveAllMultipleEdgesAttr(*digraph*)   (attribute)

    Returns: A digraph.

    If `digraph` is a digraph, then this operation returns a digraph constructed from `digraph` by removing all multiple edges. The result is the largest subdigraph of `digraph` which does not contain multiple edges.

    If `digraph` is immutable, then a new immutable digraph is returned. Otherwise, the multiple edges of the mutable digraph `digraph` are removed in-place.

```
――――――――――――――――――――――――――― Example ―――――――――――――――――――――――――――
gap> D1 := Digraph([[1, 2, 3, 2], [1, 1, 3], [2, 2, 2]]);
<immutable multidigraph with 3 vertices, 10 edges>
gap> D2 := DigraphRemoveAllMultipleEdges(D1);
<immutable digraph with 3 vertices, 6 edges>
gap> OutNeighbours(D2);
[ [ 1, 2, 3 ], [ 1, 3 ], [ 2 ] ]
gap> D := Digraph(IsMutableDigraph, [[2, 2], [1]]);
<mutable multidigraph with 2 vertices, 3 edges>
gap> DigraphRemoveAllMultipleEdges(D);
<mutable digraph with 2 vertices, 2 edges>
gap> D;
<mutable digraph with 2 vertices, 2 edges>
```

3.3.27 DigraphContractEdge (for a digraph and a list of positive integers)

▷ DigraphContractEdge(*digraph*, *edge*)   (operation)
▷ DigraphContractEdge(*digraph*, *src*, *ran*)   (operation)

    Returns: A digraph.

    If `edge` is a pair of vertices of `digraph`, or `src` and `ran` are vertices of `digraph`, where `ran <> src`, then then this operation merges the two vertices of the edge given into one. Edges incident to `src` and `ran` will now be incident to v, the new vertex, with their direction preserved.

    A new digraph constructed from `digraph` is returned, unless `digraph` belongs to `IsMutableDigraph` (3.1.2); in this case changes are made directly to `digraph`, which is then returned. The `digraph` must not belong to `IsMultiDigraph` (6.2.11).

    The labels of any remaining edges will be preserved.

    Assigned vertex labels for `src` and `ran` are combined into a list, and assigned to the new vertex v.

    If an edge `[src, src]` or `[ran, ran]` exists, a singular edge `[v, v]` is created. If edge `[ran, src]` exists, this is also removed.

```
――――――――――――――――――――――――――― Example ―――――――――――――――――――――――――――
gap> D := DigraphByEdges([[1, 2], [2, 1]]);
<immutable digraph with 2 vertices, 2 edges>
gap> D2 := DigraphContractEdge(D, 1, 2);
<immutable empty digraph with 1 vertex>
gap> DigraphEdges(D2);
[ ]
gap> D := DigraphByEdges(IsMutableDigraph, [[1, 2], [2, 3], [3, 4]]);
<mutable digraph with 4 vertices, 3 edges>
```

```
gap> DigraphVertexLabels(D);;  # setting vertex labels
gap> DigraphContractEdge(D, [2, 3]);
<mutable digraph with 3 vertices, 2 edges>
gap> DigraphEdges(D);
[ [ 1, 3 ], [ 3, 2 ] ]
gap> DigraphVertexLabels(D);
[ 1, 4, [ 2, 3 ] ]
```

3.3.28   DigraphReverseEdges (for a digraph and a list of edges)

▷ DigraphReverseEdges(*digraph, edges*)   (operation)
▷ DigraphReverseEdge(*digraph, edge*)   (operation)
▷ DigraphReverseEdge(*digraph, src, ran*)   (operation)
    Returns: A digraph.

If *digraph* is a digraph without multiple edges, and *edges* is a list of pairs of vertices of *digraph* (the entries of each pair corresponding to the source and the range of an edge, respectively), then DigraphReverseEdges returns a digraph constructed from *digraph* by reversing the orientation of every edge specified by *edges*. If only one edge is to be reversed, then DigraphReverseEdge can be used instead. In this case, the second argument should just be a single vertex-pair, or the second and third arguments should be the source and range of an edge respectively.

Note that even though *digraph* cannot have multiple edges, the output may have multiple edges.

If *digraph* belongs to IsMutableDigraph (3.1.2), then the edges are reversed in *digraph*. If *digraph* belongs to IsImmutableDigraph (3.1.3), an immutable copy of *digraph* with the specified edges reversed is returned.

```
_____ Example _____
gap> D := DigraphFromDiSparse6String(".Tg?i@s?t_e?_qEsC");
<immutable digraph with 21 vertices, 8 edges>
gap> DigraphEdges(D);
[ [ 1, 2 ], [ 1, 7 ], [ 1, 8 ], [ 5, 21 ], [ 7, 19 ], [ 9, 1 ],
  [ 11, 2 ], [ 21, 1 ] ]
gap> new := DigraphReverseEdge(D, [7, 19]);
<immutable digraph with 21 vertices, 8 edges>
gap> DigraphEdges(new);
[ [ 1, 2 ], [ 1, 7 ], [ 1, 8 ], [ 5, 21 ], [ 9, 1 ], [ 11, 2 ],
  [ 19, 7 ], [ 21, 1 ] ]
gap> D2 := DigraphMutableCopy(new);;
gap> new := DigraphReverseEdges(D2, [[19, 7]]);;
gap> D2 = new;
true
gap> D = new;
true
```

3.3.29   DigraphDisjointUnion (for an arbitrary number of digraphs)

▷ DigraphDisjointUnion(*D1, D2, ...*)   (function)
▷ DigraphDisjointUnion(*list*)   (function)
    Returns: A digraph.

In the first form, if *D1*, *D2*, etc. are digraphs, then `DigraphDisjointUnion` returns their disjoint union. In the second form, if *list* is a non-empty list of digraphs, then `DigraphDisjointUnion` returns the disjoint union of the digraphs contained in the list.

For a disjoint union of digraphs, the vertex set is the disjoint union of the vertex sets, and the edge list is the disjoint union of the edge lists.

More specifically, for a collection of digraphs *D1*, *D2*, ..., the disjoint union with have `DigraphNrVertices(D1)` + `DigraphNrVertices(D2)` + ... vertices. The edges of *D1* will remain unchanged, whilst the edges of the *i*th digraph, *D*[i], will be changed so that they belong to the vertices of the disjoint union corresponding to *D*[i]. In particular, the edges of *D*[i] will have their source and range increased by `DigraphNrVertices(D1)` + ... + `DigraphNrVertices(D[i-1])`.

Note that previously set `DigraphVertexLabels` (5.1.12) will be lost.

If the first digraph *D1* [*list[1]*] belongs to `IsMutableDigraph` (3.1.2), then *D1* [*list[1]*] is modified in place to contain the appropriate vertices and edges. If *digraph* belongs to `IsImmutableDigraph` (3.1.3), a new immutable digraph containing the appropriate vertices and edges is returned.

```
——————————————————————————— Example ———————————————————————————
gap> D1 := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> OutNeighbours(D1);
[ [ 2 ], [ 3 ], [ 1 ] ]
gap> D2 := CompleteDigraph(3);
<immutable complete digraph with 3 vertices>
gap> OutNeighbours(D2);
[ [ 2, 3 ], [ 1, 3 ], [ 1, 2 ] ]
gap> union := DigraphDisjointUnion(D1, D2);
<immutable digraph with 6 vertices, 9 edges>
gap> OutNeighbours(union);
[ [ 2 ], [ 3 ], [ 1 ], [ 5, 6 ], [ 4, 6 ], [ 4, 5 ] ]
```

3.3.30 DigraphEdgeUnion (for a positive number of digraphs)

▷ `DigraphEdgeUnion(D1, D2, ...)`   (function)
▷ `DigraphEdgeUnion(list)`   (function)

Returns: A digraph.

In the first form, if *D1*, *D2*, etc. are digraphs, then `DigraphEdgeUnion` returns their edge union. In the second form, if *list* is a non-empty list of digraphs, then `DigraphEdgeUnion` returns the edge union of the digraphs contained in the list.

The vertex set of the edge union of a collection of digraphs is the *union* of the vertex sets, whilst the edge list of the edge union is the *concatenation* of the edge lists. The number of vertices of the edge union is equal to the *maximum* number of vertices of one of the digraphs, whilst the number of edges of the edge union will equal the *sum* of the number of edges of each digraph.

Note that previously set `DigraphVertexLabels` (5.1.12) will be lost.

If the first digraph *D1* [*list[1]*] belongs to `IsMutableDigraph` (3.1.2), then *D1* [*list[1]*] is modified in place to contain the appropriate vertices and edges. If *digraph* belongs to `IsImmutableDigraph` (3.1.3), a new immutable digraph containing the appropriate vertices and edges is returned.

```
——————————————————————————— Example ———————————————————————————
gap> D := CycleDigraph(10);
<immutable cycle digraph with 10 vertices>
```

```
gap> DigraphEdgeUnion(D, D);
<immutable multidigraph with 10 vertices, 20 edges>
gap> D1 := Digraph([[2], [1]]);
<immutable digraph with 2 vertices, 2 edges>
gap> D2 := Digraph([[2, 3], [2], [1]]);
<immutable digraph with 3 vertices, 4 edges>
gap> union := DigraphEdgeUnion(D1, D2);
<immutable multidigraph with 3 vertices, 6 edges>
gap> OutNeighbours(union);
[ [ 2, 2, 3 ], [ 1, 2 ], [ 1 ] ]
gap> union = DigraphByEdges(
> Concatenation(DigraphEdges(D1), DigraphEdges(D2)));
true
```

3.3.31 DigraphJoin (for a positive number of digraphs)

▷ DigraphJoin(*D1, D2, ...*)   (function)
▷ DigraphJoin(*list*)   (function)
   Returns: A digraph.

In the first form, if *D1*, *D2*, etc. are digraphs, then DigraphJoin returns their join. In the second form, if *list* is a non-empty list of digraphs, then DigraphJoin returns the join of the digraphs contained in the list.

The join of a collection of digraphs *D1*, *D2*, ... is formed by first taking the DigraphDisjointUnion (3.3.29) of the collection. In the disjoint union, if $i \neq j$ then there are no edges between vertices corresponding to digraphs *D*[i] and *D*[j] in the collection; the join is created by including all such edges.

For example, the join of two empty digraphs is a complete bipartite digraph.

Note that previously set DigraphVertexLabels (5.1.12) will be lost.

If the first digraph *D1* [*list[1]*] belongs to IsMutableDigraph (3.1.2), then *D1* [*list[1]*] is modified in place to contain the appropriate vertices and edges. If *digraph* belongs to IsImmutableDigraph (3.1.3), a new immutable digraph containing the appropriate vertices and edges is returned.

```
─────────────── Example ───────────────
gap> D := CompleteDigraph(3);
<immutable complete digraph with 3 vertices>
gap> IsCompleteDigraph(DigraphJoin(D, D));
true
gap> D2 := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> DigraphJoin(D, D2);
<immutable digraph with 6 vertices, 27 edges>
```

3.3.32 DigraphCartesianProduct (for a positive number of digraphs)

▷ DigraphCartesianProduct(*gr1, gr2, ...*)   (function)
▷ DigraphCartesianProduct(*list*)   (function)
   Returns: A digraph.

In the first form, if *gr1*, *gr2*, etc. are digraphs, then DigraphCartesianProduct returns a digraph isomorphic to their Cartesian product.

In the second form, if `list` is a non-empty list of digraphs, then `DigraphCartesianProduct` returns a digraph isomorphic to the Cartesian product of the digraphs contained in the list.

Mathematically, the Cartesian product of two digraphs G, H is a digraph with vertex set `Cartesian(DigraphVertices(G), DigraphVertices(H))` such that there is an edge from [u, u'] to [v, v'] iff u = v and there is an edge from u' to v' in H or u' = v' and there is an edge from u to v in G.

Due to technical reasons, the digraph D returned by `DigraphCartesianProduct` has vertex set [1 .. `DigraphNrVertices(G)*DigraphNrVertices(H)`] instead, and the exact method of encoding pairs of vertices into integers is implementation specific. The original vertex pair can be somewhat regained by using `DigraphCartesianProductProjections` (3.3.39). In addition, `DigraphVertexLabels` (5.1.12) are preserved: if vertex pair [u,u'] was encoded as i then the vertex label of i will be the pair of vertex labels of u and u' i.e. `DigraphVertexLabel(D,i)` = [`DigraphVertexLabel(G,u)`, `DigraphVertexLabel(H,u')`].

As the Cartesian product is associative, the Cartesian product of a collection of digraphs *gr1*, *gr2*, ... is computed in the obvious fashion.

```
_____ Example _____
gap> gr := ChainDigraph(4);
<immutable chain digraph with 4 vertices>
gap> gr2 := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> gr3 := DigraphCartesianProduct(gr, gr2);
<immutable digraph with 12 vertices, 21 edges>
gap> IsIsomorphicDigraph(gr3,
> Digraph([[2, 5], [3, 6], [4, 7], [8],
>          [6, 9], [7, 10], [8, 11], [12],
>          [10, 1], [11, 2], [12, 3], [4]]));
true
```

3.3.33 DigraphDirectProduct (for a positive number of digraphs)

▷ `DigraphDirectProduct(gr1, gr2, ...)`   (function)
▷ `DigraphDirectProduct(list)`   (function)

Returns: A digraph.

In the first form, if *gr1*, *gr2*, etc. are digraphs, then `DigraphDirectProduct` returns a digraph isomorphic to their direct product.

In the second form, if `list` is a non-empty list of digraphs, then `DigraphDirectProduct` returns a digraph isomorphic to the direct product of the digraphs contained in the list.

Mathematically, the direct product of two digraphs G, H is a digraph with vertex set `Cartesian(DigraphVertices(G), DigraphVertices(H))` such that there is an edge from [u, u'] to [v, v'] iff there is an edge from u to v in G and an edge from u' to v' in H.

Due to technical reasons, the digraph D returned by `DigraphDirectProduct` has vertex set [1 .. `DigraphNrVertices(G)*DigraphNrVertices(H)`] instead, and the exact method of encoding pairs of vertices into integers is implementation specific. The original vertex pair can be somewhat regained by using `DigraphDirectProductProjections` (3.3.40). In addition `DigraphVertexLabels` (5.1.12) are preserved: if vertex pair [u,u'] was encoded as i then the vertex label of i will be the pair of vertex labels of u and u' i.e. `DigraphVertexLabel(D,i)` = [`DigraphVertexLabel(G,u)`, `DigraphVertexLabel(H,u')`].

As the direct product is associative, the direct product of a collection of digraphs *gr1*, *gr2*, ...
is computed in the obvious fashion.

```
 ──────────────────── Example ────────────────────
 gap> gr := ChainDigraph(4);
 <immutable chain digraph with 4 vertices>
 gap> gr2 := CycleDigraph(3);
 <immutable cycle digraph with 3 vertices>
 gap> gr3 := DigraphDirectProduct(gr, gr2);
 <immutable digraph with 12 vertices, 9 edges>
 gap> IsIsomorphicDigraph(gr3,
 > Digraph([[6], [7], [8], [],
 >          [10], [11], [12], [],
 >          [2], [3], [4], []]));
 true
```

3.3.34 ConormalProduct

▷ ConormalProduct(*D1, D2*)   (operation)

Returns: A digraph.

If *D1* and *D2* are digraphs without multiple edges, then ConormalProduct calculates the *conormal product digraph* (CNPD) of *D1* and *D2*. CNPD has vertex set V1 x V2 where V1 is the vertex set of *D1* and V2 is the vertex set of *D2* (a vertex [a, b] has label (a - 1) * |V2| + b in the output). There is an edge from [a, b] to [c, d] when at least one of the following two conditions are satisfied:

- There is an edge from a to c in *D1*.

- There is an edge from b to d in *D2*.

```
 ──────────────────── Example ────────────────────
 gap> ConormalProduct(DigraphSymmetricClosure(CycleDigraph(7)),
 > DigraphSymmetricClosure(CycleDigraph(4)));
 <immutable digraph with 28 vertices, 504 edges>
 gap> ConormalProduct(NullDigraph(0), CompleteDigraph(10));
 <immutable empty digraph with 0 vertices>
```

3.3.35 HomomorphicProduct

▷ HomomorphicProduct(*D1, D2*)   (operation)

Returns: A digraph.

If *D1* and *D2* are digraphs without multiple edges, then HomomorphicProduct calculates the *homomorphic product digraph* (HPD) of *D1* and *D2*. HPD has vertex set V1 x V2 where V1 is the vertex set of *D1* and V2 is the vertex set of *D2* (a vertex [a, b] has label (a - 1) * |V2| + b in the output). There is an edge from [a, b] to [c, d] when at least one of the following two conditions are satisfied:

- The vertices a and c of *D1* are equal.

- There is an edge from a to c in *D1* and no edge from b to d in *D2*.

```
┌─────────────────────────── Example ───────────────────────────┐
 gap> HomomorphicProduct(PetersenGraph(),
 > DigraphSymmetricClosure(ChainDigraph(4)));
 <immutable digraph with 40 vertices, 460 edges>
 gap> D1 := Digraph([[2], [1, 3, 4], [2, 5], [2, 5], [3, 4]]);
 <immutable digraph with 5 vertices, 10 edges>
 gap> D2 := Digraph([[2], [1, 3], [2, 4], [3]]);
 <immutable digraph with 4 vertices, 6 edges>
 gap> HomomorphicProduct(D1, D2);
 <immutable digraph with 20 vertices, 180 edges>
```

3.3.36 LexicographicProduct

▷ LexicographicProduct(*D1*, *D2*)   (operation)

Returns: A digraph.

If *D1* and *D2* are digraphs without multiple edges, then LexicographicProduct calculates the *lexicographic product digraph* (LPD) of *D1* and *D2*. LPD has vertex set V1 x V2 where V1 is the vertex set of *D1* and V2 is the vertex set of *D2* (a vertex [a, b] has label (a - 1) * |V2| + b in the output). There is an edge from [a, b] to [c, d] when at least one of the following two conditions are satisfied:

- There is an edge from a to c in *D1*.

- The vertices a and c of *D1* are equal and there is an edge from b to d in *D2*.

```
┌─────────────────────────── Example ───────────────────────────┐
 gap> LexicographicProduct(DigraphSymmetricClosure(CycleDigraph(3)),
 > DigraphSymmetricClosure(ChainDigraph(2)));
 <immutable digraph with 6 vertices, 30 edges>
 gap> OutNeighbours(last);
 [ [ 2, 3, 4, 5, 6 ], [ 1, 3, 4, 5, 6 ], [ 1, 2, 4, 5, 6 ],
   [ 1, 2, 3, 5, 6 ], [ 1, 2, 3, 4, 6 ], [ 1, 2, 3, 4, 5 ] ]
 gap> LexicographicProduct(NullDigraph(0), CompleteDigraph(10));
 <immutable empty digraph with 0 vertices>
```

3.3.37 ModularProduct

▷ ModularProduct(*D1*, *D2*)   (operation)

Returns: A digraph.

If *D1* and *D2* are digraphs without multiple edges, then ModularProduct calculates the *modular product digraph* (MPD) of *D1* and *D2*. MPD has vertex set V1 x V2 where V1 is the vertex set of *D1* and V2 is the vertex set of *D2* (a vertex [a, b] has label (a - 1) * |V2| + b in the output). There is an edge from [a, b] to [c, d] precisely when the following two conditions are satisfied:

- The vertices a and c of *D1* are equal if and only if the vertices b and d of *D2* are equal.

- There is an edge from a to c in *D1* if and only if there is an edge from b to d in *D2*.

Notably, the complete (with loops) subdigraphs of MPD are precisely the partial isomorphisms from *D1* to *D2*.

```
─────────────── Example ───────────────
gap> ModularProduct(Digraph([[1], [1, 2]]), Digraph([[], [2]]));
<immutable digraph with 4 vertices, 4 edges>
gap> OutNeighbours(last);
[ [ 4 ], [ 2, 3 ], [ ], [ 4 ] ]
gap> ModularProduct(PetersenGraph(),
> DigraphSymmetricClosure(CycleDigraph(5)));
<immutable digraph with 50 vertices, 950 edges>
gap> ModularProduct(NullDigraph(0), CompleteDigraph(10));
<immutable empty digraph with 0 vertices>
```

3.3.38   StrongProduct

▷ StrongProduct(`D1, D2`)   (operation)

Returns: A digraph.

If `D1` and `D2` are digraphs without multiple edges, then StrongProduct calculates the *strong product digraph* (SPD) of `D1` and `D2`. SPD has vertex set V1 x V2 where V1 is the vertex set of `D1` and V2 is the vertex set of `D2` (a vertex [a, b] has label (a - 1) * |V2| + b in the output). There is an edge from [a, b] to [c, d] when at least one of the following three conditions are satisfied:

- The vertices a and c of `D1` are equal and there is an edge from b to d in `D2`.

- The vertices b and d of `D2` are equal and there is an edge from a to c in `D1`.

- There is an edge from a to c in `D1` and there is an edge from b to d in `D2`.

The SPD of two paths of lengths m and n is also the king's graph for an m by n board.

```
─────────────── Example ───────────────
gap> D1 := Digraph([[2], [1, 3, 4], [2, 5], [2, 5], [3, 4]]);
<immutable digraph with 5 vertices, 10 edges>
gap> D2 := Digraph([[2], [1, 3, 4], [2], [2]]);
<immutable digraph with 4 vertices, 6 edges>
gap> StrongProduct(D1, D2);
<immutable digraph with 20 vertices, 130 edges>
gap> StrongProduct(DigraphSymmetricClosure(ChainDigraph(3)),
> DigraphSymmetricClosure(ChainDigraph(4)));
<immutable digraph with 12 vertices, 58 edges>
```

3.3.39   DigraphCartesianProductProjections

▷ DigraphCartesianProductProjections(`digraph`)   (attribute)

Returns: A list of transformations.

If `digraph` is a Cartesian product digraph, digraph = DigraphCartesianProduct(gr_1, gr_2, ...), then DigraphCartesianProductProjections returns a list proj such that proj[i] is the projection onto the i-th coordinate of the product.

A projection is an idempotent endomorphism of `digraph`. If gr1, gr2, ... are all loopless digraphs, then the induced subdigraph of `digraph` on the image of proj[i] is isomorphic to gr_i.

Currently this attribute is only set upon creating an immutable digraph via DigraphCartesianProduct and there is no way of calculating it for other digraphs.

For more information see DigraphCartesianProduct (3.3.32)

```
_____ Example _____
gap> D := DigraphCartesianProduct(ChainDigraph(3), CycleDigraph(4),
> Digraph([[2], [2]]));;
gap> HasDigraphCartesianProductProjections(D);
true
gap> proj := DigraphCartesianProductProjections(D);; Length(proj);
3
gap> IsIdempotent(proj[2]);
true
gap> RankOfTransformation(proj[3]);
2
gap> S := ImageSetOfTransformation(proj[2]);;
gap> IsIsomorphicDigraph(CycleDigraph(4), InducedSubdigraph(D, S));
true
```

3.3.40 DigraphDirectProductProjections

▷ DigraphDirectProductProjections(*digraph*)   (attribute)

Returns: A list of transformations.

If *digraph* is a direct product digraph, digraph = DigraphDirectProduct(gr_1, gr_2, ...), then DigraphDirectProductProjections returns a list proj such that proj[i] is the projection onto the i-th coordinate of the product.

A projection is an idempotent endomorphism of *digraph*. If gr1, gr2, ... are all loopless digraphs, then the image of *digraph* under proj[i] is isomorphic to gr_i.

Currently this attribute is only set upon creating an immutable digraph via DigraphDirectProduct and there is no way of calculating it for other digraphs.

For more information, see DigraphDirectProduct (3.3.33)

```
_____ Example _____
gap> D := DigraphDirectProduct(ChainDigraph(3), CycleDigraph(4),
> Digraph([[2], [2]]));;
gap> HasDigraphDirectProductProjections(D);
true
gap> proj := DigraphDirectProductProjections(D);; Length(proj);
3
gap> IsIdempotent(proj[2]);
true
gap> RankOfTransformation(proj[3]);
2
gap> P := DigraphRemoveAllMultipleEdges(
> ReducedDigraph(OnDigraphs(D, proj[2])));;
gap> IsIsomorphicDigraph(CycleDigraph(4), P);
true
```

3.3.41 LineDigraph

▷ LineDigraph(*digraph*)   (operation)
▷ EdgeDigraph(*digraph*)   (operation)

Returns: A digraph.

Given a digraph `digraph`, the operation returns the digraph obtained by associating a vertex with each edge of `digraph`, and creating an edge from a vertex v to a vertex u if and only if the terminal vertex of the edge associated with v is the start vertex of the edge associated with u.

Note that the returned digraph is always a new immutable digraph, and the argument `digraph` is never modified.

```
 ───────────────────── Example ─────────────────────
  gap> LineDigraph(CompleteDigraph(3));
  <immutable digraph with 6 vertices, 12 edges>
  gap> LineDigraph(ChainDigraph(3));
  <immutable digraph with 2 vertices, 1 edge>
```

3.3.42 LineUndirectedDigraph

▷ LineUndirectedDigraph(*digraph*)   (operation)
▷ EdgeUndirectedDigraph(*digraph*)   (operation)
  Returns: A digraph.

Given a symmetric digraph `digraph`, the operation returns the symmetric digraph obtained by associating a vertex with each edge of `digraph`, ignoring directions and multiplicities, and adding an edge between two vertices if and only if the corresponding edges have a vertex in common.

Note that the returned digraph is always a new immutable digraph, and the argument `digraph` is never modified.

```
 ───────────────────── Example ─────────────────────
  gap> LineUndirectedDigraph(CompleteDigraph(3));
  <immutable digraph with 3 vertices, 6 edges>
  gap> LineUndirectedDigraph(DigraphSymmetricClosure(ChainDigraph(3)));
  <immutable digraph with 2 vertices, 2 edges>
```

3.3.43 DoubleDigraph

▷ DoubleDigraph(*digraph*)   (operation)
  Returns: A digraph.

Let `digraph` be a digraph with vertex set V. This function returns the double digraph of `digraph`. The vertex set of the double digraph is the original vertex set together with a duplicate. The edges are [u_1, v_2] and [u_2, v_1] if and only if [u, v] is an edge in `digraph`, together with the original edges and their duplicates.

If `digraph` is mutable, then `digraph` is modified in-place. If `digraph` is immutable, then a new immutable digraph constructed as described above is returned.

```
 ───────────────────── Example ─────────────────────
  gap> gamma := Digraph([[2], [3], [1]]);
  <immutable digraph with 3 vertices, 3 edges>
  gap> DoubleDigraph(gamma);
  <immutable digraph with 6 vertices, 12 edges>
```

3.3.44 BipartiteDoubleDigraph

▷ BipartiteDoubleDigraph(*digraph*)   (operation)
  Returns: A digraph.

Let `digraph` be a digraph with vertex set `V`. This function returns the bipartite double digraph of `digraph`. The vertex set of the double digraph is the original vertex set together with a duplicate. The edges are [u_1, v_2] and [u_2, v_1] if and only if [u, v] is an edge in `digraph`. The resulting graph is bipartite, since the original edges are not included in the resulting digraph.

If `digraph` is mutable, then `digraph` is modified in-place. If `digraph` is immutable, then a new immutable digraph constructed as described above is returned.

```
————————— Example —————————
gap> gamma := Digraph([[2], [3], [1]]);
<immutable digraph with 3 vertices, 3 edges>
gap> BipartiteDoubleDigraph(gamma);
<immutable digraph with 6 vertices, 6 edges>
```

3.3.45 DigraphAddAllLoops

▷ DigraphAddAllLoops(`digraph`)   (operation)
▷ DigraphAddAllLoopsAttr(`digraph`)   (attribute)
    Returns: A digraph.

For a digraph `digraph` this operation returns a new digraph constructed from `digraph`, such that a loop is added for every vertex which did not have a loop in `digraph`.

If `digraph` is immutable, then a new immutable digraph is returned. Otherwise, the loops are added to the loopless vertices of the mutable digraph `digraph` in-place.

```
————————— Example —————————
gap> D := EmptyDigraph(13);
<immutable empty digraph with 13 vertices>
gap> D := DigraphAddAllLoops(D);
<immutable reflexive digraph with 13 vertices, 13 edges>
gap> OutNeighbours(D);
[ [ 1 ], [ 2 ], [ 3 ], [ 4 ], [ 5 ], [ 6 ], [ 7 ], [ 8 ], [ 9 ],
  [ 10 ], [ 11 ], [ 12 ], [ 13 ] ]
gap> D := Digraph([[1, 2, 3], [1, 3], [1]]);
<immutable digraph with 3 vertices, 6 edges>
gap> D := DigraphAddAllLoops(D);
<immutable reflexive digraph with 3 vertices, 8 edges>
gap> OutNeighbours(D);
[ [ 1, 2, 3 ], [ 1, 3, 2 ], [ 1, 3 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphAddAllLoops(D);
<mutable digraph with 3 vertices, 6 edges>
gap> D;
<mutable digraph with 3 vertices, 6 edges>
```

3.3.46 DistanceDigraph (for digraph and int)

▷ DistanceDigraph(`digraph`, `i`)   (operation)
▷ DistanceDigraph(`digraph`, `list`)   (operation)
    Returns: A digraph.

The first argument is a digraph, the second argument is a non-negative integer or a list of positive integers. This operation returns a digraph on the same set of vertices as `digraph`, with two vertices

being adjacent if and only if the distance between them in `digraph` equals `i` or is a number in `list`. See DigraphShortestDistance (5.4.2).

If `digraph` is mutable, then `digraph` is modified in-place. If `digraph` is immutable, then a new immutable digraph constructed as described above is returned.

```
─────────────── Example ───────────────
gap> digraph := DigraphFromSparse6String(
> ":]n?AL'BC_DEbEF'GIaGHdIJeGKcKL_@McDHfILaBJfHMjKM");
<immutable symmetric digraph with 30 vertices, 90 edges>
gap> DistanceDigraph(digraph, 1);
<immutable digraph with 30 vertices, 90 edges>
gap> DistanceDigraph(digraph, [1, 2]);
<immutable digraph with 30 vertices, 270 edges>
```

3.3.47 DigraphClosure

▷ DigraphClosure(`digraph, k`) (operation)

Returns: A digraph.

Given a symmetric loopless digraph with no multiple edges `digraph`, the *k-closure of* `digraph` is defined to be the unique smallest symmetric loopless digraph `C` with no multiple edges on the vertices of `digraph` that contains all the edges of `digraph` and satisfies the property that the sum of the degrees of every two non-adjacenct vertices in `C` is less than `k`. See IsSymmetricDigraph (6.2.14), DigraphHasLoops (6.2.1), IsMultiDigraph (6.2.11), and OutDegreeOfVertex (5.2.10).

The operation DigraphClosure returns the `k`-closure of `digraph`.

```
─────────────── Example ───────────────
gap> D := CompleteDigraph(6);
<immutable complete digraph with 6 vertices>
gap> D := DigraphRemoveEdges(D, [[1, 2], [2, 1]]);
<immutable digraph with 6 vertices, 28 edges>
gap> closure := DigraphClosure(D, 6);
<immutable digraph with 6 vertices, 30 edges>
gap> IsCompleteDigraph(closure);
true
```

3.3.48 DigraphMycielskian

▷ DigraphMycielskian(`digraph`) (operation)
▷ DigraphMycielskianAttr(`digraph`) (attribute)

Returns: A digraph.

If `digraph` is a symmetric digraph, then DigraphMycielskian returns the Mycielskian of `digraph`.

The Mycielskian of a symmetric digraph is a larger symmetric digraph constructed from it, which has a larger chromatic number. For further information, see https://en.wikipedia.org/wiki/Mycielskian.

If `digraph` is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph `digraph` is changed in-place into its Mycielskian.

```
─────────────── Example ───────────────
gap> D := CycleDigraph(2);
<immutable cycle digraph with 2 vertices>
gap> ChromaticNumber(D);
```

```
2
gap> D := DigraphMycielskian(D);
<immutable digraph with 5 vertices, 10 edges>
gap> ChromaticNumber(D);
3
gap> D := DigraphMycielskian(D);
<immutable digraph with 11 vertices, 40 edges>
gap> ChromaticNumber(D);
4
gap> D := CompleteBipartiteDigraph(IsMutable, 2, 3);
<mutable digraph with 5 vertices, 12 edges>
gap> DigraphMycielskian(D);
<mutable digraph with 11 vertices, 46 edges>
gap> D;
<mutable digraph with 11 vertices, 46 edges>
```

3.4 Random digraphs

3.4.1 RandomDigraph

▷ RandomDigraph([filt,]n[, p])   (operation)
  Returns: A digraph.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

The other implemented filters are as follows: IsConnectedDigraph (6.6.3), IsSymmetricDigraph (6.2.14), IsAcyclicDigraph (6.6.1), IsEulerianDigraph (6.6.10), IsHamiltonianDigraph (6.6.11).

For IsConnectedDigraph (6.6.3), a random tree is first created independent of the value of p, guaranteeing connectivity (with $n - 1$ edges), and then edges are added between the remaining pairs of vertices with probability approximately p.

For IsHamiltonianDigraph (6.6.11), a random Hamiltonian cycle is first created independent of the value of p (with n edges), and then edges are added between the remaining pairs of vertices with probability approximately p.

For IsEulerianDigraph (6.6.10), a random Eulerian cycle is created where p influences how long the cycle will be. The cycle grows by randomly considering edges that extend the cycle, and adding an edge with probability approximately p. The cycle stops when we get back to the start vertex and have no more edges left to consider from it that extend the cycle further (any possible edge from the start vertex has either been added to the cycle, or rejected, leaving no more edges to consider). Thus $p = 1$ does not necessarily guarantee a complete digraph. Instead, it guarantees that all edges considered up to the point where the cycle stops, are added.

For IsAcyclicDigraph (6.6.1) and IsSymmetricDigraph (6.2.14), edges are added between any pairs of vertices with probability approximately p.

If n is a positive integer, then this function returns a random digraph with n vertices and without multiple edges. The result may or may not have loops. If using IsAcyclicDigraph (6.6.1), the resulting graph will not have any loops by definition.

If the optional second argument p is a float with value $0 \le p \le 1$, then an edge will exist between each pair of vertices with probability approximately p. If p is not specified, then a random probability will be assumed (chosen with uniform probability).

```
————————————————————————— Example —————————————————————————
  gap> RandomDigraph(1000);
  <immutable digraph with 1000 vertices, 364444 edges>
  gap> RandomDigraph(10000, 0.023);
  <immutable digraph with 10000 vertices, 2300438 edges>
  gap> RandomDigraph(IsMutableDigraph, 1000, 1 / 2);
  <mutable digraph with 1000 vertices, 499739 edges>
  gap> RandomDigraph(IsConnectedDigraph, 1000, 0.75);
  <immutable digraph with 1000 vertices, 750265 edges>
  gap> RandomDigraph(IsSymmetricDigraph, 1000);
  <immutable digraph with 1000 vertices, 329690 edges>
  gap> RandomDigraph(IsAcyclicDigraph, 1000, 0.25);
  <immutable digraph with 1000 vertices, 125070 edges>
  gap> RandomDigraph(IsHamiltonianDigraph, 1000, 0.5);
  <immutable digraph with 1000 vertices, 500327 edges>
  gap> RandomDigraph(IsEulerianDigraph, 1000, 0.5);
  <immutable digraph with 1000 vertices, 433869 edges>
```

3.4.2 RandomMultiDigraph

▷ RandomMultiDigraph(n[, m])   (operation)

    Returns: A digraph.

If n is a positive integer, then this function returns a random digraph with n vertices. If the optional second argument m is a positive integer, then the digraph will have m edges. If m is not specified, then the number of edges will be chosen randomly (with uniform probability) from the range $[1 \ .. \ \binom{n}{2}]$.

The method used by this function chooses each edge from the set of all possible edges with uniform probability. No effort is made to avoid creating multiple edges, so it is possible (but not guaranteed) that the result will have multiple edges. The result may or may not have loops.

```
————————————————————————— Example —————————————————————————
  gap> RandomMultiDigraph(1000);
  <immutable multidigraph with 1000 vertices, 216659 edges>
  gap> RandomMultiDigraph(1000, 950);
  <immutable multidigraph with 1000 vertices, 950 edges>
```

3.4.3 RandomTournament

▷ RandomTournament([filt,]n)   (operation)

    Returns: A digraph.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

If n is a non-negative integer, this function returns a random tournament with n vertices. See IsTournament (6.2.15).

```
                              Example
  gap> RandomTournament(10);
  <immutable tournament with 10 vertices>
  gap> RandomTournament(IsMutableDigraph, 10);
  <mutable digraph with 1000 vertices, 500601 edges>
```

3.4.4 RandomLattice

▷ RandomLattice(*n*)   (operation)

   Returns: A digraph.

   If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

   If `n` is a positive integer, this function return a random lattice with m vertices, where it is guaranteed that m is between `n` and 2 * `n`. See IsLatticeDigraph (6.4.3).

```
                              Example
  gap> RandomLattice(10);
  <immutable lattice digraph with 10 vertices, 39 edges>
  gap> RandomLattice(IsMutableDigraph, 10);
  <mutable digraph with 12 vertices, 52 edges>
```

3.5 Standard examples

3.5.1 AndrasfaiGraph

▷ AndrasfaiGraph([*filt,*]*n*)   (operation)

   Returns: A digraph.

   If `n` is an integer greater than 0, then this operation returns the `n`th *Andrasfai graph*. The Andrasfai graph is a circulant graph with 3`n` - 1 vertices. The indices of the Andrasfai graph are given by the numbers between 1 and 3`n` - 1 that are congruent to 1 mod 3 (that is, for each index j, vertex i is adjacent to the $i + j$th and $i - j$ vertices). The graph has 6(3`n` - 1) edges. The graph is triangle free.

   As a circulant graph, the Andrasfai graph is biconnected, cyclic, Hamiltonian, regular, and vertex transitive.

   See https://mathworld.wolfram.com/AndrasfaiGraph.html for further details.

   If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
                              Example
  gap> D := AndrasfaiGraph(4);
  <immutable Hamiltonian biconnected vertex-transitive symmetric digraph\
   with 11 vertices, 44 edges>
  gap> IsBiconnectedDigraph(D);
  true
```

```
gap> IsIsomorphicDigraph(D, CirculantGraph(11, [1, 4, 7, 10]));
true
```

3.5.2 BananaTree

▷ BananaTree([filt,]n, k) (operation)

Returns: A digraph

If `n` and `k` are positive integers with `k` greater than 1, then this operation returns the *banana tree* with parameters `n` and `k`, as defined below.

From https://mathworld.wolfram.com/BananaTree.html:

"An (`n`,`k`)-banana tree, as defined by Chen et al. (1997), is a graph obtained by connecting one leaf of each of `n` copies of an `k`-star graph with a single root vertex that is distinct from all the stars."

Specifically, in the resulting digraph, vertex 1 is the 'root', and for each m in `[1 .. k]`, the mth star is on the vertices `[((m - 1) * n) + 2 .. (m * n) + 1]`, with the first of these being the 'centre' of the star, and the second being the 'leaf' adjacent 1.

See also `StarGraph` (3.5.43) and `IsUndirectedTree` (6.6.9).

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
  _____ Example _____
  gap> D := BananaTree(2, 4);
  <immutable undirected tree digraph with 9 vertices>
  gap> D := BananaTree(3, 3);
  <immutable undirected tree digraph with 10 vertices>
  gap> D := BananaTree(5, 2);
  <immutable undirected tree digraph with 11 vertices>
  gap> D := BananaTree(3, 4);
  <immutable undirected tree digraph with 13 vertices>
```

3.5.3 BinaryTree

▷ BinaryTree([filt,]m) (operation)

Returns: A digraph.

This function returns a binary tree of depth `m` with $2 \char` m - 1$ vertices. All edges are directed towards the root of the tree, which is vertex 1.

Note that `BinaryTree(m)` is the induced subdigraph of `BinaryTree(m+1)` on the vertices `[1..2^(m-1)]`.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
  _____ Example _____
  gap> BinaryTree(1);
  <immutable empty digraph with 1 vertex>
```

```
gap> BinaryTree(8);
<immutable digraph with 255 vertices, 254 edges>
gap> BinaryTree(IsMutableDigraph, 8);
<mutable digraph with 255 vertices, 254 edges>
```

3.5.4 BinomialTreeGraph

▷ BinomialTreeGraph([filt,]n)   (operation)

Returns: A digraph.

If `n` is a positive integer then this operation returns the `n`th *binomial tree graph*. The binomial tree graph has `n` vertices and `n-1` undirected edges. The vertices of the binomial tree graph are the numbers from 1 to `n` in binary representation, with a vertex v having as a direct parent the vertex with binary representation the same as v but with the lowest 1-bit cleared. For example, the vertex 011 has parent 010, and the vertex 010 has parent 000.

The binomial tree graph is an undirected tree, and is symmetric as a digraph.

See https://metacpan.org/pod/Graph::Maker::BinomialTree for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
———————————————————— Example ————————————————————
gap> D := BinomialTreeGraph(9);
<immutable undirected tree digraph with 9 vertices>
```

3.5.5 BishopsGraph

▷ BishopsGraph([filt,][color,]m, n)   (operation)
▷ BishopGraph([filt,][color,]m, n)   (operation)

Returns: A digraph.

If `m` and `n` are positive integers, then this operation returns the *bishop's graph* of an `m` by `n` chessboard, as a symmetric digraph.

A bishop's graph represents all possible moves of the bishop chess piece across a chessboard. An `m` by `n` chessboard is a grid of `m` columns ("files") and `n` rows ("ranks") that intersect in squares. Orthogonally adjacent squares are alternately colored light and dark, with the square in the first rank and file being dark.

The `m * n` vertices of the bishop's graph can be placed onto the `m * n` squares of an `m` by `n` chessboard, such that two vertices are adjacent in the digraph if and only if a bishop can move between the corresponding squares in a single turn. A legal bishop's move is defined as any move which moves the bishop piece to a diagonally adjacent square or to a square which can be reached through a series of diagonally adjacent squares, with all of these small moves being in the same direction.

The chosen correspondence between vertices and chess squares is given by DigraphVertexLabels (5.1.12). In more detail, the vertices of the digraph are labelled by elements of the Cartesian product [1..m] x [1..n], where the first entry indexes the column (file) of the square in the chessboard, and the second entry indexes the row (rank) of the square. (Note that the files are traditionally indexed by the lowercase letters of the alphabet). The vertices are sorted in ascending order, first by row (second component) and then column (first component). If the

optional second argument `color` is present, then this should be one of the strings `"dark"`, `"light"`, or `"both"`. The default is `"both"`. A bishop on a light square can only move to light squares, and a bishop on a dark square can only move to dark squares. This optional argument controls which bishops are represented in the resulting digraph. If `color` is `"both"`, then the resulting digraph will show all the vertices of an `m` by `n` chessboard, and will be disconnected (unless `m = n = 1`). Otherwise, `BishopsGraph` returns the induced subdigraph of this on the vertices that correspond to either the dark squares or the light squares, according to the value of `color`.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
──────────────── Example ────────────────
gap> BishopsGraph(8, 8);
<immutable symmetric digraph with 64 vertices, 560 edges>
gap> D := BishopsGraph("dark", 3, 5);
<immutable connected symmetric digraph with 8 vertices, 24 edges>
gap> IsConnectedDigraph(D);
true
gap> BishopsGraph("light", 4, 4);
<immutable connected symmetric digraph with 8 vertices, 28 edges>
gap> D := BishopsGraph("both", 1, 5);
<immutable empty digraph with 5 vertices>
```

3.5.6 BondyGraph

▷ BondyGraph(*[filt,]n*)   (operation)

Returns: A digraph.

If `n` is a non-negative integer then this operation returns the `n`th *Bondy graph*. The Bondy graphs are a family of hypohamiltonian graphs: a graph which is not Hamiltonian itself but the removal of any vertex produces a Hamiltonian graph. The Bondy graphs are the `(3(2n + 1) + 2, 2)`-th Generalised Petersen graphs, and have `12n + 10` vertices and `15 + 18n` undirected edges.

See https://mathworld.wolfram.com/HypohamiltonianGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
──────────────── Example ────────────────
gap> D := BondyGraph(1);
<immutable symmetric digraph with 22 vertices, 66 edges>
gap> IsHamiltonianDigraph(D);
false
gap> G := List([1 .. 22], x -> DigraphRemoveVertex(D, x));;
gap> ForAll(G, IsHamiltonianDigraph);
true
```

3.5.7 BookGraph

▷ BookGraph([*filt*,]*m*)   (operation)

Returns: A digraph.

The *book graph* is the Cartesian product of a complete digraph with two vertices (as the "book spine") and the $m + 1$ star graph (as the "pages"). For more details on the book graph please refer to https://mathworld.wolfram.com/BookGraph.html.

See DigraphCartesianProduct (3.3.32), CompleteDigraph (3.5.11), and StarGraph (3.5.43).

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
───────────────────── Example ─────────────────────
  gap> BookGraph(1);
  <immutable bipartite symmetric digraph with bicomponent sizes 2 and 2>
  gap> BookGraph(2);
  <immutable bipartite symmetric digraph with bicomponent sizes 3 and 3>
  gap> BookGraph(IsMutable, 12);
  <mutable digraph with 26 vertices, 74 edges>
  gap> BookGraph(7);
  <immutable bipartite symmetric digraph with bicomponent sizes 8 and 8>
  gap> IsSymmetricDigraph(BookGraph(24));
  true
  gap> IsBipartiteDigraph(BookGraph(24));
  true
```

3.5.8 StackedBookGraph

▷ StackedBookGraph([*filt*,]*m*, *n*)   (operation)

Returns: A digraph.

The *stacked book graph* is the Cartesian product of the symmetric closure of the chain digraph with n vertices (as the "book spine") and the $m + 1$ star graph (as the "pages"). For more details on the stacked book graph please refer to https://mathworld.wolfram.com/StackedBookGraph.html.

See DigraphCartesianProduct (3.3.32), DigraphSymmetricClosure (3.3.12), ChainDigraph (3.5.9), and StarGraph (3.5.43).

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
───────────────────── Example ─────────────────────
  gap> StackedBookGraph(1, 1);
  <immutable bipartite symmetric digraph with bicomponent sizes 1 and 1>
  gap> StackedBookGraph(1, 2);
  <immutable bipartite symmetric digraph with bicomponent sizes 2 and 2>
  gap> StackedBookGraph(3, 4);
  <immutable bipartite symmetric digraph with bicomponent sizes 8 and 8>
  gap> StackedBookGraph(IsMutable, 12, 5);
```

```
<mutable digraph with 65 vertices, 224 edges>
gap> StackedBookGraph(5, 5);
<immutable bipartite symmetric digraph with bicomponent sizes 13 and 1\
7>
gap> IsSymmetricDigraph(StackedBookGraph(24, 8));
true
gap> IsBipartiteDigraph(StackedBookGraph(24, 8));
true
```

3.5.9 ChainDigraph

▷ ChainDigraph([filt,]n)   (operation)

Returns: A digraph.

If `n` is a positive integer, this function returns a chain with `n` vertices and `n` - 1 edges. Specifically, for each vertex i (with i < n), there is a directed edge with source i and range i + 1.

The DigraphReflexiveTransitiveClosure (3.3.13) of a chain represents a total order.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
─────────────────────── Example ───────────────────────
gap> ChainDigraph(42);
<immutable chain digraph with 42 vertices>
gap> ChainDigraph(IsMutableDigraph, 10);
<mutable digraph with 10 vertices, 9 edges>
```

3.5.10 CirculantGraph

▷ CirculantGraph([filt,]n, par)   (operation)

Returns: A digraph.

If `n` is an integer greater than 1, and `par` is a list of integers that are contained in [1..n] then this operation returns a *circulant graph*. The circulant graph is a graph on `n` vertices, where for each element j of `par`, the ith vertex is adjacent to the $(i+j)$th and $(i-j)$th vertices.

If `par` is [1], then the graph is the `n`th cyclic graph. If `par` is [1,2,...,Int(n/2)], then the graph is the complete graph on `n` vertices. If `n` is at least 4 and `par` is [1,n] then the graph is the `n`th Mobius ladder graph.

A circulant graph is biconnected, cyclic, Hamiltonian, regular, and vertex transitive.

See https://mathworld.wolfram.com/CirculantGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
─────────────────────── Example ───────────────────────
gap> D := CirculantGraph(6, [1, 2, 3]);
<immutable Hamiltonian biconnected vertex-transitive symmetric digraph\
 with 6 vertices, 30 edges>
```

```
gap> IsIsomorphicDigraph(D, CompleteDigraph(6));
true
```

3.5.11   CompleteDigraph

▷ CompleteDigraph([*filt, *]*n*)   (operation)

   Returns: A digraph.

   If *n* is a non-negative integer, this function returns the complete digraph with *n* vertices. See IsCompleteDigraph (6.2.5).

   If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
———————————————————— Example ————————————————————
gap> CompleteDigraph(20);
<immutable complete digraph with 20 vertices>
gap> CompleteDigraph(IsMutableDigraph, 10);
<mutable digraph with 10 vertices, 90 edges>
```

3.5.12   CompleteBipartiteDigraph

▷ CompleteBipartiteDigraph([*filt, *]*m, n*)   (operation)

   Returns: A digraph.

   A complete bipartite digraph is a digraph whose vertices can be partitioned into two non-empty vertex sets, such there exists a unique edge with source i and range j if and only if i and j lie in different vertex sets.

   If *m* and *n* are positive integers, this function returns the complete bipartite digraph with vertex sets of sizes *m* (containing the vertices [1 .. m]) and *n* (containing the vertices [m + 1 .. m + n]).

   If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
———————————————————— Example ————————————————————
gap> CompleteBipartiteDigraph(2, 3);
<immutable complete bipartite digraph with bicomponent sizes 2 and 3>
gap> CompleteBipartiteDigraph(IsMutableDigraph, 3, 2);
<mutable digraph with 5 vertices, 12 edges>
```

3.5.13   CompleteMultipartiteDigraph

▷ CompleteMultipartiteDigraph([*filt, *]*orders*)   (operation)

   Returns: A digraph.

For a list `orders` of n positive integers, this function returns the digraph containing n independent sets of vertices of orders `[l[1] .. l[n]]`. Moreover, each vertex is adjacent to every other not contained in the same independent set.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
─────────── Example ───────────
gap> CompleteMultipartiteDigraph([5, 4, 2]);
<immutable complete multipartite digraph with 11 vertices, 76 edges>
gap> CompleteMultipartiteDigraph(IsMutableDigraph, [5, 4, 2]);
<mutable digraph with 11 vertices, 76 edges>
```

3.5.14 CycleDigraph

▷ CycleDigraph(`[filt, ]n`)   (operation)
▷ DigraphCycle(`[filt, ]n`)   (operation)
    Returns: A digraph.

If n is a positive integer, then these functions return a *cycle digraph* with n vertices and n edges. Specifically, for each vertex i (with i < n), there is a directed edge with source i and range i + 1. In addition, there is an edge with source n and range 1.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
─────────── Example ───────────
gap> CycleDigraph(1);
<immutable digraph with 1 vertex, 1 edge>
gap> CycleDigraph(123);
<immutable cycle digraph with 123 vertices>
gap> CycleDigraph(IsMutableDigraph, 10);
<mutable digraph with 10 vertices, 10 edges>
gap> DigraphCycle(4) = CycleDigraph(4);
true
```

3.5.15 CycleGraph

▷ CycleGraph(`[filt, ]n`)   (operation)
    Returns: A digraph.

If n is an integer greater than 2 then this operation returns the nth *cycle graph*, consisting of the cycle on n vertices. The cycle graph, unlike the cycle digraph, is symmetric. The cycle graph has n vertices and n undirected edges. The cycle graph is simple so the non-simple graphs with a single vertex and single loop and with two vertices and two edges between them are excluded.

See https://mathworld.wolfram.com/CycleGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph

(3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
────────────────────────── Example ──────────────────────────
  gap> D := CycleGraph(7);
  <immutable symmetric digraph with 7 vertices, 14 edges>
```

3.5.16 EmptyDigraph

▷ EmptyDigraph([filt,]n)   (operation)
▷ NullDigraph([filt,]n)   (operation)
    Returns: A digraph.

If `n` is a non-negative integer, this function returns the *empty* or *null* digraph with `n` vertices. An empty digraph is one with no edges.

NullDigraph is a synonym for EmptyDigraph.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
────────────────────────── Example ──────────────────────────
  gap> EmptyDigraph(20);
  <immutable empty digraph with 20 vertices>
  gap> NullDigraph(10);
  <immutable empty digraph with 10 vertices>
  gap> EmptyDigraph(IsMutableDigraph, 10);
  <mutable empty digraph with 10 vertices>
```

3.5.17 GearGraph

▷ GearGraph([filt,]n)   (operation)
    Returns: A digraph.

If `n` is a positive integer at least 3, then this operation returns the *gear graph* with $2n + 1$ vertices and $3n$ undirected edges. The `n`th gear graph is the 2th cycle graph with one additional central vertex, to which every other vertex of the cycle is connected. The gear graph is a symmetric digraph. A gear graph is a Matchstick graph, that is it is simple with a planar graph embedding, and is a unit-distance graph (that is, it can be embedded into the Euclidean plane with vertices being distinct points and edges having length 1).

See https://mathworld.wolfram.com/GearGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
────────────────────────── Example ──────────────────────────
  gap> D := GearGraph(4);
  <immutable symmetric digraph with 9 vertices, 24 edges>
```

```
gap> ChromaticNumber(D);
2
gap> IsVertexTransitive(D);
false
```

3.5.18 HaarGraph

▷ HaarGraph([filt,]n)   (operation)

Returns: A digraph.

If `n` is a positive integer then this operation returns the *Haar graph H(n)*.

The number of vertices in the Haar graph $H(n)$ is equal to twice m, where m is the number of digits required to represent `n` in binary. These vertices are arranged into bicomponents `[1..m]` and `[m+1..2*m]`. Vertices i and j in different bicomponents are adjacent by a symmetric pair of edges if and only if the binary representation of `n` has a 1 in position ($j - i$ modulo m) + 1 from the left.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
                                   Example
gap> HaarGraph(3);
<immutable bipartite vertex-transitive symmetric digraph with bicompon\
ent sizes 2 and 2>
gap> D := HaarGraph(16);
<immutable bipartite vertex-transitive symmetric digraph with bicompon\
ent sizes 5 and 5>
```

3.5.19 HalvedCubeGraph

▷ HalvedCubeGraph([filt,]n)   (operation)

Returns: A digraph.

If `n` is a positive integer at least 1, then this operation returns the `n`th *halved cube graph*, the graph of the `n`- demihypercube. The vertices of the graph are those of the `n`th- hypercube, with two vertices adjacent if and only if they are at distance 1 or 2 from each other. Equivalent constructions are as the second graph power of the `n`-1th hypercube graph, or as with vertices labelled as the binary numbers where two vertices are adjacent if they differ in a single bit, or with vertices labelled with the subset of binary numbers with even Hamming weight, with edges connecting vertices with Hamming distance exactly 2. The Halved Cube graph is distance regular and contains a Hamiltonian cycle.

See https://mathworld.wolfram.com/HalvedCubeGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
                                   Example
gap> D := HalvedCubeGraph(3);
<immutable Hamiltonian connected symmetric digraph with 4 vertices, 12\
```

```
  edges>
gap> IsDistanceRegularDigraph(D);
true
gap> IsHamiltonianDigraph(D);
true
```

3.5.20 HanoiGraph

▷ HanoiGraph([*filt,*]*n*)   (operation)

Returns: A digraph.

If *n* is positive integer then this operation returns the *n*th *Hanoi graph*. The Hanoi graph's vertices represent the possible states of the 'Tower of Hanoi' puzzle on three 'towers', while its edges represent possible moves. The Hanoi graph has `3^n` vertices, and `3 * (3^n - 1) / 2` undirected edges.

The Hanoi graph is Hamiltonian. The graph superficially resembles the Sierpinski triangle. The graph is also a 'penny graph' - a graph whose vertices can be considered to be non-overlapping unit circles on a flat surface, with two vertices adjacent only if the unit circles touch at a single point. Thus the Hanoi graph is planar.

See https://mathworld.wolfram.com/HanoiGraph.html for further details.

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

―――――――――――――――――― Example ――――――――――――――――――
```
gap> D := HanoiGraph(5);
<immutable Hamiltonian connected symmetric digraph with 243 vertices, \
726 edges>
gap> IsPlanarDigraph(D);
true
```

3.5.21 HelmGraph

▷ HelmGraph([*filt,*]*n*)   (operation)

Returns: A digraph.

If *n* is a positive integer at least 3, then this operation returns the *n*th *helm graph*. The helm graph is the *n*-1th wheel graph with, for each external vertex of the 'wheel', adjoining a new vertex incident only to the first vertex. That is, the graph looks similar to a ship's helm.

See https://mathworld.wolfram.com/WheelGraph.html for further details.

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

―――――――――――――――――― Example ――――――――――――――――――
```
gap> D := HelmGraph(4);
<immutable symmetric digraph with 9 vertices, 24 edges>
```

3.5.22 HypercubeGraph

▷ HypercubeGraph(`[filt, ]n`)   (operation)

Returns: A digraph.

If `n` is a non-negative integer, then this operation returns the `n`th *hypercube graph*. The graph has `2^n` vertices and `2^(n-1)n` edges. It is formed from the vertices and edges of the `n`-dimensional hypercube. Alternatively, the graph can be constructed by labelling each vertex with the binary numbers, with two vertices adjacent if they have Hamming distance exactly one. The hypercube graphs are Hamiltonian, distance-transitive and therefore distance-regular, and bipartite.

See https://mathworld.wolfram.com/HypercubeGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

––––––––––––––– Example –––––––––––––––
```
gap> D := HypercubeGraph(5);
<immutable Hamiltonian connected bipartite symmetric digraph with bico\
mponent sizes 16 and 16>
```

3.5.23 JohnsonDigraph

▷ JohnsonDigraph(`[filt, ]n, k`)   (operation)

Returns: A digraph.

If `n` and `k` are non-negative integers, then this operation returns a symmetric digraph which corresponds to the undirected *Johnson graph J(n,k)*.

The *Johnson graph J(n,k)* has vertices given by all the `k`-subsets of the range [1 .. `n`], and two vertices are connected by an edge if and only if their intersection has size $k - 1$.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

––––––––––––––– Example –––––––––––––––
```
gap> gr := JohnsonDigraph(3, 1);
<immutable symmetric digraph with 3 vertices, 6 edges>
gap> OutNeighbours(gr);
[ [ 2, 3 ], [ 1, 3 ], [ 1, 2 ] ]
gap> gr := JohnsonDigraph(4, 2);
<immutable symmetric digraph with 6 vertices, 24 edges>
gap> OutNeighbours(gr);
[ [ 2, 3, 4, 5 ], [ 1, 3, 4, 6 ], [ 1, 2, 5, 6 ], [ 1, 2, 5, 6 ],
  [ 1, 3, 4, 6 ], [ 2, 3, 4, 5 ] ]
gap> JohnsonDigraph(1, 0);
<immutable empty digraph with 1 vertex>
gap> JohnsonDigraph(IsMutableDigraph, 1, 0);
<mutable empty digraph with 1 vertex>
```

3.5.24 KellerGraph

▷ KellerGraph([*filt, *]*n*)   (operation)

    Returns: A digraph.

    If `n` is a nonnegative integer then this operation returns the `n`th or `n`-dimensional *Keller graph*. The graph has vertices given by the `n`-tuples on the set $[0, 1, 2, 3]$. Two vertices are adjacent if their respective tuples are such that they differ in at least two coordinates and in at least one coordinate the difference between the two is 2 mod 4. The Keller graph has `4^n` vertices.

    The Keller graphs were constructed with the intention of finding counterexamples to Keller's conjecture (`https://mathworld.wolfram.com/KellersConjecture.html`), and has been used since for testing maximum clique algorithms.

    If `n` is 1 then the graph is empty, for `n` greater than 1 the chromatic number of the Keller graph is `2^n` and the graph is Hamiltonian.

    See `https://mathworld.wolfram.com/KellerGraph.html` for further details.

    If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is `IsMutableDigraph` (3.1.2), then the digraph being created will be mutable, if `filt` is `IsImmutableDigraph` (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then `IsImmutableDigraph` (3.1.3) is used by default.

```
 ─────────────────────── Example ───────────────────────
 gap> D := KellerGraph(3);
 <immutable Hamiltonian connected symmetric digraph with 64 vertices, 2\
 176 edges>
 gap> ChromaticNumber(D);
 8
```

3.5.25 KingsGraph

▷ KingsGraph([*filt, *]*m, n*)   (operation)

    Returns: A digraph.

    If `m` and `n` are positive integers, then this operation returns the *king's graph* of an `m` by `n` chessboard, as a symmetric digraph.

    The king's graph represents all possible moves of the king chess piece across a chessboard. An `m` by `n` chessboard is a grid of `m` columns ("files") and `n` rows ("ranks") that intersect in squares. Orthogonally adjacent squares are alternately colored light and dark, with the square in the first rank and file being dark.

    The king can move only to any orthogonally or diagonally adjacent square. Thus the `m * n` vertices of the king's graph can be placed onto the `m * n` squares of an `m` by `n` chessboard, such that two vertices are adjacent in the digraph if and only if the corresponding squares are orthogonally or diagonally adjacent on the chessboard.

    The chosen correspondence between vertices and chess squares is given by `DigraphVertexLabels` (5.1.12). In more detail, the vertices of the digraph are labelled by elements of the Cartesian product `[1..m] x [1..n]`, where the first entry indexes the column (file) of the square in the chessboard, and the second entry indexes the row (rank) of the square. (Note that the files are traditionally indexed by the lowercase letters of the alphabet). The vertices are sorted in ascending order, first by row (second component) and then column (first component). See Wikipedia for further information. See also `SquareGridGraph` (3.5.41), `TriangularGridGraph` (3.5.42), and `StrongProduct` (3.3.38).

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
 ──────────────────────── Example ────────────────────────
 gap> KingsGraph(8, 8);
 <immutable connected symmetric digraph with 64 vertices, 420 edges>
 gap> D := KingsGraph(IsMutable, 2, 7);
 <mutable digraph with 14 vertices, 62 edges>
 gap> IsPlanarDigraph(D);
 true
 gap> D := KingsGraph(3, 3);
 <immutable connected symmetric digraph with 9 vertices, 40 edges>
 gap> OutNeighbors(D);
 [ [ 2, 4, 5 ], [ 1, 3, 5, 4, 6 ], [ 2, 6, 5 ], [ 5, 1, 7, 2, 8 ],
   [ 4, 6, 2, 8, 3, 7, 1, 9 ], [ 5, 3, 9, 8, 2 ], [ 8, 4, 5 ],
   [ 7, 9, 5, 6, 4 ], [ 8, 6, 5 ] ]
 gap> IsSubdigraph(QueensGraph(3, 4), KingsGraph(3, 4));
 true
```

3.5.26 KneserGraph

▷ KneserGraph([filt,]n, k)   (operation)

Returns: A digraph.

If `n` and `k` are integers greater than 0, with `k` less than `n`, then this operation returns the (`n`,`k`)-th *Kneser graph*. The Kneser graph's vertices correspond to the `k`- element subsets of a set of `n` elements, with two vertices being adjacent if and only if the subsets are disjoint. The graph has Binomial(`n`, `k`) vertices and Binomial(`n`, `k`) * Binomial(`n` - `k`, `k`) edges. Kneser graphs are regular, edge transitive, and vertex transitive. If `k` is 1, then the graph is the complete graph on `n` vertices. If (`n`, `k`) is (2m-1, `n`-1), then the graph is the mth Odd graph. The Petersen graph is the (5,2)th Kneser graph.

If `n` >= 2`k` then the graph's chromatic number is `n` - 2`k` + 2, and otherwise is 1. The Kneser graph contains a Hamiltonian cycle if `n` >= ((3 + 5 ^ 0.5) / 2)`k` + 1. The graph has clique number equal to the floor of `n` / `k`.

See https://mathworld.wolfram.com/KneserGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
 ──────────────────────── Example ────────────────────────
 gap> D := KneserGraph(7, 3);
 <immutable edge- and vertex-transitive symmetric digraph with 35 verti\
 ces, 140 edges>
 gap> IsRegularDigraph(D);
 true
 gap> ChromaticNumber(D);
 3
```

```
gap> CliqueNumber(D);
2
```

3.5.27 KnightsGraph

▷ KnightsGraph([filt,]m, n)   (operation)

    Returns: A digraph.

If `m` and `n` are positive integers, then this operation returns the *knight's graph* of an `m` by `n` chessboard, as a symmetric digraph.

A knight's graph represents all possible moves of the knight chess piece across a chessboard. An `m` by `n` chessboard is a grid of `m` columns ("files") and `n` rows ("ranks") that intersect in squares. Orthogonally adjacent squares are alternately colored light and dark, with the square in the first rank and file being dark.

The `m * n` vertices of the knight's graph can be placed onto the `m * n` squares of an `m` by `n` chessboard, such that two vertices are adjacent in the digraph if and only if a knight can move between the corresponding squares in a single turn.

The chosen correspondence between vertices and chess squares is given by `DigraphVertexLabels` (5.1.12). In more detail, the vertices of the digraph are labelled by elements of the Cartesian product `[1..m] x [1..n]`, where the first entry indexes the column (file) of the square in the chessboard, and the second entry indexes the row (rank) of the square. (Note that the files are traditionally indexed by the lowercase letters of the alphabet). The vertices are sorted in ascending order, first by row (second component) and then column (first component). See Wikipedia for further information.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is `IsMutableDigraph` (3.1.2), then the digraph being created will be mutable, if `filt` is `IsImmutableDigraph` (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then `IsImmutableDigraph` (3.1.3) is used by default.

―――――――――――――――――――――― Example ――――――――――――――――――――――
```
gap> D := KnightsGraph(8, 8);
<immutable connected symmetric digraph with 64 vertices, 336 edges>
gap> IsConnectedDigraph(D);
true
gap> D := KnightsGraph(3, 3);
<immutable symmetric digraph with 9 vertices, 16 edges>
gap> IsConnectedDigraph(D);
false
gap> KnightsGraph(IsMutable, 3, 9);
<mutable digraph with 27 vertices, 88 edges>
```

3.5.28 LindgrenSousselierGraph

▷ LindgrenSousselierGraph([filt,]n)   (operation)

    Returns: A digraph.

If `n` is an integer greater than 0, then this operation returns the `n`th *Lindgren-Sousselier graph*, an infinite family of hyophamiltonian graphs - a graph that is non-Hamiltonian but removing any vector gives a Hamiltonian graph. The graph has `6n+4` vertices and `15 + 10(n-1)` undirected edges.

The first Lindgren-Sousselier graph is the Petersen graph, and is in fact the smallest hyophamiltonian graph.

See <https://mathworld.wolfram.com/HypohamiltonianGraph.html> for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
──────────────── Example ────────────────
gap> D := LindgrenSousselierGraph(3);
<immutable symmetric digraph with 22 vertices, 70 edges>
gap> IsHamiltonianDigraph(D);
false
gap> IsHamiltonianDigraph(DigraphRemoveVertex(D, 1));
true
gap> IsIsomorphicDigraph(LindgrenSousselierGraph(1), PetersenGraph());
true
```

3.5.29 LollipopGraph

▷ LollipopGraph([filt,]m, n)   (operation)

Returns: A digraph.

If `m` and `n` are positive integers, then this operation returns the *(m,n)-lollipop graph*. As defined at <https://en.wikipedia.org/wiki/Lollipop_graph>, this consists of a complete digraph on the vertices `[1..m]` (the 'head' of the lollipop), and the symmetric closure of a chain digraph on the remaining `n` vertices (the 'stick'), connected by a bridge (the edge `[m, m+1]` and its reverse).

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
──────────────── Example ────────────────
gap> D := LollipopGraph(5, 3);
<immutable connected symmetric digraph with 8 vertices, 26 edges>
gap> CliqueNumber(D);
5
gap> DigraphUndirectedGirth(D);
3
gap> LollipopGraph(IsMutableDigraph, 3, 8);
<mutable digraph with 11 vertices, 22 edges>
```

3.5.30 MobiusLadderGraph

▷ MobiusLadderGraph([filt,]n)   (operation)

Returns: A digraph.

If `n` is a positive integer at least 4, then this operation returns the *Mobius ladder graph* that is obtained by introducing a 'twist' in the `n`th prism graph, similar to the construction of a Mobius strip. The Mobius ladder graph is isomorphic to the circulant graph `Ci(2n, [1, n])`. The Mobius ladders

are cubic, symmetric, Hamiltonian, vertex-transitive, and graceful. They are also non-planar and apex, meaning removing a single vertex produces a planar graph.

See https://mathworld.wolfram.com/MoebiusLadder.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
─────────────────── Example ───────────────────
 gap> D := MobiusLadderGraph(7);
 <immutable symmetric digraph with 14 vertices, 42 edges>
 gap> IsHamiltonianDigraph(D);
 true
 gap> IsVertexTransitive(D);
 true
 gap> IsPlanarDigraph(D);
 false
 gap> D2 := DigraphRemoveVertex(D, 1);
 <immutable digraph with 13 vertices, 36 edges>
 gap> IsPlanarDigraph(D2);
 true
```

3.5.31 MycielskiGraph

▷ MycielskiGraph(`[filt, ]n`)   (operation)

    Returns: A digraph.

If `n` is an integer greater than 1, then this operation returns the `n`th *Mycielski graph*. The Mycielskian of a triangle-free graph is a construction that adds vertices and edges to produce a new graph that is still triangle-free but has a larger chromatic number. The Mycielski graphs are a series of graphs with this construction repeated, starting with the complete graph on two vertices. The graph has $3 * 2^{(n-2)} - 1$ vertices, with the number of edges being $(18 - 27 * 2^n + 14 * 3^n) / 36$.

The Mycielski graph has chromatic number equal to `n`, clique number equal to 2, and is Hamiltonian. The graph is in fact the graph with chromatic number `n` with the least possible vertices.

See https://mathworld.wolfram.com/MycielskiGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
─────────────────── Example ───────────────────
 gap> D := MycielskiGraph(4);
 <immutable Hamiltonian connected symmetric digraph with 11 vertices, 4\
 0 edges>
 gap> ChromaticNumber(D);
 4
 gap> CliqueNumber(D);
 2
```

3.5.32   OddGraph

▷ OddGraph([*filt*,]*n*)   (operation)

Returns: A digraph.

If *n* is an integer greater than 0, then this operation returns the *n*th *odd graph*. The odd graph has vertices labelled with the *n*-1-subsets of a 2*n*-1-set, with two vertices adjacent if and only if their subsets are disjoint. The *n*th odd graph is the (2*n*-1, *n*-1)-th Kneser graph. The graph has Binomial(2*n*-1, *n*-1) vertices and *n* * Binomial(2*n*-1, *n*-1) / 2 edges.

The odd graph is regular and distance transitive (and therefore distance regular). They have chromatic number equal to 3, and all Odd graphs with *n* greater than 3 are Hamiltonian. They are also vertex and edge transitive.

See https://mathworld.wolfram.com/OddGraph.html for further details.

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
_____ Example _____
 gap> D := OddGraph(4);
 <immutable edge- and vertex-transitive symmetric digraph with 35 verti\
 ces, 140 edges>
 gap> IsIsomorphicDigraph(D, KneserGraph(7, 3));
 true
 gap> ChromaticNumber(D);
 3
```

3.5.33   PathGraph

▷ PathGraph([*filt*,]*n*)   (operation)

Returns: A digraph.

If *n* is a non-negative integer then this operation returns the *n*th *path graph*, consisting of the path on *n* vertices. This is the symmetric closure of the ChainDigraph (3.5.9). The path graph has *n* vertices and *n*-1 edges. The path graph is an undirected tree.

See https://mathworld.wolfram.com/PathGraph.html for further details.

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
_____ Example _____
 gap> D := PathGraph(12);
 <immutable undirected tree digraph with 12 vertices>
```

3.5.34   PermutationStarGraph

▷ PermutationStarGraph([*filt*,]*n*, *k*)   (operation)

Returns: A digraph.

If `n` is an integer at greater than 0 and `k` is an integer greater than 1, and `k` less than or equal to `n`, then this operation returns the `(n, k)`-th *permutation star graph*. The vertices of the graph are given by the `k`-length ordered subsets of an `n`-set, with two vertices being adjacent if one is labelled `p1 p2 p3 ... pi ... pk`, and the other is either labelled `pi p2 p3 ... p1 ... pk`, or labelled `x p2 p3 ... pi ... pk` where x is in `[1..n]` and is not equal to `p1`. The graph has `n! / (n - k)!` vertices.

The permutation star graph is regular and vertex transitive. It has diameter `2k - 1` if `k` less than or equal to `Int(n / 2)`, and diameter `Int((n - 1) / 2) + k` if `k >= Int(n / 2) + 1`.

See https://mathworld.wolfram.com/PermutationStarGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
_____ Example _____
  gap> D := PermutationStarGraph(6, 2);
  <immutable vertex-transitive symmetric digraph with 30 vertices, 150 e\
  dges>
  gap> DigraphDiameter(D);
  3
```

3.5.35 PetersenGraph

▷ PetersenGraph([filt]) (operation)

Returns: A digraph.

From https://en.wikipedia.org/wiki/Petersen_graph:

"The Petersen graph is an undirected graph with 10 vertices and 15 edges. It is a small graph that serves as a useful example and counterexample for many problems in graph theory. The Petersen graph is named after Julius Petersen, who in 1898 constructed it to be the smallest bridgeless cubic graph with no three-edge-coloring."

See also GeneralisedPetersenGraph (3.5.36).

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
_____ Example _____
  gap> ChromaticNumber(PetersenGraph());
  3
  gap> PetersenGraph(IsMutableDigraph);
  <mutable digraph with 10 vertices, 30 edges>
```

3.5.36 GeneralisedPetersenGraph

▷ GeneralisedPetersenGraph([filt,]n, k) (operation)

Returns: A digraph.

If `n` is a positive integer and `k` is a non-negative integer less than `n` / 2, then this operation returns the *generalised Petersen graph GPG(n,k)*.

From https://en.wikipedia.org/wiki/Generalized_Petersen_graph:

"The generalized Petersen graphs are a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon. They include the Petersen graph and generalize one of the ways of constructing the Petersen graph. The generalized Petersen graph family was introduced in 1950 by H. S. M. Coxeter and was given its name in 1969 by Mark Watkins."

See also `PetersenGraph` (3.5.35).

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is `IsMutableDigraph` (3.1.2), then the digraph being created will be mutable, if `filt` is `IsImmutableDigraph` (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then `IsImmutableDigraph` (3.1.3) is used by default.

```
───────────────────────── Example ─────────────────────────
gap> GeneralisedPetersenGraph(7, 2);
<immutable symmetric digraph with 14 vertices, 42 edges>
gap> GeneralisedPetersenGraph(40, 1);
<immutable symmetric digraph with 80 vertices, 240 edges>
gap> D := GeneralisedPetersenGraph(5, 2);
<immutable symmetric digraph with 10 vertices, 30 edges>
gap> IsIsomorphicDigraph(D, PetersenGraph());
true
gap> GeneralisedPetersenGraph(IsMutableDigraph, 9, 4);
<mutable digraph with 18 vertices, 54 edges>
```

3.5.37 PrismGraph

▷ `PrismGraph([filt, ]n)` (operation)

Returns: A digraph.

If `n` is a positive integer at least 3, then this operation returns the *prism graph* that is the skeleton of the `n`-prism. It has `2n` vertices and `3n` undirected edges. The prism graph is a symmetric digraph. The `n`th prism graph is isomorphic to the graph Cartesian product of the second path graph and the `n`th cycle graph, isomorphic to the generalised Petersen graph `GP(n,1)`. If `n` is odd then the prism graph is isomorphic to the Circulant graph `Ci(2n, [2,n])`. The prism graph is Hamiltonian.

See https://mathworld.wolfram.com/PrismGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is `IsMutableDigraph` (3.1.2), then the digraph being created will be mutable, if `filt` is `IsImmutableDigraph` (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then `IsImmutableDigraph` (3.1.3) is used by default.

```
───────────────────────── Example ─────────────────────────
gap> D := PrismGraph(4);
<immutable symmetric digraph with 8 vertices, 24 edges>
gap> IsHamiltonianDigraph(D);
true
gap> D := PrismGraph(5);
<immutable symmetric digraph with 10 vertices, 30 edges>
gap> IsIsomorphicDigraph(D, CirculantGraph(10, [2, 5]));
true
```

3.5.38 StackedPrismGraph

▷ StackedPrismGraph([*filt*,]*n*, *k*)   (operation)

    Returns: A digraph.

If `n` is an integer at least 3 and `k` is a positive integer then this operation returns the (`n`,`k`)-th *stacked prism graph*. The graph is `k` concentric `n`-Cycle graphs connected by spokes. The stacked prism is the graph Cartesian product of the `n`th cycle graph and the `k`th path graph. The graph has `n k` vertices and `n`(`2k - 1`) undirected edges. If `k` is 1 then the graph is the `n`th cycle graph, if `k` is 2 then the graph is the prism graph.

See https://mathworld.wolfram.com/StackedPrismGraph.html for further details.

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
                                   Example
  gap> D := StackedPrismGraph(5, 2);
  <immutable symmetric digraph with 10 vertices, 30 edges>
  gap> IsIsomorphicDigraph(D, PrismGraph(5));
  true
```

3.5.39 QueensGraph

▷ QueensGraph([*filt*,]*m*, *n*)   (operation)
▷ QueenGraph([*filt*,]*m*, *n*)   (operation)

    Returns: A digraph.

If `m` and `n` are positive integers, then this operation returns the *queen's graph* of an `m` by `n` chessboard, as a symmetric digraph.

The queen's graph represents all possible moves of the queen chess piece across a chessboard. An `m` by `n` chessboard is a grid of `m` columns ("files") and `n` rows ("ranks") that intersect in squares. Orthogonally adjacent squares are alternately colored light and dark, with the square in the first rank and file being dark.

The `m * n` vertices of the queen's graph can be placed onto the `m * n` squares of an `m` by `n` chessboard, such that two vertices are adjacent in the digraph if and only if the queen can move between the corresponding squares in a single turn. A legal queen's move is defined as one which moves the queen to an (orthogonally or diagonally) adjacent square, or to a square which can be reached through a series of such moves, with all of the small moves being in the same direction.

Note that the QueensGraph is the DigraphEdgeUnion (3.3.30) of the RooksGraph (3.5.40) and the BishopsGraph (3.5.5) of the same dimensions.

The chosen correspondence between vertices and chess squares is given by DigraphVertexLabels (5.1.12). In more detail, the vertices of the digraph are labelled by elements of the Cartesian product [1..`m`] x [1..`n`], where the first entry indexes the column (file) of the square in the chessboard, and the second entry indexes the row (rank) of the square. (Note that the files are traditionally indexed by the lowercase letters of the alphabet). The vertices are sorted in ascending order, first by row (second component) and then column (first component). If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph

being created will be mutable, if `filt` is `IsImmutableDigraph` (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then `IsImmutableDigraph` (3.1.3) is used by default.

```
_____ Example _____
gap> QueensGraph(2, 5);
<immutable connected symmetric digraph with 10 vertices, 66 edges>
gap> D := QueensGraph(4, 3);
<immutable connected symmetric digraph with 12 vertices, 92 edges>
gap> IsRegularDigraph(D);
false
gap> QueensGraph(6, 9) =
>    DigraphEdgeUnion(RooksGraph(6, 9), BishopsGraph(6, 9));
true
```

3.5.40 RooksGraph

▷ RooksGraph(`[filt, ]m, n`) (operation)
▷ RookGraph(`[filt, ]m, n`) (operation)

Returns: A digraph.

If `m` and `n` are positive integers, then this operation returns the *rook's graph* of an `m` by `n` chessboard, as a symmetric digraph.

A rook's graph represents all possible moves of the rook chess piece across a chessboard. An `m` by `n` chessboard is a grid of `m` columns ("files") and `n` rows ("ranks") that intersect in squares. Orthogonally adjacent squares are alternately colored light and dark, with the square in the first rank and file being dark.

The `m * n` vertices of the rook's graph can be placed onto the `m * n` squares of an `m` by `n` chessboard, such that two vertices are adjacent in the digraph if and only if a rook can move between the corresponding squares in a single turn. A legal rook's move is defined as one which moves the rook to an orthogonally adjacent square, or to a square which can be reached through a series of such moves, with all of the small moves being in the same direction.

The chosen correspondence between vertices and chess squares is given by `DigraphVertexLabels` (5.1.12). In more detail, the vertices of the digraph are labelled by elements of the Cartesian product `[1..m] x [1..n]`, where the first entry indexes the column (file) of the square in the chessboard, and the second entry indexes the row (rank) of the square. (Note that the files are traditionally indexed by the lowercase letters of the alphabet). The vertices are sorted in ascending order, first by row (second component) and then column (first component). See Wikipedia for further information. See also `DigraphCartesianProduct` (3.3.32) and `LineDigraph` (3.3.41).

If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is `IsMutableDigraph` (3.1.2), then the digraph being created will be mutable, if `filt` is `IsImmutableDigraph` (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then `IsImmutableDigraph` (3.1.3) is used by default.

```
_____ Example _____
gap> D := RooksGraph(7, 4);
<immutable connected regular symmetric digraph with 28 vertices, 252 e\
dges>
gap> RooksGraph(1, 8);
```

```
  <immutable connected regular symmetric digraph with 8 vertices, 56 edg\
  es>
```

3.5.41 SquareGridGraph

▷ SquareGridGraph([filt,]n, k)   (operation)
▷ GridGraph([filt,]n, k)   (operation)

    Returns: A digraph.

    If `n` and `k` are positive integers, then this operation returns a square grid graph of dimension `n` by `k`.

    A *square grid graph* of dimension `n` by `k` is the DigraphCartesianProduct (3.3.32) of the symmetric closures of the chain digraphs with `n` and `k` vertices; see DigraphSymmetricClosure (3.3.12) and ChainDigraph (3.5.9).

    In particular, the `n * k` vertices can be arranged into an `n` by `k` grid such that two vertices are adjacent in the digraph if and only if they are orthogonally adjacent in the grid. The correspondence between vertices and grid positions is given by DigraphVertexLabels (5.1.12).

    See https://en.wikipedia.org/wiki/Lattice_graph for more information.

    If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
─────────────── Example ───────────────
gap> SquareGridGraph(5, 5);
<immutable connected bipartite symmetric digraph with bicomponent size\
s 13 and 12>
gap> GridGraph(IsMutable, 3, 4);
<mutable digraph with 12 vertices, 34 edges>
```

3.5.42 TriangularGridGraph

▷ TriangularGridGraph([filt,]n, k)   (operation)

    Returns: A digraph.

    If `n` and `k` are positive integers, then this operation returns a triangular grid graph of dimension `n` by `k`.

    A *triangular grid graph* of dimension `n` by `k` is a symmetric digraph constructed from the SquareGridGraph (3.5.41) of the same dimensions, where additionally two vertices are adjacent in the digraph if they are diagonally adjacent in the grid, on a particular one of the diagonals. The correspondence between vertices and grid positions is given by DigraphVertexLabels (5.1.12). More specifically, the particular diagonal is the one such that, the vertices corresponding to the grid positions [2,1] and [1,2] are adjacent (if they exist), but those corresponding to [1,1] and [2,2] are not.

    See https://en.wikipedia.org/wiki/Lattice_graph#Other_kinds for more information.

    If the optional first argument `filt` is present, then this should specify the category or representation the digraph being created will belong to. For example, if `filt` is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if `filt` is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument `filt` is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
                            Example
  gap> TriangularGridGraph(3, 3);
  <immutable connected symmetric digraph with 9 vertices, 32 edges>
  gap> TriangularGridGraph(IsMutable, 3, 3);
  <mutable digraph with 9 vertices, 32 edges>
```

3.5.43 StarGraph

▷ StarGraph([*filt,]k*)   (operation)

Returns: A digraph.

If *k* is a positive integer, then this operation returns the *star graph* with *k* vertices, which is the undirected tree in which vertex 1 is adjacent to all other vertices. If *k* is at least 2, then this is the complete bipartite digraph with bicomponents [1] and [2 .. *k*].

See IsUndirectedTree (6.6.9), IsCompleteBipartiteDigraph (6.2.4), and DigraphBicomponents (5.4.13).

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
                            Example
  gap> StarGraph(IsMutable, 10);
  <mutable digraph with 10 vertices, 18 edges>
  gap> StarGraph(5);
  <immutable complete bipartite digraph with bicomponent sizes 1 and 4>
  gap> IsSymmetricDigraph(StarGraph(3));
  true
  gap> IsUndirectedTree(StarGraph(3));
  true
```

3.5.44 TadpoleGraph

▷ TadpoleGraph([*filt,]m, n*)   (operation)

Returns: A digraph.

The *tadpole graph* is the symmetric closure of the disjoint union of the cycle digraph on [1..*m*] (the 'head' of the tadpole) and the chain digraph on [*m*+1..*m*+*n*] (the 'tail' of the tadpole), along with the additional edges [1, *m*+1] and [1, *m*+1] which connect the 'head' and the 'tail'. For more details on the tadpole graph please refer to https://en.wikipedia.org/wiki/Tadpole_graph.

See DigraphSymmetricClosure (3.3.12), DigraphDisjointUnion (3.3.29), CycleDigraph (3.5.14), and ChainDigraph (3.5.9).

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
                            Example
  gap> TadpoleGraph(10, 15);
  <immutable symmetric digraph with 25 vertices, 50 edges>
```

```
gap> TadpoleGraph(IsMutableDigraph, 5, 6);
<mutable digraph with 11 vertices, 22 edges>
gap> IsSymmetricDigraph(TadpoleGraph(3, 5));
true
```

3.5.45   WalshHadamardGraph

▷ WalshHadamardGraph([*filt*,]*n*)   (operation)
    Returns: A digraph.

If *n* is a positive integer at least 1, then this operation returns the *Hadamard graph* constructed from the *n*th Hadamard matrix (of dimension $2\verb|^|n$) as constructed by Joseph Walsh. A Hadamard matrix is a square matrix with entries either 1 or -1, such that all the rows are mutually orthogonal. The *n*th Walsh Hadamard graph is a graph on $4n$ matrices split into four categories r_i+, r_i-, c_i+, c_i-. If h_ij are the elements of the *n*th Walsh matrix, then if $h_ij = 1$ then (r_i+, c_j+) and (r_i-, c_j-) are edges, if $h_ij = -1$ then (r_i+, c_j-) and (r_i-, c_j+) are edges. Walsh Hadamard graphs are distance transitive and distance regular.

See <https://mathworld.wolfram.com/HadamardGraph.html> for further details.

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

———————————————— Example ————————————————
```
gap> D := WalshHadamardGraph(5);
<immutable symmetric digraph with 64 vertices, 1024 edges>
gap> IsDistanceRegularDigraph(D);
true
```

3.5.46   WebGraph

▷ WebGraph([*filt*,]*n*)   (operation)
    Returns: A digraph.

If *n* is an integer at least 3 then this operation returns the *n*th *web graph*. The web graph is the (*n*,3)-th stacked prism graph with the edges of the outer cycle removed. The graph has $3n$ vertices and $4n$ undirected edges.

See <https://mathworld.wolfram.com/WebGraph.html> for further details.

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

———————————————— Example ————————————————
```
gap> D := WebGraph(5);
<immutable symmetric digraph with 15 vertices, 40 edges>
```

3.5.47 WheelGraph

▷ WheelGraph([*filt, *]*n*) (operation)

    Returns: A digraph.

If *n* is a positive integer at least 4, then this operation returns the *n*th *wheel graph*. The Wheel graph is formed from an *n*-1 cycle graph with a single additional vertex adjacent to all vertices of the cycle. The graph has *n* vertices and 2(*n*-1) edges. Wheel graphs are the skeletons of *n*-1 pyramids, and are self-dual. If *n* is odd, then the chromatic number of the wheel graph is 3 and the Wheel graph is perfect, and it is 4 otherwise. The wheel graph is also Hamiltonian and planar.

See https://mathworld.wolfram.com/WheelGraph.html for further details.

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
———————————————————————— Example ————————————————————————
gap> D := WheelGraph(8);
<immutable Hamiltonian connected symmetric digraph with 8 vertices, 28\
 edges>
gap> ChromaticNumber(D);
4
gap> IsHamiltonianDigraph(D);
true
```

3.5.48 WindmillGraph

▷ WindmillGraph([*filt, *]*n, m*) (operation)

    Returns: A digraph.

If *n* and *m* are integers greater than 1 then this operation returns the (*n*, *m*)-th *windmill graph*. The windmill graph is formed from *m* copies of the complete graph on *n* vertices with one shared vertex. The graph has *m*(*n* - 1) + 1 vertices and *m* * *n* * (*n* - 1) / 2 undirected edges. The windmill graph has chromatic number *n* and diameter 2.

See https://mathworld.wolfram.com/WindmillGraph.html for further details.

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
———————————————————————— Example ————————————————————————
gap> D := WindmillGraph(4, 3);
<immutable symmetric digraph with 10 vertices, 36 edges>
gap> ChromaticNumber(D);
4
```

Chapter 4

Operators

4.1 Operators for digraphs

digraph1 = *digraph2*

> returns true if *digraph1* and *digraph2* have the same vertices, and DigraphEdges(*digraph1*) = DigraphEdges(*digraph2*), up to some re-ordering of the edge lists.

> Note that this operator does not compare the vertex labels of *digraph1* and *digraph2*.

digraph1 < *digraph2*

> This operator returns true if one of the following holds:

> - The number n_1 of vertices in *digraph1* is less than the number n_2 of vertices in *digraph2*;

> - $n_1 = n_2$, and the number m_1 of edges in *digraph1* is less than the number m_2 of edges in *digraph2*;

> - $n_1 = n_2$, $m_1 = m_2$, and DigraphEdges(*digraph1*) is less than DigraphEdges(*digraph2*) after having both of these sets have been sorted with respect to the lexicographical order.

4.1.1 IsSubdigraph

▷ IsSubdigraph(*super, sub*) (operation)

Returns: true or false.

If *super* and *sub* are digraphs, then this operation returns true if *sub* is a subdigraph of *super*, and false if it is not.

A digraph *sub* is a *subdigraph* of a digraph *super* if *sub* and *super* share the same number of vertices, and the collection of edges of *super* (including repeats) contains the collection of edges of *sub* (including repeats).

In other words, *sub* is a subdigraph of *super* if and only if DigraphNrVertices(*sub*) = DigraphNrVertices(*super*), and for each pair of vertices i and j, there are at least as many edges of the form [i, j] in *super* as there are in *sub*.

```
─────── Example ───────
gap> g := Digraph([[2, 3], [1], [2, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> h := Digraph([[2, 3], [], [2]]);
```

```
<immutable digraph with 3 vertices, 3 edges>
gap> IsSubdigraph(g, h);
true
gap> IsSubdigraph(h, g);
false
gap> IsSubdigraph(CompleteDigraph(4), CycleDigraph(4));
true
gap> IsSubdigraph(CycleDigraph(4), ChainDigraph(4));
true
gap> g := Digraph([[2, 2], [1]]);
<immutable multidigraph with 2 vertices, 3 edges>
gap> h := Digraph([[2], [1]]);
<immutable digraph with 2 vertices, 2 edges>
gap> IsSubdigraph(g, h);
true
gap> IsSubdigraph(h, g);
false
```

4.1.2 IsUndirectedSpanningTree

▷ IsUndirectedSpanningTree(*super, sub*)   (operation)
▷ IsUndirectedSpanningForest(*super, sub*)   (operation)

Returns: true or false.

The operation IsUndirectedSpanningTree returns true if the digraph *sub* is an undirected spanning tree of the digraph *super*, and the operation IsUndirectedSpanningForest returns true if the digraph *sub* is an undirected spanning forest of the digraph *super*.

An *undirected spanning tree* of a digraph *super* is a subdigraph of *super* that is an undirected tree (see IsSubdigraph (4.1.1) and IsUndirectedTree (6.6.9)). Note that a digraph whose MaximalSymmetricSubdigraph (3.3.5) is not connected has no undirected spanning trees (see IsConnectedDigraph (6.6.3)).

An *undirected spanning forest* of a digraph *super* is a subdigraph of *super* that is an undirected forest (see IsSubdigraph (4.1.1) and IsUndirectedForest (6.6.9)), and is not contained in any larger such subdigraph of *super*. Equivalently, an undirected spanning forest is a subdigraph of *super* whose connected components coincide with those of the MaximalSymmetricSubdigraph (3.3.5) of *super* (see DigraphConnectedComponents (5.4.9)).

Note that an undirected spanning tree is an undirected spanning forest that is connected.

```
————————————————— Example —————————————————
gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>
gap> tree := Digraph([[3], [4], [1, 4], [2, 3]]);
<immutable digraph with 4 vertices, 6 edges>
gap> IsSubdigraph(D, tree) and IsUndirectedTree(tree);
true
gap> IsUndirectedSpanningTree(D, tree);
true
gap> forest := EmptyDigraph(4);
<immutable empty digraph with 4 vertices>
gap> IsSubdigraph(D, forest) and IsUndirectedForest(forest);
true
gap> IsUndirectedSpanningForest(D, forest);
```

```
false
gap> IsSubdigraph(tree, forest);
true
gap> D := DigraphDisjointUnion(CycleDigraph(2), CycleDigraph(2));
<immutable digraph with 4 vertices, 4 edges>
gap> IsUndirectedTree(D);
false
gap> IsUndirectedForest(D) and IsUndirectedSpanningForest(D, D);
true
```

Chapter 5

Attributes and operations

5.1 Vertices and edges

5.1.1 DigraphVertices

▷ DigraphVertices(*digraph*) (attribute)

Returns: A list of positive integers.

Returns the vertices of the digraph `digraph`.

Note that the vertices of a digraph are always the range of positive integers from 1 to the number of vertices of the graph, DigraphNrVertices (5.1.2). Arbitrary *labels* can be assigned to the vertices of a digraph; see DigraphVertexLabels (5.1.12) for more information about this.

```
──────────────────── Example ────────────────────
gap> gr := Digraph(["a", "b", "c"],
>                   ["a", "b", "b"],
>                   ["b", "c", "a"]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphVertices(gr);
[ 1 .. 3 ]
gap> gr := Digraph([1, 2, 3, 4, 5, 7],
>                  [1, 2, 2, 4, 4],
>                  [2, 7, 5, 3, 7]);
<immutable digraph with 6 vertices, 5 edges>
gap> DigraphVertices(gr);
[ 1 .. 6 ]
gap> DigraphVertices(RandomDigraph(100));
[ 1 .. 100 ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphVertices(D);
[ 1 .. 3 ]
```

5.1.2 DigraphNrVertices

▷ DigraphNrVertices(*digraph*) (attribute)

Returns: An non-negative integer.

Returns the number of vertices of the digraph `digraph`.

```
                              Example
  gap> gr := Digraph(["a", "b", "c"],
  >                  ["a", "b", "b"],
  >                  ["b", "c", "a"]);
  <immutable digraph with 3 vertices, 3 edges>
  gap> DigraphNrVertices(gr);
  3
  gap> gr := Digraph([1, 2, 3, 4, 5, 7],
  >                  [1, 2, 2, 4, 4],
  >                  [2, 7, 5, 3, 7]);
  <immutable digraph with 6 vertices, 5 edges>
  gap> DigraphNrVertices(gr);
  6
  gap> DigraphNrVertices(RandomDigraph(100));
  100
  gap> D := CycleDigraph(IsMutableDigraph, 3);
  <mutable digraph with 3 vertices, 3 edges>
  gap> DigraphNrVertices(D);
  3
```

5.1.3 DigraphEdges

▷ DigraphEdges(*digraph*)   (attribute)

Returns: A list of lists of two positive integers.

Returns a list of edges of the digraph *digraph*, where each edge is a pair of elements of DigraphVertices (5.1.1) of the form [source,range].

The entries of DigraphEdges(*digraph*) are in one-to-one correspondence with the edges of *digraph*. Hence DigraphEdges(*digraph*) is duplicate-free if and only if *digraph* contains no multiple edges.

The entries of DigraphEdges are guaranteed to be sorted by their first component (i.e. by the source of each edge), but they are not necessarily then sorted by the second component.

```
                              Example
  gap> gr := DigraphFromDiSparse6String(".DaXbOe?EAM@G~");
  <immutable multidigraph with 5 vertices, 16 edges>
  gap> edges := ShallowCopy(DigraphEdges(gr));; Sort(edges);
  gap> edges;
  [ [ 1, 1 ], [ 1, 3 ], [ 1, 3 ], [ 1, 4 ], [ 1, 5 ], [ 2, 1 ],
    [ 2, 2 ], [ 2, 3 ], [ 2, 5 ], [ 3, 2 ], [ 3, 4 ], [ 3, 5 ],
    [ 4, 2 ], [ 4, 4 ], [ 4, 5 ], [ 5, 1 ] ]
  gap> D := CycleDigraph(IsMutableDigraph, 3);
  <mutable digraph with 3 vertices, 3 edges>
  gap> DigraphEdges(D);
  [ [ 1, 2 ], [ 2, 3 ], [ 3, 1 ] ]
```

5.1.4 DigraphNrEdges

▷ DigraphNrEdges(*digraph*)   (attribute)

Returns: An integer.

Returns the number of edges of the digraph *digraph*.

```
─────────────── Example ───────────────
gap> gr := Digraph([
> [1, 3, 4, 5], [1, 2, 3, 5], [2, 4, 5], [2, 4, 5], [1]]);;
gap> DigraphNrEdges(gr);
15
gap> gr := Digraph(["a", "b", "c"],
>                  ["a", "b", "b"],
>                  ["b", "a", "a"]);
<immutable multidigraph with 3 vertices, 3 edges>
gap> DigraphNrEdges(gr);
3
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphNrEdges(D);
3
```

5.1.5 DigraphNrAdjacencies

▷ DigraphNrAdjacencies(*digraph*)   (attribute)

    Returns: An integer.

    Returns the number of sets $\{u, v\}$ of vertices of the digraph *digraph*, such that either (u, v) or (v, u) is an edge. The following equality holds for any digraph D with no multiple edges: `DigraphNrAdjacencies(D) * 2 - DigraphNrLoops(D) =` `DigraphNrEdges(DigraphSymmetricClosure(D))`.

```
─────────────── Example ───────────────
gap> gr := Digraph([
> [1, 3, 4, 5], [1, 2, 3, 5], [2, 4, 5], [2, 4, 5], [1]]);;
gap> DigraphNrAdjacencies(gr);
13
gap> DigraphNrAdjacencies(gr) * 2 - DigraphNrLoops(gr) =
> DigraphNrEdges(DigraphSymmetricClosure(gr));
true
```

5.1.6 DigraphNrAdjacenciesWithoutLoops

▷ DigraphNrAdjacenciesWithoutLoops(*digraph*)   (attribute)

    Returns: An integer.

    Returns the number of sets $\{u, v\}$ of vertices of the digraph *digraph*, such that $u \neq v$ and either (u, v) or (v, u) is an edge. The following equality holds for any digraph D with no multiple edges: `DigraphNrAdjacenciesWithoutLoops(D) * 2 + DigraphNrLoops(D) =` `DigraphNrEdges(DigraphSymmetricClosure(D))`.

```
─────────────── Example ───────────────
gap> gr := Digraph([
> [1, 3, 4, 5], [1, 2, 3, 5], [2, 4, 5], [2, 4, 5], [1]]);;
gap> DigraphNrAdjacenciesWithoutLoops(gr);
10
gap> DigraphNrAdjacenciesWithoutLoops(gr) * 2 + DigraphNrLoops(gr) =
> DigraphNrEdges(DigraphSymmetricClosure(gr));
true
```

5.1.7 DigraphNrLoops

▷ DigraphNrLoops(*digraph*)   (attribute)

Returns: An integer.

This function returns the number of loops of the digraph *digraph*. See DigraphHasLoops (6.2.1).

```
——————————— Example ———————————
gap> D := Digraph([[2, 3], [1, 4], [3, 3, 5], [], [2, 5]]);
<immutable multidigraph with 5 vertices, 9 edges>
gap> DigraphNrLoops(D);
3
gap> D := EmptyDigraph(5);
<immutable empty digraph with 5 vertices>
gap> DigraphNrLoops(D);
0
gap> D := CompleteDigraph(5);
<immutable complete digraph with 5 vertices>
gap> DigraphNrLoops(D);
0
```

5.1.8 DigraphSinks

▷ DigraphSinks(*digraph*)   (attribute)

Returns: A list of vertices.

This function returns a list of the sinks of the digraph *digraph*. A sink of a digraph is a vertex with out-degree zero. See OutDegreeOfVertex (5.2.10).

```
——————————— Example ———————————
gap> gr := Digraph([[3, 5, 2, 2], [3], [], [5, 2, 5, 3], []]);
<immutable multidigraph with 5 vertices, 9 edges>
gap> DigraphSinks(gr);
[ 3, 5 ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphSinks(D);
[  ]
```

5.1.9 DigraphSources

▷ DigraphSources(*digraph*)   (attribute)

Returns: A list of vertices.

This function returns an immutable list of the sources of the digraph *digraph*. A source of a digraph is a vertex with in-degree zero. See InDegreeOfVertex (5.2.12).

```
——————————— Example ———————————
gap> gr := Digraph([[3, 5, 2, 2], [3], [], [5, 2, 5, 3], []]);
<immutable multidigraph with 5 vertices, 9 edges>
gap> DigraphSources(gr);
[ 1, 4 ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphSources(D);
[  ]
```

5.1.10 DigraphTopologicalSort

▷ DigraphTopologicalSort(*digraph*)   (attribute)

    Returns: A list of positive integers, or `fail`.

    If *digraph* is a digraph whose only directed cycles are loops, then DigraphTopologicalSort returns the vertices of *digraph* ordered so that every edge's source appears no earlier in the list than its range. If the digraph *digraph* contains directed cycles of length greater than 1, then this operation returns `fail`.

    See Section 1.1.1 for the definition of a directed cycle, and the definition of a loop.

    The method used for this attribute has complexity $O(m+n)$ where m is the number of edges (counting multiple edges as one) and n is the number of vertices in the digraph.

```
 —————————————————— Example ——————————————————
 gap> D := Digraph([
 > [2, 3], [], [4, 6], [5], [], [7, 8, 9], [], [], []]);
 <immutable digraph with 9 vertices, 8 edges>
 gap> DigraphTopologicalSort(D);
 [ 2, 5, 4, 7, 8, 9, 6, 3, 1 ]
 gap> D := Digraph(IsMutableDigraph, [[2, 3], [3], [4], []]);
 <mutable digraph with 4 vertices, 4 edges>
 gap> DigraphTopologicalSort(D);
 [ 4, 3, 2, 1 ]
```

5.1.11 DigraphVertexLabel

▷ DigraphVertexLabel(*digraph, i*)   (operation)
▷ SetDigraphVertexLabel(*digraph, i, obj*)   (operation)

    If *digraph* is a digraph, then the first operation returns the label of the vertex *i*. The second operation can be used to set the label of the vertex *i* in *digraph* to the arbitrary **GAP** object *obj*.

    The label of a vertex can be changed an arbitrary number of times. If no label has been set for the vertex *i*, then the default value is *i*.

    If *digraph* is a digraph created from a record with a component DigraphVertices, then the labels of the vertices are set to the value of this component.

    Induced subdigraphs, and some other operations which create new digraphs from old ones, inherit their labels from their parents.

```
 —————————————————— Example ——————————————————
 gap> D := DigraphFromDigraph6String("&DHUEe_");
 <immutable digraph with 5 vertices, 11 edges>
 gap> DigraphVertexLabel(D, 3);
 3
 gap> D := Digraph(["a", "b", "c"], [], []);
 <immutable empty digraph with 3 vertices>
 gap> DigraphVertexLabel(D, 2);
 "b"
 gap> SetDigraphVertexLabel(D, 2, "d");
 gap> DigraphVertexLabel(D, 2);
 "d"
 gap> D := InducedSubdigraph(D, [1, 2]);
 <immutable empty digraph with 2 vertices>
 gap> DigraphVertexLabel(D, 2);
```

```
"d"
gap> D := Digraph(IsMutableDigraph, ["e", "f", "g"], [], []);
<mutable empty digraph with 3 vertices>
gap> DigraphVertexLabel(D, 1);
"e"
gap> SetDigraphVertexLabel(D, 1, "h");
gap> DigraphVertexLabel(D, 1);
"h"
gap> InducedSubdigraph(D, [1, 2]);
<mutable empty digraph with 2 vertices>
gap> DigraphVertexLabel(D, 1);
"h"
```

5.1.12 DigraphVertexLabels

▷ DigraphVertexLabels(*digraph*)   (operation)
▷ SetDigraphVertexLabels(*digraph, list*)   (operation)

If *digraph* is a digraph, then DigraphVertexLabels returns a copy of the labels of the vertices in *digraph*. SetDigraphVertexLabels can be used to set the labels of the vertices in *digraph* to the list of arbitrary GAP objects *list*, which must be of the same length as the number of vertices of *digraph*.

If the list *list* is immutable, then the vertex labels are set to a mutable copy of *list*. Otherwise, the labels are set to exactly *list*.

The label of a vertex can be changed an arbitrary number of times. If no label has been set for the vertex *i*, then the default value is *i*.

If *digraph* is a digraph created from a record with a component DigraphVertices, then the labels of the vertices are set to the value of this component. As in the above, if the component is immutable then the digraph's vertex labels are set to a mutable copy of DigraphVertices. Otherwise, they are set to exactly DigraphVertices.

Induced subdigraphs, and other operations which create new digraphs from old ones, inherit their labels from their parents.

```
 _____ Example _____
gap> D := DigraphFromDigraph6String("&DHUEe_");
<immutable digraph with 5 vertices, 11 edges>
gap> DigraphVertexLabels(D);
[ 1 .. 5 ]
gap> D := Digraph(["a", "b", "c"], [], []);
<immutable empty digraph with 3 vertices>
gap> DigraphVertexLabels(D);
[ "a", "b", "c" ]
gap> SetDigraphVertexLabel(D, 2, "d");
gap> DigraphVertexLabels(D);
[ "a", "d", "c" ]
gap> D := InducedSubdigraph(D, [1, 3]);
<immutable empty digraph with 2 vertices>
gap> DigraphVertexLabels(D);
[ "a", "c" ]
gap> D := Digraph(IsMutableDigraph, ["e", "f", "g"], [], []);
<mutable empty digraph with 3 vertices>
```

```
gap> SetDigraphVertexLabels(D, ["h", "i", "j"]);
gap> DigraphVertexLabels(D);
[ "h", "i", "j" ]
gap> InducedSubdigraph(D, [1, 3]);
<mutable empty digraph with 2 vertices>
gap> DigraphVertexLabels(D);
[ "h", "j" ]
```

5.1.13 DigraphEdgeLabel

▷ DigraphEdgeLabel(*digraph, i, j*)   (operation)
▷ SetDigraphEdgeLabel(*digraph, i, j, obj*)   (operation)

If `digraph` is a digraph without multiple edges, then the first operation returns the label of the edge from vertex `i` to vertex `j`. The second operation can be used to set the label of the edge between vertex `i` and vertex `j` to the arbitrary GAP object `obj`.

The label of an edge can be changed an arbitrary number of times. If no label has been set for the edge, then the default value is `1`.

Induced subdigraphs, and some other operations which create new digraphs from old ones, inherit their edge labels from their parents. See also DigraphEdgeLabels (5.1.14).

```
                              ─ Example ─
  gap> D := DigraphFromDigraph6String("&DHUEe_");
  <immutable digraph with 5 vertices, 11 edges>
  gap> DigraphEdgeLabel(D, 3, 1);
  1
  gap> SetDigraphEdgeLabel(D, 2, 5, [42]);
  gap> DigraphEdgeLabel(D, 2, 5);
  [ 42 ]
  gap> D := InducedSubdigraph(D, [2, 5]);
  <immutable digraph with 2 vertices, 3 edges>
  gap> DigraphEdgeLabel(D, 1, 2);
  [ 42 ]
  gap> D := ChainDigraph(IsMutableDigraph, 5);
  <mutable digraph with 5 vertices, 4 edges>
  gap> DigraphEdgeLabel(D, 2, 3);
  1
  gap> SetDigraphEdgeLabel(D, 4, 5, [1729]);
  gap> DigraphEdgeLabel(D, 4, 5);
  [ 1729 ]
  gap> InducedSubdigraph(D, [4, 5]);
  <mutable digraph with 2 vertices, 1 edge>
  gap> DigraphEdgeLabel(D, 1, 2);
  [ 1729 ]
```

5.1.14 DigraphEdgeLabels

▷ DigraphEdgeLabels(*digraph*)   (operation)
▷ SetDigraphEdgeLabels(*digraph, labels*)   (operation)
▷ SetDigraphEdgeLabels(*digraph, func*)   (operation)

If `digraph` is a digraph without multiple edges, then `DigraphEdgeLabels` returns a copy of the labels of the edges in `digraph` as a list of lists `labels` such that `labels[i][j]` is the label on the edge from vertex `i` to vertex `OutNeighbours(digraph)[i][j]`. `SetDigraphEdgeLabels` can be used to set the labels of the edges in `digraph` without multiple edges to the list `labels` of lists of arbitrary GAP objects such that `list[i][j]` is the label on the edge from vertex `i` to the vertex `OutNeighbours(digraph>[i][j]`. Alternatively `SetDigraphEdgeLabels` can be called with binary function `func` that as its second argument that when passed two vertices `i` and `j` returns the label for the edge between vertex `i` and vertex `j`.

The label of an edge can be changed an arbitrary number of times. If no label has been set for an edge, then the default value is 1.

Induced subdigraphs, and some other operations which create new digraphs from old ones, inherit their labels from their parents.

```
_____ Example _____
gap> D := DigraphFromDigraph6String("&DHUEe_");
<immutable digraph with 5 vertices, 11 edges>
gap> DigraphEdgeLabels(D);
[ [ 1 ], [ 1, 1, 1 ], [ 1 ], [ 1, 1, 1 ], [ 1, 1, 1 ] ]
gap> SetDigraphEdgeLabel(D, 2, 1, "d");
gap> DigraphEdgeLabels(D);
[ [ 1 ], [ "d", 1, 1 ], [ 1 ], [ 1, 1, 1 ], [ 1, 1, 1 ] ]
gap> D := InducedSubdigraph(D, [1, 2, 3]);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphEdgeLabels(D);
[ [ 1 ], [ "d", 1 ], [ 1 ] ]
gap> OutNeighbours(D);
[ [ 3 ], [ 1, 3 ], [ 1 ] ]
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 2, 3);
<mutable digraph with 5 vertices, 12 edges>
gap> DigraphEdgeLabels(D);
[ [ 1, 1, 1 ], [ 1, 1, 1 ], [ 1, 1 ], [ 1, 1 ], [ 1, 1 ] ]
gap> SetDigraphEdgeLabel(D, 2, 4, "a");
gap> DigraphEdgeLabels(D);
[ [ 1, 1, 1 ], [ 1, "a", 1 ], [ 1, 1 ], [ 1, 1 ], [ 1, 1 ] ]
gap> InducedSubdigraph(D, [1, 2, 3, 4]);
<mutable digraph with 4 vertices, 8 edges>
gap> DigraphEdgeLabels(D);
[ [ 1, 1 ], [ 1, "a" ], [ 1, 1 ], [ 1, 1 ] ]
gap> OutNeighbors(D);
[ [ 3, 4 ], [ 3, 4 ], [ 1, 2 ], [ 1, 2 ] ]
```

5.1.15 DigraphInEdges

▷ `DigraphInEdges(digraph, vertex)`　　　　　　　　　　　　　　　　　　　　　(operation)

Returns: A list of edges.

`DigraphInEdges` returns the list of all edges of `digraph` which have `vertex` as their range.

```
_____ Example _____
gap> D := Digraph([[2, 2], [3, 3], [4, 4], [1, 1]]);
<immutable multidigraph with 4 vertices, 8 edges>
gap> DigraphInEdges(D, 2);
[ [ 1, 2 ], [ 1, 2 ] ]
```

5.1.16 DigraphOutEdges

▷ DigraphOutEdges(*digraph, vertex*) (operation)

   Returns: A list of edges.

   DigraphOutEdges returns the list of all edges of *digraph* which have *vertex* as their source.

```
 ─────────────────────── Example ───────────────────────
 gap> D := Digraph([[2, 2], [3, 3], [4, 4], [1, 1]]);
 <immutable multidigraph with 4 vertices, 8 edges>
 gap> DigraphOutEdges(D, 2);
 [ [ 2, 3 ], [ 2, 3 ] ]
```

5.1.17 IsDigraphEdge (for digraph and list)

▷ IsDigraphEdge(*digraph, list*) (operation)
▷ IsDigraphEdge(*digraph, u, v*) (operation)

   Returns: true or false.

   In the first form, this function returns true if and only if the list *list* specifies an edge in the digraph *digraph*. Specifically, this operation returns true if *list* is a pair of positive integers where *list*[1] is the source and *list*[2] is the range of an edge in *digraph*, and false otherwise.

   The second form simply returns true if [*u*, *v*] is an edge in *digraph*, and false otherwise.

```
 ─────────────────────── Example ───────────────────────
 gap> D := Digraph([[2, 2], [6], [], [3], [], [1]]);
 <immutable multidigraph with 6 vertices, 5 edges>
 gap> IsDigraphEdge(D, [1, 1]);
 false
 gap> IsDigraphEdge(D, [1, 2]);
 true
 gap> IsDigraphEdge(D, [1, 8]);
 false
```

5.1.18 IsMatching

▷ IsMatching(*digraph, list*) (operation)
▷ IsMaximalMatching(*digraph, list*) (operation)
▷ IsMaximumMatching(*digraph, list*) (operation)
▷ IsPerfectMatching(*digraph, list*) (operation)

   Returns: true or false.

   If *digraph* is a digraph and *list* is a list of pairs of vertices of *digraph*, then IsMatching returns true if *list* is a matching of *digraph*. The operation IsMaximalMatching returns true if *list* is a maximal matching, IsMaximumMatching returns true if *list* is a maximum matching and IsPerfectMatching returns true if *list* is a perfect, matching of *digraph*, respectively. Otherwise, each of these operations return false.

   A *matching* M of a digraph *digraph* is a subset of the edges of *digraph*, i.e. DigraphEdges(*digraph*), such that no pair of distinct edges in M are incident to the same vertex of *digraph*. Note that this definition allows a matching to contain loops. See DigraphHasLoops (6.2.1). The matching M is *maximal* if it is contained in no larger matching of the digraph, is *maximum* if it has the greatest cardinality among all matchings and is *perfect* if every vertex of the digraph is incident to an edge in the matching. Every maximum or perfect matching is maximal. Note, however, that not every perfect matching of digraphs with loops is maximum.

```
 _____ Example _____
  gap> D := Digraph([[1, 2], [1, 2], [2, 3, 4], [3, 5], [1]]);
  <immutable digraph with 5 vertices, 10 edges>
  gap> IsMatching(D, [[2, 1], [3, 2]]);
  false
  gap> edges := [[3, 2]];;
  gap> IsMatching(D, edges);
  true
  gap> IsMaximalMatching(D, edges);
  false
  gap> edges := [[2, 1], [3, 4]];;
  gap> IsMaximalMatching(D, edges);
  true
  gap> IsPerfectMatching(D, edges);
  false
  gap> edges := [[1, 2], [3, 3], [4, 5]];;
  gap> IsPerfectMatching(D, edges);
  true
  gap> IsMaximumMatching(D, edges);
  false
  gap> edges := [[1, 1], [2, 2], [3, 3], [4, 5]];;
  gap> IsMaximumMatching(D, edges);
  true
```

5.1.19 DigraphMaximalMatching

▷ DigraphMaximalMatching(`digraph`)   (attribute)

   Returns: A list of pairs of vertices.

   This function returns a maximal matching of the digraph `digraph`.

   For the definition of a maximal matching, see IsMaximalMatching (5.1.18).

```
 _____ Example _____
  gap> D := DigraphFromDiSparse6String(".IeAoXCJU@|SHAe?d");
  <immutable digraph with 10 vertices, 13 edges>
  gap> M := DigraphMaximalMatching(D);; IsMaximalMatching(D, M);
  true
  gap> D := RandomDigraph(100);;
  gap> IsMaximalMatching(D, DigraphMaximalMatching(D));
  true
  gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 9, 2);
  <mutable digraph with 18 vertices, 54 edges>
  gap> IsMaximalMatching(D, DigraphMaximalMatching(D));
  true
```

5.1.20 DigraphMaximumMatching

▷ DigraphMaximumMatching(`digraph`)   (attribute)

   Returns: A list of pairs of vertices.

   This function returns a maximum matching of the digraph `digraph`.

   For the definition of a maximum matching, see IsMaximumMatching (5.1.18). If `digraph` is
bipartite (see IsBipartiteDigraph (6.2.3)), then the algorithm used has complexity O(m*sqrt(n)).

Otherwise for general graphs the complexity is `O(m*n*log(n))`. Here `n` is the number of vertices and `m` is the number of edges.

```
_____ Example _____
gap> D := DigraphFromDigraph6String("&I@EA_A?AdDp[_c??OO");
<immutable digraph with 10 vertices, 23 edges>
gap> M := DigraphMaximumMatching(D);; IsMaximalMatching(D, M);
true
gap> Length(M);
5
gap> D := Digraph([[5, 6, 7, 8], [6, 7, 8], [7, 8], [8],
>                   [], [], [], []]);;
gap> M := DigraphMaximumMatching(D);
[ [ 1, 5 ], [ 2, 6 ], [ 3, 7 ], [ 4, 8 ] ]
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 9, 2);
<mutable digraph with 18 vertices, 54 edges>
gap> M := DigraphMaximumMatching(D);;
gap> IsMaximalMatching(D, M);
true
gap> Length(M);
9
```

5.2 Neighbours and degree

5.2.1 AdjacencyMatrix

▷ AdjacencyMatrix(*digraph*)   (attribute)

▷ AdjacencyMatrixMutableCopy(*digraph*)   (operation)

   Returns: A square matrix of non-negative integers.

   This function returns the adjacency matrix mat of the digraph `digraph`. The value of the matrix entry `mat[i][j]` is the number of edges in `digraph` with source i and range j. If `digraph` has no vertices, then the empty list is returned.

   The function `AdjacencyMatrix` returns an immutable list of lists, whereas the function `AdjacencyMatrixMutableCopy` returns a copy of `AdjacencyMatrix` that is a mutable list of mutable lists.

```
_____ Example _____
gap> gr := Digraph([
> [2, 2, 2], [1, 3, 6, 8, 9, 10], [4, 6, 8],
> [1, 2, 3, 9], [3, 3], [3, 5, 6, 10], [1, 2, 7],
> [1, 2, 3, 10, 5, 6, 10], [1, 3, 4, 5, 8, 10],
> [2, 3, 4, 6, 7, 10]]);
<immutable multidigraph with 10 vertices, 44 edges>
gap> mat := AdjacencyMatrix(gr);;
gap> Display(mat);
[ [  0,  3,  0,  0,  0,  0,  0,  0,  0,  0 ],
  [  1,  0,  1,  0,  0,  1,  0,  1,  1,  1 ],
  [  0,  0,  0,  1,  0,  1,  0,  1,  0,  0 ],
  [  1,  1,  1,  0,  0,  0,  0,  0,  1,  0 ],
  [  0,  0,  2,  0,  0,  0,  0,  0,  0,  0 ],
  [  0,  0,  1,  0,  1,  1,  0,  0,  0,  1 ],
  [  1,  1,  0,  0,  0,  0,  1,  0,  0,  0 ],
```

```
  [ 1,  1,  1,  0,  1,  1,  0,  0,  0,  2 ],
  [ 1,  0,  1,  1,  1,  0,  0,  1,  0,  1 ],
  [ 0,  1,  1,  1,  0,  1,  1,  0,  0,  1 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> Display(AdjacencyMatrix(D));
[ [ 0,  1,  0 ],
  [ 0,  0,  1 ],
  [ 1,  0,  0 ] ]
```

5.2.2 CharacteristicPolynomial

▷ CharacteristicPolynomial(*digraph*)   (attribute)

Returns: A polynomial with integer coefficients.

This function returns the characteristic polynomial of the digraph *digraph*. That is it returns the characteristic polynomial of the adjacency matrix of the digraph *digraph*

———————————— Example ————————————
```
gap> D := Digraph([
> [2, 2, 2], [1, 3, 6, 8, 9, 10], [4, 6, 8],
> [1, 2, 3, 9], [3, 3], [3, 5, 6, 10], [1, 2, 7],
> [1, 2, 3, 10, 5, 6, 10], [1, 3, 4, 5, 8, 10],
> [2, 3, 4, 6, 7, 10]]);
<immutable multidigraph with 10 vertices, 44 edges>
gap> CharacteristicPolynomial(D);
x_1^10-3*x_1^9-7*x_1^8-x_1^7+14*x_1^6+x_1^5-26*x_1^4+51*x_1^3-10*x_1^2\
+18*x_1-30
gap> D := CompleteDigraph(5);
<immutable complete digraph with 5 vertices>
gap> CharacteristicPolynomial(D);
x_1^5-10*x_1^3-20*x_1^2-15*x_1-4
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> CharacteristicPolynomial(D);
x_1^3-1
```

5.2.3 BooleanAdjacencyMatrix

▷ BooleanAdjacencyMatrix(*digraph*)   (attribute)
▷ BooleanAdjacencyMatrixMutableCopy(*digraph*)   (operation)

Returns: A square matrix of booleans.

If *digraph* is a digraph with a positive number of vertices n, then BooleanAdjacencyMatrix(*digraph*) returns the boolean adjacency matrix mat of *digraph*. The value of the matrix entry mat[j][i] is true if and only if there exists an edge in *digraph* with source j and range i. If *digraph* has no vertices, then the empty list is returned.

Note that the boolean adjacency matrix loses information about multiple edges.

The attribute BooleanAdjacencyMatrix returns an immutable list of immutable lists, whereas the function BooleanAdjacencyMatrixMutableCopy returns a copy of the BooleanAdjacencyMatrix that is a mutable list of mutable lists.

```
─────────── Example ───────────
gap> gr := Digraph([[3, 4], [2, 3], [1, 2, 4], [4]]);
<immutable digraph with 4 vertices, 8 edges>
gap> PrintArray(BooleanAdjacencyMatrix(gr));
[ [ false,  false,   true,   true ],
  [ false,   true,   true,  false ],
  [  true,   true,  false,   true ],
  [ false,  false,  false,   true ] ]
gap> gr := CycleDigraph(4);;
gap> PrintArray(BooleanAdjacencyMatrix(gr));
[ [ false,   true,  false,  false ],
  [ false,  false,   true,  false ],
  [ false,  false,  false,   true ],
  [  true,  false,  false,  false ] ]
gap> BooleanAdjacencyMatrix(EmptyDigraph(0));
[  ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> PrintArray(BooleanAdjacencyMatrix(D));
[ [ false,   true,  false ],
  [ false,  false,   true ],
  [  true,  false,  false ] ]
```

5.2.4 DigraphAdjacencyFunction

▷ DigraphAdjacencyFunction(*digraph*)   (attribute)

    Returns: A function.

    If *digraph* is a digraph, then DigraphAdjacencyFunction returns a function which takes two integer parameters x, y and returns true if there exists an edge from vertex x to vertex y in *digraph* and false if not.

```
─────────── Example ───────────
gap> digraph := Digraph([[1, 2], [3], []]);
<immutable digraph with 3 vertices, 3 edges>
gap> foo := DigraphAdjacencyFunction(digraph);
function( u, v ) ... end
gap> foo(1, 1);
true
gap> foo(1, 2);
true
gap> foo(1, 3);
false
gap> foo(3, 1);
false
gap> gr := Digraph(["a", "b", "c"],
>                  ["a", "b", "b"],
>                  ["b", "a", "a"]);
<immutable multidigraph with 3 vertices, 3 edges>
gap> foo := DigraphAdjacencyFunction(gr);
function( u, v ) ... end
gap> foo(1, 2);
true
gap> foo(3, 2);
```

```
false
gap> foo(3, 1);
false
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> foo := DigraphAdjacencyFunction(D);
function( u, v ) ... end
gap> foo(1, 2);
true
gap> foo(2, 1);
false
```

5.2.5 DigraphRange

▷ DigraphRange(*digraph*)   (attribute)

▷ DigraphSource(*digraph*)   (attribute)

    Returns: A list of positive integers.

    DigraphRange and DigraphSource return the range and source of the digraph *digraph*. More precisely, position i in DigraphSource(*digraph*) and DigraphRange(*digraph*) give, respectively, the source and range of the ith edge of *digraph*.

```
―――――――――――――――――― Example ――――――――――――――――――
gap> gr := Digraph([
> [2, 1, 3, 5], [1, 3, 4], [2, 3], [2], [1, 2, 3, 4]]);
<immutable digraph with 5 vertices, 14 edges>
gap> DigraphSource(gr);
[ 1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 5, 5, 5, 5 ]
gap> DigraphRange(gr);
[ 2, 1, 3, 5, 1, 3, 4, 2, 3, 2, 1, 2, 3, 4 ]
gap> DigraphEdges(gr);
[ [ 1, 2 ], [ 1, 1 ], [ 1, 3 ], [ 1, 5 ], [ 2, 1 ], [ 2, 3 ],
  [ 2, 4 ], [ 3, 2 ], [ 3, 3 ], [ 4, 2 ], [ 5, 1 ], [ 5, 2 ],
  [ 5, 3 ], [ 5, 4 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphRange(D);
[ 2, 3, 1 ]
gap> DigraphSource(D);
[ 1, 2, 3 ]
```

5.2.6 OutNeighbours

▷ OutNeighbours(*digraph*)   (attribute)

▷ OutNeighbors(*digraph*)   (attribute)

▷ OutNeighboursMutableCopy(*digraph*)   (operation)

▷ OutNeighborsMutableCopy(*digraph*)   (operation)

    Returns: The adjacencies of a digraph.

    OutNeighbours returns the list out of out-neighbours of each vertex of the digraph *digraph*. More specifically, a vertex j appears in out[i] each time there exists the edge [i, j] in *digraph*.

The function `OutNeighbours` returns an immutable list of lists, whereas the function `OutNeighboursMutableCopy` returns a copy of `OutNeighbours` which is a mutable list of mutable lists.

Note that the entries of out are not guaranteed to be sorted in any particular order.

```
 _____ Example _____
gap> gr := Digraph(["a", "b", "c"],
>                   ["a", "b", "b"],
>                   ["b", "a", "c"]);
<immutable digraph with 3 vertices, 3 edges>
gap> OutNeighbours(gr);
[ [ 2 ], [ 1, 3 ], [  ] ]
gap> gr := Digraph([[1, 2, 3], [2, 1], [3]]);
<immutable digraph with 3 vertices, 6 edges>
gap> OutNeighbours(gr);
[ [ 1, 2, 3 ], [ 2, 1 ], [ 3 ] ]
gap> gr := DigraphByAdjacencyMatrix([
>   [1, 2, 1],
>   [1, 1, 0],
>   [0, 0, 1]]);
<immutable multidigraph with 3 vertices, 7 edges>
gap> OutNeighbours(gr);
[ [ 1, 2, 2, 3 ], [ 1, 2 ], [ 3 ] ]
gap> OutNeighboursMutableCopy(gr);
[ [ 1, 2, 2, 3 ], [ 1, 2 ], [ 3 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> OutNeighbours(D);
[ [ 2 ], [ 3 ], [ 1 ] ]
```

5.2.7 InNeighbours

▷ InNeighbours(*digraph*)   (attribute)
▷ InNeighbors(*digraph*)   (attribute)
▷ InNeighboursMutableCopy(*digraph*)   (operation)
▷ InNeighborsMutableCopy(*digraph*)   (operation)

Returns: A list of lists of vertices.

`InNeighbours` returns the list inn of in-neighbours of each vertex of the digraph *digraph*. More specifically, a vertex j appears in inn[i] each time there exists an edge [j,i] in *digraph*.

The function `InNeighbours` returns an immutable list of lists, whereas the function `InNeighboursMutableCopy` returns a copy of `InNeighbours` which is a mutable list of mutable lists.

Note that the entries of inn are not necessarily sorted into ascending order, particularly if *digraph* was constructed via `DigraphByInNeighbours` (3.1.11).

```
 _____ Example _____
gap> gr := Digraph(["a", "b", "c"],
>                   ["a", "b", "b"],
>                   ["b", "a", "c"]);
<immutable digraph with 3 vertices, 3 edges>
gap> InNeighbours(gr);
[ [ 2 ], [ 1 ], [ 2 ] ]
```

```
gap> gr := Digraph([[1, 2, 3], [2, 1], [3]]);
<immutable digraph with 3 vertices, 6 edges>
gap> InNeighbours(gr);
[ [ 1, 2 ], [ 1, 2 ], [ 1, 3 ] ]
gap> gr := DigraphByAdjacencyMatrix([
>   [1, 2, 1],
>   [1, 1, 0],
>   [0, 0, 1]]);
<immutable multidigraph with 3 vertices, 7 edges>
gap> InNeighbours(gr);
[ [ 1, 2 ], [ 1, 1, 2 ], [ 1, 3 ] ]
gap> InNeighboursMutableCopy(gr);
[ [ 1, 2 ], [ 1, 1, 2 ], [ 1, 3 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> InNeighbours(D);
[ [ 3 ], [ 1 ], [ 2 ] ]
```

5.2.8 OutDegrees

▷ OutDegrees(*digraph*) (attribute)
▷ OutDegreeSequence(*digraph*) (attribute)
▷ OutDegreeSet(*digraph*) (attribute)

Returns: A list of non-negative integers.

Given a digraph *digraph* with *n* vertices, the function OutDegrees returns an immutable list out of length *n*, such that for a vertex i in *digraph*, the value of out[i] is the out-degree of vertex i. See OutDegreeOfVertex (5.2.10).

The function OutDegreeSequence returns the same list, after it has been sorted into non-increasing order.

The function OutDegreeSet returns the same list, sorted into increasing order with duplicate entries removed.

```
 ─────────────────── Example ───────────────────
gap> D := Digraph([[1, 3, 2, 2], [], [2, 1], []]);
<immutable multidigraph with 4 vertices, 6 edges>
gap> OutDegrees(D);
[ 4, 0, 2, 0 ]
gap> OutDegreeSequence(D);
[ 4, 2, 0, 0 ]
gap> OutDegreeSet(D);
[ 0, 2, 4 ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> OutDegrees(D);
[ 1, 1, 1 ]
```

5.2.9 InDegrees

▷ InDegrees(*digraph*) (attribute)
▷ InDegreeSequence(*digraph*) (attribute)

▷ InDegreeSet(*digraph*) (attribute)

Returns: A list of non-negative integers.

Given a digraph *digraph* with *n* vertices, the function InDegrees returns an immutable list inn of length *n*, such that for a vertex i in *digraph*, the value of inn[i] is the in-degree of vertex i. See InDegreeOfVertex (5.2.12).

The function InDegreeSequence returns the same list, after it has been sorted into non-increasing order.

The function InDegreeSet returns the same list, sorted into increasing order with duplicate entries removed.

```
——————————————— Example ———————————————
gap> D := Digraph([[1, 3, 2, 2, 4], [], [2, 1, 4], []]);
<immutable multidigraph with 4 vertices, 8 edges>
gap> InDegrees(D);
[ 2, 3, 1, 2 ]
gap> InDegreeSequence(D);
[ 3, 2, 2, 1 ]
gap> InDegreeSet(D);
[ 1, 2, 3 ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> InDegrees(D);
[ 1, 1, 1 ]
```

5.2.10 OutDegreeOfVertex

▷ OutDegreeOfVertex(*digraph*, *vertex*) (operation)

Returns: The non-negative integer.

This operation returns the out-degree of the vertex *vertex* in the digraph *digraph*. The out-degree of *vertex* is the number of edges in *digraph* whose source is *vertex*.

```
——————————————— Example ———————————————
gap> D := Digraph([
> [2, 2, 1], [1, 4], [2, 2, 4, 2], [1, 1, 2, 2, 1, 2, 2]]);
<immutable multidigraph with 4 vertices, 16 edges>
gap> OutDegreeOfVertex(D, 1);
3
gap> OutDegreeOfVertex(D, 2);
2
gap> OutDegreeOfVertex(D, 3);
4
gap> OutDegreeOfVertex(D, 4);
7
```

5.2.11 OutNeighboursOfVertex

▷ OutNeighboursOfVertex(*digraph*, *vertex*) (operation)
▷ OutNeighborsOfVertex(*digraph*, *vertex*) (operation)

Returns: A list of vertices.

This operation returns the list out of vertices of the digraph *digraph*. A vertex i appears in the list out each time there exists an edge with source *vertex* and range i in *digraph*; in particular, this means that out may contain duplicates.

```
                        ___ Example ___
  gap> D := Digraph([
  > [2, 2, 3], [1, 3, 4], [2, 2, 3], [1, 1, 2, 2, 1, 2, 2]]);
  <immutable multidigraph with 4 vertices, 16 edges>
  gap> OutNeighboursOfVertex(D, 1);
  [ 2, 2, 3 ]
  gap> OutNeighboursOfVertex(D, 3);
  [ 2, 2, 3 ]
```

5.2.12 InDegreeOfVertex

▷ InDegreeOfVertex(`digraph, vertex`)   (operation)

Returns: A non-negative integer.

This operation returns the in-degree of the vertex `vertex` in the digraph `digraph`. The in-degree of `vertex` is the number of edges in `digraph` whose range is `vertex`.

```
                        ___ Example ___
  gap> D := Digraph([
  > [2, 2, 1], [1, 4], [2, 2, 4, 2], [1, 1, 2, 2, 1, 2, 2]]);
  <immutable multidigraph with 4 vertices, 16 edges>
  gap> InDegreeOfVertex(D, 1);
  5
  gap> InDegreeOfVertex(D, 2);
  9
  gap> InDegreeOfVertex(D, 3);
  0
  gap> InDegreeOfVertex(D, 4);
  2
```

5.2.13 InNeighboursOfVertex

▷ InNeighboursOfVertex(`digraph, vertex`)   (operation)
▷ InNeighborsOfVertex(`digraph, vertex`)   (operation)

Returns: A list of positive vertices.

This operation returns the list `inn` of vertices of the digraph `digraph`. A vertex i appears in the list `inn` each time there exists an edge with source i and range `vertex` in `digraph`; in particular, this means that `inn` may contain duplicates.

```
                        ___ Example ___
  gap> D := Digraph([
  > [2, 2, 3], [1, 3, 4], [2, 2, 3], [1, 1, 2, 2, 1, 2, 2]]);
  <immutable multidigraph with 4 vertices, 16 edges>
  gap> InNeighboursOfVertex(D, 1);
  [ 2, 4, 4, 4 ]
  gap> InNeighboursOfVertex(D, 2);
  [ 1, 1, 3, 3, 4, 4, 4, 4 ]
```

5.2.14 DigraphLoops

▷ DigraphLoops(`digraph`)   (attribute)

Returns: A list of vertices.

If `digraph` is a digraph, then `DigraphLoops` returns the list consisting of the `DigraphVertices` (5.1.1) of `digraph` at which there is a loop. See `DigraphHasLoops` (6.2.1).

```
───────────────────── Example ─────────────────────
gap> D := Digraph([[2], [3], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> DigraphHasLoops(D);
false
gap> DigraphLoops(D);
[  ]
gap> D := Digraph([[3, 5], [1], [2, 4, 3], [4], [2, 1]]);
<immutable digraph with 5 vertices, 9 edges>
gap> DigraphLoops(D);
[ 3, 4 ]
gap> D := Digraph(IsMutableDigraph, [[1], [1]]);
<mutable digraph with 2 vertices, 2 edges>
gap> DigraphLoops(D);
[ 1 ]
```

5.2.15 DegreeMatrix

▷ DegreeMatrix(`digraph`)   (attribute)

Returns: A square matrix of non-negative integers.

Returns the out-degree matrix `mat` of the digraph `digraph`. The value of the `i`th diagonal matrix entry is the out-degree of the vertex `i` in `digraph`. All off-diagonal entries are 0. If `digraph` has no vertices, then the empty list is returned.

See `OutDegrees` (5.2.8) for more information.

```
───────────────────── Example ─────────────────────
gap> D := Digraph([[1, 2, 3], [4], [1, 3, 4], []]);
<immutable digraph with 4 vertices, 7 edges>
gap> PrintArray(DegreeMatrix(D));
[ [  3,  0,  0,  0 ],
  [  0,  1,  0,  0 ],
  [  0,  0,  3,  0 ],
  [  0,  0,  0,  0 ] ]
gap> D := CycleDigraph(5);;
gap> PrintArray(DegreeMatrix(D));
[ [  1,  0,  0,  0,  0 ],
  [  0,  1,  0,  0,  0 ],
  [  0,  0,  1,  0,  0 ],
  [  0,  0,  0,  1,  0 ],
  [  0,  0,  0,  0,  1 ] ]
gap> DegreeMatrix(EmptyDigraph(0));
[  ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> Display(DegreeMatrix(D));
[ [  1,  0,  0 ],
  [  0,  1,  0 ],
  [  0,  0,  1 ] ]
```

5.2.16 LaplacianMatrix

▷ LaplacianMatrix(*digraph*)   (attribute)

    Returns: A square matrix of integers.

    Returns the out-degree Laplacian matrix mat of the digraph *digraph*. The out-degree Laplacian matrix is defined as DegreeMatrix(digraph) - AdjacencyMatrix(digraph). If *digraph* has no vertices, then the empty list is returned.

    See DegreeMatrix (5.2.15) and AdjacencyMatrix (5.2.1) for more information.

```
———————————————————— Example ————————————————————
gap> gr := Digraph([[1, 2, 3], [4], [1, 3, 4], []]);
<immutable digraph with 4 vertices, 7 edges>
gap> PrintArray(LaplacianMatrix(gr));
[ [   2,  -1,  -1,   0 ],
  [   0,   1,   0,  -1 ],
  [  -1,   0,   2,  -1 ],
  [   0,   0,   0,   0 ] ]
gap> LaplacianMatrix(EmptyDigraph(0));
[  ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> Display(LaplacianMatrix(D));
[ [   1,  -1,   0 ],
  [   0,   1,  -1 ],
  [  -1,   0,   1 ] ]
```

5.3 Orders

5.3.1 PartialOrderDigraphMeetOfVertices

▷ PartialOrderDigraphMeetOfVertices(*digraph, u, v*)   (operation)
▷ PartialOrderDigraphJoinOfVertices(*digraph, u, v*)   (operation)

    Returns: A positive integer or fail

    If the first argument is a partial order digraph IsPartialOrderDigraph (6.4.2) then these operations return the meet, or the join, of the two input vertices. If the meet (or join) is does not exist then fail is returned. The meet (or join) is guaranteed to exist when the first argument satisfies IsMeetSemilatticeDigraph (6.4.3) (or IsJoinSemilatticeDigraph (6.4.3)) - see the documentation for these properties for the definition of the meet (or the join).

```
———————————————————— Example ————————————————————
gap> D := Digraph([[1], [1, 2], [1, 3], [1, 2, 3, 4]]);
<immutable digraph with 4 vertices, 9 edges>
gap> PartialOrderDigraphMeetOfVertices(D, 2, 3);
4
gap> PartialOrderDigraphJoinOfVertices(D, 2, 3);
1
gap> PartialOrderDigraphMeetOfVertices(D, 1, 2);
2
gap> PartialOrderDigraphJoinOfVertices(D, 3, 4);
3
gap> D := Digraph([[1], [2], [1, 2, 3], [1, 2, 4]]);
<immutable digraph with 4 vertices, 8 edges>
```

```
gap> PartialOrderDigraphMeetOfVertices(D, 3, 4);
fail
gap> PartialOrderDigraphJoinOfVertices(D, 3, 4);
fail
gap> PartialOrderDigraphMeetOfVertices(D, 1, 2);
fail
gap> PartialOrderDigraphJoinOfVertices(D, 1, 2);
fail
```

5.3.2 NonUpperSemimodularPair

▷ NonUpperSemimodularPair(*D*)   (attribute)

▷ NonLowerSemimodularPair(*D*)   (attribute)

Returns: A pair of vertices or `fail`.

NonUpperSemimodularPair returns a pair of vertices in the digraph *D* that witnesses the fact that *D* does not represent an upper semimodular lattice, if such a pair exists.

If the digraph *D* does not satisfy IsLatticeDigraph (6.4.3), then an error is given. Otherwise if the digraph *D* does satisfy IsLatticeDigraph (6.4.3), then either a non-upper semimodular pair of vertices is returned, or `fail` is returned if no such pair exists (meaning that *D* is an upper semimodular lattice.

NonLowerSemimodularPair behaves in the analogous way to NonUpperSemimodularPair with respect to lower semimodularity.

See IsUpperSemimodularDigraph (6.4.5) and IsLowerSemimodularDigraph (6.4.5) for the definition of upper semimodularity of a lattice.

```
_____ Example _____
gap> D := DigraphFromDigraph6String(
> "&M~~sc'lYUZO__KIBboC_@h?U_?_GL?A_?c");
<immutable digraph with 14 vertices, 66 edges>
gap> NonLowerSemimodularPair(D);
[ 10, 9 ]
gap> NonUpperSemimodularPair(D);
fail
```

5.4 Reachability and connectivity

5.4.1 DigraphDiameter

▷ DigraphDiameter(*digraph*)   (attribute)

Returns: An integer or `fail`.

This function returns the diameter of the digraph *digraph*.

If a digraph *digraph* is strongly connected and has at least 1 vertex, then the *diameter* is the maximum shortest distance between any pair of distinct vertices. Otherwise then the diameter of *digraph* is undefined, and this function returns the value `fail`.

See DigraphShortestDistances (5.4.3).

```
_____ Example _____
gap> D := Digraph([[2], [3], [4, 5], [5], [1, 2, 3, 4, 5]]);
<immutable digraph with 5 vertices, 10 edges>
gap> DigraphDiameter(D);
```

```
3
gap> D := ChainDigraph(2);
<immutable chain digraph with 2 vertices>
gap> DigraphDiameter(D);
fail
gap> IsStronglyConnectedDigraph(D);
false
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 6, 2);
<mutable digraph with 12 vertices, 36 edges>
gap> DigraphDiameter(D);
4
```

5.4.2 DigraphShortestDistance (for a digraph and two vertices)

▷ DigraphShortestDistance(*digraph, u, v*)   (operation)

▷ DigraphShortestDistance(*digraph, list*)   (operation)

▷ DigraphShortestDistance(*digraph, list1, list2*)   (operation)

    Returns: An integer or `fail`

If there is a directed path in the digraph `digraph` between vertex `u` and vertex `v`, then this operation returns the length of the shortest such directed path. If no such directed path exists, then this operation returns `fail`. See Section 1.1.1 for the definition of a directed path.

If the second form is used, then `list` should be a list of length two, containing two positive integers which correspond to the vertices `u` and `v`.

Note that as usual, a vertex is considered to be at distance 0 from itself .

If the third form is used, then `list1` and `list2` are both lists of vertices. The shortest directed path between `list1` and `list2` is then the length of the shortest directed path which starts with a vertex in `list1` and terminates at a vertex in `list2`, if such directed path exists. If `list1` and `list2` have non-empty intersection, the operation returns 0.

```
────────────────────── Example ──────────────────────
gap> D := Digraph([[2], [3], [1, 4], [1, 3], [5]]);
<immutable digraph with 5 vertices, 7 edges>
gap> DigraphShortestDistance(D, 1, 3);
2
gap> DigraphShortestDistance(D, [3, 3]);
0
gap> DigraphShortestDistance(D, 5, 2);
fail
gap> DigraphShortestDistance(D, [1, 2], [4, 5]);
2
gap> DigraphShortestDistance(D, [1, 3], [3, 5]);
0
```

5.4.3 DigraphShortestDistances

▷ DigraphShortestDistances(*digraph*)   (attribute)

    Returns: A square matrix.

If `digraph` is a digraph with *n* vertices, then this function returns an $n \times n$ matrix `mat`, where each entry is either a non-negative integer, or `fail`. If $n = 0$, then an empty list is returned.

If there is a directed path from vertex i to vertex j, then the value of mat[i][j] is the length of the shortest such directed path. If no such directed path exists, then the value of mat[i][j] is fail. We use the convention that the distance from every vertex to itself is 0, i.e. mat[i][i] = 0 for all vertices i.

The method used in this function is a version of the Floyd-Warshall algorithm, and has complexity $O(n^3)$.

```
_____ Example _____
gap> D := Digraph([[1, 2], [3], [1, 2], [4]]);
<immutable digraph with 4 vertices, 6 edges>
gap> mat := DigraphShortestDistances(D);;
gap> PrintArray(mat);
[ [    0,    1,    2,  fail ],
  [    2,    0,    1,  fail ],
  [    1,    1,    0,  fail ],
  [ fail, fail, fail,    0 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphShortestDistances(D);
[ [ 0, 1, 2 ], [ 2, 0, 1 ], [ 1, 2, 0 ] ]
```

5.4.4 DigraphLongestDistanceFromVertex

▷ DigraphLongestDistanceFromVertex(*digraph, v*)   (operation)
   Returns: An integer, or infinity.

If *digraph* is a digraph and *v* is a vertex in *digraph*, then this operation returns the length of the longest directed walk in *digraph* which begins at vertex *v*. See Section 1.1.1 for the definitions of directed walk, directed cycle, and loop.

- If there exists a directed walk starting at vertex *v* which traverses a loop or a directed cycle, then we consider there to be a walk of infinite length from *v* (realised by repeatedly traversing the loop/directed cycle), and so the result is infinity. To disallow walks using loops, try using DigraphRemoveLoops (3.3.25):

  DigraphLongestDistanceFromVertex(DigraphRemoveLoops(*digraph*,*v*)).

- Otherwise, if all directed walks starting at vertex *v* have finite length, then the length of the longest such walk is returned.

Note that the result is 0 if and only if *v* is a sink of *digraph*. See DigraphSinks (5.1.8).

```
_____ Example _____
gap> D := Digraph([[2], [3, 4], [], [5], [], [6]]);
<immutable digraph with 6 vertices, 5 edges>
gap> DigraphLongestDistanceFromVertex(D, 1);
3
gap> DigraphLongestDistanceFromVertex(D, 3);
0
gap> 3 in DigraphSinks(D);
true
gap> DigraphLongestDistanceFromVertex(D, 6);
infinity
```

5.4.5 DigraphDistanceSet (for a digraph, a pos int, and an int)

▷ DigraphDistanceSet(*digraph, vertex, distance*)   (operation)
▷ DigraphDistanceSet(*digraph, vertex, distances*)   (operation)

    Returns: A list of vertices

    This operation returns the list of all vertices in digraph `digraph` such that the shortest distance to a vertex `vertex` is `distance` or is in the list `distances`.

    `digraph` should be a digraph, `vertex` should be a positive integer, `distance` should be a non-negative integer, and `distances` should be a list of non-negative integers.

```
——————————————————— Example ———————————————————
gap> D := Digraph([[2], [3], [1, 4], [1, 3]]);
<immutable digraph with 4 vertices, 6 edges>
gap> DigraphDistanceSet(D, 2, [1, 2]);
[ 3, 1, 4 ]
gap> DigraphDistanceSet(D, 3, 1);
[ 1, 4 ]
gap> DigraphDistanceSet(D, 2, 0);
[ 2 ]
```

5.4.6 DigraphGirth

▷ DigraphGirth(*digraph*)   (attribute)

    Returns: An integer, or `infinity`.

    This attribute returns the *girth* of the digraph `digraph`. The *girth* of a digraph is the length of its shortest simple circuit. See Section 1.1.1 for the definitions of simple circuit, directed cycle, and loop.

    If `digraph` has no directed cycles, then this function will return `infinity`. If `digraph` contains a loop, then this function will return 1.

    In the worst case, the method used in this function is a version of the Floyd-Warshall algorithm, and has complexity `O(n ^ 3)`, where `n` is the number of vertices in `digraph`. If the digraph has known automorphisms [see `DigraphGroup` (7.2.10)], then the performance is likely to be better.

    For symmetric digraphs, see also `DigraphUndirectedGirth` (5.4.8).

```
——————————————————— Example ———————————————————
gap> D := Digraph([[1], [1]]);
<immutable digraph with 2 vertices, 2 edges>
gap> DigraphGirth(D);
1
gap> D := Digraph([[2, 3], [3], [4], []]);
<immutable digraph with 4 vertices, 4 edges>
gap> DigraphGirth(D);
infinity
gap> D := Digraph([[2, 3], [3], [4], [1]]);
<immutable digraph with 4 vertices, 5 edges>
gap> DigraphGirth(D);
3
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 6, 2);
<mutable digraph with 12 vertices, 36 edges>
gap> DigraphGirth(D);
2
```

5.4.7 DigraphOddGirth

▷ DigraphOddGirth(*digraph*)   (attribute)

    Returns: An integer, or `infinity`.

    This attribute returns the *odd girth* of the digraph `digraph`. The *odd girth* of a digraph is the length of its shortest simple circuit of odd length. See Section 1.1.1 for the definitions of simple circuit, directed cycle, and loop.

    If `digraph` has no directed cycles of odd length, then this function will return `infinity`, even if `digraph` has a directed cycle of even length. If `digraph` contains a loop, then this function will return 1.

    See also `DigraphGirth` (5.4.6).

—————————— Example ——————————
```
gap> D := Digraph([[2], [3, 1], [1]]);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphOddGirth(D);
3
gap> D := CycleDigraph(4);
<immutable cycle digraph with 4 vertices>
gap> DigraphOddGirth(D);
infinity
gap> D := Digraph([[2], [3], [], [3], [4]]);
<immutable digraph with 5 vertices, 4 edges>
gap> DigraphOddGirth(D);
infinity
gap> D := CycleDigraph(IsMutableDigraph, 4);
<mutable digraph with 4 vertices, 4 edges>
gap> DigraphDisjointUnion(D, CycleDigraph(5));
<mutable digraph with 9 vertices, 9 edges>
gap> DigraphOddGirth(D);
5
```

5.4.8 DigraphUndirectedGirth

▷ DigraphUndirectedGirth(*digraph*)   (attribute)

    Returns: An integer or `infinity`.

    If `digraph` is a symmetric digraph, then this function returns the girth of `digraph` when treated as an undirected graph (i.e. each pair of edges $[i, j]$ and $[j, i]$ is treated as a single edge between i and j).

    The *girth* of an undirected graph is the length of its shortest simple cycle, i.e. the shortest non-trivial path starting and ending at the same vertex and passing through no vertex or edge more than once.

    If `digraph` has no cycles, then this function will return `infinity`.

—————————— Example ——————————
```
gap> D := Digraph([[2, 4], [1, 3], [2, 4], [1, 3]]);
<immutable digraph with 4 vertices, 8 edges>
gap> DigraphUndirectedGirth(D);
4
gap> D := Digraph([[2], [1, 3], [2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphUndirectedGirth(D);
```

```
    infinity
gap> D := Digraph([[1], [], [4], [3]]);
<immutable digraph with 4 vertices, 3 edges>
gap> DigraphUndirectedGirth(D);
1
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 9, 2);
<mutable digraph with 18 vertices, 54 edges>
gap> DigraphUndirectedGirth(D);
5
```

5.4.9 DigraphConnectedComponents

▷ DigraphConnectedComponents(*digraph*)　　　　　　　　　　　　　　　　　(attribute)

▷ DigraphNrConnectedComponents(*digraph*)　　　　　　　　　　　　　　　　(attribute)

　　　Returns: A record.

　　This function returns the record `wcc` corresponding to the weakly connected components of the digraph *digraph*. Two vertices of *digraph* are in the same weakly connected component whenever they are equal, or there exists a directed path (ignoring the orientation of edges) between them. More formally, two vertices are in the same weakly connected component of *digraph* if and only if they are in the same strongly connected component (see DigraphStronglyConnectedComponents (5.4.11)) of the DigraphSymmetricClosure (3.3.12) of *digraph*.

　　The set of weakly connected components is a partition of the vertex set of *digraph*.

　　The record `wcc` has 2 components: `comps` and `id`. The component `comps` is a list of the weakly connected components of *digraph* (each of which is a list of vertices). The component `id` is a list such that the vertex `i` is an element of the weakly connected component `comps[id[i]]`.

　　The method used in this function has complexity $O(m+n)$, where m is the number of edges and n is the number of vertices in the digraph.

　　DigraphNrConnectedComponents(*digraph*)　　is　　simply　　a　　shortcut　　for Length(DigraphConnectedComponents(*digraph*).comps), and is no more efficient.

```
————————————————————————————— Example —————————————————————————————
gap> gr := Digraph([[2], [3, 1], []]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphConnectedComponents(gr);
rec( comps := [ [ 1, 2, 3 ] ], id := [ 1, 1, 1 ] )
gap> gr := Digraph([[1], [1, 2], []]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphConnectedComponents(gr);
rec( comps := [ [ 1, 2 ], [ 3 ] ], id := [ 1, 1, 2 ] )
gap> DigraphNrConnectedComponents(gr);
2
gap> gr := EmptyDigraph(0);
<immutable empty digraph with 0 vertices>
gap> DigraphConnectedComponents(gr);
rec( comps := [  ], id := [  ] )
gap> D := CycleDigraph(IsMutableDigraph, 2);
<mutable digraph with 2 vertices, 2 edges>
gap> G := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> DigraphDisjointUnion(D, G);
<mutable digraph with 5 vertices, 5 edges>
```

```
gap> DigraphConnectedComponents(D);
rec( comps := [ [ 1, 2 ], [ 3, 4, 5 ] ], id := [ 1, 1, 2, 2, 2 ] )
```

5.4.10 DigraphConnectedComponent

▷ DigraphConnectedComponent(`digraph`, `vertex`)   (operation)

    Returns: A list of vertices.

    If `vertex` is a vertex in the digraph `digraph`, then this operation returns the connected component of `vertex` in `digraph`. See DigraphConnectedComponents (5.4.9) for more information.

```
──────────────────── Example ────────────────────
gap> D := Digraph([[3], [2], [1, 2], [4]]);
<immutable digraph with 4 vertices, 5 edges>
gap> DigraphConnectedComponent(D, 3);
[ 1, 2, 3 ]
gap> DigraphConnectedComponent(D, 2);
[ 1, 2, 3 ]
gap> DigraphConnectedComponent(D, 4);
[ 4 ]
```

5.4.11 DigraphStronglyConnectedComponents

▷ DigraphStronglyConnectedComponents(`digraph`)   (attribute)
▷ DigraphNrStronglyConnectedComponents(`digraph`)   (attribute)

    Returns: A record.

    This function returns the record `scc` corresponding to the strongly connected components of the digraph `digraph`. Two vertices of `digraph` are in the same strongly connected component whenever they are equal, or there is a directed path from each vertex to the other. The set of strongly connected components is a partition of the vertex set of `digraph`.

    The record `scc` has 2 components: `comps` and `id`. The component `comps` is a list of the strongly connected components of `digraph` (each of which is a list of vertices). The component `id` is a list such that the vertex `i` is an element of the strongly connected component `comps[id[i]]`.

    The method used in this function is a non-recursive version of Gabow's Algorithm [Gab00] and has complexity $O(m + n)$ where m is the number of edges (counting multiple edges as one) and n is the number of vertices in the digraph.

    DigraphNrStronglyConnectedComponents(`digraph`) is simply a shortcut for Length(DigraphStronglyConnectedComponents(`digraph`).comps), and is no more efficient.

```
──────────────────── Example ────────────────────
gap> gr := Digraph([[2], [3, 1], []]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphStronglyConnectedComponents(gr);
rec( comps := [ [ 3 ], [ 1, 2 ] ], id := [ 2, 2, 1 ] )
gap> DigraphNrStronglyConnectedComponents(gr);
2
gap> D := DigraphDisjointUnion(CycleDigraph(4), CycleDigraph(5));
<immutable digraph with 9 vertices, 9 edges>
gap> DigraphStronglyConnectedComponents(D);
rec( comps := [ [ 1, 2, 3, 4 ], [ 5, 6, 7, 8, 9 ] ],
  id := [ 1, 1, 1, 1, 2, 2, 2, 2, 2 ] )
```

```
gap> DigraphNrStronglyConnectedComponents(D);
2
gap> D := CycleDigraph(IsMutableDigraph, 2);
<mutable digraph with 2 vertices, 2 edges>
gap> G := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> DigraphDisjointUnion(D, G);
<mutable digraph with 5 vertices, 5 edges>
gap> DigraphStronglyConnectedComponents(D);
rec( comps := [ [ 1, 2 ], [ 3, 4, 5 ] ], id := [ 1, 1, 2, 2, 2 ] )
```

5.4.12 DigraphStronglyConnectedComponent

▷ DigraphStronglyConnectedComponent(*digraph, vertex*) (operation)

    Returns: A list of vertices.

    If *vertex* is a vertex in the digraph *digraph*, then this operation returns the strongly connected component of *vertex* in *digraph*. See DigraphStronglyConnectedComponents (5.4.11) for more information.

```
------------------------ Example ------------------------
gap> D := Digraph([[3], [2], [1, 2], [3]]);
<immutable digraph with 4 vertices, 5 edges>
gap> DigraphStronglyConnectedComponent(D, 3);
[ 1, 3 ]
gap> DigraphStronglyConnectedComponent(D, 2);
[ 2 ]
gap> DigraphStronglyConnectedComponent(D, 4);
[ 4 ]
```

5.4.13 DigraphBicomponents

▷ DigraphBicomponents(*digraph*) (attribute)

    Returns: A pair of lists of vertices, or fail.

    If *digraph* is a bipartite digraph, i.e. if it satisfies IsBipartiteDigraph (6.2.3), then DigraphBicomponents returns a pair of bicomponents of *digraph*. Otherwise, DigraphBicomponents returns fail.

    For a bipartite digraph, the vertices can be partitioned into two non-empty sets such that the source and range of any edge are in distinct sets. The parts of this partition are called *bicomponents* of *digraph*. Equivalently, a pair of bicomponents of *digraph* consists of the color-classes of a 2-coloring of *digraph*.

    For a bipartite digraph with at least 3 vertices, there is a unique pair of bicomponents of bipartite if and only if the digraph is connected. See IsConnectedDigraph (6.6.3) for more information.

```
------------------------ Example ------------------------
gap> D := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> DigraphBicomponents(D);
fail
gap> D := ChainDigraph(5);
<immutable chain digraph with 5 vertices>
gap> DigraphBicomponents(D);
[ [ 1, 3, 5 ], [ 2, 4 ] ]
```

```
gap> D := Digraph([[5], [1, 4], [5], [5], []]);
<immutable digraph with 5 vertices, 5 edges>
gap> DigraphBicomponents(D);
[ [ 1, 3, 4 ], [ 2, 5 ] ]
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 2, 3);
<mutable digraph with 5 vertices, 12 edges>
gap> DigraphBicomponents(D);
[ [ 1, 2 ], [ 3, 4, 5 ] ]
```

5.4.14 ArticulationPoints

▷ ArticulationPoints(`digraph`) (attribute)

Returns: A list of vertices.

A connected digraph is *biconnected* if it is still connected (in the sense of `IsConnectedDigraph` (6.6.3)) when any vertex is removed. If the digraph `digraph` is not biconnected but is connected, then any vertex v of `digraph` whose removal makes the resulting digraph disconnected is called an *articulation point*.

`ArticulationPoints` returns a list of the articulation points of `digraph`, if any, and, in particular, returns the empty list if `digraph` is not connected.

Multiple edges are ignored by this method.

The method used in this operation has complexity $O(m + n)$ where m is the number of edges and n is the number of vertices in the digraph.

If `D` has a bridge (see `Bridges` (5.4.15)), then a node incident to the bridge is an articulation point if and only if it has degree at least 2. It follows that if `D` has a bridge and at least 3 nodes, then at least one of the nodes incident to the bridge is an articulation point. The converse does not hold, there are digraphs with articulation points, but no bridges.

See also `IsBiconnectedDigraph` (6.6.4) and `IsBridgelessDigraph` (6.6.5).

```
─────────────────── Example ───────────────────
gap> ArticulationPoints(CycleDigraph(5));
[  ]
gap> D := Digraph([[2, 7], [3, 5], [4], [2], [6], [1], []]);;
gap> ArticulationPoints(D);
[ 2, 1 ]
gap> ArticulationPoints(ChainDigraph(5));
[ 4, 3, 2 ]
gap> ArticulationPoints(NullDigraph(5));
[  ]
gap> D := ChainDigraph(IsMutableDigraph, 4);
<mutable digraph with 4 vertices, 3 edges>
gap> ArticulationPoints(D);
[ 3, 2 ]
```

5.4.15 Bridges

▷ Bridges(`D`) (attribute)

Returns: A (possibly empty) list of edges.

A connected digraph is *2-edge-connected* if it is still connected (in the sense of `IsConnectedDigraph` (6.6.3)) when any edge is removed. If the digraph `D` is not 2-edge-connected

but is connected, then any edge [u, v] of *D* whose removal makes the resulting digraph disconnected is called a *bridge*.

Bridges returns a list of the bridges of *D*, if any, and, in particular, returns the empty list if *D* is not connected.

Multiple edges are ignored by this method.

The method used in this operation has complexity $O(m+n)$ where *m* is the number of edges and *n* is the number of vertices in the digraph.

If *D* has a bridge, then a node incident to the bridge is an articulation point (see ArticulationPoints (5.4.14)) if and only if it has degree at least 2. It follows that if *D* has a bridge and at least 3 nodes, then at least one of the nodes incident to the bridge is an articulation point. The converse does not hold, there are digraphs with articulation points, but no bridges.

See also IsBiconnectedDigraph (6.6.4) and IsBridgelessDigraph (6.6.5).

```
────────────────────── Example ──────────────────────
gap> D := Digraph([[2, 5], [1, 3, 4, 5], [2, 4], [2, 3], [1, 2]]);
<immutable digraph with 5 vertices, 12 edges>
gap> Bridges(D);
[  ]
gap> D := Digraph([[2], [3], [4], [2]]);
<immutable digraph with 4 vertices, 4 edges>
gap> Bridges(D);
[ [ 1, 2 ] ]
gap> Bridges(ChainDigraph(2));
[ [ 1, 2 ] ]
gap> ArticulationPoints(ChainDigraph(2));
[  ]
```

5.4.16 StrongOrientation

▷ StrongOrientation(*D*)   (operation)

▷ StrongOrientationAttr(*D*)   (attribute)

Returns: A digraph or fail.

A *strong orientation* of a connected symmetric digraph *D* (if it exists) is a strongly connected subdigraph C of *D* such that for every edge [u, v] of *D* either [u, v] or [v, u] is an edge of C but not both. Robbin's Theorem states that a digraph admits a strong orientation if and only if it is bridgeless (see IsBridgelessDigraph (6.6.5)).

This operation returns a strong orientation of the digraph *D* if *D* is symmetric and *D* admits a strong orientation. If *D* is symmetric but does not admit a strong orientation, then fail is returned. If *D* is not symmetric, then an error is given.

If *D* is immutable, StrongOrientation(*D*) returns an immutable digraph, and if *D* is mutable, then StrongOrientation(*D*) returns a mutable digraph.

The method used in this operation has complexity $O(m+n)$ where *m* is the number of edges and *n* is the number of vertices in the digraph.

```
────────────────────── Example ──────────────────────
gap> StrongOrientation(DigraphSymmetricClosure(CycleDigraph(5)))
> = CycleDigraph(5);
true
gap> D := DigraphSymmetricClosure(Digraph(
> [[2, 7], [3, 5], [4], [2], [6], [1], []]));;
gap> IsBridgelessDigraph(D);
```

```
false
gap> StrongOrientation(D);
fail
gap> StrongOrientation(NullDigraph(0));
<immutable empty digraph with 0 vertices>
gap> StrongOrientation(DigraphDisjointUnion(CompleteDigraph(3),
>                                           CompleteDigraph(3)));
fail
```

5.4.17 DigraphPeriod

▷ DigraphPeriod(*digraph*)   (attribute)

    Returns: An integer.

    This function returns the period of the digraph *digraph*.

    If a digraph *digraph* has at least one directed cycle, then the period is the greatest positive integer which divides the lengths of all directed cycles of *digraph*. If *digraph* has no directed cycles, then this function returns 0. See Section 1.1.1 for the definition of a directed cycle.

    A digraph with a period of 1 is said to be *aperiodic*. See IsAperiodicDigraph (6.6.7).

```
_____ Example _____
gap> D := Digraph([[6], [1], [2], [3], [4, 4], [5]]);
<immutable multidigraph with 6 vertices, 7 edges>
gap> DigraphPeriod(D);
6
gap> D := Digraph([[2], [3, 5], [4], [5], [1, 2]]);
<immutable digraph with 5 vertices, 7 edges>
gap> DigraphPeriod(D);
1
gap> D := ChainDigraph(2);
<immutable chain digraph with 2 vertices>
gap> DigraphPeriod(D);
0
gap> IsAcyclicDigraph(D);
true
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 5, 2);
<mutable digraph with 10 vertices, 30 edges>
gap> DigraphPeriod(D);
1
```

5.4.18 DigraphFloydWarshall

▷ DigraphFloydWarshall(*digraph, func, nopath, edge*)   (operation)

    Returns: A matrix.

    If *digraph* is a digraph with *n* vertices, then this operation returns an $n \times n$ matrix mat containing the output of a generalised version of the Floyd-Warshall algorithm, applied to *digraph*.

    The operation DigraphFloydWarshall is customised by the arguments *func*, *nopath*, and *edge*. The arguments *nopath* and *edge* can be arbitrary GAP objects. The argument *func* must be a function which accepts 4 arguments: the matrix mat, followed by 3 positive integers. The function *func* is where the work to calculate the desired outcome must be performed.

This method initialises the matrix mat by setting entry mat[i][j] to equal *edge* if there is an edge with source i and range j, and by setting entry mat[i][j] to equal *nopath* otherwise. The final part of DigraphFloydWarshall then calls the function *func* inside three nested for loops, over the vertices of *digraph*:

```
for i in DigraphsVertices(digraph) do
  for j in DigraphsVertices(digraph) do
    for k in DigraphsVertices(digraph) do
      func(mat, i, j, k);
    od;
  od;
od;
```

The matrix mat is then returned as the result. An example of using DigraphFloydWarshall to calculate the shortest (non-zero) distances between the vertices of a digraph is shown below:

```
 ─────────────────── Example ───────────────────
gap> D := DigraphFromDigraph6String("&EAHQeDB");
<immutable digraph with 6 vertices, 12 edges>
gap> func := function(mat, i, j, k)
>   if mat[i][k] <> -1 and mat[k][j] <> -1 then
>     if (mat[i][j] = -1) or (mat[i][j] > mat[i][k] + mat[k][j]) then
>       mat[i][j] := mat[i][k] + mat[k][j];
>     fi;
>   fi;
> end;
function( mat, i, j, k ) ... end
gap> shortest_distances := DigraphFloydWarshall(D, func, -1, 1);;
gap> Display(shortest_distances);
[ [   3,  -1,  -1,   2,   1,   2 ],
  [   4,   2,   1,   3,   2,   1 ],
  [   3,   1,   2,   2,   1,   2 ],
  [   1,  -1,  -1,   1,   1,   2 ],
  [   2,  -1,  -1,   1,   2,   1 ],
  [   3,  -1,  -1,   2,   1,   1 ] ]
```

5.4.19 IsReachable

▷ IsReachable(*digraph, u, v*)　　　　　　　　　　　　　　　　　　　　　(operation)

Returns: true or false.

This operation returns true if there exists a non-trivial directed walk from vertex *u* to vertex *v* in the digraph *digraph*, and false if there does not exist such a directed walk. See Section 1.1.1 for the definition of a non-trivial directed walk.

The method for IsReachable has worst case complexity of $O(m+n)$ where m is the number of edges and n the number of vertices in *digraph*.

```
 ─────────────────── Example ───────────────────
gap> D := Digraph([[2], [3], [2, 3]]);
<immutable digraph with 3 vertices, 4 edges>
gap> IsReachable(D, 1, 3);
true
gap> IsReachable(D, 2, 1);
```

```
  false
gap> IsReachable(D, 3, 3);
  true
gap> IsReachable(D, 1, 1);
  false
```

5.4.20 IsDigraphPath

▷ IsDigraphPath(*D, v, a*)   (operation)
▷ IsDigraphPath(*D, list*)   (operation)
    Returns: true or false.

This function returns true if the arguments `v` and `a` describe a path in the digraph `D`. A directed path (or directed cycle) of non-zero length n-1, $(v_1, e_1, v_2, e_2, ..., e_{n-1}, v_n)$, is represented by a pair of lists [v, a] as follows:

- `v` is the list $[v_1, v_2, ..., v_n]$.

- `a` is the list of positive integers $[a_1, a_2, ..., a_{n-1}]$ where for each each $i < n$, a_i is the position of v_{i+1} in OutNeighboursOfVertex(`D`,v_i) corresponding to the edge e_i. This can be useful if the position of a vertex in a list of out-neighours is significant, for example in orbit digraphs.

If the arguments to IsDigraphPath are a digraph `D` and list `list`, then this is equivalent to calling IsDigraphPath(`D`, *list*[1], *list*[2]).

─────────────────────────── Example ───────────────────────────
```
gap> D := Digraph(IsMutableDigraph, Combinations([1 .. 5]), IsSubset);
<mutable digraph with 32 vertices, 243 edges>
gap> DigraphReflexiveTransitiveReduction(D);
<mutable digraph with 32 vertices, 80 edges>
gap> MakeImmutable(D);
<immutable digraph with 32 vertices, 80 edges>
gap> IsDigraphPath(D, [32, 31, 33], [1, 1]);
false
gap> IsDigraphPath(D, [1], []);
true
gap> IsDigraphPath(D, [6, 9, 16, 17], [3, 3, 2]);
true
gap> IsDigraphPath(D, DigraphPath(D, 6, 1));
true
```

5.4.21 VerticesReachableFrom (for a digraph and vertex)

▷ VerticesReachableFrom(*digraph, root*)   (operation)
▷ VerticesReachableFrom(*digraph, list*)   (operation)
    Returns: A list.

This operation returns a list of the vertices v, for which there exists a non-trivial directed walk from the vertex *root*, or any of the list of vertices *list*, to vertex v in the digraph *digraph*. See Section 1.1.1 for the definition of a non-trivial directed walk.

The method for VerticesReachableFrom has worst case complexity of $O(m+n)$ where m is the number of edges and n the number of vertices in *digraph*.

This function returns an error if *root*, or any vertex in *list*, is not a vertices of *digraph*.

```
                              ____ Example ____
  gap> D := CompleteDigraph(5);
  <immutable complete digraph with 5 vertices>
  gap> VerticesReachableFrom(D, 1);
  [ 1, 2, 3, 4, 5 ]
  gap> VerticesReachableFrom(D, 3);
  [ 1, 2, 3, 4, 5 ]
  gap> D := EmptyDigraph(5);
  <immutable empty digraph with 5 vertices>
  gap> VerticesReachableFrom(D, 1);
  [  ]
  gap> VerticesReachableFrom(D, 3);
  [  ]
  gap> VerticesReachableFrom(D, [1, 2, 3, 4]);
  [  ]
  gap> VerticesReachableFrom(D, [3, 4, 5]);
  [  ]
  gap> D := CycleDigraph(4);
  <immutable cycle digraph with 4 vertices>
  gap> VerticesReachableFrom(D, 1);
  [ 1, 2, 3, 4 ]
  gap> VerticesReachableFrom(D, 3);
  [ 1, 2, 3, 4 ]
  gap> D := ChainDigraph(5);
  <immutable chain digraph with 5 vertices>
  gap> VerticesReachableFrom(D, 1);
  [ 2, 3, 4, 5 ]
  gap> VerticesReachableFrom(D, 3);
  [ 4, 5 ]
  gap> VerticesReachableFrom(D, 5);
  [  ]
  gap> VerticesReachableFrom(D, [3, 4]);
  [ 4, 5 ]
```

5.4.22 DigraphPath

▷ DigraphPath(*digraph, u, v*)   (operation)

Returns: A pair of lists, or fail.

If there exists a non-trivial directed path (or a non-trivial cycle, in the case that $u = v$) from vertex u to vertex v in the digraph `digraph`, then this operation returns such a directed path (or directed cycle). Otherwise, this operation returns fail. See Section 'Definitions' for the definition of a directed path and a directed cycle.

A directed path (or directed cycle) of non-zero length n-1, $(v_1, e_1, v_2, e_2, ..., e_{n-1}, v_n)$, is represented by a pair of lists [v,a] as follows:

- v is the list $[v_1, v_2, ..., v_n]$.

- a is the list of positive integers $[a_1, a_2, ..., a_{n-1}]$ where for each each $i < n$, a_i is the position of v_{i+1} in OutNeighboursOfVertex(`digraph`,v_i) corresponding to the edge e_i. This can be useful if the position of a vertex in a list of out-neighours is significant, for example in orbit digraphs.

The method for `DigraphPath` has worst case complexity of $O(m+n)$ where m is the number of edges and n the number of vertices in `digraph`.

See also `IsDigraphPath` (5.4.20).

```
——————————————— Example ———————————————
gap> D := Digraph([[2], [3], [2, 3]]);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphPath(D, 1, 3);
[ [ 1, 2, 3 ], [ 1, 1 ] ]
gap> DigraphPath(D, 2, 1);
fail
gap> DigraphPath(D, 3, 3);
[ [ 3, 3 ], [ 2 ] ]
gap> DigraphPath(D, 1, 1);
fail
```

5.4.23   DigraphShortestPath

▷ DigraphShortestPath(`digraph, u, v`)   (operation)

Returns: A pair of lists, or `fail`.

Returns a shortest directed path in the digraph `digraph` from vertex `u` to vertex `v`, if such a path exists. If `u = v`, then the shortest non-trivial cycle is returned, again, if it exists. Otherwise, this operation returns `fail`. See Section 'Definitions' for the definition of a directed path and a directed cycle.

See `DigraphPath` (5.4.22) for details on the output. The method for `DigraphShortestPath` has worst case complexity of $O(m+n)$ where m is the number of edges and n the number of vertices in `digraph`.

```
——————————————— Example ———————————————
gap> D := Digraph([[1, 2], [3], [2, 4], [1], [2, 4]]);
<immutable digraph with 5 vertices, 8 edges>
gap> DigraphShortestPath(D, 5, 1);
[ [ 5, 4, 1 ], [ 2, 1 ] ]
gap> DigraphShortestPath(D, 3, 3);
[ [ 3, 2, 3 ], [ 1, 1 ] ]
gap> DigraphShortestPath(D, 5, 5);
fail
gap> DigraphShortestPath(D, 1, 1);
[ [ 1, 1 ], [ 1 ] ]
```

5.4.24   DigraphRandomWalk

▷ DigraphRandomWalk(`digraph, v, t`)   (operation)

Returns: A pair of lists.

Returns a directed path corresponding to a *random walk* in the digraph `digraph`, starting at vertex `v` and having length no more than `t`.

A random walk is defined as follows. The path begins at `v`, and at each step it follows a random edge leaving the current vertex. It continues through the digraph in this way until it has traversed `t` edges, or until it reaches a vertex with no out-edges (a *sink*) and therefore cannot continue.

The output has the same form as that of `DigraphPath` (5.4.22).

```
_____ Example _____
gap> D := Digraph([[1, 2], [3], [2, 4], [1], [2, 4]]);
<immutable digraph with 5 vertices, 8 edges>
gap> DigraphRandomWalk(D, 1, 4);
[ [ 1, 2, 3, 2, 3 ], [ 2, 1, 1, 1 ] ]
```

5.4.25 DigraphAbsorptionProbabilities

▷ DigraphAbsorptionProbabilities(*digraph*)   (attribute)

Returns: A matrix of rational numbers.

A random walk of infinite length through a finite digraph may pass through several different strongly connected components (SCCs). However, with probability 1 it must eventually reach an SCC which it can never leave, because the SCC has no out-edges leading to other SCCs. We may say that such an SCC has *absorbed* the random walk, and we can calculate the probability of each SCC absorbing a walk starting at a given vertex.

DigraphAbsorptionProbabilities returns an $m \times n$ matrix mat, where m is the number of vertices in digraph and n is the number of strongly connected components. Each entry mat[i][j] is a rational number representing the probability that an unbounded random walk starting at vertex i will be absorbed by strongly connected component j – that is, the probability that it will reach the component and never leave.

Strongly connected components are indexed in the order given by DigraphStronglyConnectedComponents (5.4.11). See also DigraphRandomWalk (5.4.24) and DigraphAbsorptionExpectedSteps (5.4.26).

```
_____ Example _____
gap> gr := Digraph([[2, 3, 4], [3], [2], []]);
<immutable digraph with 4 vertices, 5 edges>
gap> DigraphStronglyConnectedComponents(gr).comps;
[ [ 2, 3 ], [ 4 ], [ 1 ] ]
gap> DigraphAbsorptionProbabilities(gr);
[ [ 2/3, 1/3, 0 ], [ 1, 0, 0 ], [ 1, 0, 0 ], [ 0, 1, 0 ] ]
```

5.4.26 DigraphAbsorptionExpectedSteps

▷ DigraphAbsorptionExpectedSteps(*digraph*)   (attribute)

Returns: A list of rational numbers.

A random walk of unbounded length through a finite digraph may pass through several different strongly connected components (SCCs). However, with probability 1 it must eventually reach an SCC which it can never leave, because the SCC has no out-edges leading to other SCCs. When this happens, we say the walk has been *absorbed*.

DigraphAbsorptionExpectedSteps returns a list L with length equal to the number of vertices of *digraph*, where L[v] is the expected number of steps a random walk starting at vertex v should take before it is absorbed – that is, the expected number of steps to reach an SCC that it can never leave.

See also DigraphRandomWalk (5.4.24) and DigraphAbsorptionProbabilities (5.4.25).

```
_____ Example _____
gap> gr := Digraph([[2], [3, 4], [], [2]]);
<immutable digraph with 4 vertices, 4 edges>
```

```
  gap> DigraphAbsorptionExpectedSteps(gr);
  [ 4, 3, 0, 4 ]
```

5.4.27 Dominators

▷ Dominators(*digraph, root*)   (operation)

    Returns: A list of lists.

    Dominators takes a `digraph` and a root `root` and returns the dominators of each vertex with respect to the root. The output is returned as a list of length `DigraphNrVertices(Digraph)`, whose `i`th entry is a list with the dominators of vertex `i` of the `digraph`. If there is no path from the root to a specific vertex, the output will contain a hole in the corresponding position. The *dominators* of a vertex u are the vertices that are contained in every path from the *root* to u, not including u itself. The method for this operation is an implementation of an algorithm by Thomas Lengauer and Robert Endre Tarjan [LT79]. The complexity of this algorithm is $O(mlogn)$ where m is the number of edges and n is the number of nodes in the subdigraph induced by the nodes in `digraph` reachable from `root`.

————— Example —————
```
  gap> D := Digraph([[2], [3, 6], [2, 4], [1], [], [3]]);
  <immutable digraph with 6 vertices, 7 edges>
  gap> Dominators(D, 1);
  [ , [ 1 ], [ 2, 1 ], [ 3, 2, 1 ],, [ 2, 1 ] ]
  gap> Dominators(D, 2);
  [ [ 4, 3, 2 ],, [ 2 ], [ 3, 2 ],, [ 2 ] ]
  gap> Dominators(D, 3);
  [ [ 4, 3 ], [ 3 ],, [ 3 ],, [ 2, 3 ] ]
  gap> Dominators(D, 4);
  [ [ 4 ], [ 1, 4 ], [ 2, 1, 4 ],,, [ 2, 1, 4 ] ]
  gap> Dominators(D, 5);
  [  ]
  gap> Dominators(D, 6);
  [ [ 4, 3, 6 ], [ 3, 6 ], [ 6 ], [ 3, 6 ] ]
```

5.4.28 DominatorTree

▷ DominatorTree(*digraph, root*)   (operation)

    Returns: A record.

    DominatorTree takes a `digraph` and a `root` vertex and returns a record with the following components:

idom

    the immediate dominators of the vertices with respect to the root.

preorder

    the preorder values of the vertices defined by the depth first search executed on the digraph.

The *immediate dominator* of a vertex u is the unique dominator of u that is dominated by all other dominators of u. The algorithm is an implementation of the fast algorithm written by Thomas Lengauer and Robert Endre Tarjan [LT79]. The complexity of this algorithm is $O(mlogn)$ where m is the number of edges and n is the number of nodes in the subdigraph induced by the nodes in `digraph` reachable from `root`.

```
                            ─── Example ───
 gap> D := Digraph([[2, 3], [4, 6], [4, 5], [3, 5], [1, 6], [2, 3]]);
 <immutable digraph with 6 vertices, 12 edges>
 gap> DominatorTree(D, 1);
 rec( idom := [ fail, 1, 1, 1, 1, 1 ],
   preorder := [ 1, 2, 4, 3, 5, 6 ] )
 gap> DominatorTree(D, 5);
 rec( idom := [ 5, 5, 5, 5, fail, 5 ],
   preorder := [ 5, 1, 2, 4, 3, 6 ] )
 gap> D := CompleteDigraph(5);
 <immutable complete digraph with 5 vertices>
 gap> DominatorTree(D, 1);
 rec( idom := [ fail, 1, 1, 1, 1 ], preorder := [ 1, 2, 3, 4, 5 ] )
 gap> DominatorTree(D, 2);
 rec( idom := [ 2, fail, 2, 2, 2 ], preorder := [ 2, 1, 3, 4, 5 ] )
```

5.4.29 IteratorOfPaths

▷ IteratorOfPaths(`digraph, u, v`)   (operation)

  Returns: An iterator.

  If `digraph` is a digraph or a list of adjacencies which defines a digraph - see `OutNeighbours` (5.2.6) - then this operation returns an iterator of the non-trivial directed paths (or directed cycles, in the case that $u = v$) in `digraph` from the vertex u to the vertex v.

  See DigraphPath (5.4.22) for more information about the representation of a directed path or directed cycle which is used, and see (**Reference: Iterators**) for more information about iterators. See Section 'Definitions' for the definition of a directed path and a directed cycle.

```
                            ─── Example ───
 gap> D := Digraph([[1, 4, 4, 2], [3, 5], [2, 3], [1, 2], [4]]);
 <immutable multidigraph with 5 vertices, 11 edges>
 gap> iter := IteratorOfPaths(D, 1, 4);
 <iterator>
 gap> NextIterator(iter);
 [ [ 1, 4 ], [ 2 ] ]
 gap> NextIterator(iter);
 [ [ 1, 4 ], [ 3 ] ]
 gap> NextIterator(iter);
 [ [ 1, 2, 5, 4 ], [ 4, 2, 1 ] ]
 gap> IsDoneIterator(iter);
 true
 gap> iter := IteratorOfPaths(D, 4, 3);
 <iterator>
 gap> NextIterator(iter);
 [ [ 4, 1, 2, 3 ], [ 1, 4, 1 ] ]
```

5.4.30 DigraphAllSimpleCircuits

▷ DigraphAllSimpleCircuits(`digraph`)   (attribute)

  Returns: A list of lists of vertices.

  If `digraph` is a digraph, then DigraphAllSimpleCircuits returns a list of the *simple circuits* in `digraph`.

See Section 1.1.1 for the definition of a simple circuit, and related notions. Note that a loop is a simple circuit.

For a digraph without multiple edges, a simple circuit is uniquely determined by its subsequence of vertices. However this is not the case for a multidigraph. The attribute `DigraphAllSimpleCircuits` ignores multiple edges, and identifies a simple circuit using only its subsequence of vertices. For example, although the simple circuits (v, e, v) and (v, e', v) (for distinct edges e and e') are mathematically distinct, `DigraphAllSimpleCircuits` considers them to be the same.

With this approach, a directed circuit of length `n` can be determined by a list of its first `n` vertices. Thus a simple circuit $(v_1, e_1, v_2, e_2, ..., e_{n-1}, v_n, e_{n+1}, v_1)$ can be represented as the list $[v_1, \ldots, v_n]$, or any cyclic permutation thereof. For each simple circuit of `digraph`, `DigraphAllSimpleCircuits(digraph)` includes precisely one such list to represent the circuit.

```
─────────────────────────── Example ───────────────────────────
gap> D := Digraph([[], [3], [2, 4], [5, 4], [4]]);
<immutable digraph with 5 vertices, 6 edges>
gap> DigraphAllSimpleCircuits(D);
[ [ 4 ], [ 4, 5 ], [ 2, 3 ] ]
gap> D := ChainDigraph(10);;
gap> DigraphAllSimpleCircuits(D);
[ ]
gap> D := Digraph([[3], [1], [1]]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphAllSimpleCircuits(D);
[ [ 1, 3 ] ]
gap> D := Digraph([[1, 1]]);
<immutable multidigraph with 1 vertex, 2 edges>
gap> DigraphAllSimpleCircuits(D);
[ [ 1 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphAllSimpleCircuits(D);
[ [ 1, 2, 3 ] ]
```

5.4.31 DigraphLongestSimpleCircuit

▷ DigraphLongestSimpleCircuit(`digraph`)   (attribute)

Returns: A list of vertices, or `fail`.

If `digraph` is a digraph, then `DigraphLongestSimpleCircuit` returns the longest *simple circuit* in `digraph`. See Section 1.1.1 for the definition of simple circuit, and the definition of length for a simple circuit.

This attribute computes `DigraphAllSimpleCircuits(digraph)` to find all the simple circuits of `digraph`, and returns one of maximal length. A simple circuit is represented as a list of vertices, in the same way as described in `DigraphAllSimpleCircuits` (5.4.30).

If `digraph` has no simple circuits, then this attribute returns `fail`. If `digraph` has multiple simple circuits of maximal length, then this attribute returns one of them.

```
─────────────────────────── Example ───────────────────────────
gap> D := Digraph([[], [3], [2, 4], [5, 4], [4]]);;
gap> DigraphLongestSimpleCircuit(D);
[ 4, 5 ]
gap> D := ChainDigraph(10);;
```

```
gap> DigraphLongestSimpleCircuit(D);
fail
gap> D := Digraph([[3], [1], [1, 4], [1, 1]]);;
gap> DigraphLongestSimpleCircuit(D);
[ 1, 3, 4 ]
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 4, 1);
<mutable digraph with 8 vertices, 24 edges>
gap> DigraphLongestSimpleCircuit(D);
[ 1, 2, 3, 4, 8, 7, 6, 5 ]
```

5.4.32 DigraphAllUndirectedSimpleCircuits

▷ DigraphAllUndirectedSimpleCircuits(`digraph`)   (attribute)

Returns: A list of lists of vertices.

If `digraph` is a digraph, then DigraphAllUndirectedSimpleCircuits returns a list of the *undirected simple circuits* in `digraph`.

See Section 1.1.1 for the definition of a simple circuit, and related notions. Note that a loop is a simple circuit. A simple circuit is undirected if the orientation of the edges in the circuit does not matter.

The attribute DigraphAllUndirectedSimpleCircuitss ignores multiple edges, and identifies a undirected simple circuit using only its subsequence of vertices. See Section 5.4.30 for more details on simple circuits.

```
─────────────── Example ───────────────
gap> D := ChainDigraph(10);;
gap> DigraphAllUndirectedSimpleCircuits(D);
[  ]
gap> D := Digraph([[1, 1]]);
<immutable multidigraph with 1 vertex, 2 edges>
gap> DigraphAllUndirectedSimpleCircuits(D);
[ [ 1 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphAllUndirectedSimpleCircuits(D);
[ [ 1, 2, 3 ] ]
gap> D := DigraphSymmetricClosure(CycleDigraph(IsMutableDigraph, 3));
<mutable digraph with 3 vertices, 6 edges>
gap> DigraphAllUndirectedSimpleCircuits(D);
[ [ 1, 2, 3 ] ]
```

5.4.33 DigraphAllChordlessCycles

▷ DigraphAllChordlessCycles(`digraph`)   (attribute)

Returns: A list of lists of vertices.

If `digraph` is a digraph, then DigraphAllChordlessCycles returns a list of the *chordless cycles* in `digraph`.

A chordless cycle *C* is a undirected simple circuit (see Section 1.1.1) where each pair of vertices in *C* are not connected by an edge not in *C*. Here, cycles of length two are ignored.

For a digraph without multiple edges, a simple circuit is uniquely determined by its subsequence of vertices. However this is not the case for a multidigraph. The attribute

DigraphAllChordlessCycles ignores multiple edges, and identifies a simple circuit using only its subsequence of vertices. See Section 5.4.30 for more details.

This method uses the algorithms described in [US14].

```
——————————————————— Example ———————————————————
gap> D := ChainDigraph(10);;
gap> DigraphAllChordlessCycles(D);
[  ]
gap> D := Digraph([[1, 1]]);
<immutable multidigraph with 1 vertex, 2 edges>
gap> DigraphAllChordlessCycles(D);
[  ]
gap> D := CycleDigraph(3);;
gap> DigraphAllChordlessCycles(D);
[ [ 2, 1, 3 ] ]
gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>
gap> DigraphAllChordlessCycles(D);
[ [ 2, 1, 3 ], [ 2, 1, 4 ], [ 3, 1, 4 ], [ 3, 2, 4 ] ]
```

5.4.34 DigraphLayers

▷ DigraphLayers(`digraph, vertex`)   (operation)

Returns: A list.

This operation returns a list `list` such that `list[i]` is the list of vertices whose minimum distance from the vertex `vertex` in `digraph` is `i - 1`. Vertex `vertex` is assumed to be at distance 0 from itself.

```
——————————————————— Example ———————————————————
gap> D := CompleteDigraph(4);;
gap> DigraphLayers(D, 1);
[ [ 1 ], [ 2, 3, 4 ] ]
```

5.4.35 DigraphDegeneracy

▷ DigraphDegeneracy(`digraph`)   (attribute)

Returns: A non-negative integer, or `fail`.

If `digraph` is a symmetric digraph without multiple edges - see IsSymmetricDigraph (6.2.14) and IsMultiDigraph (6.2.11) - then this attribute returns the degeneracy of `digraph`.

The degeneracy of a digraph is the least integer `k` such that every induced of `digraph` contains a vertex whose number of neighbours (excluding itself) is at most `k`. Note that this means that loops are ignored.

If `digraph` is not symmetric or has multiple edges then this attribute returns `fail`.

```
——————————————————— Example ———————————————————
gap> D := DigraphSymmetricClosure(ChainDigraph(5));;
gap> DigraphDegeneracy(D);
1
gap> D := CompleteDigraph(5);;
gap> DigraphDegeneracy(D);
4
gap> D := Digraph([[1], [2, 4, 5], [3, 4], [2, 3, 4], [2], []]);
<immutable digraph with 6 vertices, 10 edges>
```

```
gap> DigraphDegeneracy(D);
1
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 10, 3);
<mutable digraph with 20 vertices, 60 edges>
gap> DigraphDegeneracy(D);
3
```

5.4.36 DigraphDegeneracyOrdering

▷ DigraphDegeneracyOrdering(*digraph*)   (attribute)

    Returns: A list of integers, or `fail`.

    If *digraph* is a digraph for which `DigraphDegeneracy(digraph)` is a non-negative integer `k` - see `DigraphDegeneracy` (5.4.35) - then this attribute returns a degeneracy ordering of the vertices of the vertices of *digraph*.

    A degeneracy ordering of *digraph* is a list `ordering` of the vertices of *digraph* ordered such that for any position `i` of the list, the vertex `ordering[i]` has at most `k` neighbours in later position of the list.

    If `DigraphDegeneracy(digraph)` returns `fail`, then this attribute returns `fail`.

```
─────────────────────── Example ───────────────────────
gap> D := DigraphSymmetricClosure(ChainDigraph(5));;
gap> DigraphDegeneracyOrdering(D);
[ 5, 4, 3, 2, 1 ]
gap> D := CompleteDigraph(5);;
gap> DigraphDegeneracyOrdering(D);
[ 5, 4, 3, 2, 1 ]
gap> D := Digraph([[1], [2, 4, 5], [3, 4], [2, 3, 4], [2], []]);
<immutable digraph with 6 vertices, 10 edges>
gap> DigraphDegeneracyOrdering(D);
[ 1, 6, 5, 2, 4, 3 ]
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 3, 1);
<mutable digraph with 6 vertices, 18 edges>
gap> DigraphDegeneracyOrdering(D);
[ 6, 5, 4, 1, 3, 2 ]
```

5.4.37 HamiltonianPath

▷ HamiltonianPath(*digraph*)   (attribute)

    Returns: A list or `fail`.

    Returns a Hamiltonian path if one exists, `fail` if not.

    A *Hamiltonian path* of a digraph with n vertices is directed cycle of length n. If *digraph* is a digraph that contains a Hamiltonian path, then this function returns one, described in the form used by `DigraphAllSimpleCircuits` (5.4.30). Note if *digraph* has 0 or 1 vertices, then `HamiltonianPath` returns `[]` or `[1]`, respectively.

    The method used in this attribute has the same worst case complexity as `DigraphMonomorphism` (7.3.4).

```
─────────────────────── Example ───────────────────────
gap> D := Digraph([[]]);
<immutable empty digraph with 1 vertex>
gap> HamiltonianPath(D);
```

```
[ 1 ]
gap> D := Digraph([[2], [1]]);
<immutable digraph with 2 vertices, 2 edges>
gap> HamiltonianPath(D);
[ 1, 2 ]
gap> D := Digraph([[3], [], [2]]);
<immutable digraph with 3 vertices, 2 edges>
gap> HamiltonianPath(D);
fail
gap> D := Digraph([[2], [3], [1]]);
<immutable digraph with 3 vertices, 3 edges>
gap> HamiltonianPath(D);
[ 1, 2, 3 ]
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 5, 2);
<mutable digraph with 10 vertices, 30 edges>
gap> HamiltonianPath(D);
fail
```

5.4.38 NrSpanningTrees

▷ NrSpanningTrees(`digraph`)   (attribute)

    Returns: An integer.

    Returns the number of spanning trees of the symmetric digraph `digraph`. NrSpanningTrees will return an error if `digraph` is not a symmetric digraph.

    See IsSymmetricDigraph (6.2.14) and IsUndirectedSpanningTree (4.1.2) for more information.

```
_____ Example _____
gap> D := CompleteDigraph(5);
<immutable complete digraph with 5 vertices>
gap> NrSpanningTrees(D);
125
gap> D := DigraphSymmetricClosure(CycleDigraph(24));;
gap> NrSpanningTrees(D);
24
gap> NrSpanningTrees(EmptyDigraph(0));
0
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 9, 2);
<mutable digraph with 18 vertices, 54 edges>
gap> NrSpanningTrees(D);
1134225
```

5.4.39 DigraphDijkstra (for a source and target)

▷ DigraphDijkstra(`digraph, source, target`)   (operation)
▷ DigraphDijkstra(`digraph, source`)   (operation)

    Returns: Two lists.

    If `digraph` is a digraph and `source` and `target` are vertices of `digraph`, then DigraphDijkstra calculates the length of the shortest path from `source` to `target` and returns two lists. Each element of the first list is the distance of the corresponding element from `source`. If a vertex was not visited in the process of calculating the shortest distance to `target` or if there is no

path connecting that vertex with `source`, then the corresponding distance is `infinity`. Each element of the second list gives the previous vertex in the shortest path from `source` to the corresponding vertex. For `source` and for any vertices that remained unvisited this will be `-1`.

If the optional second argument `target` is not present, then `DigraphDijkstra` returns the shortest path from `source` to every vertex that is reachable from `source`.

```
_____ Example _____
gap> mat := [[0, 1, 1], [0, 0, 1], [0, 0, 0]];
[ [ 0, 1, 1 ], [ 0, 0, 1 ], [ 0, 0, 0 ] ]
gap> D := DigraphByAdjacencyMatrix(mat);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphDijkstra(D, 2, 3);
[ [ infinity, 0, 1 ], [ -1, -1, 2 ] ]
gap> DigraphDijkstra(D, 1, 3);
[ [ 0, 1, 1 ], [ -1, 1, 1 ] ]
gap> DigraphDijkstra(D, 1, 2);
[ [ 0, 1, 1 ], [ -1, 1, 1 ] ]
```

5.4.40 DigraphCycleBasis

▷ DigraphCycleBasis(*digraph*)   (operation)

Returns: A list and a list of GF(2) vectors

If *digraph* is a symmetric loopless digraph with no multiple edges, then `DigraphCycleBasis` calculates the *fundamental cycle basis* of *digraph* and returns a list of edges and a list of a basis vectors. If *digraph* contains a loop, this function will return an error. If *digraph* is a multi-digraph, then multiple edges incident to the same vertices are treated as a single edge by this function. See `IsSymmetricDigraph` (6.2.14), `DigraphHasLoops` (6.2.1), and `IsMultiDigraph` (6.2.11). The first list returned contains the out-neighbours that provide the ordering on the edges with the symmetric edges appearing only once; these are the same as *OutNeighbours(MaximalAntiSymmetricSubdigraph(G))*. See `MaximalAntiSymmetricSubdigraph` (3.3.6) `OutNeighbours` (5.2.6). The second list returned contains the basis vectors of the cycle space of the digraph. These vectors belongs to $(GF(2))^m$ where m is the length of the list of edges.

A graph is *eulerian* if every vertex has an even degree. A *cycle space* of a graph is a subspace of $(GF(2))^m$ representing the set of all *eulerian* subgraphs without isolated vertices. An eulerian subgraph without isolated vertices can be uniquely identified using the edges that is contains. Therefore, given some ordering on the edges of the graph, a subgraph can be represented by a vector in $(GF(2))^m$ where the i-th entry is 1 if the i-th edge is in the subgraph and 0 otherwise. In this function, the ordering of the edges is returned as the first list which corresponds to `OutNeighbours(MaximalAntiSymmetricSubdigraph(G))`. See `MaximalAntiSymmetricSubdigraph` (3.3.6) and `OutNeighbours` (5.2.6). The first basis vector for the complete digraph with 4 vertices shown below represents the edges [1, 2], [1, 3] and [2, 3] i.e. cycle subgraph between the vertices 1, 2 and 3 The cycle space is closed under the symmetric difference of the edges of the graph. This nicely corresponds to the addition of vectors in $(GF(2))^m$ which makes the vector space formulation of the cycle space very natural.

A *cycle basis* is a basis of the cycle space. A *fundamental cycle basis* is a special kind of basis of the cycle space where there is a specific spanning tree (forest, when the graph is disconnected) of the graph and each basis corresponds to the unique cycle created by adding an edge outside the spanning tree of the graph to the spanning tree. In this function, the spanning forest is rooted in the vertex with the smallest label for each connected component of the graph.

The fundamental cycle basis is unique up to reordering of the basis vectors. The number of basis vectors in the fundamental cycle basis is $m - n + c$, where m is the number of edges, n is the number of vertices, and c is the number of connected components. See `DigraphConnectedComponents` (5.4.9).

This function performs a depth first traversal of the input digraph with complexity $O(m + n)$ and the complexity of the computation of the basis is $O(m^2)$ where m is the number of edges in the input digraph.

```
_____ Example _____
  gap> D := CycleGraph(4);
  <immutable symmetric digraph with 4 vertices, 8 edges>
  gap> res := DigraphCycleBasis(D);
  [ [ [ 2, 4 ], [ 3 ], [ 4 ], [ ] ], [ <a GF2 vector of length 4> ] ]
  gap> List(res[2][1]);
  [ Z(2)^0, Z(2)^0, Z(2)^0, Z(2)^0 ]
  gap> D := CompleteDigraph(4);
  <immutable complete digraph with 4 vertices>
  gap> res := DigraphCycleBasis(D);
  [ [ [ 2, 3, 4 ], [ 3, 4 ], [ 4 ], [ ] ],
    [ <a GF2 vector of length 6>, <a GF2 vector of length 6>,
        <a GF2 vector of length 6> ] ]
  gap> List(res[2][1]);
  [ Z(2)^0, 0*Z(2), Z(2)^0, 0*Z(2), Z(2)^0, 0*Z(2) ]
  gap> List(res[2][2]);
  [ 0*Z(2), Z(2)^0, Z(2)^0, 0*Z(2), 0*Z(2), Z(2)^0 ]
  gap> List(res[2][3]);
  [ Z(2)^0, Z(2)^0, 0*Z(2), Z(2)^0, 0*Z(2), 0*Z(2) ]
```

5.5 Cayley graphs of groups

5.5.1 GroupOfCayleyDigraph

▷ GroupOfCayleyDigraph(*digraph*)   (attribute)

▷ SemigroupOfCayleyDigraph(*digraph*)   (attribute)

Returns: A group or semigroup.

If *digraph* is an immutable Cayley graph of a group G and *digraph* belongs to the category `IsCayleyDigraph` (3.1.4), then `GroupOfCayleyDigraph` returns G.

If *digraph* is a Cayley graph of a semigroup S and *digraph* belongs to the category `IsCayleyDigraph` (3.1.4), then `SemigroupOfCayleyDigraph` returns S.

See also `GeneratorsOfCayleyDigraph` (5.5.2).

```
_____ Example _____
  gap> G := DihedralGroup(IsPermGroup, 8);
  Group([ (1,2,3,4), (2,4) ])
  gap> digraph := CayleyDigraph(G);
  <immutable digraph with 8 vertices, 16 edges>
  gap> GroupOfCayleyDigraph(digraph) = G;
  true
```

5.5.2 GeneratorsOfCayleyDigraph

▷ GeneratorsOfCayleyDigraph(*digraph*)   (attribute)

Returns: A list of generators.

If `digraph` is an immutable Cayley graph of a group or semigroup with respect to a set of generators gens and `digraph` belongs to the category IsCayleyDigraph (3.1.4), then GeneratorsOfCayleyDigraph return the list of generators gens over which `digraph` is defined.

See also GroupOfCayleyDigraph (5.5.1) or SemigroupOfCayleyDigraph (5.5.1).

```
─────────────────────────── Example ───────────────────────────
gap> G := DihedralGroup(IsPermGroup, 8);
Group([ (1,2,3,4), (2,4) ])
gap> digraph := CayleyDigraph(G);
<immutable digraph with 8 vertices, 16 edges>
gap> GeneratorsOfCayleyDigraph(digraph) = GeneratorsOfGroup(G);
true
gap> digraph := CayleyDigraph(G, [()]);
<immutable digraph with 8 vertices, 8 edges>
gap> GeneratorsOfCayleyDigraph(digraph) = [()];
true
```

5.6 Associated semigroups

5.6.1 AsSemigroup (for a filter and a digraph)

▷ AsSemigroup(`filt, digraph`)   (operation)

▷ AsMonoid(`filt, digraph`)   (operation)

Returns: A semilattice of partial perms.

The operation AsSemigroup requires that `filt` be equal to IsPartialPermSemigroup (**Reference: IsPartialPermSemigroup**). If `digraph` is a IsJoinSemilatticeDigraph (6.4.3) or IsLatticeDigraph (6.4.3) then AsSemigroup returns a semigroup of partial perms which is isomorphic to the semigroup whose elements are the vertices of `digraph` with the binary operation PartialOrderDigraphJoinOfVertices (5.3.1). If `digraph` satisfies IsMeetSemilatticeDigraph (6.4.3) but not IsJoinSemilatticeDigraph (6.4.3) then AsSemigroup returns a semigroup of partial perms which is isomorphic to the semigroup whose elements are the vertices of `digraph` with the binary operation PartialOrderDigraphMeetOfVertices (5.3.1).

The operation AsMonoid behaves similarly to AsSemigroup except that `filt` may also be equal to IsPartialPermMonoid (**Reference: IsPartialPermMonoid**), `digraph` must satisfy IsLatticeDigraph (6.4.3), and the output satisfies IsMonoid (**Reference: IsMonoid**).

The output of both of these operations is guaranteed to be of minimal degree (see DegreeOfPartialPermSemigroup (**Reference: DegreeOfPartialPermSemigroup**)). Furthermore the GeneratorsOfSemigroup (**Reference: GeneratorsOfSemigroup**) of the output is guaranteed to be the unique generating set of minimal size.

```
─────────────────────────── Example ───────────────────────────
gap> di := Digraph([[1], [1, 2], [1, 3], [1, 4], [1, 2, 3, 5]]);
<immutable digraph with 5 vertices, 11 edges>
gap> S := AsSemigroup(IsPartialPermSemigroup, di);
<partial perm semigroup of rank 3 with 4 generators>
gap> ForAll(Elements(S), IsIdempotent);
true
gap> IsInverseSemigroup(S);
true
gap> Size(S);
```

```
 5
gap> di := Digraph([[1], [1, 2], [1, 2, 3]]);
<immutable digraph with 3 vertices, 6 edges>
gap> M := AsMonoid(IsPartialPermMonoid, di);
<partial perm monoid of rank 2 with 3 generators>
gap> Size(M);
 3
```

5.6.2 AsSemigroup (for a filter, semilattice digraph, and two lists)

▷ AsSemigroup(*filt, Y, gps, homs*)   (operation)

Returns: A Clifford semigroup of partial perms.

The operation AsSemigroup requires that *filt* be equal to IsPartialPermSemigroup (**Reference: IsPartialPermSemigroup**). If *Y* is a IsJoinSemilatticeDigraph (6.4.3) or IsMeetSemilatticeDigraph (6.4.3), *gps* is a list of groups corresponding to each vertex, and *homs* is a list containing for each edge (i, j) in the transitive reduction of *digraph* a triple [i, j, hom] where hom is a group homomorphism from gps[i] to gps[j], and the diagram of homomorphisms commutes, then AsSemigroup returns a semigroup of partial perms which is isomorphic to the strong semilattice of groups $S[Y; gps; homs]$.

```
                              ─── Example ───
  gap> G1 := AlternatingGroup(4);;
  gap> G2 := SymmetricGroup(2);;
  gap> G3 := SymmetricGroup(3);;
  gap> gr := Digraph([[1, 3], [2, 3], [3]]);;
  gap> sgn := function(x)
  > if SignPerm(x) = 1 then
  > return ();
  > fi;
  > return (1, 2);
  > end;;
  gap> hom13 := GroupHomomorphismByFunction(G1, G3, sgn);;
  gap> hom23 := GroupHomomorphismByFunction(G2, G3, sgn);;
  gap> T := AsSemigroup(IsPartialPermSemigroup,
  > gr,
  > [G1, G2, G3], [[1, 3, hom13], [2, 3, hom23]]);;
  gap> Size(T);
  20
  gap> D := GreensDClasses(T);;
  gap> List(D, Size);
  [ 6, 12, 2 ]
```

5.7 Planarity

5.7.1 KuratowskiPlanarSubdigraph

▷ KuratowskiPlanarSubdigraph(*digraph*)   (attribute)

Returns: A list or fail.

KuratowskiPlanarSubdigraph returns the immutable list of lists of out-neighbours of an induced subdigraph (excluding multiple edges and loops) of the digraph *digraph* that wit-

nesses the fact that `digraph` is not planar, or `fail` if `digraph` is planar. In other words, KuratowskiPlanarSubdigraph returns the out-neighbours of a subdigraph of `digraph` that is homeomorphic to the complete graph with 5 vertices, or to the complete bipartite graph with vertex sets of sizes 3 and 3.

The directions and multiplicities of any edges in `digraph` are ignored when considering whether or not `digraph` is planar.

See also IsPlanarDigraph (6.7.1) and SubdigraphHomeomorphicToK33 (5.7.5).

This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

```
_____ Example _____
gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6],
> [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<immutable digraph with 11 vertices, 25 edges>
gap> KuratowskiPlanarSubdigraph(D);
fail
gap> D := Digraph([[2, 4, 7, 9, 10], [1, 3, 4, 6, 9, 10], [6, 10],
> [2, 5, 8, 9], [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
> [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<immutable digraph with 10 vertices, 50 edges>
gap> IsPlanarDigraph(D);
false
gap> KuratowskiPlanarSubdigraph(D);
[ [ 2, 9, 7 ], [ 1, 3 ], [ 6 ], [ 5, 9 ], [ 6, 4 ], [ 3, 5 ], [ 4 ],
  [ 7, 9, 3 ], [ 8 ], [  ] ]
gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
> [3, 6], [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<mutable digraph with 11 vertices, 25 edges>
gap> KuratowskiPlanarSubdigraph(D);
fail
gap> D := Digraph(IsMutableDigraph, [[2, 4, 7, 9, 10],
> [1, 3, 4, 6, 9, 10], [6, 10], [2, 5, 8, 9],
> [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
> [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<mutable digraph with 10 vertices, 50 edges>
gap> IsPlanarDigraph(D);
false
gap> KuratowskiPlanarSubdigraph(D);
[ [ 2, 9, 7 ], [ 1, 3 ], [ 6 ], [ 5, 9 ], [ 6, 4 ], [ 3, 5 ], [ 4 ],
  [ 7, 9, 3 ], [ 8 ], [  ] ]
```

5.7.2 KuratowskiOuterPlanarSubdigraph

▷ KuratowskiOuterPlanarSubdigraph(*digraph*)   (attribute)

Returns: A list or `fail`.

KuratowskiOuterPlanarSubdigraph returns the immutable list of immutable lists of out-neighbours of an induced subdigraph (excluding multiple edges and loops) of the digraph `digraph` that witnesses the fact that `digraph` is not outer planar, or `fail` if `digraph` is outer planar. In other words, KuratowskiOuterPlanarSubdigraph returns the out-neighbours of a subdigraph of `digraph` that is homeomorphic to the complete graph with 4 vertices, or to the complete bipartite graph with vertex sets of sizes 2 and 3.

The directions and multiplicities of any edges in `digraph` are ignored when considering whether or not `digraph` is outer planar.

See also `IsOuterPlanarDigraph` (6.7.2), `SubdigraphHomeomorphicToK4` (5.7.5), and `SubdigraphHomeomorphicToK23` (5.7.5).

This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

```
 ───────────────────────── Example ─────────────────────────
 gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6],
 > [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
 <immutable digraph with 11 vertices, 25 edges>
 gap> KuratowskiOuterPlanarSubdigraph(D);
 [ [ 3, 5, 10 ], [ 9, 8, 10 ], [ 4, 1 ], [ 6, 3 ], [ 1, 11 ],
   [ 7, 4 ], [ 8, 6 ], [ 2, 7 ], [ 11, 2 ], [ 2, 1 ], [ 5, 9 ] ]
 gap> D := Digraph([[2, 4, 7, 9, 10], [1, 3, 4, 6, 9, 10], [6, 10],
 > [2, 5, 8, 9], [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
 > [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
 <immutable digraph with 10 vertices, 50 edges>
 gap> IsOuterPlanarDigraph(D);
 false
 gap> KuratowskiOuterPlanarSubdigraph(D);
 [ [  ], [  ], [  ], [ 8, 9 ], [  ], [  ], [ 9, 4 ], [ 7, 9, 4 ],
   [ 8, 7 ], [  ] ]
 gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
 > [3, 6], [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
 <mutable digraph with 11 vertices, 25 edges>
 gap> KuratowskiOuterPlanarSubdigraph(D);
 [ [ 3, 5, 10 ], [ 9, 8, 10 ], [ 4, 1 ], [ 6, 3 ], [ 1, 11 ],
   [ 7, 4 ], [ 8, 6 ], [ 2, 7 ], [ 11, 2 ], [ 2, 1 ], [ 5, 9 ] ]
 gap> D := Digraph(IsMutableDigraph, [[2, 4, 7, 9, 10],
 > [1, 3, 4, 6, 9, 10], [6, 10], [2, 5, 8, 9],
 > [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
 > [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
 <mutable digraph with 10 vertices, 50 edges>
 gap> IsOuterPlanarDigraph(D);
 false
 gap> KuratowskiOuterPlanarSubdigraph(D);
 [ [  ], [  ], [  ], [ 8, 9 ], [  ], [  ], [ 9, 4 ], [ 7, 9, 4 ],
   [ 8, 7 ], [  ] ]
```

5.7.3 PlanarEmbedding

▷ PlanarEmbedding(*digraph*)   (attribute)

Returns: A list or `fail`.

If `digraph` is a planar digraph, then `PlanarEmbedding` returns the immutable list of lists of out-neighbours of `digraph` (excluding multiple edges and loops) such that each vertex's neighbours are given in clockwise order. If `digraph` is not planar, then `fail` is returned.

The directions and multiplicities of any edges in `digraph` are ignored by `PlanarEmbedding`.

See also `IsPlanarDigraph` (6.7.1).

This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

```
                              ─── Example ───
gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6],
> [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<immutable digraph with 11 vertices, 25 edges>
gap> PlanarEmbedding(D);
[ [ 3, 10, 5 ], [ 10, 8, 9 ], [ 4, 1 ], [ 6, 3 ], [ 1, 11, 7 ],
  [ 7, 4 ], [ 8, 6 ], [ 7, 2 ], [ 2, 11 ], [ 1, 2 ], [ 9, 5 ] ]
gap> D := Digraph([[2, 4, 7, 9, 10], [1, 3, 4, 6, 9, 10], [6, 10],
> [2, 5, 8, 9], [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
> [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<immutable digraph with 10 vertices, 50 edges>
gap> PlanarEmbedding(D);
fail
gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
> [3, 6], [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<mutable digraph with 11 vertices, 25 edges>
gap> PlanarEmbedding(D);
[ [ 3, 10, 5 ], [ 10, 8, 9 ], [ 4, 1 ], [ 6, 3 ], [ 1, 11, 7 ],
  [ 7, 4 ], [ 8, 6 ], [ 7, 2 ], [ 2, 11 ], [ 1, 2 ], [ 9, 5 ] ]
gap> D := Digraph(IsMutableDigraph, [[2, 4, 7, 9, 10],
> [1, 3, 4, 6, 9, 10], [6, 10], [2, 5, 8, 9],
> [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
> [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<mutable digraph with 10 vertices, 50 edges>
gap> PlanarEmbedding(D);
fail
```

5.7.4 OuterPlanarEmbedding

▷ OuterPlanarEmbedding(*digraph*)   (attribute)

   Returns: A list or `fail`.

   If *digraph* is an outer planar digraph, then `OuterPlanarEmbedding` returns the immutable list of lists of out-neighbours of *digraph* (excluding multiple edges and loops) such that each vertex's neighbours are given in clockwise order. If *digraph* is not outer planar, then `fail` is returned.

   The directions and multiplicities of any edges in *digraph* are ignored by `OuterPlanarEmbedding`.

   See also `IsOuterPlanarDigraph` (6.7.2).

   This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

```
                              ─── Example ───
gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6],
> [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<immutable digraph with 11 vertices, 25 edges>
gap> OuterPlanarEmbedding(D);
fail
gap> D := Digraph([[2, 4, 7, 9, 10], [1, 3, 4, 6, 9, 10], [6, 10],
> [2, 5, 8, 9], [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
> [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<immutable digraph with 10 vertices, 50 edges>
gap> OuterPlanarEmbedding(D);
fail
```

```
gap> OuterPlanarEmbedding(CompleteBipartiteDigraph(2, 2));
[ [ 3, 4 ], [ 4, 3 ], [ 2, 1 ], [ 1, 2 ] ]
gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
> [3, 6], [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<mutable digraph with 11 vertices, 25 edges>
gap> OuterPlanarEmbedding(D);
fail
gap> D := Digraph(IsMutableDigraph, [[2, 4, 7, 9, 10],
> [1, 3, 4, 6, 9, 10], [6, 10], [2, 5, 8, 9],
> [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
> [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<mutable digraph with 10 vertices, 50 edges>
gap> OuterPlanarEmbedding(D);
fail
gap> OuterPlanarEmbedding(CompleteBipartiteDigraph(2, 2));
[ [ 3, 4 ], [ 4, 3 ], [ 2, 1 ], [ 1, 2 ] ]
```

5.7.5 SubdigraphHomeomorphicToK23

▷ SubdigraphHomeomorphicToK23(*digraph*)   (attribute)
▷ SubdigraphHomeomorphicToK33(*digraph*)   (attribute)
▷ SubdigraphHomeomorphicToK4(*digraph*)   (attribute)

    Returns: A list or `fail`.

    These attributes return the immutable list of lists of out-neighbours of a subdigraph of the digraph *digraph* which is homeomorphic to one of the following: the complete bipartite graph with vertex sets of sizes 2 and 3; the complete bipartite graph with vertex sets of sizes 3 and 3; or the complete graph with 4 vertices. If *digraph* has no such subdigraphs, then `fail` is returned.

    See also `IsPlanarDigraph` (6.7.1) and `IsOuterPlanarDigraph` (6.7.2) for more details.

    This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

```
_____ Example _____
gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6], [1, 7, 11],
> [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<immutable digraph with 11 vertices, 25 edges>
gap> SubdigraphHomeomorphicToK4(D);
[ [ 3, 5, 10 ], [ 9, 8, 10 ], [ 4, 1 ], [ 6, 3 ], [ 1, 7, 11 ],
  [ 7, 4 ], [ 8, 6 ], [ 2, 7 ], [ 11, 2 ], [ 2, 1 ], [ 5, 9 ] ]
gap> SubdigraphHomeomorphicToK23(D);
[ [ 3, 5, 10 ], [ 9, 8, 10 ], [ 4, 1 ], [ 6, 3 ], [ 1, 11 ],
  [ 7, 4 ], [ 8, 6 ], [ 2, 7 ], [ 11, 2 ], [ 2, 1 ], [ 5, 9 ] ]
gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6], [1, 11],
>  [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<immutable digraph with 11 vertices, 24 edges>
gap> SubdigraphHomeomorphicToK4(D);
fail
gap> SubdigraphHomeomorphicToK23(D);
[ [ 3, 10, 5 ], [ 10, 8, 9 ], [ 4, 1 ], [ 6, 3 ], [ 11, 1 ],
  [ 7, 4 ], [ 8, 6 ], [ 2, 7 ], [ 2, 11 ], [ 1, 2 ], [ 9, 5 ] ]
gap> SubdigraphHomeomorphicToK33(D);
fail
gap> SubdigraphHomeomorphicToK23(NullDigraph(0));
```

```
fail
gap> SubdigraphHomeomorphicToK33(CompleteDigraph(5));
fail
gap> SubdigraphHomeomorphicToK33(CompleteBipartiteDigraph(3, 3));
[ [ 4, 6, 5 ], [ 4, 5, 6 ], [ 6, 5, 4 ], [ 1, 2, 3 ], [ 3, 2, 1 ],
  [ 2, 3, 1 ] ]
gap> SubdigraphHomeomorphicToK4(CompleteDigraph(3));
fail
gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
> [3, 6], [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<mutable digraph with 11 vertices, 25 edges>
gap> SubdigraphHomeomorphicToK4(D);
[ [ 3, 5, 10 ], [ 9, 8, 10 ], [ 4, 1 ], [ 6, 3 ], [ 1, 7, 11 ],
  [ 7, 4 ], [ 8, 6 ], [ 2, 7 ], [ 11, 2 ], [ 2, 1 ], [ 5, 9 ] ]
gap> SubdigraphHomeomorphicToK23(D);
[ [ 3, 5, 10 ], [ 9, 8, 10 ], [ 4, 1 ], [ 6, 3 ], [ 1, 11 ],
  [ 7, 4 ], [ 8, 6 ], [ 2, 7 ], [ 11, 2 ], [ 2, 1 ], [ 5, 9 ] ]
gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
> [3, 6], [1, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<mutable digraph with 11 vertices, 24 edges>
gap> SubdigraphHomeomorphicToK4(D);
fail
gap> SubdigraphHomeomorphicToK23(D);
[ [ 3, 10, 5 ], [ 10, 8, 9 ], [ 4, 1 ], [ 6, 3 ], [ 11, 1 ],
  [ 7, 4 ], [ 8, 6 ], [ 2, 7 ], [ 2, 11 ], [ 1, 2 ], [ 9, 5 ] ]
gap> SubdigraphHomeomorphicToK33(D);
fail
gap> SubdigraphHomeomorphicToK23(NullDigraph(0));
fail
gap> SubdigraphHomeomorphicToK33(CompleteDigraph(5));
fail
gap> SubdigraphHomeomorphicToK33(CompleteBipartiteDigraph(3, 3));
[ [ 4, 6, 5 ], [ 4, 5, 6 ], [ 6, 5, 4 ], [ 1, 2, 3 ], [ 3, 2, 1 ],
  [ 2, 3, 1 ] ]
gap> SubdigraphHomeomorphicToK4(CompleteDigraph(3));
fail
```

5.8 Hashing

5.8.1 DigraphHash

▷ DigraphHash(*digraph*) (attribute)

Returns: The hash of a digraph.

This function returns a hash of *digraph*.

Note that the underlying hashing function is system dependent, and so the value of this attribute is not guaranteed to be the same on different systems.

Chapter 6

Properties of digraphs

6.1 Vertex properties

6.1.1 DigraphHasAVertex

▷ DigraphHasAVertex(*digraph*) (property)

    Returns: `true` or `false`.

    Returns `true` if the digraph *digraph* has at least one vertex, and `false` if it does not.

    See also `DigraphHasNoVertices` (6.1.2).

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
───────────────── Example ─────────────────
gap> D := Digraph([]);
<immutable empty digraph with 0 vertices>
gap> DigraphHasAVertex(D);
false
gap> D := Digraph([[]]);
<immutable empty digraph with 1 vertex>
gap> DigraphHasAVertex(D);
true
gap> D := Digraph([[], [1]]);
<immutable digraph with 2 vertices, 1 edge>
gap> DigraphHasAVertex(D);
true
```

6.1.2 DigraphHasNoVertices

▷ DigraphHasNoVertices(*digraph*) (property)

    Returns: `true` or `false`.

    Returns `true` if the digraph *digraph* is the unique digraph with zero vertices, and `false` otherwise.

    See also `DigraphHasAVertex` (6.1.1).

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
───────────────── Example ─────────────────
gap> D := Digraph([]);
<immutable empty digraph with 0 vertices>
```

```
gap> DigraphHasNoVertices(D);
true
gap> D := Digraph([[]]);
<immutable empty digraph with 1 vertex>
gap> DigraphHasNoVertices(D);
false
gap> D := Digraph([[], [1]]);
<immutable digraph with 2 vertices, 1 edge>
gap> DigraphHasNoVertices(D);
false
```

6.2 Edge properties

6.2.1 DigraphHasLoops

▷ DigraphHasLoops(*digraph*) (property)

Returns: true or false.

Returns true if the digraph *digraph* has loops, and false if it does not. A loop is an edge with equal source and range.

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
———————————————————— Example ————————————————————
gap> D := Digraph([[1, 2], [2]]);
<immutable digraph with 2 vertices, 3 edges>
gap> DigraphEdges(D);
[ [ 1, 1 ], [ 1, 2 ], [ 2, 2 ] ]
gap> DigraphHasLoops(D);
true
gap> D := Digraph([[2, 3], [1], [2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphEdges(D);
[ [ 1, 2 ], [ 1, 3 ], [ 2, 1 ], [ 3, 2 ] ]
gap> DigraphHasLoops(D);
false
gap> D := CompleteDigraph(IsMutableDigraph, 4);
<mutable digraph with 4 vertices, 12 edges>
gap> DigraphHasLoops(D);
false
```

6.2.2 IsAntiSymmetricDigraph

▷ IsAntiSymmetricDigraph(*digraph*) (property)
▷ IsAntisymmetricDigraph(*digraph*) (property)

Returns: true or false.

This property is true if the digraph *digraph* is antisymmetric, and false if it is not.

A digraph is *antisymmetric* if whenever there is an edge with source u and range v, and an edge with source v and range u, then the vertices u and v are equal.

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
────────────────────────── Example ──────────────────────────
  gap> gr1 := Digraph([[2], [1, 3], [2, 3]]);
  <immutable digraph with 3 vertices, 5 edges>
  gap> IsAntisymmetricDigraph(gr1);
  false
  gap> DigraphEdges(gr1){[1, 2]};
  [ [ 1, 2 ], [ 2, 1 ] ]
  gap> gr2 := Digraph([[1, 2], [3, 3], [1]]);
  <immutable multidigraph with 3 vertices, 5 edges>
  gap> IsAntisymmetricDigraph(gr2);
  true
  gap> DigraphEdges(gr2);
  [ [ 1, 1 ], [ 1, 2 ], [ 2, 3 ], [ 2, 3 ], [ 3, 1 ] ]
```

6.2.3 IsBipartiteDigraph

▷ IsBipartiteDigraph(*digraph*)   (property)

    Returns: `true` or `false`.

    This property is `true` if the digraph *digraph* is bipartite, and `false` if it is not. A digraph is bipartite if and only if the vertices of *digraph* can be partitioned into two non-empty sets such that the source and range of any edge of *digraph* lie in distinct sets. Equivalently, a digraph is bipartite if and only if it is 2-colorable; see `DigraphGreedyColouring` (7.3.16).

    See also `DigraphBicomponents` (5.4.13).

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
────────────────────────── Example ──────────────────────────
  gap> D := ChainDigraph(4);
  <immutable chain digraph with 4 vertices>
  gap> IsBipartiteDigraph(D);
  true
  gap> D := CycleDigraph(3);
  <immutable cycle digraph with 3 vertices>
  gap> IsBipartiteDigraph(D);
  false
  gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
  <mutable digraph with 9 vertices, 40 edges>
  gap> IsBipartiteDigraph(D);
  true
```

6.2.4 IsCompleteBipartiteDigraph

▷ IsCompleteBipartiteDigraph(*digraph*)   (property)

    Returns: `true` or `false`.

    Returns `true` if the digraph *digraph* is a complete bipartite digraph, and `false` if it is not.

    A digraph is a *complete bipartite digraph* if it is bipartite, see `IsBipartiteDigraph` (6.2.3), and there exists a unique edge with source i and range j if and only if i and j lie in different bicomponents of *digraph*, see `DigraphBicomponents` (5.4.13).

    Equivalently, a bipartite digraph with bicomponents of size m and n is complete precisely when it has $2mn$ edges, none of which are multiple edges.

See also CompleteBipartiteDigraph (3.5.12).

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
─────────────────────────── Example ───────────────────────────
gap> D := CycleDigraph(2);
<immutable cycle digraph with 2 vertices>
gap> IsCompleteBipartiteDigraph(D);
true
gap> D := CycleDigraph(4);
<immutable cycle digraph with 4 vertices>
gap> IsBipartiteDigraph(D);
true
gap> IsCompleteBipartiteDigraph(D);
false
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
<mutable digraph with 9 vertices, 40 edges>
gap> IsCompleteBipartiteDigraph(D);
true
```

6.2.5 IsCompleteDigraph

▷ IsCompleteDigraph(*digraph*)   (property)

Returns: `true` or `false`.

Returns `true` if the digraph `digraph` is complete, and `false` if it is not.

A digraph is *complete* if it has no loops, and for all *distinct* vertices i and j, there is exactly one edge with source i and range j. Equivalently, a digraph with n vertices is complete precisely when it has $n(n-1)$ edges, no loops, and no multiple edges.

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
─────────────────────────── Example ───────────────────────────
gap> D := Digraph([[2, 3], [1, 3], [1, 2]]);
<immutable digraph with 3 vertices, 6 edges>
gap> IsCompleteDigraph(D);
true
gap> D := Digraph([[2, 2], [1]]);
<immutable multidigraph with 2 vertices, 3 edges>
gap> IsCompleteDigraph(D);
false
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
<mutable digraph with 9 vertices, 40 edges>
gap> IsCompleteDigraph(D);
false
```

6.2.6 IsCompleteMultipartiteDigraph

▷ IsCompleteMultipartiteDigraph(*digraph*)   (property)

Returns: `true` or `false`.

This property returns `true` if `digraph` is a complete multipartite digraph, and `false` if not.

A digraph is a *complete multipartite digraph* if and only if its vertices can be partitioned into at least two maximal independent sets, where every possible edge between these independent sets occurs in the digraph exactly once.

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
──────────────────────── Example ────────────────────────
  gap> D := CompleteMultipartiteDigraph([2, 4, 6]);
  <immutable complete multipartite digraph with 12 vertices, 88 edges>
  gap> IsCompleteMultipartiteDigraph(D);
  true
  gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
  <mutable digraph with 9 vertices, 40 edges>
  gap> IsCompleteMultipartiteDigraph(D);
  true
```

6.2.7 IsEmptyDigraph

▷ IsEmptyDigraph(`digraph`)   (property)
▷ IsNullDigraph(`digraph`)   (property)
     Returns: `true` or `false`.

Returns `true` if the digraph `digraph` is empty, and `false` if it is not. A digraph is *empty* if it has no edges.

IsNullDigraph is a synonym for IsEmptyDigraph. See also IsNonemptyDigraph (6.2.12).

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
──────────────────────── Example ────────────────────────
  gap> D := Digraph([[], []]);
  <immutable empty digraph with 2 vertices>
  gap> IsEmptyDigraph(D);
  true
  gap> IsNullDigraph(D);
  true
  gap> D := Digraph([[], [1]]);
  <immutable digraph with 2 vertices, 1 edge>
  gap> IsEmptyDigraph(D);
  false
  gap> IsNullDigraph(D);
  false
```

6.2.8 IsEquivalenceDigraph

▷ IsEquivalenceDigraph(`digraph`)   (property)
     Returns: `true` or `false`.

A digraph is an equivalence digraph if and only if the digraph satisfies all of IsReflexiveDigraph (6.2.13), IsSymmetricDigraph (6.2.14) and IsTransitiveDigraph (6.2.16). A partial order `digraph` corresponds to an equivalence relation.

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
                            Example
gap> D := Digraph([[1, 3], [2], [1, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsEquivalenceDigraph(D);
true
```

6.2.9 IsFunctionalDigraph

▷ IsFunctionalDigraph(*digraph*)   (property)

    Returns: true or false.

    This property is true if the digraph *digraph* is functional.

    A digraph is *functional* if every vertex is the source of a unique edge.

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
                            Example
gap> gr1 := Digraph([[3], [2], [2], [1], [6], [5]]);
<immutable digraph with 6 vertices, 6 edges>
gap> IsFunctionalDigraph(gr1);
true
gap> gr2 := Digraph([[1, 2], [1]]);
<immutable digraph with 2 vertices, 3 edges>
gap> IsFunctionalDigraph(gr2);
false
gap> gr3 := Digraph(3, [1, 2, 3], [2, 3, 1]);
<immutable digraph with 3 vertices, 3 edges>
gap> IsFunctionalDigraph(gr3);
true
```

6.2.10 IsPermutationDigraph

▷ IsPermutationDigraph(*digraph*)   (property)

    Returns: true or false.

    This property is true if the digraph *digraph* is functional and each node has only one in-neighbour.

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
                            Example
gap> gr1 := Digraph([[3], [2], [2], [1], [6], [5]]);
<immutable digraph with 6 vertices, 6 edges>
gap> IsPermutationDigraph(gr1);
false
gap> gr2 := Digraph([[1, 2], [1]]);
<immutable digraph with 2 vertices, 3 edges>
gap> IsPermutationDigraph(gr2);
false
gap> gr3 := Digraph(3, [1, 2, 3], [2, 3, 1]);
<immutable digraph with 3 vertices, 3 edges>
gap> IsPermutationDigraph(gr3);
true
```

6.2.11 IsMultiDigraph

▷ IsMultiDigraph(`digraph`) (property)

    Returns: `true` or `false`.

    A *multidigraph* is one that has at least two edges with equal source and range.

    If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
——————————————— Example ———————————————
gap> D := Digraph(["a", "b", "c"], ["a", "b", "b"], ["b", "c", "a"]);
<immutable digraph with 3 vertices, 3 edges>
gap> IsMultiDigraph(D);
false
gap> D := DigraphFromDigraph6String("&Bug");
<immutable digraph with 3 vertices, 6 edges>
gap> IsDuplicateFree(DigraphEdges(D));
true
gap> IsMultiDigraph(D);
false
gap> D := Digraph([[1, 2, 3, 2], [2, 1], [3]]);
<immutable multidigraph with 3 vertices, 7 edges>
gap> IsDuplicateFree(DigraphEdges(D));
false
gap> IsMultiDigraph(D);
true
gap> D := DigraphMutableCopy(D);
<mutable multidigraph with 3 vertices, 7 edges>
gap> IsMultiDigraph(D);
true
```

6.2.12 IsNonemptyDigraph

▷ IsNonemptyDigraph(`digraph`) (property)

    Returns: `true` or `false`.

    Returns `true` if the digraph `digraph` is nonempty, and `false` if it is not. A digraph is *nonempty* if it has at least one edge.

    See also `IsEmptyDigraph` (6.2.7).

    If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
——————————————— Example ———————————————
gap> D := Digraph([[], []]);
<immutable empty digraph with 2 vertices>
gap> IsNonemptyDigraph(D);
false
gap> D := Digraph([[], [1]]);
<immutable digraph with 2 vertices, 1 edge>
gap> IsNonemptyDigraph(D);
true
```

6.2.13   IsReflexiveDigraph

▷ IsReflexiveDigraph(*digraph*)   (property)

   Returns: `true` or `false`.

   This property is `true` if the digraph `digraph` is reflexive, and `false` if it is not. A digraph is *reflexive* if it has a loop at every vertex.

   If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
                            ——— Example ———
gap> D := Digraph([[1, 2], [2]]);
<immutable digraph with 2 vertices, 3 edges>
gap> IsReflexiveDigraph(D);
true
gap> D := Digraph([[3, 1], [4, 2], [3], [2, 1]]);
<immutable digraph with 4 vertices, 7 edges>
gap> IsReflexiveDigraph(D);
false
```

6.2.14   IsSymmetricDigraph

▷ IsSymmetricDigraph(*digraph*)   (property)

   Returns: `true` or `false`.

   This property is `true` if the digraph `digraph` is symmetric, and `false` if it is not.

   A *symmetric digraph* is one where for each non-loop edge, having source u and range v, there is a corresponding edge with source v and range u. If there are n edges with source u and range v, then there must be precisely n edges with source v and range u. In other words, a symmetric digraph has a symmetric adjacency matrix `AdjacencyMatrix` (5.2.1).

   If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
                            ——— Example ———
gap> gr1 := Digraph([[2], [1, 3], [2, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsSymmetricDigraph(gr1);
true
gap> adj1 := AdjacencyMatrix(gr1);;
gap> Display(adj1);
[ [  0,  1,  0 ],
  [  1,  0,  1 ],
  [  0,  1,  1 ] ]
gap> adj1 = TransposedMat(adj1);
true
gap> gr1 = DigraphReverse(gr1);
true
gap> gr2 := Digraph([[2, 3], [1, 3], [2, 3]]);
<immutable digraph with 3 vertices, 6 edges>
gap> IsSymmetricDigraph(gr2);
false
gap> adj2 := AdjacencyMatrix(gr2);;
gap> Display(adj2);
[ [  0,  1,  1 ],
  [  1,  0,  1 ],
```

```
   [ 0,  1,  1 ] ]
gap> adj2 = TransposedMat(adj2);
false
```

6.2.15 IsTournament

▷ IsTournament(*digraph*)   (property)

    Returns: true or false.

    This property is true if the digraph *digraph* is a tournament, and false if it is not.

    A tournament is an orientation of a complete (undirected) graph. Specifically, a tournament is a digraph which has a unique directed edge (of some orientation) between any pair of distinct vertices, and no loops.

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
 ─────────────────────────────── Example ───────────────────────────────
gap> D := Digraph([[2, 3, 4], [3, 4], [4], []]);
<immutable digraph with 4 vertices, 6 edges>
gap> IsTournament(D);
true
gap> D := Digraph([[2], [1], [3]]);
<immutable digraph with 3 vertices, 3 edges>
gap> IsTournament(D);
false
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> IsTournament(D);
true
gap> DigraphRemoveEdge(D, 1, 2);
<mutable digraph with 3 vertices, 2 edges>
gap> IsTournament(D);
false
```

6.2.16 IsTransitiveDigraph

▷ IsTransitiveDigraph(*digraph*)   (property)

    Returns: true or false.

    This property is true if the digraph *digraph* is transitive, and false if it is not. A digraph is *transitive* if whenever [i, j] and [j, k] are edges of the digraph, then [i, k] is also an edge of the digraph.

    Let n be the number of vertices of an arbitrary digraph, and let m be the number of edges. For general digraphs, the methods used for this property use a version of the Floyd-Warshall algorithm, and have complexity $O(n^3)$. However for digraphs which are topologically sortable [DigraphTopologicalSort (5.1.10)], then methods with complexity $O(m + n + m \cdot n)$ will be used when appropriate.

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
 ─────────────────────────────── Example ───────────────────────────────
gap> D := Digraph([[1, 2], [3], [3]]);
<immutable digraph with 3 vertices, 4 edges>
```

```
gap> IsTransitiveDigraph(D);
false
gap> gr2 := Digraph([[1, 2, 3], [3], [3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsTransitiveDigraph(gr2);
true
gap> gr2 = DigraphTransitiveClosure(D);
true
gap> gr3 := Digraph([[1, 2, 2, 3], [3, 3], [3]]);
<immutable multidigraph with 3 vertices, 7 edges>
gap> IsTransitiveDigraph(gr3);
true
```

6.3 Edge Weights

6.3.1 EdgeWeights

▷ EdgeWeights(*digraph*)   (attribute)
▷ EdgeWeightsMutableCopy(*digraph*)   (operation)

Returns: A list of lists of integers, floats or rationals.

EdgeWeights returns the list of lists of edge weights of the edges of the digraph *digraph*.

More specifically, `weights[i][j]` is the weight given to the jth edge from vertex i, according to the ordering of edges given by `OutNeighbours(digraph)[i]`.

The function EdgeWeights returns an immutable list of immutable lists, whereas the function EdgeWeightsMutableCopy returns a copy of EdgeWeights which is a mutable list of mutable lists.

The edge weights of a digraph cannot be computed and must be set either using SetEdgeWeights or EdgeWeightedDigraph (6.3.2).

```
                            —— Example ——
 gap> gr := EdgeWeightedDigraph([[2], [3], [1]], [[5], [10], [15]]);
 <immutable digraph with 3 vertices, 3 edges>
 gap> EdgeWeights(gr);
 [ [ 5 ], [ 10 ], [ 15 ] ]
 gap> a := EdgeWeightsMutableCopy(gr);
 [ [ 5 ], [ 10 ], [ 15 ] ]
 gap> a[1][1] := 100;
 100
 gap> a;
 [ [ 100 ], [ 10 ], [ 15 ] ]
 gap> b := EdgeWeights(gr);
 [ [ 5 ], [ 10 ], [ 15 ] ]
 gap> b[1][1] := 534;
 Error, List Assignment: <list> must be a mutable list
```

6.3.2 EdgeWeightedDigraph

▷ EdgeWeightedDigraph(*digraph, weights*)   (function)

Returns: A digraph or fail

The argument *digraph* may be a digraph or a list of lists of integers, floats or rationals.

`weights` must be a list of lists of integers, floats or rationals of an equal size and shape to
`OutNeighbours(digraph)`, otherwise it will fail.

This will create a digraph and set the EdgeWeights to `weights`.

See `EdgeWeights` (6.3.1).

```
――――――――――――――――――――――――― Example ―――――――――――――――――――――――――
gap> g := EdgeWeightedDigraph(Digraph([[2], [1]]), [[5], [15]]);
<immutable digraph with 2 vertices, 2 edges>
gap> g := EdgeWeightedDigraph([[2], [1]], [[5], [15]]);
<immutable digraph with 2 vertices, 2 edges>
gap> EdgeWeights(g);
[ [ 5 ], [ 15 ] ]
```

6.3.3 EdgeWeightedDigraphTotalWeight

▷ EdgeWeightedDigraphTotalWeight(`digraph`)   (attribute)

Returns: An integer, float or rational.

If `digraph` is a digraph with edge weights, then this attribute returns the sum of the weights of its
edges.

If the argument `digraph` is mutable, then the return value of this attribute is recomputed every
time it is called.

See `EdgeWeights` (6.3.1).

```
――――――――――――――――――――――――― Example ―――――――――――――――――――――――――
gap> D := EdgeWeightedDigraph([[2], [1], [1, 2]],
>                             [[12], [5], [6, 9]]);
<immutable digraph with 3 vertices, 4 edges>
gap> EdgeWeightedDigraphTotalWeight(D);
32
```

6.3.4 EdgeWeightedDigraphMinimumSpanningTree

▷ EdgeWeightedDigraphMinimumSpanningTree(`digraph`)   (attribute)

Returns: A digraph.

If `digraph` is a connected digraph with edge weights, then this attribute returns a digraph which
is a minimum spanning tree of `digraph`.

A *spanning tree* of a digraph is a subdigraph with the same vertices but a subset of its edges that
form an undirected tree. It is *minimum* if it has the smallest possible total weight for a spanning tree
of that digraph.

If the argument `digraph` is mutable, then the return value of this attribute is recomputed every
time it is called.

See   `EdgeWeights`   (6.3.1),   `EdgeWeightedDigraphTotalWeight`   (6.3.3)   and
`IsConnectedDigraph` (6.6.3).

```
――――――――――――――――――――――――― Example ―――――――――――――――――――――――――
gap> D := EdgeWeightedDigraph([[2], [1], [1, 2]],
>                             [[12], [5], [6, 9]]);
<immutable digraph with 3 vertices, 4 edges>
gap> T := EdgeWeightedDigraphMinimumSpanningTree(D);
<immutable digraph with 3 vertices, 2 edges>
gap> EdgeWeights(T);
[ [  ], [ 5 ], [ 6 ] ]
```

6.4 Orders

6.4.1 IsPreorderDigraph

▷ IsPreorderDigraph(*digraph*) (property)
▷ IsQuasiorderDigraph(*digraph*) (property)
    Returns: true or false.

A digraph is a preorder digraph if and only if the digraph satisfies both IsReflexiveDigraph (6.2.13) and IsTransitiveDigraph (6.2.16). A preorder digraph (or quasiorder digraph) *digraph* corresponds to the preorder relation $\leq$ defined by $x \leq y$ if and only if [x, y] is an edge of *digraph*.

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
_____ Example _____
gap> D := Digraph([[1], [2, 3], [2, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsPreorderDigraph(D);
true
gap> D := Digraph([[1 .. 4], [1 .. 4], [1 .. 4], [1 .. 4]]);
<immutable digraph with 4 vertices, 16 edges>
gap> IsPreorderDigraph(D);
true
gap> D := Digraph([[2], [3], [4], [5], [1]]);
<immutable digraph with 5 vertices, 5 edges>
gap> IsPreorderDigraph(D);
false
gap> D := Digraph([[1], [1, 2], [2, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsQuasiorderDigraph(D);
false
```

6.4.2 IsPartialOrderDigraph

▷ IsPartialOrderDigraph(*digraph*) (property)
    Returns: true or false.

A digraph is a partial order digraph if and only if the digraph satisfies all of IsReflexiveDigraph (6.2.13), IsAntisymmetricDigraph (6.2.2) and IsTransitiveDigraph (6.2.16). A partial order *digraph* corresponds to the partial order relation $\leq$ defined by $x \leq y$ if and only if [x, y] is an edge of *digraph*.

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
_____ Example _____
gap> D := Digraph([[1, 3], [2, 3], [3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsPartialOrderDigraph(D);
true
gap> D := CycleDigraph(5);
<immutable cycle digraph with 5 vertices>
gap> IsPartialOrderDigraph(D);
false
gap> D := Digraph([[1, 1], [1, 1, 2], [3], [3, 3, 4, 4]]);
```

```
<immutable multidigraph with 4 vertices, 10 edges>
gap> IsPartialOrderDigraph(D);
true
```

6.4.3 IsMeetSemilatticeDigraph

▷ IsMeetSemilatticeDigraph(*digraph*)   (property)
▷ IsJoinSemilatticeDigraph(*digraph*)   (property)
▷ IsLatticeDigraph(*digraph*)   (property)

Returns: true or false.

IsMeetSemilatticeDigraph returns true if the digraph *digraph* is a meet semilattice; IsJoinSemilatticeDigraph returns true if the digraph *digraph* is a join semilattice; and IsLatticeDigraph returns true if the digraph *digraph* is both a meet and a join semilattice.

For a partial order digraph IsPartialOrderDigraph (6.4.2) the corresponding partial order is the relation $\leq$, defined by $x \leq y$ if and only if [x, y] is an edge. A digraph is a *meet semilattice* if it is a partial order and every pair of vertices has a greatest lower bound (meet) with respect to the aforementioned relation. A *join semilattice* is a partial order where every pair of vertices has a least upper bound (join) with respect to the relation.

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
———————————————— Example ————————————————
gap> D := Digraph([[1, 3], [2, 3], [3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsMeetSemilatticeDigraph(D);
false
gap> IsJoinSemilatticeDigraph(D);
true
gap> IsLatticeDigraph(D);
false
gap> D := Digraph([[1], [2], [1 .. 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsJoinSemilatticeDigraph(D);
false
gap> IsMeetSemilatticeDigraph(D);
true
gap> IsLatticeDigraph(D);
false
gap> D := Digraph([[1 .. 4], [2, 4], [3, 4], [4]]);
<immutable digraph with 4 vertices, 9 edges>
gap> IsMeetSemilatticeDigraph(D);
true
gap> IsJoinSemilatticeDigraph(D);
true
gap> IsLatticeDigraph(D);
true
```

6.4.4 DigraphMeetTable

▷ DigraphMeetTable(*digraph*)   (attribute)
▷ DigraphJoinTable(*digraph*)   (attribute)

Returns: A matrix or `fail`.

DigraphMeetTable returns the *meet table* of `digraph` if `digraph` is a meet semilattice digraph IsMeetSemilatticeDigraph (6.4.3). Similarly, DigraphJoinTable returns the *join table* of `digraph` if `digraph` is a join semilattice digraph. IsJoinSemilatticeDigraph (6.4.3).

When `digraph` is a meet semilattice digraph with *n* vertices, the *meet table* of `digraph` is the matrix *A* such that the (i, j) entry of *A* is the meet of vertices *i* and *j*.

Similarly, when `digraph` is a join semilattice digraph with *n* vertices, the *join table* of `digraph` is the matrix *A* such that the (i, j) entry of *A* is the join of vertices *i* and *j*. Otherwise, each function returns `fail`.

```
――――――――――――――――――――― Example ―――――――――――――――――――――
gap> D := Digraph([[1, 2, 3, 4], [2, 4], [3, 4], [4]]);
<immutable digraph with 4 vertices, 9 edges>
gap> IsJoinSemilatticeDigraph(D);
true
gap> Display(DigraphJoinTable(D));
[ [  1,  2,  3,  4 ],
  [  2,  2,  4,  4 ],
  [  3,  4,  3,  4 ],
  [  4,  4,  4,  4 ] ]
gap> D := CycleDigraph(5);
<immutable cycle digraph with 5 vertices>
gap> DigraphJoinTable(D);
fail
```

6.4.5 IsUpperSemimodularDigraph

▷ IsUpperSemimodularDigraph(*D*)   (property)
▷ IsLowerSemimodularDigraph(*D*)   (property)

Returns: `true` or `false`.

IsUpperSemimodularDigraph returns `true` if the digraph `D` represents an upper semimodular lattice and `false` if it does not. Similarly, IsLowerSemimodularDigraph returns `true` if `D` represents a lower semimodular lattice and `false` if it does not.

In a lattice we say that a vertex `a` *covers* a vertex `b` if `a` is greater than `b`, and there are no further vertices between `a` and `b`. A lattice is *upper semimodular* if whenever the meet of `a` and `b` is covered by `a`, the join of `a` and `b` covers `b`. *Lower semimodularity* is defined analogously.

See also IsLatticeDigraph (6.4.3), NonUpperSemimodularPair (5.3.2), and NonLowerSemimodularPair (5.3.2). If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
――――――――――――――――――――― Example ―――――――――――――――――――――
gap> D := DigraphFromDigraph6String(
> "&M~~sc'lYUZO__KIBboC_@h?U_?_GL?A_?c");
<immutable digraph with 14 vertices, 66 edges>
gap> IsUpperSemimodularDigraph(D);
true
gap> IsLowerSemimodularDigraph(D);
false
```

6.4.6 IsDistributiveLatticeDigraph

▷ IsDistributiveLatticeDigraph(*digraph*)   (property)

Returns: true or false.

IsDistributiveLatticeDigraph returns true if the digraph *digraph* is a distributive lattice digraph.

A *distributive lattice digraph* is a lattice digraph (IsLatticeDigraph (6.4.3)) which is distributive. That is to say, the functions PartialOrderDigraphMeetOfVertices (5.3.1) and PartialOrderDigraphJoinOfVertices (5.3.1) distribute over each other.

Equivalently, a distributive lattice digraph is a lattice digraph in which the *lattice digraphs* representing $M3$ and $N5$ are not embeddable as lattices (see https://en.wikipedia.org/wiki/Distributive_lattice and IsLatticeEmbedding (7.3.21)).

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
─────────────── Example ───────────────
 gap> D := Digraph([[2, 3], [4], [4], []]);
 <immutable digraph with 4 vertices, 4 edges>
 gap> D := DigraphReflexiveTransitiveClosure(D);
 <immutable preorder digraph with 4 vertices, 9 edges>
 gap> IsDistributiveLatticeDigraph(D);
 true
 gap> N5 := Digraph([[2, 4], [3], [5], [5], []]);
 <immutable digraph with 5 vertices, 5 edges>
 gap> N5 := DigraphReflexiveTransitiveClosure(N5);
 <immutable preorder digraph with 5 vertices, 13 edges>
 gap> IsDistributiveLatticeDigraph(N5);
 false
```

6.4.7 IsModularLatticeDigraph

▷ IsModularLatticeDigraph(*digraph*)   (property)

Returns: true or false.

IsModularLatticeDigraph returns true if the digraph *digraph* is a modular lattice digraph.

A *modular lattice digraph* is a lattice digraph (IsLatticeDigraph (6.4.3)) which is modular. That is to say, the lattice digraph representing $N5$ is not embeddable as a lattice (see https://en.wikipedia.org/wiki/Modular_lattice and IsLatticeEmbedding (7.3.21)).

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
─────────────── Example ───────────────
 gap> D := ChainDigraph(5);
 <immutable chain digraph with 5 vertices>
 gap> D := DigraphReflexiveTransitiveClosure(D);
 <immutable preorder digraph with 5 vertices, 15 edges>
 gap> IsModularLatticeDigraph(D);
 true
 gap> N5 := Digraph([[2, 3], [4], [4], []]);
 <immutable digraph with 4 vertices, 4 edges>
 gap> DigraphReflexiveTransitiveClosure(N5);
 <immutable preorder digraph with 4 vertices, 9 edges>
```

```
gap> IsModularLatticeDigraph(N5);
false
```

6.5 Regularity

6.5.1 IsInRegularDigraph

▷ IsInRegularDigraph(*digraph*) (property)

Returns: true or false.

This property is true if there is an integer n such that for every vertex v of digraph *digraph* there are exactly n edges terminating in v. See also IsOutRegularDigraph (6.5.2) and IsRegularDigraph (6.5.3).

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
_____ Example _____
gap> IsInRegularDigraph(CompleteDigraph(4));
true
gap> IsInRegularDigraph(ChainDigraph(4));
false
```

6.5.2 IsOutRegularDigraph

▷ IsOutRegularDigraph(*digraph*) (property)

Returns: true or false.

This property is true if there is an integer n such that for every vertex v of digraph *digraph* there are exactly n edges starting at v.

See also IsInRegularDigraph (6.5.1) and IsRegularDigraph (6.5.3).

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
_____ Example _____
gap> IsOutRegularDigraph(CompleteDigraph(4));
true
gap> IsOutRegularDigraph(ChainDigraph(4));
false
```

6.5.3 IsRegularDigraph

▷ IsRegularDigraph(*digraph*) (property)

Returns: true or false.

This property is true if there is an integer n such that for every vertex v of digraph *digraph* there are exactly n edges starting and terminating at v. In other words, the property is true if *digraph* is both in-regular and and out-regular. See also IsInRegularDigraph (6.5.1) and IsOutRegularDigraph (6.5.2).

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
                            ─ Example ─
  gap> IsRegularDigraph(CompleteDigraph(4));
  true
  gap> IsRegularDigraph(ChainDigraph(4));
  false
```

6.5.4 IsDistanceRegularDigraph

▷ IsDistanceRegularDigraph(*digraph*)   (property)

    Returns: true or false.

    If *digraph* is a connected symmetric graph, this property returns true if for any two vertices u and v of *digraph* and any two integers i and j between 0 and the diameter of *digraph*, the number of vertices at distance i from u and distance j from v depends only on i, j, and the distance between vertices u and v.

    Alternatively, a distance regular graph is a graph for which there exist integers b_i, c_i, and i such that for any two vertices u, v in *digraph* which are distance i apart, there are exactly b_i neighbors of v which are at distance i - 1 away from u, and c_i neighbors of v which are at distance i + 1 away from u. This definition is used to check whether *digraph* is distance regular.

    In the case where *digraph* is not symmetric or not connected, the property is false.

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
                            ─ Example ─
  gap> D := DigraphSymmetricClosure(ChainDigraph(5));;
  gap> IsDistanceRegularDigraph(D);
  false
  gap> D := Digraph([[2, 3, 4], [1, 3, 4], [1, 2, 4], [1, 2, 3]]);
  <immutable digraph with 4 vertices, 12 edges>
  gap> IsDistanceRegularDigraph(D);
  true
```

6.6 Connectivity and cycles

6.6.1 IsAcyclicDigraph

▷ IsAcyclicDigraph(*digraph*)   (property)

    Returns: true or false.

    This property is true if the digraph *digraph* is acyclic, and false if it is not. A digraph is *acyclic* if every directed cycle on the digraph is trivial. See Section 1.1.1 for the definition of a directed cycle, and of a trivial directed cycle.

    The method used in this operation has complexity $O(m + n)$ where m is the number of edges (counting multiple edges as one) and n is the number of vertices in the digraph.

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
                            ─ Example ─
  gap> Petersen := Graph(SymmetricGroup(5), [[1, 2]], OnSets,
  > function(x, y)
  >   return IsEmpty(Intersection(x, y));
  > end);;
```

```
gap> D := Digraph(Petersen);
<immutable digraph with 10 vertices, 30 edges>
gap> IsAcyclicDigraph(D);
false
gap> D := DigraphFromDiSparse6String(
> ".b_OGCIDBaPGkULEbQHCeRIdrHcuZMfRyDAbPhTi|zF");
<immutable digraph with 35 vertices, 34 edges>
gap> IsAcyclicDigraph(D);
true
gap> IsAcyclicDigraph(ChainDigraph(10));
true
gap> D := CompleteDigraph(IsMutableDigraph, 4);
<mutable digraph with 4 vertices, 12 edges>
gap> IsAcyclicDigraph(D);
false
gap> IsAcyclicDigraph(CycleDigraph(10));
false
```

6.6.2 IsChainDigraph

▷ IsChainDigraph(*digraph*)   (property)

Returns: true or false.

IsChainDigraph returns true if the digraph *digraph* is isomorphic to the chain digraph with the same number of vertices as *digraph*, and false if it is not; see ChainDigraph (3.5.9).

A digraph is a *chain* if and only if it is a directed tree, in which every vertex has out degree at most one; see IsDirectedTree (6.6.8) and OutDegrees (5.2.8).

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
———————————————————— Example ————————————————————
gap> D := Digraph([[1, 3], [2, 3], [3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsChainDigraph(D);
false
gap> D := ChainDigraph(5);
<immutable chain digraph with 5 vertices>
gap> IsChainDigraph(D);
true
gap> D := DigraphReverse(D);
<immutable digraph with 5 vertices, 4 edges>
gap> IsChainDigraph(D);
true
gap> D := ChainDigraph(IsMutableDigraph, 5);
<mutable digraph with 5 vertices, 4 edges>
gap> IsChainDigraph(D);
true
gap> DigraphReverse(D);
<mutable digraph with 5 vertices, 4 edges>
gap> IsChainDigraph(D);
true
```

6.6.3 IsConnectedDigraph

▷ IsConnectedDigraph(*digraph*) (property)

    Returns: `true` or `false`.

    This property is `true` if the digraph *digraph* is weakly connected and `false` if it is not. A digraph *digraph* is *weakly connected* if it is possible to travel from any vertex to any other vertex by traversing edges in either direction (possibly against the orientation of some of them).

    The method used in this function has complexity $O(m)$ if the digraph's `DigraphSource` (5.2.5) attribute is set, otherwise it has complexity $O(m+n)$ (where m is the number of edges and n is the number of vertices of the digraph).

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
————————————————————— Example —————————————————————
gap> D := Digraph([[2], [3], []]);;
gap> IsConnectedDigraph(D);
true
gap> D := Digraph([[1, 3], [4], [3], []]);;
gap> IsConnectedDigraph(D);
false
gap> D := Digraph(IsMutableDigraph, [[2], [3], []]);;
gap> IsConnectedDigraph(D);
true
gap> D := Digraph(IsMutableDigraph, [[1, 3], [4], [3], []]);;
gap> IsConnectedDigraph(D);
false
```

6.6.4 IsBiconnectedDigraph

▷ IsBiconnectedDigraph(*digraph*) (property)

    Returns: `true` or `false`.

    A connected digraph is *biconnected* if it is still connected (in the sense of `IsConnectedDigraph` (6.6.3)) when any vertex is removed. If *D* has at least 3 vertices, then `IsBiconnectedDigraph` implies `IsBridgelessDigraph` (6.6.5); see `ArticulationPoints` (5.4.14) or `Bridges` (5.4.15) for a more detailed explanation.

    `IsBiconnectedDigraph` returns `true` if the digraph *digraph* is biconnected, and `false` if it is not. In particular, `IsBiconnectedDigraph` returns `false` if *digraph* is not connected.

    Multiple edges are ignored by this method.

    The method used in this operation has complexity $O(m+n)$ where m is the number of edges and n is the number of vertices in the digraph.

    See also `Bridges` (5.4.15), `ArticulationPoints` (5.4.14), and `IsBridgelessDigraph` (6.6.5).

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
————————————————————— Example —————————————————————
gap> IsBiconnectedDigraph(Digraph([[1, 3], [2, 3], [3]]));
false
gap> IsBiconnectedDigraph(CycleDigraph(5));
true
gap> D := Digraph([[1, 1], [1, 1, 2], [3], [3, 3, 4, 4]]);;
gap> IsBiconnectedDigraph(D);
```

```
false
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
<mutable digraph with 9 vertices, 40 edges>
gap> IsBiconnectedDigraph(D);
true
```

6.6.5 IsBridgelessDigraph

▷ IsBridgelessDigraph(*digraph*)   (property)

Returns: true or false.

A connected digraph is *bridgeless* if it is still connected (in the sense of IsConnectedDigraph (6.6.3)) when any edge is removed. If *digraph* has at least 3 vertices, then IsBiconnectedDigraph (6.6.4) implies IsBridgelessDigraph; see ArticulationPoints (5.4.14) or Bridges (5.4.15) for a more detailed explanation.

IsBridgelessDigraph returns true if the digraph *digraph* is bridgeless, and false if it is not. In particular, IsBridgelessDigraph returns false if *digraph* is not connected.

Multiple edges are ignored by this method.

The method used in this operation has complexity $O(m+n)$ where m is the number of edges and n is the number of vertices in the digraph.

See also Bridges (5.4.15), ArticulationPoints (5.4.14), and IsBiconnectedDigraph (6.6.4).

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
_____ Example _____
gap> IsBridgelessDigraph(Digraph([[1, 3], [2, 3], [3]]));
false
gap> IsBridgelessDigraph(CycleDigraph(5));
true
gap> D := Digraph([[1, 1], [1, 1, 2], [3], [3, 3, 4, 4]]);;
gap> IsBridgelessDigraph(D);
false
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
<mutable digraph with 9 vertices, 40 edges>
gap> IsBridgelessDigraph(D);
true
gap> D := Digraph([[2, 5], [1, 3, 4, 5], [2, 4], [2, 3], [1, 2]]);
<immutable digraph with 5 vertices, 12 edges>
gap> IsBridgelessDigraph(D);
true
gap> IsBiconnectedDigraph(D);
false
gap> D := Digraph([[2], [3], [4], [2]]);
<immutable digraph with 4 vertices, 4 edges>
gap> IsBridgelessDigraph(D);
false
gap> IsBiconnectedDigraph(D);
false
gap> IsBridgelessDigraph(ChainDigraph(2));
false
```

```
gap> IsBiconnectedDigraph(ChainDigraph(2));
true
```

6.6.6 IsStronglyConnectedDigraph

▷ IsStronglyConnectedDigraph(*digraph*)   (property)

    Returns: true or false.

    This property is true if the digraph *digraph* is strongly connected and false if it is not.

    A digraph *digraph* is *strongly connected* if there is a directed path from every vertex to every other vertex. See Section 1.1.1 for the definition of a directed path.

    The method used in this operation is based on Gabow's Algorithm [Gab00] and has complexity $O(m + n)$, where m is the number of edges (counting multiple edges as one) and n is the number of vertices in the digraph.

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

———— Example ————
```
gap> D := CycleDigraph(250000);
<immutable cycle digraph with 250000 vertices>
gap> IsStronglyConnectedDigraph(D);
true
gap> D := DigraphRemoveEdges(D, [[250000, 1]]);
<immutable digraph with 250000 vertices, 249999 edges>
gap> IsStronglyConnectedDigraph(D);
false
gap> D := CycleDigraph(IsMutableDigraph, 250000);
<mutable digraph with 250000 vertices, 250000 edges>
gap> IsStronglyConnectedDigraph(D);
true
gap> DigraphRemoveEdge(D, [250000, 1]);
<mutable digraph with 250000 vertices, 249999 edges>
gap> IsStronglyConnectedDigraph(D);
false
```

6.6.7 IsAperiodicDigraph

▷ IsAperiodicDigraph(*digraph*)   (property)

    Returns: true or false.

    This property is true if the digraph *digraph* is aperiodic, i.e. if its DigraphPeriod (5.4.17) is equal to 1. Otherwise, the property is false.

    If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

———— Example ————
```
gap> D := Digraph([[6], [1], [2], [3], [4, 4], [5]]);
<immutable multidigraph with 6 vertices, 7 edges>
gap> IsAperiodicDigraph(D);
false
gap> D := Digraph([[2], [3, 5], [4], [5], [1, 2]]);
<immutable digraph with 5 vertices, 7 edges>
gap> IsAperiodicDigraph(D);
```

```
  true
gap> D := Digraph(IsMutableDigraph, [[2], [3, 5], [4], [5], [1, 2]]);
<mutable digraph with 5 vertices, 7 edges>
gap> IsAperiodicDigraph(D);
true
```

6.6.8 IsDirectedTree

▷ IsDirectedTree(*digraph*)   (property)

Returns: true or false.

Returns true if the digraph `digraph` is a directed tree, and false if it is not.

A *directed tree* is an acyclic digraph with precisely 1 source, such that no two vertices share an out-neighbour. Note that the empty digraph with zero vertices is not considered to be a directed tree, because it has no source.

See also DigraphSources (5.1.9).

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
                           ──────── Example ────────
gap> D := Digraph([[], [2]]);
<immutable digraph with 2 vertices, 1 edge>
gap> IsDirectedTree(D);
false
gap> D := Digraph([[3], [3], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> IsDirectedTree(D);
false
gap> D := Digraph([[2], [3], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> IsDirectedTree(D);
true
gap> D := Digraph([[2, 3], [6], [4, 5], [], [], []]);
<immutable digraph with 6 vertices, 5 edges>
gap> IsDirectedTree(D);
true
```

6.6.9 IsUndirectedTree

▷ IsUndirectedTree(*digraph*)   (property)
▷ IsUndirectedForest(*digraph*)   (property)

Returns: true or false.

The property IsUndirectedTree returns true if the digraph `digraph` is an undirected tree, and the property IsUndirectedForest returns true if `digraph` is an undirected forest; otherwise, these properties return false.

An *undirected tree* is a symmetric digraph without loops, in which for any pair of distinct vertices u and v, there is exactly one directed path from u to v. See IsSymmetricDigraph (6.2.14) and DigraphHasLoops (6.2.1), and see Section 1.1.1 for the definition of directed path. This definition implies that an undirected tree has no multiple edges.

An *undirected forest* is a digraph, each of whose connected components is an undirected tree. In other words, an undirected forest is isomorphic to a disjoint union of undirected trees. See

DigraphConnectedComponents (5.4.9) and DigraphDisjointUnion (3.3.29). In particular, every undirected tree is an undirected forest.

Please note that the digraph with zero vertices is considered to be neither an undirected tree nor an undirected forest.

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
 ─────────────────────────── Example ───────────────────────────
 gap> D := Digraph([[3], [3], [1, 2]]);
 <immutable digraph with 3 vertices, 4 edges>
 gap> IsUndirectedTree(D);
 true
 gap> IsSymmetricDigraph(D) and not DigraphHasLoops(D);
 true
 gap> D := Digraph([[3], [5], [1, 4], [3], [2]]);
 <immutable digraph with 5 vertices, 6 edges>
 gap> IsConnectedDigraph(D);
 false
 gap> IsUndirectedTree(D);
 false
 gap> IsUndirectedForest(D);
 true
 gap> D := Digraph([[1, 2], [1], [2]]);
 <immutable digraph with 3 vertices, 4 edges>
 gap> IsUndirectedTree(D) or IsUndirectedForest(D);
 false
 gap> IsSymmetricDigraph(D) or not DigraphHasLoops(D);
 false
```

6.6.10 IsEulerianDigraph

▷ IsEulerianDigraph(`digraph`) (property)

Returns: true or false.

This property returns true if the digraph `digraph` is Eulerian.

A connected digraph is called *Eulerian* if there exists a directed circuit on the digraph which includes every edge exactly once. See Section 1.1.1 for the definition of a directed circuit. Note that the empty digraph with at most one vertex is considered to be Eulerian.

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```
 ─────────────────────────── Example ───────────────────────────
 gap> D := Digraph([[]]);
 <immutable empty digraph with 1 vertex>
 gap> IsEulerianDigraph(D);
 true
 gap> D := Digraph([[2], []]);
 <immutable digraph with 2 vertices, 1 edge>
 gap> IsEulerianDigraph(D);
 false
 gap> D := Digraph([[3], [], [2]]);
 <immutable digraph with 3 vertices, 2 edges>
 gap> IsEulerianDigraph(D);
```

```
  false
  gap> D := Digraph([[2], [3], [1]]);
  <immutable digraph with 3 vertices, 3 edges>
  gap> IsEulerianDigraph(D);
  true
```

6.6.11 IsHamiltonianDigraph

▷ IsHamiltonianDigraph(*digraph*)   (property)
   Returns: true or false.

   If *digraph* is Hamiltonian, then this property returns true, and false if it is not.

   A digraph with n vertices is *Hamiltonian* if it has a directed cycle of length n. See Section 1.1.1 for the definition of a directed cycle. Note the empty digraphs on 0 and 1 vertices are considered to be Hamiltonian.

   The method used in this operation has the worst case complexity as DigraphMonomorphism (7.3.4).

   If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

─────────────────────────── Example ───────────────────────────
```
  gap> g := Digraph([[]]);
  <immutable empty digraph with 1 vertex>
  gap> IsHamiltonianDigraph(g);
  true
  gap> g := Digraph([[2], [1]]);
  <immutable digraph with 2 vertices, 2 edges>
  gap> IsHamiltonianDigraph(g);
  true
  gap> g := Digraph([[3], [], [2]]);
  <immutable digraph with 3 vertices, 2 edges>
  gap> IsHamiltonianDigraph(g);
  false
  gap> g := Digraph([[2], [3], [1]]);
  <immutable digraph with 3 vertices, 3 edges>
  gap> IsHamiltonianDigraph(g);
  true
```

6.6.12 IsCycleDigraph

▷ IsCycleDigraph(*digraph*)   (property)
   Returns: true or false.

   IsCycleDigraph returns true if the digraph *digraph* is isomorphic to the cycle digraph with the same number of vertices as *digraph*, and false if it is not; see CycleDigraph (3.5.14).

   A digraph is a *cycle* if and only if it is strongly connected and has the same number of edges as vertices.

   If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

─────────────────────────── Example ───────────────────────────
```
  gap> D := Digraph([[1, 3], [2, 3], [3]]);
  <immutable digraph with 3 vertices, 5 edges>
```

```
gap> IsCycleDigraph(D);
false
gap> D := CycleDigraph(5);
<immutable cycle digraph with 5 vertices>
gap> IsCycleDigraph(D);
true
gap> D := OnDigraphs(D, (1, 2, 3));
<immutable digraph with 5 vertices, 5 edges>
gap> D = CycleDigraph(5);
false
gap> IsCycleDigraph(D);
true
```

6.7 Planarity

6.7.1 IsPlanarDigraph

▷ IsPlanarDigraph(*digraph*)   (property)

Returns: true or false.

A *planar* digraph is a digraph that can be embedded in the plane in such a way that its edges do not intersect. A digraph is planar if and only if it does not have a subdigraph that is homeomorphic to either the complete graph on 5 vertices or the complete bipartite graph with vertex sets of sizes 3 and 3.

IsPlanarDigraph returns true if the digraph *digraph* is planar and false if it is not. The directions and multiplicities of any edges in *digraph* are ignored by IsPlanarDigraph.

See also IsOuterPlanarDigraph (6.7.2).

This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

─────────────────────────── Example ───────────────────────────
```
gap> IsPlanarDigraph(CompleteDigraph(4));
true
gap> IsPlanarDigraph(CompleteDigraph(5));
false
gap> IsPlanarDigraph(CompleteBipartiteDigraph(2, 3));
true
gap> IsPlanarDigraph(CompleteBipartiteDigraph(3, 3));
false
gap> IsPlanarDigraph(CompleteDigraph(IsMutableDigraph, 4));
true
gap> IsPlanarDigraph(CompleteDigraph(IsMutableDigraph, 5));
false
gap> IsPlanarDigraph(CompleteBipartiteDigraph(IsMutableDigraph, 2, 3));
true
gap> IsPlanarDigraph(CompleteBipartiteDigraph(IsMutableDigraph, 3, 3));
false
```

6.7.2 IsOuterPlanarDigraph

▷ IsOuterPlanarDigraph(*digraph*)   (property)

    Returns: true or false.

An *outer planar* digraph is a digraph that can be embedded in the plane in such a way that its edges do not intersect, and all vertices belong to the unbounded face of the embedding. A digraph is outer planar if and only if it does not have a subdigraph that is homeomorphic to either the complete graph on 4 vertices or the complete bipartite graph with vertex sets of sizes 2 and 3.

    IsOuterPlanarDigraph returns true if the digraph *digraph* is outer planar and false if it is not. The directions and multiplicities of any edges in *digraph* are ignored by IsPlanarDigraph.

    See also IsPlanarDigraph (6.7.1). This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

```
———— Example ————
gap> IsOuterPlanarDigraph(CompleteDigraph(4));
false
gap> IsOuterPlanarDigraph(CompleteDigraph(5));
false
gap> IsOuterPlanarDigraph(CompleteBipartiteDigraph(2, 3));
false
gap> IsOuterPlanarDigraph(CompleteBipartiteDigraph(3, 3));
false
gap> IsOuterPlanarDigraph(CycleDigraph(10));
true
gap> IsOuterPlanarDigraph(CompleteDigraph(IsMutableDigraph, 4));
false
gap> IsOuterPlanarDigraph(CompleteDigraph(IsMutableDigraph, 5));
false
gap> IsOuterPlanarDigraph(CompleteBipartiteDigraph(IsMutableDigraph,
>                                                  2, 3));
false
gap> IsOuterPlanarDigraph(CompleteBipartiteDigraph(IsMutableDigraph,
>                                                  3, 3));
false
gap> IsOuterPlanarDigraph(CycleDigraph(IsMutableDigraph, 10));
true
```

6.8 Homomorphisms and transformations

6.8.1 IsDigraphCore

▷ IsDigraphCore(*digraph*)   (property)

    Returns: true or false.

This property returns true if *digraph* is a core, and false if it is not.

A digraph D is a *core* if and only if it has no proper subdigraphs A such that there exists a homomorphism from D to A. In other words, a digraph D is a core if and only if every endomorphism on D is an automorphism on D.

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
  ───────────────────────── Example ─────────────────────────
  gap> D := CompleteDigraph(6);
  <immutable complete digraph with 6 vertices>
  gap> IsDigraphCore(D);
  true
  gap> D := DigraphSymmetricClosure(CycleDigraph(6));
  <immutable symmetric digraph with 6 vertices, 12 edges>
  gap> DigraphHomomorphism(D, CompleteDigraph(2));
  Transformation( [ 1, 2, 1, 2, 1, 2 ] )
  gap> IsDigraphCore(D);
  false
```

6.8.2 IsEdgeTransitive

▷ IsEdgeTransitive(*digraph*)   (property)

Returns: true or false.

If *digraph* is a digraph without multiple edges, then IsEdgeTransitive returns true if *digraph* is edge transitive, and false otherwise. A digraph is *edge transitive* if its automorphism group acts transitively on its edges (via the action OnPairs (**Reference: OnPairs**)).

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
  ───────────────────────── Example ─────────────────────────
  gap> IsEdgeTransitive(CompleteDigraph(2));
  true
  gap> IsEdgeTransitive(ChainDigraph(3));
  false
  gap> IsEdgeTransitive(Digraph([[2], [3, 3, 3], []]));
  Error, the argument <D> must be a digraph with no multiple edges,
```

6.8.3 IsVertexTransitive

▷ IsVertexTransitive(*digraph*)   (property)

Returns: true or false.

If *digraph* is a digraph, then IsVertexTransitive returns true if *digraph* is vertex transitive, and false otherwise. A digraph is *vertex transitive* if its automorphism group acts transitively on its vertices.

If the argument *digraph* is mutable, then the return value of this property is recomputed every time it is called.

```
  ───────────────────────── Example ─────────────────────────
  gap> IsVertexTransitive(CompleteDigraph(2));
  true
  gap> IsVertexTransitive(ChainDigraph(3));
  false
```

Chapter 7

Homomorphisms

7.1 Acting on digraphs

7.1.1 OnDigraphs (for a digraph and a perm)

▷ OnDigraphs(*digraph, perm*)   (operation)

▷ OnDigraphs(*digraph, trans*)   (operation)

   Returns: A digraph.

   If *digraph* is a digraph, and the second argument *perm* is a *permutation* of the vertices of *digraph*, then this operation returns a digraph constructed by relabelling the vertices of *digraph* according to *perm*. Note that for an automorphism f of a digraph, we have OnDigraphs(*digraph*, f) = *digraph*.

   If the second argument is a *transformation trans* of the vertices of *digraph*, then this operation returns a digraph constructed by transforming the source and range of each edge according to *trans*. Thus a vertex which does not appear in the image of *trans* will be isolated in the returned digraph, and the returned digraph may contain multiple edges, even if *digraph* does not. If *trans* is mathematically a permutation, then the result coincides with OnDigraphs(*digraph*, AsPermutation(*trans*)).

   The DigraphVertexLabels (5.1.12) of *digraph* will not be retained in the returned digraph.

   If *digraph* belongs to IsMutableDigraph (3.1.2), then relabelling of the vertices is performed directly on *digraph*. If *digraph* belongs to IsImmutableDigraph (3.1.3), an immutable copy of *digraph* with the vertices relabelled is returned.

```
_____ Example _____
gap> D := Digraph([[3], [1, 3, 5], [1], [1, 2, 4], [2, 3, 5]]);
<immutable digraph with 5 vertices, 11 edges>
gap> new := OnDigraphs(D, (1, 2));
<immutable digraph with 5 vertices, 11 edges>
gap> OutNeighbours(new);
[ [ 2, 3, 5 ], [ 3 ], [ 2 ], [ 2, 1, 4 ], [ 1, 3, 5 ] ]
gap> D := Digraph([[2], [], [2]]);
<immutable digraph with 3 vertices, 2 edges>
gap> t := Transformation([1, 2, 1]);;
gap> new := OnDigraphs(D, t);
<immutable multidigraph with 3 vertices, 2 edges>
gap> OutNeighbours(new);
[ [ 2, 2 ], [  ], [  ] ]
```

```
gap> ForAll(DigraphEdges(D),
>   e -> IsDigraphEdge(new, [e[1] ^ t, e[2] ^ t]));
true
```

7.1.2 OnMultiDigraphs

▷ OnMultiDigraphs(*digraph*, *pair*)   (operation)
▷ OnMultiDigraphs(*digraph*, *perm1*, *perm2*)   (operation)
    Returns: A digraph.

If *digraph* is a digraph, and *pair* is a pair consisting of a permutation of the vertices and a permutation of the edges of *digraph*, then this operation returns a digraph constructed by relabelling the vertices and edges of *digraph* according to *perm[1]* and *perm[2]*, respectively.

In its second form, OnMultiDigraphs returns a digraph with vertices and edges permuted by *perm1* and *perm2*, respectively.

Note that OnDigraphs(*digraph*, perm)=OnMultiDigraphs(*digraph*, [perm, ()]) where perm is a permutation of the vertices of *digraph*. If you are only interested in the action of a permutation on the vertices of a digraph, then you can use OnDigraphs instead of OnMultiDigraphs. If *digraph* belongs to IsMutableDigraph (3.1.2), then relabelling of the vertices is performed directly on *digraph*. If *digraph* belongs to IsImmutableDigraph (3.1.3), an immutable copy of *digraph* with the vertices relabelled is returned.

```
--------------------------- Example ----------------------------
gap> D1 := Digraph([
> [3, 6, 3], [], [3], [9, 10], [9], [],  [], [10, 4, 10], [], []]);
<immutable multidigraph with 10 vertices, 10 edges>
gap> p := BlissCanonicalLabelling(D1);
[ (1,7)(3,6)(4,10)(5,9), () ]
gap> D2 := OnMultiDigraphs(D1, p);
<immutable multidigraph with 10 vertices, 10 edges>
gap> OutNeighbours(D2);
[ [  ], [  ], [  ], [  ], [  ], [ 6 ], [ 6, 3, 6 ], [ 4, 10, 4 ],
  [ 5 ], [ 5, 4 ] ]
```

7.1.3 OnTuplesDigraphs (for a list of digraphs and a perm)

▷ OnTuplesDigraphs(*list*, *perm*)   (operation)
▷ OnSetsDigraphs(*list*, *perm*)   (operation)
    Returns: A list or set of digraphs.

If *list* is a list of digraphs, and *perm* is a *permutation* of the vertices of the digraphs in *list*, then OnTuplesDigraphs returns a new list constructed by applying *perm* via OnDigraphs (7.1.1) to a copy (with the same mutability) of each entry of *list* in turn.

More precisely, OnTuplesDigraphs(*list*,*perm*) is a list of length Length(*list*), whose i-th entry is OnDigraphs(DigraphCopy(*list*[i]), *perm*).

If *list* is moreover a GAP set (i.e. a duplicate-free sorted list), then OnSetsDigraphs returns the sorted output of OnTuplesDigraphs, which is therefore again a set.

```
--------------------------- Example ----------------------------
gap> list := [CycleDigraph(IsMutableDigraph, 6),
>             DigraphReverse(CycleDigraph(6))];
[ <mutable digraph with 6 vertices, 6 edges>,
```

```
     <immutable digraph with 6 vertices, 6 edges> ]
gap> p := (1, 6)(2, 5)(3, 4);;
gap> result_tuples := OnTuplesDigraphs(list, p);
[ <mutable digraph with 6 vertices, 6 edges>,
  <immutable digraph with 6 vertices, 6 edges> ]
gap> result_tuples[2] = OnDigraphs(list[2], p);
true
gap> result_tuples = list;
false
gap> result_tuples = Reversed(list);
true
gap> result_sets := OnSetsDigraphs(list, p);
[ <immutable digraph with 6 vertices, 6 edges>,
  <mutable digraph with 6 vertices, 6 edges> ]
gap> result_sets = list;
true
```

7.2 Isomorphisms and canonical labellings

From version 0.11.0 of Digraphs it is possible to use either bliss or nauty (via NautyTracesInterface) to calculate canonical labellings and automorphism groups of digraphs; see [JK07] and [MP14] for more details about bliss and nauty , respectively.

7.2.1 DigraphsUseNauty

▷ DigraphsUseNauty()   (function)
▷ DigraphsUseBliss()   (function)
    Returns: Nothing.

    These functions can be used to specify whether nauty or bliss should be used by default by Digraphs. If NautyTracesInterface is not available, then these functions do nothing. Otherwise, by calling DigraphsUseNauty subsequent computations will default to using nauty rather than bliss , where possible.

    You can call these functions at any point in a GAP session, as many times as you like, it is guaranteed that existing digraphs remain valid, and that comparison of existing digraphs and newly created digraphs via IsIsomorphicDigraph (7.2.15), IsIsomorphicDigraph (7.2.16), IsomorphismDigraphs (7.2.17), and IsomorphismDigraphs (7.2.18) are also valid.

    It is also possible to compute the automorphism group of a specific digraph using both nauty and bliss using NautyAutomorphismGroup (7.2.4) and BlissAutomorphismGroup (7.2.3), respectively.

7.2.2 AutomorphismGroup (for a digraph)

▷ AutomorphismGroup(*digraph*)   (attribute)
    Returns: A permutation group.

    If *digraph* is a digraph, then this attribute contains the group of automorphisms of *digraph*. An *automorphism* of *digraph* is an isomorphism from *digraph* to itself. See IsomorphismDigraphs (7.2.17) for more information about isomorphisms of digraphs.

    If *digraph* is not a multidigraph then the automorphism group is returned as a group of permutations on the set of vertices of *digraph*.

If `digraph` is a multidigraph then the automorphism group is returned as the direct product of a group of permutations on the set of vertices of `digraph` with a group of permutations on the set of edges of `digraph`. These groups can be accessed using `Projection` (**Reference: Projection for a domain and a positive integer**) on the returned group.

By default, the automorphism group is found using bliss by Tommi Junttila and Petteri Kaski. If NautyTracesInterface is available, then nauty by Brendan Mckay and Adolfo Piperno can be used instead; see `BlissAutomorphismGroup` (7.2.3), `NautyAutomorphismGroup` (7.2.4), `DigraphsUseBliss` (7.2.1), and `DigraphsUseNauty` (7.2.1).

If the argument `digraph` is mutable, then the return value of this attribute is recomputed every time it is called.

```
——————————————————— Example ———————————————————
gap> johnson := DigraphFromGraph6String("E}lw");
<immutable symmetric digraph with 6 vertices, 24 edges>
gap> G := AutomorphismGroup(johnson);
Group([ (3,4), (2,3)(4,5), (1,2)(5,6) ])
gap> cycle := CycleDigraph(9);
<immutable cycle digraph with 9 vertices>
gap> G := AutomorphismGroup(cycle);
Group([ (1,2,3,4,5,6,7,8,9) ])
gap> IsCyclic(G) and Size(G) = 9;
true
```

7.2.3 BlissAutomorphismGroup (for a digraph)

▷ BlissAutomorphismGroup(`digraph`)   (attribute)
▷ BlissAutomorphismGroup(`digraph, vertex_colours`)   (operation)
▷ BlissAutomorphismGroup(`digraph, vertex_colours, edge_colours`)   (operation)

Returns: A permutation group.

If `digraph` is a digraph, then this attribute contains the group of automorphisms of `digraph` as calculated using bliss by Tommi Junttila and Petteri Kaski.

The attribute `AutomorphismGroup` (7.2.2) and operation `AutomorphismGroup` (7.2.5) returns the value of either `BlissAutomorphismGroup` or `NautyAutomorphismGroup` (7.2.4). These groups are, of course, equal but their generating sets may differ.

The attribute `AutomorphismGroup` (7.2.6) returns the value of `BlissAutomorphismGroup` as it is not implemented for nauty The requirements for the optional arguments `vertex_colours` and `edge_colours` are documented in `AutomorphismGroup` (7.2.6). See also `DigraphsUseBliss` (7.2.1), and `DigraphsUseNauty` (7.2.1).

If the argument `digraph` is mutable, then the return value of this attribute is recomputed every time it is called.

```
——————————————————— Example ———————————————————
gap> G := BlissAutomorphismGroup(JohnsonDigraph(5, 2));;
gap> IsSymmetricGroup(G);
true
gap> Size(G);
120
```

7.2.4 NautyAutomorphismGroup

▷ NautyAutomorphismGroup(`digraph[, vert_colours]`)   (attribute)
     Returns: A permutation group.

   If `digraph` is a digraph, then this attribute contains the group of automorphisms of `digraph` as calculated using nauty by Brendan Mckay and Adolfo Piperno via NautyTracesInterface . The attribute AutomorphismGroup (7.2.2) and operation AutomorphismGroup (7.2.5) returns the value of either NautyAutomorphismGroup or BlissAutomorphismGroup (7.2.3). These groups are, of course, equal but their generating sets may differ.

   See also DigraphsUseBliss (7.2.1), and DigraphsUseNauty (7.2.1).

   If the argument `digraph` is mutable, then the return value of this attribute is recomputed every time it is called.

```
_____ Example _____
gap> NautyAutomorphismGroup(JohnsonDigraph(5, 2));
Group([ (3,4)(6,7)(8,9), (2,3)(5,6)(9,10), (2,5)(3,6)(4,7),
 (1,2)(6,8)(7,9) ])
```

7.2.5 AutomorphismGroup (for a digraph and a homogeneous list)

▷ AutomorphismGroup(`digraph, vert_colours`)   (operation)
     Returns: A permutation group.

   This operation computes the automorphism group of a vertex-coloured digraph. A vertex-coloured digraph can be specified by its underlying digraph `digraph` and its colouring `vert_colours`. Let n be the number of vertices of `digraph`. The colouring `vert_colours` may have one of the following two forms:

- a list of n integers, where `vert_colours`[i] is the colour of vertex i, using the colours [1 .. m] for some m <= n; or

- a list of non-empty disjoint lists whose union is DigraphVertices(`digraph`), such that `vert_colours`[i] is the list of all vertices with colour i.

The *automorphism group* of a coloured digraph `digraph` with colouring `vert_colours` is the group consisting of its automorphisms; an *automorphism* of `digraph` is an isomorphism of coloured digraphs from `digraph` to itself. This group is equal to the subgroup of AutomorphismGroup(`digraph`) consisting of those automorphisms that preserve the colouring specified by `vert_colours`. See AutomorphismGroup (7.2.2), and see IsomorphismDigraphs (7.2.18) for more information about isomorphisms of coloured digraphs.

   If `digraph` is not a multidigraph then the automorphism group is returned as a group of permutations on the set of vertices of `digraph`.

   If `digraph` is a multidigraph then the automorphism group is returned as the direct product of a group of permutations on the set of vertices of `digraph` with a group of permutations on the set of edges of `digraph`. These groups can be accessed using Projection (**Reference: Projection for a domain and a positive integer**) on the returned group.

   By default, the automorphism group is found using bliss by Tommi Junttila and Petteri Kaski. If NautyTracesInterface is available, then nauty by Brendan Mckay and Adolfo Piperno can be used instead; see BlissAutomorphismGroup (7.2.3), NautyAutomorphismGroup (7.2.4), DigraphsUseBliss (7.2.1), and DigraphsUseNauty (7.2.1).

```
                           ─── Example ───
  gap> cycle := CycleDigraph(9);
  <immutable cycle digraph with 9 vertices>
  gap> G := AutomorphismGroup(cycle);;
  gap> IsCyclic(G) and Size(G) = 9;
  true
  gap> colours := [[1, 4, 7], [2, 5, 8], [3, 6, 9]];;
  gap> H := AutomorphismGroup(cycle, colours);;
  gap> Size(H);
  3
  gap> H = AutomorphismGroup(cycle, [1, 2, 3, 1, 2, 3, 1, 2, 3]);
  true
  gap> H = SubgroupByProperty(G, p -> OnTuplesSets(colours, p) = colours);
  true
  gap> IsTrivial(AutomorphismGroup(cycle, [1, 1, 2, 2, 2, 2, 2, 2, 2]));
  true
```

7.2.6 AutomorphismGroup (for a digraph, homogeneous list, and list)

▷ AutomorphismGroup(`digraph, vert_colours, edge_colours`)   (operation)

Returns: A permutation group.

This operation computes the automorphism group of a vertex- and/or edge-coloured digraph. A coloured digraph can be specified by its underlying digraph `digraph` and colourings `vert_colours`, `edge_colours`. Let n be the number of vertices of `digraph`. The colourings must have the following forms:

- `vert_colours` must be `fail` or a list of n integers, where `vert_colours`[i] is the colour of vertex i, using the colours [1 .. m] for some m <= n;

- `edge_colours` must be `fail` or a list of n lists of integers of the same shape as OutNeighbours(digraph), where `edge_colours`[i][j] is the colour of the edge OutNeighbours(digraph)[i][j], using the colours [1 .. k] for some k <= n;

Giving `vert_colours` [`edge_colours`] as `fail` is equivalent to setting all vertices [edges] to be the same colour.

Unlike AutomorphismGroup (7.2.2), it is possible to obtain the automorphism group of an edge-coloured multidigraph (see IsMultiDigraph (6.2.11)) when no two edges share the same source, range, and colour. The *automorphism group* of a vertex/edge-coloured digraph `digraph` with colouring `c` is the group consisting of its vertex/edge-colour preserving automorphisms; an *automorphism* of `digraph` is an isomorphism of vertex/edge-coloured digraphs from `digraph` to itself. This group is equal to the subgroup of AutomorphismGroup(`digraph`) consisting of those automorphisms that preserve the colouring specified by `colours`. See AutomorphismGroup (7.2.2), and see IsomorphismDigraphs (7.2.18) for more information about isomorphisms of coloured digraphs.

If `digraph` is not a multidigraph then the automorphism group is returned as a group of permutations on the set of vertices of `digraph`.

If `digraph` is a multidigraph then the automorphism group is returned as the direct product of a group of permutations on the set of vertices of `digraph` with a group of permutations on the set of edges of `digraph`. These groups can be accessed using Projection (**Reference: Projection for a domain and a positive integer**) on the returned group.

By default, the automorphism group is found using bliss by Tommi Junttila and Petteri Kaski. If NautyTracesInterface is available, then nauty by Brendan Mckay and Adolfo Piperno can be used instead; see `BlissAutomorphismGroup` (7.2.3), `NautyAutomorphismGroup` (7.2.4), `DigraphsUseBliss` (7.2.1), and `DigraphsUseNauty` (7.2.1).

```
————————————————————— Example —————————————————————
gap> cycle := CycleDigraph(12);
<immutable cycle digraph with 12 vertices>
gap> vert_colours := List([1 .. 12], x -> x mod 3 + 1);;
gap> edge_colours := List([1 .. 12], x -> [x mod 2 + 1]);;
gap> Size(AutomorphismGroup(cycle));
12
gap> Size(AutomorphismGroup(cycle, vert_colours));
4
gap> Size(AutomorphismGroup(cycle, fail, edge_colours));
6
gap> Size(AutomorphismGroup(cycle, vert_colours, edge_colours));
2
gap> IsTrivial(AutomorphismGroup(cycle,
> vert_colours, List([1 .. 12], x -> [x mod 4 + 1])));
true
```

7.2.7   BlissCanonicalLabelling (for a digraph)

▷ BlissCanonicalLabelling(*digraph*)   (attribute)
▷ NautyCanonicalLabelling(*digraph*)   (attribute)

   Returns: A permutation, or a list of two permutations.

   A function ρ that maps a digraph to a digraph is a *canonical representative map* if the following two conditions hold for all digraphs G and H:

   • $\rho(G)$ and G are isomorphic as digraphs; and

   • $\rho(G) = \rho(H)$ if and only if G and H are isomorphic as digraphs.

A *canonical labelling* of a digraph G (under ρ) is an isomorphism of G onto its *canonical representative*, $\rho(G)$. See `IsomorphismDigraphs` (7.2.17) for more information about isomorphisms of digraphs.

   `BlissCanonicalLabelling` returns a canonical labelling of the digraph *digraph* found using bliss by Tommi Junttila and Petteri Kaski. `NautyCanonicalLabelling` returns a canonical labelling of the digraph *digraph* found using nauty by Brendan McKay and Adolfo Piperno. Note that the canonical labellings returned by bliss and nauty are not usually the same (and may depend of the version used).

   `BlissCanonicalLabelling` can only be computed if *digraph* has no multiple edges; see `IsMultiDigraph` (6.2.11).

```
————————————————————— Example —————————————————————
gap> digraph1 := DigraphFromDiSparse6String(".ImNS_AiB?qRN");
<immutable digraph with 10 vertices, 8 edges>
gap> BlissCanonicalLabelling(digraph1);
(1,9,5,7)(3,6,4,10)
gap> p := (1, 2, 7, 5)(3, 9)(6, 10, 8);;
gap> digraph2 := OnDigraphs(digraph1, p);
```

```
   <immutable digraph with 10 vertices, 8 edges>
   gap> digraph1 = digraph2;
   false
   gap> OnDigraphs(digraph1, BlissCanonicalLabelling(digraph1)) =
   >      OnDigraphs(digraph2, BlissCanonicalLabelling(digraph2));
   true
```

7.2.8 BlissCanonicalLabelling (for a digraph and a list)

▷ BlissCanonicalLabelling(*digraph*, *colours*)   (operation)
▷ NautyCanonicalLabelling(*digraph*, *colours*)   (operation)

Returns: A permutation.

A function ρ that maps a coloured digraph to a coloured digraph is a *canonical representative map* if the following two conditions hold for all coloured digraphs G and H:

- $\rho(G)$ and G are isomorphic as coloured digraphs; and

- $\rho(G) = \rho(H)$ if and only if G and H are isomorphic as coloured digraphs.

A *canonical labelling* of a coloured digraph G (under ρ) is an isomorphism of G onto its *canonical representative*, $\rho(G)$. See IsomorphismDigraphs (7.2.18) for more information about isomorphisms of coloured digraphs.

A coloured digraph can be specified by its underlying digraph *digraph* and its colouring *colours*. Let n be the number of vertices of *digraph*. The colouring *colours* may have one of the following two forms:

- a list of n integers, where *colours*[i] is the colour of vertex i, using the colours [1 .. m] for some m <= n; or

- a list of non-empty disjoint lists whose union is DigraphVertices(*digraph*), such that *colours*[i] is the list of all vertices with colour i.

If *digraph* and *colours* together form a coloured digraph, BlissCanonicalLabelling returns a canonical labelling of the digraph *digraph* found using bliss by Tommi Junttila and Petteri Kaski. Similarly, NautyCanonicalLabelling returns a canonical labelling of the digraph *digraph* found using nauty by Brendan McKay and Adolfo Piperno. Note that the canonical labellings returned by bliss and nauty are not usually the same (and may depend of the version used).

BlissCanonicalLabelling can only be computed if *digraph* has no multiple edges; see IsMultiDigraph (6.2.11). The canonical labelling of *digraph* is given as a permutation of its vertices. The canonical representative of *digraph* can be created from *digraph* and its canonical labelling p by using the operation OnDigraphs (7.1.1):

```
─────────────────── Example ───────────────────
   gap> OnDigraphs(digraph, p);
```

The colouring of the canonical representative can easily be constructed. A vertex v (in *digraph*) has colour i if and only if the vertex v ^ p (in the canonical representative) has colour i, where p is the permutation of the canonical labelling that acts on the vertices of *digraph*. In particular, if *colours* has the first form that is described above, then the colouring of the canonical representative is given by:

```
                         ─── Example ───
  gap> List(DigraphVertices(digraph), i -> colours[i / p]);
```

On the other hand, if `colours` has the second form above, then the canonical representative has colouring:

```
                         ─── Example ───
  gap> OnTuplesSets(colours, p);
```

```
                         ─── Example ───
  gap> digraph := DigraphFromDiSparse6String(".ImNS_AiB?qRN");
  <immutable digraph with 10 vertices, 8 edges>
  gap> colours := [[1, 2, 8, 9, 10], [3, 4, 5, 6, 7]];;
  gap> p := BlissCanonicalLabelling(digraph, colours);
  (1,5,8,4,10,3,9)(6,7)
  gap> OnDigraphs(digraph, p);
  <immutable digraph with 10 vertices, 8 edges>
  gap> OnTuplesSets(colours, p);
  [ [ 1, 2, 3, 4, 5 ], [ 6, 7, 8, 9, 10 ] ]
  gap> colours := [1, 1, 1, 1, 2, 3, 1, 3, 2, 1];;
  gap> p := BlissCanonicalLabelling(digraph, colours);
  (1,6,9,7)(3,4,5,8,10)
  gap> OnDigraphs(digraph, p);
  <immutable digraph with 10 vertices, 8 edges>
  gap> List(DigraphVertices(digraph), i -> colours[i / p]);
  [ 1, 1, 1, 1, 1, 1, 2, 2, 3, 3 ]
```

7.2.9 BlissCanonicalDigraph

▷ BlissCanonicalDigraph(*digraph*)   (attribute)
▷ NautyCanonicalDigraph(*digraph*)   (attribute)
    Returns: A digraph.

    The attribute `BlissCanonicalLabelling` returns the canonical representative found by applying `BlissCanonicalLabelling` (7.2.7). The digraph returned is canonical in the sense that

- `BlissCanonicalDigraph(`*digraph*`)` and *digraph* are isomorphic as digraphs; and

- If gr is any digraph then `BlissCanonicalDigraph(gr)` and `BlissCanonicalDigraph(`*digraph*`)` are equal if and only if gr and *digraph* are isomorphic as digraphs.

Analogously, the attribute `NautyCanonicalLabelling` returns the canonical representative found by applying `NautyCanonicalLabelling` (7.2.7).

    If the argument *digraph* is mutable, then the return value of this attribute is recomputed every time it is called.

```
                         ─── Example ───
  gap> digraph := Digraph([[1], [2, 3], [3], [1, 2, 3]]);
  <immutable digraph with 4 vertices, 7 edges>
  gap> canon := BlissCanonicalDigraph(digraph);
  <immutable digraph with 4 vertices, 7 edges>
  gap> OutNeighbours(canon);
  [ [ 1 ], [ 2 ], [ 3, 2 ], [ 1, 3, 2 ] ]
```

7.2.10 DigraphGroup

▷ DigraphGroup(*digraph*) (attribute)

Returns: A permutation group.

If *digraph* is immutable and was created knowing a subgroup of its automorphism group, then this group is stored in the attribute DigraphGroup. If *digraph* is mutable, or was not created knowing a subgroup of its automorphism group, then DigraphGroup returns the entire automorphism group of *digraph*. Note that if *digraph* is mutable, then the automorphism group is recomputed every time this function is called.

Note that certain other constructor operations such as CayleyDigraph (3.1.12), BipartiteDoubleDigraph (3.3.44), and DoubleDigraph (3.3.43), may not require a group as one of the arguments, but use the standard constructor method using a group, and hence set the DigraphGroup attribute for the resulting digraph.

```
 _____ Example _____
 gap> n := 4;;
 gap> adj := function(x, y)
 >      return (((x - y) mod n) = 1) or (((x - y) mod n) = n - 1);
 >    end;;
 gap> group := CyclicGroup(IsPermGroup, n);
 Group([ (1,2,3,4) ])
 gap> D := Digraph(IsMutableDigraph, group, [1 .. n], \^, adj);
 <mutable digraph with 4 vertices, 8 edges>
 gap> HasDigraphGroup(D);
 false
 gap> DigraphGroup(D);
 Group([ (2,4), (1,2)(3,4) ])
 gap> AutomorphismGroup(D);
 Group([ (2,4), (1,2)(3,4) ])
 gap> D := Digraph(group, [1 .. n], \^, adj);
 <immutable digraph with 4 vertices, 8 edges>
 gap> HasDigraphGroup(D);
 true
 gap> DigraphGroup(D);
 Group([ (1,2,3,4) ])
 gap> D := DoubleDigraph(D);
 <immutable digraph with 8 vertices, 32 edges>
 gap> HasDigraphGroup(D);
 true
 gap> DigraphGroup(D);
 Group([ (1,2,3,4)(5,6,7,8), (1,5)(2,6)(3,7)(4,8) ])
 gap> AutomorphismGroup(D) =
 > Group([(6, 8), (5, 7), (4, 6), (3, 5), (2, 4),
 >        (1, 2)(3, 4)(5, 6)(7, 8)]);
 true
 gap> D := Digraph([[2, 3], [], []]);
 <immutable digraph with 3 vertices, 2 edges>
 gap> HasDigraphGroup(D);
 false
 gap> HasAutomorphismGroup(D);
 false
 gap> DigraphGroup(D);
 Group([ (2,3) ])
```

```
gap> HasAutomorphismGroup(D);
true
gap> group := DihedralGroup(8);
<pc group of size 8 with 3 generators>
gap> D := CayleyDigraph(group);
<immutable digraph with 8 vertices, 24 edges>
gap> HasDigraphGroup(D);
true
gap> GeneratorsOfGroup(DigraphGroup(D));
[ (1,2)(3,5)(4,6)(7,8), (1,7,4,3)(2,5,6,8), (1,4)(2,6)(3,7)(5,8) ]
```

7.2.11 DigraphOrbits

▷ DigraphOrbits(*digraph*)   (attribute)

   Returns: An immutable list of lists of integers.

   DigraphOrbits returns the orbits of the action of the DigraphGroup (7.2.10) on the set of vertices of *digraph*.

─────────────────────────── Example ───────────────────────────
```
gap> G := Group([(2, 3)(7, 8, 9), (1, 2, 3)(4, 5, 6)(8, 9)]);;
gap> D := EdgeOrbitsDigraph(G, [1, 2]);
<immutable digraph with 9 vertices, 6 edges>
gap> DigraphOrbits(D);
[ [ 1, 2, 3 ], [ 4, 5, 6 ], [ 7, 8, 9 ] ]
gap> D := DigraphMutableCopy(D);
<mutable digraph with 9 vertices, 6 edges>
gap> DigraphOrbits(D);
[ [ 1, 2, 3 ], [ 4, 5, 6, 7, 8, 9 ] ]
```

7.2.12 DigraphOrbitReps

▷ DigraphOrbitReps(*digraph*)   (attribute)

   Returns: An immutable list of integers.

   DigraphOrbitReps returns a list of orbit representatives of the action of the DigraphGroup (7.2.10) on the set of vertices of *digraph*.

─────────────────────────── Example ───────────────────────────
```
gap> D := CayleyDigraph(AlternatingGroup(4));
<immutable digraph with 12 vertices, 24 edges>
gap> DigraphOrbitReps(D);
[ 1 ]
gap> D := DigraphMutableCopy(D);
<mutable digraph with 12 vertices, 24 edges>
gap> DigraphOrbitReps(D);
[ 1 ]
gap> D := DigraphFromDigraph6String("&IGO??S?'?_@?a?CK?O");
<immutable digraph with 10 vertices, 14 edges>
gap> DigraphOrbitReps(D);
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ]
gap> DigraphOrbitReps(DigraphMutableCopy(D));
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ]
```

7.2.13 DigraphSchreierVector

▷ DigraphSchreierVector(`digraph`) (attribute)

    Returns: An immutable list of integers.

    DigraphSchreierVector returns the so-called *Schreier vector* of the action of the DigraphGroup (7.2.10) on the set of vertices of `digraph`. The Schreier vector is a list sch of integers with length DigraphNrVertices(`digraph`) where:

sch[i] < 0:

    implies that i is an orbit representative and DigraphOrbitReps(`digraph`)[-sch[i]] = i.

sch[i] > 0:

    implies that i / gens[sch[i]] is one step closer to the root (or representative) of the tree, where gens is the generators of DigraphGroup(`digraph`).

```
_____ Example _____
gap> n := 4;;
gap> adj := function(x, y)
>       return (((x - y) mod n) = 1) or (((x - y) mod n) = n - 1);
>     end;;
gap> group := CyclicGroup(IsPermGroup, n);
Group([ (1,2,3,4) ])
gap> D := Digraph(IsMutableDigraph, group, [1 .. n], \^, adj);
<mutable digraph with 4 vertices, 8 edges>
gap> sch := DigraphSchreierVector(D);
[ -1, 2, 2, 1 ]
gap> D := CayleyDigraph(AlternatingGroup(4));
<immutable digraph with 12 vertices, 24 edges>
gap> sch := DigraphSchreierVector(D);
[ -1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 1, 1 ]
gap> DigraphOrbitReps(D);
[ 1 ]
gap> gens := GeneratorsOfGroup(DigraphGroup(D));
[ (1,7,5)(2,10,9)(3,4,11)(6,8,12), (1,3,2)(4,5,6)(7,9,8)(10,11,12) ]
gap> 10 / gens[sch[10]];
2
gap> 7 / gens[sch[7]];
1
gap> 5 / gens[sch[5]];
7
```

7.2.14 DigraphStabilizer

▷ DigraphStabilizer(`digraph`, `v`) (operation)

    Returns: A permutation group.

    DigraphStabilizer returns the stabilizer of the vertex `v` under of the action of the DigraphGroup (7.2.10) on the set of vertices of `digraph`.

```
_____ Example _____
gap> D := DigraphFromDigraph6String("&GYHPQgWTIIPW");
<immutable digraph with 8 vertices, 24 edges>
gap> DigraphStabilizer(D, 8);
Group(())
```

```
gap> DigraphStabilizer(D, 2);
Group(())
gap> D := DigraphMutableCopy(D);
<mutable digraph with 8 vertices, 24 edges>
gap> DigraphStabilizer(D, 8);
Group(())
gap> DigraphStabilizer(D, 2);
Group(())
```

7.2.15 IsIsomorphicDigraph (for digraphs)

▷ IsIsomorphicDigraph(*digraph1, digraph2*)   (operation)

    Returns: true or false.

    This operation returns true if there exists an isomorphism from the digraph *digraph1* to the digraph *digraph2*. See IsomorphismDigraphs (7.2.17) for more information about isomorphisms of digraphs.

    By default, an isomorphism is found using the canonical labellings of the digraphs obtained from bliss by Tommi Junttila and Petteri Kaski. If NautyTracesInterface is available, then nauty by Brendan Mckay and Adolfo Piperno can be used instead; see DigraphsUseBliss (7.2.1), and DigraphsUseNauty (7.2.1).

```
_____ Example _____
gap> digraph1 := CycleDigraph(4);
<immutable cycle digraph with 4 vertices>
gap> digraph2 := CycleDigraph(5);
<immutable cycle digraph with 5 vertices>
gap> IsIsomorphicDigraph(digraph1, digraph2);
false
gap> digraph2 := DigraphReverse(digraph1);
<immutable digraph with 4 vertices, 4 edges>
gap> IsIsomorphicDigraph(digraph1, digraph2);
true
gap> digraph1 := Digraph([[3], [], []]);
<immutable digraph with 3 vertices, 1 edge>
gap> digraph2 := Digraph([[], [], [2]]);
<immutable digraph with 3 vertices, 1 edge>
gap> IsIsomorphicDigraph(digraph1, digraph2);
true
```

7.2.16 IsIsomorphicDigraph (for digraphs and homogeneous lists)

▷ IsIsomorphicDigraph(*digraph1, digraph2, colours1, colours2*)   (operation)

    Returns: true or false.

    This operation tests for isomorphism of coloured digraphs. A coloured digraph can be specified by its underlying digraph *digraph1* and its colouring *colours1*. Let n be the number of vertices of *digraph1*. The colouring *colours1* may have one of the following two forms:

- a list of n integers, where *colours*[i] is the colour of vertex i, using the colours [1 .. m] for some m <= n; or

- a list of non-empty disjoint lists whose union is DigraphVertices(*digraph*), such that *colours*[i] is the list of all vertices with colour i.

If *digraph1* and *digraph2* are digraphs without multiple edges, and *colours1* and *colours2* are colourings of *digraph1* and *digraph2*, respectively, then this operation returns true if there exists an isomorphism between these two coloured digraphs. See IsomorphismDigraphs (7.2.18) for more information about isomorphisms of coloured digraphs.

By default, an isomorphism is found using the canonical labellings of the digraphs obtained from bliss by Tommi Junttila and Petteri Kaski. If NautyTracesInterface is available, then nauty by Brendan Mckay and Adolfo Piperno can be used instead; see DigraphsUseBliss (7.2.1), and DigraphsUseNauty (7.2.1).

```
──────────────── Example ────────────────
gap> digraph1 := ChainDigraph(4);
<immutable chain digraph with 4 vertices>
gap> digraph2 := ChainDigraph(3);
<immutable chain digraph with 3 vertices>
gap> IsIsomorphicDigraph(digraph1, digraph2,
>  [[1, 4], [2, 3]], [[1, 2], [3]]);
false
gap> digraph2 := DigraphReverse(digraph1);
<immutable digraph with 4 vertices, 3 edges>
gap> IsIsomorphicDigraph(digraph1, digraph2,
>  [1, 1, 1, 1], [1, 1, 1, 1]);
true
gap> IsIsomorphicDigraph(digraph1, digraph2,
>  [1, 2, 2, 1], [1, 2, 2, 1]);
true
gap> IsIsomorphicDigraph(digraph1, digraph2,
>  [1, 1, 2, 2], [1, 1, 2, 2]);
false
```

7.2.17 IsomorphismDigraphs (for digraphs)

▷ IsomorphismDigraphs(*digraph1, digraph2*) (operation)

Returns: A permutation, or a pair of permutations, or fail.

This operation returns an isomorphism between the digraphs *digraph1* and *digraph2* if one exists, else this operation returns fail.

An *isomorphism* from a digraph *digraph1* to a digraph *digraph2* is a bijection p from the vertices of *digraph1* to the vertices of *digraph2* with the following property: for all vertices i and j of *digraph1*, [i, j] is an edge of *digraph1* if and only if [i ^ p, j ^ p] is an edge of *digraph2*.

If there exists such an isomorphism, then this operation returns one. The form of this isomorphism is a permutation p of the vertices of *digraph1* such that

OnDigraphs(*digraph1*, p) = digraph2. By default, an isomorphism is found using the canonical labellings of the digraphs obtained from bliss by Tommi Junttila and Petteri Kaski. If NautyTracesInterface is available, then nauty by Brendan Mckay and Adolfo Piperno can be used instead; see DigraphsUseBliss (7.2.1), and DigraphsUseNauty (7.2.1).

```
──────────────── Example ────────────────
gap> digraph1 := CycleDigraph(4);
<immutable cycle digraph with 4 vertices>
gap> digraph2 := CycleDigraph(5);
```

```
<immutable cycle digraph with 5 vertices>
gap> IsomorphismDigraphs(digraph1, digraph2);
fail
gap> digraph1 := CompleteBipartiteDigraph(10, 5);
<immutable complete bipartite digraph with bicomponent sizes 10 and 5>
gap> digraph2 := CompleteBipartiteDigraph(5, 10);
<immutable complete bipartite digraph with bicomponent sizes 5 and 10>
gap> p := IsomorphismDigraphs(digraph1, digraph2);
(1,6,11)(2,7,12)(3,8,13)(4,9,14)(5,10,15)
gap> OnDigraphs(digraph1, p) = digraph2;
true
```

7.2.18 IsomorphismDigraphs (for digraphs and homogeneous lists)

▷ IsomorphismDigraphs(*digraph1, digraph2, colours1, colours2*)   (operation)
    Returns: A permutation, or `fail`.

This operation searches for an isomorphism between coloured digraphs. A coloured digraph can be specified by its underlying digraph *digraph1* and its colouring *colours1*. Let n be the number of vertices of *digraph1*. The colouring *colours1* may have one of the following two forms:

- a list of n integers, where *colours*[i] is the colour of vertex i, using the colours [1 .. m] for some m <= n; or

- a list of non-empty disjoint lists whose union is DigraphVertices(*digraph*), such that *colours*[i] is the list of all vertices with colour i.

An *isomorphism* between coloured digraphs is an isomorphism between the underlying digraphs that preserves the colourings. See IsomorphismDigraphs (7.2.17) for more information about isomorphisms of digraphs. More precisely, let f be an isomorphism of digraphs from the digraph *digraph1* (with colouring *colours1*) to the digraph *digraph2* (with colouring *colours2*), and let p be the permutation of the vertices of *digraph1* that corresponds to f. Then f preserves the colourings of *digraph1* and *digraph2* – and hence is an isomorphism of coloured digraphs – if *colours1*[i] = *colours2*[i ^ p] for all vertices i in *digraph1*.

This operation returns such an isomorphism if one exists, else it returns `fail`.

By default, an isomorphism is found using the canonical labellings of the digraphs obtained from bliss by Tommi Junttila and Petteri Kaski. If NautyTracesInterface is available, then nauty by Brendan Mckay and Adolfo Piperno can be used instead; see DigraphsUseBliss (7.2.1), and DigraphsUseNauty (7.2.1).

```
——————————— Example ———————————
gap> digraph1 := ChainDigraph(4);
<immutable chain digraph with 4 vertices>
gap> digraph2 := ChainDigraph(3);
<immutable chain digraph with 3 vertices>
gap> IsomorphismDigraphs(digraph1, digraph2,
>   [[1, 4], [2, 3]], [[1, 2], [3]]);
fail
gap> digraph2 := DigraphReverse(digraph1);
<immutable digraph with 4 vertices, 3 edges>
gap> colours1 := [1, 1, 1, 1];;
gap> colours2 := [1, 1, 1, 1];;
```

```
gap> p := IsomorphismDigraphs(digraph1, digraph2, colours1, colours2);
(1,4)(2,3)
gap> OnDigraphs(digraph1, p) = digraph2;
true
gap> List(DigraphVertices(digraph1), i -> colours1[i ^ p]) = colours2;
true
gap> colours1 := [1, 1, 2, 2];;
gap> colours2 := [2, 2, 1, 1];;
gap> p := IsomorphismDigraphs(digraph1, digraph2, colours1, colours2);
(1,4)(2,3)
gap> OnDigraphs(digraph1, p) = digraph2;
true
gap> List(DigraphVertices(digraph1), i -> colours1[i ^ p]) = colours2;
true
gap> IsomorphismDigraphs(digraph1, digraph2,
>  [1, 1, 2, 2], [1, 1, 2, 2]);
fail
```

7.2.19   RepresentativeOutNeighbours

▷ RepresentativeOutNeighbours(*digraph*)   (attribute)

   Returns: An immutable list of lists.

   This function returns the list out of *out-neighbours* of each representative of the orbits of the action of DigraphGroup (7.2.10) on the vertex set of the digraph *digraph*.

   More specifically, if reps is the list of orbit representatives, then a vertex j appears in out[i] each time there exists an edge with source reps[i] and range j in *digraph*.

   If DigraphGroup (7.2.10) is trivial, then OutNeighbours (5.2.6) is returned.

```
                                   Example
gap> D := Digraph([
>  [2, 1, 3, 4, 5], [3, 5], [2], [1, 2, 3, 5], [1, 2, 3, 4]]);
<immutable digraph with 5 vertices, 16 edges>
gap> DigraphGroup(D);
Group(())
gap> RepresentativeOutNeighbours(D);
[ [ 2, 1, 3, 4, 5 ], [ 3, 5 ], [ 2 ], [ 1, 2, 3, 5 ], [ 1, 2, 3, 4 ] ]
gap> D := Digraph(IsMutableDigraph, [
>  [2, 1, 3, 4, 5], [3, 5], [2], [1, 2, 3, 5], [1, 2, 3, 4]]);
<mutable digraph with 5 vertices, 16 edges>
gap> DigraphGroup(D);
Group(())
gap> RepresentativeOutNeighbours(D);
[ [ 2, 1, 3, 4, 5 ], [ 3, 5 ], [ 2 ], [ 1, 2, 3, 5 ], [ 1, 2, 3, 4 ] ]
gap> D := DigraphFromDigraph6String("&GYHPQgWTIIPW");
<immutable digraph with 8 vertices, 24 edges>
gap> G := DigraphGroup(D);;
gap> GeneratorsOfGroup(G);
[ (1,2)(3,4)(5,6)(7,8), (1,3,2,4)(5,7,6,8), (1,5)(2,6)(3,8)(4,7) ]
gap> Set(RepresentativeOutNeighbours(D), Set);
[ [ 2, 3, 5 ] ]
```

7.2.20 IsDigraphIsomorphism (for digraphs and transformation or permutation)

▷ IsDigraphIsomorphism(*src, ran, x*)   (operation)
▷ IsDigraphIsomorphism(*src, ran, x, col1, col2*)   (operation)
▷ IsDigraphAutomorphism(*digraph, x*)   (operation)
▷ IsDigraphAutomorphism(*digraph, x, col*)   (operation)

Returns: true or false.

IsDigraphIsomorphism returns true if the permutation or transformation *x* is an isomorphism from the digraph *src* to the digraph *ran*.

IsDigraphAutomorphism returns true if the permutation or transformation *x* is an automorphism of the digraph *digraph*.

A permutation or transformation *x* is an *isomorphism* from a digraph *src* to a digraph *ran* if the following hold:

- *x* is a bijection from the vertices of *src* to those of *ran*;

- [u ^ x, v ^ x] is an edge of *ran* if and only if [u, v] is an edge of *src*; and

- *x* fixes every i which is not a vertex of *src*.

See also AutomorphismGroup (7.2.2).

If *col1* and *col2*, or *col*, are given, then they must represent vertex colourings; see AutomorphismGroup (7.2.5) for details of the permissible values for these arguments. The homomorphism must then also have the property:

- col1[i] = col2[i ^ x] for all vertices i of *src*, for IsDigraphIsomorphism.

- col[i] = col[i ^ x] for all vertices i of *digraph*, for IsDigraphAutomorphism.

For some digraphs, it can be faster to use IsDigraphAutomorphism than to test membership in the automorphism group of *digraph*.

```
─────────────────────────── Example ───────────────────────────
gap> src := Digraph([[1], [1, 2], [1, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsDigraphAutomorphism(src, (1, 2, 3));
false
gap> IsDigraphAutomorphism(src, (2, 3));
true
gap> IsDigraphAutomorphism(src, (2, 3), [2, 1, 1]);
true
gap> IsDigraphAutomorphism(src, (2, 3), [2, 2, 1]);
false
gap> IsDigraphAutomorphism(src, (2, 3)(4, 5));
true
gap> IsDigraphAutomorphism(src, (1, 4));
false
gap> IsDigraphAutomorphism(src, ());
true
gap> ran := Digraph([[2, 1], [2], [2, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsDigraphIsomorphism(src, ran, (1, 2));
true
```

```
gap> IsDigraphIsomorphism(ran, src, (1, 2));
true
gap> IsDigraphIsomorphism(ran, src, (1, 2));
true
gap> IsDigraphIsomorphism(src, Digraph([[3], [1, 3], [2]]), (1, 2, 3));
false
gap> IsDigraphIsomorphism(src, ran, (1, 2), [1, 2, 3], [2, 1, 3]);
true
gap> IsDigraphIsomorphism(src, ran, (1, 2), [1, 2, 2], [2, 1, 3]);
false
```

7.2.21 IsDigraphColouring

▷ IsDigraphColouring(*digraph*, *list*)   (operation)
▷ IsDigraphColouring(*digraph*, *t*)   (operation)
  Returns: true or false.

  The operation IsDigraphColouring verifies whether or not the list *list* describes a proper colouring of the digraph *digraph*.

  A list *list* describes a *proper colouring* of a digraph *digraph* if *list* consists of positive integers, the length of *list* equals the number of vertices in *digraph*, and for any vertices u, v of *digraph* if u and v are adjacent, then *list*[u] >< *list*[v].

  A transformation *t* describes a proper colouring of a digraph *digraph*, if ImageListOfTransformation(*t*, DigraphNrVertices(*digraph*)) is a proper colouring of *digraph*.

  See also IsDigraphHomomorphism (7.3.10).

```
───────────────────── Example ─────────────────────
gap> D := JohnsonDigraph(5, 3);
<immutable symmetric digraph with 10 vertices, 60 edges>
gap> IsDigraphColouring(D, [1, 2, 3, 3, 2, 1, 4, 5, 6, 7]);
true
gap> IsDigraphColouring(D, [1, 2, 3, 3, 2, 1, 2, 5, 6, 7]);
false
gap> IsDigraphColouring(D, [1, 2, 3, 3, 2, 1, 2, 5, 6, -1]);
false
gap> IsDigraphColouring(D, [1, 2, 3]);
false
gap> IsDigraphColouring(D, IdentityTransformation);
true
```

7.2.22 MaximalCommonSubdigraph

▷ MaximalCommonSubdigraph(*D1*, *D2*)   (operation)
  Returns: A list containing a digraph and two transformations.

  If *D1* and *D2* are digraphs without multiple edges, then MaximalCommonSubdigraph returns a maximal common subgraph M of *D1* and *D2* with the maximum number of vertices. So M is a digraph which embeds into both *D1* and *D2* and has the largest number of vertices among such digraphs. It returns a list [M, t1, t2] where M is the maximal common subdigraph and t1, t2 are transformations embedding M into *D1* and *D2* respectively.

```
  _____ Example _____
  gap> MaximalCommonSubdigraph(PetersenGraph(), CompleteDigraph(10));
  [ <immutable digraph with 2 vertices, 2 edges>,
    IdentityTransformation, IdentityTransformation ]
  gap> MaximalCommonSubdigraph(PetersenGraph(),
  > DigraphSymmetricClosure(CycleDigraph(5)));
  [ <immutable digraph with 5 vertices, 10 edges>,
    IdentityTransformation, IdentityTransformation ]
  gap> MaximalCommonSubdigraph(NullDigraph(0), CompleteDigraph(10));
  [ <immutable empty digraph with 0 vertices>, IdentityTransformation,
    IdentityTransformation ]
```

7.2.23 MinimalCommonSuperdigraph

▷ MinimalCommonSuperdigraph(*D1, D2*)   (operation)

Returns: A list containing a digraph and two transformations.

If *D1* and *D2* are digraphs without multiple edges, then `MinimalCommonSuperdigraph` returns a minimal common superdigraph M of *D1* and *D2* with the minimum number of vertices. So M is a digraph into which both *D1* and *D2* embed and has the smallest number of vertices among such digraphs. It returns a list [M, t1, t2] where M is the minimal common superdigraph and t1, t2 are transformations embedding *D1* and *D2* respectively into M.

```
  _____ Example _____
  gap> MinimalCommonSuperdigraph(PetersenGraph(), CompleteDigraph(10));
  [ <immutable digraph with 18 vertices, 118 edges>,
    IdentityTransformation,
    Transformation( [ 1, 2, 11, 12, 13, 14, 15, 16, 17, 18, 11, 12, 13,
        14, 15, 16, 17, 18 ] ) ]
  gap> MinimalCommonSuperdigraph(PetersenGraph(),
  > DigraphSymmetricClosure(CycleDigraph(5)));
  [ <immutable digraph with 10 vertices, 30 edges>,
    IdentityTransformation, IdentityTransformation ]
  gap> MinimalCommonSuperdigraph(NullDigraph(0), CompleteDigraph(10));
  [ <immutable digraph with 10 vertices, 90 edges>,
    IdentityTransformation, IdentityTransformation ]
```

7.3 Homomorphisms of digraphs

The following methods exist to find homomorphisms between digraphs. If an argument to one of these methods is a digraph with multiple edges, then the multiplicity of edges will be ignored in order to perform the calculation; the digraph will be treated as if it has no multiple edges.

7.3.1 HomomorphismDigraphsFinder

▷ HomomorphismDigraphsFinder(*D1, D2, hook, user_param, max_results, hint, injective, image, partial_map, colors1, colors2[, order, aut_grp]*)   (function)

Returns: The argument `user_param`.

This function finds homomorphisms from the digraph *D1* to the digraph *D2* subject to the conditions imposed by the other arguments as described below.

If `f` and `g` are homomorphisms found by `HomomorphismDigraphsFinder`, then `f` cannot be obtained from `g` by right multiplying by an automorphism of *D2* in `aut_grp`.

hook

This argument should be a function or `fail`.

If *hook* is a function, then it must have two arguments *user_param* (see below) and a transformation `t`. The function *hook*(*user_param*, `t`) is called every time a new homomorphism `t` is found by `HomomorphismDigraphsFinder`. If the function returns `true`, then `HomomorphismDigraphsFinder` stops and does not find any further homomorphisms. This feature might be useful if you are searching for a homomorphism that satisfies some condition that you cannot specify via the other arguments to `HomomorphismDigraphsFinder`.

If *hook* is `fail`, then a default function is used which simply adds every new homomorphism found by `HomomorphismDigraphsFinder` to *user_param*, which must be a mutable list in this case.

user_param

If *hook* is a function, then *user_param* can be any **GAP** object. The object *user_param* is used as the first argument of the function *hook*. For example, *user_param* might be a transformation semigroup, and *hook*(*user_param*, `t`) might set *user_param* to be the closure of *user_param* and `t`.

If the value of *hook* is `fail`, then the value of *user_param* must be a mutable list.

max_results

This argument should be a positive integer or `infinity`. `HomomorphismDigraphsFinder` will return after it has found *max_results* homomorphisms or the search is complete, whichever happens first.

hint

This argument should be a positive integer or `fail`.

If *hint* is a positive integer, then only homorphisms of rank *hint* are found.

If *hint* is `fail`, then no restriction is put on the rank of homomorphisms found.

injective

This argument should be 0, 1, or 2. If it is 2, then only embeddings are found, if it is 1, then only injective homomorphisms are found, and if it is 0 there are no restrictions imposed by this argument.

For backwards compatibility, *injective* can also be `false` or `true` which correspond to the values 0 and 1 described in the previous paragraph, respectively.

image

This argument should be a subset of the vertices of the graph *D2*. `HomomorphismDigraphsFinder` only finds homomorphisms from *D1* to the subgraph of *D2* induced by the vertices *image*.

The returned homomorphisms (if any) are still "up to the action" of the group specified by `aut_grp` (which is the entire automorphism group by default). This might generate unexpected results. For example, if *D1* has automorphism group where one orbit consists of, say, 1 and

2, then `HomomorphismDigraphsFinder` will only attempt to find homomorphisms mapping 1 to 1, and if there are no such homomorphisms with image set equal to *image*, then no homomorphisms will be returned (even if there is a homomorphism from *D1* to *D2* mapping 1 to 2). To ensure that that ALL homomorphisms with image set equal to *image* are considered it is necessary for the last argument `aut_grp` to be the trivial permutation group.

partial_map

This argument should be a partial map from *D1* to *D2*, that is, a (not necessarily dense) list of vertices of the digraph *D2* of length no greater than the number vertices in the digraph *D1*. `HomomorphismDigraphsFinder` only finds homomorphisms extending *partial_map* (if any).

colors1

This should be a list representing possible colours of vertices in the digraph *D1*; see `AutomorphismGroup` (7.2.5) for details of the permissible values for this argument.

colors2

This should be a list representing possible colours of vertices in the digraph *D2*; see `AutomorphismGroup` (7.2.5) for details of the permissible values for this argument.

order

The optional argument *order* specifies the order the vertices in *D1* appear in the search for homomorphisms. The value of this parameter can have a large impact on the runtime of the function. It seems in many cases to be a good idea for this to be the `DigraphWelshPowellOrder` (7.3.17), i.e. vertices ordered from highest to lowest degree.

aut_grp

The optional argument `aut_grp` should be a subgroup of the automorphism group of *D2*. This function returns unique representatives of the homomorphisms found up to right multiplication by `aut_grp`. If this argument is not specific, it defaults to the full automorphism group of *D2*, which may be costly to calculate.

```
──────────────────── Example ────────────────────
gap> D := ChainDigraph(10);
<immutable chain digraph with 10 vertices>
gap> D := DigraphSymmetricClosure(D);
<immutable symmetric digraph with 10 vertices, 18 edges>
gap> HomomorphismDigraphsFinder(D, D, fail, [], infinity, 2, 0,
> [3, 4], [], fail, fail);
#I  WARNING you are trying to find homomorphisms by specifying a subset
of the vertices of the target digraph. This might lead to unexpected
results! If this happens, try passing Group(()) as the last argument.
Please see the documentation of HomomorphismDigraphsFinder for details.
[ Transformation( [ 3, 4, 3, 4, 3, 4, 3, 4, 3, 4 ] ),
  Transformation( [ 4, 3, 4, 3, 4, 3, 4, 3, 4, 3 ] ) ]
gap> D2 := CompleteDigraph(6);;
gap> HomomorphismDigraphsFinder(D, D2, fail, [], 1, fail, 0,
> [1 .. 6], [1, 2, 1], fail, fail);
[ Transformation( [ 1, 2, 1, 3, 4, 5, 6, 1, 2, 1 ] ) ]
gap> func := function(user_param, t)
> Add(user_param, t * user_param[1]);
> end;;
gap> HomomorphismDigraphsFinder(D, D2, func, [Transformation([2, 2])],
```

```
> 3, fail, 0, [1 .. 6], [1, 2, 1], fail, fail);
[ Transformation( [ 2, 2 ] ),
  Transformation( [ 2, 2, 2, 3, 4, 5, 6, 2, 2, 2 ] ),
  Transformation( [ 2, 2, 2, 3, 4, 5, 6, 2, 2, 3 ] ),
  Transformation( [ 2, 2, 2, 3, 4, 5, 6, 2, 2, 4 ] ) ]
gap> HomomorphismDigraphsFinder(NullDigraph(2), NullDigraph(3), fail,
> [], infinity, fail, 1, [1, 2, 3], fail, fail, fail, fail,
> Group(()));
[ IdentityTransformation, Transformation( [ 1, 3, 3 ] ),
  Transformation( [ 2, 1 ] ), Transformation( [ 2, 3, 3 ] ),
  Transformation( [ 3, 1, 3 ] ), Transformation( [ 3, 2, 3 ] ) ]
gap> HomomorphismDigraphsFinder(NullDigraph(2), NullDigraph(3), fail,
> [], infinity, fail, 1, [1, 2, 3], fail, fail, fail, fail,
> Group((1, 2)));
[ IdentityTransformation, Transformation( [ 1, 3, 3 ] ),
  Transformation( [ 3, 1, 3 ] ) ]
```

7.3.2 DigraphHomomorphism

▷ DigraphHomomorphism(*digraph1, digraph2*)   (operation)

 Returns: A transformation, or `fail`.

A homomorphism from *digraph1* to *digraph2* is a mapping from the vertex set of *digraph1* to a subset of the vertices of *digraph2*, such that every pair of vertices [i,j] which has an edge i->j is mapped to a pair of vertices [a,b] which has an edge a->b. Note that non-adjacent vertices can still be mapped to adjacent vertices.

DigraphHomomorphism returns a single homomorphism between *digraph1* and *digraph2* if it exists, otherwise it returns `fail`.

```
─────────────────────── Example ───────────────────────
gap> gr1 := ChainDigraph(3);;
gap> gr2 := Digraph([[3, 5], [2], [3, 1], [], [4]]);
<immutable digraph with 5 vertices, 6 edges>
gap> DigraphHomomorphism(gr1, gr1);
IdentityTransformation
gap> map := DigraphHomomorphism(gr1, gr2);
Transformation( [ 3, 1, 5, 4, 5 ] )
gap> IsDigraphHomomorphism(gr1, gr2, map);
true
```

7.3.3 HomomorphismsDigraphs

▷ HomomorphismsDigraphs(*digraph1, digraph2*)   (operation)
▷ HomomorphismsDigraphsRepresentatives(*digraph1, digraph2*)   (operation)

 Returns: A list of transformations.

HomomorphismsDigraphsRepresentatives finds every DigraphHomomorphism (7.3.2) between *digraph1* and *digraph2*, up to right multiplication by an element of the AutomorphismGroup (7.2.2) of *digraph2*. In other words, every homomorphism f between *digraph1* and *digraph2* can be written as the composition f = g * x, where g is one of the HomomorphismsDigraphsRepresentatives and x is an automorphism of *digraph2*.

HomomorphismsDigraphs returns all homomorphisms between *digraph1* and *digraph2*.

```
                         Example
gap> gr1 := ChainDigraph(3);;
gap> gr2 := Digraph([[3, 5], [2], [3, 1], [], [4]]);
<immutable digraph with 5 vertices, 6 edges>
gap> HomomorphismsDigraphs(gr1, gr2);
[ Transformation( [ 1, 3, 1 ] ), Transformation( [ 1, 3, 3 ] ),
  Transformation( [ 1, 5, 4, 4, 5 ] ), Transformation( [ 2, 2, 2 ] ),
  Transformation( [ 3, 1, 3 ] ), Transformation( [ 3, 1, 5, 4, 5 ] ),
  Transformation( [ 3, 3, 1 ] ), Transformation( [ 3, 3, 3 ] ) ]
gap> HomomorphismsDigraphsRepresentatives(gr1, CompleteDigraph(3));
[ Transformation( [ 2, 1 ] ), Transformation( [ 2, 1, 2 ] ) ]
```

7.3.4 DigraphMonomorphism

▷ DigraphMonomorphism(*digraph1, digraph2*)   (operation)

  Returns: A transformation, or `fail`.

  DigraphMonomorphism returns a single *injective* DigraphHomomorphism (7.3.2) between *digraph1* and *digraph2* if one exists, otherwise it returns `fail`.

```
                         Example
gap> gr1 := ChainDigraph(3);;
gap> gr2 := Digraph([[3, 5], [2], [3, 1], [], [4]]);
<immutable digraph with 5 vertices, 6 edges>
gap> DigraphMonomorphism(gr1, gr1);
IdentityTransformation
gap> DigraphMonomorphism(gr1, gr2);
Transformation( [ 3, 1, 5, 4, 5 ] )
```

7.3.5 MonomorphismsDigraphs

▷ MonomorphismsDigraphs(*digraph1, digraph2*)   (operation)
▷ MonomorphismsDigraphsRepresentatives(*digraph1, digraph2*)   (operation)

  Returns: A list of transformations.

  These operations behave the same as HomomorphismsDigraphs (7.3.3) and HomomorphismsDigraphsRepresentatives (7.3.3), except they only return *injective* homomorphisms.

```
                         Example
gap> gr1 := ChainDigraph(3);;
gap> gr2 := Digraph([[3, 5], [2], [3, 1], [], [4]]);
<immutable digraph with 5 vertices, 6 edges>
gap> MonomorphismsDigraphs(gr1, gr2);
[ Transformation( [ 1, 5, 4, 4, 5 ] ),
  Transformation( [ 3, 1, 5, 4, 5 ] ) ]
gap> MonomorphismsDigraphsRepresentatives(gr1, CompleteDigraph(3));
[ Transformation( [ 2, 1 ] ) ]
```

7.3.6 DigraphEpimorphism

▷ DigraphEpimorphism(*digraph1, digraph2*)   (operation)

  Returns: A transformation, or `fail`.

DigraphEpimorphism returns a single *surjective* DigraphHomomorphism (7.3.2) between `digraph1` and `digraph2` if one exists, otherwise it returns `fail`.

```
─────────────────────────── Example ───────────────────────────
gap> gr1 := DigraphReverse(ChainDigraph(4));
<immutable digraph with 4 vertices, 3 edges>
gap> gr2 := DigraphRemoveEdge(CompleteDigraph(3), [1, 2]);
<immutable digraph with 3 vertices, 5 edges>
gap> DigraphEpimorphism(gr2, gr1);
fail
gap> DigraphEpimorphism(gr1, gr2);
Transformation( [ 3, 1, 2, 3 ] )
```

7.3.7 EpimorphismsDigraphs

▷ EpimorphismsDigraphs(`digraph1, digraph2`)   (operation)
▷ EpimorphismsDigraphsRepresentatives(`digraph1, digraph2`)   (operation)

Returns: A list of transformations.

These operations behave the same as HomomorphismsDigraphs (7.3.3) and HomomorphismsDigraphsRepresentatives (7.3.3), except they only return *surjective* homomorphisms.

```
─────────────────────────── Example ───────────────────────────
gap> gr1 := DigraphReverse(ChainDigraph(4));
<immutable digraph with 4 vertices, 3 edges>
gap> gr2 := DigraphSymmetricClosure(CycleDigraph(3));
<immutable symmetric digraph with 3 vertices, 6 edges>
gap> EpimorphismsDigraphsRepresentatives(gr1, gr2);
[ Transformation( [ 3, 1, 2, 1 ] ), Transformation( [ 3, 1, 2, 3 ] ),
  Transformation( [ 2, 1, 2, 3 ] ) ]
gap> EpimorphismsDigraphs(gr1, gr2);
[ Transformation( [ 1, 2, 1, 3 ] ), Transformation( [ 1, 2, 3, 1 ] ),
  Transformation( [ 1, 2, 3, 2 ] ), Transformation( [ 1, 3, 1, 2 ] ),
  Transformation( [ 1, 3, 2, 1 ] ), Transformation( [ 1, 3, 2, 3 ] ),
  Transformation( [ 2, 1, 2, 3 ] ), Transformation( [ 2, 1, 3, 1 ] ),
  Transformation( [ 2, 1, 3, 2 ] ), Transformation( [ 2, 3, 1, 2 ] ),
  Transformation( [ 2, 3, 1, 3 ] ), Transformation( [ 2, 3, 2, 1 ] ),
  Transformation( [ 3, 1, 2, 1 ] ), Transformation( [ 3, 1, 2, 3 ] ),
  Transformation( [ 3, 1, 3, 2 ] ), Transformation( [ 3, 2, 1, 2 ] ),
  Transformation( [ 3, 2, 1, 3 ] ), Transformation( [ 3, 2, 3, 1 ] ) ]
```

7.3.8 DigraphEmbedding

▷ DigraphEmbedding(`digraph1, digraph2`)   (operation)

Returns: A transformation, or `fail`.

An embedding of a digraph `digraph1` into another digraph `digraph2` is a DigraphMonomorphism (7.3.4) from `digraph1` to `digraph2` which has the additional property that a pair of vertices `[i, j]` which have no edge i -> j in `digraph1` are mapped to a pair of vertices `[a, b]` which have no edge a->b in `digraph2`.

In other words, an embedding t is an isomorphism from `digraph1` to the InducedSubdigraph (3.3.3) of `digraph2` on the image of t.

DigraphEmbedding returns a single embedding if one exists, otherwise it returns `fail`.

```
                              ── Example ──
  gap> gr := ChainDigraph(3);
  <immutable chain digraph with 3 vertices>
  gap> DigraphEmbedding(gr, CompleteDigraph(4));
  fail
  gap> DigraphEmbedding(gr, Digraph([[3], [1, 4], [1], [3]]));
  Transformation( [ 2, 4, 3, 4 ] )
```

7.3.9 EmbeddingsDigraphs

▷ EmbeddingsDigraphs(*D1, D2*)   (operation)
▷ EmbeddingsDigraphsRepresentatives(*D1, D2*)   (operation)

    Returns: A list of transformations.

    These operations behave the same as HomomorphismsDigraphs (7.3.3) and HomomorphismsDigraphsRepresentatives (7.3.3), except they only return embeddings of *D1* into *D2*.

    See also IsDigraphEmbedding (7.3.11).

```
                              ── Example ──
  gap> D1 := NullDigraph(2);
  <immutable empty digraph with 2 vertices>
  gap> D2 := CycleDigraph(5);
  <immutable cycle digraph with 5 vertices>
  gap> EmbeddingsDigraphsRepresentatives(D1, D2);
  [ Transformation( [ 1, 3, 3 ] ), Transformation( [ 1, 4, 3, 4 ] ) ]
  gap> EmbeddingsDigraphs(D1, D2);
  [ Transformation( [ 1, 3, 3 ] ), Transformation( [ 1, 4, 3, 4 ] ),
    Transformation( [ 2, 4, 4, 5, 1 ] ),
    Transformation( [ 2, 5, 4, 5, 1 ] ),
    Transformation( [ 3, 1, 5, 1, 2 ] ),
    Transformation( [ 3, 5, 5, 1, 2 ] ),
    Transformation( [ 4, 1, 1, 2, 3 ] ),
    Transformation( [ 4, 2, 1, 2, 3 ] ),
    Transformation( [ 5, 2, 2, 3, 4 ] ),
    Transformation( [ 5, 3, 2, 3, 4 ] ) ]
```

7.3.10 IsDigraphHomomorphism (for digraphs and a permutation or transformation)

▷ IsDigraphHomomorphism(*src, ran, x*)   (operation)
▷ IsDigraphHomomorphism(*src, ran, x, col1, col2*)   (operation)
▷ IsDigraphEpimorphism(*src, ran, x*)   (operation)
▷ IsDigraphEpimorphism(*src, ran, x, col1, col2*)   (operation)
▷ IsDigraphMonomorphism(*src, ran, x*)   (operation)
▷ IsDigraphMonomorphism(*src, ran, x, col1, col2*)   (operation)
▷ IsDigraphEndomorphism(*digraph, x*)   (operation)
▷ IsDigraphEndomorphism(*digraph, x, col*)   (operation)

    Returns: true or false.

    IsDigraphHomomorphism returns true if the permutation or transformation *x* is a homomorphism from the digraph *src* to the digraph *ran*.

IsDigraphEpimorphism returns `true` if the permutation or transformation `x` is a surjective homomorphism from the digraph `src` to the digraph `ran`.

IsDigraphMonomorphism returns `true` if the permutation or transformation `x` is an injective homomorphism from the digraph `src` to the digraph `ran`.

IsDigraphEndomorphism returns `true` if the permutation or transformation `x` is an endomorphism of the digraph `digraph`.

A permutation or transformation `x` is a *homomorphism* from a digraph `src` to a digraph `ran` if the following hold:

- `[u ^ x, v ^ x]` is an edge of `ran` whenever `[u, v]` is an edge of `src`; and

- `x` maps the vertices of `src` to a subset of the vertices of `ran`, i.e. `IsSubset(DigraphVertices(ran), OnSets(DigraphVertices(src), x))` is true.

Note that if `i` is any integer greater than `DigraphNrVertice(src)`, then the action of `x` on `i` is ignored by this function. One consequence of this is that distinct transformations or permutations might represent the same homomorphism. For example, if `src` and `ran` are `CycleDigraph(2)`, then both the permutations `(1, 2)` and `(1, 2)(3, 4)` represent the same automorphism of `src`.

See also GeneratorsOfEndomorphismMonoid (7.3.14).

If `col1` and `col2`, or `col`, are given, then they must represent vertex colourings; see AutomorphismGroup (7.2.5) for details of the permissible values for these argument. The homomorphism must then also have the property:

- `col[i] = col[i ^ x]` for all vertices `i` of `digraph`, in the case of IsDigraphEndomorphism.

- `col1[i] = col2[i ^ x]` for all vertices `i` of `src`, in the cases of the other operations.

See also DigraphsRespectsColouring (7.3.13).

```
 ─────────────────────────── Example ───────────────────────────
gap> src := Digraph([[1], [1, 2], [1, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> ran := Digraph([[1], [1, 2]]);
<immutable digraph with 2 vertices, 3 edges>
gap> IsDigraphHomomorphism(src, ran, Transformation([1, 2, 2]));
true
gap> IsDigraphHomomorphism(src, ran, Transformation([2, 1, 2]));
false
gap> IsDigraphHomomorphism(src, ran, Transformation([3, 3, 3]));
false
gap> IsDigraphHomomorphism(src, src, Transformation([3, 3, 3]));
true
gap> IsDigraphHomomorphism(src, ran, Transformation([1, 2, 2]),
>                          [1, 2, 2], [1, 2]);
true
gap> IsDigraphHomomorphism(src, ran, Transformation([1, 2, 2]),
>                          [2, 1, 1], [1, 2]);
false
gap> IsDigraphEndomorphism(src, Transformation([3, 3, 3]));
true
gap> IsDigraphEndomorphism(src, Transformation([3, 3, 3]), [1, 1, 1]);
true
```

```
gap> IsDigraphEndomorphism(src, Transformation([3, 3, 3]), [1, 1, 2]);
false
gap> IsDigraphEpimorphism(src, ran, Transformation([3, 3, 3]));
false
gap> IsDigraphMonomorphism(src, ran, Transformation([1, 2, 2]));
false
gap> IsDigraphEpimorphism(src, ran, Transformation([1, 2, 2]));
true
gap> IsDigraphMonomorphism(ran, src, ());
true
```

7.3.11   IsDigraphEmbedding (for digraphs and a permutation or transformation)

▷ IsDigraphEmbedding(*src, ran, x*)   (operation)
▷ IsDigraphEmbedding(*src, ran, x, col1, col2*)   (operation)
   Returns: true or false.

   IsDigraphEmbedding returns true if the permutation or transformation *x* is a embedding of the digraph *src* into the digraph *ran*, while respecting the colourings *col1* and *col2* if given.

   A permutation or transformation *x* is a *embedding* of a digraph *src* into a digraph *ran* if *x* is a monomorphism from *src* to *ran* and the inverse of *x* is a monomorphism from the subdigraph of *ran* induced by the image of *x* to *src*. See also IsDigraphHomomorphism (7.3.10).

```
─────────────────────────── Example ───────────────────────────
gap> src := Digraph([[1], [1, 2]]);
<immutable digraph with 2 vertices, 3 edges>
gap> ran := Digraph([[1], [1, 2], [1, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsDigraphMonomorphism(src, ran, ());
true
gap> IsDigraphEmbedding(src, ran, ());
true
gap> IsDigraphEmbedding(src, ran, (), [2, 1], [2, 1, 1]);
true
gap> IsDigraphEmbedding(src, ran, (), [2, 1], [1, 2, 1]);
false
gap> ran := Digraph([[1, 2], [1, 2], [1, 3]]);
<immutable digraph with 3 vertices, 6 edges>
gap> IsDigraphMonomorphism(src, ran, IdentityTransformation);
true
gap> IsDigraphEmbedding(src, ran, IdentityTransformation);
false
```

7.3.12   SubdigraphsMonomorphisms

▷ SubdigraphsMonomorphisms(*digraph1, digraph2*)   (operation)
▷ SubdigraphsMonomorphismsRepresentatives(*digraph1, digraph2*)   (operation)
   Returns: A list of transformations.

   These operations behave the same as HomomorphismsDigraphs (7.3.3) and HomomorphismsDigraphsRepresentatives (7.3.3), except they only return *injective* homomorphisms with the following property: the (not necessarily induced) subdigraphs defined by the

images of these monomorphisms are all of the subdigraphs of `digraph2` that are isomorphic to `digraph1`. Note that the subdigraphs of the previous sentence are those obtained by applying the corresponding monomorphism to the vertices and the edges of `digraph1`, and are therefore possibly strictly contained in the induced subdigraph on the same vertex set.

```
_____ Example _____
  gap> SubdigraphsMonomorphisms(CompleteBipartiteDigraph(2, 2),
  > CompleteDigraph(4));
  [ Transformation( [ 1, 3, 2 ] ), Transformation( [ 2, 3, 1 ] ),
    Transformation( [ 3, 4, 2, 1 ] ) ]
  gap> SubdigraphsMonomorphismsRepresentatives(
  > CompleteBipartiteDigraph(2, 2), CompleteDigraph(4));
  [ Transformation( [ 1, 3, 2 ] ) ]
```

7.3.13 DigraphsRespectsColouring

▷ DigraphsRespectsColouring(*src, ran, x, col1, col2*)   (operation)

Returns: `true` or `false`.

The operation `DigraphsRespectsColouring` verifies whether or not the permutation or transformation `x` respects the vertex colourings `col1` and `col2` of the digraphs `src` and `range`. That is, `DigraphsRespectsColouring` returns `true` if and only if for all vertices i of `src`, col1[i] = col2[i ^ x].

```
_____ Example _____
  gap> src := Digraph([[1], [1, 2]]);
  <immutable digraph with 2 vertices, 3 edges>
  gap> ran := Digraph([[1], [1, 2], [1, 3]]);
  <immutable digraph with 3 vertices, 5 edges>
  gap> DigraphsRespectsColouring(src, ran, (1, 2), [2, 1], [1, 2, 1]);
  true
  gap> DigraphsRespectsColouring(src, ran, (1, 2), [2, 1], [2, 1, 1]);
  false
```

7.3.14 GeneratorsOfEndomorphismMonoid

▷ GeneratorsOfEndomorphismMonoid(*digraph[, colors][, limit]*)   (function)
▷ GeneratorsOfEndomorphismMonoidAttr(*digraph*)   (attribute)

Returns: A list of transformations.

An endomorphism of `digraph` is a homomorphism DigraphHomomorphism (7.3.2) from `digraph` back to itself. `GeneratorsOfEndomorphismMonoid`, called with a single argument, returns a generating set for the monoid of all endomorphisms of `digraph`. If `digraph` belongs to IsImmutableDigraph (3.1.3), then the value of `GeneratorsOfEndomorphismMonoid` will not be recomputed on future calls.

If the `colors` argument is specified, then `GeneratorsOfEndomorphismMonoid` will return a generating set for the monoid of endomorphisms which respect the given colouring. The colouring `colors` can be in one of two forms:

- A list of positive integers of size the number of vertices of `digraph`, where `colors[i]` is the colour of vertex i.

- A list of lists, such that `colors[i]` is a list of all vertices with colour i.

If the `limit` argument is specified, then it will return only the first `limit` homomorphisms, where `limit` must be a positive integer or `infinity`.

```
_____ Example _____
gap> gr := Digraph(List([1 .. 3], x -> [1 .. 3]));;
gap> GeneratorsOfEndomorphismMonoid(gr);
[ Transformation( [ 1, 3, 2 ] ), Transformation( [ 2, 1 ] ),
  IdentityTransformation, Transformation( [ 1, 2, 1 ] ),
  Transformation( [ 1, 2, 2 ] ), Transformation( [ 1, 1, 2 ] ),
  Transformation( [ 1, 1, 1 ] ) ]
gap> GeneratorsOfEndomorphismMonoid(gr, 3);
[ Transformation( [ 1, 3, 2 ] ), Transformation( [ 2, 1 ] ),
  IdentityTransformation ]
gap> gr := CompleteDigraph(3);;
gap> GeneratorsOfEndomorphismMonoid(gr);
[ Transformation( [ 2, 3, 1 ] ), Transformation( [ 2, 1 ] ),
  IdentityTransformation ]
gap> GeneratorsOfEndomorphismMonoid(gr, [1, 2, 2]);
[ Transformation( [ 1, 3, 2 ] ), IdentityTransformation ]
gap> GeneratorsOfEndomorphismMonoid(gr, [[1], [2, 3]]);
[ Transformation( [ 1, 3, 2 ] ), IdentityTransformation ]
```

7.3.15 DigraphColouring (for a digraph and a number of colours)

▷ DigraphColouring(*digraph*, *n*)   (operation)

    Returns: A transformation, or `fail`.

    A *proper colouring* of a digraph is a labelling of its vertices in such a way that adjacent vertices have different labels. A *proper n-colouring* is a proper colouring that uses exactly n colours. Equivalently, a proper (n-)colouring of a digraph can be defined to be a DigraphEpimorphism (7.3.6) from a digraph onto the complete digraph (with n vertices); see CompleteDigraph (3.5.11). Note that a digraph with loops (DigraphHasLoops (6.2.1)) does not have a proper n-colouring for any value n.

    If *digraph* is a digraph and *n* is a non-negative integer, then DigraphColouring(*digraph*, *n*) returns an epimorphism from *digraph* onto the complete digraph with *n* vertices if one exists, else it returns `fail`.

    See also DigraphGreedyColouring (7.3.16) and

    Note that a digraph with at least two vertices has a 2-colouring if and only if it is bipartite, see IsBipartiteDigraph (6.2.3).

```
_____ Example _____
gap> DigraphColouring(CompleteDigraph(5), 4);
fail
gap> DigraphColouring(ChainDigraph(10), 1);
fail
gap> D := ChainDigraph(10);;
gap> t := DigraphColouring(D, 2);
Transformation( [ 1, 2, 1, 2, 1, 2, 1, 2, 1, 2 ] )
gap> IsDigraphColouring(D, t);
true
gap> DigraphGreedyColouring(D);
Transformation( [ 2, 1, 2, 1, 2, 1, 2, 1, 2, 1 ] )
```

7.3.16 DigraphGreedyColouring (for a digraph and vertex order)

▷ DigraphGreedyColouring(*digraph, order*) (operation)
▷ DigraphGreedyColouring(*digraph, func*) (operation)
▷ DigraphGreedyColouring(*digraph*) (attribute)

   Returns: A transformation, or `fail`.

   A *proper colouring* of a digraph is a labelling of its vertices in such a way that adjacent vertices have different labels. Note that a digraph with loops (DigraphHasLoops (6.2.1)) does not have any proper colouring.

   If *digraph* is a digraph and *order* is a dense list consisting of all of the vertices of *digraph* (in any order), then DigraphGreedyColouring uses a greedy algorithm with the specified order to obtain some proper colouring of *digraph*, which may not use the minimal number of colours.

   If *digraph* is a digraph and *func* is a function whose argument is a digraph, and that returns a dense list *order*, then DigraphGreedyColouring(*digraph, func*) returns DigraphGreedyColouring(*digraph, func*(*digraph*)).

   If the optional second argument (either a list or a function), is not specified, then DigraphWelshPowellOrder (7.3.17) is used by default.

   See also DigraphColouring (7.3.15).

```
 _____ Example _____
 gap> DigraphGreedyColouring(ChainDigraph(10));
 Transformation( [ 2, 1, 2, 1, 2, 1, 2, 1, 2, 1 ] )
 gap> DigraphGreedyColouring(ChainDigraph(10), [1 .. 10]);
 Transformation( [ 1, 2, 1, 2, 1, 2, 1, 2, 1, 2 ] )
```

7.3.17 DigraphWelshPowellOrder

▷ DigraphWelshPowellOrder(*digraph*) (attribute)

   Returns: A list of the vertices.

   DigraphWelshPowellOrder returns a list of all of the vertices of the digraph *digraph* ordered according to the sum of the number of out- and in-neighbours, from highest to lowest.

```
 _____ Example _____
 gap> DigraphWelshPowellOrder(Digraph([[4], [9], [9], [],
 >                                      [4, 6, 9], [1], [], [],
 >                                      [4, 5], [4, 5]]));
 [ 5, 9, 4, 1, 6, 10, 2, 3, 7, 8 ]
```

7.3.18 ChromaticNumber

▷ ChromaticNumber(*digraph*) (attribute)

   Returns: A non-negative integer.

   A *proper colouring* of a digraph is a labelling of its vertices in such a way that adjacent vertices have different labels. Equivalently, a proper digraph colouring can be defined to be a DigraphEpimorphism (7.3.6) from a digraph onto a complete digraph.

   If *digraph* is a digraph without loops (see DigraphHasLoops (6.2.1), then ChromaticNumber returns the least non-negative integer n such that there is a proper colouring of *digraph* with n colours. In other words, for a digraph with at least one vertex, ChromaticNumber returns the least number n such that DigraphColouring(*digraph*, n) does not return `fail`. See DigraphColouring (7.3.15).

It is possible to select the algorithm to compute the chromatic number via the use of value options. The permitted algorithms and values to pass as options are:

- `lawler` - Lawler's Algorithm [Law76]

- `byskov` - Byskov's Algorithm [Bys02]

```
_____ Example _____
gap> ChromaticNumber(NullDigraph(10));
1
gap> ChromaticNumber(CompleteDigraph(10));
10
gap> ChromaticNumber(CompleteBipartiteDigraph(5, 5));
2
gap> ChromaticNumber(Digraph([[], [3], [5], [2, 3], [4]]));
3
gap> ChromaticNumber(NullDigraph(0));
0
gap> D := PetersenGraph(IsMutableDigraph);
<mutable digraph with 10 vertices, 30 edges>
gap> ChromaticNumber(D);
3
gap> ChromaticNumber(CompleteDigraph(10) : lawler);
10
gap> ChromaticNumber(CompleteDigraph(10) : byskov);
10
```

7.3.19 DigraphCore

▷ DigraphCore(*D*)   (attribute)

Returns: A list of positive integers.

If *D* is a digraph, then `DigraphCore` returns a list of vertices corresponding to the core of *D*. In particular, the subdigraph of *D* induced by this list is isomorphic to the core of *D*.

The *core* of a digraph D is the minimal subdigraph *C* of D which is a homomorphic image of D. The core of a digraph is unique up to isomorphism.

```
_____ Example _____
gap> D := DigraphSymmetricClosure(CycleDigraph(8));
<immutable symmetric digraph with 8 vertices, 16 edges>
gap> DigraphCore(D);
[ 1, 2 ]
gap> D := PetersenGraph();
<immutable digraph with 10 vertices, 30 edges>
gap> DigraphCore(D);
[ 1 .. 10 ]
gap> D := Digraph(IsMutableDigraph, [[3], [3], [4], [5], [2]]);
<mutable digraph with 5 vertices, 5 edges>
gap> DigraphCore(D);
[ 2, 3, 4, 5 ]
```

7.3.20 LatticeDigraphEmbedding

▷ LatticeDigraphEmbedding(*L1*, *L2*)   (operation)

Returns: A transformation, or `fail`.

If *L1* and *L2* are lattice digraphs (IsLatticeDigraph (6.4.3) returns true, then LatticeDigraphEmbedding returns a single *injective* DigraphHomomorphism (7.3.2) between *L1* and *L2*, with the property that it is a *lattice homomorphism*. If no such homomorphism exists, fail is returned.

A *lattice homomorphism* is a digraph homomorphism which respects meets and joins of every pair of vertices. Note that every injective lattice homomorphism map is an embedding, in the sense that the inverse of map is a lattice homomorphism also.

```
_____ Example _____
gap> D := DigraphReflexiveTransitiveClosure(ChainDigraph(5));
<immutable preorder digraph with 5 vertices, 15 edges>
gap> L1 := DigraphReflexiveTransitiveClosure(ChainDigraph(5));
<immutable preorder digraph with 5 vertices, 15 edges>
gap> L2 := DigraphReflexiveTransitiveClosure(ChainDigraph(6));
<immutable preorder digraph with 6 vertices, 21 edges>
gap> LatticeDigraphEmbedding(L1, L2);
IdentityTransformation
gap> LatticeDigraphEmbedding(L2, L1);
fail
```

7.3.21 IsLatticeHomomorphism (for digraphs and a permutation or transformation)

▷ IsLatticeHomomorphism(*L1, L2, map*)   (operation)
▷ IsLatticeEpimorphism(*L1, L2, map*)   (operation)
▷ IsLatticeEmbedding(*L1, L2, map*)   (operation)
▷ IsLatticeMonomorphism(*L1, L2, map*)   (operation)
▷ IsLatticeEndomorphism(*L, map*)   (operation)

Returns: true or false.

Each of the function described in this section (except IsLatticeEndomorphism) takes a pair of digraphs *L1* and *L2*, and a transformation *map*, returning true if *map* is a *lattice homomorphism* from *L1* to *L2*, and false otherwise. If *L1* or *L2* is not a lattice, then false is returned.

A transformation or permutation *map* is a *lattice homomorphism* if *map* respects meets and joins of every pair of vertices, and *map* fixes every i which is not a vertex of *L1*.

IsLatticeHomomorphism returns true if the permutation or transformation *map* is a lattice homomorphism from the lattice digraph *L1* to the lattice digraph *L2*.

IsLatticeEpimorphism returns true if the permutation or transformation *map* is a surjective lattice homomorphism from the lattice digraph *L1* to the lattice digraph *L2*.

IsLatticeEmbedding returns true if the permutation or transformation *map* is an injective lattice homomorphism from the lattice digraph *L1* to the lattice digraph *L2*. The function IsLatticeMonomorphism is a synonym of IsLatticeEmbedding.

IsLatticeEndomorphism returns true if the permutation or transformation *map* is an lattice endomorphism of the lattice digraph *L*.

```
_____ Example _____
gap> G := Digraph([[2, 4], [3, 7], [6], [5, 7], [6], [], [6]]);
<immutable digraph with 7 vertices, 9 edges>
gap> D := DigraphRemoveVertex(G, 7);
<immutable digraph with 6 vertices, 6 edges>
gap> G := DigraphReflexiveTransitiveClosure(G);
<immutable preorder digraph with 7 vertices, 22 edges>
gap> D := DigraphReflexiveTransitiveClosure(D);
```

```
<immutable preorder digraph with 6 vertices, 17 edges>
gap> IsDigraphEmbedding(D, G, IdentityTransformation);
true
gap> IsLatticeHomomorphism(D, G, IdentityTransformation);
false
gap> D := Digraph([[2, 3], [4], [4], []]);
<immutable digraph with 4 vertices, 4 edges>
gap> G := Digraph([[2, 3], [4], [4], [5], []]);
<immutable digraph with 5 vertices, 5 edges>
gap> D := DigraphReflexiveTransitiveClosure(D);
<immutable preorder digraph with 4 vertices, 9 edges>
gap> G := DigraphReflexiveTransitiveClosure(G);
<immutable preorder digraph with 5 vertices, 14 edges>
gap> IsLatticeEmbedding(D, G, IdentityTransformation);
true
gap> IsLatticeMonomorphism(D, G, IdentityTransformation);
true
gap> f := Transformation([1, 2, 3, 4, 4]);
Transformation( [ 1, 2, 3, 4, 4 ] )
gap> IsLatticeEpimorphism(G, D, f);
true
gap> IsLatticeEndomorphism(D, (2, 3));
true
```

Chapter 8

Finding cliques and independent sets

In Digraphs, a *clique* of a digraph is a set of mutually adjacent vertices of the digraph, and an *independent set* is a set of mutually non-adjacent vertices of the digraph. A *maximal clique* is a clique which is not properly contained in another clique, and a *maximal independent set* is an independent set which is not properly contained in another independent set. Using this definition in Digraphs, cliques and independent sets are both permitted, but not required, to contain vertices at which there is a loop. Another name for a clique is a *complete subgraph*.

Digraphs provides extensive functionality for computing cliques and independent sets of a digraph, whether maximal or not. The fundamental algorithm used in most of the methods in Digraphs to calculate cliques and independent sets is a version of the Bron-Kerbosch algorithm. Calculating the cliques and independent sets of a digraph is a well-known and hard problem, and searching for cliques or independent sets in a digraph can be very lengthy, even for a digraph with a small number of vertices. Digraphs uses several strategies to increase the performance of these calculations.

From the definition of cliques and independent sets, it follows that the presence of loops and multiple edges in a digraph is irrelevant to the existence of cliques and independent sets in the digraph. See DigraphHasLoops (6.2.1) and IsMultiDigraph (6.2.11) for more information about these properties. Therefore given a digraph *digraph*, the cliques and independent sets of *digraph* are equal to the cliques and independent sets of the digraph:

- DigraphRemoveLoops(DigraphRemoveAllMultipleEdges(*digraph*)).

See DigraphRemoveLoops (3.3.25) and DigraphRemoveAllMultipleEdges (3.3.26) for more information about these attributes. Furthermore, the cliques of this digraph are equal to the cliques of the digraph formed by removing any edge [u,v] for which the corresponding reverse edge [v,u] is not present. Therefore, overall, the cliques of *digraph* are equal to the cliques of the symmetric digraph:

- MaximalSymmetricSubdigraphWithoutLoops(*digraph*).

See MaximalSymmetricSubdigraphWithoutLoops (3.3.5) for more information about this. The AutomorphismGroup (7.2.2) of this symmetric digraph contains the automorphism group of *digraph* as a subgroup. By performing the search for maximal cliques with the help of this larger automorphism group to reduce the search space, the computation time may be reduced. The functions and attributes which return representatives of cliques of *digraph* will return orbit representatives of cliques under the action of the automorphism group of the *maximal symmetric subdigraph without loops* on sets of vertices.

The independent sets of a digraph are equal to the independent sets of the DigraphSymmetricClosure (3.3.12). Therefore, overall, the independent sets of `digraph` are equal to the independent sets of the symmetric digraph:

• `DigraphSymmetricClosure(DigraphRemoveLoops(DigraphRemoveAllMultipleEdges(digraph)))`.

Again, the automorphism group of this symmetric digraph contains the automorphism group of `digraph`. Since a search for independent sets is equal to a search for cliques in the DigraphDual (3.3.11), the methods used in Digraphs usually transform a search for independent sets into a search for cliques, as described above. The functions and attributes which return representatives of independent sets of `digraph` will return orbit representatives of independent sets under the action of the automorphism group of the *symmetric closure* of the digraph formed by removing loops and multiple edges.

Please note that in Digraphs, cliques and independent sets are not required to be maximal. Some authors use the word clique to mean *maximal* clique, and some authors use the phrase independent set to mean *maximal* independent set, but please note that Digraphs does not use this definition.

8.1 Finding cliques

8.1.1 IsClique

▷ IsClique(*digraph, l*)   (operation)
▷ IsMaximalClique(*digraph, l*)   (operation)
   Returns: `true` or `false`.

If `digraph` is a digraph and `l` is a duplicate-free list of vertices of `digraph`, then IsClique(`digraph`,`l`) returns `true` if `l` is a *clique* of `digraph` and `false` if it is not. Similarly, IsMaximalClique(`digraph`,`l`) returns `true` if `l` is a *maximal clique* of `digraph` and `false` if it is not.

A *clique* of a digraph is a set of mutually adjacent vertices of the digraph. A *maximal clique* is a clique that is not properly contained in another clique. A clique is permitted, but not required, to contain vertices at which there is a loop.

```
————————————————————— Example —————————————————————
gap> D := CompleteDigraph(4);;
gap> IsClique(D, [1, 3, 2]);
true
gap> IsMaximalClique(D, [1, 3, 2]);
false
gap> IsMaximalClique(D, DigraphVertices(D));
true
gap> D := Digraph([[1, 2, 4, 4], [1, 3, 4], [2, 1], [1, 2]]);
<immutable multidigraph with 4 vertices, 11 edges>
gap> IsClique(D, [2, 3, 4]);
false
gap> IsMaximalClique(D, [1, 2, 4]);
true
gap> D := CompleteDigraph(IsMutableDigraph, 4);;
gap> IsClique(D, [1, 3, 2]);
true
```

8.1.2 CliquesFinder

▷ CliquesFinder(*digraph, hook, user_param, limit, include, exclude, max, size, reps*) (function)

    Returns: The argument *user_param*.

    This function finds cliques of the digraph *digraph* subject to the conditions imposed by the other arguments as described below. Note that a clique is represented by the immutable list of the vertices that it contains.

    Let `G` denote the automorphism group of the maximal symmetric subdigraph of *digraph* without loops (see `AutomorphismGroup` (7.2.2) and `MaximalSymmetricSubdigraphWithoutLoops` (3.3.5)).

hook

    This argument should be a function or `fail`.

    If *hook* is a function, then it should have two arguments *user_param* (see below) and a clique c. The function *hook*(*user_param*, c) is called every time a new clique c is found by `CliquesFinder`.

    If *hook* is `fail`, then a default function is used that simply adds every new clique found by `CliquesFinder` to *user_param*, which must be a list in this case.

user_param

    If *hook* is a function, then *user_param* can be any GAP object. The object *user_param* is used as the first argument for the function *hook*. For example, *user_param* might be a list, and *hook*(*user_param*, c) might add the size of the clique c to the list *user_param*.

    If the value of *hook* is `fail`, then the value of *user_param* must be a list.

limit

    This argument should be a positive integer or `infinity`. `CliquesFinder` will return after it has found *limit* cliques or the search is complete.

include **and** *exclude*

    These arguments should each be a (possibly empty) duplicate-free list of vertices of *digraph* (i.e. positive integers less than the number of vertices of *digraph*).

    `CliquesFinder` will only look for cliques containing all of the vertices in *include* and containing none of the vertices in *exclude*.

    Note that the search may be much more efficient if each of these lists is invariant under the action of `G` on sets of vertices.

max This argument should be `true` or `false`. If *max* is true then `CliquesFinder` will only search for *maximal* cliques. If `max` is `false` then non-maximal cliques may be found.

size

    This argument should be `fail` or a positive integer. If *size* is a positive integer then `CliquesFinder` will only search for cliques that contain precisely *size* vertices. If *size* is `fail` then cliques of any size may be found.

reps

    This argument should be `true` or `false`.

If `reps` is true then the arguments `include` and `exclude` are each required to be invariant under the action of G on sets of vertices. In this case, CliquesFinder will find representatives of the orbits of the desired cliques under the action of G, *although representatives may be returned that are in the same orbit*. If `reps` is false then CliquesFinder will not take this into consideration.

For a digraph such that G is non-trivial, the search for clique representatives can be much more efficient than the search for all cliques.

This function uses a version of the Bron-Kerbosch algorithm.

```
———————————————————— Example ————————————————————
gap> D := CompleteDigraph(5);
<immutable complete digraph with 5 vertices>
gap> user_param := [];;
gap> f := function(a, b)  # Calculate size of clique
>   AddSet(user_param, Size(b));
> end;;
gap> CliquesFinder(D, f, user_param, infinity, [], [], false, fail,
>                      true);
[ 1, 2, 3, 4, 5 ]
gap> CliquesFinder(D, fail, [], 5, [2, 4], [3], false, fail, false);
[ [ 2, 4 ], [ 1, 2, 4 ], [ 2, 4, 5 ], [ 1, 2, 4, 5 ] ]
gap> CliquesFinder(D, fail, [], 2, [2, 4], [3], false, fail, false);
[ [ 2, 4 ], [ 1, 2, 4 ] ]
gap> CliquesFinder(D, fail, [], infinity, [], [], true, 5, false);
[ [ 1, 2, 3, 4, 5 ] ]
gap> CliquesFinder(D, fail, [], infinity, [1, 3], [], false, 3, false);
[ [ 1, 2, 3 ], [ 1, 3, 4 ], [ 1, 3, 5 ] ]
gap> CliquesFinder(D, fail, [], infinity, [1, 3], [], true, 3, false);
[  ]
gap> D := CompleteDigraph(IsMutableDigraph, 5);
<mutable digraph with 5 vertices, 20 edges>
gap> user_param := [];;
gap> f := function(a, b)  # Calculate size of clique
>   AddSet(user_param, Size(b));
> end;;
gap> CliquesFinder(D, f, user_param, infinity, [], [], false, fail,
>                      true);
[ 1, 2, 3, 4, 5 ]
gap> CliquesFinder(D, fail, [], 5, [2, 4], [3], false, fail, false);
[ [ 2, 4 ], [ 1, 2, 4 ], [ 2, 4, 5 ], [ 1, 2, 4, 5 ] ]
gap> CliquesFinder(D, fail, [], 2, [2, 4], [3], false, fail, false);
[ [ 2, 4 ], [ 1, 2, 4 ] ]
gap> CliquesFinder(D, fail, [], infinity, [], [], true, 5, false);
[ [ 1, 2, 3, 4, 5 ] ]
gap> CliquesFinder(D, fail, [], infinity, [1, 3], [], false, 3, false);
[ [ 1, 2, 3 ], [ 1, 3, 4 ], [ 1, 3, 5 ] ]
gap> CliquesFinder(D, fail, [], infinity, [1, 3], [], true, 3, false);
[  ]
```

8.1.3 DigraphClique

▷ DigraphClique(*digraph[, include[, exclude[, size]]]*)   (function)
▷ DigraphMaximalClique(*digraph[, include[, exclude[, size]]]*)   (function)

    Returns: An immutable list of positive integers, or `fail`.

    If *digraph* is a digraph, then these functions returns a clique of *digraph* if one exists that satisfies the arguments, else it returns `fail`. A clique is defined by the set of vertices that it contains; see `IsClique` (8.1.1) and `IsMaximalClique` (8.1.1).

    The optional arguments *include* and *exclude* must each be a (possibly empty) duplicate-free list of vertices of *digraph*, and the optional argument *size* must be a positive integer. By default, *include* and *exclude* are empty. These functions will search for a clique of *digraph* that includes the vertices of *include* but does not include any vertices in *exclude*; if the argument *size* is supplied, then additionally the clique will be required to contain precisely *size* vertices.

    If *include* is not a clique, then these functions return `fail`. Otherwise, the functions behave in the following way, depending on the number of arguments:

One or two arguments

    If one or two arguments are supplied, then `DigraphClique` and `DigraphMaximalClique` greedily enlarge the clique *include* until it cannot continue. The result is guaranteed to be a maximal clique. This may or may not return an answer more quickly than using `DigraphMaximalCliques` (8.1.4) with a limit of 1.

Three arguments

    If three arguments are supplied, then `DigraphClique` greedily enlarges the clique *include* until it cannot continue, although this clique may not be maximal.

    Given three arguments, `DigraphMaximalClique` returns the maximal clique returned by `DigraphMaximalCliques(`*digraph*`, `*include*`, `*exclude*`, 1)` if one exists, else `fail`.

Four arguments

    If four arguments are supplied, then `DigraphClique` returns the clique returned by `DigraphCliques(`*digraph*`, `*include*`, `*exclude*`, 1, `*size*`)` if one exists, else `fail`. This clique may not be maximal.

    Given four arguments, `DigraphMaximalClique` returns the maximal clique returned by `DigraphMaximalCliques(`*digraph*`, `*include*`, `*exclude*`, 1, `*size*`)` if one exists, else `fail`.

```
_____ Example _____
gap> D := Digraph([[2, 3, 4], [1, 3], [1, 2], [1, 5], []]);
<immutable digraph with 5 vertices, 9 edges>
gap> IsSymmetricDigraph(D);
false
gap> DigraphClique(D);
[ 5 ]
gap> DigraphMaximalClique(D);
[ 5 ]
gap> DigraphClique(D, [1, 2]);
[ 1, 2, 3 ]
gap> DigraphMaximalClique(D, [1, 3]);
[ 1, 3, 2 ]
gap> DigraphClique(D, [1], [2]);
```

```
[ 1, 4 ]
gap> DigraphMaximalClique(D, [1], [3, 4]);
fail
gap> DigraphClique(D, [1, 5]);
fail
gap> DigraphClique(D, [], [], 2);
[ 1, 2 ]
gap> D := Digraph(IsMutableDigraph,
>                 [[2, 3, 4], [1, 3], [1, 2], [1, 5], []]);
<mutable digraph with 5 vertices, 9 edges>
gap> IsSymmetricDigraph(D);
false
gap> DigraphClique(D);
[ 5 ]
```

8.1.4   DigraphMaximalCliques

▷ DigraphMaximalCliques(*digraph*[, *include*[, *exclude*[, *limit*[, *size*]]]])   (function)
▷ DigraphMaximalCliquesReps(*digraph*[, *include*[, *exclude*[, *limit*[, *size*]]]])

(function)

▷ DigraphCliques(*digraph*[, *include*[, *exclude*[, *limit*[, *size*]]]])   (function)
▷ DigraphCliquesReps(*digraph*[, *include*[, *exclude*[, *limit*[, *size*]]]])   (function)
▷ DigraphMaximalCliquesAttr(*digraph*)   (attribute)
▷ DigraphMaximalCliquesRepsAttr(*digraph*)   (attribute)

   Returns: An immutable list of lists of positive integers.

   If `digraph` is digraph, then these functions and attributes use `CliquesFinder` (8.1.2) to return cliques of `digraph`. A clique is defined by the set of vertices that it contains; see `IsClique` (8.1.1) and `IsMaximalClique` (8.1.1).

   The optional arguments `include` and `exclude` must each be a (possibly empty) list of vertices of `digraph`, the optional argument `limit` must be either a positive integer or `infinity`, and the optional argument `size` must be a positive integer. If not specified, then `include` and `exclude` are chosen to be empty lists, and `limit` is set to `infinity`.

   The functions will return as many suitable cliques as possible, up to the number `limit`. These functions will find cliques that contain all of the vertices of `include` but do not contain any of the vertices of `exclude`. The argument `size` restricts the search to those cliques that contain precisely `size` vertices. If the function or attribute has `Maximal` in its name, then only maximal cliques will be returned; otherwise non-maximal cliques may be returned.

   Let `G` denote the automorphism group of maximal symmetric subdigraph of `digraph` without loops (see `AutomorphismGroup` (7.2.2) and `MaximalSymmetricSubdigraphWithoutLoops` (3.3.5)).

Distinct cliques

   `DigraphMaximalCliques` and `DigraphCliques` each return a duplicate-free list of at most `limit` cliques of `digraph` that satisfy the arguments.

   The computation may be significantly faster if `include` and `exclude` are invariant under the action of `G` on sets of vertices.

Orbit representatives of cliques

   To use `DigraphMaximalCliquesReps` or `DigraphCliquesReps`, the arguments `include` and

exclude must each be invariant under the action of G on sets of vertices.

If this is the case, then DigraphMaximalCliquesReps and DigraphCliquesReps each return a duplicate-free list of at most `limit` orbits representatives (under the action of G on sets vertices) of cliques of `digraph` that satisfy the arguments.

The representatives are not guaranteed to be in distinct orbits. However, if fewer than `lim` results are returned, then there will be at least one representative from each orbit of maximal cliques.

```
─────────────────────────── Example ───────────────────────────
gap> D := Digraph([
> [2, 3], [1, 3], [1, 2, 4], [3, 5, 6], [4, 6], [4, 5]]);
<immutable digraph with 6 vertices, 14 edges>
gap> IsSymmetricDigraph(D);
true
gap> G := AutomorphismGroup(D);
Group([ (5,6), (1,2), (1,5)(2,6)(3,4) ])
gap> AsSet(DigraphMaximalCliques(D));
[ [ 1, 2, 3 ], [ 3, 4 ], [ 4, 5, 6 ] ]
gap> DigraphMaximalCliquesReps(D);
[ [ 1, 2, 3 ], [ 3, 4 ] ]
gap> Orbit(G, [1, 2, 3], OnSets);
[ [ 1, 2, 3 ], [ 4, 5, 6 ] ]
gap> Orbit(G, [3, 4], OnSets);
[ [ 3, 4 ] ]
gap> DigraphMaximalCliquesReps(D, [3, 4], [], 1);
[ [ 3, 4 ] ]
gap> DigraphMaximalCliques(D, [1, 2], [5, 6], 1, 2);
[ ]
gap> DigraphCliques(D, [1], [5, 6], infinity, 2);
[ [ 1, 2 ], [ 1, 3 ] ]
gap> D := Digraph(IsMutableDigraph, [
> [2, 3], [1, 3], [1, 2, 4], [3, 5, 6], [4, 6], [4, 5]]);
<mutable digraph with 6 vertices, 14 edges>
gap> IsSymmetricDigraph(D);
true
gap> G := AutomorphismGroup(D);
Group([ (5,6), (1,2), (1,5)(2,6)(3,4) ])
gap> AsSet(DigraphMaximalCliques(D));
[ [ 1, 2, 3 ], [ 3, 4 ], [ 4, 5, 6 ] ]
```

8.1.5 CliqueNumber

▷ CliqueNumber(`digraph`)   (attribute)

Returns: A non-negative integer.

If `digraph` is a digraph, then CliqueNumber(`digraph`) returns the largest integer n such that `digraph` contains a clique with n vertices as an induced subdigraph.

A *clique* of a digraph is a set of mutually adjacent vertices of the digraph. Loops and multiple edges are ignored for the purpose of determining the clique number of a digraph.

```
─────────────────────────── Example ───────────────────────────
gap> D := CompleteDigraph(4);;
gap> CliqueNumber(D);
```

```
4
gap> D := Digraph([[1, 2, 4, 4], [1, 3, 4], [2, 1], [1, 2]]);
<immutable multidigraph with 4 vertices, 11 edges>
gap> CliqueNumber(D);
3
gap> D := CompleteDigraph(IsMutableDigraph, 4);;
gap> CliqueNumber(D);
4
```

8.2 Finding independent sets

8.2.1 IsIndependentSet

▷ IsIndependentSet(*digraph, l*)   (operation)
▷ IsMaximalIndependentSet(*digraph, l*)   (operation)

Returns: true or false.

If *digraph* is a digraph and *l* is a duplicate-free list of vertices of *digraph*, then
IsIndependentSet(*digraph*,*l*) returns true if *l* is an *independent set* of *digraph* and false
if it is not. Similarly, IsMaximalIndependentSet(*digraph*,*l*) returns true if *l* is a *maximal
independent set* of *digraph* and false if it is not.

An *independent set* of a digraph is a set of mutually non-adjacent vertices of the digraph. A
maximal independent set is an independent set that is not properly contained in another independent
set. An independent set is permitted, but not required, to contain vertices at which there is a loop.

```
───────────────────── Example ─────────────────────
gap> D := CycleDigraph(4);;
gap> IsIndependentSet(D, [1]);
true
gap> IsMaximalIndependentSet(D, [1]);
false
gap> IsIndependentSet(D, [1, 4, 3]);
false
gap> IsIndependentSet(D, [2, 4]);
true
gap> IsMaximalIndependentSet(D, [2, 4]);
true
gap> D := CycleDigraph(IsMutableDigraph, 4);;
gap> IsIndependentSet(D, [1]);
true
```

8.2.2 DigraphIndependentSet

▷ DigraphIndependentSet(*digraph[, include[, exclude[, size]]]*)   (function)
▷ DigraphMaximalIndependentSet(*digraph[, include[, exclude[, size]]]*)   (function)

Returns: An immutable list of positive integers, or fail.

If *digraph* is a digraph, then these functions returns an independent set of *digraph* if one exists
that satisfies the arguments, else it returns fail. An independent set is defined by the set of vertices
that it contains; see IsIndependentSet (8.2.1) and IsMaximalIndependentSet (8.2.1).

The optional arguments *include* and *exclude* must each be a (possibly empty) duplicate-free
list of vertices of *digraph*, and the optional argument *size* must be a positive integer. By default,

include and *exclude* are empty. These functions will search for an independent set of *digraph* that includes the vertices of *include* but does not include any vertices in *exclude*; if the argument *size* is supplied, then additionally the independent set will be required to contain precisely *size* vertices.

If *include* is not an independent set, then these functions return `fail`. Otherwise, the functions behave in the following way, depending on the number of arguments:

One or two arguments

If one or two arguments are supplied, then `DigraphIndependentSet` and `DigraphMaximalIndependentSet` greedily enlarge the independent set *include* until it cannot continue. The result is guaranteed to be a maximal independent set. This may or may not return an answer more quickly than using `DigraphMaximalIndependentSets` (8.2.3) with a limit of 1.

Three arguments

If three arguments are supplied, then `DigraphIndependentSet` greedily enlarges the independent set *include* until it cannot continue, although this independent set may not be maximal.

Given three arguments, `DigraphMaximalIndependentSet` returns the maximal independent set returned by `DigraphMaximalIndependentSets(`*digraph*`, `*include*`, `*exclude*`, 1)` if one exists, else `fail`.

Four arguments

If four arguments are supplied, then `DigraphIndependentSet` returns the independent set returned by `DigraphIndependentSets(`*digraph*`, `*include*`, `*exclude*`, 1, `*size*`)` if one exists, else `fail`. This independent set may not be maximal.

Given four arguments, `DigraphMaximalIndependentSet` returns the maximal independent set returned by `DigraphMaximalIndependentSets(`*digraph*`, `*include*`, `*exclude*`, 1, `*size*`)` if one exists, else `fail`.

```
————————————————— Example —————————————————
gap> D := ChainDigraph(6);
<immutable chain digraph with 6 vertices>
gap> DigraphIndependentSet(D);
[ 6, 4, 2 ]
gap> DigraphMaximalIndependentSet(D);
[ 6, 4, 2 ]
gap> DigraphIndependentSet(D, [2, 4]);
[ 2, 4, 6 ]
gap> DigraphMaximalIndependentSet(D, [1, 3]);
[ 1, 3, 6 ]
gap> DigraphIndependentSet(D, [2, 4], [6]);
[ 2, 4 ]
gap> DigraphMaximalIndependentSet(D, [2, 4], [6]);
fail
gap> DigraphIndependentSet(D, [1], [], 2);
[ 1, 3 ]
gap> DigraphMaximalIndependentSet(D, [1], [], 2);
fail
gap> DigraphMaximalIndependentSet(D, [1], [], 3);
[ 1, 3, 5 ]
gap> D := ChainDigraph(IsMutableDigraph, 6);
<mutable digraph with 6 vertices, 5 edges>
```

```
gap> DigraphIndependentSet(D);
[ 6, 4, 2 ]
gap> DigraphMaximalIndependentSet(D);
[ 6, 4, 2 ]
gap> DigraphIndependentSet(D, [2, 4]);
[ 2, 4, 6 ]
gap> DigraphMaximalIndependentSet(D, [1, 3]);
[ 1, 3, 6 ]
gap> DigraphIndependentSet(D, [2, 4], [6]);
[ 2, 4 ]
gap> DigraphMaximalIndependentSet(D, [2, 4], [6]);
fail
gap> DigraphIndependentSet(D, [1], [], 2);
[ 1, 3 ]
gap> DigraphMaximalIndependentSet(D, [1], [], 2);
fail
gap> DigraphMaximalIndependentSet(D, [1], [], 3);
[ 1, 3, 5 ]
```

8.2.3 DigraphMaximalIndependentSets

▷ DigraphMaximalIndependentSets(`digraph[, include[, exclude[, limit[,
size]]]]`)   (function)
▷ DigraphMaximalIndependentSetsReps(`digraph[, include[, exclude[, limit[,
size]]]]`)   (function)
▷ DigraphIndependentSets(`digraph[, include[, exclude[, limit[, size]]]]`)   (function)
▷ DigraphIndependentSetsReps(`digraph[, include[, exclude[, limit[, size]]]]`)
                                                                              (function)
▷ DigraphMaximalIndependentSetsAttr(`digraph`)   (attribute)
▷ DigraphMaximalIndependentSetsRepsAttr(`digraph`)   (attribute)

Returns: An immutable list of lists of positive integers.

If `digraph` is digraph, then these functions and attributes use CliquesFinder (8.1.2) to return independent sets of `digraph`. An independent set is defined by the set of vertices that it contains; see IsMaximalIndependentSet (8.2.1) and IsIndependentSet (8.2.1).

The optional arguments `include` and `exclude` must each be a (possibly empty) list of vertices of `digraph`, the optional argument `limit` must be either a positive integer or `infinity`, and the optional argument `size` must be a positive integer. If not specified, then `include` and `exclude` are chosen to be empty lists, and `limit` is set to `infinity`.

The functions will return as many suitable independent sets as possible, up to the number `limit`. These functions will find independent sets that contain all of the vertices of `include` but do not contain any of the vertices of `exclude` The argument `size` restricts the search to those cliques that contain precisely `size` vertices. If the function or attribute has Maximal in its name, then only maximal independent sets will be returned; otherwise non-maximal independent sets may be returned.

Let G denote the AutomorphismGroup (7.2.2) of the DigraphSymmetricClosure (3.3.12) of the digraph formed from `digraph` by removing loops and ignoring the multiplicity of edges.

Distinct independent sets

DigraphMaximalIndependentSets and DigraphIndependentSets each return a duplicate-

free list of at most *limit* independent sets of `digraph` that satisfy the arguments.

The computation may be significantly faster if `include` and `exclude` are invariant under the action of G on sets of vertices.

Representatives of distinct orbits of independent sets

To use DigraphMaximalIndependentSetsReps or DigraphIndependentSetsReps, the arguments *include* and *exclude* must each be invariant under the action of G on sets of vertices.

If this is the case, then DigraphMaximalIndependentSetsReps and DigraphIndependentSetsReps each return a list of at most *limit* orbits representatives (under the action of G on sets of vertices) of independent sets of `digraph` that satisfy the arguments.

The representatives are not guaranteed to be in distinct orbits. However, if *lim* is not specified, or fewer than *lim* results are returned, then there will be at least one representative from each orbit of maximal independent sets.

```
─────────────── Example ───────────────
gap> D := CycleDigraph(5);
<immutable cycle digraph with 5 vertices>
gap> DigraphMaximalIndependentSetsReps(D);
[ [ 1, 3 ] ]
gap> DigraphIndependentSetsReps(D);
[ [ 1 ], [ 1, 3 ] ]
gap> Set(DigraphMaximalIndependentSets(D));
[ [ 1, 3 ], [ 1, 4 ], [ 2, 4 ], [ 2, 5 ], [ 3, 5 ] ]
gap> DigraphMaximalIndependentSets(D, [1]);
[ [ 1, 3 ], [ 1, 4 ] ]
gap> DigraphIndependentSets(D, [], [4, 5]);
[ [ 1 ], [ 2 ], [ 3 ], [ 1, 3 ] ]
gap> DigraphIndependentSets(D, [], [4, 5], 1, 2);
[ [ 1, 3 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 5);
<mutable digraph with 5 vertices, 5 edges>
gap> DigraphMaximalIndependentSetsReps(D);
[ [ 1, 3 ] ]
gap> DigraphIndependentSetsReps(D);
[ [ 1 ], [ 1, 3 ] ]
gap> Set(DigraphMaximalIndependentSets(D));
[ [ 1, 3 ], [ 1, 4 ], [ 2, 4 ], [ 2, 5 ], [ 3, 5 ] ]
gap> DigraphMaximalIndependentSets(D, [1]);
[ [ 1, 3 ], [ 1, 4 ] ]
gap> DigraphIndependentSets(D, [], [4, 5]);
[ [ 1 ], [ 2 ], [ 3 ], [ 1, 3 ] ]
gap> DigraphIndependentSets(D, [], [4, 5], 1, 2);
[ [ 1, 3 ] ]
```

Chapter 9

Visualising and IO

9.1 Visualising a digraph

9.1.1 Splash

▷ Splash(*str[, opts]*) (function)

Returns: Nothing.

This function attempts to convert the string *str* into a pdf document and open this document, i.e. to splash it all over your monitor.

The string *str* must correspond to a valid dot or LaTeX text file and you must have have GraphViz and pdflatex installed on your computer. For details about these file formats, see https://www.latex-project.org and https://www.graphviz.org.

This function is provided to allow convenient, immediate viewing of the pictures produced by the function DotDigraph (9.1.2).

The optional second argument *opts* should be a record with components corresponding to various options, given below.

path this should be a string representing the path to the directory where you want Splash to do its work. The default value of this option is "~/".

directory
> this should be a string representing the name of the directory in path where you want Splash to do its work. This function will create this directory if does not already exist.
>
> The default value of this option is "tmp.viz" if the option path is present, and the result of DirectoryTemporary (**Reference: DirectoryTemporary**) is used otherwise.

filename
> this should be a string representing the name of the file where *str* will be written. The default value of this option is "vizpicture".

viewer
> this should be a string representing the name of the program which should open the files produced by GraphViz or pdflatex.

type this option can be used to specify that the string *str* contains a LaTeX or dot document. You can specify this option in *str* directly by making the first line "%latex" or "//dot". There is no default value for this option, this option must be specified in *str* or in *opt.type*.

engine

this option can be used to specify the GraphViz engine to use to render a `dot` document. The valid choices are `"dot"`, `"neato"`, `"circo"`, `"twopi"`, `"fdp"`, `"sfdp"`, and `"patchwork"`. Please refer to the GraphViz documentation for details on these engines. The default value for this option is `"dot"`, and it must be specified in `opt.engine`.

filetype

this should be a string representing the type of file which `Splash` should produce. For LaTeX files, this option is ignored and the default value `"pdf"` is used.

This function was written by Attila Egri-Nagy and Manuel Delgado with some minor changes by J. D. Mitchell.

────────────────── Example ──────────────────
```
gap> Splash(DotDigraph(RandomDigraph(4)));
```

9.1.2 DotDigraph

▷ DotDigraph(*digraph*)   (attribute)
▷ DotColoredDigraph(*digraph, vert, edge*)   (operation)
▷ DotVertexLabelledDigraph(*digraph*)   (operation)
▷ DotVertexColoredDigraph(*digraph, vert*)   (operation)
▷ DotEdgeColoredDigraph(*digraph, edge*)   (operation)

Returns: A string.

`DotDigraph` produces a graphical representation of the digraph `digraph`. Vertices are displayed as circles, numbered consistently with `digraph`. Edges are displayed as arrowed lines between vertices, with the arrowhead of each line pointing towards the range of the edge.

`DotColoredDigraph` differs from `DotDigraph` only in that the values in given in the two lists are used to color the vertices and edges of the graph when displayed. The list for vertex colours should be a list of length equal to the number of vertices, containing strings that are accepted by the graphviz software, which is the one used for graph representation. The list for edge colours should be a list of lists with the same shape of the outneighbours of the digraph that contains strings that correspond to colours accepted by the graphviz software. If the lists are not the appropriate size, or have holes then the function will return an error.

`DotVertexColoredDigraph` differs from `DotDigraph` only in that the values in given in the list are used to color the vertices of the graph when displayed. The list for vertex colours should be a list of length equal to the number of vertices, containing strings that are accepted by the graphviz software, which is the one used for graph representation. If the list is not the appropriate size, or has holes then the function will return an error.

`DotEdgeColoredDigraph` differs from `DotDigraph` only in that the values in given in the list are used to color the vertices and edges of the graph when displayed. The list for edge colours should be a list of lists with the same shape of the outneighbours of the digraph that contains strings that correspond to colours accepted by the graphviz software. If the list is not the appropriate size, or has holes then the function will return an error.

`DotVertexLabelledDigraph` differs from `DotDigraph` only in that the values in `DigraphVertexLabels` (5.1.12) are used to label the vertices in the produced picture rather than the numbers 1 to the number of vertices of the digraph.

The output is in `dot` format (also known as `GraphViz`) format. For details about this file format, and information about how to display or edit this format see `https://www.graphviz.org`.

The string returned by `DotDigraph` or `DotVertexLabelledDigraph` can be written to a file using the command `FileString` (**GAPDoc: FileString**).

```
                           Example
gap> adj := List([1 .. 4], x -> [1, 1, 1, 1]);
[ [ 1, 1, 1, 1 ], [ 1, 1, 1, 1 ], [ 1, 1, 1, 1 ], [ 1, 1, 1, 1 ] ]
gap> adj[1][3] := 0;
0
gap> gr := DigraphByAdjacencyMatrix(adj);
<immutable digraph with 4 vertices, 15 edges>
gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>
gap> vertcolors := [];;
gap> vertcolors[1] := "blue";; vertcolors[2] := "red";;
gap> vertcolors[3] := "green";; vertcolors[4] := "yellow";;
gap> edgecolors := [];;
gap> edgecolors[1] := [];; edgecolors[2] := [];;
gap> edgecolors[3] := [];; edgecolors[4] := [];;
gap> edgecolors[1][2] := "lightblue";;
gap> edgecolors[1][3] := "pink";;
gap> edgecolors[1][4] := "purple";;
gap> edgecolors[2][1] := "lightblue";;
gap> edgecolors[2][3] := "pink";;
gap> edgecolors[2][4] := "purple";;
gap> edgecolors[3][1] := "lightblue";;
gap> edgecolors[3][2] := "pink";;
gap> edgecolors[3][4] := "purple";;
gap> edgecolors[4][1] := "lightblue";;
gap> edgecolors[4][2] := "pink";;
gap> edgecolors[4][3] := "purple";;
gap> Print(DotColoredDigraph(D, vertcolors, edgecolors));
//dot
digraph hgn{
node [shape=circle]
1[color=blue, style=filled]
2[color=red, style=filled]
3[color=green, style=filled]
4[color=yellow, style=filled]
1 -> 2[color=lightblue]
1 -> 3[color=pink]
1 -> 4[color=purple]
2 -> 1[color=lightblue]
2 -> 3[color=pink]
2 -> 4[color=purple]
3 -> 1[color=lightblue]
3 -> 2[color=pink]
3 -> 4[color=purple]
4 -> 1[color=lightblue]
4 -> 2[color=pink]
4 -> 3[color=purple]
}
gap> D := EmptyDigraph(3);
<immutable empty digraph with 3 vertices>
gap> vertcolors := [];;
```

```
gap> vertcolors[1] := "blue";; vertcolors[2] := "red";;
gap> vertcolors[3] := "green";;
gap> edgecolors := [];;
gap> edgecolors[1] := [];; edgecolors[2] := [];;
gap> edgecolors[3] := [];;
gap> Print(DotColoredDigraph(D, vertcolors, edgecolors));
//dot
digraph hgn{
node [shape=circle]
1[color=blue, style=filled]
2[color=red, style=filled]
3[color=green, style=filled]
}
gap> D := Digraph([[2], [1, 3], [2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> vertcolors := [];;
gap> vertcolors[1] := "blue";;
gap> vertcolors[2] := "pink";;
gap> vertcolors[3] := "purple";;
gap> edgecolors := [];;
gap> edgecolors[1] := [];; edgecolors[2] := [];;
gap> edgecolors[3] := [];;
gap> edgecolors[1][2] := "green";; edgecolors[2][1] := "green";;
gap> edgecolors[2][3] := "red";; edgecolors[3][2] := "red";;
gap> Print(DotSymmetricColoredDigraph(D, vertcolors, edgecolors));
//dot
graph hgn{
node [shape=circle]

1[color=blue, style=filled]
2[color=pink, style=filled]
3[color=purple, style=filled]
1 -- 2[color=green]
2 -- 3[color=red]
}
gap> D := Digraph([[2, 3], [1, 3], [1]]);
<immutable digraph with 3 vertices, 5 edges>
gap> vertcolors := [];;
gap> vertcolors[1] := "blue";; vertcolors[2] := "red";;
gap> vertcolors[3] := "green";;
gap> edgecolors := [];;
gap> edgecolors[1] := [];; edgecolors[2] := [];;
gap> edgecolors[3] := [];;
gap> edgecolors[1][2] := "orange";; edgecolors[1][3] := "yellow";;
gap> edgecolors[2][1] := "orange";; edgecolors[2][3] := "pink";;
gap> edgecolors[3][1] := "yellow";;
gap> Print(DotColoredDigraph(D, vertcolors, edgecolors));;
//dot
digraph hgn{
node [shape=circle]
1[color=blue, style=filled]
2[color=red, style=filled]
```

```
3[color=green, style=filled]
1 -> 2[color=orange]
1 -> 3[color=yellow]
2 -> 1[color=orange]
2 -> 3[color=pink]
3 -> 1[color=yellow]
}
gap> D := Digraph(IsMutableDigraph, [[2, 3], [1, 3], [1]]);
<mutable digraph with 3 vertices, 5 edges>
gap> vertcolors := [];;
gap> vertcolors[1] := "blue";; vertcolors[2] := "red";;
gap> vertcolors[3] := "green";;
gap> edgecolors := [];;
gap> edgecolors[1] := [];; edgecolors[2] := [];;
gap> edgecolors[3] := [];;
gap> edgecolors[1][2] := "orange";; edgecolors[1][3] := "yellow";;
gap> edgecolors[2][1] := "orange";; edgecolors[2][3] := "pink";;
gap> edgecolors[3][1] := "yellow";;
gap> Print(DotColoredDigraph(D, vertcolors, edgecolors));;
//dot
digraph hgn{
node [shape=circle]
1[color=blue, style=filled]
2[color=red, style=filled]
3[color=green, style=filled]
1 -> 2[color=orange]
1 -> 3[color=yellow]
2 -> 1[color=orange]
2 -> 3[color=pink]
3 -> 1[color=yellow]
}
gap> D;
<mutable digraph with 3 vertices, 5 edges>
gap> DotSymmetricDigraph(gr2){[12 .. 70]};
" hgn{\n\nnode [shape=circle]\n\n1\n2\n3\n4\n1 -- 2\n2 -- 3\n3 -- 3\n3 -"
gap> DotSymmetricDigraph(gr1);
Error, the argument <D> must be a symmetric digraph,
gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>
gap> vertcolors := [];;
gap> vertcolors[1] := "blue";; vertcolors[2] := "red";;
gap> vertcolors[3] := "green";; vertcolors[4] := "yellow";;
gap> Print(DotVertexColoredDigraph(D, vertcolors));
//dot
digraph hgn{
node [shape=circle]
1[color=blue, style=filled]
2[color=red, style=filled]
3[color=green, style=filled]
4[color=yellow, style=filled]
1 -> 2
1 -> 3
```

```
1 -> 4
2 -> 1
2 -> 3
2 -> 4
3 -> 1
3 -> 2
3 -> 4
4 -> 1
4 -> 2
4 -> 3
}
gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>
gap> edgecolors := [];;
gap> edgecolors[1] := [];; edgecolors[2] := [];;
gap> edgecolors[3] := [];; edgecolors[4] := [];;
gap> edgecolors[1][2] := "lightblue";;
gap> edgecolors[1][3] := "pink";;
gap> edgecolors[1][4] := "purple";;
gap> edgecolors[2][1] := "lightblue";;
gap> edgecolors[2][3] := "pink";;
gap> edgecolors[2][4] := "purple";;
gap> edgecolors[3][1] := "lightblue";;
gap> edgecolors[3][2] := "pink";;
gap> edgecolors[3][4] := "purple";;
gap> edgecolors[4][1] := "lightblue";;
gap> edgecolors[4][2] := "pink";;
gap> edgecolors[4][3] := "purple";;
gap> Print(DotEdgeColoredDigraph(D, edgecolors));
//dot
digraph hgn{
node [shape=circle]
1
2
3
4
1 -> 2[color=lightblue]
1 -> 3[color=pink]
1 -> 4[color=purple]
2 -> 1[color=lightblue]
2 -> 3[color=pink]
2 -> 4[color=purple]
3 -> 1[color=lightblue]
3 -> 2[color=pink]
3 -> 4[color=purple]
4 -> 1[color=lightblue]
4 -> 2[color=pink]
4 -> 3[color=purple]
}
gap> FileString("dot/k4.dot", DotDigraph(gr));
154
```

9.1.3 DotSymmetricDigraph

▷ DotSymmetricDigraph(*digraph*)   (attribute)
▷ DotSymmetricColoredDigraph(*digraph, vert, edge*)   (operation)
▷ DotSymmetricVertexColoredDigraph(*digraph, vert*)   (operation)
▷ DotSymmetricEdgeColoredDigraph(*digraph, edge*)   (operation)

    Returns: A string.

    This function produces a graphical representation of the symmetric digraph *digraph*. DotSymmetricDigraph will return an error if *digraph* is not a symmetric digraph. See IsSymmetricDigraph (6.2.14).

    The function DotSymmetricColoredDigraph differs from DotDigraph only in that the values given in the two lists are used to color the vertices and edges of the graph when displayed. The list for vertex colours should be a list of length equal to the number of vertices, containing strings that are accepted by the graphviz software, which is the one used for graph representation. The list for edge colours should be a list of lists with the same shape of the outneighbours of the digraph that contains strings that correspond to colours accepted by the graphviz software. If the list is not the appropriate size, or has holes then the function will return an error.

    The function DotSymmetricVertexColoredDigraph differs from DotDigraph only in that the values in given in the list is used to color the vertices of the graph when displayed. The list for vertex colours should be a list of length equal to the number of vertices, containing strings that are accepted by the graphviz software, which is the one used for graph representation. If the list is not the appropriate size, or has holes then the function will return an error.

    The function DotSymmetricEdgeColoredDigraph differs from DotDigraph only in that the values given in the list are used to color the edges of the graph when displayed. The list for edge colours should be a list of lists with the same shape of the outneighbours, containing strings that are accepted by the graphviz software, which is the one used for graph representation. If the list is not the appropriate size, or has holes then the function will return an error.

    Vertices are displayed as circles, numbered consistently with *digraph*. Since *digraph* is symmetric, for every non-loop edge there is a complementary edge with opposite source and range. DotSymmetricDigraph displays each pair of complementary edges as a single line between the relevant vertices, with no arrowhead.

    The output is in dot format (also known as GraphViz) format. For details about this file format, and information about how to display or edit this format see https://www.graphviz.org.

    The string returned by DotSymmetricDigraph can be written to a file using the command FileString (**GAPDoc: FileString**).

```
_____ Example _____
gap> star := Digraph([[2, 2, 3, 4], [1, 1], [1], [1, 4]]);
<immutable multidigraph with 4 vertices, 9 edges>
gap> IsSymmetricDigraph(star);
true
gap> FileString("dot/star.dot", DotSymmetricDigraph(gr));
gap> D := Digraph([[2], [1, 3], [2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> vertcolors := [];;
gap> vertcolors[1] := "blue";;
gap> vertcolors[2] := "pink";;
gap> vertcolors[3] := "purple";;
gap> edgecolors := [];;
gap> edgecolors[1] := [];; edgecolors[2] := [];;
```

```
gap> edgecolors[3] := [];;
gap> edgecolors[1][2] := "green";; edgecolors[2][1] := "green";;
gap> edgecolors[2][3] := "red";; edgecolors[3][2] := "red";;
gap> Print(DotSymmetricColoredDigraph(D, vertcolors, edgecolors));
//dot
graph hgn{
node [shape=circle]

1[color=blue, style=filled]
2[color=pink, style=filled]
3[color=purple, style=filled]
1 -- 2[color=green]
2 -- 3[color=red]
}
gap> D := Digraph([[2], [1, 3], [2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> vertcolors := [];;
gap> vertcolors[1] := "blue";;
gap> vertcolors[2] := "pink";;
gap> vertcolors[3] := "purple";;
gap> Print(DotSymmetricVertexColoredDigraph(D, vertcolors));
//dot
graph hgn{
node [shape=circle]

1[color=blue, style=filled]
2[color=pink, style=filled]
3[color=purple, style=filled]
1 -- 2
2 -- 3
}
gap> D := Digraph([[2], [1, 3], [2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> edgecolors := [];;
gap> edgecolors[1] := [];; edgecolors[2] := [];;
gap> edgecolors[3] := [];;
gap> edgecolors[1][2] := "green";; edgecolors[2][1] := "green";;
gap> edgecolors[2][3] := "red";; edgecolors[3][2] := "red";;
gap> Print(DotSymmetricEdgeColoredDigraph(D, edgecolors));
//dot
graph hgn{
node [shape=circle]

1
2
3
1 -- 2[color=green]
2 -- 3[color=red]
}
83
```

9.1.4 DotPartialOrderDigraph

▷ DotPartialOrderDigraph(*digraph*) (attribute)

Returns: A string.

This function produces a graphical representation of a partial order digraph *digraph*. DotPartialOrderDigraph will return an error if *digraph* is not a partial order digraph. See IsPartialOrderDigraph (6.4.2).

Since *digraph* is a partial order, it is both reflexive and transitive. The output of DotPartialOrderDigraph is the DotDigraph (9.1.2) of the DigraphReflexiveTransitiveReduction (3.3.14) of *digraph*.

The output is in dot format (also known as GraphViz) format. For details about this file format, and information about how to display or edit this format see https://www.graphviz.org.

The string returned by DotPartialOrderDigraph can be written to a file using the command FileString (**GAPDoc: FileString**).

```
_____ Example _____
gap> poset := Digraph([[1, 4], [2], [2, 3, 4], [4]);
gap> IsPartialOrderDigraph(gr);
true
gap> FileString("dot/poset.dot", DotPartialOrderDigraph(gr));
83
```

9.1.5 DotPreorderDigraph

▷ DotPreorderDigraph(*digraph*) (attribute)
▷ DotQuasiorderDigraph(*digraph*) (attribute)

Returns: A string.

This function produces a graphical representation of a preorder digraph *digraph*. DotPreorderDigraph will return an error if *digraph* is not a preorder digraph. See IsPreorderDigraph (6.4.1).

A preorder digraph is reflexive and transitive but in general it is not anti-symmetric and may have strongly connected components containing more than one vertex. The QuotientDigraph (3.3.9) Q obtained by forming the quotient of *digraph* by the partition of its vertices into the strongly connected components satisfies IsPartialOrderDigraph (6.4.2). Thus every vertex of Q corresponds to a strongly connected component of *digraph*. The output of DotPreorderDigraph displays the DigraphReflexiveTransitiveReduction (3.3.14) of Q with vertices displayed as rounded rectangles labelled by all of the vertices of *digraph* in the corresponding strongly connected component.

The output is in dot format (also known as GraphViz) format. For details about this file format, and information about how to display or edit this format see https://www.graphviz.org.

The string returned by DotPreorderDigraph can be written to a file using the command FileString (**GAPDoc: FileString**).

```
_____ Example _____
gap> preset := Digraph([[1, 2, 4, 5], [1, 2, 4, 5], [3, 4], [4], [1, 2, 4, 5]]);
gap> IsPreorderDigraph(gr);
true
gap> FileString("dot/preset.dot", DotProrderDigraph(gr));
83
```

9.1.6 DotHighlightedDigraph

▷ DotHighlightedDigraph(*digraph, verts[, colour1, colour2]*) (operation)

Returns: A string.

DotHighlightedDigraph produces a graphical representation of the digraph *digraph*, where the vertices in the list *verts*, and edges between them, are drawn with colour *colour1* and all other vertices and edges in *digraph* are drawn with colour *colour2*. If *colour1* and *colour2* are not given then DotHighlightedDigraph uses black and grey respectively.

Note that DotHighlightedDigraph does not validate the colours *colour1* and *colour2* - consult the GraphViz documentation to see what is available. See DotDigraph (9.1.2) for more details on the output.

```
_____ Example _____
gap> digraph := Digraph([[2, 3], [2], [1, 3]]);
<digraph with 3 vertices, 5 edges>
gap> FileString("dot/my_digraph.dot",
> DotHighlightedDigraph(digraph, [1, 2], "red", "black"));
264
```

9.2 Reading and writing digraphs to a file

This section describes different ways to store and read graphs from a file in the Digraphs package.

Graph6

Graph6 is a graph data format for storing undirected graphs with no multiple edges nor loops of size up to $2^{36} - 1$ in printable characters. The format consists of two parts. The first part uses one to eight bytes to store the number of vertices. And the second part is the upper half of the adjacency matrix converted into ASCII characters. For a more detail description see Graph6.

Sparse6

Sparse6 is a graph data format for storing undirected graphs with possibly multiple edges or loops. The maximal number of vertices allowed is $2^{36} - 1$. The format consists of two parts. The first part uses one to eight bytes to store the number of vertices. And the second part only stores information about the edges. Therefore, the *Sparse6* format return a more compact encoding then *Graph6* for sparse graph, i.e. graphs where the number of edges is much less than the number of vertices squared. For a more detail description see Sparse6.

Digraph6

Digraph6 is a new format based on *Graph6*, but designed for digraphs. The entire adjacency matrix is stored, and therefore there is support for directed edges and single-vertex loops. However, multiple edges are not supported.

DiSparse6

DiSparse6 is a new format based on *Sparse6*, but designed for digraphs. In this format the list of edges is partitioned into increasing and decreasing edges, depending whether the edge has its source bigger than the range. Then both sets of edges are written separately in *Sparse6* format with a separation symbol in between.

9.2.1 String

▷ String(*digraph*)   (attribute)

▷ PrintString(*digraph*)   (operation)

    Returns: A string.

    Returns a string `string` such that `EvalString(string)` is equal to `digraph`, and has the same mutability. See `EvalString` (**Reference: EvalString**).

    The methods installed for `String` make some attempts to ensure that `string` has as short a length as possible, but there may exist shorter strings that also evaluate to `digraph`.

    It is possible that `string` may contain escaped special characters. To obtain a representation of `digraph` that can be entered as GAP input, please use `Print` (**Reference: Print**). Note that `Print` for a digraph delegates to `PrintString`, which delegates to `String`.

```
 _____ Example _____
gap> D := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> Print(D);
CycleDigraph(3);
gap> G := PetersenGraph(IsMutableDigraph);
<mutable digraph with 10 vertices, 30 edges>
gap> String(G);
"DigraphFromGraph6String(IsMutableDigraph, \"IheA@GUAo\");"
gap> Print(last);
DigraphFromGraph6String(IsMutableDigraph, "IheA@GUAo");
gap> DigraphFromGraph6String(IsMutableDigraph, "IheA@GUAo");
<mutable digraph with 10 vertices, 30 edges>
```

9.2.2 DigraphFromGraph6String

▷ DigraphFromGraph6String(*[filt,]str*)   (operation)

▷ DigraphFromDigraph6String(*[filt,]str*)   (operation)

▷ DigraphFromSparse6String(*[filt,]str*)   (operation)

▷ DigraphFromDiSparse6String(*[filt,]str*)   (operation)

    Returns: A digraph.

    If `str` is a string encoding a graph in Graph6, Digraph6, Sparse6 or DiSparse6 format, then the corresponding function returns a digraph. In the case of either Graph6 or Sparse6, formats which do not support directed edges, this will be a digraph such that for every edge, the edge going in the opposite direction is also present.

    Each of these functions takes an optional first argument `filt`, which should be either `IsMutableDigraph` (3.1.2) or `IsImmutableDigraph` (3.1.3), and which specifies whether the output digraph shall be mutable or immutable. If no first argument is provided, then an immutable digraph is returned by default.

```
 _____ Example _____
gap> DigraphFromGraph6String("?");
<immutable empty digraph with 0 vertices>
gap> DigraphFromGraph6String("C]");
<immutable symmetric digraph with 4 vertices, 8 edges>
gap> DigraphFromGraph6String("H?AAEM{");
<immutable symmetric digraph with 9 vertices, 22 edges>
gap> DigraphFromDigraph6String("&?");
<immutable empty digraph with 0 vertices>
```

```
gap> DigraphFromDigraph6String(IsMutableDigraph, "&DOOOW?");
<mutable digraph with 5 vertices, 5 edges>
gap> DigraphFromDigraph6String("&CQFG");
<immutable digraph with 4 vertices, 6 edges>
gap> DigraphFromDigraph6String("&IM[SrKLc~lhesbU[F_");
<immutable digraph with 10 vertices, 51 edges>
gap> DigraphFromDiSparse6String(".CaWBGA?b");
<immutable multidigraph with 4 vertices, 9 edges>
```

9.2.3 Graph6String

▷ Graph6String(*digraph*) (operation)
▷ Digraph6String(*digraph*) (operation)
▷ Sparse6String(*digraph*) (operation)
▷ DiSparse6String(*digraph*) (operation)

Returns: A string.

These four functions return a highly compressed string fully describing the digraph *digraph*.

Graph6 and Digraph6 are formats best used on small, dense graphs, if applicable. For larger, sparse graphs use *Sparse6* and *Disparse6* (this latter also preserves multiple edges).

See WriteDigraphs (9.2.6).

```
─────────────── Example ───────────────
gap> gr := Digraph([[2, 3], [1], [1]]);
<immutable digraph with 3 vertices, 4 edges>
gap> Sparse6String(gr);
":Bc"
gap> DiSparse6String(gr);
".Bc{f"
```

9.2.4 DigraphFile

▷ DigraphFile(*filename[, coder][, mode]*) (function)

Returns: An IO file object.

If *filename* is a string representing the name of a file, then DigraphFile returns an IO package file object for that file.

If the optional argument *coder* is specified and is a function which either encodes a digraph as a string, or decodes a string into a digraph, then this function will be used when reading or writing to the returned file object. If the optional argument *coder* is not specified, then the encoding of the digraphs in the returned file object must be specified in the the file extension. The file extension must be one of: .g6, .s6, .d6, .ds6, .txt, .p, or .pickle; more details of these file formats is given below.

If the optional argument *mode* is specified, then it must be one of: "w" (for write), "a" (for append), or "r" (for read). If *mode* is not specified, then "r" is used by default.

If *filename* ends in one of: .gz, .bz2, or .xz, then the digraphs which are read from, or written to, the returned file object are decompressed, or compressed, appropriately.

The file object returned by DigraphFile can be given as the first argument for either of the functions ReadDigraphs (9.2.5) or WriteDigraphs (9.2.6). The purpose of this is to reduce the overhead of recreating the file object inside the functions ReadDigraphs (9.2.5) or WriteDigraphs (9.2.6) when, for example, reading or writing many digraphs in a loop.

The currently supported file formats, and associated filename extensions, are:

graph6 (.g6)

A standard and widely-used format for undirected graphs, with no support for loops or multiple edges. Only symmetric graphs are allowed – each edge is combined with its converse edge to produce a single undirected edge. This format is best used for "dense" graphs – those with many edges per vertex.

sparse6 (.s6)

Unlike graph6, sparse6 has support for loops and multiple edges. However, its use is still limited to symmetric graphs. This format is better-suited to "sparse" graphs – those with few edges per vertex.

digraph6 (.d6)

This format is based on graph6, but stores direction information - therefore is not limited to symmetric graphs. Loops are allowed, but multiple edges are not. Best compression with "dense" graphs.

disparse6 (.ds6)

Any type of digraph can be encoded in disparse6: directions, loops, and multiple edges are all allowed. Similar to sparse6, this has the best compression rate with "sparse" graphs.

plain text (.txt)

This is a human-readable format which stores graphs in the form 0 7 0 8 1 7 2 8 3 8 4 8 5 8 6 8 i.e. pairs of vertices describing edges in a graph. More specifically, the vertices making up one edge must be separated by a single space, and pairs of vertices must be separated by two spaces.

See `ReadPlainTextDigraph` (9.2.13) for a more flexible way to store digraphs in a plain text file.

pickled (`.p` or `.pickle`)

Digraphs are pickled using the IO package. This is particularly good when the `DigraphGroup` (7.2.10) is non-trivial.

```
_____ Example _____
gap> filename := Concatenation(DIGRAPHS_Dir(), "/tst/out/man.d6.gz");;
gap> file := DigraphFile(filename, "w");;
gap> for i in [1 .. 10] do
> WriteDigraphs(file, Digraph([[1, 3], [2], [1, 2]]));
> od;
gap> IO_Close(file);;
gap> file := DigraphFile(filename, "r");;
gap> ReadDigraphs(file, 9);
<immutable digraph with 3 vertices, 5 edges>
gap> IO_Close(file);;
```

9.2.5 ReadDigraphs

▷ `ReadDigraphs(filename[, decoder][, n])` (function)

  Returns: A digraph, or a list of digraphs.

  If `filename` is a string containing the name of a file containing encoded digraphs or an IO file object created using `DigraphFile` (9.2.4), then `ReadDigraphs` returns the digraphs encoded in the

file as a list. Note that if `filename` is a compressed file, which has been compressed appropriately to give a filename extension of .gz, .bz2, or .xz, then ReadDigraphs can read `filename` without it first needing to be decompressed.

If the optional argument *decoder* is specified and is a function which decodes a string into a digraph, then ReadDigraphs will use *decoder* to decode the digraphs contained in `filename`.

If the optional argument `n` is specified, then ReadDigraphs returns the `n`th digraph encoded in the file `filename`.

If the optional argument *decoder* is not specified, then ReadDigraphs will deduce which decoder to use based on the filename extension of `filename` (after removing the compression-related filename extensions .gz, .bz2, and .xz). For example, if the filename extension is .g6, then ReadDigraphs will use the graph6 decoder DigraphFromGraph6String (9.2.2).

The currently supported file formats, and associated filename extensions, are:

graph6 (.g6)
> A standard and widely-used format for undirected graphs, with no support for loops or multiple edges. Only symmetric graphs are allowed – each edge is combined with its converse edge to produce a single undirected edge. This format is best used for "dense" graphs – those with many edges per vertex.

sparse6 (.s6)
> Unlike graph6, sparse6 has support for loops and multiple edges. However, its use is still limited to symmetric graphs. This format is better-suited to "sparse" graphs – those with few edges per vertex.

digraph6 (.d6)
> This format is based on graph6, but stores direction information - therefore is not limited to symmetric graphs. Loops are allowed, but multiple edges are not. Best compression with "dense" graphs.

disparse6 (.ds6)
> Any type of digraph can be encoded in disparse6: directions, loops, and multiple edges are all allowed. Similar to sparse6, this has the best compression rate with "sparse" graphs.

plain text (.txt)
> This is a human-readable format which stores graphs in the form 0 7 0 8 1 7 2 8 3 8 4 8 5 8 6 8 i.e. pairs of vertices describing edges in a graph. More specifically, the vertices making up one edge must be separated by a single space, and pairs of vertices must be separated by two spaces.
>
> See ReadPlainTextDigraph (9.2.13) for a more flexible way to store digraphs in a plain text file.

pickled (.p or .pickle)
> Digraphs are pickled using the IO package. This is particularly good when the DigraphGroup (7.2.10) is non-trivial.

```
_____ Example _____
  gap> ReadDigraphs(
  > Concatenation(DIGRAPHS_Dir(), "/data/graph5.g6.gz"), 10);
  <immutable symmetric digraph with 5 vertices, 8 edges>
  gap> ReadDigraphs(
```

```
 > Concatenation(DIGRAPHS_Dir(), "/data/graph5.g6.gz"), 17);
<immutable symmetric digraph with 5 vertices, 12 edges>
gap> ReadDigraphs(
> Concatenation(DIGRAPHS_Dir(), "/data/tree9.4.txt"));
[ <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges>,
  <immutable digraph with 9 vertices, 8 edges> ]
```

9.2.6 WriteDigraphs

▷ WriteDigraphs(*filename, digraphs[, encoder][, mode]*)   (function)

If `digraphs` is a list of digraphs or a digraph and `filename` is a string or an IO file object created using `DigraphFile` (9.2.4), then `WriteDigraphs` writes the digraphs to the file represented by `filename`. If the supplied filename ends in one of the extensions `.gz`, `.bz2`, or `.xz`, then the file will be compressed appropriately. Excluding these extensions, if the file ends with an extension in the list below, the corresponding graph format will be used to encode it. If such an extension is not included, an appropriate format will be chosen intelligently, and an extension appended, to minimise file size.

For more verbose information on the progress of the function, set the info level of `InfoDigraphs` to 1 or higher, using `SetInfoLevel`.

The currently supported file formats are:

graph6 (.g6)

A standard and widely-used format for undirected graphs, with no support for loops or multiple edges. Only symmetric graphs are allowed – each edge is combined with its converse edge to produce a single undirected edge. This format is best used for "dense" graphs – those with many edges per vertex.

sparse6 (.s6)

Unlike graph6, sparse6 has support for loops and multiple edges. However, its use is still limited to symmetric graphs. This format is better-suited to "sparse" graphs – those with few edges per vertex.

digraph6 (.d6)

This format is based on graph6, but stores direction information - therefore is not limited to symmetric graphs. Loops are allowed, but multiple edges are not. Best compression with "dense" graphs.

disparse6 (.ds6)

Any type of digraph can be encoded in disparse6: directions, loops, and multiple edges are all allowed. Similar to sparse6, this has the best compression rate with "sparse" graphs.

plain text (.txt)

This is a human-readable format which stores graphs in the form 0 7 0 8 1 7 2 8 3 8 4 8 5 8 6 8 i.e. pairs of vertices describing edges in a graph. More specifically, the vertices making up one edge must be separated by a single space, and pairs of vertices must be separated by two spaces.

See `ReadPlainTextDigraph` (9.2.13) for a more flexible way to store digraphs in a plain text file.

pickled (`.p` or `.pickle`)

Digraphs are pickled using the IO package. This is particularly good when the `DigraphGroup` (7.2.10) is non-trivial.

```
———————————————————————— Example ————————————————————————
  gap> grs := [];;
  gap> grs[1] := Digraph([]);
  <immutable empty digraph with 0 vertices>
  gap> grs[2] := Digraph([[1, 3], [2], [1, 2]]);
  <immutable digraph with 3 vertices, 5 edges>
  gap> grs[3] := Digraph([
  > [6, 7], [6, 9], [1, 3, 4, 5, 8, 9],
  > [1, 2, 3, 4, 5, 6, 7, 10], [1, 5, 6, 7, 10], [2, 4, 5, 9, 10],
  > [3, 4, 5, 6, 7, 8, 9, 10], [1, 3, 5, 7, 8, 9], [1, 2, 5],
  > [1, 2, 4, 6, 7, 8]]);
  <immutable digraph with 10 vertices, 51 edges>
  gap> filename := Concatenation(DIGRAPHS_Dir(), "/tst/out/man.d6.gz");;
  gap> WriteDigraphs(filename, grs, "w");
  IO_OK
  gap> ReadDigraphs(filename);
  [ <immutable empty digraph with 0 vertices>,
    <immutable digraph with 3 vertices, 5 edges>,
    <immutable digraph with 10 vertices, 51 edges> ]
```

9.2.7 IteratorFromDigraphFile

▷ IteratorFromDigraphFile(*filename[, decoder]*)   (function)

Returns: An iterator.

If *filename* is a string representing the name of a file containing encoded digraphs, then `IteratorFromDigraphFile` returns an iterator for which the value of `NextIterator` (**Reference: NextIterator**) is the next digraph encoded in the file.

If the optional argument *decoder* is specified and is a function which decodes a string into a digraph, then `IteratorFromDigraphFile` will use *decoder* to decode the digraphs contained in *filename*.

The purpose of this function is to easily allow looping over digraphs encoded in a file when loading all of the encoded digraphs would require too much memory.

To see what file types are available, see `WriteDigraphs` (9.2.6).

```
─────────────────────────────── Example ───────────────────────────────
  gap> filename := Concatenation(DIGRAPHS_Dir(), "/tst/out/man.d6.gz");;
  gap> file := DigraphFile(filename, "w");;
  gap> for i in [1 .. 10] do
  >   WriteDigraphs(file, Digraph([[1, 3], [2], [1, 2]]));
  > od;
  gap> IO_Close(file);;
  gap> iter := IteratorFromDigraphFile(filename);
  <iterator>
  gap> for x in iter do od;
```

9.2.8 DigraphPlainTextLineEncoder

▷ DigraphPlainTextLineEncoder(*delimiter1[, delimiter2], offset*)   (function)
▷ DigraphPlainTextLineDecoder(*delimiter1[, delimiter2], offset*)   (operation)
Returns: A string.

These two functions return a function which encodes or decodes a digraph in a plain text format.

DigraphPlainTextLineEncoder returns a function which takes a single digraph as an argument. The function returns a string describing the edges of that digraph; each edge is written as a pair of integers separated by the string *delimiter2*, and the edges themselves are separated by the string *delimiter1*. *DigraphPlainTextLineDecoder* returns the corresponding decoder function, which takes a string argument in this format and returns a digraph.

If only one delimiter is passed as an argument to *DigraphPlainTextLineDecoder*, it will return a function which decodes a single edge, returning its contents as a list of integers.

The argument *offset* should be an integer, which will describe a number to be added to each vertex before it is encoded, or after it is decoded. This may be used, for example, to label vertices starting at 0 instead of 1.

Note that the number of vertices of a digraph is not stored, and so vertices which are not connected to any edge may be lost.

```
─────────────────────────────── Example ───────────────────────────────
  gap> gr := Digraph([[2, 3], [1], [1]]);
  <immutable digraph with 3 vertices, 4 edges>
  gap> enc := DigraphPlainTextLineEncoder("  ", " ", -1);;
  gap> dec := DigraphPlainTextLineDecoder("  ", " ", 1);;
  gap> enc(gr);
  "0 1  0 2  1 0  2 0"
  gap> dec(last);
  <immutable digraph with 3 vertices, 4 edges>
```

9.2.9 TournamentLineDecoder

▷ TournamentLineDecoder(*str*)   (operation)
Returns: A digraph.

This function takes a string *str*, decodes it, and then returns the tournament [see IsTournament (6.2.15)] which it defines, according to the following rules.

The characters of the string *str* represent the entries in the upper triangle of a tournament's adjacency matrix. The number of vertices n will be detected from the length of the string and will be as large as possible.

The first character represents the possible edge 1 -> 2, the second represents 1 -> 3 and so on until 1 -> n; then the following character represents 2 -> 3, and so on up to the character which represents the edge n-1 -> n.

If a character of the string with corresponding edge i -> j is equal to 1, then the edge i -> j is present in the tournament. Otherwise, the edge i -> j is present instead. In this way, all the possible edges are encoded one-by-one.

```
———————————————————— Example ————————————————————
  gap> gr := TournamentLineDecoder("100001");
  <immutable digraph with 4 vertices, 6 edges>
  gap> OutNeighbours(gr);
  [ [ 2 ], [  ], [ 1, 2, 4 ], [ 1, 2 ] ]
```

9.2.10 AdjacencyMatrixUpperTriangleLineDecoder

▷ AdjacencyMatrixUpperTriangleLineDecoder(*str*)   (operation)

Returns: A digraph.

This function takes a string *str*, decodes it, and then returns the topologically sorted digraph [see DigraphTopologicalSort (5.1.10)] which it defines, according to the following rules.

The characters of the string *str* represent the entries in the upper triangle of a digraph's adjacency matrix. The number of vertices n will be detected from the length of the string and will be as large as possible.

The first character represents the possible edge 1 -> 2, the second represents 1 -> 3 and so on until 1 -> n; then the following character represents 2 -> 3, and so on up to the character which represents the edge n-1 -> n. If a character of the string with corresponding edge i -> j is equal to 1, then this edge is present in the digraph. Otherwise, it is not present. In this way, all the possible edges are encoded one-by-one.

In particular, note that there exists no edge [i, j] if $j \leq i$. In order words, the digraph will be topologically sorted.

```
———————————————————— Example ————————————————————
  gap> gr := AdjacencyMatrixUpperTriangleLineDecoder("100001");
  <immutable digraph with 4 vertices, 2 edges>
  gap> OutNeighbours(gr);
  [ [ 2 ], [  ], [ 4 ], [  ] ]
  gap> gr := AdjacencyMatrixUpperTriangleLineDecoder("111111x111");
  <immutable digraph with 5 vertices, 9 edges>
  gap> OutNeighbours(gr);
  [ [ 2, 3, 4, 5 ], [ 3, 4 ], [ 4, 5 ], [ 5 ], [  ] ]
```

9.2.11 TCodeDecoder

▷ TCodeDecoder(*str*)   (operation)

Returns: A digraph.

If *str* is a string consisting of at least two non-negative integers separated by spaces, then this function will attempt to return the digraph which it defines as a TCode string.

The first integer of the string defines the number of vertices v in the digraph, and the second defines the number of edges e. The following 2e integers should be vertex numbers in the range [0 .. v-1]. These integers are read in pairs and define the digraph's edges. This function will return an error if *str* has fewer than 2e+2 entries.

Note that the vertex numbers will be incremented by 1 in the digraph returned. Hence the string fragment 0 6 will describe the edge [1,7].

```
─────────────────────── Example ───────────────────────
gap> gr := TCodeDecoder("3 2 0 2 2 1");
<immutable digraph with 3 vertices, 2 edges>
gap> OutNeighbours(gr);
[ [ 3 ], [  ], [ 2 ] ]
gap> gr := TCodeDecoder("12 3 0 10 5 2 8 8");
<immutable digraph with 12 vertices, 3 edges>
gap> OutNeighbours(gr);
[ [ 11 ], [  ], [  ], [  ], [  ], [ 3 ], [  ], [  ], [ 9 ], [  ],
  [  ], [  ] ]
```

9.2.12 PlainTextString

▷ PlainTextString(*digraph*)   (operation)
▷ DigraphFromPlainTextString(*s*)   (operation)

Returns: A string.

PlainTextString takes a single digraph, and returns a string describing the edges of that digraph. *DigraphFromPlainTextString* takes such a string and returns the digraph which it describes. Each edge is written as a pair of integers separated by a single space. The edges themselves are separated by a double space. Vertex numbers are reduced by 1 when they are encoded, so that vertices in the string are labelled starting at 0.

Note that the number of vertices of a digraph is not stored, and so vertices which are not connected to any edge may be lost.

The operation DigraphFromPlainTextString takes an optional first argument IsMutableDigraph (3.1.2) or IsImmutableDigraph (3.1.3), which specifies whether the output digraph shall be mutable or immutable. If no first argument is provided, then an immutable digraph is returned by default.

```
─────────────────────── Example ───────────────────────
gap> gr := Digraph([[2, 3], [1], [1]]);
<immutable digraph with 3 vertices, 4 edges>
gap> PlainTextString(gr);
"0 1  0 2  1 0  2 0"
gap> DigraphFromPlainTextString(last);
<immutable digraph with 3 vertices, 4 edges>
```

9.2.13 WritePlainTextDigraph

▷ WritePlainTextDigraph(*filename, digraph, delimiter, offset*)   (function)
▷ ReadPlainTextDigraph(*filename, delimiter, offset, ignore*)   (operation)

These functions write and read a single digraph in a human-readable plain text format as follows: each line contains a single edge, and each edge is written as a pair of integers separated by the string *delimiter*.

filename should be the name of a file which will be written to or read from, and *offset* should be an integer which is added to each vertex number as it is written or read. For example, if WritePlainTextDigraph is called with *offset* -1, then the vertices will be numbered in the file

starting from 0 instead of 1 - ReadPlainTextDigraph would then need to be called with *offset* 1 to convert back to the original graph.

ignore should be a list of characters which will be ignored when reading the graph.

```
——————————————————— Example ———————————————————
gap> gr := Digraph([[1, 2, 3], [1, 1], [2]]);
<immutable multidigraph with 3 vertices, 6 edges>
gap> filename := Concatenation(DIGRAPHS_Dir(), "/tst/out/plain.txt");;
gap> WritePlainTextDigraph(filename, gr, ",", -1);
gap> ReadPlainTextDigraph(filename, ",", 1, ['/', '%']);
<immutable multidigraph with 3 vertices, 6 edges>
```

9.2.14 WriteDIMACSDigraph

▷ WriteDIMACSDigraph(*filename, digraph*)   (operation)

▷ ReadDIMACSDigraph(*filename*)   (operation)

These operations write or read the single symmetric digraph *digraph* to or from a file in DIMACS format, as appropriate. The operation WriteDIMACSDigraph records the vertices and edges of *digraph*. The vertex labels of *digraph* will be recorded only if they are integers. See IsSymmetricDigraph (6.2.14) and DigraphVertexLabels (5.1.12).

The first argument *filename* should be the name of the file which will be written to or read from. A file can contain one symmetric digraph in DIMACS format. If *filename* ends in one of .gz, .bz2, or .xz, then the file is compressed, or decompressed, appropriately.

The DIMACS format is described as follows. Each line in the DIMACS file has one of four types:

- A line beginning with c and followed by any number of characters is a comment line, and is ignored.

- A line beginning with p defines the numbers of vertices and edges the digraph. This line has the format p edge <nr_vertices> <nr_edges>, where <nr_vertices> and <nr_edges> are replaced by the relevant integers. There must be exactly one such line in the file, and it must occur before any of the following kinds of line.

  Although it is required to be present, the value of <nr_edges> will be ignored. The correct number of edges will be deduced from the rest of the information in the file.

- A line of the form e <v> <w>, where <v> and <w> are integers in the range [1 .. <nr_vertices>], specifies that there is a (symmetric) edge in the digraph between the vertices <v> and <w>. A symmetric edge only needs to be defined once; an additional line e <v> <w>, or e <w> <v>, will be interpreted as an additional, multiple, edge. Loops are permitted.

- A line of the form n <v> <label>, where <v> is an integer in the range [1 .. <nr_vertices>] and <label> is an integer, signifies that the vertex <v> has the label <label> in the digraph. If a label is not specified for a vertex, then ReadDIMACSDigraph will assign the label 1, according to the DIMACS specification.

A detailed definition of the DIMACS format can be found at http://mat.gsia.cmu.edu/COLOR/general/ccformat.ps, in Section 2.1. Note that optional descriptor lines, as described in Section 2.1, will be ignored.

```
──────────────────── Example ────────────────────
gap> gr := Digraph([[2], [1, 3, 4], [2, 4], [2, 3]]);
<immutable digraph with 4 vertices, 8 edges>
gap> filename := Concatenation(DIGRAPHS_Dir(),
>                                   "/tst/out/dimacs.dimacs");;
gap> WriteDIMACSDigraph(filename, gr);;
gap> ReadDIMACSDigraph(filename);
<immutable digraph with 4 vertices, 8 edges>
```

Appendix A

Grape to Digraphs Command Map

Below is a table of GRAPE commands with the Digraphs counterparts. The sections in this chapter correspond to the chapters in the GRAPE manual.

A.1 Functions to construct and modify graphs

The table in this section contains more information when viewed in html format.

GRAPE command	Digraphs command	
Graph	Digraph (3.1.7)	
EdgeOrbitsGraph	EdgeOrbitsDigraph (3.1.10)	
NullGraph	NullDigraph (3.5.16)	
CompleteGraph	CompleteDigraph (3.5.11)	
JohnsonGraph	JohnsonDigraph (3.5.23)	
CayleyGraph	CayleyDigraph (3.1.12)	
AddEdgeOrbit	DigraphAddEdgeOrbit (3.3.18)	
RemoveEdgeOrbit	DigraphRemoveEdgeOrbit (3.3.23)	
AssignVertexNames	SetDigraphVertexLabels (5.1.12)	

A.2 Functions to inspect graphs, vertices and edges

The table in this section contains more information when viewed in html format.

GRAPE command	Digraphs command	
`IsGraph`	`IsDigraph` (3.1.1)	
`OrderGraph`	`DigraphNrVertices` (5.1.2)	
`IsVertex(graph, v)`	`v in DigraphVertices(digraph)`	
`VertexName`	`DigraphVertexLabel` (5.1.11)	
`VertexNames`	`DigraphVertexLabels` (5.1.12)	
`Vertices`	`DigraphVertices` (5.1.1)	
`VertexDegree`	`OutDegreeOfVertex` (5.2.10)	
`VertexDegrees`	`OutDegreeSet` (5.2.8)	
`IsLoopy`	`DigraphHasLoops` (6.2.1)	
`IsSimpleGraph`	`IsSymmetricDigraph` (6.2.14)	
`Adjacency`	`OutNeighboursOfVertex` (5.2.11)	
`IsEdge`	`IsDigraphEdge` (5.1.17)	
`DirectedEdges`	`DigraphEdges` (5.1.3)	
`UndirectedEdges`	None	
`Distance`	`DigraphShortestDistance` (5.4.2)	
`Diameter`	`DigraphDiameter` (5.4.1)	
`Girth`	`DigraphUndirectedGirth` (5.4.8)	
`IsConnectedGraph`	`IsStronglyConnectedDigraph` (6.6.6)	
`IsBipartite`	`IsBipartiteDigraph` (6.2.3)	
`IsNullGraph`	`IsNullDigraph` (6.2.7)	
`IsCompleteGraph`	`IsCompleteDigraph` (6.2.5)	

A.3 Functions to determine regularity properties of graphs

The table in this section contains more information when viewed in html format.

GRAPE command	Digraphs command	
`IsRegularGraph`	`IsOutRegularDigraph` (6.5.2)	
`LocalParameters`	None	
`GlobalParameters`	None	
`IsDistanceRegular`	`IsDistanceRegularDigraph` (6.5.4)	
`CollapsedAdjacencyMat`	None	
`OrbitalGraphColadjMats`	None	
`VertexTransitiveDRGs`	None	

A.4 Some special vertex subsets of a graph

The table in this section contains more information when viewed in html format.

GRAPE command	Digraphs command	
`ConnectedComponent`	`DigraphConnectedComponent` (5.4.10)	
`ConnectedComponents`	`DigraphConnectedComponents` (5.4.9)	
`Bicomponents`	`DigraphBicomponents` (5.4.13)	
`DistanceSet`	`DigraphDistanceSet` (5.4.5)	
`Layers`	`DigraphLayers` (5.4.34)	
`IndependentSet`	`DigraphIndependentSet` (8.2.2)	

A.5 Functions to construct new graphs from old

The table in this section contains more information when viewed in html format.

GRAPE command	Digraphs command	
InducedSubgraph	InducedSubdigraph (3.3.3)	
DistanceSetInduced	None	
DistanceGraph	DistanceDigraph (3.3.46)	
ComplementGraph	DigraphDual (3.3.11)	
PointGraph	None	
EdgeGraph	EdgeUndirectedDigraph (3.3.42)	
SwitchedGraph	None	
UnderlyingGraph	DigraphSymmetricClosure (3.3.12)	
QuotientGraph	QuotientDigraph (3.3.9)	
BipartiteDouble	BipartiteDoubleDigraph (3.3.44)	
GeodesicsGraph	None	
CollapsedIndependentOrbitsGraph	None	
CollapsedCompleteOrbitsGraph	None	
NewGroupGraph	None	

A.6 Vertex-Colouring and Complete Subgraphs

The table in this section contains more information when viewed in html format.

GRAPE command	Digraphs command	
VertexColouring	DigraphGreedyColouring (7.3.16)	
CompleteSubgraphs	DigraphCliques (8.1.4)	
CompleteSubgraphsOfGivenSize	DigraphCliques (8.1.4)	

A.7 Automorphism groups and isomorphism testing for graphs

The table in this section contains more information when viewed in html format.

GRAPE command	Digraphs command	
AutGroupGraph	AutomorphismGroup (7.2.2)	
GraphIsomorphism	IsomorphismDigraphs (7.2.17)	
IsIsomorphicGraph	IsIsomorphicDigraph (7.2.15)	
GraphIsomorphismClassRepresentatives	None	

References

[BM04]   John M. Boyer and Wendy J. Myrvold. On the cutting edge: Simplified o(n) planarity by edge addition. *Journal of Graph Algorithms and Applications*, 8(3):241–273, 2004. 6

[BM06]   John M. Boyer and Wendy J. Myrvold. On the cutting edge: Simplified o(n) planarity by edge addition. In *Graph Algorithms and Applications 5*, pages 241–273. WORLD SCIENTIFIC, Jun 2006. 6, 129, 130, 131, 132, 158, 159

[Boy06]   John M. Boyer. A new method for efficiently generating planar graph visibility representations. In Patrick Healy and Nikola S. Nikolov, editors, *Graph Drawing*, pages 508–511, Berlin, Heidelberg, 2006. Springer Berlin Heidelberg. 6

[Boy12]   John M. Boyer. Subgraph homeomorphism via the edge addition planarity algorithm. *Journal of Graph Algorithms and Applications*, 16(2):381–410, 2012. 6

[Bys02]   Jesper Byskov. Chromatic number in time o(2.4023 n) using maximal independent sets. *BRICS Report Series*, 9, 12 2002. 191

[CK86]   R. Calderbank and W. M. Kantor. The geometry of two-weight codes. *Bull. London Math. Soc.*, 18(2):97–122, 1986. 15

[Gab00]   Harold N. Gabow. Path-based depth-first search for strong and biconnected components. *Information Processing Letters*, 74(34):107 – 114, 2000. 108, 154

[JK07]   Tommi Junttila and Petteri Kaski. Engineering an efficient canonical labeling tool for large and sparse graphs. In David Applegate, Gerth Stølting Brodal, Daniel Panario, and Robert Sedgewick, editors, *Proceedings of the Ninth Workshop on Algorithm Engineering and Experiments and the Fourth Workshop on Analytic Algorithms and Combinatorics*, pages 135–149. SIAM, 2007. 6, 163

[Law76]   E. Lawler. A note on the complexity of the chromatic number problem. *Inf. Process. Lett.*, 5:66–67, 1976. 191

[LT79]   Thomas Lengauer and Robert E. Tarjan. A fast algorithm for finding dominators in a flowgraph. *ACM Transactions on Programming Languages and Systems*, 1(1):121–141, 1979. 118

[MP14]   Brendan D. McKay and Adolfo Piperno. Practical graph isomorphism, II. *Journal of Symbolic Computation*, 60(0):94 – 112, 2014. 6, 163

[US14]   Takeaki Uno and Hiroko Satoh. An efficient algorithm for enumerating chordless cycles and chordless paths. In *Discovery Science*, pages 313–324. Springer International Publishing, 2014. 122

[vLS81] J. H. van Lint and A. Schrijver. Construction of strongly regular graphs, two-weight codes and partial geometries by finite fields. *Combinatorica*, 1(1):63–73, 1981. 15

Index